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RESEARCH

Two–layer model via non–quasi–periodic normal form theory

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Abstract

The "two–layer model" is a $2+\frac{1}{2}$ degrees–of–freedom non–autonomous dynamical system consisting of a massive, ellipsoidal (possibly spheric) body made of two layers – a hard core and a viscous fluid – revolving about a major planet or a star. We assume that the rotation and the two revolution periods (of core and shell) are close to a resonance, and aim to investigate, in a rigorous way, the mathematical conditions which maintain the resonant motion. In a previous article (Pinzari et al. in Celest Mech Dyn Astron 136(5):39, 2024), we discussed the phenomenon known as "capture into resonance", via qualitative arguments supported by numerical findings. In this paper, we reframe the model along the lines of a suitable version of (which we refer to as "non–quasi–periodic") normal form theory and provide an explicit amount of the resonance trapping time, which is estimated as exponentially–long, in terms of the small parameters of the system.

Keywords Capture into resonance · Non–quasi–periodic normal form theory · Friction

Mathematics Subject Classification 37J40 · 70F40

Contents

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1 Introduction

In a previous paper, Pinzari et al. (2024) (in turn based on Goldreich and Peale (1966)), we proposed a model ("two–layer model", in what follows) for the capture into spin-orbit resonance in which the body is composed by two layer, a lighter shell and an heavier core, interacting via a liquid, or viscous, friction. The model is motivated by the study of the capture into 3:2 resonance of Mercury, in which the viscous friction can be related to a melted mantle between a solid crust and a solid kernel, and by the icy Jupiter's satellites, that will be studied in a near future by the JUICE mission (Van Hoolst et al. 2024), that are supposed to have a water ocean between the solid icy crust and a rocky core. The model includes a simple — albeit natural — description of the different friction felt by the crust and the core, which is taken proportional to relative velocity (for a simplified model see (Scoppola et al. 2022), other kinds of friction are considered in Rochester (1970); Lhotka et al. (2025)).

In Pinzari et al. (2024), we focused on the study of the equations of motions of the system (see eq. (25) of that paper) resulting from the lower nonzero terms of a time–averaged series expansion of the potential in terms of quite natural small parameters of the system: the eccentricity of the orbit, the a-sphericity of the body and the inverse distance from the sun. Notwithstanding the tailored approximations, the motion equations which we obtained are still non–trivial, due to nonlinearities. Quite surprisingly, based essentially on numerics, we found that such simplified equations provide an account of a possible mechanism of capture into resonance.

The purpose of this paper is to understand under which respect the neglected higher order terms do not interfere with such description. We remark that, by the occurrence of friction, the model is far from being Hamiltonian, whence powerful tools from perturbation theory (like Kolmogorov–Arnold–Moser or Nekhorossev; see below) are not available.

Among the recent theories which deal with friction, conformally symplectic theory is worth to be mentioned (Wojtkowski and Liverani 1998; Calleja et al. 2013; Marò and Sorrentino 2017).

The approach we follow is, in a sense, traditional, in two respects. On the one side, as in Pinzari et al. (2024), we start with a Lagrangian analysis, as we cannot do differently, due to the occurrence of friction. On the other side, we develop a new analytical tool, which we refer to as non–quasi–periodic (NQP, hereafter) normal form theory. In the 70 s of previous Century developing ideas going back to Littlewood, H. Poincaré, Moser, Kolmogorov and his mentor (Hardy et al. 1934; Poincaré 1892; Moser 1961; Birkhoff 1927; Kolmogorov 1954; Arnold 1963b, c), a young student of V.I.Arnold, named Nehorošev (1977, 1979) considered the problem of maximum variation of the action coordinates, I, under the action of an Hamiltonian

$$H(I,\varphi) = h(I) + \epsilon f(I,\varphi) \qquad 0 < \epsilon \ll 1$$

which is a slight perturbation of an integrable one, here named as $h(I)$. Hamiltonians of this kind are common in the literature (e.g., the Hamiltonian of the n–body problem, the Euler top, anharmonic interacting oscillators, etc). They are widely studied, since the discovery of the so–called Kolmogorov–Arnold–Moser (KAM) theory (Kolmogorov 1954; Moser 1962; Arnold 1963a), which originated a flow of research still not exhausted, which spreads to dissipative and infinite–dimension systems: see, e.g., Wayne (1990); Kappeler et al. (n.d.); Kuksin and Maiocchi (2018); Sevryuk (2003); Calleja et al. (2017); Berti et al. (2021); Chierchia and Procesi (2022); Calleja et al. (2024) and references therein, for an overview. Nekhorossev proved that, along the motions of H the j^{th} action coordinate I_j satisfy an inequality like

$$|I_j(t) - I_j(0)| \leq \epsilon^b \qquad \text{for} \quad |t| \leq \frac{1}{\epsilon} \exp\left(\frac{1}{\epsilon^a}\right)$$

for suitable positive numbers a, b. Nekhorossev's papers had a deep impact on the scientific community. After him, many authors thoroughly studied and progressively clarified the analytic set–up, both in the original setting (Benettin and Gallavotti 1986; Pöschel 1993; Lochak 1992; Lochak and Neĭshtadt 1992; Bounemoura and Niederman 2012; Guzzo et al. 2016) or for systems exhibiting an elliptic equilibrium (Fassò et al. 1998; Guzzo et al. 1998; Pöschel 1999; Pinzari 2013; Bounemoura et al. 2020), or, finally, for numerical approaches (Celletti and Ferrara 1996; Morbidelli and Giorgilli 1997; Froeschlé et al. 2000; Sansottera et al. 2013). An extension of normal form theory to non–autonomous Hamiltonian systems with a special decay of the remainder term f has been recently obtained in Fortunati and Wiggins (2016). Recall also Fassò (1989) and references therein for an application to non–hamiltonian vectorfields based on Lie series. Notwithstanding the variety of the recalled analytic results, the occurrence of friction makes them of no practical use to the two–layer model. Using the machinery from Pöschel (1993), we develop a theory for ODEs equipped with vector–fields where, in the lowest approximation, part (possibly, none) of the variables has a quasi–periodic motion, while the other part (possibly, all of them) affords dumped oscillations, i.e., oscillations with complex frequencies, whose real part is negative (even though the theory is meaningful for any complex value of the frequency). Previous similar statements appeared in the unpublished note[1] and, later, in Chen and Pinzari (2021); see (Pinzari 2023) for a review.

Apart from its interest from a technical point of view, we believe that our result is physically meaningful, because it allows, quite constructively (meaning that all the constants involved may be estimated through the proofs: see, e.g., Eq.s (15), (18)), to ensure that the motions of the relevant quantities in the two–layer model are close to such dumped oscillations, for exponentially–long times.

In particular, we are able to exhibit an explicit value of the time T such that for $t < T$ the solution of the linear system stays close, in a suitable norm, to a dumped oscillation, and to compare it with the characteristic time τ ruling the exponential decay, given by the inverse of the modulus of the real part of the frequencies. As a matter of fact, if $T > \tau$, the solution will never escape from the equilibrium, at all times. We remark at this respect that the specificities of the problem at hand allow us to reformulate the underlying, rather complicated, fourth–order eigenvalue equation as second–order ODE and to treat it via the min–max principle eventually (see also Remark 5.1 below).

This paper is organized as follows. In the next Sect. 2, we recall the basic framework of Pinzari et al. (2024), so as to derive the explicit form of the motion equations (2), and state our main result, Theorem 2.1. In Sect. 3 we state precisely the aforementioned NQP normal form theory (see Theorems 3.1, 3.2) and prove Theorem 2.1 and, in Sects. 4.1 and 4.2 we

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[1] *arXiv:1710.02689.*

prove Theorems 3.1, 3.2, respectively. In Sect. 5, we provide the mentioned upper and lower bounds of the size of dumping. Finally, we dedicate Appendix A to recall an abstract result on actions of change of coordinates to vector–fields, which may turn to be useful to non–expert readers.

2 Lagrangian set–up and result

2.1 Description of the model

It is usually believed (and partly proved) that, in general, resonances carry instability in conservative, close–to–be–integrable systems, up to the point of causing ejections of planets or collisions. Still, resonances are ubiquitous in planetary models and look stable over million of years. The idea underlying this section (or, more in general, this note) is to show a mechanism which stabilizes resonances through friction. In particular, we focus on a variation of the well known spin–orbit problem, which we name, as in the title, *two–layer model*. This is a $2+\frac{1}{2}$ degrees–of–freedom dynamical system, constructed as follows. With reference to (Pinzari et al., 2024, Fig. 1), we consider an extended body with total mass m (denoted as P, "planet", in what follows) moving on a plane and undergoing gravity attraction by a point-wise attracting mass M (S, "sun"). For simplicity, we assume that S is fixed in some point of the plane and that the center of P describes a Keplerian, elliptic orbit $\mathcal{E}$, with one of its foci at S and fixed semi–major axis a, perihelion direction $\mathbf{i}$ and eccentricity e (we assume the perihelion is well defined, namely, $e \neq 0$). The position of P on $\mathcal{E}$ is determined by the value of the "mean anomaly" ℓ, which evolves linearly in time, accordingly to Kepler law:

$$\dot{\ell} = \omega \qquad \omega := 2\pi\sqrt{\frac{GM}{a^3}} \tag{1}$$

with G the gravity constant. Concerning the shape and structure of P, we assume it consists of two thickless layers (called "core" and "shell" in what follows), both having elliptic shape, but possibly oriented in different directions. The different orientation of the two ellipses is physically interpreted as evidence of mutual friction between the layers, which, as well as in Pinzari et al. (2024), we aim to take into proper consideration. In addition, for the core and the shell we consider only motions which are close to be "resonant" (namely, with periods ratios close to a rational number) with the revolutions of P about S. As, due to the friction, the energy is not conserved, the Hamiltonian analysis (and its powerful machinery) is not an option. We then proceed with a Lagrangian analysis, as this allows to set friction forces in via a Reileigh function R. To choose the Lagrangian coordinates, we fix a reference frame with the first axis in the direction $\mathbf{i}$ of the perihelion of $\mathcal{E}$. We denote as $\rho = \rho(a, e, \ell)$ the position of the center of P relatively to S; as φ and ν the angles formed by ρ and the semi–major axes directions of the shell and the core. Since our purpose is to study motions for φ and ν which are close to a $2(k/2+1) : 2(k/2+1) : 2$ ratio between φ, ν and ℓ ("spin–orbit resonance"), we introduce the quantities γ and η via

$$\varphi = \frac{k\ell}{2} + \gamma \text{ and } \nu = \frac{k\ell}{2} + \eta.$$

The motion of γ and η is determined by the two second–order equations

$$\left\{\begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}}\right) = \frac{\partial \mathcal{L}}{\partial \gamma} + \frac{\partial R}{\partial \dot{\gamma}} \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) = \frac{\partial \mathcal{L}}{\partial \eta} + \frac{\partial R}{\partial \dot{\eta}}. \end{array}\right. \tag{2}$$

The explicit expressions of $\mathcal{L}$ and R are (compare with equations (Pinzari et al., 2024, (15), (16), (23), (24)). Beware that β and β' correspond to λ and λ′ in Pinzari et al. (2024).)

$$\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_\eta - \widehat{\mathcal{V}}, \qquad R = -\frac{1}{2}\beta\,(\dot\gamma - \dot\eta)^2 - \frac{1}{2}\beta'\left(\dot\eta + \frac{k\omega}{2}\right)^2,$$

with $\beta > 0$ a "viscous friction coefficient", $\beta' > 0$ a "viscoelastic friction coefficient', and

$$\begin{aligned}
\mathcal{L}_\gamma &= \frac{1}{2}C'\left[\frac{k}{2}\omega + \dot\gamma + \dot\vartheta(\ell,\omega,e)\right]^2 + \frac{3}{8}\omega^2(B'-A')a_k(e)e^k\cos 2\gamma - \widetilde{\mathcal{V}}'(\ell,\gamma,e)\\
\mathcal{L}_\eta &= \frac{1}{2}C\left[\frac{k}{2}\omega + \dot\eta + \dot\vartheta(\ell,\omega,e)\right]^2 + \frac{3}{8}\omega^2(B-A)a_k(e)e^k\cos 2\eta - \widetilde{\mathcal{V}}(\ell,\eta,e)\\
&\widetilde{\mathcal{V}},\quad \widetilde{\mathcal{V}}' = \mathrm{O}\left(\frac{r^2}{a^3}\right),\quad \widehat{\mathcal{V}} = \mathrm{O}\left(\frac{r^3}{a^4}\right)
\end{aligned} \tag{3}$$

where $\widetilde{\mathcal{V}}(\ell,\eta,e)$, $\widetilde{\mathcal{V}}'(\ell,\eta,e)$ have vanishing[2] ℓ–average and with r being the average radius of P. To lighten notations, we introduce the homogeneous quantities

$$\begin{aligned}
&c_1 := \frac{3}{4}\frac{B'-A'}{C'}\omega^2 e^k a_k(e), \quad \theta := \frac{\beta}{C'}\\
&c_2 := \frac{3}{4}\frac{B-A}{C}\omega^2 e^k a_k(e), \quad \epsilon := \frac{\beta}{C}, \quad \upsilon := \frac{\beta'}{C}, \quad -v_0 := \frac{k}{2}\\
&\widetilde{P}_\gamma := -\frac{\partial_\gamma \widetilde{\mathcal{V}}'}{C'}, \quad \widehat{P}_\gamma := -\frac{\partial_\gamma \widehat{\mathcal{V}}'}{C'}, \quad \widetilde{P}_\eta := -\frac{\partial_\eta \widetilde{\mathcal{V}}}{C}, \quad \widehat{P}_\eta := -\frac{\partial_\eta \widehat{\mathcal{V}}}{C}
\end{aligned}$$

Here, A, B, C, A', B', C' are the moments of inertia of core and shell, respectively; e, a are eccentricity and semi–major axis of the ellipse, ω is the Keplerian frequency given in (1). We recall that the functions $a_k(e)$ are[3] power series of e starting with 1. The coefficients c_1,

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[2] We sketchily justify the formulae in (3), leaving the details to the interested reader. The gravitational potential generated by the shell–core system has an expansion in powers of the distance ρ between its center and the attracting mass given by (see (Pinzari et al., 2024, Eq. (2)))

$$\mathcal{V} + \mathcal{V}' = -\frac{GM}{\rho}\left\{m + \frac{1}{4\rho^2}\Big[A + B + 3(B-A)\cos 2\nu + A' + B' + 3(B'-A')\cos 2\varphi\Big] + \ldots\right\} \tag{4}$$

where G is the gravity constant, m, M are the masses of the rotating and the attracting body, ρ is the distance among them, A, B, A', B' are the moments of inertia, φ, ν are angles describing rotations of shell and core, respectively, and finally the dots convey terms which begin with power ρ^{-3}. The lowest order term in (4), $-\frac{GMm}{\rho}$, is kicked out, because plays the rôle of orbital potential for the variable ρ (see the term E_k in (Pinzari et al. 2024, Eq. (4))). The remaining part is $O(r^2/a^3)$ (because A, B, A', $B' = O(r^2)$ while $\rho = O(a)$) and is eventually split into the sum of its ℓ–average, which here we denote as $\overline{\mathcal{V}} + \overline{\mathcal{V}}'$, plus a zero–average part $\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}'$. Proceeding similarly as in (Pinzari et al., 2024, Eq. (7)-(13)) one can show that the lowest order terms of $\overline{\mathcal{V}} + \overline{\mathcal{V}}'$ are precisely the terms with $\cos 2\gamma$, $\cos 2\eta$ in (3). Higher order terms in the expansion of $\overline{\mathcal{V}} + \overline{\mathcal{V}}'$ and terms coming from the dots in (4) correspond to the function $\widehat{\mathcal{V}}$ in (3) and are readily seen to be $O(r^3/a^4)$.

[3] As discussed in Pinzari et al. (2024), the terms $a_k(e)$ come from the mean anomaly–eccentricity expansion of $\left(\frac{a}{\rho}\right)^3$ which, as known from the literature, has the form

$$\left(\frac{a}{\rho}\right)^3 = \sum_{k=0}^{\infty} a_k(e)e^k\cos k\ell$$

with $a_k(0) = 1$.

c_2 are related to the geometry and the order of resonance, while θ, ϵ and υ are related to the friction coefficients β, β'. From the physical meaning of such coefficients, in this paper we shall always regard

$$\theta > \epsilon > \upsilon \tag{5}$$

even though more precise quantitative relations will be specified.

With the above definitions and notations, we rewrite Equations (1) and (2) in the form of first order ODEs system

$$\begin{cases} \dot\gamma = p_\gamma \\ \dot p_\gamma = -c_1 \sin 2\gamma - \theta(p_\gamma - p_\eta) + \widetilde{P}_\gamma + \widehat{P}_\gamma \\ \dot\eta = p_\eta \\ \dot p_\eta = -c_2 \sin 2\eta + \epsilon(p_\gamma - p_\eta) - \upsilon(p_\eta - v_0) + \widetilde{P}_\eta + \widehat{P}_\eta \\ \dot\ell = \omega \end{cases} \tag{6}$$

Neglecting the P's releases[4] the system considered in Pinzari et al. (2024). The system (6) will be referred to as *full system* in what follows. Such locution is to be understood as opposite to *linearized system*, which is a further simplification, not considered in Pinzari et al. (2024), and now we describe.

2.2 The linearized system and result

We consider the point $(0, 0, \eta_0, 0)$, where

$$\eta_0 := \frac{1}{2}\sin^{-1}\left(\frac{\upsilon}{c_2} v_0\right) \qquad \mathrm{mod}\ \pi.$$

The point $(0, 0, \eta_0, 0)$ is an equilibrium of the vector–field obtained from the first four equations in (6) by neglecting all the P's. It is well–defined provided that

$$\frac{\upsilon}{c_2}|v_0| < 1.$$

An expansion about such equilibrium leads to the system

$$\begin{cases} \dot\gamma = p_\gamma \\ \dot p_\gamma = -2c_1\,\gamma - \theta\, p_\gamma + \theta\, p_\psi + \breve{P}_\gamma + \widetilde{P}_\gamma + \widehat{P}_\gamma \\ \dot\psi = p_\psi \\ \dot p_\psi = -2\bar c_2\,\psi + \epsilon\, p_\gamma - \delta\, p_\psi + \breve{P}_u + \widetilde{P}_u + \widehat{P}_u \\ \dot\ell = \omega \end{cases} \tag{7}$$

where η has been changed with $\psi := \eta - \eta_0$,

$$\delta := \epsilon + \upsilon,\ \bar c_2 := c_2 \cos 2\eta_0$$

$\widetilde{P}_\gamma$, $\widehat{P}_\gamma$, $\widetilde{P}_u$, $\widehat{P}_u$ denoting (with abuse) the previous functions in the new variables, and

$$\breve{P}_\gamma(\gamma, \psi, \ell) := -c_1 \sin 2\gamma + 2c_1\gamma, \quad \breve{P}_u(\gamma, \psi, \ell) := -c_2 \sin 2(\psi + \eta_0) + \upsilon v_0 + 2\bar c_2\,\psi \tag{8}$$

being the higher order terms released from the expansion. From now on, we shall neglect to write the "bar" in (7). Neglecting all the P's the system decouples as a linear one involving

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[4] Compare Eq. (25) in Pinzari et al. (2024) and the comment below.

the "slow" variables $(\gamma, p_\gamma, \psi, p_\psi)$, and hence named *linearized system*,

$$\begin{cases} \dot\gamma = p_\gamma \\ \dot p_\gamma = -2c_1\,\gamma - \theta p_\gamma + \theta p_\psi \\ \dot\psi = p_\psi \\ \dot p_\psi = -2c_2\,\psi + \epsilon p_\gamma - \delta p_\psi \end{cases} \tag{9}$$

plus the equation (1) for the "fast" variable ℓ, left apart. We denote as L the matrix of the coefficients of (9). We have

Proposition 2.1 *For values of c_1, c_2, θ, ϵ and δ such that the resolvent of the characteristic polynomial of L does not vanish, and if the inequality in (5) and*

$$\epsilon \neq 0, \quad \theta^2 < \frac{4}{9}\min\{c_1, c_2\} \tag{10}$$

hold, then L admits two distinct complex–conjugated couples of eigenvalues

$$\lambda_1 = a_1 + \mathrm{i}\Omega_1, \quad \lambda_2 = a_2 + \mathrm{i}\Omega_2, \quad \lambda_3 = a_1 - \mathrm{i}\Omega_1, \quad \lambda_4 = a_2 - \mathrm{i}\Omega_2 \tag{11}$$

with strictly negative real parts a_i and non–vanishing imaginary parts Ω_i. More precisely, the following bound holds

$$\operatorname{Re}\lambda_j \subset \left[-\frac{3}{2}\theta, -\frac{\upsilon}{6}\right], \quad |\operatorname{Im}\lambda_j| \subset \left[\frac{\epsilon}{\theta}\sqrt{\min\{c_1, c_2\}}, \ \frac{\theta}{\epsilon}\sqrt{2\max\{c_1, c_2\}}\right]. \tag{12}$$

The proof of Proposition 2.1 is provided in Sect. 5.

Proposition 2.1 implies that the motions of γ and ψ along the solutions of the linearized system are given by

$$\gamma_*(t) := \sum_{j=1}^{4} b_{1j} e^{\lambda_j t}, \qquad \psi_*(t) := \sum_{j=1}^{4} b_{3j} e^{\lambda_j t} \tag{13}$$

where $b = (b_{ij})$ is such that $b^{-1}Lb$ is in diagonal form. Our purpose is to continue the motions (13) to the full system.

We shall be able to prove that such continuation exists under two conditions, which in a sense are rather classical. The former condition is related to the numbers a_i, Ω_i in (11), for which we require that a strong non–resonance holds. The condition we take resembles the classical "Diophantine" condition, and is stated as follows:

$$\begin{aligned} &\forall\,(\alpha, q) \in \mathbb{N} \times \mathbb{Z}: \quad \alpha - |q| \geq 0, \quad (-1)^{\alpha - |q|} = 1 \\ &|a_i\alpha - a_j| \geq \frac{\upsilon}{(\alpha+1)^\tau} \quad \text{or} \quad |\Omega_i q - \Omega_j| \geq \frac{\upsilon}{(|q|+1)^\tau} \quad \forall i \neq j \in \{1,\ 2\}. \end{aligned} \tag{14}$$

The second assumption is, as it always happens in similar contexts, is a "smallness" condition which ensures that the full system is close enough to the averaged and linearized one given in (9). It is worth remarking, at this respect, that the complexity of the problem is such that there is not a single parameter which makes the remainder small. Rather, the smallness is required by asking that a suitable combination of the quantities involved (masses, distances, moments of inertia ratio, friction coefficients, etc) is small. Compare Equations (15) and (18) below.

Theorem 2.1 *Under condition* (14) *and the assumptions of Proposition 2.1, there exist numbers* $c > 1$ *and* $\varepsilon_* < 1$ *such that for any choice of* c_1, c_2, ϵ, δ, υ, v_0, r, a, C, C' *such that, if the quantities*

$$\begin{aligned} \mu_0 &:= \max\left\{c_1,\ c_2,\ \varepsilon_0,\ \epsilon\varepsilon_0,\ \delta\varepsilon_0,\ |v_0|\upsilon,\ \frac{r^2}{a^3\varepsilon_0\min\{C,\ C'\}},\ \frac{\theta}{\epsilon}max\{c_1, c_2\}\right\} \\ \mu_1 &:= \max\left\{\frac{\mu_0^2}{\omega},\ \mu_0\frac{r}{a},\ c_1\varepsilon_0^3,\ c_2\varepsilon_0^2,\ \frac{\epsilon}{\theta}min\{c_1, c_2\}\right\} \end{aligned} \tag{15}$$

verify

$$c\frac{\mu_0}{\omega} \le 1, \qquad c\frac{\mu_1}{\upsilon} \le 1 \tag{16}$$

then all the solutions of the system of ODEs (7) *with initial datum in* B_{ε_*} *verify*

$$|\gamma(t) - \hat\gamma(t)| < c\,\varepsilon, \quad |\psi(t) - \hat\psi(t)| < c\,\varepsilon \qquad \forall\,|t| < T \tag{17}$$

where $\hat\gamma(t)$, $\hat\psi(t)$ *have the expression in* (13), *with* λ_j *replaced by suitable* $\hat\lambda_j$ *verifying*

$$\mathrm{Re}\,\hat\lambda_j < 0, \quad |\hat\lambda_j - \lambda_j| \le \frac{\mu_1}{2}$$

while ε *and* T *are given by*

$$\varepsilon := \max\left\{\frac{\mu_0}{\omega},\ \left(\frac{\mu_1}{\upsilon}\right)^{\frac{1}{\tau+1}}\right\}, \quad T \ge \frac{1}{c\upsilon}\varepsilon^{-\tau}e^{\frac{1}{c\varepsilon}} \tag{18}$$

where τ *is the number in* (14).

Remark 2.1 As a comparison with Pinzari et al. (2024), we emphasize that, while the system considered in our previous paper is time–averaged and truncated to the lowest order terms with respect to distances, Theorem 2.1 is tailored on the *full* system, however regarded as a small perturbation of the *linearized* and time–averaged equations. In particular, in the present paper we are far from proving, rigorously, the trapping mechanism naively sketched in (Pinzari et al., 2024, Figure 2). Rather, Theorem 2.1 (namely, Equation (17)) says that the motions of the full system remain ε–close to the solutions (13) of the averaged and linearized system for ε^{-1}–exponentially–long times, where ε is defined through (15) and (18).

3 Proof of Theorem 2.1 via NQP normal form theory

The proof of Theorem 2.1 uses a formulation of normal form theory for vector–fields, carefully designed around the system (7). More precisely, it is based on two results (Theorem 3.1 and Theorem 3.2 below) which here we quote together with the necessary background of notations and definitions. While the proof of such results is deferred to the next sections, here we prove how Theorem 2.1 follows from them.

3.1 Definitions and notations

- We fix a norm $|\cdot|$ in $\mathbb{C}^p$ and then: For a given set $A \subset \mathbb{R}^p$ and $r > 0$, we let

$$A_r := \bigcup_{x\in A} B_r(x)$$

where $B_r(x)$ is the complex ball with radius r centered at x:

$$B_r(x) := \Big\{z \in \mathbb{C}^p : \ |z - x| < r\Big\}.$$

- We denote as $\mathcal{O}_u$, with $u := (\varepsilon, s)$, the space of vector–fields

$$Z = (Z_1, \dots, Z_{m+n}) : \quad V_u := B^m_\varepsilon \times \mathbb{T}^n_s \to \mathbb{C}^{m+n} \tag{19}$$

which are holomorphic on V_{u_0}, $u_0 = (\varepsilon_0, s_0)$, with some $\varepsilon_0 > \varepsilon$, $s_0 > s$.

- If

$$Z_h = \sum_{\alpha,k} z^h_{\alpha k} \zeta^\alpha e^{\mathrm{i}(k\cdot\varphi)}$$

denotes the Taylor–Fourier expansion, we define the *weighted norms* as

$$|X|^w_u := \sum_{h=1}^{m+n} w_h^{-1} |Z_h|_u, \quad |||X|||^w_u := \sum_{h=1}^{m+n} w_h^{-1} \|Z_h\|_u \tag{20}$$

where

$$|Z_h|_u := \sup_{V_u} |Z_h|, \quad \|Z_h\|_u := \sum_{\alpha,k} |z^h_{\alpha k}| \varepsilon^{|\alpha|_1} e^{|k|_1 s}$$

and with $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^{m+n}_+$ the *weights.*

- For each $1 \le h \le m + n$, let $p_h \in \mathbb{N}^{m+n}$ be defined so that

$$p_h = (p_{h1}, \dots, p_{h,m+n}), \quad p_{hj} = \begin{cases} \delta_{hj} & \text{if } h = 1, \dots, m \\ 0 & \text{if } h = m+1, \dots, m+n \end{cases} \tag{21}$$

where δ_{hj} is the Kronecker symbol. The quantity

$$\mathrm{d}^h_{\alpha,k}(\lambda, \omega) := -\Big(\big(\alpha, k\big) - p_h\Big) \cdot (\lambda, \mathrm{i}\omega). \tag{22}$$

will be referred to as h^{th} *small denominator*.

- If $\gamma \ge 0$, define the $(m + n)$–ple of lattices $\Lambda^\gamma = (\Lambda^\gamma_1, \dots, \Lambda^\gamma_{m+n})$

$$\Lambda^\gamma_h(\lambda, \omega) := \Big\{(\alpha, k) \in \mathbb{N}^m \times \mathbb{Z}^n : \ |\mathrm{d}^h_{\alpha,k}(\lambda, \omega)| \le \gamma\Big\} \tag{23}$$

If $\gamma = 0$, the $(m+n)$–ple $\Lambda^0 = (\Lambda^0_1, \dots, \Lambda^0_{m+n})$ will be also referred to as *exactly resonant lattices* $(m + n)$*–ple*. If $\gamma > 0$ they will be called γ–robust resonant lattices $(m + n)$–ple $\Lambda_h \subset \mathbb{N}^m \times \mathbb{Z}^n$.

- The projectors $T_K Z$, $\Pi_\Lambda Z$ will denote the vector–fields defined via

$$(T_K Z)_h := \sum_{|(\alpha,k)|<K} z^h_{\alpha k} \zeta^\alpha e^{\mathrm{i}(k\cdot\varphi)} \qquad (\Pi_\Lambda Z)_h := \Pi_{\Lambda_h} Z_h := \sum_{(\alpha,k)\in\Lambda_h} z^h_{\alpha k} \zeta^\alpha e^{\mathrm{i}(k\cdot\varphi)}. \tag{24}$$

where the norm $|q|$ of an integer vector $q = (q_1, \dots, q_p)$ is $\sum_{i=1}^p |q_i|$.

3.1 Normal form theorems

We consider a system of ODEs

$$\dot{x} = X(x) \tag{25}$$

where $x := (\zeta, \varphi) \in B \times \mathbb{T}^n$, with $B \subset \mathbb{R}^m$ is a neighborhood of $0 = (0, \dots, 0)$, $X(x)$ is a vector–field having the form

$$X(x) = N(x) + P(x) \tag{26}$$

where $N(x)$ is φ–independent and given by

$$N(\zeta) = \begin{pmatrix} \lambda_1 \zeta_1 \\ \vdots \\ \lambda_m \zeta_m \\ \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} \tag{27}$$

with suitable $\lambda \in \mathbb{C}^m$, $\omega \in \mathbb{C}^n$. Then we have

Theorem 3.1 *Let $\gamma > 0$, $1 \leq K \in \mathbb{N}$, $X \in \mathcal{O}_u$ be as in (26), with N as in (27), $u = (\varepsilon, s)$, $w = (\rho, \sigma) < u/2$. Assume that*

$$e\gamma^{-1} |||P|||_u^w < 1 \tag{28}$$

Then there exists a holomorphic change of coordinates

$$\phi_+ : x_+ = (\zeta_+, \varphi_+) \to x = (\zeta, \varphi)$$

which carries X to $X_+ \in \mathcal{O}_{u-2w}$, with $X_+ = N + G_+ + P_+$, where

$$G_+ = \Pi_{\Lambda^\gamma} T_{2K} P. \tag{29}$$

Moreover, there exists $Y \in \mathcal{O}_u$ such that $X_+ := e^{\mathcal{L}_Y} X$ and

$$|||P_+|||_{u-2w}^w \leq \frac{1}{1 - e\gamma^{-1} |||P|||_u^w} \left(e\gamma^{-1} |||P|||_u^w |||P|||_{u-w}^w + e^{-K\tau} |||P|||_u^w \right) \tag{30}$$

with τ as in (76) and $|||Y|||_u^w \leq \gamma^{-1} |||P|||_u^w$. Finally, the transformation ϕ_+ verifies

$$|\phi_+ - \mathrm{id}|_{u-2w}^w \leq \gamma^{-1} |||P|||_u^w \tag{31}$$

Theorem 3.2 *There exists $C_* > 0$ such that the following holds. Let $\gamma > 0$, $1 \leq K \in \mathbb{N}$, $X \in \mathcal{O}_u$ be as in (26), with N as in (27), $u = (\varepsilon, s)$, $w = (\rho, \sigma) < u/4$. Put $\bar{\sigma} := \min\{\sigma, \rho/\varepsilon\}$. Assume that*

$$K\bar{\sigma} \geq \log(12) \tag{32}$$

and that P is so small that

$$C_* K \bar{\sigma} \gamma^{-1} |||P|||_u^w < 1 \tag{33}$$

Then there exists a holomorphic transformation of coordinates

$$\phi_* : \quad V_{u-4w} \to V_u$$

which carries X to

$$X_* = N + G_* + P_* \in \mathcal{O}_{u-4w}$$

with G_* verifying

$$G_* = \Pi_{\Lambda^\gamma} T_{2K} G_*, \qquad |||G_* - \Pi_{\Lambda^\gamma} T_{2K} P|||^w_{u-4w} \le 8e\gamma^{-1} \left(|||P|||^w_u\right)^2$$

and P_* "small":

$$|||P_*|||^w_{u-4w} \le e^{-K\bar{\sigma}/4} \, |||P|||^w_u$$

Finally, the transformation ϕ_* verifies

$$|\phi_* - \mathrm{id}|^w_{u-4w} \le 2\gamma^{-1} |||P|||^w_u. \tag{34}$$

3.3 Proof of Theorem 2.1

We consider the sets Λ^γ_h defined in (23), with $m = 4$, $n = 1$, $h = 1, 2, 3, 4, 5$, λ_i as in Proposition 2.1 and ω the Keplerian frequency (1). We write the product of projectors

$$\Pi_\Lambda T_{2K} = \Pi_{\Lambda^{\gamma,2K}}$$

where $\Lambda^{\gamma,2K}_h := \Lambda^\gamma_h \cap \{|(\alpha, k)| < 2K\}$, and we study the sets $\Lambda^{\gamma,2K}_h$.

Proposition 3.1 *Let p_h be as in (21), $h \in \{1, 2, 3, 4, 5\}$. If inequalities*

$$(2K+1)\frac{\theta}{\epsilon}\sqrt{2\max\{c_1, c_2\}} < \omega, \quad \gamma < \min\left\{\frac{\upsilon}{6},\ 2\frac{\epsilon}{\theta}\sqrt{\min\{c_1, c_2\}},\ \frac{\upsilon}{(2K+1)^\tau}\right\} \tag{35}$$

are satisfied, then $\Lambda^{\gamma,2K}_h = \{p_h\}$, for all h.

Proof We begin with $h = 5$. p_5 is the null vector $(0, 0, 0, 0, 0)$, hence (see definitions (21) and (22))

$$\mathrm{d}^5_{\alpha,k}(\lambda, \omega) = -(\lambda \cdot \alpha + \mathrm{i}k\omega).$$

We prove that

$$|\mathrm{d}^5_{\alpha,k}(\lambda, \omega)| \ge \frac{\upsilon}{6} \quad \forall\, (\alpha, k) \in \mathbb{N}^5 \times \mathbb{Z} \setminus \{(0, 0, 0, 0, 0)\}. \tag{36}$$

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, k) \in \mathbb{N}^4 \times \mathbb{Z}\setminus\{(0, 0, 0, 0)\}$. If $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ne (0, 0, 0, 0)$, then, as $\alpha_j \ge 0$ and $\operatorname{Re}\lambda_j < 0$ for all $1 \le j \le 4$, by (12),

$$|\mathrm{d}^5_{\alpha,k}(\lambda, \omega)| \ge \left|\sum_{j=1}^4 \alpha_j \operatorname{Re}\lambda_j\right| = \sum_{j=1}^4 \alpha_j |\operatorname{Re}\lambda_j| \ge \min_{1\le j\le 4} |\operatorname{Re}\lambda_j| \ge \frac{\upsilon}{6}.$$

If, on the other hand, if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, 0, 0)$ and $k \ne 0$, one has

$$|\alpha \cdot \lambda + \mathrm{i}\omega k| = \omega|k| \ge \omega \ge \frac{\upsilon}{6}.$$

Then (36) is completely proved.

Now we consider the case $h = 1$. p_1 is the vector $(1, 0, 0, 0, 0)$, hence,

$$\mathrm{d}^1_{\alpha,k}(\lambda, \omega) = -(\lambda \cdot \alpha - \lambda_1 + \mathrm{i}k\omega). \tag{37}$$

We prove that

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| \geq \min\left\{\frac{\upsilon}{6},\ 2|\Omega_1|,\ \frac{\upsilon}{(2K+1)^\tau}\right\} \quad \forall\,(\alpha,k)\in\mathbb{N}^3\times\mathbb{Z}\setminus\{(1,0,0,0,0)\}. \tag{38}$$

We distinguish three cases, according to the values of (α, k), which corresponds to subdivide $\Lambda_1^{\gamma,K}$ into proper regions.

Case $\alpha_1 \geq 1, (\alpha,k) \neq (1,0,0,0,0)$. Proceeding as in the previous case (but with α_1 replaced by $\alpha_1 - 1 \geq 0$), we equally conclude

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| \geq \frac{\upsilon}{6} \quad \forall\,(\alpha,k)\in\mathbb{N}^5\times\mathbb{Z}\setminus\{p_1\},\ \alpha_1\geq 1. \tag{39}$$

Case $\alpha_1 = 0, \alpha_3 \neq 0$. Using (11) into (37), we get

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| = \Big|(\alpha_2+\alpha_4)a_2 + (\alpha_3-1)a_1 + \mathrm{i}\Big(-(\alpha_3+1)\Omega_1 + (\alpha_2-\alpha_4)\Omega_2 + k\omega\Big)\Big| \quad \text{if } \alpha_1 = 0 \tag{40}$$

Then

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| \geq \begin{cases} \min_i |\Re\lambda_i| \geq \frac{\upsilon}{6} & \text{if } (\alpha_2,\alpha_3,\alpha_4)\neq(0,1,0) \\ \omega - 2|\Omega_1| \geq \frac{\omega}{2} & \text{if } (\alpha_2,\alpha_3,\alpha_4)=(0,1,0)\ k\neq 0 \\ 2\inf|\Omega_1| \geq \frac{\epsilon}{\theta}\sqrt{\min c_1, c_2} & \text{if } (\alpha_2,\alpha_3,\alpha_4)=(0,1,0),\ k=0 \end{cases} \tag{41}$$

because of (12).

Case $\alpha_1 = \alpha_3 = 0$. We obtain, from (40),

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| = \Big|(\alpha_2+\alpha_4)a_2 - a_1 + \mathrm{i}\Big(-\Omega_1 + (\alpha_2-\alpha_4)\Omega_2 + k\omega\Big)\Big| \quad \text{if} \quad \alpha_1=\alpha_3=0$$

If $k \neq 0$, from the imaginary part, we have

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| \geq \omega - (2K+1)|\Omega| \geq \frac{\omega}{2} \tag{42}$$

Otherwise, if $k = 0$, from condition (14), with $\alpha = \alpha_2 + \alpha_4$ and $q = \alpha_2 - \alpha_4$, we have

$$|\mathrm{d}^1_{\alpha,k}(\lambda,\omega)| \geq \frac{\upsilon}{(2K+1)^\tau}. \tag{43}$$

Collecting (39), (41), (42) and (43), we obtain (38). With a cyclic permutation of the α–indices, one obtains the proof for the cases $h = 2, 3, 4$. □

Proof of Theorem 2.1 We let $\zeta_0 := (\gamma, p_\gamma, \psi, p_\psi)$, $\varphi_0 := \ell$, $x_0 := (\zeta_0, \varphi_0)$.

$$N_0 = \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \quad P_0 := \begin{pmatrix} L\zeta_0 \\ 0 \end{pmatrix} + \breve{P} + \widetilde{P} + \widehat{P} \tag{44}$$

where the matrix L as well as the components of $\breve{P}$, $\widetilde{P}$ and $\widehat{P}$ are defined via the right hand side of (7). Then the vector–field at right hand side of (7) is

$$X_0(x_0) = N_0 + P_0(x_0) \tag{45}$$

We next proceed in four steps. Along the proof, $\mathrm{g}_1, \ldots, \mathrm{g}_{12}$ will denote suitably large numbers, independent of $c_1, c_2, \epsilon, \delta, \upsilon, v_0, r, a, C, C'$.

Step 1: application of Theorem 3.1 to X_0 We fix $u = u_0 = (\varepsilon_0, s_0)$ so that P_0 is real–analytic in the domain

$$\zeta_0 \in B^4_{\varepsilon_0}, \ \varphi_0 \in \mathbb{T}_{s_0}$$

and choose the weights $w_0 = \frac{u_0}{4}$. We aim to apply Theorem 3.1 to $N = N_0$, $P = P_0$ as in (45). From equations (6) and (3), we can bound

$$\vert\vert\vert P_0 \vert\vert\vert_{u_0}^{w_0} \leq \mathrm{g}_1 \max \left\{ c_1, \ c_2, \ \varepsilon_0, \ \epsilon\varepsilon_0, \ \delta\varepsilon_0, \ |\upsilon_0|\upsilon, \ \frac{r^2}{a^3 \varepsilon_0 \min\{C, \ C'\}} \right\} =: \mathrm{g}_1 \bar{\mu}_0. \quad (46)$$

Incidentally, remark that $\bar{\mu}_0$ is defined similarly as μ_0 in (15), apart for the fact that the last term in the bracket is missing, whence $\bar{\mu}_0 \leq \mu_0$. A similar observation will hold for the parameter $\bar{\mu}_1 \leq \mu_1$, which will be introduced in Step 3 below. Condition (28) is satisfied, due to (16), provided that $c \geq C_* \bar{\sigma} \mathrm{g}_1$. We prove that, with the choice

$$\gamma_0 = \frac{\omega}{2} \quad (47)$$

we obtain

$$\Lambda_{0h} := \Lambda_h^{\frac{\omega}{2}} = \mathbb{N}^4 \times \left\{ 0 \right\} \quad \forall\, h = 1, \ldots, 5. \quad (48)$$

Indeed, as all the λ_i's are zero, we have

$$\mathrm{d}^h_{\alpha,k}(0, \omega) = -\mathrm{i}k\omega \qquad \forall\, \alpha \in \mathbb{N}^4, \quad \forall\, h = 1, \ldots, 5.$$

Hence, as the Keplerian frequency ω does not vanish,

$$\Lambda_h^{\frac{\omega}{2}} = \mathbb{N}^4 \times \left\{ k \in \mathbb{Z} : \ |k|\omega \leq \frac{\omega}{2} \right\} = \mathbb{N}^4 \times \left\{ 0 \right\} \quad \forall\, h = 1, \ldots, 5.$$

as claimed. We denote as $\Lambda_0 = (\Lambda_{01}, \ldots, \Lambda_{05})$ the 5–ple of lattices (48). If τ_0 corresponds to τ in the thesis of Theorem 3.1, we choose

$$K_0 \geq {\tau_0}^{-1} \log \left(\frac{\bar{\mu}_0}{\omega} \right)^{-1}. \quad (49)$$

By the thesis of Theorem 3.1, we find a change of coordinates

$$\phi_1 : \ x_1 = (\zeta_1, \varphi_1) \in V_{u_1} \to x_0 = (\zeta_0, \varphi_0) = \phi_1(\zeta_1, \varphi_1) \in V_{u_0} \quad (50)$$

where $u_1 = \frac{u_0}{2}$, which transforms the vector–field X_0 in (45) to

$$X_1(x_1) = N_0 + \overline{P}_0(\zeta_1) + \widetilde{P}_1(x_1) \in \mathcal{O}_{u_1} \quad (51)$$

$\overline{P}_0$ is the φ_0–average of P_0 (because in this case $G_+ = \Pi_{\Lambda_0} T_{K_0} P_0 = \overline{P}_0$) and $\widetilde{P}_1(x_1)$, corresponding to P_+, verifies

$$\vert\vert\vert \widetilde{P}_1 \vert\vert\vert_{u_0/2}^{u_0/4} \leq \mathrm{g}_2 \frac{\bar{\mu}_0^2}{\omega}$$

having used (49). By (31), (46) and (47), the transformation ϕ_1 in (50) verifies

$$|\phi_1 - \mathrm{id}|_{u_0/2}^{u_0/4} \leq \frac{\bar{\mu}_0}{\omega}. \quad (52)$$

Step 2: diagonalization of the linear part Taking the decomposition (44) of P_0 into account, the vector–field $X_1(x_1)$ in (51) can be written as

$$X_1(x_1) = N_1(\zeta_1) + P_1(x_1) \quad (53)$$

where

$$N_1(\zeta_1)=\begin{pmatrix}L\zeta_1\\ \omega\end{pmatrix},\quad P_1(x_1):=\check{P}(x_1)+\widehat{\overline{P}}(x_1)+\widetilde{P}_1(x_1)$$

with $\widehat{\overline{P}}(\zeta_1)$ being the φ_0–average of $\widehat{P}$ computed in ζ_1. Here, we have used that $\widetilde{P}$ has vanishing φ_0–average and $\check{P}$, is φ_0–independent. In Sect. 5 it is shown that the eigenvalues of L are distinct and have negative real part. If b is the 4×4 matrix such that $b^{-1}Lb$ we define the change of coordinates

$$\phi_2:\quad x_2=(\zeta_2,\varphi_2)\in V_{u_2}\to x_1=(\zeta_1,\varphi_1)=\phi_2(x_2):=(b\zeta_2,\varphi_2)\in V_{u_1}.$$

with

$$u_2=(\varepsilon_2,s_2),\quad \varepsilon_2:=\frac{\varepsilon_1}{\|b\|},\qquad s_2:=s_1$$

with $\|b\|$ denoting the operator norm of b. The change ϕ_2 carries the vector–field $X_1(x_1)$ in (53) to

$$X_2(x_2):=\phi_2^{-1}X_1(\phi_2(x_2))=N_2(\zeta_2)+P_2(x_2) \tag{54}$$

where

$$N_2(\zeta_2)=\phi_2^{-1}N_1(b\zeta_2)=\begin{pmatrix}\lambda_1\zeta_{2,1}\\ \lambda_2\zeta_{2,2}\\ \lambda_3\zeta_{2,3}\\ \lambda_4\zeta_{2,4}\\ \omega\end{pmatrix},\qquad P_2(x_2):=\phi_2^{-1}P_1(\phi_2(x_2)). \tag{55}$$

with λ_j the eigenvalues of L.
Step 3: application of Theorem 3.2 to X_2 Choosing $w_2:=\frac{u_2}{8}$, from (8), (44) and (55), we have

$$|||P_2|||_{u_2}^{u_2/8}\le \mathrm{g}_3\max\left\{\frac{\bar\mu_0^2}{\omega},\ \bar\mu_0\frac{r}{a},\ c_1\varepsilon_0^3,\ c_2\varepsilon_0^2\right\}=:\ \mathrm{g}_3\bar\mu_1 \tag{56}$$

where, as anticipated in Step 1 above, $\bar\mu_1\le\mu_1$, with μ_1 given in (15). Let

$$K_1:=min\left\{\omega\frac{\epsilon}{\theta}\,(max\{c_1,c_2\})^{-1},\ \left(\upsilon\frac{\theta}{\epsilon}\,(min\{c_1,c_2\})^{-1}\right)^{\frac{1}{\tau}},\ \left(\frac{\upsilon}{\bar\mu_1}\right)^{\frac{1}{\tau+1}}\right\},$$
$$\gamma_1:=\frac{\upsilon}{(3K_1)^\tau}, \tag{57}$$

It is readily checked that that inequalities (32), (33) and (35) are satisfied if $K=K_1,\gamma=\gamma_1$, provided that c is suitably large, as a consequence of the assumptions (16). By the validity of (32) and (33), Theorem 3.2 applies. By the validity of (35), one has

$$\Lambda_{1h}:=\Lambda_h^{\gamma_1,2K_1}=\{p_h\}\qquad h=1,2,3,4,5$$

with p_h as in (21). By the thesis of Theorem 3.2, one finds a change of coordinates

$$\phi_3:\ x_3=(\zeta_3,\varphi_3)\in V_{u_3}\to x_2=(\zeta_2,\varphi_2)=\phi_3(\zeta_3,\varphi_3)\in V_{u_2} \tag{58}$$

with $u_3=\frac{u_2}{2}$, which carries the vector–field $X_2(x_2)$ in (54) to

$$X_3(x_3)=N_3(\zeta_3)+P_3(x_3) \tag{59}$$

where

$$N_3(\zeta_3) = N_2(\zeta_3) + G_3(\zeta_3) \tag{60}$$

with N_2 as in (55), and G_3 satisfies $G_3 = \Pi_{\Lambda_1} T_{K_1} G_3$. As $\Lambda_{1h} = \{p_h\}$, it follows that G_3 has the form

$$G_3(\zeta_3) = \begin{pmatrix} \tilde\lambda_1 \zeta_{3,1} \\ \tilde\lambda_2 \zeta_{3,2} \\ \tilde\lambda_3 \zeta_{3,3} \\ \tilde\lambda_4 \zeta_{3,4} \\ \tilde\omega \end{pmatrix} \tag{61}$$

Moreover, the following bounds hold:

$$\begin{aligned} &|||G_3|||_{u_3}^{u_3/4} \le |||\Pi_{\Lambda_1} T_{K_1} P_2|||_{u_3}^{u_3/4} + |||G_3 - \Pi_{\Lambda_1} T_{K_1} P_2|||_{u_3}^{u_3/4} \le g_2 \bar\mu_1 + g_4 \frac{\bar\mu_1^2}{\gamma_1} \le g_5 \bar\mu_1 \\ &|||P_3|||_{u_3}^{u_3/4} \le |||P_2|||_{u_2}^{u_2/8} e^{-K_1 \bar\sigma/4} \le g_6 \bar\mu_1 e^{-K_1 \bar\sigma/4}. \end{aligned} \tag{62}$$

By (34), (56) and (36), the transformation ϕ_3 in (58) verifies

$$|\phi_3 - \mathrm{id}|_{u_2/2}^{u_2/8} \le g_7 \frac{\bar\mu_1}{\gamma_1} = g_8 K_1^{\tau} \frac{\bar\mu_1}{\upsilon} \le \frac{g_9}{K_1}. \tag{63}$$

Step 4: conclusion By (62) and Lemma 4.1 below, the numbers $\tilde\lambda_j$ in (61) verify

$$|\tilde\lambda_j| = \left| \partial_{\zeta_{3,j}} G_3(\zeta_3)\big|_{\zeta_{3,j}=0} \right| \le g_{10} \bar\mu_1 \tag{64}$$

Using (59), (60) and (61), we have that the coordinates $\zeta_{3,i}$ satisfy the ODEs

$$\dot\zeta_{3,j} = \hat\lambda_j \zeta_{3,j} + P_{3,j}(\zeta_3, \omega t) \tag{65}$$

where $\hat\lambda_j := \lambda_j + \tilde\lambda_j$. Moreover, $\hat\lambda_j$ have negative real part, as it follows from (64) and the inequality, implied by (16),

$$\bar\mu_1 \le \frac{\upsilon}{3} \le \min_j |\operatorname{Re} \lambda_j|.$$

Rewriting (65) in the form

$$\zeta_{3,j}(t) = \zeta_j(0) e^{\hat\lambda_j t} + \int_0^t P_{3,j}(\zeta_3(\tau), \omega\tau) e^{\hat\lambda_j (t-\tau)} d\tau$$

we find (since $\operatorname{Re} \hat\lambda_j < 0$)

$$\begin{aligned} |\zeta_{3,j}(t) - \zeta_{3,j}(0) e^{\hat\lambda_j t}| &= \left| \int_0^t P_{3,j}(\zeta_3(\tau), \omega\tau) e^{\hat\lambda_j (t-\tau)} d\tau \right| \\ &\le \int_0^{|t|} |P_{3,j}(\zeta_3(\tau), \omega\tau)| d\tau \le g_{11} |t| \bar\mu_1 e^{-K_1 \bar\sigma/4} \\ &\le \varepsilon \qquad \forall\, |t| \le \frac{\varepsilon}{g_{11} \bar\mu_1} e^{K_1 \bar\sigma/4}. \end{aligned} \tag{66}$$

On the other hand, taking track of the transformations, x_0 and x_3 are related via

$$|x_0 - \phi_2(x_3)| = |\phi_1 \circ \phi_2 \circ \phi_3(x_3) - \phi_2(x_3)| \le g_{12} \max\left\{ \frac{\bar\mu_0}{\omega}, \frac{1}{K_1} \right\} \le g_{12} \varepsilon \tag{67}$$

having used (52) and (63) and the definitions of ε and K_1 in (15) and (57). Taking the projection on ζ_3, we find

$$|\zeta_0(t) - b\zeta_3(t)| \le \mathrm{g}_{12}\varepsilon \qquad \forall\, t \in \mathbb{R}.$$

Using finally (66), we arrive at (17), with $T = \frac{\varepsilon}{\mathrm{g}_{11}\bar\mu_1}\, e^{K_1\bar\sigma/4}$. On the other hand, the definition of K_1 in (57) and the last inequality in (67) imply

$$\frac{1}{\bar\mu_1} \ge \frac{K_1^{\tau+1}}{\upsilon}, \qquad K_1 \ge \frac{1}{\varepsilon}$$

whence the lower bound for T in (18) follows. The theorem is proved. □

4 Proof of Theorems 3.1 and 3.2

4.1 Proof of Theorem 3.1

Definition 4.1 • We call *time–one flow of* Y a one–parameter of diffeomorphism Φ_1^Y, where $x(\tau) := \Phi_\tau^Y(y)$ solves

$$\begin{cases} \partial_\tau x = Y(x) \\ \\ x(0) = y \end{cases}$$

- For a given C^∞ vector–field Y, we denote as

$$\mathcal{L}_Y := [Y, \cdot]$$

the *Lie operator*, where

$$(Y, X) := J_X Y - J_Y X, \quad (J_X)_{ij} := \partial_{x_j} X_i$$

denotes the *Lie brackets* of two vector–fields. The map

$$e^{\mathcal{L}_Y} := \sum_{k=0}^{+\infty} \frac{\mathcal{L}_Y^k}{k!} \tag{68}$$

is called *Lie series* generated by Y.

Proposition 4.1 *Assume that $e^{\mathcal{L}_Y}$ is well defined. Then the time–one map of Y, Φ_1^Y, carries the ODE* (25) *to $\dot y = Z(y)$, where $Z = e^{\mathcal{L}_Y} X$.*

Proposition 4.1 is a well–known result in differential geometry. A self–contained proof can be however found in Appendix A.
Our aim is now to provide conditions so that the series (68) is well defined.

Lemma 4.1 (Cauchy Inequalities) *Let $Z \in \mathcal{O}_u$, $u = (\varepsilon, s)$, $0 < \rho < \varepsilon$, $0 < \sigma < s$.*

$$\text{(i) } \|\partial^p_{\varphi_i} Z_h\|_{u-\sigma} \le \left(\frac{p}{e\sigma}\right)^p \|Z\|_u, \qquad \text{(ii) } \|\partial^p_{\zeta_i} Z_h\|_{\varepsilon-\rho,s} \le \frac{p!}{\rho^p}\|Z_h\|_u$$

Proof (i) From the formula

$$\partial^p_{\varphi_i} Z_h = \sum_{(\alpha,k)} z^h_{\alpha,k}\zeta^\alpha (\mathrm{i}k_i)^p e^{\mathrm{i}k\cdot\varphi}$$

we get

$$\begin{aligned}
\|\partial^p_{\varphi_i} Z_h\|_{u-\sigma} &= \sum_{(\alpha,k)} |z^h_{\alpha,k}|\varepsilon^{|\alpha|_1}|k_i|^p e^{|k|_1(s-\sigma)}\\
&\le \sum_{(\alpha,k)} |z^h_{\alpha,k}|\varepsilon^{|\alpha|_1}|k|_1^p e^{-|k|_1\sigma)} e^{|k|_1 s}\\
&\le \frac{1}{\sigma^p}\sup_{x\ge 0} x^p e^{-x}\sum_{(\alpha,k)} |z^h_{\alpha,k}|\varepsilon^{|\alpha|_1} e^{|k|_1 s}\\
&= \left(\frac{p}{e\sigma}\right)^p \|Z_h\|_{u},\,.
\end{aligned}$$

(ii) From the formula

$$\partial^p_{\zeta_i} Z_h = \sum_{(\alpha,k)\colon\, \alpha_i\ge p} z^h_{\alpha,k}\alpha_i(\alpha_i-1)\cdots(\alpha_i-p+1)\zeta_i^{\alpha_i-p}\prod_{j\ne i}\zeta_j^{\alpha_j} e^{\mathrm{i}k\cdot\varphi}$$

we get

$$\begin{aligned}
\|\partial^p_{\zeta_i} Z_h\|_{\varepsilon-\rho,s} &= \sum_{(\alpha,k)\colon\, \alpha_i\ge p} |z^h_{\alpha,k}|\alpha_i(\alpha_i-1)\cdots(\alpha_i-p+1)(\varepsilon-\rho)^{\alpha_i-p}(\varepsilon-\rho)^{|\hat{\alpha_i}|_1} e^{|k|_1 s}\\
&= \frac{p!}{\rho^p}\sum_{(\alpha,k)\colon\, \alpha_i\ge p} |z^h_{\alpha,k}|\frac{\alpha_i(\alpha_i-1)\cdots(\alpha_i-p+1)}{p!}(\varepsilon-\rho)^{\alpha_i-p}\rho^p\\
&\quad (\varepsilon-\rho)^{|\hat{\alpha_i}|_1} e^{|k|_1 s}
\end{aligned}$$

with $\hat{\alpha_i}$ being α deprived if α_i. Using now

$$\begin{aligned}
\frac{\alpha_i(\alpha_i-1)\cdots(\alpha_i-p+1)}{p!}(\varepsilon-\rho)^{\alpha_i-p}\rho^p &\le \sum_{p=0}^{\alpha_i}\frac{\alpha_i(\alpha_i-1)\cdots(\alpha_i-p+1)}{p!}\\
&\quad (\varepsilon-\rho)^{\alpha_i-p}\rho^p = \varepsilon^{\alpha_i}
\end{aligned}$$

we get the thesis. □

Lemma 4.2 *Let $w < u \le u_0$; $Y \in \mathcal{O}_{u_0}$, $W \in \mathcal{O}_u$. Then*

$$\vert\vert\vert\mathcal{L}_Y[W]\vert\vert\vert_{u-w}^{u_0-u+w} \le \vert\vert\vert Y\vert\vert\vert_{u-w}^{w}\vert\vert\vert W\vert\vert\vert_{u}^{u_0-u+w} + \vert\vert\vert W\vert\vert\vert_{u-w}^{u_0-u+w}\vert\vert\vert Y\vert\vert\vert_{u_0}^{u_0-u+w}.$$

Proof One has

$$\begin{aligned}
\vert\vert\vert\mathcal{L}_Y[W]\vert\vert\vert_{u-w}^{u_0-u+w} &= \vert\vert\vert J_W Y - J_Y W\vert\vert\vert_{u-w}^{u_0-u+w}\\
&\le \vert\vert\vert J_W Y\vert\vert\vert_{u-w}^{u_0-u+w} + \vert\vert\vert J_Y W\vert\vert\vert_{u-w}^{u_0-u+w}
\end{aligned}$$

Now, $(J_W Y)_i = \sum_j \partial_{x_j} W_i Y_j$, so, using Cauchy inequalities,

$$\begin{aligned}
\|(J_W Y)_i\|_{u-w} &\le \textstyle\sum_j \|\partial_{x_j} W_i\|_{u-w}\|Y_j\|_{u-w}\\
&\le \textstyle\sum_j w_j^{-1}\|W_i\|_u\|Y_j\|_{u-w}\\
&= \vert\vert\vert Y\vert\vert\vert_{u-w}^{w}\|W_i\|_u
\end{aligned}$$

Similarly,

$$\|(J_Y W)_i\|_{u-w} \le \vert\vert\vert W\vert\vert\vert_{u-w}^{u_0-u+w}\|Y_i\|_{u_0}.$$

Taking the u_0-u+w–weighted norms, the thesis follows. □

Lemma 4.3 *Let* $0 < w < u$, $Y \in \mathcal{O}_{u+w}$, $W \in \mathcal{O}_u$. *Then*

$$|||\mathcal{L}_Y^k[W]|||_{u-w}^w \le k! q^k |||W|||_u^w, \qquad q := e|||Y|||_{u+w}^w \tag{69}$$

Proof We apply Lemma 4.2 with W replaced by $\mathcal{L}_Y^{i-1}[W]$, u replaced by $u-(i-1)w/k$, w replaced by w/k and, finally, $u_0 = u+w$. With $|||\cdot|||_i^w = |||\cdot|||_{u-i\frac{w}{k}}^w$, $0 \le i \le k$, so that $|||\cdot|||_0^w = |||\cdot|||_u^w$ and $|||\cdot|||_k^w = |||\cdot|||_{u-w}^w$,

$$\begin{aligned}|||\mathcal{L}_Y^i[W]|||_i^{w+w/k} &= \left|\left|\left|\left[Y, \mathcal{L}_Y^{i-1}[W]\right]\right|\right|\right|_i^{w+w/k}\\ &\le |||Y|||_i^{w/k}|||\mathcal{L}_Y^{i-1}[W]|||_{i-1}^{w+w/k} + |||Y|||_{u+w}^{w+w/k}|||\mathcal{L}_Y^{i-1}[W]|||_i^{w+w/k}.\end{aligned}$$

Hence, de–homogenizating,

$$\begin{aligned}\frac{k}{k+1}|||\mathcal{L}_Y^i[W]|||_i^w &\le k\tfrac{k}{k+1}|||Y|||_i^w|||\mathcal{L}_Y^{i-1}[W]|||_{i-1}^w + \tfrac{k^2}{(k+1)^2}|||Y|||_{u+w}^w|||\mathcal{L}_Y^{i-1}[W]|||_i^w\\ &\le \tfrac{k^2}{k+1}\left(1+\tfrac{1}{k+1}\right)|||Y|||_{u+w}^w|||\mathcal{L}_Y^{i-1}[W]|||_{i-1}^w\end{aligned}$$

Eliminating the common factor $\frac{k}{k+1}$

$$|||\mathcal{L}_Y^i[W]|||_i^w \le k\left(1+\tfrac{1}{k+1}\right)|||Y|||_{u+w}^w|||\mathcal{L}_Y^{i-1}[W]|||_{i-1}^w$$

and iterating k times from $i=k$ to $i=1$, by Stirling, we get

$$|||\mathcal{L}_Y^k[W]|||_{u-w}^w \le k^k\left(1+\tfrac{1}{k}\right)^k\left(|||Y|||_{u+w}^w\right)^k|||W|||_u^w \le e^k k!\left(|||Y|||_{u+w}^w\right)^k|||W|||_u^w$$

as claimed. □

Lemma 4.3 has the following immediate corollary. We denote as

$$e_m^{\mathcal{L}_Y} = \sum_{k\ge m}\frac{\mathcal{L}_Y^k}{k!} \tag{70}$$

the m–tails of the Lie operator (68).

Proposition 4.2 *Let* $0 < w < u$, $Y \in \mathcal{O}_{u+w}$, q *as in* (69) *verify* $0 \le q < 1$. *Then the Lie series* $e^{\mathcal{L}_Y}$ *defines an operator*

$$e^{\mathcal{L}_Y} : \quad \mathcal{O}_u \to \mathcal{O}_{u-w}$$

and its m–tails (70) *verify*

$$\left|\left|\left|e_m^{\mathcal{L}_Y}W\right|\right|\right|_{u-w}^w \le \frac{q^m}{1-q}|||W|||_u^w \qquad \forall\ W \in \mathcal{O}_u.$$

Definition 4.2 *(Homological equation)* We call *homological equation associated to N* an equation of the form

$$(Y, N) = Z. \tag{71}$$

$$\mathcal{N}_u^\gamma := \{Z' \in \mathcal{O}_u : \ \Pi_{\Lambda^\gamma}Z' = 0\}.$$

Proposition 4.3 (i) *If* (71) *holds with* Y, $Z \in \mathcal{O}_u$, *then* $Z \in \mathcal{N}_u^0$.

(ii) *Let* $\gamma > 0$ *and* $Z \in \mathcal{N}_u^\gamma$. *Then there exists a unique* $Y \in \mathcal{N}_u^\gamma$ *verifying* (71).

(iii) *The unique vector–field* Y *in* (ii) *verifies*

$$\|Y_h\|_u \le \frac{\|Z_h\|_u}{\gamma} \qquad \forall\, h = 1, \ \ldots, m+n.$$

Proof (i) The Jacobian of N is given by

$$J_N = \begin{pmatrix} D_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times n} \end{pmatrix}$$

where D is the diagonal matrix having the λ_i's along its principal diagonal. Then we have

$$(\mathcal{L}_Y N)_h = [Y, N]_h = \Big(J_N Y - J_Y N(x)\Big)_h = a_h Y_h - \textstyle\sum_{j=1}^m \lambda_j z_j \partial_{z_j} Y_h - \sum_{i=1}^n \omega_i \partial_{\varphi_i} Y_h$$

with $a_h = \lambda_h$ if $1 \le h \le m$; $a_h = 0$ if $m+1 \le h \le m+n$. From these formulae one easily finds that the Taylor–Fourier expansion of the function $Z := \mathcal{L}_Y N$ is related to corresponding one of Y via the relation among coefficients

$$z^h_{\alpha k} = \mathrm{d}^h_{\alpha k}(\lambda, \omega)\, y^h_{\alpha k} \tag{72}$$

with $\mathrm{d}^h_{\alpha k}(\lambda, \omega)$ as in (22). But by definition of Λ^0_h, $\mathrm{d}^h_{\alpha k}(\lambda, \omega) = 0$ for all $(\alpha, k) \in \Lambda^0_h$. By (72), we then have also $z^h_{\alpha k}$ for all $(\alpha, k) \in \Lambda^0_h$, which amounts to say $\Pi_{\Lambda^0_h} Z_h = 0$ for all $h = 1$, ..., $m+n$, namely, $\Pi_{\Lambda^0} Z = 0$. As $Z \in \mathcal{O}_u$ by assumption, we then have $Z \in \mathcal{N}^0_u$.
(ii) Let $\gamma > 0$ and $Z \in \mathcal{N}_u^\gamma$. Consider the vector–field Y defined via

$$Y_h = \sum_{(\alpha,k)\in\mathbb{N}^n\times\mathbb{Z}^n\setminus\Lambda^\gamma_h} \frac{z^h_{\alpha k}}{\mathrm{d}^h_{\alpha k}} \zeta^\alpha e^{\mathrm{i}(k\cdot\varphi)} \tag{73}$$

Y_h is well defined because, due to the definition of Λ^γ_h, we have

$$||\mathrm{d}^h_{\alpha k}(\lambda, \omega)| > \gamma \quad \forall\, (\alpha, k) \in \mathbb{N}^n \times \mathbb{Z}^n \setminus \Lambda^\gamma_h. \tag{74}$$

Hence,

$$\begin{aligned} \|Y_h\|_u &= \sum_{(\alpha,k)\in\mathbb{N}^n\times\mathbb{Z}^n\setminus\Lambda^\gamma_h} \frac{|z^h_{\alpha k}|}{|\mathrm{d}^h_{\alpha k}|} \varepsilon^{|\alpha|_1} e^{|k|_1 s} \le \gamma^{-1} \sum_{(\alpha,k)\in\mathbb{N}^n\times\mathbb{Z}^n\setminus\Lambda^\gamma_h} |z^h_{\alpha k}| \varepsilon^{|\alpha|_1} e^{|k|_1 s} \\ &= \gamma^{-1}\|Z_h\|_u. \end{aligned} \tag{75}$$

which shows that $Y \in \mathcal{O}_u$. Moreover, by the discussion in (i), the Taylor–Fourier components of $Z'_h := [Y, N]_h$ are obtained from the corresponding ones of Y_h by a multiplication by $\mathrm{d}^h_{\alpha k}$. Using (73) gives

$$[Y, N]_h = \sum_{(\alpha,k)\in\mathbb{N}^n\times\mathbb{Z}^n\setminus\Lambda^\gamma_h} z^h_{\alpha k} \zeta^\alpha e^{\mathrm{i}(k\cdot\varphi)}$$

But the right hand side of this equality coincides with Z_h, as we have assumed $Z \in \mathcal{N}^\gamma$, whence $\Pi_{\Lambda_h} Z_h = 0$ for all $h = 1, \ \ldots, m+n$. Incidentally, we also proved (iii) in (75). □

Definition 4.3 *(Ultraviolet K–tail)* Let $K \in \mathbb{N}$, $K > 0$. We say that the vector–field $Z \in \mathcal{O}_u$ is a *ultraviolet K–tail* if

$$T_K Z = 0$$

with $T_K Z$ defined as in (24). The set of $Z \in \mathcal{O}_u$ for which this holds will be denoted as $\mathcal{T}^K_u$.

Lemma 4.4 (Estimate of the ultraviolet K–tail) *Let* $u = (\varepsilon, s)$, $w = (\rho, \sigma) < u$. *Let* $Z \in \mathcal{T}_u^{2K}$ *be a ultraviolet* $2K$*–tail. Then*

$$\|Z_h\|_{u-w} \le e^{-K\tau}\|Z_h\|_u, \qquad \tau := \min\left\{\sigma,\ \log(1-\frac{\rho}{\varepsilon})^{-1}\right\}. \tag{76}$$

Proof By definition,

$$\|Z_h\|_{u-w} = \sum_{|(\alpha,k)|_1 \ge 2K} |z^h_{\alpha k}|(\varepsilon-\rho)^{|\alpha|_1} e^{|k|_1(s-\sigma)}$$

Now, as $|(\alpha,k)|_1 = |\alpha|_1 + |k|_1$, either $|\alpha|_1 \ge K$, or $|k|_1 \ge K$. The terms of the summand with $|\alpha|_1 \ge K$ are above by $(1-\frac{\rho}{\varepsilon})^K |z^h_{\alpha k}|\varepsilon^{|\alpha|_1} e^{|k|_1 s}$; the ones with $|k|_1 \ge K$ are bounded by $e^{-K\sigma}|z^h_{\alpha k}|\varepsilon^{|\alpha|_1} e^{|k|_1 s}$. ☐

Lemma 4.5 *The norms* (20) *verify*

$$|X|_u^w \le |||X|||_u^w \qquad \forall\, X \in \mathcal{O}_u,\ \forall\, 0 < w < u.$$

Proof Obvious.

Theorem 4.1 *Let* $G \in \mathcal{O}_u$ *verify* $G = \Pi_{\Lambda^\gamma} T_{2K} G$. *The thesis of Theorem 3.1 holds also if X in* (26) *is replaced with*

$$X = N + G + P \in \mathcal{O}_u$$

G_+ *in* (29) *with*

$$G_+ = G + \Pi_{\Lambda^\gamma} T_{2K} P.$$

and the inequality (30) *with*

$$|||P_+|||^w_{u-2w} \le \frac{1}{1-e\gamma^{-1}\,|||P|||^w_u}\left(e\gamma^{-1}|||P|||^w_u\,|||P|||^w_{u-w} + |||[Y,\ G]|||^w_{u-w} + e^{-K\tau}\ |||P|||^w_u\right)$$

Proof We decompose

$$P = P^{<2K} + P^{\ge 2K}$$

and, further,

$$P^{<2K} = \bar{P} + \tilde{P}$$

with

$$P^{<2K} := T_{2K}P, \qquad P^{\ge 2K} := (I - T_{2K})P.$$

and

$$\bar{P} := \Pi_{\Lambda^\gamma} P^{<2K}, \quad \tilde{P} = (I - \Pi_{\Lambda^\gamma})P^{<2K}.$$

We have

$$\begin{aligned} X_+ = e^{\mathcal{L}_Y}X = e^{\mathcal{L}_Y}\left(N + G + P^{<2K} + P^{\ge 2K}\right) &= N + G + P^{<2K} + [Y, N] + P_+ \\ = N + G + \bar{P} + [Y, N] + \tilde{P} + P_+ & \end{aligned} \tag{77}$$

with

$$P_+ = e_2^{\mathcal{L}_Y}N + e_1^{\mathcal{L}_Y}P^{<2K} + e_1^{\mathcal{L}_Y}G + e_0^{\mathcal{L}_Y}P^{\ge 2K} \tag{78}$$

By definition, $\Pi_{\Lambda^\gamma}\tilde{P} = 0$. Moreover, as $P \in \mathcal{O}_u$, also $\tilde{P} \in O_u$. Then $\tilde{P} \in \mathcal{N}_u^\gamma$. By Proposition 4.3, (ii), there exists a unique $Y \in \mathcal{N}_u^\gamma$ verifying

$$(Y, N) = -\tilde{P}. \tag{79}$$

Then (77) becomes

$$X_+ = N + G_+ + P_+$$

with $G_+ := G + \bar{P}$. The time–one flow of Y is well defined as per Proposition 4.2, because.

$$q := e|||Y|||_u^w \leq e\gamma^{-1}|||\tilde{P}|||_u^w \leq e\gamma^{-1}|||P|||_u^w < 1. \tag{80}$$

By Proposition 4.2, the Lie series $e^{\mathcal{L}_Y}$ defines an operator

$$e^{\mathcal{L}_Y}: \quad W \in \mathcal{O}_{u-w} \to \mathcal{O}_{u-2w}$$

and its tails $e_m^{\mathcal{L}_Y}$ verify

$$\begin{aligned} \Big|\Big|\Big| e_m^{\mathcal{L}_Y} W \Big|\Big|\Big|_{u-2w}^w &\leq \tfrac{q^m}{1-q}|||W|||_{u-w}^w \\ &\leq \tfrac{(e\gamma^{-1}|||P|||_u^w)^m}{1-e\gamma^{-1}|||P|||_u^w}|||W|||_{u-w}^w \end{aligned}$$

for all $W \in \mathcal{O}_{u-w}$. In particular, $e^{\mathcal{L}_Y}$ is well defined on $\mathcal{O}_{u-w} \subset \mathcal{O}_u$, hence $P_+ \in \mathcal{O}_{u-w}$. The bounds on P_+ in (78) are obtained as follows. Using the homological equation (79), one finds

$$\begin{aligned} e_2^{\mathcal{L}_Y} N + e_1^{\mathcal{L}_Y} P^{<2K} &= \sum_{k=1}^{\infty} \frac{\mathcal{L}_Y^{k+1} N}{(k+1)!} + \frac{\mathcal{L}_Y^k P^{<2K}}{k!} \\ &= \sum_{k=1}^{\infty} \mathcal{L}_Y^k \left(-\frac{\tilde{P}}{(k+1)!} + \frac{P^{<2K}}{k!} \right) \\ &= \sum_{k=1}^{\infty} \mathcal{L}_Y^k \left(\frac{k}{(k+1)!}\tilde{P} + \frac{\bar{P}}{k!} \right) \end{aligned}$$

which gives

$$\begin{aligned} |||e_2^{\mathcal{L}_Y} N + e_1^{\mathcal{L}_Y} P^{<2K}|||_{u-w}^w &\leq \sum_{k=1}^{\infty} q^k k! \left|\left|\left| \frac{k}{(k+1)!}\tilde{P} + \frac{\bar{P}}{k!} \right|\right|\right|_{u-w}^w \\ &= \sum_{k=1}^{\infty} q^k k! \left(\frac{k}{(k+1)!}|||\tilde{P}|||_{u-w}^w + \frac{1}{k!}|||\bar{P}|||_{u-w}^w \right) \\ &\leq \sum_{k=1}^{\infty} q^k |||P^{<2K}|||_{u-w}^w = \frac{q}{1-q}|||P^{<2K}|||_{u-w}^w \end{aligned}$$

The other bounds

$$\begin{aligned} \Big|\Big|\Big| e_1^{\mathcal{L}_Y} G \Big|\Big|\Big|_{u-2w}^w &\leq \frac{1}{1-q}|||\mathcal{L}_Y G|||_{u-w}^w = \frac{1}{1-q}|||[Y,\ G]|||_{u-w}^w \\ \Big|\Big|\Big| e_0^{\mathcal{L}_Y} P^{\geq 2K} \Big|\Big|\Big|_{u-2w}^w &\leq \frac{1}{1-q}\Big|\Big|\Big| P^{\geq 2K} \Big|\Big|\Big|_{u-w}^w \leq \frac{1}{1-q}e^{-K\tau}|||P|||_u^w \end{aligned}$$

are similarly established. Finally, it follows from the identity

$$\phi_+(x_+) = \Phi_1^Y(x_+) = x_+ + Y(\Phi_{\tau_*}^Y(x_+)) \qquad \tau_* \in (0,1)$$

and Lemma 4.5 that

$$|\phi_+ - \mathrm{id}|_{\bar u}^{\bar w} \leq |Y|_{\bar u}^{\bar w} \leq |||Y|||_{\bar u}^{\bar w} \leq |||Y|||_{u}^{\bar w} \quad \forall\, \bar u \leq u: \ \Phi_{\tau_*}^Y(x_+) \in U_{\bar u}, \ \forall \bar w$$

Taking $\bar u = u - 2w$, $\bar w = w$ and using (80), we have

$$|\phi_+ - \mathrm{id}|_{u-2w}^{w} \leq |Y|_u^w \leq |||Y|||_u^w \leq \gamma^{-1} |||P|||_u^w$$

which is (31). □

4.2 Proof of Theorem 3.2

Put

$$x = x_0 := (\zeta_0, \varphi_0), \qquad X_0(x_0) := N(\zeta_0) + P_0(x_0).$$

We aim to apply Theorem 4.1 to X_0 hence, with $G_0 = 0$. This is possible because non–resonance condition is verified, and the inequalities (32) and (33) imply (28), provided that $C_* \log(12) \geq e$. We then find $Y_0 \in \mathcal{O}_u$ such that $\phi_1 := \Phi_1^{Y_0}$ and $\Phi_0 = e^{\mathcal{L}_{Y_0}}$ verify

$$\phi_1: \quad x_1 \in V_{u-2w} \to x_0 \in V_{u_0}, \quad \Phi_0: \mathcal{O}_u \to \mathcal{O}_{u-2w} \tag{81}$$

such that

$$X_1 := e^{\mathcal{L}_{Y_0}} X_0 = N + \bar P_0 + P_1 \tag{82}$$

where

$$\bar P_0 \in \mathcal{O}_u \tag{83}$$

and

$$\begin{aligned} |||P_1|||_{u-2w}^w &\leq \frac{1}{1 - e\gamma^{-1} |||P_0|||_u^w} \left(e\gamma^{-1} |||P_0|||_u^w |||P_0|||_{u-w}^w + e^{-K\tau} |||P_0|||_u^w \right) \\ &\leq 2 |||P_0|||_u^w \left(e\gamma^{-1} |||P_0|||_u^w + e^{-K\tau} \right) \end{aligned} \tag{84}$$

If $\gamma^{-1} |||P_0|||_u^w \leq e^{-K\tau}$, there is no much to say. Indeed, using

$$\tau = \min \left\{ \sigma, \ \log \left(1 - \frac{\rho}{\varepsilon} \right)^{-1} \right\} \geq \min \left\{ \sigma, \ \frac{\rho}{\varepsilon} \right\} = \bar\sigma$$

and (32), we have

$$|||P_1|||_{u-2w}^w \leq 4e^{-K\tau} |||P_0|||_u^w = e^{-K\tau + 2\log 2} |||P_0|||_u^w \leq e^{-K\bar\sigma + 2\log 2} |||P_0|||_u^w \leq e^{-K\bar\sigma/4} |||P_0|||_u^w$$

and the proof ends here. If, instead, $\gamma^{-1} |||P_0|||_u^w > e^{-K\tau}$, we need a recursion.

Fix

$$p \in \mathbb{N} \setminus \{0\}, \qquad p \leq \frac{K\bar\sigma}{\log(12)} \tag{85}$$

By (32), such a p exist. The number p will be used as the amount of iterations. The higher bound in the second inequality in (85) will be needed in order to guarantee a suitably fast decay of the perturbing terms. Later on, we shall choose p as the greatest natural number satisfying such inequality, but this is not needed as of now. As of now, we observe that combining such inequality with condition (33), we have

$$epC\gamma^{-1}|||P|||_u^w < 1 \tag{86}$$

with $C := e^{-1}C_* \log(12)$. A suitable $C \geq 1$ (which corresponds to a suitable $C_* \geq e/\log(12)$) will be fixed along the way.

Induction We prove that, if

$$u_0 := u, \quad w_0 := w, \quad u_j = u - 2w - 2\frac{j-1}{p}w, \quad w_j = \frac{w}{p} \qquad j \in \{1, \dots, p+1\}$$

for any $j \in \{1, \dots, p+1\}$, it is possible to find $Y_{j-1} \in \mathcal{O}_{u_{j-1}}$ such that $\phi_j = \Phi_1^{Y_{j-1}}$ and $\Phi_{j-1} := e^{\mathcal{L}_{Y_{j-1}}}$ verify

$$\phi_j : \quad x_j \in V_{u_j} \to x_{j-1} \in V_{u_{j-1}}, \quad \Phi_{j-1} : \mathcal{O}_{u_{j-1}} \to \mathcal{O}_{u_j} \tag{87}$$

and

$$X_j = \Phi_{j-1}X_{j-1} = N + \sum_{i=0}^{j-1} \bar{P}_i + P_j \tag{88}$$

where

$$\bar{P}_i \in \mathcal{O}_{u_i} \quad \forall\, 0 \leq i \leq j-1, \tag{89}$$

$$|||P_j|||_{u_j}^w \leq \frac{1}{2}|||P_{j-1}|||_{u_{j-1}}^w \tag{90}$$

and, moreover,

$$e\gamma^{-1}|||P_j|||_{u_j}^{w/p} < 1. \tag{91}$$

When $j = 1$, (87), (88) and (89) are precisely as in (81), (82) and (83). We check that also (90), (91) are true with $j = 1$. Indeed, (86) and (84) imply

$$|||P_1|||_{u-2w}^w \leq 4e\gamma^{-1}\left(|||P_0|||_u^w\right)^2 \leq \frac{1}{2}|||P_0|||_u^w \quad (C \geq 8) \tag{92}$$

and, moreover,

$$e\gamma^{-1}|||P_1|||_{u-2w}^{w/p} = e\gamma^{-1}|||P_1|||_{u-2w}^w p \leq 4\left(e\gamma^{-1}|||P_0|||_u^w\right)^2 p < \frac{4}{C^2 p} < 1 \tag{93}$$

so the base step $j = 1$ is complete. Let us now assume that (87), (88), (89), (90), (91) hold for some $j \in \{1, \dots, p\}$, and let us prove the same for $j + 1$.

By (91) and the non–resonance condition, Theorem 4.1 can be applied with $X = X_j$, $G = \sum_{i=0}^{j-1} \bar{P}_i$, $P = P_j$, $u = u_j$, w replaced by w/p and one finds Φ_j verifying (87), (88), (89) with j replaced by $j + 1$.

We prove that (90) holds with j replaced by $j + 1$. This will end the induction, after remarking that (91) with j replaced by $j + 1$ is trivially implied by (91) itself and (90) with j replaced by $j + 1$. By the thesis of Theorem 4.1, we have

$$|||P_{j+1}|||_{u_{j+1}}^{w/p} \leq \frac{1}{1 - e\gamma^{-1}\left|\left|\left|P_j\right|\right|\right|_{u_j}^{w/p}}$$

$$\left(e\gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} |\!|\!| P_j |\!|\!|_{u_j-w/p}^{w/p} + |\!|\!| [Y_j, \sum_{i=0}^{j} \bar{P}_i] |\!|\!|_{u_j-w/p}^{w/p} + e^{-K\tau(p)} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} \right)$$

$$\leq 2 \,|\!|\!| P_j |\!|\!|_{u_j}^{w/p} \left(e\gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} + e^{-K\tau(p)} \right) + 2 |\!|\!| [Y_j, \sum_{i=0}^{j} \bar{P}_i] |\!|\!|_{u_j-w/p}^{w/p}$$

with $\tau(p) := \min\left\{ \frac{\sigma}{p},\ \log\left(1 - \frac{\rho}{p\varepsilon}\right)^{-1} \right\}$ and $|\!|\!| Y_j |\!|\!|_{u_j}^{w/p} \leq \gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p}$. We check the following bounds

$$2e\gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} \leq \frac{1}{6} \tag{94}$$

$$2e^{-K\tau(p)} \leq \frac{1}{6} \tag{95}$$

$$2 |\!|\!| [Y_j, \sum_{i=0}^{j} \bar{P}_i] |\!|\!|_{u_j-w/p}^{w/p} \leq \frac{1}{6} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} \tag{96}$$

which will imply (90) with j replaced by $j+1$ after dehomogeneizating the weight. As a consequence of (93) (and (90) if $j > 1$), we have

$$2e\gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} \leq 2e\gamma^{-1} |\!|\!| P_1 |\!|\!|_{u_1}^{w/p} \leq \frac{8}{C^2} \leq \frac{1}{6} \qquad C \geq 4\sqrt{3}$$

so (94) is proved. Moreover, the choice of p in (85) guarantees that

$$K\tau(p) = K \min\left\{ \frac{\sigma}{p},\ \log\left(1 - \frac{\rho}{p\varepsilon}\right)^{-1} \right\} \geq \frac{K\bar{\sigma}}{p} \geq \log(12)$$

which gives (95). It remains to prove (96). Using Lemma 4.2 with $Y = \bar{P}_i$, $W = Y_j$, $u_0 = u_i$, $u = u_j$, w replaced by w/p, we get

$$\begin{aligned}
2 |\!|\!| [Y_j, \sum_{i=0}^{j} \bar{P}_i] |\!|\!|_{u_j-w/p}^{w/p} &\leq 2 \sum_{i=0}^{j} |\!|\!| [Y_j,\ \bar{P}_i] |\!|\!|_{u_j-w/p}^{w/p} \\
&\leq 2 \sum_{i=0}^{j} |\!|\!| P_i |\!|\!|_{u_j-w/p}^{w/p} |\!|\!| Y_j |\!|\!|_{u_j}^{2(j-i)w/p+w/p} + |\!|\!| Y_j |\!|\!|_{u_j-w/p}^{2(j-i)w/p+w/p} |\!|\!| P_i |\!|\!|_{u_i}^{w/p} \\
&= 2 \sum_{i=0}^{j} \frac{1}{2(j-i)+1} |\!|\!| P_i |\!|\!|_{u_j-w/p}^{w/p} |\!|\!| Y_j |\!|\!|_{u_j}^{w/p} + |\!|\!| Y_j |\!|\!|_{u_j-w/p}^{w/p} |\!|\!| P_i |\!|\!|_{u_i}^{w/p} \\
&\leq 4p |\!|\!| Y_j |\!|\!|_{u_j}^{w/p} \sum_{i=0}^{j} \frac{|\!|\!| P_i |\!|\!|_{u_i}^{w}}{2(j-i)+1} \\
&\leq 4p\gamma^{-1} |\!|\!| P_j |\!|\!|_{u_j}^{w/p} \sum_{i=0}^{j} \frac{|\!|\!| P_i |\!|\!|_{u_i}^{w}}{2(j-i)+1} \\
&= c |\!|\!| P_j |\!|\!|_{u_j}^{w/p}
\end{aligned}$$

with

$$c := 4p\gamma^{-1} \sum_{i=0}^{j} \frac{|\!|\!| P_i |\!|\!|_{u_i}^{w}}{2(j-i)+1} = 4p\gamma^{-1} \frac{|\!|\!| P_0 |\!|\!|_{u_0}^{w}}{2j+1} + 8p\gamma^{-1} |\!|\!| P_1 |\!|\!|_{u_1}^{w}$$

$$
\begin{aligned}
&\le 4p\gamma^{-1}\frac{|\!|\!| P_0 |\!|\!|^w_{u_0}}{2j+1} + 32pe\gamma^{-2}\left(|\!|\!| P_0 |\!|\!|^w_u\right)^2\\
&\le \frac{8}{eC} + \frac{32}{peC^2} \le \frac{1}{6} \qquad (C \ge 48)
\end{aligned}
$$

This completes the induction. Choosing now

$$
p = p_* := \left[\frac{K\bar\sigma}{\log(12)}\right] \qquad j = p_* + 1
$$

we obtain

$$
X_* := X_{p_*+1} = N + G_* + P_*
$$

with $P_* := P_{p_*+1}$ verifying

$$
|\!|\!| P_* |\!|\!|^w_{u-4w} \le \frac{1}{2^{p_*+1}} |\!|\!| P_0 |\!|\!|^w_u \le 2^{-\frac{K\bar\sigma}{\log(12)}} |\!|\!| P_0 |\!|\!|^w_u \le e^{-K\bar\sigma/4} |\!|\!| P_0 |\!|\!|^w_u
$$

and $G_* := \sum_{i=0}^{p_*} \bar P_i$ verifying (by (92) and (90))

$$
|\!|\!| G_* - \bar P_0 |\!|\!|^w_{u-4w} = |\!|\!| \sum_{i=1}^{p_*} \bar P_i |\!|\!|^w_{u-4w} \le 2 |\!|\!| P_1 |\!|\!|^w_{u-4w} \le 8e\gamma^{-1}\left(|\!|\!| P_0 |\!|\!|^w_u\right)^2 .
$$

We finally prove (34). By (31), the transformations ϕ_j in (87) verify

$$
|\phi_j - \mathrm{id}|^{w_{j-1}}_{u_{j-1}-2w_{j-1}} \le \gamma^{-1} |\!|\!| P_{j-1} |\!|\!|^{w_{j-1}}_{u_{j-1}}, \quad j = 1, \ldots, p_* + 1
$$

Then $\phi_* := \phi_1 \circ \cdots \circ \phi_{p_*+1}$

$$
\begin{aligned}
|\phi_* - \mathrm{id}|^w_{u-4w} &\le \sum_{j=1}^{p_*+1} |\phi_j - \mathrm{id}|^w_{u-4w} = |\phi_1 - \mathrm{id}|^w_{u-2w} + \sum_{j=2}^{p_*+1} |\phi_j - \mathrm{id}|^w_{u-4w}\\
&= |\phi_1 - \mathrm{id}|^w_{u-2w} + \frac{1}{p_*}\sum_{j=2}^{p_*+1} |\phi_j - \mathrm{id}|^{w_j}_{u-4w}\\
&\le \gamma^{-1} |\!|\!| P_0 |\!|\!|^{w_0}_{u_0} + \gamma^{-1}\frac{1}{p_*}\sum_{j=2}^{p_*+1} |\!|\!| P_{j-1} |\!|\!|^{w_{j-1}}_{u_{j-1}}\\
&\le \gamma^{-1} |\!|\!| P_0 |\!|\!|^{w_0}_{u_0} + 2\gamma^{-1}\frac{1}{p_*} |\!|\!| P_1 |\!|\!|^{w_1}_{u_1} = \gamma^{-1} |\!|\!| P_0 |\!|\!|^{w_0}_{u_0} + 2\gamma^{-1} |\!|\!| P_1 |\!|\!|^{w_0}_{u_0-2w_0}\\
&\le 2\gamma^{-1} |\!|\!| P_0 |\!|\!|^{w_0}_{u_0}
\end{aligned}
$$

having used (92) in the last step.

5 Proof of Proposition 2.1

The eigenvalue–eigenvector equation for the matrix L, namely,

$$
Ly = \lambda y \qquad \lambda \in \mathbb{C},\ y \in \mathbb{C}^4 \setminus \{0\}
$$

can be equivalently formulated as the request that the ODE

$$
\dot x(t) = Lx(t) \tag{97}
$$

has the solution $x(t) = e^{\lambda t} y$. In turn, writing

$$x = \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_1' \\ y_2 \\ y_2' \end{pmatrix} \tag{98}$$

and defining

$$\mathrm{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

by multiplying the first and the third equation of (97) by ϵ, θ, respectively, and taking their time–derivative, we obtain the second–order, two–dimensional ODE

$$T\ddot{\mathrm{x}} + B\dot{\mathrm{x}} + V\mathrm{x} = 0, \tag{99}$$

where

$$T := \begin{pmatrix} \epsilon & 0 \\ 0 & \theta \end{pmatrix}, \quad B := \theta \begin{pmatrix} \epsilon & -\epsilon \\ -\epsilon & \delta \end{pmatrix}, \quad V := 2 \begin{pmatrix} c_1\epsilon & 0 \\ 0 & c_2\theta \end{pmatrix}.$$

Thus, we equivalently look for solutions of (99) of the form

$$\mathrm{x}(t) = e^{\lambda t}\mathrm{y}, \quad \text{with } \mathrm{y} \in \mathbb{C}^2 \setminus \{0\} \tag{100}$$

up to recover the eigenvector y in (98) via the relations

$$\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} := \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Note that T, B and V are real and symmetric[5] and their respective minimum, maximum eigenvalues are given by/satisfy

$$\begin{aligned} \lambda_-^T &= \epsilon, \quad \lambda_+^T = \theta \\ \lambda_-^B &= \frac{\theta}{2}\left(\epsilon + \delta - \sqrt{(\epsilon+\delta)^2 - 4\epsilon\upsilon}\right) \geq \frac{\theta\epsilon\upsilon}{\epsilon+\delta} \\ \lambda_+^B &= \frac{\theta}{2}\left(\epsilon + \delta + \sqrt{(\epsilon+\delta)^2 - 4\epsilon\upsilon}\right) \leq \theta(\epsilon+\delta) \\ \lambda_-^V &= 2\min\{c_1\epsilon, c_2\theta\} \geq 2\epsilon\min\{c_1, c_2\}, \\ \lambda_+^V &= 2\max\{c_1\epsilon, c_2\theta\} \leq 2\theta\max\{c_1, c_2\}. \end{aligned} \tag{101}$$

Replacing (100) into (99) and taking the Hermitian inner product (here denoted as $(\cdot,\cdot)$) with y leads to relation:

$$\lambda^2(\mathrm{y}, T\mathrm{y}) + \lambda(\mathrm{y}, B\mathrm{y}) + (\mathrm{y}, V\mathrm{y}) = 0.$$

We solve for λ:

$$\lambda = -\frac{(\mathrm{y}, B\mathrm{y})}{2(\mathrm{y}, T\mathrm{y})} \pm \mathrm{i}\frac{\sqrt{4(\mathrm{y}, T\mathrm{y})(\mathrm{y}, V\mathrm{y}) - (\mathrm{y}, B\mathrm{y})^2}}{2(\mathrm{y}, T\mathrm{y})}. \tag{102}$$

---

[5] The multiplication by ϵ, θ allowed to have the matrix B symmetric, keeping T and V (diagonal, hence) symmetric.

As Equation (102) does not change multiplying y by an arbitrary $c \in \mathbb{C} \setminus \{0\}$, we do not loose generality if we assume $(\mathrm{y}, \mathrm{y}) = 1$. Under such assumption, by the min-max principle, the expression under the square root is bounded below by

$$\begin{aligned} 4\lambda_-^T \lambda_-^V - (\lambda_+^B)^2 &\geq 8\epsilon^2 \min\{c_1, c_2\} - \theta^2(\epsilon + \delta)^2 > 8\epsilon^2 \min\{c_1, c_2\} - 9\theta^2\epsilon^2 \\ &\geq 4\epsilon^2 \min\{c_1, c_2\} \end{aligned} \tag{103}$$

and above by

$$4\lambda_+^T \lambda_+^V \leq 8\theta^2 \max\{c_1, c_2\} \tag{104}$$

having used (5), (10) and (101). Equations (102) and (103) show that the eigenvalues of L come in complex conjugated couples with non–vanishing imaginary part. As we have assumed that the resolvent of the characteristic polynomial of L does not vanish, L has two distinct such couples. Moreover, again from (5), (101), (102) and (104), we have

$$\mathrm{Re}\,\lambda = -\frac{(\mathrm{y}, B\mathrm{y})}{2(\mathrm{y}, T\mathrm{y})} \in \left[-\frac{\lambda_+^B}{2\lambda_-^T}, -\frac{\lambda_-^B}{2\lambda_+^T}\right] \subset \left[-\frac{\theta(\epsilon+\delta)}{2\epsilon}, -\frac{\epsilon\upsilon}{2(\epsilon+\delta)}\right] \subset \left[-\frac{3}{2}\theta, -\frac{\upsilon}{6}\right]$$

$$|\,\mathrm{Im}\,\lambda| = \frac{\sqrt{4(\mathrm{y}, T\mathrm{y})(\mathrm{y}, V\mathrm{y}) - (\mathrm{y}, B\mathrm{y})^2}}{2(\mathrm{y}, T\mathrm{y})} \subset \left[\frac{\epsilon}{\theta}\sqrt{\min\{c_1, c_2\}}, \frac{\theta}{\epsilon}\sqrt{2\max\{c_1, c_2\}}\right]$$

which proves (12). □

Remark 5.1 The procedure here used to prove Proposition 2.1 is considerably simpler than a strategy based on the analysis of the characteristic polynomial of L, which is given by $P(\lambda) = (\lambda^2 + \theta\lambda + 2c_1)(\lambda^2 + \delta\lambda + 2c_2) - \theta\epsilon\lambda^2$. Remark that the same argument may be applied whenever one needs to infer algebraic properties of the eigenvalues of any $n \times n$ matrix L whose ODE (97) may be put in the form (99), with T, B and V Hermitian.

A Proof of Proposition 4.1

In general, a diffeomorphism $x = \Phi(y)$ transforms the Equation (25) to $\dot{y} = Z(y)$, where

$$Z(y) = J(y)^{-1} X\big(\Phi(y)\big)$$

with $J(y)$ being the Jacobian matrix of the transformation, i.e.,

$$J(y)_{hk} = \partial_{y_k} \Phi_h(y), \quad \text{if} \quad \Phi = (\Phi_1, \ldots, \Phi_n).$$

Applying this to the flow Φ_τ^Y in Definition 4.1, we obtain that the new vector–field is

$$Z_\tau(y) := J_\tau^Y(y)^{-1} X\big(\Phi_\tau^Y(y)\big) \quad \text{with} \quad (J_\tau^Y(y))_{hk} := \partial_{y_k}\big(\Phi_\tau^Y(y)\big)_h.$$

We stress that the thesis of Proposition 4.1 is an immediate consequence of the following identity

$$\frac{d^k}{dt^k} Z_t(y) = J_t^Y(y)^{-1} \mathcal{L}_Y^k X\big(\Phi_t^Y(y)\big) \quad \forall\, 0 \leq t \leq \tau \tag{105}$$

which we are going to prove. Indeed, (105) implies

$$\frac{d^k}{dt^k} Z_t(y)\Big|_{t=0} = \mathcal{L}_Y^k X\big(y\big)$$

which gives

$$Z(y) = Z_\tau(y) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \frac{d^k}{dt^k} Z_t(y)\Big|_{t=0} = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \mathcal{L}_Y^k X\big(y\big) = e^{\tau \mathcal{L}_Y} X(y).$$

Taking $\tau = 1$ will give the proof of Proposition 4.1.
Let us then prove (105). We use the expansion

$$\Phi_t^Y(y) = \Phi_{t_0}^Y(y) + Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0) + o(t - t_0) \tag{106}$$

and

$$J_t^Y(y) = \Big(\mathbb{I} + J_Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0)\Big) J_{t_0}^Y(y) + o(t - t_0) \qquad J_Y(z)_{hk} = \partial_{z_k} Y_h(z). \tag{107}$$

Equation (107) gives

$$(J_t^Y(\eta))^{-1} = (J_{t_0}^Y(y))^{-1} \Big(\mathbb{I} - J_Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0)\Big) + o(t - t_0). \tag{108}$$

While (106) gives

$$\begin{aligned} X\big(\Phi_t^Y(y)\big) &= X\big(\Phi_{t_0}^Y(y) + Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0) + o(t - t_0)\big) \\ &= X\big(\Phi_{t_0}^Y(y)\big) + J_X\big(\Phi_{t_0}^Y(y)\big) Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0) + o(t - t_0) \end{aligned} \tag{109}$$

Collecting (108) and (109), we then find

$$\begin{aligned} Z_t(y) &= J_t^Y(y)^{-1} X\big(\Phi_t^Y(y)\big) \\ &= J_{t_0}^Y(y)^{-1} \\ &\quad \Big(\mathbb{I} - J_Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0) + o(t - t_0)\Big)\Big(X\big(\Phi_{t_0}^Y(y)\big) + J_X\big(\Phi_{t_0}^Y(y)\big) Y\big(\Phi_{t_0}^Y(y)\big)(t - t_0)\Big) \\ &\quad + o(t - t_0) \\ &= J_{t_0}^Y(y)^{-1} X\big(\Phi_{t_0}^Y(y)\big) \\ &\quad + J_{t_0}^Y(y)^{-1} \Big(J_X\big(\Phi_{t_0}^Y(y)\big) Y\big(\Phi_{t_0}^Y(y)\big) - J_Y\big(\Phi_{t_0}^Y(y)\big) X\big(\Phi_{t_0}^Y(y)\big)\Big)(t - t_0) + o(t - t_0) \end{aligned}$$

This expansion shows that

$$\frac{d}{dt} Z_t(y) = \frac{d}{dt}\Big(J_t^Y(y)^{-1} X\big(\Phi_t^Y(y)\big)\Big) = J_t^Y(y)^{-1} \mathcal{L}_Y X\big(\Phi_t^Y(y)\big)$$

By iteration, we have (105). □

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