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On the monotonicity of weighted perimeters of convex bodies

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Abstract
We prove that, among weighted isotropic perimeters, only constant multiples of the Euclidean perimeter satisfy the monotonicity property on nested convex bodies. Although the analogous result fails for general weighted anisotropic perimeters, a similar characterization holds for radially-weighted anisotropic densities.

KEYWORDS
convex body, monotonicity property, weighted perimeter

1 | INTRODUCTION

1.1 | Monotonicity property

Let $N \geq 2$. If $A, B \subset \mathbb{R}^N$ are two nested convex bodies, that is compact convex sets with non-empty interior such that $A \subset B$, then

$$P(A) \leq P(B), \tag{1.1}$$

where $P(E) = \mathscr{H}^{N-1}(\partial E)$ denotes the Euclidean perimeter of the convex body $E \subset \mathbb{R}^N$. The monotonicity property (1.1) is well known and dates back to the ancient Greeks (Archimedes took it as a postulate in his work on the sphere and the cylinder [1, p. 36]).

Inequality (1.1) can be proved in several ways: by the *Cauchy formula* for the area surface of convex bodies [5, Sect. 7]; by the monotonicity property of *mixed volumes* [5, Sect. 8]; by the Lipschitz property of the projection on a convex closed set [6, Lem. 2.4]; by the fact that the perimeter is decreased under intersection with half-spaces [25, Ex. 15.13].

Inequality (1.1) extends to the *anisotropic* (*Wulff*) Φ*-perimeter*

$$P_\Phi(E) = \int_{\partial E} \Phi(\nu_E(x))\, \mathrm{d}\mathscr{H}^{N-1}(x),$$

where $\nu_E : \partial E \to \mathbb{S}^{N-1}$ is the inner unit normal of the convex body $E \subset \mathbb{R}^N$ (defined $\mathscr{H}^{N-1}$-a.e. on ∂E) and $\Phi : \mathbb{R}^N \to [0, +\infty]$ is a fixed lower-semicontinuous, positively 1-homogeneous and convex function. Clearly, if $\Phi = |\cdot|$, then $P_\Phi(E) = P(E)$. Similarly to Equation (1.1), the monotonicity of the Φ-perimeter is a consequence of one of the following: the *Cauchy formula* for the anisotropic perimeter [5, Sect. 7]; the monotonicity property of *mixed volumes* [5, Sect. 8]; the fact that the anisotropic perimeter is decreased under intersection with half-spaces [25, Rem. 20.3].

In passing, we mention that the monotonicity property holds even for perimeter functionals of the *non-local* type, as the *fractional perimeter* [16, Lem. B.1] and, more generally, *non-local perimeters* induced by a suitable interaction kernel [4, Cor. 2.30].

The monotonicity property of perimeters has gained increasing attention in recent years. We refer to [7, 8, 24, 31] and to the survey [20] for quantitative versions of the monotonicity inequality (see also [22] for the quantitative monotonicity in the *non-local* setting), and to [3, 9, 12, 15, 21, 23, 26] for some applications and related results.

1.2 | Main result

In this note, we are interested in studying the monotonicity property on nested convex bodies for the class of *weighted perimeters*. Given a Borel function $f : \mathbb{R}^N \to [0, +\infty]$, we let

$$P_f(E) = \int_{\partial E} f(x) d\mathscr{H}^{N-1}(x) \tag{1.2}$$

be the *weighted* (*isotropic*) *perimeter* of the convex body $E \subset \mathbb{R}^N$. Clearly, if $f \equiv c$ for some $c \in [0, +\infty)$, then $P_f = cP$, a constant multiple of the Euclidean perimeter. Weighted perimeters have been largely investigated in relation to *isoperimetric*, *cluster*, and *Cheeger problems*, see [2, 10, 11, 13, 17–19, 28–30] and the survey [27] for an account on the existing literature.

Our main result is the following rigidity property, namely, the only weighted perimeter satisfying the monotonicity property is (a constant multiple of) the Euclidean perimeter.

Theorem 1.1. *Let $f : \mathbb{R}^N \to [0, +\infty]$ be a Borel function such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. If the weighted perimeter P_f in Equation (1.2) satisfies the monotonicity property, that is,*

$$P_f(A) \le P_f(B)\ \textit{for any two nested convex bodies}\ A \subset B\ \textit{in}\ \mathbb{R}^N, \tag{1.3}$$

then $f \equiv c$ a.e. for some $c \ge 0$.

Theorem 1.1 is quite intuitive. In fact, one clearly expects that, if f is not constant in some direction, then the monotonicity property should be violated on any suitable family of convex bodies with some side (continuously) deforming along that direction. However, one should carefully keep into account the values of f on the entire boundary of each convex body of the family, which forces one to consider deformations in that direction given by graphs of concave functions fixing the boundary of the chosen side.

One may wonder whether the analog of Theorem 1.1 holds for weighted *anisotropic* perimeters. More precisely, given a non-negative Finslerian weight $f : \mathbb{R}^N \times \mathbb{S}^{N-1} \to [0, +\infty]$ (i.e., possibly depending also on the inner unit normal $\nu_E : \partial E \to \mathbb{S}^{N-1}$ of the convex body $E \subset \mathbb{R}^N$) and assuming the monotonicity of the weighted anisotropic perimeter P_f, is it true that $f = f(x, \nu)$ does not depend on x? This is in general false. As a counterexample, consider any bounded vector field $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ with constant divergence, $\operatorname{div} F \equiv \alpha$ for some $\alpha \in [0, +\infty)$, and define the anisotropic weight $f : \mathbb{R}^N \times \mathbb{S}^{N-1} \to [0, +\infty)$ as

$$f(x, \nu) = F(x) \cdot \nu + \beta \quad \text{for } x \in \mathbb{R}^N \text{ and } \nu \in \mathbb{S}^{N-1}, \tag{1.4}$$

where $\beta \in [\,\|F\|_\infty, +\infty)$ ensures the non-negativity of the weight f. By the divergence theorem, the anisotropic weighted perimeter

$$P_f(E) = \int_{\partial E} f(x, \nu_E(x)) d\mathcal{H}^{N-1}(x) \tag{1.5}$$

on the convex body $E \subset \mathbb{R}^N$ satisfies

$$\begin{aligned} P_f(E) &= \int_{\partial E} f(x, \nu_E(x)) d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial E} F(x) \cdot \nu_E(x) d\mathcal{H}^{N-1}(x) + \beta P(E) \\ &= \int_E \operatorname{div} F dx + \beta P(E) \\ &= \alpha |E| + \beta P(E), \end{aligned}$$

readily yielding the desired monotonicity property in virtue of that of the Euclidean perimeter (1.1) and that of the Lebesgue measure with respect to nestedness.

Despite the counterexample in Equation (1.4), from Theorem 1.1 we can deduce the following result, which provides a partial analog of the rigidity property in the anisotropic regime under some additional structural assumptions on the weight function.

Corollary 1.2. *Let $f : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty]$ be a Borel function such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{2N})$. Assume that there exist a radial Borel function $g : \mathbb{R}^N \to [0, +\infty]$ and a lower semicontinuous, 1-homogeneous and convex function $\Phi : \mathbb{R}^N \to (0, +\infty]$ such that*

$$f(x, v) = g(x)\,\Phi(v) \quad \text{for } x \in \mathbb{R}^N \text{ and } v \in \mathbb{R}^N. \tag{1.6}$$

If the anisotropic weighted perimeter P_f in Equation (1.5) satisfies the monotonicity property (1.3), then $g \equiv c$ a.e. for some $c \geq 0$.

The proof of Corollary 1.2 combines the invariance of the monotonicity property with respect to rotations with Theorem 1.1.

2 | PROOFS OF THE STATEMENTS

2.1 | Proof of Theorem 1.1

We begin by observing that it is not restrictive to assume that $f \in C^\infty(\mathbb{R}^N)$. Indeed, given $A \subset B$ two nested convex bodies in $\mathbb{R}^N$, the translated sets $A + y \subset B + y$ are still two nested convex bodies for any $y \in \mathbb{R}^N$. Therefore, in virtue of Equation (1.3) and changing variables, we get

$$\int_{\partial A} f(x - y) d\mathcal{H}^{N-1}(x) \leq \int_{\partial B} f(x - y) d\mathcal{H}^{N-1}(x). \tag{2.1}$$

Let now $(\varrho_\varepsilon)_{\varepsilon>0} \subset C^\infty_c(\mathbb{R}^N)$ be any family of non-negative convolution kernels (for instance, $\varrho_\varepsilon = \varepsilon^{-N}\varrho(\cdot/\varepsilon)$ for some $\varrho \in C^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varrho \subset B_1$, $\varrho \geq 0$, and $\int_{\mathbb{R}^N} \varrho dx = 1$). Multiplying Equation (2.1) by $\varrho_\varepsilon(y)$, integrating on $\mathbb{R}^N$ with respect to y, and owing to the Fubini–Tonelli theorem, we infer that

$$\begin{aligned} \int_{\partial A} f_\varepsilon(x) d\mathcal{H}^{N-1}(x) &= \int_{\partial A} \int_{\mathbb{R}^N} f(x - y)\,\varrho_\varepsilon(y) dy\, d\mathcal{H}^{N-1}(x) \\ &\leq \int_{\partial B} \int_{\mathbb{R}^N} f(x - y)\,\varrho_\varepsilon(y) dy\, d\mathcal{H}^{N-1}(x) = \int_{\partial B} f_\varepsilon(x) d\mathcal{H}^{N-1}(x), \end{aligned}$$

where $f_\varepsilon = f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^N)$ is the standard convolution. By the arbitrariness of the nested convex bodies A and B, the weight f_ε still verifies Equation (1.3) for each $\varepsilon > 0$. If we show that $\nabla f_\varepsilon \equiv 0$ for each $\varepsilon > 0$, then also $\nabla f \equiv 0$ in the sense of distributions, and thus f is equivalent to a constant function.

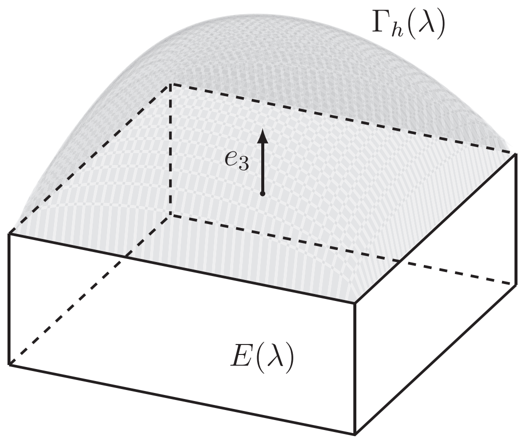

FIGURE 1 The set $E(\lambda)$ and its deformation $F_h(\lambda)$ for $\lambda > 0$ and a given concave function $h : [-\delta, \delta]^2 \to \mathbb{R}$ vanishing on the boundary of its domain.

Consequently, from now on, we assume that $f \in C^\infty(\mathbb{R}^N)$. We now claim that $\partial_{x_N} f(x) = 0$ for each $x \in \mathbb{R}^N$. By the translation invariance in Equation (2.1), we just need to show that $\partial_{x_N} f(0) = 0$.

Let $\delta > 0$ to be chosen later on. For $\lambda \in \mathbb{R}$, we define

$$E(\lambda) = \begin{cases} [-\delta, \delta]^{N-1} \times [-\delta, 0] & \text{for } \lambda \geq 0, \\ [-\delta, \delta]^{N-1} \times [0, \delta] & \text{for } \lambda < 0. \end{cases}$$

Moreover, given $h : [-\delta, \delta]^{N-1} \to \mathbb{R}$ any concave function vanishing on the boundary of $[-\delta, \delta]^{N-1} \subset \mathbb{R}^{N-1}$, we set

$$\Gamma_h(\lambda) = \begin{cases} \{x = (x', x_N) \in \mathbb{R}^N : x' \in [-\delta, \delta]^{N-1} \text{ and } 0 \leq x_N \leq \lambda h(x')\} & \text{for } \lambda \geq 0, \\ \{x = (x', x_N) \in \mathbb{R}^N : x' \in [-\delta, \delta]^{N-1} \text{ and } \lambda h(x') \leq x_N \leq 0\} & \text{for } \lambda < 0, \end{cases}$$

and we refer to Figure 1 for a visual aid in the three-dimensional case. Note that $E(\lambda)$ and $F(\lambda) = E(\lambda) \cup \Gamma_h(\lambda)$ are convex bodies in $\mathbb{R}^N$ with $E(\lambda) \subset F(\lambda)$ for all $\lambda \in \mathbb{R}$. Hence, in virtue of Equation (1.3), we get that

$$P_f(F(\lambda)) \geq P_f(E(\lambda)) \quad \text{for all } \lambda \in \mathbb{R}.$$

By the area formula, the above inequality rewrites as

$$\int_{[-\delta,\delta]^{N-1}} f(x', \lambda h(x')) \sqrt{1 + \lambda^2 |\nabla h(x')|^2} - f(x', 0) dx' \geq 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

In particular, the function $\ell : \mathbb{R} \to [0, +\infty)$, given by

$$\ell(\lambda) = \int_{[-\delta,\delta]^{N-1}} f(x', \lambda h(x')) \sqrt{1 + \lambda^2 |\nabla h(x')|^2} dx' \quad \text{for } \lambda \in \mathbb{R},$$

achieves its minimum at $\lambda = 0$, so that $\ell'(0) = 0$. Since f is a smooth function, we can exchange the differentiation and the integration signs, obtaining

$$\int_{[-\delta,\delta]^{N-1}} \partial_{x_N} f(x', 0)\, h(x') dx' = 0 \tag{2.2}$$

for any concave function $h : [-\delta, \delta]^{N-1} \to \mathbb{R}$ vanishing on the boundary of $[-\delta, \delta]^{N-1}$. In particular, $h(x') > 0$ for all $x' \in (-\delta, \delta)^{N-1}$ as soon as $h \not\equiv 0$. By contradiction, if $\partial_{x_N} f(0) \neq 0$, then, by smoothness of f, we may assume that $\partial_{x_N} f(x)$ has constant sign for each $x \in B_r(0)$ for some $r > 0$. Choosing $\delta > 0$ so small that $[-\delta, \delta]^{N-1} \times \{0\} \subset B_r(0)$, the equality (2.2) immediately yields a contradiction.

In the previous argument, the choice of fixing the Nth component does not play any role and can be repeated almost *verbatim* to show that $\partial_{x_i} f(0) = 0$ for each $i = 1, \dots, N$. Thus, again by the translation invariance (2.1), we get that $\nabla f(x) = 0$ for all $x \in \mathbb{R}^N$, yielding the conclusion.

Remark 2.1. In the above proof, one needs much less than the monotonicity of the perimeter on nested convex bodies in order to conclude that the weight is constant. Indeed, it would be enough to know that, for each direction $e_i \in \mathbb{S}^{N-1}$, $i = 1, \dots, N$, and each point $x \in \mathbb{R}^N$, the monotonicity property holds on two hypercubes (not necessarily with the same edge size) with a face containing x and orthogonal to e_i with opposite outward normals $\pm e_i$ on that face.

2.2 | Proof of Corollary 1.2

Let us denote by $\mathrm{SO}(N)$ be the special orthogonal group, and let $\mu \in \mathscr{P}(\mathrm{SO}(N))$ be the (unique) *Haar probability measure* on $\mathrm{SO}(N)$ (see [14] for a detailed exposition). Given $A \subset B$ two nested convex bodies in $\mathbb{R}^N$, the rotated sets $\mathcal{R}(A) \subset \mathcal{R}(B)$ are still two nested convex bodies for any $\mathcal{R} \in \mathrm{SO}(N)$. Therefore, in virtue of Equation (1.3) and changing variables, we get

$$\int_{\partial A} f(\mathcal{R}(x), \mathcal{R}(\nu_A(x))) d\mathscr{H}^{N-1}(x) \le \int_{\partial B} f(\mathcal{R}(x), \mathcal{R}(\nu_B(x))) d\mathscr{H}^{N-1}(x), \tag{2.3}$$

owing to the elementary facts that $\mathcal{R}(\partial E) = \partial \mathcal{R}(E)$ and that $\nu_{\mathcal{R}(E)}(\mathcal{R}(x)) = \mathcal{R}(\nu_E(x))$ for $\mathscr{H}^{N-1}$-a.e. $x \in \partial E$ whenever $E \subset \mathbb{R}^N$ is a convex body (refer to [25, Sect. 17.1] for a precise justification). Due to Equation (1.6) and the radial assumption on g, inequality (2.3) rewrites as

$$\int_{\partial A} g(x)\, \Phi(\mathcal{R}(\nu_A(x))) d\mathscr{H}^{N-1}(x) \le \int_{\partial B} g(x)\, \Phi(\mathcal{R}(\nu_B(x))) d\mathscr{H}^{N-1}(x) \tag{2.4}$$

for $\mathcal{R} \in \mathrm{SO}(N)$. We now claim that the function

$$\mathbb{S}^{N-1} \ni \nu \mapsto \int_{\mathrm{SO}(N)} \Phi(\mathcal{R}(\nu)) d\mu(\mathcal{R})$$

is constant. Indeed, given any $\nu \in \mathbb{S}^{N-1}$, we can find $\mathcal{R}_\nu \in \mathrm{SO}(N)$ such that $\nu = \mathcal{R}_\nu(\mathrm{e}_1)$. Due to the invariance properties of the Haar measure μ, we can compute

$$\begin{aligned} \int_{\mathrm{SO}(N)} \Phi\,(\mathcal{R}(\nu))\; d\mu(\mathcal{R}) &= \int_{\mathrm{SO}(N)} \Phi\,(\mathcal{R}(\mathcal{R}_\nu(e_1)))\, d\mu(\mathcal{R}) \\ &= \int_{\mathrm{SO}(N)} \Phi\,(\mathcal{Q}(e_1))\; d\mu(\mathcal{Q}\mathcal{R}_\nu^{-1}) \\ &= \int_{\mathrm{SO}(N)} \Phi\,(\mathcal{Q}(e_1))\; d\mu(\mathcal{Q}) \end{aligned} \tag{2.5}$$

where, with a slight abuse of notation, $\mathcal{Q} \mapsto \mu(\mathcal{Q}\mathcal{R}_\nu^{-1})$ stands for the push-forward of the measure μ with respect to the right translation by $\mathcal{R}_\nu^{-1}$. Hence, integrating on $\mathrm{SO}(N)$ with respect to μ, using the Fubini–Tonelli theorem, the above equality, that $\Phi > 0$, and simplifying, from Equation (2.4) we get

$$\int_{\partial A} g(x) d\mathscr{H}^{N-1}(x) \le \int_{\partial B} g(x) d\mathscr{H}^{N-1}(x)$$

for any two nested convex bodies $A \subset B$. The conclusion follows from Theorem 1.1.

Remark 2.2. One could slightly weaken the hypotheses of Corollary 1.2 by allowing Φ to also attain zero. In fact, it is enough to require that the integral in Equation (2.5) is not zero.

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