Massimiliano Carrara
Filippo Mancini
Michele Pra Baldi
Wei Zhu

A New Game Theoretic Semantics (GTS-2) for Weak Kleene Logics

Abstract. Hintikka's game theoretical approach to semantics has been successfully applied also to some non-classical logics. A recent example is Başkent (*A game theoretical semantics for logics of nonsense*, 2020. arXiv:2009.10878), where a game theoretical semantics based on three players and the notion of *dominant winning strategy* is devised to fit both Bochvar and Halldén's logics of nonsense, which represent two basic systems of the family of weak Kleene logics. In this paper, we present and discuss a new game theoretic semantics for Bochvar and Halldén's logics, GTS-2, and show how it generalizes to a broader family of logics of variable inclusions.

Keywords: Weak Kleene logics, Game theoretical semantics, Logics of variable inclusion.

1. Introduction

Traditionally, Kleene's three-valued logics divide into two families: strong and weak.[1] Weak Kleene logics, K_3^w, originate from weak tables (see Table 1 below). The two most important K_3^w are Bochvar [5][2] and Halldén's [24] logics (B and H, respectively), which differ in the designated values they take on.[3]

[1]See Kleene et al. [33].

[2]Translated in Bochvar and Bergmann [6].

[3]There is increasing interest in K_3^w. To give some examples, Coniglio and Corbalan [16] develop sequent calculi for K_3^w, Paoli and Pra Baldi [36] introduce a cut-free calculus (a hybrid system between a natural deduction calculus and a sequent calculus) for H, Ciuni [12] explores some connections between H and Graham Priest's Logic of Paradox, LP, and Ciuni and Carrara [13] focus on logical consequence in H.

[4]To mention a few more examples, Barrio et al. [1], Cobreros and Carrara [15], Bonzio et al. [8], Ciuni and Carrara [14], Da Re et al. [18], and Paoli and Pra Baldi [35].

Special Issue: Strong and weak Kleene logics
Edited by **Gavin St. John and Francesco Paoli**.

Received February 4, 2023

Studia Logica (2024) 112: 1439–1463
https://doi.org/10.1007/s11225-024-10113-5

A good deal of research has been conducted on K_3^w, both on the proof-theoretical and on the algebraic side.[4] Interestingly for us, Başkent [3] gives a Hintikkian game theoretical semantics (GTS) for both B and H. To do that, three-players non-sequential semantic games are employed, together with the notion of *dominant winning strategy*. However, it is not straightforward to give a philosophical justification of why a GTS for B and H should work in the way Başkent [3] describes. It might be possible to develop a different GTS that is semantically equivalent to Başkent's GTS, but hopefully better in some other respects. The present work aims precisely to explore such a possibility.

This paper is organized as follows. In Section 2, we introduce the logics B and H. In Section 3, we present Başkent's GTS for B and H. In Section 4, we develop GTS-2, that is our new GTS for B and H. Finally, in Section 5 we provide a generalization of GTS-2 to a broader family of logics of variable inclusion.

2. Preliminaries

B and H are three-valued logics belonging to the family of K_3^w. Their language is the same as for classical logic, thus containing a unary operation $\neg$ and two binary operations $\wedge, \vee$. This will be the only language we will refer to in the first three sections of the paper. However, in Section 5 we will work with arbitrary languages and different logics, so it is convenient to state the required logical preliminaries on a greater level of generality.

As usual, given a propositional language $\mathcal{L}$, we denote by $\mathbf{Fm}_{\mathcal{L}}$ the (absolutely free) algebra of $\mathcal{L}$-formulas built over a countable set of variables $\mathsf{Var} = \{x_1, x_2, \dots\}$, and by $Fm_{\mathcal{L}}$ its underlying universe. We omit the subscript $\mathcal{L}$ when it is clear from the context. We use $\varphi, \phi, \psi, \gamma, \delta \dots$ to denote arbitrary formulas, $\Gamma, \Phi, \Psi, \Sigma, \dots$ for sets of formulas, and $x_1, x_2, \dots, y, z, t, \dots$ for variables. $\mathsf{Var}(\phi)$ is the set of all and only the propositional variables occurring in ϕ, for all $\phi \in Fm$. We also apply Var to sets of formulas by stipulating that, for any $\Gamma \subseteq Fm$, $\mathsf{Var}(\Gamma) = \bigcup\{\mathsf{Var}(\phi) \mid \phi \in \Gamma\}$. For our purposes, we can safely identify a logic L and its associated consequence relation $\models_{\mathrm{L}}$ between sets of formulas and formulas. When the set Γ is in this relation with the formula ϕ, we write $\Gamma \models_{\mathrm{L}} \phi$, as is customary. Recall that a (logical) matrix is a pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an algebra and $F \subseteq A$. By a *valuation* we understand a homomorphism from $\mathbf{Fm}$ to an algebra $\mathbf{A}$, which of course must be of the same language as $\mathbf{Fm}$. Given a logic L and an algebra $\mathbf{A}$ in the language $\mathcal{L}$, we denote by $Val_{\mathrm{L}}(\mathbf{A})$ the set of all valuations

Table 1. The weak tables

ϕ	$\neg\phi$
t	**f**
u	**u**
f	**t**

$\phi \vee \psi$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

$\phi \wedge \psi$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathbf{A}$. A matrix $\langle \mathbf{A}, F\rangle$ is complete for a logic L when $\mathbf{A}$ is in the same language as L and the following holds:

$$\Gamma \models_{\mathrm{L}} \phi \iff \forall V \in Val_{\mathrm{L}}(\mathbf{A}),\ \text{if } V(\Gamma) \subseteq F \text{ then } V(\phi) \in F.$$

The set F represents the set of designated values and, consequently, it can be seen as the realization of a unary relation over the algebra $\mathbf{A}$. Not every logic is complete with respect to a single matrix, but in this paper we will only consider logics which have this property, unless stated otherwise.

It is well-known that the two logics we are mainly interested in, namely B and H, enjoy this property, as we proceed to recall. Let us consider the three-element algebra **WK** described by the truth tables below:

The logics B and H are known to be complete with respect to two matrices based on **WK**. But while the set of designated values of B is $F_{\mathrm{B}} = \{\mathbf{t}\}$, that of H is $F_{\mathrm{H}} = \{\mathbf{t}, \mathbf{u}\}$. In other words, B and H can be defined as follows:

Definition 2.1. $\Gamma \vDash_B \phi$ if and only if there is no valuation $V : \mathbf{Fm} \to \mathbf{WK}$ such that $V(\gamma) \in F_{\mathrm{B}}$ for all $\gamma \in \Gamma$, and $V(\phi) \notin F_{\mathrm{B}}$.

Definition 2.2. $\Gamma \vDash_H \phi$ if and only if there is no valuation $V : \mathbf{Fm} \to \mathbf{WK}$ such that $V(\gamma) \in F_{\mathrm{H}}$ for all $\gamma \in \Gamma$, and $V(\phi) \notin F_{\mathrm{H}}$.

Table 1 provides the full *weak tables* as they can be derived from Bochvar and Bergmann [6] and Halldén [24]. The element **u** is usually called *contaminating*,[5] as for each valuation V on **WK** and each formula $\phi(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ it holds that

$$V(\phi(x_1, \ldots, x_n)) = \mathbf{u} \iff V(x_i) = \mathbf{u},$$

for some $1 \leq i \leq n$. In other words, a formula is interpreted as **u** if and only if one of its arguments is so.

[5]See e.g. Ciuni [12], Ciuni and Carrara [13] and Correia [17].

Bochvar [5] and Halldén [24] interpret the third value **u** as *nonsensical*—or *meaningless*, in Bochvar's jargon.[6] This interpretation goes along with the way **u** propagates to a compound formula from its components: the sense of a compound sentence depends on that of its components, and if some component makes no sense, the sentence as a whole will make no sense either.[7] In their jargon, nonsensical sentences are expressions that are syntactically well-formed and yet fail to convey a proposition.

Here are some failures of classically valid inferences in H, where $\supset$ is the material implication defined as $\phi \supset \psi = \neg\phi \vee \psi$:

$$\phi \wedge \neg\phi \not\models_H \psi \qquad \textit{Ex Contradictione Quodlibet} \qquad \text{(ECQ)}$$
$$\phi \wedge \psi \not\models_H \psi \qquad \textit{Conjunctive Simplification} \qquad \text{(CS)}$$
$$\phi, \phi \supset \psi \not\models_H \psi \qquad \textit{Modus Ponens} \qquad \text{(MP)}$$
$$\neg\psi, \phi \supset \psi \not\models_H \neg\phi \qquad \textit{Modus Tollens} \qquad \text{(MT)}$$
$$\phi \supset \psi, \psi \supset \chi \not\models_H \phi \supset \chi \qquad \textit{Transitivity of Conditional} \qquad (\text{TR} \supset)$$
$$\phi \supset (\psi \wedge \neg\psi) \not\models_H \neg\phi \qquad \textit{Reductio ad Absurdum} \qquad \text{(RAA)}$$

The failure of ECQ makes H paraconsistent. Further, all the classical H-invalid inferences turn out to be B-valid. By contrast, all the H-valid formulas turn out to be B-invalid. More generally, B has no tautology, which makes it paracomplete—i.e., $\not\models_{\text{B}} \phi \vee \neg\phi$—even if it is not paraconsistent—i.e., ECQ is B-valid. A relevant failure of a classically valid inference in B is:

$$\phi \not\models_{\text{B}} \phi \vee \psi \qquad \textit{Disjunctive Adjunction} \qquad \text{(DA)}$$

Let us now move on and examine a different (game theoretical) semantics for B and H.

[6] Bochvar [5] and Halldén [24] use *nonsensical*—or *meaningless*—as an umbrella term including paradoxical statements such as the Liar and Russell's paradoxes, vague sentences, denotational failure, and ambiguity. Though, there are some issues concerning the interpretation of **u**. Recently, some new interpretations have been proposed. For example, see Beall [4], Szmuc [39], Carrara and Zhu [10], Boem and Bonzio [7], Carrara and Zhu [11], Ferguson [19], and Szmuc and Ferguson [38]. Here, we do not enter this topic since this would go much beyond the purpose of the present discussion.

[7] The principle is also endorsed by Goddard [22] and Goddard and Routley [23].

3. Başkent's GTS for B and H

In the standard matrix semantics introduced in the previous Section 2, a logic can be defined via preservation of designated values under all valuations. Instead, the key idea behind a game theoretic semantics (GTS) is to model a logic on the basis of the outcomes of a semantic game.[8] In GTS, some players play against each other to win the game by reaching atomic formulas with specific truth-values based on their roles, and following some well-defined rules—i.e., the rules of the game. Specifically, which truth-value ϕ obtains under a valuation depends on who, among the players, has a winning strategy, that is a set of moves which guarantees victory regardless of how the opponents play. Let us be more precise and present the GTS developed by Başkent [3] for B and H. First, we define a model M for semantic games:

Definition 3.1. A model M for semantic games for B and H is a pair $M = \langle D, V \rangle$, where $D \subseteq \mathsf{Var}$ is a non-empty domain on which the game is played and V is a valuation on **WK**.

Second, we define a semantic game for B and H, $S_{\mathrm{BH}}(\phi, M)$, for a given formula ϕ in a model M:

Definition 3.2. A semantic game $S_{\mathrm{BH}}(\phi, M)$ for a formula ϕ in a model M is a tuple $\langle \pi, \rho, \sigma, \tau, F \rangle$ where:

- π is the set of players: $\pi = \{\text{Falsifier}, \text{Verifier}, \text{Dominator}\}$.
- ρ is the set of game rules, that is inductively defined as follows:
  - (ρ_x) If $\phi \in \mathsf{Var}$, the game terminates, and Verifier wins if $V(\phi) = \mathbf{t}$, Falsifier wins if $V(\phi) = \mathbf{f}$, and Dominator wins if $V(\phi) = \mathbf{u}$;
  - ($\rho_\neg$) If $\phi = \neg\psi$, Falsifier and Verifier switch roles, Dominator keeps his role, and the game continues as $S_{\mathrm{BH}}(\psi, M)$;
  - ($\rho_\wedge$) If $\phi = \chi \wedge \psi$, Falsifier and Dominator choose between χ and ψ independently and simultaneously;
  - ($\rho_\vee$) If $\phi = \chi \vee \psi$, Verifier and Dominator choose between χ and ψ independently and simultaneously;
  - (ρ_S) Dominator's strategy strictly dominates the Verifier's and Falsifier's.
- σ is the set of positions (or moves) of the game, i.e., the set of tuples $\langle p_i, \varphi \rangle$, where $p_i \in \pi$ and φ is a sub-formula of ϕ, such that:

[8] For an introduction to Hintikka's game theoretical semantics see e.g. Hintikka [26], Hintikka and Sandu [29], and Pietarinen [37]. GTSs for some many-valued logics have been presented in Fermüller [20] and Fermüller and Majer [21].

(σ_x) If $\phi \in \mathsf{Var}$, then $\langle p_i, \phi \rangle \in \sigma$ for all $p_i \in \pi$;
($\sigma_\neg$) If $\phi = \neg\psi$, then $\langle p_i, \phi \rangle \in \sigma$ for all $p_i \in \pi$, and $\langle p_j, \psi \rangle \in \sigma$ for some $p_j \in \pi$ depending on ψ's main connective;
($\sigma_\wedge$) If $\phi = \chi \wedge \psi$, $\langle \text{Falsifier}, \phi \rangle \in \sigma$, $\langle \text{Dominator}, \phi \rangle \in \sigma$, and $\langle p_j, \chi \rangle$, $\langle p_k, \psi \rangle \in \sigma$ for some $p_j, p_k \in \pi$ depending on χ and ψ's main connectives;
($\sigma_\vee$) If $\phi = \chi \vee \psi$, $\langle \text{Verifier}, \phi \rangle \in \sigma$, $\langle \text{Dominator}, \phi \rangle \in \sigma$, and $\langle p_j, \chi \rangle$, $\langle p_k, \psi \rangle \in \sigma$ for some $p_j, p_k \in \pi$ depending on χ and ψ's main connectives.

- τ is the set of positions of the game-token in the case of concurrent play, i.e., the set of pairs of concurrent positions of two different players that is determined by the following clauses:

($\tau_\wedge$) If $\phi = \chi \wedge \psi$, $\{\langle \text{Falsifier}, \phi \rangle, \langle \text{Dominator}, \phi \rangle\} \in \tau$;
($\tau_\vee$) If $\phi = \chi \vee \psi$, $\{\langle \text{Verifier}, \phi \rangle, \langle \text{Dominator}, \phi \rangle\} \in \tau$.

- F is the set of designated values: $F_{\text{B}} = \{\mathbf{t}\}$ for B, and $F_{\text{H}} = \{\mathbf{t}, \mathbf{u}\}$ for H.

Thus, semantic games for B and H are 3-players non-collaborative non-sequential perfect-information games. On the contrary, semantic games for classical propositional logic are 2-players non-collaborative sequential[9] perfect-information games. A *run* (or *play*) of a semantic game $S_{\text{BH}}(\phi, M)$ is a sequence of moves from τ which starts with $\{\langle p_i, \phi \rangle\}$ and ends with a position with a propositional variable occurring in ϕ, e.g., $\{\langle p_i, x \rangle\}$ where $x \in \mathsf{Var}(\phi)$.

Then, we define the notion of *dominant winning strategy*, that is the key notion Başkent's GTS for B and H makes use of to evaluate the truth-value of a given formula.

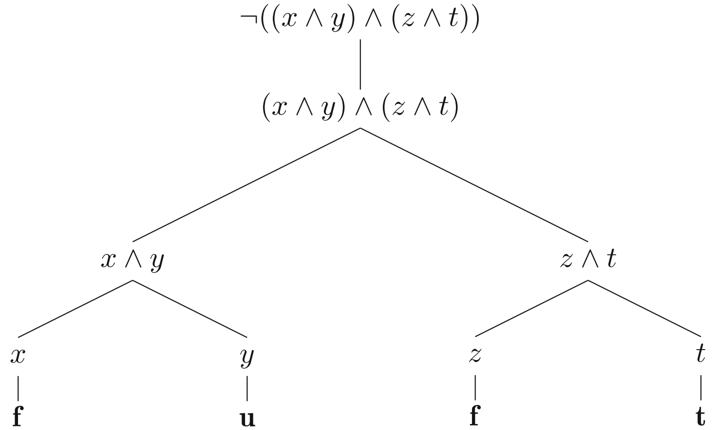
DEFINITION 3.3. A *winning strategy* for $S_{\text{BH}}(\phi, M)$ is a set of rules that guides one player throughout the play to win, regardless of how the opponents play. A winning strategy is called *dominant* if and only if it determines the truth-value of ϕ.

Since some games may allow two players to have both a winning strategy, Başkent [3] introduces dominant winning strategies which, based on (ρ_S), uniquely determine the truth-value of ϕ.

Let us make an example. Consider the formula $\neg((x \wedge y) \wedge (z \wedge t))$—let us call it ϕ. We want to evaluate ϕ in a model M such that $V(x) = V(z) = \mathbf{f}$, $V(t) = \mathbf{t}$, and $V(y) = \mathbf{u}$. A good way to visualize $S_{\text{BH}}(\phi, M)$ is by using game trees, i.e., a step-by-step decomposition of ϕ into its sub-formulas, down to

[9]That is, there are not concurrent moves—i.e., τ is a set of singletons.

its propositional variables. Here, every node correspond to one possible move of the game—i.e., it represents a point of choice for a player.



It should be clear that both the Verifier and the Dominator have a winning strategy for $S_{\mathrm{BH}}(\phi, M)$. To see that, let us go over the game. It starts from the top, with $\neg((x \wedge y) \wedge (z \wedge t))$. Since the main connective of ϕ is $\neg$, the rule to apply is $(\rho_{\neg})$, so that Falsifier and Verifier switch roles, whereas Dominator keeps his. Then, the game continues as $S_{\mathrm{BH}}((x \wedge y) \wedge (z \wedge t), M)$. Here, the main connective is $\wedge$, so that the second move is concurrently up to both Falsifier and Dominator, as ruled by $(\rho_{\wedge})$. They can choose between $x \wedge y$ and $z \wedge t$. Regardless of their choice, Falsifier and Dominator also retain control of the third move, again because of $(\rho_{\wedge})$. Now, since they both have the opportunity to end the game by reaching a propositional variable with the truth-value they want (**f** for the Falsifier, and **u** for the Dominator), they both have a winning strategy for $S_{\mathrm{BH}}((x \wedge y) \wedge (z \wedge t), M)$. Consequently, both the (initial) Verifier and Dominator possess a winning strategy for $S_{\mathrm{BH}}(\neg((x \wedge y) \wedge (z \wedge t)), M)$. However, according to (ρ_S), only the Dominator has a *dominant* winning strategy—which determines **u** as the truth-value of ϕ in the model M, based on the theorem proved by Başkent [3] that we will discuss a little further below. The following is an example of run for $S_{\mathrm{BH}}(\phi, M)$, where both the Verifier and the Dominator win the game:

$\{\{\langle \text{Verifier}, \neg((x \wedge y) \wedge (z \wedge t))\rangle, \langle \text{Falsifier}, \neg((x \wedge y) \wedge (z \wedge t))\rangle, \langle \text{Dominator}, \neg((x \wedge y) \wedge (z \wedge t))\rangle\}, \{\langle \text{Falsifier}, (x \wedge y) \wedge (z \wedge t)\rangle, \langle \text{Dominator}, (x \wedge y) \wedge (z \wedge t)\rangle\}, \{\langle \text{Falsifier}, x \wedge y\rangle, \langle \text{Dominator}, x \wedge y\rangle\}, \{\langle \text{Falsifier}, x\rangle, \langle \text{Verifier}, x\rangle, \langle \text{Dominator}, x\rangle, \langle \text{Falsifier}, y\rangle, \langle \text{Verifier}, y\rangle, \langle \text{Dominator}, y\rangle\}\}$.

Başkent [3, pp. 72–74, Theorem 3.5] proves an important correctness result. This shows that for every formula ϕ, for every model M, and for every game $S_{\rm BH}(\phi, M)$, having a dominant winning strategy for the (initial) Verifier, the (initial) Falsifier, or the Dominator, matches uniquely with being ϕ true (**t**), false (**f**), or nonsensical (**u**), respectively, under the valuation witnessed by M on **WK**. In other words, evaluations on the weak tables in table 1 and Başkent's semantic games "do exactly the same job". Then, we can take the condition of "having one player a dominant winning strategy" for $S_{\rm BH}(\phi, M)$ as defining the truth-value of ϕ, depending on who the player is. With this tools at hand, it is shown that the semantic definition of B and H introduced in Section 2 is fully captured by means of Başkent's semantic games, as follows[10]:

THEOREM 3.4. $\Delta \vDash_{\rm B} \psi$ *if and only if there is no model M such that the Verifier has a dominant winning strategy for $S_{\rm BH}(\phi, M)$ for all $\phi \in \Delta$, but he does not have it*[11] *for $S_{\rm BH}(\psi, M)$.*

THEOREM 3.5. $\Delta \vDash_{\rm H} \psi$ *if and only if there is no model M such that either the Verifier or the Dominator have a dominant winning strategy for $S_{\rm BH}(\phi, M)$ for every $\phi \in \Delta$, and the Falsifier has a dominant winning strategy for $S_{\rm BH}(\psi, M)$.*

Let us now show the invalidity of some relevant inferences in B and H by using Başkent's GTS.

EXAMPLE 3.1. (ECQ) $x \wedge \neg x \nvDash_H y$

PROOF. Let M be a model such that $V(x) = \mathbf{u}$ and $V(y) = \mathbf{f}$. Then, consider the two following games: $S_{\rm BH}(x \wedge \neg x, M)$ and $S_{\rm BH}(y, M)$. The Dominator clearly has a dominant winning strategy for $S_{\rm BH}(x \wedge \neg x, M)$, as x takes value **u** in M. Therefore, both the Verifier and the Falsifier do not have a dominant winning strategy—notice that there is no way for them to win. Thus, since every semantic game is determined in Başkent's GTS, the Dominator has a dominant winning strategy. However, the Falsifier has a dominant winning strategy for $S_{\rm BH}(y, M)$. Thus, by Theorem 3.5, $x \wedge \neg x \nvDash_H y$. ∎

[10] Def. 3.4 and 3.5 are somehow redundant since the designated values are specified in F—i.e., they are part of the definition of semantic game. However, we believe that our choice makes this introduction easier.

[11] That is, either the Falsifier or the Dominator have a dominant winning strategy for $S_{\rm BH}(\psi, M)$. This is so because in Başkent's GTS, for every semantic game there is always a dominant winning strategy (semantics games are determined, see Section 4) for exactly one player (dominant winning strategies are player-exclusive).

Example 3.2. (DA) $x \not\models_{\mathrm{B}} x \vee y$

Proof. Let M be a model such that $V(x) = \mathbf{t}$ and $V(y) = \mathbf{u}$. Then, consider the two following games: $S_{\mathrm{BH}}(x, M)$ and $S_{\mathrm{BH}}(x \vee y, M)$. Since x is true in M, the Verifier has a dominant winning strategy for $S_{\mathrm{BH}}(x, M)$. Since y obtains $\mathbf{u}$ in M, the Dominator has a dominant winning strategy for $S_{\mathrm{BH}}(x \vee y, M)$. Therefore, $x \not\models_{\mathrm{B}} x \vee y$ by Theorem 3.4. ■

This concludes our presentation of Başkent's GTS. We are now ready to develop and discuss our own game theoretic semantics for B and H.

4. GTS-2: A New GTS for B and H

In this section we present a new game theoretical semantics for B and H which differs from Başkent's GTS in the following two respects. First, it resorts to 2-players sequential semantic games instead of 3-players non-sequential games. Second, because of the previous difference, it just needs winning strategies—and not dominant winning strategies—to determine the truth-values of any formula. We will call such a new game theoretic semantics GTS-2.

The idea is simple. Informally, the Verifier and the Falsifier play a game that consists of two subgames,[12] to be played in the following order. The first game is collaborative: both players cooperate to win. Here, they aim to reach a nonsensical propositional variable by following some suitable game rules. If they have a winning strategy, the initial formula obtains $\mathbf{u}$ as truth-value, and the (whole) game ends. If they do not have a winning strategy, they start playing a second subgame, which is the standard classical semantic game. Then, if the Verifier has a winning strategy for such second subgame, the formula is true; if the Falsifier has it, the formula is false.

Let us now be more precise and present our game theoretical semantics formally. We define a model for semantics games as in Definition 3.1. Before stating the definition of semantic game, we need to introduce the following notation, which will be instrumental also for Section 5. Let ϕ be a classical formula and $M = \langle D, V \rangle$ be a model for semantic games for B and H. We define a new valuation $V^{\phi} : \mathbf{Fm} \to \mathbf{WK}$ as the unique extension of the following map V^{ϕ}_{Var}, defined for every $x \in \mathsf{Var}$ as:

$$V^{\phi}_{\mathsf{Var}}(x) = \begin{cases} V(x) \text{ if } x \in \mathsf{Var}(\phi) \\ \mathbf{f} \text{ otherwise.} \end{cases} \qquad \text{(Val-}\mathbf{WK}\text{)}$$

[12] The idea of using subgames was inspired to us by Hintikka and Carlson [28].

Given $M = \langle D, V \rangle$, we set $M^{\phi} = \langle D, V^{\phi} \rangle$. Notice that V, V^{ϕ} agree on $\mathsf{Var}(\phi)$, so $V(\phi) = V^{\phi}(\phi)$.

Then, we define a semantic game for B and H as follows:

Definition 4.1. A semantic game $G_{\mathrm{BH}}(\phi, M)$ for a formula ϕ in a model M is a triple $\langle G_1(\phi, M), G_2(\phi, M^{\phi}), R \rangle$, such that:

- $G_1(\phi, M)$ is the first subgame of $G_{\mathrm{BH}}(\phi, M)$, that is a triple $\langle \pi, \rho_1, \sigma_1 \rangle$ where:

  ■ π is the set of players: $\pi = \{\text{Falsifier}, \text{Verifier}\}$.

  ■ ρ_1 is the set of game rules. It is inductively defined as follows, where $p_a \in \pi$ is an arbitrary chosen but fixed player:

  (ρ_{1x}) If $\phi \in \mathsf{Var}$, $G_1(\phi, M)$ terminates, and both the players win if and only if $V(\phi) = \mathbf{u}$;

  ($\rho_{1\neg}$) If $\phi = \neg\psi$, p_a chooses ψ. The game continues as $G_1(\psi, M)$.

  ($\rho_{1\circ}$) If $\phi = \psi_1 \circ \psi_2$ for $\circ \in \{\wedge, \vee\}$, p_a chooses among ψ_1, ψ_2, say ψ_i, and the game continues as $G_1(\psi_i, M)$.

  ■ σ_1 is the set of positions (or moves) of the game, i.e., the set of tuples $\langle p_i, \varphi \rangle$, where $p_i \in \pi$, φ is a sub-formula of ϕ, and $p_a \in \pi$ refers to the player chosen in ρ_1, such that:

  (σ_{1x}) If $\phi \in \mathsf{Var}$, then $\langle \text{Verifier}, \phi \rangle, \langle \text{Falsifier}, \phi \rangle \in \sigma$;

  ($\sigma_{1\neg}$) If $\phi = \neg\psi$, then $\langle p_a, \phi \rangle \in \sigma$ and $\langle p_a, \psi_i \rangle \in \sigma$ for some $1 \leq i \leq n$.

  ($\sigma_{1\circ}$) If $\phi = \psi_1 \circ \psi_2$ for $\circ \in \{\wedge, \vee\}$, $\langle p_a, \psi_i \rangle \in \sigma_1$, for some $i \in \{1, 2\}$.

- $G_2(\phi, M^{\phi})$ is the second subgame of $G_{\mathrm{BH}}(\phi, M)$, that is a triple $\langle \pi, \rho_2, \sigma_2 \rangle$ where:

  ■ π is the set of players: $\pi = \{\text{Falsifier}, \text{Verifier}\}$.

  ■ ρ_2 is the set of game rules, that is inductively defined as follows:

  (ρ_{2x}) If $\phi \in \mathsf{Var}$, $G_2(\phi, M^{\phi})$ terminates, the Verifier wins if $V^{\phi}(\phi) = \mathbf{t}$, and the Falsifier wins if $V^{\phi}(\phi) = \mathbf{f}$;

  ($\rho_{2\neg}$) If $\phi = \neg\psi$, Falsifier and Verifier switch roles, and the game continues as $G_2(\psi, M^{\phi})$;

  ($\rho_{2\wedge}$) If $\phi = \chi \wedge \psi$, Falsifier chooses between χ and ψ;

  ($\rho_{2\vee}$) If $\phi = \chi \vee \psi$, Verifier chooses between χ and ψ.

  ■ σ_2 is the set of positions (or moves) of the game, i.e., the set of tuples $\langle p_i, \varphi \rangle$, where $p_i \in \pi$ and φ is a sub-formula of ϕ, such that:

  (σ_{2x}) If $\phi \in \mathsf{Var}$, then $\langle \text{Verifier}, \phi \rangle, \langle \text{Falsifier}, \phi \rangle \in \sigma$;

  ($\sigma_{2\neg}$) If $\phi = \neg\psi$, then $\langle \text{Verifier}, \phi \rangle, \langle \text{Falsifier}, \phi \rangle \in \sigma$, and $\langle p_i, \psi \rangle \in \sigma$ for some $p_i \in \pi$ depending on ψ's main connective;

($\sigma_{2\wedge}$) If $\phi = \chi \wedge \psi$, $\langle \text{Falsifier}, \phi \rangle \in \sigma$, and $\langle p_j, \chi \rangle, \langle p_k, \psi \rangle \in \sigma$ for some $p_j, p_k \in \pi$ depending on χ and ψ's main connectives;

($\sigma_{2\vee}$) If $\phi = \chi \vee \psi$, $\langle \text{Verifier}, \phi \rangle \in \sigma$, and $\langle p_j, \chi \rangle, \langle p_k, \psi \rangle \in \sigma$ for some $p_j, p_k \in \pi$ depending on χ and ψ's main connectives.

- R is the set of rules for $G_{\text{BH}}(\phi, M)$:

  (R) Verifier and Falsifier play $G_1(\phi, M)$ first. If they have a winning strategy for $G_1(\phi, M)$, then $G_{\text{BH}}(\phi, M)$ ends; otherwise, they move on and play $G_2(\phi, M^\phi)$.

Note that we do not need the set τ of positions of the game-token in the case of concurrent play, since both $G_1(\phi, M)$ and $G_2(\phi, M^\phi)$ are sequential games. Also, we did not include the set of designated values, since we are going to define validity separately.

A *run* (or *play*) of a semantic game $G_1(\phi, M)$ is a sequence of moves from σ_1 which starts with $\langle p_i, \phi \rangle$ and ends with a position with a propositional variable occurring in ϕ, e.g., $\langle p_i, x \rangle$, where $x \in \mathsf{Var}(\phi)$. Similarly for $G_2(\phi, M^\phi)$.

Then, we define the notion of *winning strategy* for every game involved in GTS-2:

Definition 4.2. A *winning strategy* for:

- $G_1(\phi, M)$ is a set of rules that guides both players throughout the play of $G_1(\phi, M)$ to win;
- $G_2(\phi, M^\phi)$ is a set of rules that guides one player throughout the play of $G_2(\phi, M^\phi)$ to win, regardless of how the opponents play;
- $G_{\text{BH}}(\phi, M)$ is a winning strategy either for $G_1(\phi, M)$ or $G_2(\phi, M^\phi)$.

An important remark is needed here. Clearly, in the above definition of $G_{\text{BH}}(\phi, M)$, the second subgame $G_2(\phi, M^\phi)$ is nothing but a classical semantic game. However, if the model $M = \langle D, V \rangle$ and the formula ϕ are such that $V(x) = \mathbf{u}$ for some $x \in \mathsf{Var}(\phi)$, the resulting M^ϕ is not a model for a classical semantic game, as V^ϕ is not a classical valuation. Thus, if there exists a run of $G_{\text{BH}}(\phi, M)$ where no player has a winning strategy for $G_1(\phi, M)$ and $V(x) = \mathbf{u}$ for some $x \in \mathsf{Var}(\phi)$, the game $G_{\text{BH}}(\phi, M)$ is not well-defined. The following fact excludes this possibility, thus showing that $G_{\text{BH}}(\phi, M)$ is well-defined. Its proof is a specialization of the one for Fact 5.1, and we omit it.

Fact 4.1. Let $G_{\mathrm{BH}}(\phi, M)$ be a semantic game. If no player has a winning strategy for $G_1(\phi, M)$, then M^ϕ is a model of a classical semantic game.

Thus, since $G_2(\phi, M^\phi)$ is a classical semantic game, it is total and determined.[13] On the other hand, it should be easy to see that $G_1(\phi, M)$ is total but not determined. We will tacitly make use the above Fact 4.1 in the remaining part of this section.

We are now ready to prove the correctness theorem for GTS-2.

THEOREM 4.3. *For every semantic game* $G_{\mathrm{BH}}(\phi, M)$*:*

(u) Both the Verifier and the Falsifier have a winning strategy for $G_{\mathrm{BH}}(\phi, M)$ *if and only if* $V(\phi) = \mathbf{u}$ *in* M*;*

(t) Only the Verifier has a winning strategy for $G_{\mathrm{BH}}(\phi, M)$ *if and only if* $V(\phi) = \mathbf{t}$ *in* M*;*

(f) Only the Falsifier has a winning strategy for $G_{\mathrm{BH}}(\phi, M)$ *if and only if* $V(\phi) = \mathbf{f}$ *in* M*.*

PROOF. (u) (Left-to-Right) Let us assume that both the Falsifier and the Verifier have a winning strategy for $G_{\mathrm{BH}}(\phi, M)$. Since $G_2(\phi, M^\phi)$ is a classical game, it is not possible they both have a winning strategy for it. This implies that they both have a winning strategy for $G_1(\phi, M)$. Then, because of (ρ_{1x}) there is $x \in \mathsf{Var}(\phi)$ such that $V(x) = \mathbf{u}$ in M, and therefore $V(\phi) = \mathbf{u}$ in M because of the contamination feature of $\mathbf{u}$.
(Right-to-Left). Let us assume that $V(\phi) = \mathbf{u}$ in M. Thus, $V(x) = \mathbf{u}$ for some $x \in \mathsf{Var}(\phi)$. Therefore, because of (ρ_{1x}) and $(\rho_{1\circ})$, both the Falsifier and the Verifier have a winning strategy for $G_1(\phi, M)$, and then they both have it for $G_{\mathrm{BH}}(\phi, M)$.

We only prove (t), as (f) relies on a similar argument.
(Left-to-Right). Suppose the Verifier is the unique player having a winning for $G_{\mathrm{BH}}(\phi, M)$. Therefore, he has it for $G_2(\phi, M^\phi)$ only, by ρ_1. Since $G_2(\phi, M^\phi)$ is a classical semantic game, $V^\phi(\phi) = \mathbf{t}$. Moreover, since V, V^ϕ agree on $\mathsf{Var}(\phi)$, it follows that $V(\phi) = \mathbf{t}$.
(Right-to-Left). If $V(\phi) = \mathbf{t}$, there is no variable occurring in ϕ which maps to $\mathbf{u}$. This entails that no player has a winning strategy for $G_1(\phi, M)$ so, by (R), the players move on and play $G_2(\phi, M^\phi)$, which is a classical semantic game. Since V^ϕ and V agree on $\mathsf{Var}(\phi)$, we have that $V^\phi(\phi) = \mathbf{t}$. Thus, the Verifier only has a winning strategy for $G_2(\phi, M^\phi)$, being also the unique player having a winning strategy for $G_{\mathrm{BH}}(\phi, M)$. ∎

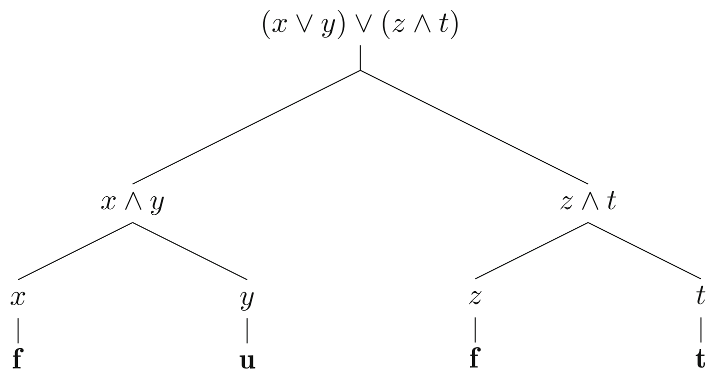
[13] A game is *total* if players always win or lose, so that there are no draws, and it is *determined* if one or other, or both of the players, have a winning strategy.

In light of the above theorem, a characterization of B and H in terms of semantic games immediately follows. The following results can also be obtained as particular cases of Theorems 5.5 and 5.4 of Section 5.

COROLLARY 4.1. $\Delta \vDash_{\mathrm{B}} \psi$ *if and only if there is no model M such that only the Verifier has a winning strategy for $G_{\mathrm{BH}}(\phi, M)$ for all $\phi \in \Delta$, and the Falsifier has a winning strategy for $G_{\mathrm{BH}}(\psi, M)$.*

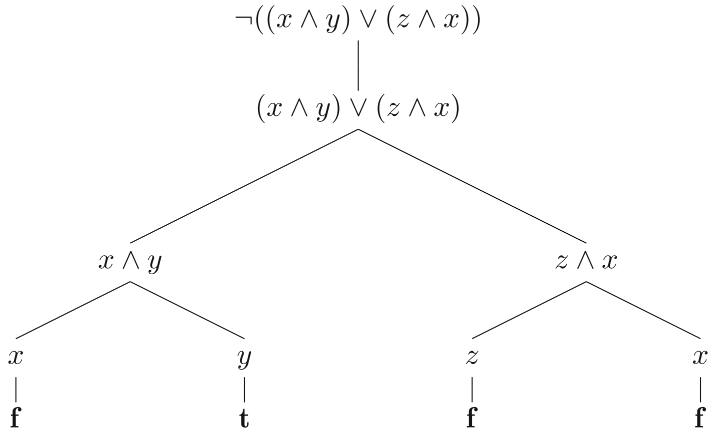
COROLLARY 4.2. $\Delta \vDash_{\mathrm{H}} \psi$ *if and only if there is no model M such that the Verifier has a winning strategy for $G_{\mathrm{BH}}(\phi, M)$ for every $\phi \in \Delta$, but he has not it for $G_{\mathrm{BH}}(\psi, M)$.*

Let us make some examples. Consider the formula $(x \vee y) \vee (z \wedge t)$—let us call it ϕ. We want to evaluate ϕ in a model M such that $V(x) = V(z) = \mathbf{f}$, $V(t) = \mathbf{t}$, and $V(y) = \mathbf{u}$. The game tree of ϕ is as follows:



Then, let us examine $G_{\mathrm{BH}}(\phi, M)$. According to (R), Verifier and Falsifier have to play $G_1(\phi, M)$ first. Here, since there is a propositional variable that is $\mathbf{u}$ in M, y, they have a winning strategy. Thus, $G_{\mathrm{BH}}(\phi, M)$ ends, and ϕ obtains $\mathbf{u}$ in M.

As a further example, consider $\neg((x \wedge y) \vee (z \wedge x))$ - let us call it ψ. We want to evaluate ψ in a model M such that $V(x) = V(z) = \mathbf{f}$, and $V(y) = \mathbf{t}$. ψ's game tree is as follows:



The players start playing $G_1(\psi, M)$. However, since no propositional variables occurring in ψ obtains **u**, Verifier and Falsifier do not have a winning strategy. Therefore, they move to $G_2(\psi, M^{\psi})$ according to (R). Here, the Verifier has a winning strategy. To see that, let us go over this second subgame. In the first step, Verifier and Falsifier switch role. Then, the new Verifier has to move. Since both z and x are false, there is no victory for him on the right branch of the tree. Then, he can go to the left. But again, this is a losing move, since now it is the Falsifier's move, and he can win by choosing x. Therefore, the (initial) Verifier has a winning strategy for $G_2(\psi, M^{\psi})$. Thus, only the Verifier has a winning strategy for $G_{\mathrm{BH}}(\psi, M)$. Consequently, ψ is true in M.

We conclude this section by using GTS-2 to prove the validity of some inference rules we already discussed in Section 2.

EXAMPLE 4.1. $x \wedge \neg x \models_H x \wedge y$

PROOF. We have to show that there is no model M such that the Verifier has a winning strategy for $G_{\mathrm{BH}}(x \wedge \neg x, M)$, but not for $G_{\mathrm{BH}}(x \wedge y, M)$. Let M be a model such that the Verifier has a winning strategy for $G_{\mathrm{BH}}(x \wedge \neg x, M)$. Notice that $V(x \wedge \neg x) \in \{\mathbf{f}, \mathbf{u}\}$, so from our assumption and Theorem 4.3 it follows that $V(x) = \mathbf{u}$. Thus, $V(x) = \mathbf{u} = V(x \wedge \neg x) = V(x \wedge y)$, which entails that the Verifier has a winning strategy for $G_{\mathrm{BH}}(x \wedge y, M)$. By Corollary 4.2 we conclude $x \wedge \neg x \models_H x \wedge y$. ■

EXAMPLE 4.2. $x \vee y \models_{\mathrm{B}} x \vee \neg x$

PROOF. Let M be a model such that only the Verifier has a winning strategy for $G_{\mathrm{BH}}(x \vee y)$. This entails $V(x) \neq \mathbf{u}$, so $V(x \vee \neg x) = \mathbf{t}$. By Theorem 4.3 the Verifier only has a winning strategy for $G_{\mathrm{BH}}(x \vee \neg x)$. Thus, $x \vee y \models_{\mathrm{B}} x \vee \neg x$ by Corollary 4.1. ■

4.1. GTS-2 and Başkent's GTS: A Brief Comparison

Theorem 4.3 and Başkent [3, pp. 72–74, Theorem 3.5] make our and Başkent's GTSs semantically equivalent. However, they differ in some other respects. To recap the differences, Başkent's semantic games for B and H are 3-players (the Verifier, the Falsifier, and the Dominator) non-collaborative concurrent games, and he employs both the notions of winning strategy and dominant winning strategy. Instead, GTS-2 semantic games are 2-players (the Verifier and the Falsifier) games consisting of two sequential subgames (the first of which is collaborative, whereas the second is non-collaborative), and we just need winning strategies to determine the truth value of any formula. In light of this, one might ask which of the two game theoretical semantics is better. In this section, we sketch two reasons why we should prefer ours over Başkent's GTS: its theoretical simplicity and philosophical justification.

First, we claim that GTS-2 is simpler than Başkent's GTS. Simplicity can be a rather ambiguous and sometimes elusive concept, which is why it requires to be better specified. The reason why we think our semantics is simpler is that it requires both fewer players and fewer notions to be developed, since, e.g., the concept of *dominant winning strategy* and simultaneous moves (i.e., the set τ as in Definition 3.2) are not needed for GTS-2. Second, we argue that GTS-2 may have a better philosophical justification than Başkent's GTS. No doubt, these two technical mechanisms give exactly the same results. But this is only part of the matter. Consider, first, logical games, in general. A convincing philosophical account of logical games is required, as raised by Wilfrid Hodges in the *Dawkins question*: "we should be able to tell a realistic story of a situation in which some agent [...] is trying to do something intelligible, and doing it is the same thing as winning in the game" (Hodges and Väänänen, [32], §2). In other words, is there a natural explanation for seeing logic in terms of game theoretical interactions between players? Why should we prefer playing logical games instead of using a more standard Tarskian semantics? As for classical game theoretical semantics, some answers have been put forth. Famously, Hintikka [27] extended a game-semantic reading of the quantifiers suggested by Henkin [25]. Instead, Lorenzen interprets semantic games as 'dialogical games', played by two people, a *proponent* and an *opponent*, over a given statement.[14] Recently, a more robust proposal based on the theory of assertions developed by Dummett and Brandom has been put forward by Marion [34]. But unfortunately, such issue is still an open problem even in the classical case, let alone in the non-classical ones.[15]

As for Başkent's GTS, we find it difficult to make sense of the 3-players setting. For it is not clear who (or what) the players should be associated with. The (initial) Verifier and Falsifier might represent the proponent and the opponent of Lorenzen's dialogical games; or 'Myself' and 'Nature' of Hintikka's semantic games. But who is the Dominator? Başkent [3, p. 70] introduces such third player "to force the three truth values" of B and H. But except for this technical reason, we are given no clue about any possible philosophical interpretation of the Dominator.

Instead, GTS-2 could be in a better position. For it requires only two players, who can be associated to two discussant evaluating the truth-value of a given formula. In the first game, they cooperate to check whether the formula, say ϕ, is meaningful. For a necessary condition for ϕ being true or false is that it is not nonsensical. Thus, together they first check whether such a condition is satisfied. If it is—i.e., if there is no winning strategy for the first subgame—then they move to the second subgame and play against each other to determine whether ϕ is true or false. If it is not, they end the game and 'agree' that the formula is nonsensical, for which reason there is no point in asking whether the proposition is true or false. Of course, this is just a preliminary first attempt to offer a suitable philosophical justification of GTS-2, and a more in-depth investigation has to be done to settle this issue.

Finally, GTS-2 has another important virtue that could put it in a better position: with a little effort it can be generalized to other non-classical logics which share some distinctive features with B and H. This is the topic of the next section.

---

[14]The proponent can be thought of as defending the claim "The given statement is true" against any attempts of the opponent to refute it. Similarly, the opponent defends the claim "The given statement is false" against any proponent's attempted refutations. Thus, the proponent starts the game by asserting a given statement and the game develops according to some game rules—which are basically those of classical semantic games —, thought as a regimentation of conversational moves from the life-world.

[15]For instance, Hodges [30], Hodges and Väänänen [32], and Hodges [31] have provided criticisms for both Hintikka and Lorenzen's interpretations.

5. A Generalization of GTS-2 for Logics of Variable Inclusion

The underlying idea at the core of GTS-2 is to specify two subgames, G_1 and G_2, to be played sequentially. The goal was to encode the matrix-based definitions of B and H, whose underlying algebra **WK** is built by "adding" a third element, denoted as **u**, to the classical two-elements Boolean algebra. The aim of the first subgame is to determine if the input-formula is interpreted classically, namely if its value belongs to the two element Boolean algebra with universe $\{\mathbf{f}, \mathbf{t}\}$. If this is the case, the classical subgame G_2 ensures that this value is correctly detected by the players. As we shall prove, this construction can be significantly generalized, so to be applied to a wide family of non-classical logics belonging to the so-called *logics of variable inclusion.*[16] B and H, indeed, are two of the most paradigmatic examples of logics in this family. Before proceeding further, we now briefly review the essential information on the topic.

Given a logic L, it is always possible to single out two specific sublogics of L, namely its *left* and *right* variable inclusion companions, denoted as L^l and L^r respectively. The syntactic definitions of these logics obtain as follows, where $\Gamma \cup \{\phi\}$ is a set of formulas in the language of L. Notice that L^l and L^r are logics in the same language as L.

Definition 5.1.

$$\Gamma \models_{\mathrm{L}^r} \phi \iff \begin{cases} \Gamma \models_{\mathrm{L}} \phi \text{ and } \mathsf{Var}(\phi) \subseteq \mathsf{Var}(\Gamma) \text{ or} \\ \Gamma \text{ is inconsistent in L.} \end{cases}$$

$$\Gamma \models_{\mathrm{L}^l} \phi \iff \Delta \models_{\mathrm{L}} \phi \text{ for some } \Delta \subseteq \Gamma \text{ such that } \mathsf{Var}(\Delta) \subseteq \mathsf{Var}(\phi).$$

Recall that a set of formulas Γ is inconsistent in L if $\Gamma \models_{\mathrm{L}} \phi$ for every formula ϕ. Weak Kleene logics are examples of logics of variable inclusion, as they can be derived from the above definitions by setting $\mathrm{L} = \mathrm{CL}$. In this case, $\mathrm{L}^l = \mathrm{H}$ and $\mathrm{L}^r = \mathrm{B}$.

Several semantic properties of logics of variable inclusion are well-known, and an extensive discussion can be found in Bonzio et al. [9]. For instance, if L is complete with respect to a matrix $\langle \mathbf{A}, F\rangle$, then L^l is complete with respect to $\langle \mathbf{A}^{\mathbf{u}}, F \cup \{\mathbf{u}\}\rangle$, while L^r is complete with respect to $\langle \mathbf{A}^{\mathbf{u}}, F\rangle$. Here, for simplicity, $\{\mathbf{u}\}$ can be seen as the universe of a trivial algebra, and $\mathbf{A}^{\mathbf{u}}$ is the algebra, in the same language as L, defined as follows:

- the universe of $\mathbf{A}^{\mathbf{u}}$ is $A \cup \{\mathbf{u}\}$

[16]We take the opportunity to thank one of the referees for pointing this out.

- for each n-ary operation f in the language of L and $a_1, \ldots, a_n \in A \cup \{\mathbf{u}\}$

$$f^{\mathbf{A}^u}(a_1, \ldots, a_n) = \begin{cases} \mathbf{u} \text{ if } a_i = \mathbf{u} \text{ for some } 1 \leq i \leq n \\ f^{\mathbf{A}}(a_1, \ldots, a_n) \text{ otherwise.} \end{cases}$$

Notice that **WK** obtains when **A** is the two element Boolean algebra in the above display. In light of this, we can ask whether the strategy we applied to develop GTS-2 may be generalized to a broader family of logics of variable inclusion.

Our next task is to single out the general features that allowed us to extract semantic games for B and H from a classical semantic game. In that context, a key aspect is the availability of an efficient notion of semantic game for classical logic. Thus, informally, our final goal is to provide a 'receipt' such that, in presence of an efficient notion of semantic game for a logic L, it provides an efficient notion of semantic game for $\mathrm{L}^l, \mathrm{L}^r$. Let us now translate this intuition formally. From now on, L will denote an arbitrary logic which is complete with respect to a single matrix $\langle \mathbf{A}, F \rangle$, unless stated otherwise. We will always assume that $\mathrm{GTS_L} = \langle \pi, \rho, \sigma \rangle$ is a game theoretic semantics for L based on semantic games $G_{\mathrm{L}}(\phi, M)$, for a formula ϕ in the language of L in a model $M = \langle D, V \rangle$, where V is a valuation on **A**. Moreover, G_{L} is assumed to be *weakly correct* for L, namely that there exist two subsets of players $\pi_V, \pi_F \subseteq \pi$ such that, for each semantic game $G_{\mathrm{L}}(\phi, M)$:

$$V(\phi) \in F \iff \exists p \in \pi_V \text{ such that } p \text{ has a winning strategy for } G_{\mathrm{L}}(\phi, M)$$
$$V(\phi) \notin F \iff \exists p \in \pi_F \text{ such that } p \text{ has a winning strategy for } G_{\mathrm{L}}(\phi, M).$$

Intuitively, this means that the unary relation F, which denotes the set of designated values on **A**, is well-represented by the notion of winning strategy in the semantic game G_{L}. Notice that for each run of the game, π_V, π_F are disjoint sets.

If we restrict our attention to classical logic, where $F = \{\mathbf{t}\}$, the two equivalences in the above display specialize as follows:

$$V(\phi) = \mathbf{t} \iff \exists p \in \pi_V \text{ such that } p \text{ has a winning strategy for } G_2(\phi, M)$$
$$V(\phi) = \mathbf{f} \iff \exists p \in \pi_F \text{ such that } p \text{ has a winning strategy for } G_2(\phi, M)$$

where G_2 is a classical semantic game. Moreover, it is easy to see that classical semantic games fulfill these two conditions, which realize by letting π_V, π_F to be the singletons whose elements are the Verifier and the Falsifier, respectively. When these reasonable constraints on L and GTS_L are met, a generalization of the idea developed in the previous section allows us to build a game theoretic semantics for L^l and L^r. The next definitions specify how.

Definition 5.2. A model M for semantic games for L^l and L^r is a pair $M = \langle D, V\rangle$, where $D \subseteq \mathsf{Var}$ is a non-empty domain on which the game is played and V is a valuation on $\mathbf{A^u}$.

Observe that if $M = \langle D, V\rangle$ is a model for semantic games for L and $M' = \langle D, V'\rangle$ is a model for semantic games for $\mathrm{L}^l, \mathrm{L}^r$, they (possibly) differ for the values given by V, V', whose codomains are $\mathbf{A}$ and $\mathbf{A^u}$ respectively. Moreover, if $V'(x) \neq \mathbf{u}$ for all $x \in \mathsf{Var}$, then M' is a model for semantic games for L. We now generalize a construction we used in the previous Section 4. Let ϕ be an L-formula and $M = \langle D, V\rangle$ be a model for semantic games for $\mathrm{L}^l, \mathrm{L}^r$. We define a new valuation V^ϕ on $\mathbf{A^u}$ as the unique extension of the following map $V^\phi_{\mathsf{Var}} : Fm \to A^{\mathbf{u}}$:

$$V^\phi(x) = \begin{cases} V(x) \text{ if } x \in \mathsf{Var}(\phi) \\ a \text{ otherwise,} \end{cases} \tag{Val}$$

where a in an arbitrary element of $\mathbf{A}$. Given $M = \langle D, V\rangle$, we set $M^\phi = \langle D, V^\phi\rangle$. We can now define our notion of semantic game for L^l and L^r.

Definition 5.3. A semantic game $G_{\mathrm{L}^l\mathrm{L}^r}(\phi, M)$ for a formula ϕ in a model M is a triple $\langle G_u(\phi, M), G_{\mathrm{L}}(\phi, M^\phi), R\rangle$, such that:

- $G_u(\phi, M)$ is the first subgame of $G_{\mathrm{L}^l\mathrm{L}^r}(\phi, M)$, that is a triple $\langle \pi, \rho_u, \sigma_u\rangle$ where π is the same set of players of G_{L}, and:
  - ρ_u is the set of game rules, defined for each n-ary operation f in the language of L as:

    (ρ_{ux}) if $\phi \in \mathsf{Var}$, then $G_u(\phi, M)$ terminates, and all players win if and only if $V(\phi) = \mathbf{u}$;

    (ρ_{uf}) if $\phi = f(\psi_1, \ldots, \psi_n)$, an arbitrarily chosen player $p_a \in \pi$ chooses among $\psi_1, \ldots, \psi_n$.
  - σ_u is the set of positions of the game, given by:

    (σ_{ux}) if $\phi \in \mathsf{Var}$, $\langle p_i, \phi\rangle \in \sigma$, for each $p_i \in \pi$;

    (σ_{uf}) if f is an n-ary operation in the language of L and $\phi = f(\psi_1, \ldots, \psi_n)$, then $\langle p_a, \phi\rangle \in \sigma$ and $\langle p_a, \psi_i\rangle \in \sigma$ for some $1 \leq i \leq n$.
- $G_{\mathrm{L}}(\phi, M^\phi)$ is the second subgame, which is the game for L.
- R is the set of rules for $G_{\mathrm{L}^l\mathrm{L}^r}(\phi, M)$:

  (R) All players in π play $G_u(\phi, M)$ first. If they have a winning strategy for it, then $G_{\mathrm{L}^l\mathrm{L}^r}(\phi, M)$ ends. Otherwise, they move on and play $G_{\mathrm{L}}(\phi, M^\phi)$.

Next, we define a *run* of the game $G_{\text{L}^l\text{L}^r}(\phi, M)$ and the notion of *winning strategy* as in Section 4. It is easy to see that the following statements are equivalent for each model $M = \langle D, V\rangle$ and $\phi \in Fm$:

(i) a player has a winning strategy for $G_u(\phi, M)$;

(ii) all players have a winning strategy for $G_u(\phi, M)$;

(iii) $V(\phi) = \mathbf{u}$.

Observe moreover that, in general, given a model $M = \langle D, V\rangle$ for a semantic game for L^l, L^r and a formula ϕ, the game $G_{\text{L}}(\phi, M^{\phi})$ is not well-defined: this happens whenever $V(x) = \mathbf{u}$ for some $x \in \mathsf{Var}(\phi)$. The following fact shows how to overcome this inconvenience. Notice that Fact 4.1 can be seen as an instance of Fact 5.1.

Fact 5.1. Let $G_{\text{L}^l\text{L}^r}(\phi, M)$ be a semantic game. If no player has a winning strategy for $G_u(\phi, M)$, then M^{ϕ} is a model for semantic games for L.

Proof. Suppose no player has a winning strategy for $G_u(\phi, M)$. Thus, by (ρ_{ux}) in Definition 5.1, for each $x \in \mathsf{Var}(\phi)$ we have that $V(x) \neq \mathbf{u}$. Thus, by (Val), V^{ϕ}_{Var} is a map from Var to A, which entails that V^{ϕ} is a valuation on $\mathbf{A}$. This ensures that M^{ϕ} is a model for semantic games for L. ■

The above considerations ensure that $G_{\text{L}^l\text{L}^r}(\phi, M)$ is well-defined: by (R), $G_{\text{L}}(\phi, M^{\phi})$ has to be played if and only if nobody has a winning strategy for the first game, $G_u(\phi, M)$. This, together with Fact 5.1, entails $G_{\text{L}}(\phi, M^{\phi})$ is well-defined. We will tacitly use this fact throughout this section.

In what follows, we identify two teams: the teams of Verifiers (π_V) and the team of Falsifiers (π_F), as for G_{L}. Notice that the existence of such sets of players is guaranteed by our assumption on the weakly correctness of G_{L}. We say that the team π_V of Verifiers has a winning strategy if one of its members has one. Similarly for the team π_F of Falsifiers.

We are now ready to show that the GTSs so defined for L^l and L^r are correct.

Theorem 5.4. *The following are equivalent for any* $\Gamma \cup \{\phi\} \subseteq Fm$*:*

(i) $\Gamma \models_{\text{L}^l} \phi$

(ii) for every model M, IF, for every $\gamma \in \Gamma$, π_T has a winning strategy for $G_{\text{L}^l,\text{L}^r}(\gamma, M)$, THEN π_T has a winning strategy for $G_{\text{L}^l,\text{L}^r}(\phi, M)$.

Proof. $(i) \Rightarrow (ii)$. Suppose $\Gamma \models_{\text{L}^l} \phi$ and let $M = \langle D, V\rangle$ be a model such that, for each $\gamma \in \Gamma$, π_T has a winning strategy for $G_{\text{L}^l,\text{L}^r}(\gamma, M)$. We claim $V(\gamma) \in F \cup \{\mathbf{u}\}$ for each $\gamma \in \Gamma$. To this end, suppose towards a contradiction that there exists $\gamma \in \Gamma$ such that $V(\gamma) \in A \smallsetminus F$. Thus, the game does not end with $G_u(\gamma, M)$ and its outcome relies on the fact that π_V has a winning strategy for $G_{\mathrm{L}}(\gamma, M^\gamma)$. Since V, V^γ agree on γ, the weakly correctness of G_{L} for L entails that $V^\gamma(\gamma) = V(\gamma) \in F$. This is a contradiction and it establishes our claim. By (i), we obtain $V(\phi) \in F \cup \{\mathbf{u}\}$. If $V(\phi) = \mathbf{u}$, then π_V has a winning strategy for $G_u(\phi, M)$. Otherwise, since G_{L} is weakly correct for L, π_V has a winning strategy for $G_{\mathrm{L}}(\phi, M^\phi)$. In both cases, π_V has a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\phi, M)$, as desired.

$(ii) \Rightarrow (i)$. By contraposition, assume $\Gamma \not\models_{\mathrm{L}^l} \phi$. So, for some valuation V on $\mathbf{A^u}$ and each $\gamma \in \Gamma$ it holds $V(\gamma) \in F \cup \{\mathbf{u}\}$ and $V(\phi) \in A \smallsetminus F$. Let M be a model for $G_{\mathrm{L}^l,\mathrm{L}^r}$ built over V. For each $\gamma \in \Gamma$ either $V(\gamma) = \mathbf{u}$ or $V(\gamma) \in F$. In the first case, π_V clearly has a winning strategy for $G_u(\gamma, M)$. Otherwise, π_V has a winning strategy for $G_{\mathrm{L}}(\gamma, M^\gamma)$. This is true because V, V^γ agree on γ and G_{L} is weakly correct for L. This proves that π_V has a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\gamma, M)$, for every $\gamma \in \Gamma$. Moreover, $V(\phi) \in A \smallsetminus F$ entails that no player has a winning strategy for $G_u(\phi, M)$. However, upon noticing that $V(\phi) = V^\phi(\phi)$, the weakly correctness of G_{L} entails that only π_F has a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\phi, M)$. ■

THEOREM 5.5. *The following are equivalent for any* $\Gamma \cup \{\phi\} \subseteq Fm$*:*

(i) $\Gamma \models_{\mathrm{L}^r} \phi$

(ii) for every model M*, IF for every* $\gamma \in \Gamma$*,* π_T *only has a winning strategy for* $G_{\mathrm{L}^l,\mathrm{L}^r}(\gamma, M)$*, THEN* π_T *only has a winning strategy for* $G_{\mathrm{L}^l,\mathrm{L}^r}(\phi, M)$*.*

PROOF. $(i) \Rightarrow (ii)$. Assume $\Gamma \models_{\mathrm{L}^r} \phi$ and let $M = \langle D, V\rangle$ such that, for each $\gamma \in \Gamma$, π_V only has a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\gamma, M)$. For any such γ, this entails $V(\gamma) \neq \mathbf{u}$, thus no player has a winning strategy for $G_u(\gamma, M)$. So, π_V is the only team with a winning strategy for $G_{\mathrm{L}}(\gamma, M^\gamma)$. Since G_{L} is weakly correct for L, $V^\gamma(\gamma) = V(\gamma) \in F$. By (i), $V(\phi) \in F$, so $V(\phi) \neq \mathbf{u}$ and no player has a winning strategy for $G_u(\phi, M)$. However, since G_{L} is weakly correct for L, π_V has a winning strategy for $G_{\mathrm{L}}(\phi, M^\phi)$, while this is not true for π_F. We conclude π_V is the only team with a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\phi, M)$.

$(ii) \Rightarrow (i)$. We reason by contraposition, so assume $\Gamma \not\models_{\mathrm{L}} \phi$. This entails that for a valuation V on $\mathbf{A^u}$, $V(\gamma) \in F$ for each $\gamma \in \Gamma$, but $V(\phi) \notin F$. Let M be a model for $G_{\mathrm{L}^l,\mathrm{L}^r}$ having V as underlying valuation. For each $\gamma \in \Gamma$, no player has a winning strategy for $G_u(\gamma, M)$, thus the outcome of $G_{\mathrm{L}^l,\mathrm{L}^r}(\gamma, M)$ relies on the outcome of $G_{\mathrm{L}}(\gamma, M^\gamma)$. Since V and V^γ agree on the value of γ, $V^\gamma(\gamma) \in F$ and, by the weakly correctness of the game for L, this shows that π_V only has a winning strategy for it. So, π_V is the unique team having a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\gamma, M)$, for each $\gamma \in \Gamma$. On the other hand, since $V(\phi) \notin F$, either $V(\phi) = \mathbf{u}$ or $V(\phi) \in A \setminus F$. In the first case, π_F has a winning strategy for $G_u(\phi, M)$. Otherwise, a similar argument as above shows that π_F has a winning strategy for $G_{\mathrm{L}}(\phi, M^{\phi})$. Therefore, π_F has a winning strategy for $G_{\mathrm{L}^l,\mathrm{L}^r}(\phi, M)$. This concludes the proof. ■

6. Concluding Remarks

In this paper we have developed a new GTS for B and H, i.e. GTS-2, that is simpler and possibly more welcoming toward a satisfactory philosophical interpretation than Başkent's GTS. Moreover, GTS-2 provides a general strategy to develop a suitable GTS for logics of variable inclusion, L^l and L^r, once a suitable GTS for L is available.

Until now, relatively little effort has been made in developing game theoretic semantics for non-classical logics. We hope that this work will help to draw attention to this topic. Given the large number of existing non-classical logics, there is much to be done. As for future developments, we intend to pursue two directions. On the one hand, an in-depth investigation on the philosophical interpretation of GTS-2 is necessary, as the discussion outline in Section 4.1 shows. On the other hand, we aim to explore whether the approach we employed in developing GTS-2 can be applied to get suitable GTSs also for different non-classical logics, such as Strong Kleene logics. In this case, the contaminating feature of the non-classical value is lost. Therefore, some adjustments are necessary—at best—to develop a semantic game which includes a classical sub-game. In this respect, Başkent [2, §3] already made an important contribution, showing that his strategy can be implemented to develop a suitable GTS for LP. Whether ours can be also applied in that case is then an interesting matter for future investigations.

Acknowledgements. An earlier version of the paper was presented at the XXII Trends in Logic workshop, entitled S*trong & Weak Kleene Logics*, (Cagliari, Italy—18–20 July 2022). Massimiliano Carrara and Michele Pra Baldi acknowledge the CARIPARO Foundation excellence project (2020–2024): *Polarization of irrational collective beliefs in post-truth societies*. Filippo Mancini's research is supported by the DFG (*Deutsche Forschungsgemeinschaft*) funded research group *Inductive Metaphysics* (Grant No. FOR 2495). Wei Zhu was funded by the Postdoctoral Stipend from the *Fritz*

Thyssen Foundation, Germany, for the project (Ref. 40.22.0.015PH) *Suspension and Computational Errors in Paraconsistent Weak Kleene Belief Update*. We would like to thank the anonymous referees for their precious comments and suggestions on earlier versions of the paper.

Funding Open access funding provided by Università degli Studi di Padova within the CRUI-CARE Agreement.

References

[1] Barrio, E., F. Pailos, and D. Szmuc, A cartography of logics of formal inconsistency and truth. *Unpublished Manuscript*, 2016.

[2] Başkent, C., Game theoretical semantics for some non-classical logics, *Journal of Applied Non-Classical Logics* 26(3):208–239, 2016.

[3] Başkent, C., A game theoretical semantics for logics of nonsense. arXiv preprint arXiv:2009.10878, 2020.

[4] Beall, J., Off-topic: A new interpretation of weak-kleene logic, *The Australasian Journal of Logic* 13(6):136–142, 2016.

[5] Bochvar, D., On a three-valued calculus and its application in the analysis of the paradoxes of the extended functional calculus, *Matamaticheskii Sbornik* 4:287–308, 1938.

[6] Bochvar, D. A., and M. Bergmann, On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus, *History and Philosophy of Logic* 2(1–2):87–112, 1981.

[7] Boem, F., and S. Bonzio, A logic for a critical attitude? *Logic and Logical Philosophy* 31(2):261–288, 2022.

[8] Bonzio, S., J. Gil-Ferez, F. Paoli, and L. Peruzzi, On paraconsistent weak Kleene logic, *Studia Logica* 105(2):253–297, 2017.

[9] Bonzio, S., F. Paoli, and M. Pra Baldi, *Logics of Variable Inclusion*. Springer, Berlin, 2022.

[10] Carrara, M., and W. Zhu, On a new reading of nonsense in weak kleene logics, *Manuscript*, 2019.

[11] Carrara, M., and W. Zhu, Computational errors and suspension in a PWK epistemic agent, *Journal of Logic and Computation* 31(7):1740–1757, 2021.

[12] Ciuni, R., Conjunction in paraconsistent weak Kleene logic, in Arazim, P., and M. Dancák, (eds.), *Logica Yearbook 2014*, College Publications, London, 2015, pp. 61–76.

[13] Ciuni, R., and M. Carrara, Characterizing logical consequence in paraconsistent weak Kleene, in Felline, L., A. Ledda, F. Paoli, and E. Rossanese, (eds.), *New Developments in Logic and the Philosophy of Science*, College Publications, London, 2016, pp. 165–176.

[14] Ciuni, R., and M. Carrara, Semantical analysis of weak kleene logics, *Journal of Applied Non-Classical Logics* 29(1):1–36, 2019.

[15] Cobreros, P., and M. Carrara, Tree-based analysis of paraconsistent weak Kleene. *MS*, 2016.

[16] Coniglio, M. E., and M. Corbalan, Sequent calculi for the classical fragment of Bochvar and Halldén's nonsense logic, in Kesner, D., and V. Petrucio, (eds.), *Proceedings of the 7th LSFA Workshop, Electronic Proceedings in Computer Science*, 2012, pp. 125–136.

[17] Correia, F., Weak necessity on weak Kleene matrices, in Wolter, F., H. Wansing, M. de Rijke, and M. Zakharyaschev, (eds.), *Advances in Modal Logic 3*, World Scientific Publishing Co., Singapore, 2002, pp. 73–90.

[18] Da Re, B., F. Pailos, and D. Szmuc, Theories of truth based on four-valued infectious logics, *Logic Journal of the IGPL* 28:712–746, 2020.

[19] Ferguson, T. M., *Meaning and Proscription in Formal Logic*, Springer, Cham, 2019.

[20] Fermüller, C. G., Hintikka-style semantic games for fuzzy logics, in Beierle, C., and C. Meghini, (eds.), *Foundations of Information and Knowledge Systems—8th International Symposium, FoIKS 2014, Bordeaux, France, March 3–7, 2014. Proceedings*, vol. 8367 of *Lecture Notes in Computer Science*, Springer, 2014, pp. 193–210.

[21] Fermüller, C. G., and O. Majer, On semantic games for łukasiewicz logic, in *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*, Springer, 2018, pp. 263–278.

[22] Goddard, L., Towards a logic of significance, *Notre Dame Journal of Formal Logic* 9:233–264, 1968.

[23] Goddard, L., and Routley, *The Logic of Significance and Context*, vol. 1, Scottish Academic Press, Edinburgh, 1973.

[24] Halldén, S., *The Logic of Nonsense*, Uppsala Universitets Arsskrift, Uppsala, 1949.

[25] Henkin, L., Some remarks on infinitely long formulas, infinitistic methods, in *Proceedings of the Symposium on Foundations of Mathematics, Warsaw*, 1961, pp. 167–183.

[26] Hintikka, J., *Language-Games and Information: Kantian Themes in the Philosophy of Logic*, Clarendon Press, Oxford, 1973a.

[27] Hintikka, J., Logic, language-games and information, kantian themes in the philosophy of logic, *Revue Philosophique de la France Et de l*, 163, 1973b.

[28] Hintikka, J., and L. Carlson, Conditionals, generic quantifiers, and other applications of subgames, *Game-Theoretical Semantics: Essays on Semantics by Hintikka, Carlson, Peacocke, Rantala, and Saarinen*, 1979, pp. 179–214.

[29] Hintikka, J. G., and G. Sandu, Game-theoretic semantics, in Benthem and Meulen, (eds.), *Handbook of Logic and Language*, MIT Press, 1997.

[30] Hodges, W., Dialogue foundations: A sceptical look: Wilfrid hodges, in *Aristotelian Society Supplementary Volume*, vol. 75, Wiley Online Library, 2001, pp. 17–32.

[31] Hodges, W., The logic of quantifiers, *The Philosophy of Jaakko Hintikka* 521–534, 2006.

[32] Hodges, W., and J. Väänänen, Logic and games, in Zalta, E. N., (eds.), *The Stanford Encyclopedia of Philosophy*, Metaphysics Research Lab, Stanford University, Fall 2019 edition, 2019.

[33] Kleene, S. C., N. de Bruijn, J. de Groot, and A. C. Zaanen, *Introduction to Metamathematics*, vol. 483, van Nostrand, New York, 1952.

[34] Marion, M., Why play logical games?, in O. Majer, A.-V. Pietarinen, and T. Tulenheimo, (eds.), *Games: Unifying Logic, Language, and Philosophy*, Springer Dordrecht, 2009, pp. 3–26.

[35] Paoli, F., and M. Pra Baldi, Extensions of paraconsistent weak Kleene logic, *Logic Journal of the IGPL* 29(5):798–822, 2020a.

[36] Paoli, F., and M. Pra Baldi, Proof theory of paraconsistent weak kleene logic, *Studia Logica* 108(4):779–802, 2020b.

[37] Pietarinen, A.-V., Semantic games in logic and epistemology, in Rahman, S., J. Symons, D. M. Gabbay, and J. P. V. Bendegem, (eds.), *Logic, Epistemology, and the Unity of Science*, Springer Netherlands, Dordrecht, 2004, pp. 57–103.

[38] Szmuc, D., and T. M. Ferguson, Meaningless divisions, *Notre Dame Journal of Formal Logic* 62:399–424, 2021.

[39] Szmuc, D. E., An epistemic interpretation of paraconsistent weak kleene logic, *Logic and Logical Philosophy* 28(2):277–330, 2019.

M. Carrara, M. Pra Baldi
FISPPA Department
University of Padua
Padua
Italy
massimiliano.carrara@unipd.it

F. Mancini
Department of Philosophy
University of Bonn
Bonn
Germany

W. Zhu
Department of Philosophy
University of Regensburg
Regensburg
Germany