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New exact solutions to a class of coupled nonlinear PDEs

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Abstract

We decomposed a coupled system of nonlinear partial differential equations (NLPDEs) into a set of algebraic equations as well as an ordinary differential equation (ODE) which is solved by using the Exp-function method. This approach not only does not restrict us to a special ODE, but also provides us with new and more general exact travelling wave solutions. The validity and reliability of that is tested by its application to a class of coupled NLPDEs, where the corresponding ODEs are the generalized Riccati equation or the auxiliary ordinary differential equation.

Keywords: Coupled NLPDEs, Exp-function method, exact travelling wave solutions, generalized Riccati equation, auxiliary ordinary differential equation.

1. Introduction

Coupled NLPDEs arise in a variety of scientific fields, especially in fluid mechanics, solid state physics, chemical physics, plasma physics, plasma waves, capillary-gravity waves, etc. Many powerful methods such as the tanh-function method [1], the extended tanh-function method [2-6], the modified extended tanh-function method [13-15], the variational iteration method [4], the Exp-function method [12], the sine-cosine method [3, 16], the extended Fan's sub-equation method [17], the homotopy perturbation method [18], the homogeneous balance method [19], the Adomian decomposition method [20] and some algebraic methods by using the Riccati equation, the generalized Riccati equation and the general elliptic equation [21-27], have been proposed to seek exact travelling wave solutions of the coupled NLPDEs.

Recently, the Exp-function method proposed by He [7] which is a straightforward and effective method, has been successfully applied to find generalized solitary solutions, compact-like solutions and periodic solutions for nonlinear wave equations. Some illustrative examples in Refs. [8-11] show that this method presents a wide application for handling nonlinear wave equations. Since, apparently, this method cannot be readily applied to coupled NLPDEs in a direct manner [12], we decompose a coupled system into a set of algebraic equations and an ODE, which is solved by using the Exp-function method, in order to find generalized solitary solutions of coupled NLPDEs.

In what follows, we first give the steps of the proposed approach in the next section. In section 3, we solve the generalized Riccati equation by using the Exp-function method, which enable us to find new exact travelling wave solutions of some coupled NLPDEs such as the coupled Burgers equations in two different forms, the quasi-nonlinear hyperbolic equations and the (2 +1)-dimensional Boiti-Leon-Pempinelle equation. Section 4 is devoted to find exact solutions of the auxiliary ordinary differential equation by using the Exp-function method which provides us with new exact travelling wave solutions of some coupled NLPDEs such as the coupled equal width wave equations, the coupled KdV equations and two types of the generalized Hirota-Satsuma coupled KdV equations. We solve the coupled Burgers equations in the modified form in section

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5. Finally, a brief conclusion is given in the last section of the paper.

2. The proposed method

The following system of NLPDEs

$$F_1(u,v,u_t,v_t,u_x,v_x,u_y,v_y,u_{xx},v_{xx},\ldots)=0,$$
$$F_2(u,v,u_t,v_t,u_x,v_x,u_y,v_y,u_{xx},v_{xx},\ldots)=0,$$

can be converted to the corresponding system of ODEs

$$G_1(u,v,u',v',u'',v'',\ldots)=0, \quad (1)$$

$$G_2(u,v,u',v',u'',v'',\ldots)=0, \quad (2)$$

by using the transformations
$u=u(\xi)$,
$v=v(\xi)$,
and on the supposition that ξ is a linear combination of t, x and y. As far as possible, Eqs. (1) and (2) are integrated as long as all terms contain derivatives where usually integration constants are considered zero. We use the appropriate ansatz

$u=a_0+a_1v$,
or
$v=a_0+a_1u$,

where a_0 and a_1 are unknown constants, to convert the system of ordinary differential equations (1) and (2) into a system of algebraic equations and an ODE. Having the parameters a_0 and a_1 through solving the system of algebraic equations and exact solutions of the corresponding ODE by using the Exp-function method, we obtain generalized solitary solutions of u and v in a closed form.

3. The generalized Riccati equation

Now, we intend to apply the method to a class of coupled NLPDEs which lead to the generalized Riccati equation (GRE) as the corresponding ODE in the method. In Ref. [28], Xie et al, introduced the GRE

$$\varphi'(\xi)=q\varphi^2(\xi)+p\varphi(\xi)+r \quad (3)$$

where p, q and r are three constants and the prime denotes differentiation with respect to ξ. Since

$$\varphi'(\xi)=q\varphi^2(\xi)+p\varphi(\xi)+r$$
$$=q\left((\varphi+\frac{p}{2q})^2+\frac{4qr-p^2}{4q^2}\right),$$

by the new variables

$$\psi=\varphi+\frac{p}{2q},\ \sigma=\frac{4qr-p^2}{4q^2},\ \eta=q\xi, \quad (4)$$

the solutions of Eq. (3) can be found through the solutions of the Riccati equation

$$\psi'(\eta)=\psi^2(\eta)+\sigma. \quad (5)$$

Recently, Dai et al. in Ref. [21] used the Exp-function method to obtain exact solutions of the Eq. (5), where σ is a constant. As a special case, they found these three sets of solutions

$$\psi_1(\eta)=\frac{f_{-1}\exp(-2\sqrt{-\sigma}\eta)-\sqrt{-\sigma}g_0}{\sqrt{-\frac{1}{\sigma}f_{-1}\exp(-2\sqrt{-\sigma}\eta)+g_0}}, \quad (6)$$

$$\psi_2(\eta)=\frac{-\sqrt{-\sigma}g_{-1}\exp(\sqrt{-\sigma}\eta)+f_1\exp(-\sqrt{-\sigma}\eta)}{g_{-1}\exp(\sqrt{-\sigma}\eta)+\sqrt{-\frac{1}{\sigma}}f_1\exp(-\sqrt{-\sigma}\eta)}, \quad (7)$$

$$\psi_3(\eta)=\frac{\Pi}{\Omega}, \quad (8)$$

where

$$\Pi=-\sqrt{-\sigma}g_1\exp(2\sqrt{-\sigma}\eta)+f_0-\sqrt{-\sigma}g_{-1}\exp(-2\sqrt{-\sigma}\eta)$$

$$\Omega=\frac{\sigma g_0{}^2+f_0{}^2}{4\sigma g_{-1}}\exp(2\sqrt{-\sigma}\eta)+g_0+g_{-1}\exp(-2\sqrt{-\sigma}\eta)$$

f f_0, f_1, f_{-1}, g_0, g_1, g_{-1} are arbitrary constants, but the solution (8) in [21] is not a correct solution of the Eq. (5). We express the correct form of it as:

$$\psi_3(\eta)=\frac{\Pi_1}{\Omega_1}. \tag{9}$$

$$\Pi_1=\frac{-(\sigma g_0^2+f_0^2)}{4\sqrt{-\sigma}g_{-1}}\exp(-2\sqrt{-\sigma}\eta)+f_0 -\frac{g_1(\sigma g_0+\sqrt{-\sigma}f_0)}{f_0-\sqrt{-\sigma}g_0}\exp(2\sqrt{-\sigma}\eta)$$

$$\Omega_1=\frac{\sigma g_0^2+f_0^2}{4\sigma g_{-1}}\exp(-2\sqrt{-\sigma}\eta)+g_0 +g_1\exp(2\sqrt{-\sigma}\eta)$$

In this section, we generate new exact travelling solutions of some coupled NLPDEs based on the given solutions of the Riccati equation (5) by (6), (7) and (9).

3.1 The coupled Burgers equations

We examine the coupled Burgers system of the form

$$u_t-u_{xx}+2uu_x+\alpha(uv)_x=0, \tag{10}$$

$$v_t-v_{xx}+2vv_x+\beta(uv)_x=0, \tag{11}$$

derived by Esipov [30], where α and β are constants. It is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity. Using $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ and the wave variable $\xi=x-ct$, Eqs. (10) and (11) change to

$$-cu'-u''+2uu'+\alpha(uv)'=0, \tag{12}$$

$$-cv'-v''+2vv'+\beta(uv)'=0. \tag{13}$$

Integrating Eqs. (12) and (13) once with respect to ξ and setting the constants of integration to zero, we obtain

$$-cu-u'+u^2+\alpha(uv)=0, \tag{14}$$

$$-cv+v'+v^2+\beta(uv)=0. \tag{15}$$

Let

$$u=a_0+a_1v, \tag{16}$$

where a_0 and a_1 are unknown constants. Substituting Eq. (16) into Eq. (15), we have the GRE

$$v'=(1+\beta a_1)v^2+(-c+\beta a_0)v. \tag{17}$$

Inserting Eqs. (16) and (17) into Eq. (14), yields an equation in terms of powers of v which equating the coefficients of them to zero gets an algebraic system of equations. Solving this system leads to

$$a_0=0,\quad a_1=\frac{\alpha-1}{\beta-1}.$$

Then, Eq. (17) can be converted to

$$v'=\frac{\beta\alpha-1}{\beta-1}v^2-cv, \tag{18}$$

which is a GRE with $q=\frac{\beta\alpha-1}{\beta-1}$, $p=-c$, $r=0$. On account of the relations in (4), (5) and the obtained solutions of Eq. (18), we get the following sets of solutions

(i) The first set

$$v_1=\frac{f_{-1}\exp(-c\xi)-\frac{c(\beta-1)g_0}{2(\beta\alpha-1)}}{\frac{2(\beta\alpha-1)f_{-1}}{c(\beta-1)}\exp(-c\xi)+g_0}+\frac{c(\beta-1)}{2(\beta\alpha-1)},$$

$$u_1=\frac{\alpha-1}{\beta-1}\times \times\left(\frac{f_{-1}\exp(-c\xi)-\frac{c(\beta-1)g_0}{2(\beta\alpha-1)}}{\frac{2(\beta\alpha-1)f_{-1}}{c(\beta-1)}\exp(-c\xi)+g_0}+\frac{c(\beta-1)}{2(\beta\alpha-1)}\right),$$

(ii) The second set

$$v_2 = \frac{\frac{-c(\beta-1)g_{-1}}{2(\beta\alpha-1)}\exp(\frac{c}{2}\xi) + f_1\exp(-\frac{c}{2}\xi)}{g_{-1}\exp(\frac{c}{2}\xi) + \frac{2(\beta\alpha-1)f_1}{c(\beta-1)}\exp(-\frac{c}{2}\xi)} + \frac{c(\beta-1)}{2(\beta\alpha-1)},$$

$$u_2 = \frac{\alpha-1}{\beta-1}\left(\frac{\frac{-c(\beta-1)g_{-1}}{2(\beta\alpha-1)}\exp(\frac{c}{2}\xi) + f_1\exp(-\frac{c}{2}\xi)}{g_{-1}\exp(\frac{c}{2}\xi) + \frac{2(\beta\alpha-1)f_1}{c(\beta-1)}\exp(-\frac{c}{2}\xi)} + \frac{c(\beta-1)}{2(\beta\alpha-1)}\right),$$

(iii) The third set

$$v_3 = \frac{\frac{-(\sigma g_0^2+f_0^2)}{4\sqrt{-\sigma}g_{-1}}\exp(-c\xi) + f_0 - \frac{g_1(\sigma g_0+\sqrt{-\sigma}f_0)}{f_0-\sqrt{-\sigma}g_0}\exp(c\xi)}{\frac{\sigma g_0^2+f_0^2}{4\sigma g_{-1}}\exp(-c\xi) + g_0 + g_1\exp(c\xi)} + \frac{c(\beta-1)}{2(\beta\alpha-1)},$$

$$u_3 = \frac{\alpha-1}{\beta-1}\left(\frac{\frac{-(\sigma g_0^2+f_0^2)}{4\sqrt{-\sigma}g_{-1}}\exp(-c\xi) + f_0 - \frac{g_1(\sigma g_0+\sqrt{-\sigma}f_0)}{f_0-\sqrt{-\sigma}g_0}\exp(c\xi)}{\frac{\sigma g_0^2+f_0^2}{4\sigma g_{-1}}\exp(-c\xi) + g_0 + g_1\exp(c\xi)} + \frac{c(\beta-1)}{2(\beta\alpha-1)}\right),$$

where $f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, c$ are arbitrary constants, $\sigma = \frac{-c^2(\beta-1)^2}{4(\beta\alpha-1)^2}$ and $\xi = x - ct$.

3.2 The (2+1)-dimensional coupled Burgers equations

Consider the following coupled Burgers system of the form [31]

$$u_t - 2uu_x - u_{xx} - u_{yy} - 2vu_y = 0, \quad (19)$$

$$v_t - 2uv_x - v_{xx} - v_{yy} - 2vv_y = 0. \quad (20)$$

Detailed physical descriptions of these coupled Burgers equations can be found in [32]. Using $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ and the wave variable $\xi = x+ky+ct$, Eqs. (19) and (20) change to

$$cu' - 2uu' - u'' - k^2u'' - 2kvu' = 0, \quad (21)$$

$$cv' - 2uv' - v'' - k^2v'' - 2kvv' = 0. \quad (22)$$

Let

$$u = a_0 + a_1 v, \quad (23)$$

where a_0 and a_1 are unknown constants. Substituting Eq. (23) into Eq. (22) and integrating once with setting the constant of integration to zero, we obtain the equation

$$v' = -\left(\frac{a_1+k}{k^2+1}\right)v^2 + \left(\frac{c-2a_0}{k^2+1}\right)v, \quad (24)$$

which is a GRE with $q = \frac{-a_1-k}{k^2+1}$, $p = \frac{c-2a_0}{k^2+1}$, $r=0$. Inserting Eqs. (23) and (24) into Eq. (21), yields an algebraic system of equations in powers of v which equating the coefficients of them to zero gets an algebraic system of equations that leads to the free values for the parameters a_0 and a_1. On account of the relations in (4), (5) and the obtained solutions of Eq. (24), we get the following sets of solutions

(i) The first set

$$v_1 = \frac{f_{-1}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right) - \frac{(c-2a_0)g_0}{2(a_1+k)}}{\frac{2(a_1+k)f_{-1}}{c-2a_0}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right)+g_0} + \frac{c-2a_0}{2(a_1+k)},$$

$$u_1 = a_0 + a_1\left(\frac{f_{-1}\exp(\frac{c-2a_0}{k^2+1}\xi) - \frac{(c-2a_0)g_0}{2(a_1+k)}}{\frac{2(a_1+k)f_{-1}}{c-2a_0}\exp(\frac{c-2a_0}{k^2+1}\xi)+g_0} + \frac{c-2a_0}{2(a_1+k)}\right),$$

(ii) The second set

$$v_2 = \frac{\frac{-(c-2a_0)g_{-1}}{2(a_1+k)}\exp\left(\frac{-(c-2a_0)}{2(k^2+1)}\xi\right) + f_1\exp\left(\frac{c-2a_0}{2(k^2+1)}\xi\right)}{g_{-1}\exp\left(\frac{-(c-2a_0)}{2(k^2+1)}\xi\right) + \frac{2(a_1+k)f_1}{c-2a_0}\exp\left(\frac{c-2a_0}{2(k^2+1)}\xi\right)} + \frac{c-2a_0}{2(a_1+k)},$$

$$u_2 = a_0 + a_1\left(\frac{\frac{-(c-2a_0)g_{-1}}{2(a_1+k)}\exp\left(\frac{-(c-2a_0)}{2(k^2+1)}\xi\right) + f_1\exp\left(\frac{c-2a_0}{2(k^2+1)}\xi\right)}{g_{-1}\exp\left(\frac{-(c-2a_0)}{2(k^2+1)}\xi\right) + \frac{2(a_1+k)f_1}{c-2a_0}\exp\left(\frac{c-2a_0}{2(k^2+1)}\xi\right)} + \frac{c-2a_0}{2(a_1+k)}\right),$$

(iii) The third set

$$v_3 = \frac{\frac{-(\sigma g_0{}^2+f_0{}^2)}{4\sqrt{-\sigma}g_{-1}}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right) + f_0 - \frac{g_1(\sigma g_0+\sqrt{-\sigma}f_0)}{f_0-\sqrt{-\sigma}g_0}\exp\left(\frac{-(c-2a_0)}{k^2+1}\xi\right)}{\frac{\sigma g_0{}^2+f_0^2}{4\sigma g_{-1}}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right) + g_0 + g_1\exp\left(\frac{-(c-2a_0)}{k^2+1}\xi\right)} + \frac{c-2a_0}{2(a_1+k)},$$

$$u_3 = a_0 + a_1\left(\frac{\frac{-(\sigma g_0{}^2+f_0{}^2)}{4\sqrt{-\sigma}g_{-1}}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right) + f_0 - \frac{g_1(\sigma g_0+\sqrt{-\sigma}f_0)}{f_0-\sqrt{-\sigma}g_0}\exp\left(\frac{-(c-2a_0)}{k^2+1}\xi\right)}{\frac{\sigma g_0{}^2+f_0^2}{4\sigma g_{-1}}\exp\left(\frac{c-2a_0}{k^2+1}\xi\right) + g_0 + g_1\exp\left(\frac{-(c-2a_0)}{k^2+1}\xi\right)} + \frac{c-2a_0}{2(a_1+k)}\right),$$

where $f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, c$ are arbitrary constants, $\sigma = \frac{-(c-2a_0)^2}{4(a_1+k)^2}$ and $\xi = x + ky + ct$.

3.3 The system of nonlinear hyperbolic equations

Consider the nonlinear hyperbolic system [33]

$$u_t + u_x + \alpha uv = 0, \tag{25}$$

$$v_t - v_x + \alpha uv = 0, \tag{26}$$

which represents interaction of the two waves travelling in the opposite directions, where α is a constant. Using $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ and the wave variable $\xi=x-ct$, Eqs. (25) and (26) change to

$$-cu'+u'+\alpha uv=0, \quad (27)$$

$$-cv'-v'+\alpha uv=0. \quad (28)$$

Let

$$u=a_0+a_1 v, \quad (29)$$

where a_0 and a_1 are unknown constants. Substituting Eq. (29) into Eq. (28), we have the GRE

$$v'=\frac{ca_1}{c+1}v^2+\frac{\alpha a_0}{c+1}v. \quad (30)$$

Inserting Eqs. (29) and (30) into Eq. (27), yields an equation in terms of powers of v which equating the coefficients of them to zero gets an algebraic system of equations. Solving this system leads to

$$a_1=\frac{c+1}{c-1}.$$

Then, Eq. (30) can be converted to the equation

$$v'=\frac{\alpha}{c-1}v^2+\frac{\alpha a_0}{c+1}v, \quad (31)$$

which is a GRE with $q=\frac{\alpha}{c-1}$, $p=\frac{\alpha a_0}{c+1}$, $r=0$. On account of the relations in (4), (5) and the obtained solutions of Eq. (31), we get the following sets of solutions

(i) The first set

$$v_1=\frac{f_{-1}\exp\left(\frac{-\alpha a_0}{c+1}\xi\right)-\frac{a_0(c-1)g_0}{2(c+1)}}{\frac{2(c+1)f_{-1}}{a_0(c-1)}\exp\left(\frac{-\alpha a_0}{c+1}\xi\right)+g_0}-\frac{a_0(c-1)}{2(c+1)},$$

$$u_1=a_0+\frac{c+1}{c-1}\left(\frac{f_{-1}\exp\left(\frac{-\alpha a_0}{c+1}\xi\right)-\frac{a_0(c-1)g_0}{2(c+1)}}{\frac{2(c+1)f_{-1}}{a_0(c-1)}\exp\left(\frac{-\alpha a_0}{c+1}\xi\right)+g_0}-\frac{a_0(c-1)}{2(c+1)}\right),$$

(ii) The second set

$$v_2=\frac{\frac{-a_0(c-1)g_{-1}}{2(c+1)}\exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)+f_1\exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right)}{g_{-1}\exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)+\frac{2(c+1)}{a_0(c-1)}f_1\exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right)}-\frac{a_0(c-1)}{2(c+1)},$$

$$u_2=a_0+\frac{c+1}{c-1}\left(\frac{\frac{-a_0(c-1)g_{-1}}{2(c+1)}\exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)+f_1\exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right)}{g_{-1}\exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)+\frac{2(c+1)}{a_0(c-1)}f_1\exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right)}-\frac{a_0(c-1)}{2(c+1)}\right),$$

(iii) The third set

$$v_3 = \frac{\frac{-(\sigma g_0^2 + f_0^2)}{4\sqrt{-\sigma} g_{-1}} \exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right) + f_0 - \frac{g_1(\sigma g_0 + \sqrt{-\sigma} f_0)}{f_0 - \sqrt{-\sigma} g_0} \exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)}{\frac{\sigma g_0^2 + f_0^2}{4\sigma g_{-1}} \exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right) + g_0 + g_1 \exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)} - \frac{a_0(c-1)}{2(c+1)},$$

$$u_3 = a_0 + \frac{c+1}{c-1}\left(\frac{\frac{-(\sigma g_0^2 + f_0^2)}{4\sqrt{-\sigma} g_{-1}} \exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right) + f_0 - \frac{g_1(\sigma g_0 + \sqrt{-\sigma} f_0)}{f_0 - \sqrt{-\sigma} g_0} \exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)}{\frac{\sigma g_0^2 + f_0^2}{4\sigma g_{-1}} \exp\left(\frac{-\alpha a_0}{2(c+1)}\xi\right) + g_0 + g_1 \exp\left(\frac{\alpha a_0}{2(c+1)}\xi\right)} - \frac{a_0(c-1)}{2(c+1)}\right),$$

where $a_0, f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, c$ are arbitrary constants, $\sigma = \frac{-a_0^2(c-1)^2}{4(c+1)^2}$ and $\xi = x - ct$.

3.4 The (2 + 1)-dimensional Boiti-Leon-Pempinelle equation

Now, we examine the (2 + 1)-dimensional Boiti-Leon-Pempinelle equation of the form [34]

$$u_{yt} - (u^2)_{xy} + u_{xxy} - 2v_{xxx} = 0 \quad (32)$$

$$v_t - v_{xx} - 2uv_x = 0. \quad (33)$$

Using $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$ and the wave variable $\xi = x + y - ct$, Eqs. (32) and (33) change to

$$-cu'' - (u^2)'' + u''' - 2v''' = 0, \quad (34)$$

$$-cv' - v'' - 2uv' = 0. \quad (35)$$

Integrating Eq. (34) twice with respect to ξ and setting constants of integration to zero, we obtain

$$-cu - u^2 + u' - 2v' = 0, \quad (36)$$

$$-cv' - v'' - 2uv' = 0. \quad (37)$$

Let

$$v = a_0 + a_1 u, \quad (38)$$

where a_0 and a_1 are unknown constants. Substituting Eq. (38) into Eq. (36), we have the GRE

$$u' = \frac{1}{2a-1}u^2 + \frac{c}{1-2a_1}u. \quad (39)$$

Inserting Eqs. (38) and (39) into Eq. (37), yields an equation in terms of powers of u which equating the coefficients of them to zero gets an algebraic system of equations. Solving this system leads to

$$a_1 = 1.$$

Then, Eq. (39) can be converted to the equation

$$u' = -u^2 - cu, \quad (40)$$

which is a GRE with $q=-1$, $p = -c$, $r = 0$. On account of the relations in (4), (5) and the obtained solutions of Eq. (40), we get the following sets of solutions

(i) The first set

$$u_1 = \frac{f_{-1}\exp(c\xi) - \frac{cg_0}{2}}{\frac{2f_{-1}}{c}\exp(c\xi) + g_0} - \frac{c}{2},$$

$$v_1 = a_0 + \frac{f_{-1}\exp(c\xi) - \frac{cg_0}{2}}{\frac{2f_{-1}}{c}\exp(c\xi) + g_0} - \frac{c}{2},$$

(ii) The second set

$$u_2 = \frac{\frac{-cg_{-1}}{2}\exp(\frac{-c}{2}\xi) + f_1 \exp(\frac{c}{2}\xi)}{g_{-1}\exp(\frac{-c}{2}\xi) + \frac{2f_1}{c}\exp(\frac{c}{2}\xi)} - \frac{c}{2},$$

$$v_2 = a_0 + \frac{\frac{-cg_{-1}}{2}\exp(\frac{-c}{2}\xi) + f_1 \exp(\frac{c}{2}\xi)}{g_{-1}\exp(\frac{-c}{2}\xi) + \frac{2f_1}{c}\exp(\frac{c}{2}\xi)} - \frac{c}{2},$$

(iii) The third set

$$u_3 = \frac{\frac{-(\sigma g_0^2 + f_0^2)}{4\sqrt{-\sigma}g_{-1}}\exp(c\xi) + f_0 - \frac{g_1(\sigma g_0 + \sqrt{-\sigma}f_0)}{f_0 - \sqrt{-\sigma}g_0}\exp(-c\xi)}{\frac{\sigma g_0^2 + f_0^2}{4\sigma g_{-1}}\exp(c\xi) + g_0 + g_1\exp(-c\xi)} - \frac{c}{2},$$

$$v_3 = a_0 + \frac{\frac{-(\sigma g_0^2 + f_0^2)}{4\sqrt{-\sigma}g_{-1}}\exp(c\xi) + f_0 - \frac{g_1(\sigma g_0 + \sqrt{-\sigma}f_0)}{f_0 - \sqrt{-\sigma}g_0}\exp(-c\xi)}{\frac{\sigma g_0^2 + f_0^2}{4\sigma g_{-1}}\exp(c\xi) + g_0 + g_1\exp(-c\xi)} - \frac{c}{2},$$

where $a_0, f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, c$ are arbitrary constants, $\sigma = \frac{-c^2}{4}$ and $\xi = x + y - ct$.

Remark. We only considered the one set of solutions reported in [21] for brevity.

4. The auxiliary ordinary differential equation

In the sequel, we would like to apply the method to a class of coupled NLPDEs which lead to the auxiliary ordinary differential equation (AODE) [29]

$$(\varphi'(\xi))^2 = h_2\varphi^2(\xi) + h_3\varphi^3(\xi) + h_4\varphi^4(\xi), \quad (41)$$

as the corresponding ODE in the proposed method, where h_2, h_3 and h_4 are arbitrary constants and the prime denotes differentiation with respect to ξ. Hence, we use the Exp-function method to obtain exact solutions of the AODE which is a special case of the general elliptic equation. By introducing the variable η as

$$\eta = k\xi, \quad (42)$$

where k is a constant to be determined later, the Eq. (41) becomes

$$k^2\varphi'^2 - h_2\varphi^2 - h_3\varphi^3 - h_4\varphi^4 = 0, \quad (43)$$

where prime denotes the derivative with respect to η. According to the Exp-function method, we assume that the solution of Eq. (43) can be expressed in the following form

$$\varphi(\eta) = \frac{\sum_{n=-d}^{e} f_n \exp(n\eta)}{\sum_{m=-p}^{q} g_m \exp(m\eta)}, \quad (44)$$

where f_n and g_m are unknown constants to be determined, p, q, e amd d are positive integers which are given by the homogeneous balance principle. To determine the values of q and e, we balance the linear term of highest order with the highest order nonlinear term in Eq. (43) to have

$$\varphi'^2 = \frac{b_1 \exp[(2q+2e)\eta] + \cdots}{b_2 \exp(4q\eta) + \cdots}, \qquad (45)$$

and

$$\varphi^4 = \frac{b_3 \exp(4e\eta) + \cdots}{b_4 \exp(4q\eta) + \cdots}, \qquad (46)$$

where b_i are coefficients for simplicity. Balancing highest order of Exp-function in Eqs. (45) and (46), we have $e = q$. In the same manner as illustrated above, we find $d = p$. For simplicity, we set $e = q = 1$ and $d = p = 1$, then Eq. (44) reduces to

$$\varphi(\eta) = \frac{f_1 \exp(\eta) + f_0 + f_{-1}\exp(-\eta)}{g_1 \exp(\eta) + g_0 + g_{-1}\exp(-\eta)}. \qquad (47)$$

Substituting Eq. (47) into Eq. (43), and equating to zero the coefficients of all powers of $\exp(\eta)$ yields a set of algebraic equations for f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, c and k. Solving this system gets this set of solution

$$\begin{cases} f_1 = f_{-1} = 0, & f_0 = -\dfrac{2h_2 g_0}{h_3}, \\ g_1 = -\dfrac{1}{4}\dfrac{g_0^2(4h_4 h_2 - h_3^2)}{g_{-1}h_3^2}, & k = \sqrt{h_2}, \end{cases}$$

where g_0 and g_{-1} are arbitrary constants. In view of the above set, we have the following exact solution of the AODE (41)

$$\varphi(\xi) = \frac{-\dfrac{2h_2 g_0}{h_3}}{-\dfrac{1}{4}\dfrac{g_0^2(4h_2 h_3 - h_3^2)}{g_{-1}h_3^2}\exp(\sqrt{h_2}\,\xi) + g_0 + g_{-1}\exp(-\sqrt{h_2}\,\xi)}. \qquad (48)$$

4.1 The coupled equal width wave equations

Now, we examine the coupled equal width wave system, in the normalized form [16]

$$u_t + uu_x - u_{xxt} + vv_x = 0, \qquad (49)$$

$$v_t + vv_x - v_{xxt} = 0, \qquad (50)$$

where the boundary conditions u, u', $u'' \to b_1$ and v, v', $v'' \to b_2$ as $x \to \mp\infty$ such that b_1, b_2 are arbitrary constants. Using $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$ and the wave variable $\xi = x - ct$, Eqs. (49) and (50) change to

$$-cu' + uu' + cu''' + vv' = 0, \qquad (51)$$

$$-cv' + vv' + cv''' = 0. \qquad (52)$$

Integrating Eqs. (51) and (52) once with respect to ξ and setting the constants of integration to zero, we obtain

$$-cu + \frac{u^2}{2} + cu'' + \frac{v^2}{2} = 0, \qquad (53)$$

$$-cv + \frac{v^2}{2} + cv'' = 0. \qquad (54)$$

Let

$$v = a_0 + a_1 u, \qquad (55)$$

where a_0 and a_1 are unknown constants. Substituting Eq. (55) into Eq. (54), we have

$$u'' = -\frac{a_1}{2c}u^2 + (1 - \frac{a_0}{c})u + (\frac{a_0}{a_1} - \frac{a_0^2}{2ca_1}), \qquad (56)$$

or

$$(u')^2 = (\frac{2a_0}{a_1} - \frac{a_0^2}{ca_1})u + (1 - \frac{a_0}{c})u^2 - \frac{a_1}{3c}u^3. \qquad (57)$$

Inserting Eqs. (56) and (55) into Eq. (53) yields an equation in terms of powers of u which, equating the coefficients of them to zero, gets an algebraic system of equations. Solving this system leads to

$$a_0 = 0, \quad a_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Then, Eq. (57) can be converted to

$$(u')^2 = u^2 - \frac{1 + i\sqrt{3}}{6c} u^3, \tag{58}$$

which is an AODE with $h_2 = 1$, $h_3 = -\frac{1 + i\sqrt{3}}{6c}$ and $h_4 = 0$. On account of the obtained solutions of Eq. (58) based on Eq. (48), we get the following sets of solutions

$$u = \frac{6cg_0}{(\frac{1}{2} + \frac{i\sqrt{3}}{2})\left(\frac{1}{4}\frac{g_0{}^2}{g_{-1}}\exp(\pm\xi) + g_0 + g_{-1}\exp(\mp\xi)\right)},$$

$$v = \frac{6cg_0}{\frac{1}{4}\frac{g_0{}^2}{g_{-1}}\exp(\pm\xi) + g_0 + g_{-1}\exp(\mp\xi)},$$

where g_0, g_{-1} c are arbitrary constants and $\xi = x - ct$.

4.2 The coupled KdV equations

Consider the coupled KdV system of the form [35]

$$u_t + 6\alpha u u_x - 2bvv_x + \alpha u_{xxx} = 0, \tag{59}$$

$$v_t + 3\beta u v_x + \beta v_{xxx} = 0, \tag{60}$$

derived by Hirota and Satsuma to model the interaction of two long water waves with different dispersion relation, where α, β and b are constants. Using $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$ and the wave variable $\xi = x - ct$, Eqs. (59) and (60) change to

$$-cu' + 6\alpha uu' - 2bvv' + \alpha u''' = 0, \tag{61}$$

$$-cv' + 3\beta uv' + \beta v''' = 0. \tag{62}$$

Integrating Eq. (61) once with respect to ξ, we obtain

$$-cu + 3\alpha u^2 - bv^2 + \alpha u'' = l_1, \tag{63}$$

$$-cv' + 3\beta uv' + \beta v''' = 0, \tag{64}$$

where l_1 is an integration constant. Let

$$u = a_0 + a_1 v, \tag{65}$$

where a_0 and a_1 are unknown constants. Substituting Eq. (65) into Eq. (64) and integrating once with respect to ξ, where l_2 is an integration constant, we have

$$v'' = \frac{-3a_1}{2} v^2 + \frac{(c - 3\beta a_0)}{\beta} v + l_2, \tag{66}$$

or

$$(v')^2 = 2l_2 v + \frac{(c - 3\beta a_0)}{\beta} v^2 - a_1 v^3. \tag{67}$$

Inserting Eqs. (66) and (65) into Eq. (63) and on the supposition that $l_2 = 0$, yields an equation in terms of powers of v which, equating the coefficients of them to zero, gets an algebraic system of equations. Solving this system leads to

$$a_0 = \frac{c(\beta - \alpha)}{3\alpha\beta}, \quad a_1 = \sqrt{\frac{2b}{3\alpha}}, \quad l_1 = \frac{c^2(\alpha - \beta)}{3\beta^2}.$$

Then, Eq. (67) can be converted to

$$(v')^2 = \frac{c(2\alpha - \beta)}{\alpha\beta} v^2 - \sqrt{\frac{2b}{3\alpha}} v^3, \tag{68}$$

which is an AODE with $h_2 = \frac{c(2\alpha - \beta)}{\alpha\beta}$, $h_3 = -\sqrt{\frac{2b}{3\alpha}}$ and $h_4 = 0$. On account of the obtained solutions of Eq. (68) based on Eq. (48), we get the following sets of solutions

$$v=\frac{c\sqrt{6}(2\alpha-\beta)g_0}{\beta\sqrt{\alpha b}\left(\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{\frac{c(2\alpha-\beta)}{\alpha\beta}}\xi)+g_0+g_{-1}\exp(-\sqrt{\frac{c(2\alpha-\beta)}{\alpha\beta}}\xi)\right)},$$

$$u=\frac{c(\beta-\alpha)}{3\alpha\beta}+\frac{2c(2\alpha-\beta)g_0}{\alpha\beta\left(\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{\frac{c(2\alpha-\beta)}{\alpha\beta}}\xi)+g_0+g_{-1}\exp(-\sqrt{\frac{c(2\alpha-\beta)}{\alpha\beta}}\xi)\right)},$$

where g_0, g_{-1}, c are arbitrary constants and $\xi=x-ct$.

4.3 The generalized Hirota–Satsuma coupled KdV equations I

Consider the generalized Hirota-Satsuma coupled KdV system [26]

$$u_t-\frac{1}{2}u_{xxx}+3uu_x-3(vw)_x=0, \tag{69}$$

$$v_t+v_{xxx}-3uv_x=0, \tag{70}$$

$$w_t+w_{xxx}-3uw_x=0, \tag{71}$$

given by Wu et al. This system reduces to complex coupled KdV system and Hirota-Satsuma equation with $w=v^*$ and $w=v$, respectively. Using $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ and the wave variable $\xi=x-ct$, Eqs. (69), (70) and (71) change to

$$-cu'-\frac{1}{2}u'''+3uu'-3(vw)'=0, \tag{72}$$

$$-cv'+v'''-3uv'=0, \tag{73}$$

$$-cw'+w'''-3uw'=0. \tag{74}$$

Integrating Eq. (72) once with respect to ξ and setting the constant of integration to zero, we obtain

$$-cu-\frac{1}{2}u''+\frac{3}{2}u^2-3vw=0, \tag{75}$$

$$-cv'+v'''-3uv'=0, \tag{76}$$

$$-cw'+w'''-3uw'=0. \tag{77}$$

Let

$$v=a_0+a_1u, \tag{78}$$

and

$$w=b_0+b_1u, \tag{79}$$

where a_0, a_1, b_0 and b_1 are unknown constants.

Substituting Eq. (78) into Eq. (76) or Eq. (79) into Eq. (77), we have

$$u''=\frac{3}{2}u^2+cu, \tag{80}$$

or

$$(u')^2=cu^2+u^3, \tag{81}$$

which is an AODE with $h_2=c$, $h_3=1$ and $h_4=0$. Inserting Eqs. (80), (79) and (78) into Eq. (75) yields an equation in terms of powers of u which equating the coefficients of them to zero gets an algebraic system of equations. Solving this system leads to

$$a_0=0,\quad b_0=\frac{-c}{2a_1},\quad b_1=\frac{1}{4a_1},$$

and

$$a_0=-2a_1c,\ b_0=0,\quad b_1=\frac{1}{4a_1}.$$

On account of the obtained solutions of Eq. (81) based on Eq. (48), we get the following sets of solutions

(i) The first set

$$u_1 = \frac{-2cg_0}{\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)},$$

$$v_1 = \frac{-2ca_1g_0}{\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)},$$

$$w_1 = -\frac{c}{2a_1} - \frac{cg_0}{2a_1\left(\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)\right)},$$

(ii) The second set

$$u_2 = \frac{-2cg_0}{\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)},$$

$$v_2 = -2a_1c - \frac{2ca_1g_0}{\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)},$$

$$w_2 = \frac{-cg_0}{2a_1\left(\frac{1}{4}\frac{{g_0}^2}{g_{-1}}\exp(\sqrt{c}\xi) + g_0 + g_{-1}\exp(-\sqrt{c}\xi)\right)},$$

where g_0, g_{-1}, c are arbitrary constants and $\xi = x - ct$.

4.4 The generalized Hirota–Satsuma coupled KdV equations II

Now, consider the following generalized Hirota-Satsuma coupled KdV system [26]

$$u_t - \frac{1}{4}u_{xxx} + 3uu_x - 3(w - v^2)_x = 0, \quad (82)$$

$$v_t + \frac{1}{2}v_{xxx} + 3uv_x = 0, \quad (83)$$

$$w_t + \frac{1}{2}w_{xxx} + 3uw_x = 0, \quad (84)$$

proposed by Satsuma and Hirota. They found its three-soliton solutions and showed that the Hirota-Satsuma equation is a special case of it with $w = 0$ and scaling transformation $x \to \sqrt{2}x$, $t \to \sqrt{2}t$. Using $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$ and the wave variable $\xi = x - ct$, Eqs. (82), (83) and (84) change to

$$-cu' - \frac{1}{4}u''' + 3uu' - 3(w - v^2)' = 0, \quad (85)$$

$$-cv' + \frac{1}{2}v''' + 3uv' = 0, \quad (86)$$

$$-cw' + \frac{1}{2}w''' + 3uw' = 0. \quad (87)$$

Integrating Eq. (85) once with respect to ξ and setting the constant of integration to zero, we obtain

$$-cu-\frac{1}{4}u''+\frac{3}{2}u^2-3(w-v^2)=0, \quad (88)$$

$$-cv'+\frac{1}{2}v'''+3uv'=0, \quad (89)$$

$$-cw'+\frac{1}{2}w'''+3uw'=0. \quad (90)$$

Let

$$v=a_0+a_1u, \quad (91)$$

and

$$w=b_0+b_1u, \quad (92)$$

where a_0, a_1, b_0 and b_1 are unknown constants.

Substituting Eq. (91) into Eq. (89) or Eq. (92) into Eq. (90), we have

$$u''=-3u^2+2cu, \quad (93)$$

or

$$(u')^2=2cu^2-2u^3, \quad (94)$$

which is an AODE with $h_2=2c$, $h_3=-2$ and $h_4=0$. Inserting Eqs. (93), (92) and (91) into Eq. (88), yields an equation in terms of powers of u which equating the coefficients of them to zero gets an algebraic system of equations. Solving this system leads to

$$a_1=\frac{1}{2}i\sqrt{3},\quad b_0=a_0{}^2,\quad b_1=-\frac{c}{2}+a_0 i\sqrt{3}.$$

On account of the obtained solutions of Eq. (94) based on Eq. (48), we get the following set of solutions

$$u=\frac{2cg_0}{g_{-1}\exp(-\sqrt{2c}\xi)+g_0+\frac{1}{4}\frac{g_0{}^2}{g_{-1}}\exp(\sqrt{2c}\xi)},$$

$$v=a_0+\frac{1}{2}i\sqrt{3}\left(\frac{2cg_0}{g_{-1}\exp(-\sqrt{2c}\xi)+g_0+\frac{1}{4}\frac{g_0{}^2}{g_{-1}}\exp(\sqrt{2c}\xi)}\right),$$

$$w=a_0{}^2+(-\frac{c}{2}+a_0 i\sqrt{3})\left(\frac{2cg_0}{g_{-1}\exp(-\sqrt{2c}\xi)+g_0+\frac{1}{4}\frac{g_0{}^2}{g_{-1}}\exp(\sqrt{2c}\xi)}\right),$$

where g_0, g_{-1}, c are arbitrary constants and $\xi=x-ct$.

4.4 The modified coupled Burgers equations

Now, we consider the coupled Burgers equations in the modified form

$$u_t-u_{xx}+3u^2u_x+\alpha(uv)_x=0, \quad (95)$$

$$v_t-v_{xx}+3v^2v_x+\beta(uv)_x=0, \quad (96)$$

where α and β are constants. Using $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$ and the wave variable $\xi=x-ct$, Eqs. (95) and (96) change to

$$-cu'-u''+3u^2u'+\alpha(uv)'=0, \quad (97)$$

$$-cv'-v''+3v^2v'+\beta(uv)'=0. \quad (98)$$

Integrating Eqs. (97) and (98) once with respect to ξ, we obtain

$$-cu-u'+u^3+\alpha(uv)=l_1, \quad (99)$$

$$-cv-v'+v^3+\beta(uv)=l_2, \quad (100)$$

where l_1 and l_2 are integration constants. Now, let

$$u = a_0 + a_1 v, \tag{101}$$

where a_0 and a_1 are unknown constants. Substituting Eq. (101) into Eq. (100) yields the following equation

$$v' = v^3 + \beta a_1 v^2 + (-c + \beta a_0)v - l_2. \tag{102}$$

Inserting Eqs. (101) and (102) into Eq. (99), and on the supposition that $l_1 = 0$, we get an equation in terms of powers of v which, equating the coefficients of them to zero, gets an algebraic system of equations. Solving this system leads to

$$a_0 = -\frac{\alpha}{3} + \frac{\beta}{3}, \quad a_1 = 1, \quad l_2 = -\frac{c\alpha}{3} + \frac{c\beta}{3} + \frac{\alpha^3}{27} - \frac{\alpha^2\beta}{9} + \frac{\alpha\beta^2}{9} - \frac{\beta^3}{27}.$$

Then, Eq. (102) can be converted to the equation

$$v' = v^3 + \beta v^2 + (-c - \frac{\alpha\beta}{3} + \frac{\beta^2}{3})v - l_2. \tag{103}$$

Now, suppose

$$v = w + A. \tag{104}$$

Substituting Eq. (104) into Eq. (103), we get

$$w' = w^3 + (3A + \beta)w^2 + \left(3A^2 - c - \frac{\alpha\beta}{3} + \frac{\beta^2}{3} + 2\beta A\right)w + A^3 + \beta A^2 - cA - \frac{\alpha\beta A}{3} + \frac{\beta^2 A}{3} - l_2. \tag{105}$$

Now, setting the coefficients of w_2 and w_3 to zero in Eq. (105) yields

$$A = -\frac{\beta}{3}, \qquad c = -\frac{\alpha\beta}{3}.$$

Therefore, Eq. (105) changes to

$$w' = w^3 - \frac{\alpha^3}{27}. \tag{106}$$

In order to find the solutions of Eq. (106), we seek the solutions of the following equation

$$w' = w^3 + \sigma, \tag{107}$$

where σ is a constant and the prime denotes differentiation with respect to ξ. By introducing the variable η as

$$\eta = k\xi, \tag{108}$$

where k is a constant to be determined later, the Eq. (107) becomes

$$kw' - w^3 - \sigma = 0, \tag{109}$$

where prime denotes the derivative with respect to η. According to the Exp-function method, we assume that the solution of Eq. (109) can be expressed in the following form

$$w(\eta) = \frac{\sum_{n=-d}^{e} f_n \exp(n\eta)}{\sum_{m=-p}^{q} g_m \exp(m\eta)}, \tag{110}$$

where f_n and g_m are unknown constants to be determined, p, q, e and d are positive integers which are given by the homogeneous balance principle. To determine the values of q and e, we balance the linear term of highest order with the highest order nonlinear term in Eq. (109) to have

$$w' = \frac{b_1 \exp[(2q + e)\eta] + \cdots}{b_2 \exp(3q\eta) + \cdots}, \tag{111}$$

and

$$w^3 = \frac{b_3 \exp(3e\eta) + \cdots}{b_4 \exp(3q\eta) + \cdots}, \tag{112}$$

where b_i are coefficients for simplicity. Balancing highest order of Exp-function in Eqs. (111) and (112), we have $e = q$. In the same manner as illustrated above, we find $d = p$. For simplicity, we set $e = q = 1$ and $d = p = 1$, then Eq. (110) reduces to

$$w(\eta)=\frac{f_1\exp(\eta)+f_0+f_{-1}\exp(-\eta)}{g_1\exp(\eta)+g_0+g_{-1}\exp(-\eta)}. \quad (113)$$

Substituting Eq. (113) into Eq. (109), and equating to zero the coefficients of all powers of $\exp(\eta)$ yields a set of algebraic equations for f_0, f_1, f_{-1}, g_0, g_1, g_{-1}, and k. Solving this system, gets four sets of solutions

$$\begin{cases} f_1=\dfrac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_1}{2}, & g_0=\dfrac{-(1+i\sqrt{3})^2 f_0}{4\sqrt[3]{\sigma}}, \\ g_{-1}=\dfrac{-(1+i\sqrt{3})^2 f_{-1}}{4\sqrt[3]{\sigma}}, & k=\dfrac{3\sqrt[3]{\sigma^2}(1+i\sqrt{3})^2}{4}, \end{cases}$$

and

$$\begin{cases} f_1=\dfrac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_1}{4}, & g_0=\dfrac{(1+i\sqrt{3})f_0}{2\sqrt[3]{\sigma}}, \\ g_{-1}=\dfrac{(1+i\sqrt{3})f_{-1}}{2\sqrt[3]{\sigma}}, & k=\dfrac{-3\sqrt[3]{\sigma^2}(1+i\sqrt{3})}{2}. \end{cases}$$

and

$$\begin{cases} f_0=g_0=0, & f_{-1}=\dfrac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_{-1}}{2}, \\ g_1=\dfrac{-(1+i\sqrt{3})^2 f_1}{4\sqrt[3]{\sigma}}, & k=\dfrac{-3\sqrt[3]{\sigma^2}(1+i\sqrt{3})^2}{8}, \end{cases}$$

and

$$\begin{cases} f_0=g_0=0, & f_{-1}=\dfrac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_{-1}}{4}, \\ g_1=\dfrac{(1+i\sqrt{3})f_1}{2\sqrt[3]{\sigma}}, & k=\dfrac{3\sqrt[3]{\sigma}(1+i\sqrt{3})}{4}. \end{cases}$$

On account of Eqs. (113), (106), (101) and the above mentioned solutions, we get exact solutions of u and v in the following closed forms

(i) The first set

$$v_1=\frac{\dfrac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_1}{2}\exp(k\xi)+f_0+f_{-1}\exp(-k\xi)}{g_1\exp(k\xi)+\dfrac{-(1+i\sqrt{3})^2 f_0}{4\sqrt[3]{\sigma}}+\dfrac{-(1+i\sqrt{3})^2 f_{-1}}{4\sqrt[3]{\sigma}}\exp(-k\xi)},$$

$$u_1=-\frac{\alpha}{3}+\frac{\beta}{3}+\frac{\dfrac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_1}{2}\exp(k\xi)+f_0+f_{-1}\exp(-k\xi)}{g_1\exp(k\xi)+\dfrac{-(1+i\sqrt{3})^2 f_0}{4\sqrt[3]{\sigma}}+\dfrac{-(1+i\sqrt{3})^2 f_{-1}}{4\sqrt[3]{\sigma}}\exp(-k\xi)},$$

where $k=\dfrac{3\sqrt[3]{\sigma^2}(1+i\sqrt{3})^2}{4}$.

(ii) The second set

$$v_2 = \frac{\frac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_1}{4}\exp(k\xi) + f_0 + f_{-1}\exp(-k\xi)}{g_1\exp(k\xi) + \frac{(1+i\sqrt{3})f_0}{2\sqrt[3]{\sigma}} + \frac{(1+i\sqrt{3})f_{-1}}{2\sqrt[3]{\sigma}}\exp(-k\xi)},$$

$$u_2 = -\frac{\alpha}{3} + \frac{\beta}{3} + \frac{\frac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_1}{4}\exp(k\xi) + f_0 + f_{-1}\exp(-k\xi)}{g_1\exp(k\xi) + \frac{(1+i\sqrt{3})f_0}{2\sqrt[3]{\sigma}} + \frac{(1+i\sqrt{3})f_{-1}}{2\sqrt[3]{\sigma}}\exp(-k\xi)},$$

where $k = \frac{-3\sqrt[3]{\sigma^2}(1+i\sqrt{3})}{2}$.

(iii) The third set

$$v_3 = \frac{f_1\exp(k\xi) + \frac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_{-1}}{2}\exp(-k\xi)}{\frac{-(1+i\sqrt{3})^2 f_1}{4\sqrt[3]{\sigma}}\exp(k\xi) + g_{-1}\exp(-k\xi)},$$

$$u_3 = -\frac{\alpha}{3} + \frac{\beta}{3} + \frac{f_1\exp(k\xi) + \frac{\sqrt[3]{\sigma}(1+i\sqrt{3})g_{-1}}{2}\exp(-k\xi)}{\frac{-(1+i\sqrt{3})^2 f_1}{4\sqrt[3]{\sigma}}\exp(k\xi) + g_{-1}\exp(-k\xi)},$$

where $k = \frac{-3\sqrt[3]{\sigma^2}(1+i\sqrt{3})^2}{8}$.

(iv) The fourth set

$$v_4 = \frac{f_1\exp(k\xi) + \frac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_{-1}}{4}\exp(-k\xi)}{\frac{(1+i\sqrt{3})f_1}{2\sqrt[3]{\sigma}}\exp(k\xi) + g_{-1}\exp(-k\xi)},$$

$$u_4 = -\frac{\alpha}{3} + \frac{\beta}{3} + \frac{f_1\exp(k\xi) + \frac{-\sqrt[3]{\sigma}(1+i\sqrt{3})^2 g_{-1}}{4}\exp(-k\xi)}{\frac{(1+i\sqrt{3})f_1}{2\sqrt[3]{\sigma}}\exp(k\xi) + g_{-1}\exp(-k\xi)},$$

where $k = \frac{3\sqrt[3]{\sigma}(1+i\sqrt{3})}{4}$. In all the given solutions, f_0, f_1, f_{-1}, g_0, g_1, g_{-1} are arbitrary constants, $\sigma = \frac{-\alpha^3}{27}$, $c = -\frac{\alpha\beta}{3}$ and $\xi = x - ct$.

6. Conclusion

We extract the generalized solitary solutions of the coupled NLPDEs via decomposing them into a system of algebraic equations as well as an ordinary differential equation and solving the corresponding ODE through Exp-function method. We have found some new exact travelling wave solutions for the coupled Burgers equations in two different forms, the system of nonlinear hyperbolic equations, the (2 + 1)-dimensional Boiti-Leon-Pempinelle equation, the coupled equal width wave equations, the coupled KdV equations, two types of the generalized Hirota-Satsuma coupled KdV equations and the coupled Burgers equations in the modified form.

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