

EXISTENCE AND UNIQUENESS THEOREMS FOR SOME SEMI-LINEAR EQUATIONS ON LOCALLY FINITE GRAPHS

ANDREA PINAMONTI AND GIORGIO STEFANI

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ABSTRACT. We study some semi-linear equations for the (m, p) -Laplacian operator on locally finite weighted graphs. We prove existence of weak solutions for all $m \in \mathbb{N}$ and $p \in (1, +\infty)$ via a variational method already known in the literature by exploiting the continuity properties of the energy functionals involved. When $m = 1$, we also establish a uniqueness result in the spirit of the Brezis–Strauss Theorem. We finally provide some applications of our main results by dealing with some Yamabe-type and Kazdan–Warner-type equations on locally finite weighted graphs.

1. INTRODUCTION

1.1. Framework. When dealing with PDEs coming from the Euler–Lagrange equations of some energy functional, existence and multiplicity results of weak solutions are usually achieved via the so-called Variational Method.

In the recent years, this approach has been employed by many authors in order to deal with a large variety of interesting PDEs on graphs, see [5–8, 10–23, 25–29] and the references therein. Of particular interest for the scopes of the present paper is the work [19], where the authors proved existence of weak solutions for a Yamabe-type equation on locally finite weighted graphs via the celebrated Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1].

The main aim of this note is twofold. On the one hand, by exploiting some ideas developed in [9, 24] in the context of Carnot groups, we prove the existence of weak solutions for a Yamabe-type equation on locally finite weighted graphs. Our result is similar to the one of [19] but holds under a different set of assumptions. On the other hand, we adapt the strategy of [3] developed in the Euclidean setting to establish a uniqueness result for the weak solutions of Yamabe-type equations on locally finite weighted graphs in the spirit of the celebrated Brezis–Strauss Theorem [4].

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1.2. Main notation. Before stating our main results, we need to recall some notation, see section 2 for the precise definitions.

Given $G = (V, E)$ a locally finite non-oriented graph, the *vertex boundary* $\partial\Omega$ and the *vertex interior* Ω° of a connected subgraph $\Omega \subset V$ are defined as

$$\partial\Omega = \{x \in \Omega : \exists y \notin \Omega \text{ such that } xy \in E\}, \quad \Omega^\circ = \Omega \setminus \partial\Omega.$$

We say that Ω is *bounded* if it is a bounded subset of V with respect to the usual *vertex distance* $d: V \times V \rightarrow [0, +\infty)$.

Once a symmetric *weight function* $w: V \times V \rightarrow [0, \infty)$ is given, we can define the *Laplacian* of a function $u: V \rightarrow \mathbb{R}$ as

$$(1.1) \quad \Delta u(x) = \frac{1}{\mathbf{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x)) \quad \text{for } x \in V,$$

where $\mathbf{m}: V \rightarrow [0, +\infty)$ is the measure function

$$(1.2) \quad \mathbf{m}(x) = \sum_{y \in V} w_{xy} \quad \text{for all } x \in V.$$

The *gradient form* associated to the Laplacian operator is the bilinear symmetric form

$$\Gamma(u, v)(x) = \frac{1}{2\mathbf{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x))(v(y) - v(x)), \quad x \in V,$$

defined for any couple of functions $u, v: V \rightarrow \mathbb{R}$. As a consequence, the *slope* of the function $u: V \rightarrow \mathbb{R}$ is given by

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left(\frac{1}{2\mathbf{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}} \quad \text{for } x \in V.$$

Note that $|\Gamma(u, v)| \leq |\nabla u| |\nabla v|$ for any couple of functions $u, v: V \rightarrow \mathbb{R}$. In analogy with the Euclidean framework, for any $m \in \mathbb{N}$ we recursively define the *m-slope* of the function u as

$$|\nabla^m u| = \begin{cases} |\nabla(\Delta^{\frac{m-1}{2}} u)| & \text{if } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}} u| & \text{if } m \text{ is even,} \end{cases}$$

where $|\Delta^{\frac{m}{2}} u|$ denotes the usual absolute value of the function $\Delta^{\frac{m}{2}} u$. The natural operator associated to the Sobolev spaces $(W_0^{m,p}(\Omega), \|\cdot\|_{W_0^{m,p}(\Omega)})$ (see (2.1) for the precise definition) is the *(m, p)-Laplacian operator*

$$\mathcal{L}_{m,p}: W_0^{m,p}(\Omega) \rightarrow L^p(\Omega)$$

defined in the distributional sense for all $u \in W_0^{m,p}(\Omega)$ as

$$(1.3) \quad \int_{\Omega} \mathcal{L}_{m,p} u \varphi \, d\mathbf{m} = \begin{cases} \int_{\Omega} |\nabla^m u|^{p-2} \Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \varphi) \, d\mathbf{m} & \text{if } m \text{ is odd,} \\ \int_{\Omega} |\nabla^m u|^{p-2} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \varphi \, d\mathbf{m} & \text{if } m \text{ is even,} \end{cases}$$

whenever $\varphi \in W_0^{m,p}(\Omega)$.

The (m, p) -Laplacian $\mathcal{L}_{m,p}u$ can be explicitly computed at any point of Ω . In particular, $\mathcal{L}_{1,p}$ is the p -Laplacian operator, given by

$$(1.4) \quad \Delta_p u(x) = \frac{1}{\mathfrak{m}(x)} \sum_{y \in \Omega} (|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x)) w_{xy}(u(y) - u(x)), \quad x \in \Omega,$$

for all $u \in W_0^{1,p}(\Omega)$. When $p = 2$, we recover the usual Laplacian operator defined in (1.1).

1.3. Main results. We are now ready to state our main results. Our first main theorem is the following existence result for a Yamabe-type equation for the (m, p) -Laplacian operator on locally finite weighted graphs.

Theorem 1.1. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $m \in \mathbb{N}$, $p \in (1, +\infty)$ and $q \in [p - 1, +\infty)$. Let $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that*

$$(1.5) \quad |f(x, t)| \leq a(x) + b(x) |t|^q \quad \text{for every } (x, t) \in \Omega \times \mathbb{R}$$

for some non-negative $a, b \in L^1(\Omega)$ with $\|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)} > 0$. There exists

$$(1.6) \quad \Lambda = \Lambda(m, p, q, \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)}) > 0$$

such that the Yamabe-type problem

$$(1.7) \quad \begin{cases} \mathcal{L}_{m,p}u = \lambda f(x, u) & \text{in } \Omega^\circ, \\ |\nabla^j u| = 0 & \text{on } \partial\Omega, \quad 0 \leq j \leq m - 1, \end{cases}$$

admits at least one non-trivial solution $u_\lambda \in W_0^{m,p}(\Omega)$ for every $0 < \lambda < \Lambda$.

We observe that the growth condition of the function f assumed in (1.5) of Theorem 1.1 is different from the one assumed in [19, Theorem 3]. In particular, we do not assume that $f(x, 0) = 0$ for all $x \in \Omega$. We also underline that the existence threshold (1.6) depends uniquely on the growth of the function f and not on the first eigenvalue of the (m, p) -Laplacian, as instead it happens in [19, Theorem 3].

Our second main result is the following uniqueness theorem for a Yamabe-type equation for the p -Laplacian operator on locally finite weighted graphs in the spirit of the famous Brezis–Strauss Theorem, see [3, 4].

Theorem 1.2. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $p \in [1, +\infty)$ and let $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $g(x, 0) = 0$ and $t \mapsto g(x, t)$ is non-decreasing for all $x \in \Omega$. If $f_1, f_2 \in L^1(\Omega)$, $h \in L^1(\partial\Omega)$ and $u_1, u_2 \in W^{1,p}(\Omega)$ solve the problems*

$$\begin{cases} -\Delta_p u_i + g(x, u_i) = f_i & \text{in } \Omega^\circ \\ u_i = h & \text{on } \partial\Omega \end{cases} \quad \text{for } i = 1, 2,$$

then

$$(1.8) \quad \int_{\Omega} |g(x, u_1) - g(x, u_2)| \, d\mathfrak{m} \leq \int_{\Omega} |f_1 - f_2| \, d\mathfrak{m}.$$

As a consequence, for every $f \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$ the problem

$$(1.9) \quad \begin{cases} -\Delta_p u + g(x, u) = f & \text{in } \Omega^\circ, \\ u = h & \text{on } \partial\Omega \end{cases}$$

admits at most one solution $u \in W^{1,p}(\Omega)$.

By combining Theorem 1.1 and Theorem 1.2, we get the following well-posedness result for a Yamabe-type problem for the p -Laplacian on locally finite weighted graphs.

Proposition 1.3. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $p \in (1, +\infty)$, $q \in [p-1, +\infty)$ and $a, b \in L^1(\Omega)$ with $\inf_{\Omega} b \geq 0$. There exists*

$$\Lambda = \Lambda(m, p, q, \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)}) > 0$$

such that the Yamabe-type problem

$$(1.10) \quad \begin{cases} -\Delta_p u + b|u|^{q-1}u = a & \text{in } \Omega^\circ, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$.

1.4. Organization of the paper. The structure of the paper is the following. In Section 2 we recall the preliminary definitions and notions needed in the paper. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2 respectively. Finally, in Section 5 we provide some applications of our main results, along with the proof of Proposition 1.3.

2. PRELIMINARIES

In this section, we introduce the main notation and some preliminary results we will need in the sequel of the paper.

2.1. Non-oriented graphs. Let V be a non-empty set and let $E \subset V \times V$. We write

$$x \sim y \iff xy = (x, y) \in E.$$

We will always assume that

$$xy \in E \iff yx \in E.$$

We say that the couple $G = (V, E)$ is a *non-oriented graph* with *vertices* V and *edges* E .

The non-oriented graph G is *locally finite* if

$$\#\{y \in V : xy \in E\} < +\infty \quad \text{for all } x \in V,$$

that is, each vertex in V belongs to a finite number of edges in E .

Given $n \in \mathbb{N}$, a *path* on G is any finite sequence of vertices $\{x_k\}_{k=1, \dots, n} \subset V$ such that

$$x_k x_{k+1} \in E \quad \text{for all } k = 1, \dots, n-1.$$

The *length* of a path on G is the number of edges in the path. We say that G is *connected* if, for any two vertices $x, y \in V$, there is a path connecting x and y . If G is connected, then the function $d: V \times V \rightarrow [0, +\infty)$ given by

$$d(x, y) = \min \{n \in \mathbb{N}_0 : x \text{ and } y \text{ can be connected by a path of length } n\},$$

for $x, y \in V$, is a distance on V . As a consequence, any connected locally finite non-oriented graph has at most countable many vertices.

Let $G = (V, E)$ be a locally finite non-oriented graph. A *weight* on G is a function $w: V \times V \rightarrow [0, +\infty)$, $w(x, y) = w_{xy}$ for $x, y \in V$, such that

$$w_{xy} = w_{yx} \quad \text{and} \quad w_{xy} > 0 \iff xy \in E$$

for all $x, y \in V$. We conclude this section by pointing out that the function $\mathbf{m}: V \rightarrow [0, +\infty)$ defined in (1.2) can be interpreted as a measure on the graph by simply setting

$$\int_V u \, d\mathbf{m} = \int_V u(x) \, d\mathbf{m}(x) = \sum_{y \in V} u(y) \mathbf{m}(y) \in [0, +\infty]$$

for any function $u: V \rightarrow [0, +\infty)$.

2.2. Sobolev spaces on bounded domains. Let $G = (V, E)$ be a weighted locally finite graph and let $\Omega \subset V$ be a bounded domain. Note that the integral

$$\int_\Omega u \, d\mathbf{m} = \int_\Omega u(x) \, d\mathbf{m}(x) = \sum_{x \in \Omega} u(x) \mathbf{m}(x)$$

of a function $u: \Omega \rightarrow \mathbb{R}$ is well-defined, since Ω is a finite set. Let $p \in [1, +\infty]$ and $m \in \mathbb{N}_0$. The *Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions $u: \Omega \rightarrow \mathbb{R}$ such that

$$(2.1) \quad \|u\|_{W^{m,p}(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{L^p(\Omega)} < +\infty.$$

When $m = 0$, this space is simply the *Lebesgue space* $L^p(\Omega)$. Since Ω is a finite set, the Banach space $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is finite dimensional and, actually, coincides with the set of all real-valued functions on Ω .

For $m \in \mathbb{N}$, we define

$$(2.2) \quad C_0^m(\Omega) := \{u: \Omega \rightarrow \mathbb{R} : |\nabla^k u| = 0 \text{ on } \partial\Omega \text{ for all } 0 \leq k \leq m - 1\}$$

and we let $W_0^{m,p}(\Omega)$ be the completion of $C_0^m(\Omega)$ with respect to the Sobolev norm (2.1). The following result is proved in [19, Theorem 7].

Theorem 2.1 (Sobolev embedding). *Let $G = (V, E)$ be a locally finite graph and let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $m \in \mathbb{N}$ and $p \in [1, +\infty)$. The space $W_0^{m,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $q \in [1, +\infty]$, i.e. there exists a constant $C_{m,p} > 0$, depending only on m, p and Ω , such that*

$$(2.3) \quad \|u\|_{L^q(\Omega)} \leq C_{m,p} \|\nabla^m u\|_{L^p(\Omega)}$$

for all $q \in [1, +\infty]$ and $u \in W_0^{m,p}(\Omega)$.

By Theorem 2.1, the space $(W_0^{m,p}(\Omega), \|\cdot\|_{W_0^{m,p}(\Omega)})$ is a finite dimensional Banach space, where

$$(2.4) \quad \|u\|_{W_0^{m,p}(\Omega)} = \|\nabla^m u\|_{L^p(\Omega)}$$

is a norm on $W_0^{m,p}(\Omega)$ equivalent to the norm (2.1). Since Ω is a finite set, the Banach space $(W_0^{m,p}(\Omega), \|\cdot\|_{W_0^{m,p}(\Omega)})$ is finite dimensional and, actually, coincides with the set $C_0^m(\Omega)$ defined in (2.2).

3. PROOF OF THEOREM 1.1

In this section, we prove our first main result following the strategy outlined in [9]. Given $\lambda > 0$, we define

$$(3.1) \quad \Phi(u) = \|u\|_{W_0^{m,p}(\Omega)}, \quad \Psi_\lambda(u) = \lambda \int_\Omega F(x, u) \, dm,$$

for all $u \in W_0^{m,p}(\Omega)$, where

$$(3.2) \quad F(x, t) = \int_0^t f(x, \tau) \, d\tau \quad \text{for all } t \in \mathbb{R}.$$

Note that, thanks to the assumption in (1.5), the functional Ψ_λ is well-defined and (strongly) continuous on $W_0^{m,p}(\Omega)$. Indeed, we can estimate

$$|F(x, t)| \leq \int_0^{|t|} |f(x, \tau)| \, d\tau \leq \int_0^{|t|} a(x) + b(x) |\tau|^q \, d\tau = a(x) |t| + b(x) \frac{|t|^{1+q}}{1+q}$$

for all $(x, t) \in \Omega \times \mathbb{R}$, so that

$$|\Psi_\lambda(u)| \leq \lambda \left(\|a\|_{L^1(\Omega)} \|u\|_{L^\infty(\Omega)} + \|b\|_{L^1(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}^{1+q}}{1+q} \right)$$

which is finite for all $u \in W_0^{m,p}(\Omega)$ by Theorem 2.1. In addition, if $(u_n)_{n \in \mathbb{N}} \subset W_0^{m,p}(\Omega)$ is converging to some $u \in W_0^{m,p}(\Omega)$, then $u_n \rightarrow u$ in $L^\infty(\Omega)$ as $n \rightarrow +\infty$ and thus

$$\lim_{n \rightarrow +\infty} \Psi_\lambda(u_n) = \lim_{n \rightarrow +\infty} \sum_{x \in \Omega} F(x, u_n(x)) \, m(x) = \sum_{x \in \Omega} F(x, u(x)) \, m(x) = \Psi_\lambda(u)$$

by the continuity of the function $t \mapsto F(x, t)$ for $x \in \Omega$ fixed.

The following two results are proved in [9, Lemma 3.2 and Lemma 3.3] respectively for the case $p = 2$. Here we reproduce the proofs in our setting in the more general case $p \in (1, +\infty)$ for the reader's ease.

Lemma 3.1. *Let $p \in (1, +\infty)$ and $\lambda > 0$. If*

$$(3.3) \quad \limsup_{\varepsilon \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \varrho - \varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \varrho^{p-1}$$

for some $\varrho > 0$, then

$$(3.4) \quad \inf_{\sigma < \varrho} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} < \frac{1}{p}.$$

Proof. Let $\varepsilon \in (0, \varrho)$ and note that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\varrho^p - (\varrho - \varepsilon)^p} = \frac{1}{p\varrho^{p-1}}.$$

Therefore, in virtue of (3.3), we get that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \varrho - \varepsilon])} \Psi_\lambda(u)}{\varrho^p - (\varrho - \varepsilon)^p} \\ &= \limsup_{\varepsilon \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \varrho - \varepsilon])} \Psi_\lambda(u)}{\varepsilon} \cdot \frac{\varepsilon}{\varrho^p - (\varrho - \varepsilon)^p} \\ &= \frac{1}{p\varrho^{p-1}} \limsup_{\varepsilon \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \varrho - \varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \frac{1}{p}. \end{aligned}$$

Thus we can find $\bar{\varepsilon} \in (0, \varrho)$ such that

$$\frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \varrho - \bar{\varepsilon}])} \Psi_\lambda(u)}{\varrho^p - (\varrho - \bar{\varepsilon})^p} < \frac{1}{p}$$

and so $\bar{\sigma} = \varrho - \bar{\varepsilon} < \varrho$ gives

$$\begin{aligned} \inf_{\sigma < \varrho} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} \\ < \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \bar{\sigma}])} \Psi_\lambda(u)}{\varrho^p - \bar{\sigma}^p} < \frac{1}{p} \end{aligned}$$

proving (3.4). The proof is complete. □

Lemma 3.2. *Let $p \in (1, +\infty)$ and $\lambda > 0$. If (3.4) holds for some $\varrho > 0$, then*

$$(3.5) \quad \inf_{u \in \Phi^{-1}([0, \varrho])} \frac{\sup_{v \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|_{W_0^{m,p}(\Omega)}^p} < \frac{1}{p}.$$

Proof. In virtue of (3.4), we can find $\bar{\sigma} \in (0, \varrho)$ such that

$$\sup_{u \in \Phi^{-1}([0, \bar{\sigma}])} \Psi_\lambda(u) > \sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \frac{1}{p}(\varrho^p - \bar{\sigma}^p).$$

Since the functional Ψ_λ is continuous on $W_0^{m,p}(\Omega)$, we can find $\bar{u} \in W_0^{m,p}(\Omega)$ with $\|\bar{u}\|_{W_0^{m,p}(\Omega)} = \bar{\sigma}$ such that

$$\sup_{u \in \Phi^{-1}([0, \bar{\sigma}])} \Psi_\lambda(u) = \sup_{\|u\|_{W_0^{m,p}(\Omega)} = \bar{\sigma}} \Psi_\lambda(u) = \Psi_\lambda(\bar{u})$$

and so

$$\Psi_\lambda(\bar{u}) > \sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \frac{1}{p}(\varrho^p - \bar{\sigma}^p).$$

We thus conclude that

$$\inf_{u \in \Phi^{-1}([0, \varrho])} \frac{\sup_{v \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|_{W_0^{m,p}(\Omega)}^p} < \frac{\sup_{v \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(v) - \Psi_\lambda(\bar{u})}{\varrho^p - \|\bar{u}\|_{W_0^{m,p}(\Omega)}^p} < \frac{1}{p}$$

proving (3.5). The proof is complete. □

We are now ready to prove our first main result, in analogy with [9, Theorem 3.1].

Proof of Theorem 1.1. Let $\lambda > 0$ and consider the energy functional $\mathcal{E}_\lambda: W_0^{m,p}(\Omega) \rightarrow \mathbb{R}$ defined as

$$\mathcal{E}_\lambda(u) = \frac{\Phi(u)^p}{p} - \Psi_\lambda(u) \quad \text{for all } u \in W_0^{m,p}(\Omega),$$

where Φ and Ψ_λ are as in (3.1). By the growth condition (1.5) and Theorem 2.1, we have that $\mathcal{E}_\lambda \in C^1(W_0^{m,p}(\Omega); \mathbb{R})$, with derivative at $u \in W_0^{m,p}(\Omega)$ given by

$$\mathcal{E}'_\lambda(u)[\varphi] = \begin{cases} \int_\Omega |\nabla^m u|^{p-2} \Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \varphi) \, dm - \lambda \int_\Omega f(x, u) \varphi \, dm & \text{if } m \text{ is odd,} \\ \int_\Omega |\nabla^m u|^{p-2} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \varphi \, dm - \lambda \int_\Omega f(x, u) \varphi \, dm & \text{if } m \text{ is even,} \end{cases}$$

for any $\varphi \in W_0^{m,p}(\Omega)$. In particular, the solutions of the problem (1.7) are exactly the critical points of the functional \mathcal{E}_λ . Now let $\varrho > 0$ to be fixed later. Since \mathcal{E}_λ is a continuous functional on $W_0^{m,p}(\Omega)$, there exists $u_{\lambda,\varrho} \in \Phi^{-1}([0, \varrho])$ such that

$$(3.6) \quad \mathcal{E}_\lambda(u_{\lambda,\varrho}) = \inf_{u \in \Phi^{-1}([0,\varrho])} \mathcal{E}_\lambda(u).$$

To conclude the proof, we just need to show that $\|u_{\lambda,\varrho}\|_{W_0^{m,p}(\Omega)} < \varrho$. To this aim, for $\varepsilon \in (0, \varrho)$ we consider

$$\Lambda(\varrho, \varepsilon) = \frac{\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon}.$$

Recalling the definition of Ψ_λ in (3.1), we have

$$\begin{aligned} \Lambda(\varrho, \varepsilon) &= \frac{1}{\varepsilon} \left(\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u) \right) \\ &\leq \frac{\lambda}{\varepsilon} \sup_{u \in \Phi^{-1}([0,1])} \int_\Omega \left| \int_{(\varrho-\varepsilon)u(x)}^{\varrho u(x)} |f(x, t)| dt \right| dm(x). \end{aligned}$$

Thanks to the growth condition (1.5), we can estimate

$$\begin{aligned} \frac{\lambda}{\varepsilon} \int_\Omega \left| \int_{(\varrho-\varepsilon)u(x)}^{\varrho u(x)} |f(x, t)| dt \right| dm(x) &\leq \frac{\lambda}{\varepsilon} \int_\Omega \varepsilon a(x)|u(x)| + b(x) \left(\frac{\varrho^{q+1} - (\varrho - \varepsilon)^{q+1}}{q + 1} \right) |u(x)|^{q+1} dm(x) \\ &\leq \lambda \|u\|_{L^\infty(\Omega)} \|a\|_{L^1(\Omega)} + \frac{\lambda \|u\|_{L^\infty(\Omega)}^{q+1} \|b\|_{L^1(\Omega)}}{q + 1} \left(\frac{\varrho^{q+1} - (\varrho - \varepsilon)^{q+1}}{\varepsilon} \right) \end{aligned}$$

for all $u \in W_0^{m,p}(\Omega)$. Thus, by the embedding inequality (2.3), we get

$$\Lambda(\varrho, \varepsilon) \leq \lambda C_{m,p} \|a\|_{L^1(\Omega)} + \frac{\lambda C_{m,p}^{q+1} \|b\|_{L^1(\Omega)}}{q + 1} \left(\frac{\varrho^{q+1} - (\varrho - \varepsilon)^{q+1}}{\varepsilon} \right)$$

and so

$$\limsup_{\varepsilon \rightarrow 0^+} \Lambda(\varrho, \varepsilon) \leq \lambda (C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q).$$

We now define

$$(3.7) \quad \lambda_\varrho = \frac{\varrho^{p-1}}{C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q} \in (0, +\infty)$$

and, consequently,

$$\Lambda = \sup_{\varrho > 0} \lambda_\varrho \in (0, +\infty)$$

(note that $\Lambda < +\infty$ is ensured by the fact that $q \geq p - 1$). Now fix $\lambda < \Lambda$ and choose the parameter $\varrho > 0$ in such a way that $\lambda < \lambda_\varrho < \Lambda$. This choice implies that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \Lambda(\varrho, \varepsilon) &\leq \lambda (C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q) \\ &< \lambda_\varrho (C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q) < \varrho^{p-1}, \end{aligned}$$

so that

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \varrho^{p-1}.$$

We can now apply Lemma 3.1 to get that

$$\inf_{\sigma < \varrho} \frac{\sup_{u \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0, \sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} < \frac{1}{p}$$

and so, by Lemma 3.2, we infer that

$$\inf_{u \in \Phi^{-1}([0, \varrho])} \frac{\sup_{v \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|_{W_0^{m,p}(\Omega)}^p} < \frac{1}{p}.$$

The above inequality implies that there exists $w_{\lambda, \varrho} \in \Phi^{-1}([0, \varrho])$ such that

$$\sup_{v \in \Phi^{-1}([0, \varrho])} \Psi_\lambda(v) < \Psi_\lambda(w_{\lambda, \varrho}) + \frac{\varrho^p - \|w_{\lambda, \varrho}\|_{W_0^{m,p}(\Omega)}^p}{p}.$$

Now, if by contradiction we assume that $\|u_{\lambda, \varrho}\|_{W_0^{m,p}(\Omega)} = \varrho$, then the previous inequality implies that

$$\Psi_\lambda(u_{\lambda, \varrho}) < \Psi_\lambda(w_{\lambda, \varrho}) + \frac{\varrho^p - \|w_{\lambda, \varrho}\|_{W_0^{m,p}(\Omega)}^p}{p}$$

which is equivalent to $\mathcal{E}_\lambda(u_{\lambda, \varrho}) > \mathcal{E}_\lambda(w_{\lambda, \varrho})$, contradicting (3.6). The proof is complete. \square

Remark 3.3 (The precise value of Λ in Theorem 1.1). Note that the above proof allows to give a precise value to the existence threshold $\Lambda > 0$ in Theorem 1.1. Indeed, one just need to find the maximal value of the function defined in (3.7), which is explicitly computable in term of $p, q, \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)}$ and $C_{m,p}$. In particular, in the limiting case $q = p - 1$, one has

$$\Lambda = \lim_{\varrho \rightarrow +\infty} \frac{\varrho^{p-1}}{C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^p \|b\|_{L^1(\Omega)} \varrho^{p-1}} = \frac{1}{C_{m,p}^p \|b\|_{L^1(\Omega)}},$$

which does not depend on $\|a\|_{L^1(\Omega)}$.

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. The overall strategy is to adapt the line developed in [3, Appendix B] for the Euclidean setting to the present framework. Note that [3] is focused on the case $p = 2$ only. Nonetheless, exploiting the explicit expression (1.4) of the p -Laplacian, we are able to extend the approach of [3] also to the case $p \neq 2$.

We begin with the following result, analogous to [3, Lemma B.1].

Lemma 4.1. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $p \in [1, +\infty)$ and $f \in L^1(\Omega)$. If $u \in W_0^{1,p}(\Omega)$ is a solution of the problem*

$$(4.1) \quad \begin{cases} -\Delta_p u = f & \text{in } \Omega^\circ, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$\int_{\Omega} f H(u) \, dm \geq 0$$

for every non-decreasing locally Lipschitz function $H: \mathbb{R} \rightarrow \mathbb{R}$ such that $H(0) = 0$.

Proof. We start by observing that $H(u) \in W_0^{1,p}(\Omega)$. Indeed, $H(v) \in C_0^0(\Omega)$ for all $v \in C_0^0(\Omega)$ with $|\nabla H(v)| \leq L|\nabla v|$ on Ω , where $L = \text{Lip}(H, [-c, c])$, $c = \|v\|_{L^\infty(\Omega)}$. Using $H(u)$ as a test function in (4.1), we get

$$\int_{\Omega} f H(u) \, dm = - \int_{\Omega} \Delta_p u H(u) \, dm = \int_{\Omega} |\nabla u|^{p-2} \Gamma(u, H(u)) \, dm \geq 0,$$

because

$$\Gamma(u, H(u))(x) = \frac{1}{2m(x)} \sum_{y \in \Omega} w_{xy}(u(y) - u(x))(H(u(y)) - H(u(x))) \geq 0$$

since H is non-decreasing. The proof is complete. □

As a consequence, and in analogy with [3, Proposition B.2], from Lemma 4.1 we deduce the following result.

Corollary 4.2. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $p \in [1, +\infty)$, $M > 0$ and $f \in L^1(\Omega)$. If $u \in W_0^{1,p}(\Omega)$ is a solution of the problem*

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega^\circ, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$\int_{\Omega \cap \{u \geq M\}} f \, dm \geq 0, \quad \int_{\Omega \cap \{u \leq -M\}} f \, dm \geq 0.$$

In particular,

$$\int_{\Omega \cap \{|u| \geq M\}} f \, \text{sgn}(u) \, dm \geq 0,$$

where $\text{sgn}: \mathbb{R} \rightarrow \mathbb{R}$ is the sign function defined by $\text{sgn}(t) = \frac{t}{|t|}$ for $t \neq 0$ and $\text{sgn}(0) = 0$.

Proof. For every $n \in \mathbb{N}$ such that $n > \frac{1}{M}$, we let $H_n: \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$H_n(t) = \begin{cases} 0 & \text{for } t \leq M - \frac{1}{n}, \\ nt - nM + 1 & \text{for } M - \frac{1}{n} < t < M, \\ 1 & \text{for } t \geq M. \end{cases}$$

Since H_n is Lipschitz, non-decreasing and such that $H_n(0) = 0$, by Lemma 4.1 we get that

$$\int_{\Omega} f H_n(u) \, dm \geq 0.$$

Passing to the limit as $n \rightarrow +\infty$, we find that

$$\int_{\Omega \cap \{u \geq M\}} f \, dm \geq 0,$$

as desired. The conclusion thus follows by linearity. □

We are now ready to prove our second main result, in analogy with [3, Corollary B.1].

Proof of Theorem 1.2. The function $v = u_1 - u_2 \in W_0^{1,p}(\Omega)$ solves the problem

$$(4.2) \quad \begin{cases} -\Delta_p v = F & \text{in } \Omega^\circ, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

with $F = f_1 - f_2 - g(x, u_1) + g(x, u_2) \in L^1(\Omega)$. By Corollary 4.2, we have that

$$\int_{\Omega} F \operatorname{sgn}(v) \, dm \geq 0,$$

which is equivalent to

$$\int_{\Omega} (g(x, u_1) - g(x, u_2)) \operatorname{sgn}(u_1 - u_2) \, dm \leq \int_{\Omega} (f_1 - f_2) \operatorname{sgn}(u_1 - u_2) \, dm$$

and (1.8) immediately follows. As a consequence, if $f_1 = f_2$ then also $g(x, u_1) = g(x, u_2)$ and thus $F = 0$ in (4.2). Therefore $\Delta_p v = 0$ in Ω and thus

$$\sup_{\Omega} |v| \leq C_{1,p} \int_{\Omega} |\nabla v|^p \, dm = -C_{1,p} \int_{\Omega} v \Delta_p v \, dm = 0$$

by Theorem 2.1 and (1.3), so that $u_1 = u_2$. The proof is complete. □

5. APPLICATIONS

In this last section we briefly discuss some applications of our main results.

We begin by stating the following result, which shows that the Dirichlet problem in $W_0^{1,2}(\Omega)$ for the Laplacian operator with sufficiently well-behaved non-linearity admits a unique solution.

Corollary 5.1. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $t \mapsto g(x, t)$ is C^1 and non-decreasing with $g(x, 0) = \partial_t g(x, 0) = 0$ for all $x \in \Omega$. Let us set $f(x) = g(x, 0)$ for all $x \in \Omega$. There exists $\delta > 0$ with the following property: if $f \in L^2(\Omega)$ with $\|f - \bar{f}\|_{L^2(\Omega)} < \delta$, then the problem*

$$\begin{cases} -\Delta u + g(x, u) = f & \text{in } \Omega^\circ, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in W_0^{1,2}(\Omega)$.

The uniqueness part in Corollary 5.1 is clearly immediately achieved by Theorem 1.2, while the existence part follows from the following result, which is inspired by the work [2].

Lemma 5.2. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $t \mapsto g(x, t)$ is of class C^1 with $\partial_t g(x, 0) = 0$ for all $x \in \Omega$. Let us set $\bar{f}(x) = g(x, 0)$ for all $x \in \Omega$. There exist $\delta, \varepsilon > 0$ with the following property: if $f \in L^2(\Omega)$ with $\|f - \bar{f}\|_{L^2(\Omega)} < \delta$, then the problem*

$$\begin{cases} -\Delta u + g(x, u) = f & \text{in } \Omega^\circ, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in W_0^{1,2}(\Omega)$ with $\|u\|_{W_0^{1,2}(\Omega)} < \varepsilon$.

Proof. Let us consider the function $\mathcal{F}: W_0^{1,2}(\Omega) \rightarrow L^2(\Omega)$ defined by

$$\mathcal{F}(u) = -\Delta u + g(x, u)$$

for all $u \in W_0^{1,2}(\Omega)$. Note that the map \mathcal{F} is well-defined, since $W_0^{1,2}(\Omega) \subset L^\infty(\Omega)$ with continuous embedding by Theorem 2.1 and thus also $x \mapsto g(x, u(x)) \in L^\infty(\Omega)$ by the continuity property of g and by the fact that the number of vertices in Ω is finite. We additionally note that $\mathcal{F} \in C^1(W_0^{1,2}(\Omega), L^2(\Omega))$. Indeed, the Laplacian Δ is linear and the map $u \mapsto g(x, u)$ is of class C^1 thanks to the continuity properties of g . Finally, we observe that the map $\mathcal{F}'(0): W_0^{1,2}(\Omega) \rightarrow L^2(\Omega)$ is invertible, since $\mathcal{F}'(0) = -\Delta$ by the assumption that $\partial_t g(x, 0) = 0$ for all $x \in \Omega$. Since $\mathcal{F}(0) = \bar{f}$, the conclusion follows by the Inverse Function Theorem and the proof is complete. \square

Our two main results Theorems 1.1 and 1.2 can be combined in order to achieve the well-posedness of a Yamabe-type problem on bounded domains, namely Proposition 1.3.

Proof of Proposition 1.3. The function $g(x, t) = b(x)|t|^{q-1}t$, defined for $(x, t) \in \Omega \times \mathbb{R}$, satisfies the assumptions of Theorem 1.2, so that problem (1.10) admits at most one solution and we just need to deal with the existence issue. If $\|a\|_{L^1(\Omega)} = 0$, then clearly $a = 0$ and thus the null function $u = 0$ is the unique solution of problem (1.10). If $\|a\|_{L^1(\Omega)} > 0$ instead, then we apply Theorem 1.1. Indeed, the function $f(x, t) = a(x) - b(x)|t|^{q-1}t$, defined for $(x, t) \in \Omega \times \mathbb{R}$, satisfies the assumptions of Theorem 1.1, and the conclusion thus follows in virtue of Remark 3.3. \square

We conclude our paper with the following uniqueness result for a Kazdan–Warner-type problem on bounded domains. Its proof is a simple application of Theorem 1.2 and is thus left to the reader.

Corollary 5.3. *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \emptyset$ and $\partial\Omega \neq \emptyset$. Let $p \in [1, +\infty)$ and let $\alpha, \beta \in L^1(\Omega)$ be two non-negative functions. For every $f \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$, the Kazdan–Warner-type problem*

$$(5.1) \quad \begin{cases} -\Delta_p u + \alpha e^{\beta u} = f & \text{in } \Omega^\circ, \\ u = h & \text{on } \partial\Omega \end{cases}$$

admits at most one solution $u \in W^{1,p}(\Omega)$.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI TRENTO, VIA SOMMARIVE 14,
38123 POVO (TN), ITALY

Email address: `andrea.pinamonti@unitn.it`

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI (SISSA), VIA BONOMEA 265, 34136
TRIESTE (TS), ITALY

Email address: `giorgio.stefani.math@gmail.com`