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Deformations of Hypersurfaces with Nonconstant Alexander Polynomial

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Let $X \subset \mathbf{P}^n$ be an irreducible hypersurface of degree $d \geq 3$ with only isolated semi-weighted homogeneous singularities, such that $\exp(\frac{2\pi i}{k})$ is a zero of its Alexander polynomial. Then we show that the equianalytic deformation space of X is not T-smooth except for a finite list of triples (n, d, k). This result captures the very classical examples by B. Segre of families of degree $6m$ plane curves with $6m^2$, $7m^2$, $8m^2$, and $9m^2$ cusps, where $m \geq 3$. Moreover, we argue that many of the hypersurfaces with nontrivial Alexander polynomial are limits of constructions of hypersurfaces with not T-smooth deformation spaces. In many instances, this description can be used to find candidates for Alexander-equivalent Zariski pairs.

1 Introduction

Let $X \subset \mathbf{P}^n$ be a hypersurface with isolated singularities, and let $\Delta_X \in \mathbf{Z}[t]$ be its Alexander polynomial (cf. Definition 3.1).

The 1st example in the literature of a hypersurface with nonconstant Alexander polynomial is a plane sextic with six cusps on a conic, due to Zariski [14]. One easily checks that there exist sextic curves with six cusps such that these six cusps are not on a conic. Such a sextic has a constant Alexander polynomial. Hence, we obtain a Zariski pair, a pair of singular hypersurfaces $X_1, X_2 \subset \mathbf{P}^n$ with the same combinatorial data, but

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such that there is no homeomorphism between the pairs $(\mathbf{P}^n, X_1)$ and $(\mathbf{P}^n, X_2)$. One easily checks that in this case both curves have a T-smooth equianalytic deformation space; see [15, Section VIII.5].

B. Segre noted that one can easily generalise this example to higher degree and considered two families of degree $6m$ curves with $6m^2$ cusps. For the 1st construction, pick two sufficiently general homogenous polynomials $f, g \in \mathbf{C}[x_0, x_1, x_2]$ of degree $2m$ and $3m$, respectively. Then the curve of degree $6m$ given by

$$f^3 + g^2 = 0$$

has $6m^2$ cusps. Within the space of degree $6m$ curves the expected codimension of the locus of curves with $6m^2$ cusps is $12m^2$; however, one easily shows that this family has codimension $12m^2 - \frac{1}{2}(m-1)(m-2)$. That is, for $m > 2$, this family has a larger equisingular deformation space than expected. Similar examples are due to Segre for degree $6m$ curves with either $7m^2$, $8m^2$, or $9m^2$ cusps. (For more on this, see [15, Section VIII.5].) These examples have nonconstant Alexander polynomial and their equisingular deformation spaces have larger dimension than expected. We will come back to these examples in Example 4.1.

Take now a sextic curve C with six cusps not on a conic and pull this curve back under a general self-map of $\mathbf{P}^2$ of degree m. Then the pullback of C has $6m^2$ cusps. One easily checks that the saturation of the Jacobian ideal J^{sat} is $6m$-regular, and therefore, the equianalytic deformation space is T-smooth. For ordinary cusps, the equianalytic and equisingular deformation space coincide; hence also the latter space is T-smooth. Hence, for every $m > 2$, we have two ways to show that the space parametrizing curves of degree $6m$ with $6m^2$ cusps is reducible, that is, we can detect this both by the Alexander polynomial and by the dimension of the equisingular deformation space.

There have been more "recent" attempts to explain this excess dimension of the deformation space (e.g., [12]). We will give an explanation different from those we were able to locate in the literature. Our main result states that Segre's construction is part of a rather frequently occurring phenomenon.

Theorem 1.1. Let (n, d, k) be integers such that $d \geq 3, n \geq 2$, and $k \geq 1$. Suppose $X \subset \mathbf{P}^n$ is an irreducible hypersurface of degree d with isolated semi-weighted homogeneous singularities (cf. Definition 2.8) such that $\exp(2\pi i/k)$ is a zero of its Alexander polynomial of X.

Moreover, assume that we are *not* in one of the following cases:

1. $n = 2$, $d \in \{6, 12\}$, and $k = 6$;
2. $n \in \{3, 4, 6\}$, $d = 3$, and $k = 1$;
3. $n \in \{3, 4, 5\}$, $d = 3$, and $k = 3$;
4. $n = 3$, $d \in \{4, 6\}$, and $k = 2$;
5. $n = 3$, $d = k = 4$;
6. $n = 4$, $d = 4$, and $k = 1$.

Then the equianalytic deformation space of X is not T-smooth.

If $d = 1$, then X is a hyperplane and therefore X is smooth. If $d = 2$ and X has isolated singularities, then X is a quadric of rank one less than the maximal rank. In this case, the deformation theory is very simple. For this reason, we considered only the case $d \geq 3$.

Unfortunately, our methods only show that the tangent space has dimension larger than expected and our method can only be applied to the equianalytic deformation space. In particular, this result does not show that the equisingular deformation space has dimension larger than expected, but it is strong evidence of it. More precisely, let X be a hypersurface with nonconstant Alexander polynomial, and let X' be an equisingular deformation of X. Then the Alexander polynomials of X and X' are the same. Hence, the equianalytic deformation space of every equisingular deformation of X is nowhere T-smooth, and therefore, each of these spaces is nonreduced or has dimension larger than expected. In case there exists a further hypersurface X'' with the same combinatorial data as X, but with a T-smooth equianalytic deformation space, then the space of hypersurfaces with this combinatorial data has at least two irreducible components.

In the final section, we will provide constructions of hypersurfaces with deformation spaces with dimension larger than expected. These constructions depend on choices of several parameters, each of which are integers, subject to several inequalities. For most choices of parameters, the resulting hypersurface has constant Alexander polynomial, but for a few choices of these parameters, the corresponding hypersurface has nonconstant Alexander polynomial. Moreover, in the latter case, at least one of the before-mentioned inequalities turns out to be an equality, that is, the examples with nonconstant Alexander polynomials can be considered to be boundary cases or limit cases. This strongly suggests that the Alexander polynomial is not an optimal invariant to detect examples of reducible spaces parametrizing singular hypersurfaces with fixed combinatorial data. The dimension of the equisingular deformation space seems a better invariant. On the other hand, certain geometric phenomena (e.g., quasi-torus structures, the relation with Mordell–Weil ranks of isotrivial fibrations) can only occur for hypersurfaces with nonconstant Alexander polynomials; see [1, 8].

The assumption that all singularities are semi-weighted homogeneous is needed in our proof, as we heavily use the fact that for each singular point the pole-order filtration and the Hodge filtration on the cohomology of the Milnor fibre coincide. If one could control the difference between these two filtrations for other types of singularities, then one might be able to extend our approach to larger classes of singularities.

The proof of the main result consists of two parts. In the 1st part, we reconsider Dimca's approach [3, 5] to calculate the Alexander polynomial of a hypersurface with isolated semi-weighted homogenous singularities. This is done in Section 2. The upshot of this method is that if $\exp(2\pi i/k)$ is a zero of its Alexander polynomial and J is the Jacobian ideal of X, then its saturation J^{sat} has defect in every degree $\leq \alpha(n,d,k)d-n-1$. The rational number $\alpha(n,d,k)$ will be introduced in Section 3, but for the rest of this introduction, it suffices to know that $\alpha(n,d,k)$ lies in the interval $[\frac{n}{2},\frac{n+1}{2}]$.

In Section 3, we determine all values (n,d,k) for which $\alpha(n,d,k)d-n+1<d$. Except for the case $n=2, k=1$ (reducible plane curves) and $d=2$ (quadric cones), there are only finitely many triples (n,d,k) for which this inequality holds. This part is a purely combinatorial exercise. It then easily follows that the tangent space of the equisingular deformation space is larger than expected, except for these exceptional values of (n,k,d).

In Section 4, we discuss some examples and explain some of the remarks made above in more detail.

2 Calculation of $H^n(X)$

In this section, we discuss Dimca's method to calculate the mixed Hodge structure on the cohomology of hypersurfaces with isolated semi-weighted homogeneous singularities; see [3] and [5, Section 6.3]. At certain instances, we differ slightly from Dimca's approach and for that reason we recall large part of the construction.

Notation 2.1. Let $n\geq 2$, and let $R=\mathbf{C}[x_0,\dots,x_n]$ be the polynomial ring in $n+1$ variables, with its natural grading. Let $d\geq 1$ be an integer. For $f\in R_d$, let $X=V(f)\subset\mathbf{P}^n$ be the associated hypersurface. Let $U=\mathbf{P}^n\setminus X$, and let $X^*=X\setminus S$, where $S=X_{\text{sing}}$.

Assumption 2.2. For the rest of this section, we assume that f is chosen such that X has isolated singularities, that is, that S is finite.

Remark 2.3. Using the Lefschetz hyperplane theorem and a result by Kato-Matsumoto [5, Theorems 5.2.6 and 5.2.11], we can determine $H^j(X,\mathbf{C})$ for all $j \neq n-1, n$. In the rest of this section, our focus will be on the case $j = n$, since this group is used to determine the Alexander polynomial of X, as we will see in the next section. More precisely, we aim to give an upper bound for the dimension of the graded pieces $\mathrm{Gr}^s_F(H^n(X))$ of the Hodge filtration F.

If $n = 2$, then $H^2(X)$ is of pure type $(1,1)$ and the dimension equals the number of irreducible components of X. Hence, we may assume for the moment that $n \geq 3$.

Notation 2.4. Denote with $H^j(X)_{\mathrm{prim}}$ and $H^j(X^*)_{\mathrm{prim}}$ the primitive cohomology as defined in [3, Section 2].

By construction, the groups $H^j(X)_{\mathrm{prim}}$ and $H^j(X^*)_{\mathrm{prim}}$ are sub-Hodge structures of $H^j(X)$ and $H^j(X^*)$, respectively. The following result can be found at [3, page 291].

Proposition 2.5. Let $\iota : X^* \to X$ be the inclusion map. Then, for all $j \neq 2n-2$, we have that the kernel (respectively the cokernel) of $\iota^* : H^j(X) \to H^j(X^*)$ equals the kernel (respectively the cokernel) of $\iota^* : H^j(X)_{\mathrm{prim}} \to H^j(X^*)_{\mathrm{prim}}$.

Moreover, $H^n(X^*)_{\mathrm{prim}} = 0$.

Notation 2.6. For a proper subset W of X, denote with $H^n_W(X)$ the cohomology of X with support in W. If $W = \{p_1, \ldots, p_l\}$ is a finite set, then using excision, it follows easily that $H^n_W(X) = \oplus_{i=1}^l H^n_{p_i}(X)$.

Let $\vartheta : H^n(U)(1) \to H^n_S(X)$ be the composition of the following maps:

$$H^n(U)(1) \xrightarrow{\cong} H^{n-1}(X^*)_{\mathrm{prim}} \to H^{n-1}(X^*) \xrightarrow{\delta^{n-1}} H^n(X,X^*) \xrightarrow{\cong} H^n_S(X),$$

where the 1st map is the Poincaré residue map, the 2nd map is the natural inclusion, the 3rd map is the connecting homomorphism of the sequence of the pair (X,X^*)

$$H^{n-1}(X,X^*) \to H^{n-1}(X) \to H^{n-1}(X^*) \xrightarrow{\delta^{n-1}} H^n(X,X^*) \to H^n(X) \to H^n(X^*),$$

and the 4th map is the natural isomorphism $H^n(X^*)_{\mathrm{prim}} \to H^n_S(X)$.

Lemma 2.7. [3, Equation (2.3)] Suppose $n \geq 3$. The map $\vartheta : H^n(U)(1) \to H^n_S(X)$ is a natural morphism of MHS. Moreover, $\mathrm{coker}(\vartheta) \cong H^n(X)_{\mathrm{prim}}$ as MHS.

Proof. The Poincaré residue map $H^n(U)(1) \xrightarrow{\cong} H^n(X^*)_{\text{prim}}$ is a morphism of MHS [3, Lemma 2.2]. The inclusion $H^n(X)_{\text{prim}} \to H^n(X)$ is a morphism of MHS. The exact sequence of the pair (X, X^*) is an exact sequence of MHS; see [3, page 291] or [10]. Therefore also, δ^{n-1} is a morphism of MHS. Hence, ϑ is a composition of morphisms of MHS and is itself a morphism of MHS.

We have the following identifications:

$$\begin{aligned} H^n(X)_{\text{prim}} &= \ker(H^n(X)_{\text{prim}} \to H^n(X^*)_{\text{prim}}) \\ &= \ker(H^n(X) \to H^n(X^*)) \\ &= \operatorname{im}(H^n_S(X) \to H^n(X)) \\ &\cong H^n_S(X)/\operatorname{im}(\delta^{n-1}) \\ &= H^n_S(X)/\operatorname{im}(\delta^{n-1}_{\text{prim}}) \\ &= H^n_S(X)/\operatorname{im}(\vartheta) = \operatorname{coker}(\vartheta). \end{aligned}$$

The 1st equality holds since $H^n(X^*)_{\text{prim}} = 0$, the 2nd follows from Proposition 2.5, the 3rd and 4th follow from the long exact sequence of the pair (X, X^*). The 5th equality follows from Proposition 2.5 and the 6th follows from the fact that the Poincaré residue map is an isomorphism (see [3, Lemma 2.2]). ■

In order to determine the cokernel of ϑ, we start by identifying generators for $H^n(U)(1)$. This is relatively straightforward since U is affine, and therefore, its cohomology is the cohomology of its algebraic de Rham complex. Let us define the following n-form on $\mathbf{C}^{n+1} \setminus \{(0, 0, \ldots, 0)\}$:

$$\Omega = \left(\prod_{j=0}^{n} x_j\right) \sum_{j=0}^{n} (-1)^j \frac{dx_0}{x_0} \wedge \cdots \wedge \widehat{\frac{dx_j}{x_j}} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

This form can be used to define n-forms on U. Let us consider $H^n(U)$. Using that U is affine, it is easy to show that $H^n(U)$ is spanned by classes $\frac{g}{f^s}\Omega$, with $g \in R_{sd-n-1}$, $s \in \mathbf{Z}_{>0}$; see [3, Equation 1.3]. We can filter $H^n(U)$ by the order of the pole, by setting $P^s H^n(U)$ to be the subspace of classes that can be represented by elements of the form

$$\frac{g}{f^{n-s}}\Omega.$$

Let $F^\bullet$ be the Hodge filtration on $H^n(U)$. Deligne and Dimca [2] showed that $F^\bullet \subset P^\bullet$.

On the local side, we can proceed similarly. Let $p \in S$. Let $V_p \subset \mathbf{C}^n$ be a neighbourhood of p. Assume that we choose local coordinates $z_1, \dots, z_n$ such that $p = (0, \dots, 0)$. Let $f_p = 0$ be a local equation for X in a neighbourhood of p. Let $\Omega_p = dz_1 \wedge \dots \wedge dz_n$. Let $U_p = V_p \setminus Z(f_p)$. Then the n-forms on U_p can be written as

$$\frac{g}{f_p^{n-s}} \Omega_p.$$

We can define analogously a pole order filtration on $H^n(U_p)$; see [3, page 288]. Again, this is a decreasing filtration satisfying $F^\bullet \subset P^\bullet$ [5, Proposition 6.1.39], and therefore, we can always find a representative for a given cohomology class such that $0 \le s \le n$. There is a stronger result in the case of semi-weighted homogeneous singularities. First, we recall the definition of semi-weighted homogeneous singularities.

Definition 2.8. Let $g : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be an analytic function germ. Let $(Y, 0) = (g^{-1}(0), 0)$ be the associated hypersurface singularity. We say that $(Y, 0)$ is a *weighted homogeneous singularity* if there exists a weighted homogeneous polynomial $h \in \mathbf{C}[z_1, \dots, z_n]$ and an analytic isomorphism $\varphi : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $(\varphi(h^{-1}(0)), 0) = (X, 0)$.

We say that $(Y, 0)$ is a *semi-weighted homogeneous singularity* if there exist a polynomial $h \in \mathbf{C}[z_1, \dots, z_n]$, integers $w_1, \dots, w_n$ and an analytic isomorphism $\varphi : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $(\varphi(h^{-1}(0)), 0) = (Y, 0)$ and such that we can write $h = h_0 + h_1$ with

1. h_0 has an isolated singularity at the origin;
2. h_0 is a weighted homogeneous polynomial with respect to $w_1, \dots, w_n$;
3. each monomial in h_1 has weighted degree strictly larger than the weighted degree of h_0.

Let $X \subset \mathbf{P}^n$ be a hypersurface. Then we say that a point $p \in X$ is a *weighted homogeneous singularity*, respectively a *semi-weighted homogeneous singularity* if there exists an analytic neighbourhood V of p in $\mathbf{P}^n$ such that $(V \cap X, p)$ is a weighted homogeneous singularity, respectively a semi-weighted homogeneous singularity.

Suppose now that V_p is chosen sufficiently small such that $Z(f_p)$ is contractible. Then the local Poincaré residue map and the long exact sequence for the pair $(Z(f_p), Z(f_p) \setminus \{p\})$ yield isomorphisms of MHS

$$H^n(U_p)(1) \to H^{n-1}((Z(f_p) \setminus \{p\}) \cap V_p) \to H^n_p(X).$$

We can use the above isomorphism to define the P-filtration on $H^n_p(X)$; see [3, page 288]. If f_p is semi-weighted homogeneous, then the filtrations $F^\bullet$ and $P^\bullet$ on $H^n(U_p)$ coincide by [3, page 289].

Lemma 2.9. Suppose $n \geq 3$. Suppose all singularities of X are semi-weighted homogeneous. We have that $\mathrm{Gr}^s_F H^n(X)_{\mathrm{prim}}$ is isomorphic with the cokernel of

$$\overline{F^s\vartheta} : F^s H^n(U)(1) \to \oplus_{p\in S} \mathrm{Gr}^s_P H^n_p(X).$$

Proof. The morphism $\vartheta : H^n(U)(1) \to \oplus_{p\in S} H^n_p(X)$ is strict for the Hodge filtration by Lemma 2.7 and [5, Remark C16]. Hence, Gr^s_F of the cokernel equals the cokernel of

$$F^s H^n(U)(1) \to \oplus_{p\in S} \mathrm{Gr}^s_F H^n_p(X).$$

Since all singularities are semi-weighted homogeneous, we obtain that $F^\bullet$ and $P^\bullet$ coincide on $H^n_p(X)$ by [3, page 289]. ■

We introduced a P-filtration on $H^n_p(X)$. The direct sum of these filtrations yields a P-filtration on $H^n_S(X)$. We also introduced a P-filtration on $H^n(U)(1)$. The following lemma shows that the morphism of MHS $\vartheta : H^n(U)(1) \to H^n_S(X)$ respects these P-filtrations. However, ϑ is strict for the Hodge-filtration but does not need to be strict for the P-filtration.

Lemma 2.10. Suppose $n \geq 3$. The morphism $\vartheta : H^n(U)(1) \to H^n_S(X)$ respects the P-filtrations on $H^n(U)$ and $H^n_S(X)$.

Proof. Let $p \in S$. Without loss of generality, we may assume $p = (1 : 0 : 0 : \cdots : 0)$ and that we have local coordinates $z_j = x_j/x_0$, for $j = 1, \ldots n$.

Consider now composition of ϑ with the natural projection map $H^n_S(X) \to H^n_p(X)$:

$$H^n(U)(1) \to H^n_p(X).$$

We aim to make this map explicit. Consider the affine chart $x_0 \neq 0$. Let $\omega \in P^s H^n(U)(1)$. Pick some representative $\frac{g}{f^{n-s}}\Omega$ for ω. Let f_p be a local equation for (X, p). Then, locally, we can write this form as

$$\frac{g_p}{f_p^{n-s}} dz_1 \wedge dz_2 \wedge \ldots dz_n.$$

Hence, $P^s(H^n(U)(1)) \subset P^s H^n_p(X)$. ■

Proposition 2.11. Suppose $n \geq 3$. We have that $\mathrm{Gr}_F^s H^n(X)_{\mathrm{prim}}$ is isomorphic to the cokernel of

$$\mathrm{Gr}_P^s H^n(U)(1) \to \oplus_{p\in S}\mathrm{Gr}_P^s H_p^n(X).$$

Proof. Let $\tau : H^n(U) \to \oplus_{p\in S} H_p^n(X)$. From Lemma 2.9, it follows that $\mathrm{Gr}_F^s H^n(X)_{\mathrm{prim}}$ equals the cokernel of

$$\overline{\tau} : F^s H^n(U)(1) \to \oplus_{p\in S}\mathrm{Gr}_P^s H_p^n(X).$$

From Lemma 2.10, it follows that this map can be extended to $P^s H^n(U)(1) \supset F^s H^n(U)(1)$, that is, the map

$$\overline{\tau} : P^s H^n(U)(1) \to \oplus_{p\in S}\mathrm{Gr}_P^s H_p^n(X)$$

is well defined. It remains to show that the image of P^s is contained in the image of F^s.

From Lemma 2.10, it follows that

$$\tau(P^s H^n(U)(1)) \subset P^s H_p^n(X) = F^s H_p^n(X).$$

Hence,

$$\tau(P^s H^n(U)(1)) \subset F^s \tau(H^n(U)).$$

Since τ is strict for F [5, Remark C16], we find

$$F^s \tau(H^n(U)(1)) = \tau(F^s H^n(U)(1))$$

and we are done. ■

Notation 2.12. Let $f \in R$, then $J(f) \subset R$ is the ideal generated by the partials $\partial f/\partial x_i$ for $i = 0, \ldots, n$ and $J^{\mathrm{sat}}(f)$ its saturation with respect to the irrelevant ideal $(x_0, \ldots, x_n)$. If no confusion arises, then we will write J and J^{sat} for $J(f)$ and $J^{\mathrm{sat}}(f)$, respectively.

Lemma 2.13. Suppose $n \geq 3$ and $s < n-1$. There is a natural surjective map

$$(R/J)_{(n-s)d-n-1} \to \mathrm{Gr}_P^s H^n(U).$$

Proof. By the definition of the $P^\bullet$-filtration, there is a surjective map

$$R_{(n-s)d-n-1} \to P^s H^n(U)$$

for any $s \in \{0, \dots, n-1\}$, sending g to $\frac{g}{f^{n-s}}\Omega$. Consider for $i \in \{0, \dots, n\}$ the $n-1$-form

$$\frac{g}{f^{n-s-1}} \sum_{j \neq i} \epsilon x_i dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

with $\epsilon = (-1)^{i+j+1}$ for $i < j$ and $(-1)^{i+j}$ for $i > j$. Differentiating this form shows that

$$\frac{fg_{x_i} - (n-1-s)gf_{x_i}}{f^{n-s}}\Omega$$

is zero in cohomology, in particular, the image of $J_{(n-s)d-n-1}$ in $P^s H^n(U)(1)$ is contained in P^{s-1}. Therefore, there is a surjection

$$(R/J)_{(n-s)d-n-1} \to \mathrm{Gr}_P^s H^n(U).$$

■

For $p \in S$, let $\mathcal{T}_p$ be the Tjurina algebra of X at p. Let f_p and U_p as above.

Lemma 2.14. Suppose $n \geq 3$. There is a natural surjective map $\mathcal{T}_p \to \mathrm{Gr}_P^s H_p^n(X)$.

Proof. Consider now the map $\mathbf{C}\{z_1, \dots, z_n\} \to P^s H^n(U_p)$ sending g to

$$\frac{g}{f_p^{n-s}}\Omega_p.$$

Obviously, the ideal generated by f_p lands in P^{s+1}. Differentiating for $j \in \{1, \dots, n\}$ the $(n-1)$-form

$$\frac{1}{f_p^{n-s-1}} dz_1 \wedge dz_2 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

yields that $J(f_p)$ lands in P^{s+1}. Hence, we obtain a well-defined map from the Tjurina algebra of X at p to $\mathrm{Gr}_P^s H^n(U_p)$. ■

Remark 2.15. If f_p is weighted homogeneous with weights w_i and degree d_p, then the map

$$\mathcal{T}_p \to \mathrm{Gr}_P^s H_p^n(X)$$

has a natural section. This allows us to identify

$$\mathrm{Gr}_P^s H^n(U_p)(1) \text{ with } (\mathcal{T}_p)_{(n-s)d_p - \sum w_i}.$$

However, this latter fact is only used in the examples. For more details, see [4, Example 3.6].

Proposition 2.16. Suppose $n \geq 3$. Let $s \in \{0, \dots, n-2\}$. The dimension of $\mathrm{Gr}_F^s H^n(X)$ is at most the defect of J^{sat} in degree $(n-s)d - n - 1$.

Proof. Recall that there is a natural map from the global coordinate ring to each of the local rings. The Jacobian ideal of f is generated by the $n+1$ partials of f, whereas ideal defining the Tjurina algebra is generated by n partials of f_p and f_p itself. Assume for the moment that $p = (1 : 0 : \cdots : 0)$. Let f_i be the partial of f with respect to x_i. Then the global Jacobian ideal is generated by $(f_0, \dots, f_n)$, whereas the ideal generating the Tjurina algebra is generated by $(f, f_1, \dots, f_n)$ where we substitute $x_0 = 1$ and write in local coordinates.

Writing the Euler relation $df(x_0, \dots, x_n) = \sum x_i f_i$ in local coordinates, we obtain that $df(1, z_1, \dots, z_n) \equiv f_0 \bmod (f_1, \dots, f_n)$ in $\mathbf{C}\{z_1, \dots, z_n\}$. Therefore, there is a well-defined natural map $(R/J)_{(n-s)d-n-1} \to \mathcal{T}_p$.

Consider now

$$H^n(U)(1) \to H^n(U_p) \to H^{n-1}(X^* \cap U_p) \to H^n_p(X).$$

This map factors through the natural restriction map of forms $H^n(U) \to H^n(U_p)$, which respects the P-filtration; hence, we have a commutative diagram

$$\begin{array}{ccc} (R/J)_{(n-s)d-n-1} & \xrightarrow{\tau} & \oplus_{p\in S}\mathcal{T}_p \\ \downarrow & & \downarrow \\ \mathrm{Gr}_P^s H^n(U)(1) & \xrightarrow{\overline{\tau}} & \oplus_{p\in S}\mathrm{Gr}_P^s H^n_p(X). \end{array}$$

Since both vertical maps are surjective, the cokernel of the bottom row is a quotient of the cokernel of the top row. Moreover, in a neighbourhood of p, one can identify $\mathrm{Proj}(R/J)$ with $\mathrm{Spec}(\mathcal{T}_p)$. In particular, the scheme $V(J)$ is just $V(\oplus \mathcal{T}_p)$, and therefore, the kernel of the map in the top row equals J^{sat}/J. Let ξ be the length of $V(J) = V(J^{\mathrm{sat}})$, the total Tjurina number of the singularities of X. Then the cokernel of the top row has dimension

$$\xi - h_{J^{\mathrm{sat}}}((n-s)d - n - 1),$$

that is, the defect of J^{sat} in degree $(n-s)d - n - 1$. ∎

The above method calculates (or bounds) the dimension of $H^n(X)_{\text{prim}}$ for an $n-1$-dimensional hypersurface. The latter dimension equals the vanishing order of 1 as a zero of its Alexander polynomial. To find the other vanishing orders, one has to consider the d-fold cover of $\mathbf{P}^n$ ramified along X, which is a hypersurface in $\mathbf{P}^{n+1}$. (See also [5, Remark 6.2.23].) At this stage, we will again include the case $n = 2$, that is, assume now that $n \geq 2$.

Let $\zeta_d = \exp(2\pi i/d)$. We consider now the hypersurface $\tilde{X} \subset \mathbf{P}^{n+1}$ given by $y^d + f = 0$. Let $\tilde{U} = \mathbf{P}^{n+1} \setminus \tilde{X}$. Let T be the map $y \mapsto \zeta_d^{-1} y$. Then T^* acts on $H^n(\tilde{U})$. Let $\tilde{S} = \tilde{X}_{\text{sing}}$. We have that $q = (x_0 : \cdots : x_n : y) \in \tilde{S}$ if and only if $p = (x_0 : \cdots : x_n) \in S$ and $y = 0$. In particular, there is a natural bijection between $\tilde{S}$ and S. The fix locus of the automorphism T contains $\tilde{S}$. Moreover, for $q \in S$, then the induced linear map T^* acts on $\mathbf{C}[x_0, \ldots, x_n, y]$ and maps the Jacobian ideal of $y^d + f_p$ to itself; hence, T^* acts on $\mathcal{T}_q$.

Proposition 2.17. Suppose $k \in \{1, \ldots, d-1\}$ and $s \in \{0, \ldots, n-1\}$. Then the ζ_d^k eigenspace for T^* acting on $\text{Gr}_F^s H^{n+1}(\tilde{X})_{\text{prim}}$ has dimension at most the defect of $J^{\text{sat}}(f)$ in degree $(n+1-s)d - n - 1 - k$. The 1-eigenspace of T^* acting on $H^{n+1}(\tilde{X})_{\text{prim}}$ is zero.

Proof. Recall that y^{d-1} is in the Jacobian ideal both on the local and the global sides, and all other partials do not involve y. Let $q \in \tilde{S}$, and let p be the corresponding point in S, then $\mathcal{T}_q = \oplus_{r=0}^{d-2} y^r \mathcal{T}_p$ as $\mathbf{C}$-algebras. Recall that T^* maps Ω_q to $\zeta_d^{-1}\Omega_q$. Obviously, T^* acts on $H^n(\tilde{U})$ and sends Ω to $\zeta_d^{-1}\Omega$. Since $y^{d-1} \in J(y^d + f)$, we can decompose the Jacobian ring of $\tilde{X}$ as follows:

$$R[z]/J(y^d + f) \cong \oplus_{r=0}^{d-2} y^r R/J(f).$$

Consider now

$$H^{n+1}(\tilde{U})(1) \to H_q^{n+1}(\tilde{X}).$$

As above, we find a commutative diagram

$$\begin{array}{ccc}
(R[z]/J(y^d+f))_{(n+1-s)d-n-2} & \xrightarrow{\tau} & \oplus_{q\in\tilde{S}}\mathcal{T}_q \\
\downarrow & & \downarrow \\
\text{Gr}_P^s H^{n+1}(\tilde{U})(1) & \xrightarrow{\overline{\tau}} & \oplus_{q\in\tilde{S}} \text{Gr}_P^s H_q^{n+1}(\tilde{X}).
\end{array}$$

Both vertical maps are surjective. Each of the above maps is equivariant for T^*.

In particular, the eigenvalues of T^* on $H_q^{n+1}(\tilde{X})$ are all d-th roots of unity, but different from 1 and therefore the 1-eigenspace of T^* acting on $H^{n+1}(\tilde{X})_{\text{prim}}$ is zero.

From the above diagram, it follows that the cokernel of the bottom row is a quotient of the cokernel of the top row. Moreover, the dimension of the ζ_d^{-k}-eigenspace is at most the cokernel of

$$\tau_k : y^{k-1}R/J(f)_{(n+1-s)d-n-2-(k-1)} \to \oplus_{p\in S}y^{k-1}\mathcal{T}_p.$$

In a neighbourhood of p, one can identify $\text{Proj}(R/J)$ with $\text{Spec}(\mathcal{T}_p)$. In particular, the scheme $V(J)$ is just $V(\oplus\mathcal{T}_p)$. In particular, the kernel of the map of τ is just $(J^{\text{sat}}/J)_{(n+1-s)d-n-2}$. Let ξ be the length of $V(J) = V(J^{\text{sat}})$, the total Tjurina number of the singularities. Then the cokernel of τ_k has dimension

$$\xi - h_{J^{\text{sat}}}((n-s)d - n - 1 - k),$$

which equals the defect of J^{sat} in degree $(n+1-s)d - n - 1 - k$, by definition. ■

3 Calculation of Alexander Polynomial

In this section, we will use the results of the previous section to calculate the Alexander polynomial to identify a range of degrees for which the ideal J^{sat} has defect.

Definition 3.1. [5, Definition 1.1.19] Let $n \geq 2$. Let $f \in R$ be a homogeneous polynomial of degree d such that $X = V(f) \subset \mathbf{P}^n$ has isolated singularities. Let $F = Z(f+1) \subset \mathbf{C}^{n+1}$ be the affine Milnor fibre of the singularity $(f, 0)$. Then the *Alexander polynomial* of X is the characteristic polynomial of the monodromy operator acting on $H^{n-1}(F)$ and denoted by $\Delta_X(t)$.

Consider $\tilde{X} = Z(y^d + f) \subset \mathbf{P}^{n+1}$. Then the map $(x_0, \ldots, x_n) \to (x_0 : \cdots : x_n : 1)$ maps F onto $\tilde{X} \setminus (\tilde{X} \cap Z(y))$. The set $\tilde{X} \cap Z(y)$ equals X. In this way, we find an exact sequence

$$0 \to H_c^n(X)_{\text{prim}} \to H_c^{n+1}(F) \to H_c^{n+1}(\tilde{X})_{\text{prim}} \to 0.$$

The map $F \to \tilde{X} \setminus (\tilde{X} \cap Z(y))$ is an isomorphism. Using Poincaré duality on F, we obtain that

$$H^{n-1}(F) \cong H_c^{n+1}(F)^* \cong H_c^{n+1}(\tilde{X} \setminus (\tilde{X} \cap Z(y)))^*.$$

In particular, $H^{n-1}(F)$ is an extension of $H_{\text{prim}}^{n+1}(\tilde{X})^*$ by $H^n(X)^*_{\text{prim}}$.

As is shown in [5, Remark 6.2.23], we have that the monodromy operator on $H_c^{n+1}(F)$ is just the extension of the operator T on $H^{n+1}(\tilde{X})_{\text{prim}}$ and the identity map on $H^n(X)_{\text{prim}}$. Therefore, the Alexander polynomial of X is $(t-1)^a\varphi(t)$ with $a = h^n(X)_{\text{prim}}$ and $\varphi(t)$ the characteristic polynomial of T on $H^{n+1}(\tilde{X})_{\text{prim}}$. Since T^d is the identity operator, we have that all zeroes of the Alexander polynomial are d-th roots of unity, and from Proposition 2.17, it follows that $\varphi(1) \neq 0$.

In the case that all singularities of X are semi-weighted homogeneous, $H^n(X)$ has a pure weight n Hodge structure and $H^{n+1}(\tilde{X})$ has a pure weight $n+1$ Hodge structure; see [10].

Steenbrink [11] studied extensively the spectrum of polynomials with isolated singularities. The polynomial f has a one-dimensional singular locus; hence, there are two spectra: one associated with $H^n(F)$ and one with $H^{n-1}(F)$ (see [9, Section II.8.10]).

In the sequel, we will call the spectrum associated with $H^{n-1}(F)$ the spectrum of f. We will use the spectrum merely for bookkeeping reasons.

Definition 3.2. Let $\mathbf{Z}[\mathbf{Q}]$ be the group of formal sums of rational numbers, that is, the set of expressions of the form $\sum_{\alpha\in\mathbf{Q}} n_\alpha[\alpha]$, with $n_\alpha \in \mathbf{Z}$ for all α and such that the set $\{\alpha \mid n_\alpha \neq 0\}$ is finite. The group law on $\mathbf{Z}[\mathbf{Q}]$ is the natural addition.

The *spectrum* $sp(f)$ of f is the element $\sum n_\alpha[\alpha]$ of $\mathbf{Z}[\mathbf{Q}]$ such that

1. If $\alpha \notin [0,n] \cap \frac{1}{d}\mathbf{Z}$, then $n_\alpha = 0$.
2. If α is an integer, then $n_\alpha = \dim \operatorname{Gr}_F^{n-\alpha} H^n(X)_{\text{prim}}$.
3. If α is not an integer, but $d\alpha$ is integer, then let $s = \lceil\alpha\rceil$ and $k = d(s-\alpha)$. Then n_α equals the dimension of ζ_d^{-k} eigenspace for T^* acting on $\operatorname{Gr}_F^{n+1-s} H^{n+1}(\tilde{X})$.

Lemma 3.3. We have $n_\alpha = n_{n-\alpha}$ and $\sum_\alpha n_\alpha = \deg(\Delta_X)$.

Proof. Suppose first that α is some integer. The Hodge structure on $H^n(X)$ is pure of weight n; see [10]. Hence, $h_{\text{prim}}^{\alpha,n-\alpha} = h_{\text{prim}}^{n-\alpha,\alpha}$. In particular, we find that $n_\alpha = n_{n-\alpha}$.

Suppose now that α is not an integer. Let $s = \lceil\alpha\rceil$ and $k = d(s-\alpha)$. Let $\zeta_d = \exp(2\pi i/d)$. The Hodge structure on $H^{n+1}(\tilde{X})$ is pure of weight $n+1$ by [10]. Complex conjugation maps the ζ_d^e-eigenspace to the ζ_d^{d-e}-eigenspace. In particular, the ζ_d^k-eigenspace on $H^{s,n+1-s}$ and the ζ_d^{d-k} eigenspace of $H^{n+1-s,s}$ have the same dimension, hence $n_{s-\frac{k}{d}} = n_{n+1-s-\frac{(d-k)}{d}}$. Since we have

$$n+1-s-\frac{(d-k)}{d} = n - \left(s - \frac{k}{d}\right),$$

the statement follows.

Since $(T^*)^d$ is the identity, it follows that T^* is diagonalizable on $H^{n+1}(\tilde{X})$ and all eigenvalues of T^* are d-th roots of unity. Moreover, 1 is not an eigenvalue by Proposition 2.17.

Hence, the sum of the dimensions of the eigenspaces of $H^n(X)_{\text{prim}}$ and of the eigenspaces of $H^{n+1}(\tilde{X})_{\text{prim}}$ equal the total dimension that in turn equals the degree of the Alexander polynomial. Hence,

$$\sum_{\alpha} n_\alpha = \deg(\Delta_X).$$

■

Let J be the Jacobian ideal of f. Then J^{sat} is the ideal of the scheme $V(J)$. Let ξ be the length of this scheme. Propositions 2.16 and 2.17 imply the following result.

Proposition 3.4. Suppose $\alpha > 1$. We have

$$n_\alpha \leq \xi - h_{J^{\text{sat}}}(\alpha d - n - 1).$$

Proof. Suppose first that α is an integer. Then,

$$n_\alpha = \dim \operatorname{Gr}_F^{n-\alpha} H^n(X)_{\text{prim}}$$

Proposition 2.16 implies that the latter is at most the defect of J^{sat} in degree $(n-(n-\alpha))d-n-1$. Suppose now that α is not an integer, then write $s = \lceil \alpha \rceil$ and $k = d(s-\alpha)$. Then n_α equals the dimension of ζ_d^{-k} eigenspace for T^* acting on $\operatorname{Gr}_F^{n+1-s} H^{n+1}(\tilde{X})$. Proposition 2.17 implies that this at most the defect of J^{sat} in degree $(n+1-(n+1-s))d-n-1-k = (s-\frac{k}{d})d-n-1$. ■

Lemma 3.5. Let $\Sigma \subset \mathbf{P}^n$ be a zero-dimensional scheme of length m. Then,

$$\delta(k) := m - h_{I(\Sigma)}(k)$$

is decreasing as a function in k.

Proof. Choose coordinates on $\mathbf{P}^n$ such that $V(x_0) \cap \Sigma = \emptyset$. The number $\delta(k)$ equals the dimension of the cokernel of the evaluation map

$$ev : R_k \to \oplus_{p \in \Sigma} A_p,$$

where A_p is the affine coordinate ring of Σ in an affine neighbourhood of p, that is, obtained by setting $x_0 = 1$.

Let $f_1, \dots, f_m$ be a basis for the image in degree k, and let $F_1, \dots, F_k$ be elements such that $ev(F_i) = f_i$.

Then in degree $k+1$, we have that $ev(x_0 F_i) = ev(F_i) = f_i$; hence, the dimension of the image in degree $k+1$ is at least the dimension of the image in degree k. ■

Proposition 3.6. Suppose $\alpha > 1$ and $n_\alpha > 0$. Then J^{sat} has defect in every degree $\leq \alpha d - n - 1$.

Proof. If $n_\alpha > 0$, then Proposition 3.4 implies that J^{sat} has defect in degree $\alpha d - n - 1$.

However, J^{sat} is the ideal of a zero-dimensional projective scheme, and for such a scheme, one has that the defect is a decreasing function in the degree by the previous lemma; hence, J^{sat} has defect in every degree up to $\alpha d - n - 1$. ■

In order to show that J^{sat} has defect in degree d, we need to find an α such that $n_\alpha > 0$ and $\alpha d - n - 1 \geq d$. Using the symmetry of the spectrum, we know that if for some α, we have $n_\alpha > 0$, then we can find an $\alpha \geq \frac{n}{2}$ with $n_\alpha > 0$. However, for $n = 2$ and for $n \geq 3$ and d small this is insufficient to show that J^{sat} has defect in degree d. If we take into account the k such that Δ_X has a primitive k-th root of unity as a zero, then we find a slightly larger α contained in the interval $[\frac{n}{2}, \frac{n+1}{2}]$. To identify such an α, we use the following notation.

Definition 3.7. Let $k > 2$ be an integer, such that $k \mid d$. Let $\psi(k)$ be the largest integer m such that $\gcd(m, k) = 1$ and $m < \frac{k}{2}$. Define

$$\alpha(n, d, k) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even and } k = 1 \text{ or } n \text{ is odd and } k = 2 \\ \frac{n+1}{2} & \text{if } n \text{ is even and } k = 2 \text{ or } n \text{ is odd and } k = 1 \\ \frac{n+1}{2} - \frac{1}{k} & \text{if } n \text{ is odd and } k > 2. \\ \frac{n}{2} + \frac{\psi(k)}{k} & \text{if } n \text{ is even and } k > 2. \end{cases}$$

Note that $\alpha(n, d, k) \geq \frac{n}{2}$.

For an integer k, let $\zeta_k := \exp(2\pi i/k)$.

Proposition 3.8. Suppose ζ_k is a root of the Alexander polynomial, then n_α is nonzero for some α at least $\alpha(n, d, k)$.

Proof. If $k = 1$, then by the symmetry property (Lemma 3.3) $n_\alpha > 0$ for some integer $\alpha \geq \frac{n}{2}$.

If $k = 2$, then by the symmetry property $n_\alpha > 0$ for some $p + \frac{1}{2}$ with p an integer $p \geq \frac{n-1}{2}$.

Suppose now $k > 2$. Since the Alexander polynomial is in $\mathbf{Q}[t]$, we have for each i with $0 < i < k$ and $\gcd(i, k) = 1$ that the sum

$$\sum_{j=0}^{n} n_{j+\frac{i}{k}}$$

is independent of i. Using the symmetry, we find that there at least $\varphi(k)/2$ values of α occurring in the spectrum, which are of the form $\frac{n}{2} + \frac{i}{k}$ with $\gcd(i, k) = 1$ and $i > 0$.

Recall that there are precisely $\varphi(k)/2$ such values of α in the interval $[n/2, (n+1)/2]$. The largest one equals $\frac{n}{2} + \frac{\psi(k)}{k}$ if n is even and $\frac{n-1}{2} + \frac{k-1}{k}$ if n is odd. ■

We will now identify the values of (n, d, k) such that $\alpha(n, d, k)d - n - 1 \geq d$.

Lemma 3.9. Let $n \geq 3$. Suppose that one of the following conditions hold:

1. $d \geq 8, n = 3$;
2. $d \geq 5, n = 4$;
3. $d \geq 4, n \geq 5$.

Then $(\alpha(n, d, k) - 1)d \geq n + 1$.

Proof. By definition, we have $\alpha(n, d, k) \geq \frac{n}{2}$. Hence, we are fine if $d \geq \frac{2n+2}{n-2} = 2 + \frac{6}{n-2}$. ■

Lemma 3.10. Suppose $n = d = 4$. If $k \neq 1$, then $(\alpha(n, d, k) - 1)d \geq n + 1$.

Proof. Since k divides d and $k \neq 1$, we know that $k \in \{2, 4\}$. The claim follows from $\alpha(4, 4, 2) = \frac{5}{2}, \alpha(4, 4, 4) = \frac{9}{4}$. ■

Lemma 3.11. Suppose $n = 3$ and either

1. $k = 1$ and $d \geq 4$ or
2. $k \geq 3$ and $k \neq d$ (then $d \geq 2k \geq 6$) or
3. $k = d$ and $d \geq 5$.

Then $(\alpha(n, d, k) - 1)d \geq n + 1$.

Proof. Suppose that $k = 1$. Recall that $\alpha(3,d,1) = 2$. For $d \geq 4$, we have $(\alpha(3,d,1) - 1)d = d \geq 4 = n+1$.

If $k \geq 3$, then $d = kj$ for some positive integer j. Recall that $\alpha(3,kj,j) = \frac{2k-1}{k}$. Hence, $(\alpha - 1)d = (k-1)j$. This is at least 4 if $j \geq 2$ or $j = 1$ and $k \geq 5$. Hence, we have to exclude the case $k = d$ and $k \in \{3,4\}$. ■

Remark 3.12. Suppose now that $n = 3$ and $k = 2$ and that we have $d = 2j$ for some $j \geq 2$ (since we excluded $d = 2$). Recall that $\alpha(3,2j,2) = \frac{3}{2}$. Hence, $(\alpha(3,2j,2) - 1)d = j$. In particular, for $j = 2,3$ (hence $d = 4,6$), we have to exclude $k = 2$. These are the only even values of d between 3 and 7.

Lemma 3.13. Suppose $d = 3$ and $n \geq 3$. Moreover, suppose that

$$(n,k) \notin \{(3,1),(4,1),(6,1),(3,3),(4,3),(5,3)\}.$$

Then $(\alpha(n,d,k) - 1)d \geq n+1$.

Proof. Since k is a divisor d, we know $k \in \{1,3\}$.

Suppose first that $k = 1$. If n is odd, then $(\alpha(n,d,k) - 1)d = \frac{3(n-1)}{2}$. This is at least $n+1$ for $n \geq 5$. If n is even, then $(\alpha(n,d,k) - 1)d = \frac{3n-6}{2}$. This is at least $n+1$ for $n \geq 7$.

Suppose now that $k = 3$. If n is odd, then $(\alpha(n,d,k) - 1)d = \frac{3n-5}{2}$. This is at least $n+1$ for $n \geq 7$. If n is even, then $(\alpha(n,d,k) - 1)d = \frac{3n-4}{2}$. This is at least $n+1$ for $n \geq 6$. ■

Lemma 3.14. Suppose $n = 2$, $d \geq 3$. Suppose that k is not a pure prime power and that $(d,k) \notin \{(6,6),(12,6)\}$. Then $(\alpha(n,d,k) - 1)d \geq n+1$.

Proof. Write $d = kj$. The smallest k that is a not a pure prime power is 6. In particular, $(\alpha(2,d,k) - 1)d = j\psi(k)$.

If $\psi(k) = 1$, then $\varphi(k) = 2$. The only $k \geq 6$ for which this is possible is $k = 6$. If $k = 6$, then $d \geq 18$ and therefore $j \geq 3$. If $j = 1$ and $\psi(k) = 2$, then $\varphi(k) = 4$ and k is odd. In particular, k would be equal to 5, which we excluded. Hence, one of $\psi(k) > 2$, $j > 2$ or $\psi(k) = j = 2$ holds and $(\alpha(2,d,k) - 1)d = j\psi(k) \geq 3$. ■

Proposition 3.15. Let (n,d,k) be integers such that $d \geq 3, n \geq 2$, and $k \geq 1$ is a divisor of d. Suppose $X \subset \mathbf{P}^n$ is an irreducible hypersurface of degree d with isolated semi-weighted homogeneous singularities such that ζ_k is a zero of the Alexander polynomial of X.

Moreover, assume that we are not in one of the following cases:

1. $n = 2$, $d \in \{6, 12\}$, and $k = 6$;
2. $n \in \{3, 4, 6\}$, $d = 3$, and $k = 1$;
3. $n \in \{3, 4, 5\}$, $d = 3$, and $k = 3$;
4. $n = 3$, $d \in \{4, 6\}$, and $k = 2$;
5. $n = 3$, $d = k = 4$;
6. $n = 4$, $d = 4$, and $k = 1$.

Then J^{sat} has defect in degree d.

Proof. The main result of [13] implies that if $n = 2$, then $\Delta_X(\zeta_{p^r}) \neq 0$ for any prime number p and nonnegative integer r. Hence, if $n = 2$, then k is not a prime power.

Since ζ_k is a zero of the Alexander polynomial, we know by Propositions 3.6 and 3.8 that J^{sat} has defect in any degree up to $\alpha(n, d, k)d - n - 1$. From Lemmata 3.9– 3.11, 3.13, and 3.14, it follows that $\alpha(n, d, k)d - n - 1 \geq d$. ■

Theorem 3.16. Let (n, d, k) be integers such that $d \geq 3, n \geq 2$, and $k \geq 1$ is a divisor of d. Suppose $X \subset \mathbf{P}^n$ is an irreducible hypersurface of degree d with isolated semi-weighted homogeneous singularities such that ζ_k is a zero of the Alexander polynomial of X.

Moreover, assume that we are not in one of the cases (1)–(6) of the previous proposition. Then the equianalytic deformation space of X is not T-smooth.

Proof. From [7, Section 1.1.4.1], it follows that the equianalytic deformation space is T-smooth if and only if J^{sat} has no defect in degree d. ■

4 Examples

We start with a general construction.

Example 4.1. Let $f \in \mathbf{C}[y_1, \ldots, y_n]$ be a weighted homogeneous polynomial, smooth outside the origin, with rational weights $w_1, \ldots, w_n$, such that $\deg(f) = 1$. Assume that the Tjurina algebra is not trivial. Let v be the smallest positive integer such that vw_i is an integer for all i.

Let $m \geq 1$ be an integer, and let g_i be a general form of degree mvw_i. Then,

$$f(g_1, \ldots, g_n)$$

is a homogeneous polynomial of degree $d = mv$. Let $X = V(f(g_1, \ldots, g_n)) \subset \mathbf{P}^n$.

Assume now that the g_i are chosen such that $g_1, \ldots, g_n$ form a regular sequence. Then the singular locus of X contains the complete intersection $S_0 = V(g_1, \ldots, g_n)$. Moreover, if the g_i are sufficiently general, then at each point of S_0 the local equation for the singular point for some choice of coordinates is $f = 0$. Note that S_0 consists of

$$(mv)^n \prod w_i$$

points. We claim that the Alexander polynomial of X is nontrivial.

In Proposition 2.17, we showed that n_α is *at most* the dimension of the cokernel of

$$R_{\alpha d-n-1} \to \oplus_{p\in S} \mathcal{T}_p.$$

However, if all singularities are weighted homogeneous, then $\mathcal{T}_p$ is a graded algebra. We can use this to determine n_α precisely. That is, Proposition 2.11 together with Remark 2.15 yield that n_α *equals* the dimension of the cokernel of

$$R_{\alpha d-n-1} \to \oplus_{p\in S} (\mathcal{T}_p)_\alpha. \tag{1}$$

The choice of local coordinates to obtain the correct grading on $\mathcal{T}_p$ is very tricky, basically because one has to pick a particular part of a Taylor expansion and this is very sensitive to coordinate changes. However, this is not an issue for the smallest α occurring in the spectrum of the singularity f. For such an α, we have that $(\mathcal{T}_p)_\alpha = \mathcal{T}_p/m_p$, where m_p is the maximal ideal of p. Changing coordinates would yield an automorphism given by multiplication by a nonzero number.

The smallest number in the spectrum of the isolated singularity $f = 0$ is the sum of the weights $\alpha = \sum w_i$. We want to show that $n_\alpha > 0$ for this α. For each $p \in S_0 := Z(g_1, \ldots g_n)$ we have $(\mathcal{T}_p)_\alpha = \mathcal{T}_p/m_p$. Hence, the cokernel of (1) equals the cokernel of the evaluation map

$$R_{\alpha d-n-1} \to \oplus_{p\in S_0} \mathbf{C}.$$

Hence,

$$n_\alpha = \left((mv)^n \prod w_i\right) - h_{I(S_0)}(\alpha d - n - 1).$$

Since $g_1, \ldots, g_n$ define a scheme-theoretic complete intersection, the ideal generated by them has the Koszul complex on $g_1, \ldots, g_n$ as its resolution. The highest degree of any generator in the resolution is $\sum \deg(g_i) = d \sum w_i$. Hence, the largest degree for which there is defect equals $(d \sum w_i) - n - 1$, that is, we know that there is defect in degree

$$d \sum w_i - n - 1 = \alpha d - n - 1.$$

Hence, $n_\alpha > 0$ and $\exp(2\pi i \sum w_i)$ is a zero of the Alexander polynomial. From $n_\alpha > 0$, for $\alpha = \sum w_i$, it follows that $n_\alpha > 0$ for $\alpha = n - \sum w_i$ by Lemma 3.3. Hence, J^{sat} has defect in degree $(n - \sum w_i)d - n - 1 = (n - \sum w_i)mv - n - 1$.

If we additionally assume that all the weights w_i are of the form $1/k_i$, with $k_i \in \mathbf{Z}$, then the Tjurina number of each singularity is $\prod(k_i - 1) = \prod \frac{1-w_i}{w_i}$. Hence, the total Tjurina number equals

$$(mv)^n \prod_i (1 - w_i) = d^n \prod_i (1 - w_i).$$

This number is so large that it is not clear whether for fixed $(n, k_1, \ldots, k_n, m)$, there exists a component of the space of degree mv-curves with $(mv)^n \prod w_i$ singularities analytically equivalent with $f = 0$ and constant Alexander polynomial.

If we instead assume that $n = 2, f = y_1^2 + y_2^3, w_1 = \frac{1}{2}, w_2 = \frac{1}{3}, \alpha = \frac{5}{6}$, then we recover the example of B. Segre of degree $6m$ curves with $6m^2$ cusps.

We will now give two examples to illustrate how the above construction is the limit of known constructions of hypersurfaces with deformation space whose dimension is larger than expected. The 1st example below is due to Greuel *et al.* [6, Proposition 3.4].

Example 4.2. Fix $k \in \mathbf{Z}_{>0}$. Let $d = 6m$, pick nonnegative integers $a_1, b_1 \le 6m$, such that a_1 and b_1 are divisible by 2 and by 3, respectively. Let $a_2 = 3m - \frac{a_1}{2}, b_2 = 2m - \frac{b_1}{3}$.

Pick general homogeneous forms f_1, f_2, g_1, g_2 in x_0, x_1, x_2 of degree a_1, a_2, b_1, b_2. Consider the curve

$$f_1 f_2^2 + g_1 g_2^3 = 0.$$

Then $J^{\mathrm{sat}} \supset (f_2, g_2^2)$. The latter ideal is a complete intersection ideal. This ideal has defect in degree d if and only if

$$a_2 + 2b_2 \geq 6m + 3,$$

which happens if and only if

$$\frac{a_1}{2} + \frac{2b_1}{3} \leq m - 3.$$

If the forms are chosen sufficiently general, then the singular locus is $f_2 = g_2 = 0$ and all singularities are ordinary cusps. The curve has nonconstant Alexander polynomial if and only if (f_2, g_2) has defect in degree $5m - 3$, that is, if $a_2 + b_2 \geq 5m$. However, $a_2 \leq 3m, b_2 \leq 2m$; therefore, we have $a_1 = b_1 = 0$.

Hence, for each choice of $a_1, b_1 \geq 0$ satisfying

$$\frac{a_1}{2} + \frac{2b_1}{3} \leq m - 3,$$

we find examples with non-T-smooth deformation space, but only for $a_1 = b_1 = 0$, we find examples with nonconstant Alexander polynomial.

We can apply the idea behind Example 4.2 to Example 4.1.

Example 4.3. Again, let $f \in \mathbf{C}[y_1, \ldots, y_n]$ be a weighted homogeneous polynomial, smooth outside the origin, with weights $w_1, \ldots, w_n$, such that $\deg(f) = 1$. Assume that the Tjurina algebra is not trivial.

Write $f = \sum_{i=1}^{s} M_i(y_1, \ldots, y_n)$, where each M_i is some $\mathbf{C}^*$-multiple of a monomial in $y_1, \ldots, y_n$. Consider

$$F(\beta_1, \ldots, \beta_s; y_1, \ldots, y_n) := \sum_{i=1}^{s} \beta_i M_i(y_1, \ldots, y_n).$$

Let v be the smallest positive integer such that vw_i is an integer for all i.

Let $m_1 \geq 1$ be an integer, and let g_i be a general form of degree $m_1 v w_i$. Let m_2 be another integer, and let $h_1, \ldots, h_s$ be general forms of degree m_2. Consider the hypersurface

$$F(h_1, \ldots, h_s; g_1, \ldots, g_n) := \sum_{i=1}^{s} h_i M_i(g_1, \ldots, g_n).$$

If the forms are sufficiently general, then one has singularities with local equation $f = 0$ along $g_1 = \cdots = g_n = 0$ and no further singularities. In this case, J^{sat} has defect in degree

$$\left(n - \sum w_i\right) m_1 v - n - 1.$$

The degree of the hypersurface equals $m_1 v + m_2$. Hence, if

$$m_2 \leq (n - w - 1)m_1 v - n - 1,$$

then the deformation space is not T-smooth. For $m_2 = 0$, we recover the example with a nonconstant Alexander polynomial.

If we can separate variables in f, that is, suppose we can write

$$f = \sum_{j=1}^{s} f_j(x_{i_j}, x_{i_j+1}, \ldots, x_{i_j+k_j})$$

with $i_{j+1} = i_j + k_j + 1$, then we can apply this construction for each summand with different choices of m_2, under the condition that $m_1 v + m_2$ is the same for each summand.

In Example 4.2, we did this for $f = y_1^2 + y_2^3$ and we took $f_1 = y_1^2$ and $f_2 = y_2^3$.

In the case of plane curves with A_2-singularities, Segre considered a family with $6m^2$ cusps on a curve of degree $6m$. In this case, the expected dimension of the deformation space equals

$$\frac{(6m+1)(6m+2)}{2} - 1 - 12m^2 = 6m^2 + 9m,$$

which is definitely positive. However, in the case of m^n ordinary r-fold points on a degree rm hypersurface in $\mathbf{P}^n$, we obtain that the expected dimension is

$$d_{r,n}(m) = \binom{mr+n}{n} - m^n(r-1)^n - 1.$$

The leading coefficient of $d_{r,n}(m)$ equals $r^n(\frac{1}{n!} - 1)$. Hence, for m sufficiently large, we have that the expected dimension is negative. Therefore, the mere existence is sufficient to prove that the deformation space is not T-smooth. We will now give an example of hypersurfaces with ordinary r-fold points, for which the Alexander polynomial is nonconstant and the expected dimension is positive.

Example 4.4. Fix integers ℓ, r both at least 2. Let $n=2\ell$. Fix another positive integer m.

Fix t polynomials $g_1, \ldots, g_\ell$ of degree m such that $g_1, \ldots, g_\ell, x_{\ell+1}, \ldots, x_{2\ell}$ define a complete intersection, which, as a scheme, is reduced. Then this complete intersection consists of m^ℓ points.

Pick now t generic forms $h_1, \ldots, h_\ell$ of degree $m(r-1)$ from the ideal

$$(g_1, \ldots, g_\ell, x_{\ell+1}, \ldots, x_{2\ell})^{r-1}.$$

Consider now $X = V(\sum_{i=1}^{\ell} x_{t+i} h_i)$. Then at each point in

$$Z(g_1, \ldots, g_\ell, x_{\ell+1}, \ldots, x_{2\ell}),$$

we have an r-fold point, and if the h_i are chosen sufficiently general, then these points are ordinary r-fold points.

In this way, we have m^ℓ points of order r. The Milnor number of an r-fold point is $(r-1)^{2\ell}$; hence, the expected codimension equals $m^\ell(r-1)^{2\ell}$, whereas the space of polynomials of degree $m(r-1)+1$ has dimension $\binom{}{m(r-1)+1+2\ell 2\ell}$. The former polynomial is a polynomial of degree ℓ in m, whereas the latter polynomial is a polynomial of degree 2ℓ in m with positive leading coefficient. Hence, for m sufficiently large, the expected dimension

$$\binom{m(r-1)+1+2\ell}{2\ell} - m^\ell(r-1)^{2\ell} - 1$$

is positive.

In this case, we have that the ℓ-plane $x_{\ell+1} = \cdots = x_{2\ell} = 0$ defines a nonzero class of Hodge type (ℓ, ℓ) in $H^{2\ell}(X, \mathbf{C})_{\text{prim}}$. In particular, $n_\ell \neq 0$. For this reason, we have that J^{sat} has defect in any degree $\leq \ell m(r-1) - \ell - 1$. The latter quantity is at least $d = m(r-1)+1$.

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References

[1] Cogolludo-Agustín, J.-I. and A. Libgober. "Mordell–Weil groups of elliptic threefolds and the Alexander module of plane curves." *J. Reine Angew. Math.* 697 (2014): 15–55.

[2] Deligne, P. and A. Dimca. “Filtrations de Hodge et par l’ordre du pôle pour les hypersurfaces singulières.” *Ann. Sci. Éc. Norm. Supér. (4)* 4, no. 23 (1990): 645–56.

[3] Dimca, A. “Betti numbers of hypersurfaces and defects of linear systems.” *Duke Math. J.* 60 (1990): 285–98.

[4] Dimca, A. “On the Milnor fibrations of weighted homogeneous polynomials.” *Compos. Math.* 76 (1990): 19–47. Algebraic geometry (Berlin, 1988).

[5] Dimca, A. *Singularities and Topology of Hypersurfaces*. Universitext. New York: Springer, 1992.

[6] Greuel, G.-M., C. Lossen, and E. Shustin. “Castelnuovo function, zero-dimensional schemes and singular plane curves.” *J. Algebraic Geom.* 9 (2000): 663–710.

[7] Greuel, G.-M., C. Lossen, and E. Shustin. *Singular Algebraic Curves*. Springer Monographs in Mathematics. With an appendix by Oleg Viro. Cham: Springer, 2018.

[8] Kloosterman, R. “Mordell–Weil lattices and toric decompositions of plane curves.” *Math. Ann.* 367 (2017): 755–83. https://doi.org/10.1007/s00208-016-1399-9.

[9] Kulikov, V. S. *Mixed Hodge Structures and Singularities*, vol. 132. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1998.

[10] Steenbrink, J. H. M. “Intersection form for quasi-homogeneous singularities.” *Compos. Math.* 34 (1977): 211–23.

[11] Steenbrink, J. H. M. “The Spectrum of Hypersurface Singularities.” In *Actes du Colloque de Théorie de Hodge*, vol. 11. Number 179–180 in Astérisque. 163–84. SMF, 1989. Actes du Colloque de Théorie de Hodge (Luminy, 1987).

[12] Tannenbaum, A. “On the classical characteristic linear series of plane curves with nodes and cuspidal points: two examples of Beniamino Segre.” *Compos. Math.* 51 (1984): 169–83.

[13] Zariski, O. “On the linear connection index of the algebraic surfaces $z^n = f(x, y)$.” *Proc. Natl. Acad. Sci. USA* 15 (1929): 494–501.

[14] Zariski, O. “On the irregularity of cyclic multiple planes.” *Ann. of Math. (2)* 2, no. 32 (1931): 485–511.

[15] Zariski, O. *Algebraic Surfaces*, vol. Band 61. Ergebnisse der Mathematik und ihrer Grenzgebiete. New York–Heidelberg: Springer, 1971.