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RESEARCH ARTICLE

Accessible Mathematics: Representation of Functions Through Sound and Touch

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ABSTRACT This paper presents a new tool to help people with visual impairments learn mathematical functions through the extensive use of sounds and touch. STEM subjects like mathematics are difficult for this kind of user, since concepts like lines or curves heavily depend on visual content to convey essential information. *GraficiAccessibili* is a mobile app that allows to draw lines and parabolas in the cartesian plane, incorporating sound and haptic feedback to make them perceivable by all users. Mathematical functions are sonified enabling users to discern the position of their points on the quarters of the cartesian plane through sound cues. Furthermore, the application assists users in exploring the cartesian plane through touch giving haptic feedback for curves and axes. In this way, the user can mentally visualize the graphical representation of the function. Tests were conducted with real users to assess the effectiveness of this approach.

INDEX TERMS Accessibility, mathematics, sonification, sound, haptic feedback, education, functions, lines, parabola, unity, android, iOS, iPad, cross-platform, musical notes, visual impairments.

I. INTRODUCTION

People with visual impairments face significant challenges in accessing education due to the predominance of visual-based learning resources. Visual impairment is a medical condition that indicates a person's eyesight or ability to see is significantly compromised or conventional techniques such as eyeglasses or contact lenses cannot fully correct it. There are different ways of describing the condition of visual loss in an individual, including:

- **Blindness** is the most severe form of visual impairment, i.e., the total or almost total loss of functional vision. Individuals who are blind may rely on Braille, guide dogs, or assistive technology to navigate the world;
- **Low Vision** refers to the medical condition of having some degree of usable vision but with limitations. People with low vision need specific tools such as magnifiers or screen readers to read or perform daily tasks;
- **Legal blindness** is a specific classification used for eligibility purposes. In the US, a person is legally blind if their best-corrected visual acuity is 20/200 or worse, or if their visual field is restricted to 20 degrees or less.

About 940 million people suffer some degree of vision loss, 246 million have low vision and 39 million are blind [1].

The fields of science, technology, engineering, and mathematics (STEM) have made invaluable contributions to technological advancement and societal progress. However, these fields have historically been dominated by visually able individuals, which has created a significant barrier to participation for those with visual impairments [16]. The first significant challenge pertains to the visual nature of STEM subjects. The use of diagrams, graphs, and complex equations, which are fundamental to understanding scientific concepts, are often inaccessible to those without sight. Additionally, laboratory experiments that rely heavily on visual observation and manipulation present further obstacles. The lack of accessible textbooks and learning materials, coupled with limited access to assistive technology and insufficient training in its use, exacerbates these difficulties. Consequently, individuals with visual impairments often experience lower mathematical performance compared to

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their peers [17]. Furthermore, they also find difficulties in reading in writing mathematical notations and functions due to the lack of accessible resources for the representation of symbols [18]. The World Health Organization (WHO) stated these challenges have a big impact on the lives of young children which bring them to have lifelong consequences, affecting educational achievement, career choices and the prevalence of depression and anxiety in these individuals. Consequently, the development of new tools to facilitate STEM subjects is crucial in overcoming social barriers and promoting equal opportunities for all [21].

The efforts for a more inclusive education began in recent decades, with the invention of innovative tools such as the Braille Code (1950s), and the subsequent emergence of electronic devices. This paper goes in this direction and presents an innovative tool for students with visual impairments that facilitates learning of concepts which strongly rely on their graphical representation. A multimodal interaction is implemented to perceive mathematical functions drawn in a Cartesian plane through sonification. This uses sound as the main transmission medium of information, and the haptic vibration of tablets and smartphones to provide information by touch. The growing integration of electronic devices in educational contexts led us to choose a mobile application as a tool for our goal. Furthermore, the cross-platform framework Unity was selected for development purposes, to reach both the most popular operating systems: Android and iOS. In particular, the tool was designed for tablet devices, in response to the need to provide e-learning tools that are user-friendly for both students and teachers, whether they are in the classroom or at home, without requiring advanced expertise. Furthermore, this decision aims to prevent students from feeling isolated during the app use by creating a supportive environment where assistance is readily available.

This study is in line with the objectives of the UN 2030 Agenda for Sustainable Development, an ambitious plan to eradicate poverty and promote economic prosperity, social development, and environmental protection on a global scale [19]. In particular with SDGs:

- Goal 4: Promoting inclusive and quality education by providing accessible learning tools for people with visual impairments.
- Goal 10: Reducing inequality by enabling access to STEM education for individuals with visual impairments.
- Goal 11: Making cities inclusive and sustainable through digital education tools, reducing reliance on paper-based resources.
- Goal 17: Strengthening global partnerships for sustainable development by contributing to inclusive solutions.

The paper is organized as follows: Section II describes the background and the related works, the design and implementation of our application are discussed in Section III, Section IV presents the experiments conducted to evaluate the effectiveness of our system. We conclude in Section V, discussing the results and the future works.

II. RELATED WORKS

A. TRADITIONAL TOOLS FOR PEOPLE VISUALLY IMPAIRED

Many other works designed educational materials and tools to help people with visual impairments to study mathematics. Brawand and Johnson defined the use of tools such as the abacus, braille codes, and tactile materials as the most effective methodologies [22]. Cranmber abacus is a valuable tool for teaching primary operations of mathematics, such as addition, subtractions, setting and counting. It is typically introduced to students at the primary level and is still used in many schools until the second grade. However, it is less commonly used for more advanced mathematical skills, such as multiplication with digits and working with fractions.

The Nemeth Code is employed to convey advanced mathematical operations. The Nemeth Code represents the standard Braille notation system for mathematics and science. It is of particular importance for visually impaired students, as it provides access to complex mathematical and scientific content [22]. Developed by Dr. Abraham Nemeth, the Nemeth Code is regularly updated to ensure the accurate representation of intricate mathematical and scientific notations in Braille. It encompasses a wide array of mathematical symbols, operations, and functions, allowing for detailed and precise expression of these elements in Braille [23]. One significant drawback of the Nemeth Code is its considerable learning curve, which can present a significant challenge for both students and educators with regard to learning and comprehension.

Tactile graphics, which include tactile pictures, tactile diagrams, tactile maps, and tactile graphs, are images that employ raised surfaces to enable visually impaired individuals to perceive them [39]. These graphics are employed to convey non-textual information, such as maps, paintings, graphs, and diagrams. They are a valuable tool for teaching mathematics, despite some drawbacks since they require particular skills to both students and teachers and the creation of tactile graphics often involves manual efforts by experts, making the process lasting from one to several hours [41].

B. SOUND AND PEOPLE WITH VISUAL IMPAIRMENTS

Sound plays an essential role for people with visual impairments by providing auditory cues for navigation, environmental awareness, and information comprehension. Research in this area led to considering sound as one of the most suitable tools to enable individuals with visual disabilities to learn and discern elements that typically require extensive use of sight. In smartphone and tablet devices, speech is used as a guide to assist users in navigating interfaces: *TalkBack* is the screen reader provided by Google and *VoiceOver* is the one provided by Apple. They both provide spoken descriptions of the elements available on the current interface, gesture-based navigation, and assistance with keyboard commands [31].

Flowers and Hauer studied how to present the non-visual presentation of data by combining various dimensions of sound, such as pitch and loudness for the first time in 1995 [28]. In 2011, Neff and Pitt suggested providing spatial audio to allow for a more accurate mapping of non-linear content [29]. Lastly, in 2022 Wang et al. showed how the different auditory channels, such as pitch and volume, impact the data and visualization perception among people with visual impairments [25].

1) SONIFICATION AND THE STUDY OF GRAPH FUNCTIONS

Sonification is the use of non-speech sounds to convey information. More specifically, sonification is the transformation of data relations into relations perceived through an acoustic signal to facilitate communication or interpretation [33]. To convey information, data are represented using auditory means and sound parameters such as pitch, volume, timbre, or sound duration. The main drawback of sonification is that there is no consolidated framework that can prescribe to designers how to create an audio chart given data or convert a visualization into an audio chart [25]. Despite this, there have been many studies and experiments over the past decades that have all agreed on the effectiveness of using sound for visually impaired individuals to learn concepts more rapidly than with traditional instruments and with a reduced workload. While people with visual impairments are usually trained in the use of tactile devices and systems, sonification can provide more flexible and less costly solutions.

In 1985, Mansur et al. [32] conducted a research on using sound to teach mathematics to people with visual impairments. Their research showed greater effectiveness of sound methods on tactile ones due to their wider bandwidth. The experiments involved sonifying cartesian coordinates in order to develop a prototype system that generated sound graphs within 90 seconds. Test subjects accurately distinguished between different types of graph data and tended to associate higher pitches with higher spatial positions. Both visually impaired and sighted users were able to accurately recognize graph features such as slope, symmetry, and monotonicity.

Brewster [34] explored the use of non-speech sounds to improve the accessibility of graphs and tables for visually impaired users. They used MIDI synthesis for non-speech sounds and Microsoft Speech Synthesis for speech audio. In their experiments, y-axis values were mapped to MIDI note numbers (0-127), while the x-axis represented time. The experiments carried out showed positive effects, allowing participants for faster responding.

Auditory mappings for data perception, including pitch, frequency, tap, duration, volume, and timbre, were examined by Wang et al. [25]. They conducted experiments comparing these mappings and found that frequency was the most intuitive, while tap sounds were the most accurate. Different features were preferred for different types of data. Participants showed an inclination to reason visually when listening to auditory representations. In the experiments, specific frequencies and volume intensities were chosen, with taps lasting 0.5 seconds. Continuous tones represented data duration proportionally and timbre conveyed qualitative data. The results showed the intuitiveness of frequency across all data types, with taps being more accurate. The results of the study influenced our project implementation decisions.

Audiograph is an example of a desktop application that uses sound to teach mathematical functions to people with visual impairments [42]. The application has a screen reader-activated interface centered on a cartesian plane and a control options window to handle function entry and a wide range of customization options which include audio parameters, graphic settings, and analysis of mathematical points (i.e., its minimum, and maximum points). However, it may require users to read extensive documentation.

C. HAPTIC DEVICES

The rise of haptic devices encouraged the development of innovative tools that rely on the use of touch to convey information and overcome the limits of tactile graphs. In 1994, Massie et al. [35] designed the PHANTOM haptic interface, a force feedback device that allows users to interact with virtual objects. It consists of a robotic arm that tracks hand gestures in three dimensions and applies forces to mimic real-world touch. PHANTOM haptic interface is useful for people with visual impairments as it translates visual data into tactile understanding. Ramloll et al. [36] used the PHANTHOM haptic interface's ability to provide tactile feedback. Their research focused on the development of haptic and sonified line graphs to improve the education experience of people with visual impairment.

Gorlewicz and Webster [37] developed an Android application which leverages the haptic technology of tablets to assist visually impaired students in understanding algebra and geometry. Their idea consists on translating mathematical concepts into sensory feedback distinguishable through touch and sound. They demonstrated how tablet features could be used in an educational context with positive results.

D. BACKGROUND

Our research is influenced by [40]. The authors focused on the development of a Virtual Reality Application tailored for students with visual impairments and children with Attention Deficit/Hyperactivity Disorder (ADHD) with the ultimate goal of enhancing immersion and inclusivity in the learning of mathematics.

It was developed for Meta Quest 2 devices to provide an immersive and rich VR experience. To discern functions, the application considers a cartesian plane and it sonifies functions by assigning a unique frequency to each y-value on the graph. In terms of the tactile aspect, researchers leveraged the hand-tracking capabilities of the Meta Quest

2 and employed the two controllers to enable users to virtually interact with elements on the cartesian plane. This interaction provides tactile feedback, which allows users to perceive the elements in question. The experiments conducted showed some key limitations. 28 students were asked to recognize mathematical functions through sound, yielding an average score of 2.57 correct answers out of 5. This result fell below expectations, maybe due to potential gaps in students' prior knowledge. The experiments also showed the occurrence of *cybersickness*, and the authors highlighted the necessity for trained people to guide students in using the tool. This project was essential for our research and inspired us to incorporate a multimodal approach in our application. Challenges, like user dizziness and prior mathematical knowledge, led us to consider a tablet app more suitable for education context. Additionally, we decided to focus on initially covering two simple function types: lines and parabolas, thus facilitating a correct use of sounds and haptic feedback.

III. DESIGN AND REQUIREMENTS OF THE APPLICATION

The development of our application started with a definition a list of goals and functionalities:

- the use of Android tablets or iPads;
- the use of a cross-platform framework to develop the application;
- the correct representations of mathematical functions;
- a deep study about the use of haptic feedback and sound to make points recognizable in a cartesian plane;
- the application must be fully accessible.

In terms of user experience, it was crucial to develop an application that is intuitive, easily accessible, and appealing to users. In education, clarity is important for teachers and students to feel comfortable using the app. V-Bny difficulty, whether due to poorly developed features or a bad user interface, can discourage usage. The user interface has been designed to be suitable for use in educational contexts and to enrich conventional educational tools such as Geogebra. Rich in color and featuring cartoon elements, the aesthetics enhance the user experience, making the app enjoyable to use.

Regarding accessibility, the application fully support Talkback and VoiceOver. The app also allows users, in particular blind users, to explore the cartesian plane. Haptic and sound cues are provided to orientate users, telling them the values along the x and y-axis. Users can also interact with the functions drafted on the cartesian plane by touching them. The touch interaction is characterized by a haptic vibration and by the sound of the function on that specific point.

As mentioned before, the application is designed for Android and iOS devices, with particular attention to tablets, since they provide more space for interaction with the whole hand. Moreover, from our previous experience (see Section II-D), we identified the tablet as the optimal device for use in an educational context. The use of tablets in the classroom allows for the easy assistance of students and facilitates interaction with teachers and peers. Additionally, tablets do not cause the adverse effects commonly associated with virtual reality devices, such as cybersickness. Furthermore, tablets are significantly more cost-effective than VR devices. Unfortunately, the multimodal approach implemented can be fully enjoyable only with Android tablets since iPads do not provide haptic feedback.

We chose Unity as development platform for this project for several reasons. Firstly, it is cross-platform, which allows developers to create games and applications that can be deployed across different devices and operating systems. Secondly, it enables developers to save time and resources that would otherwise be spent developing separate versions of the application for each platform. To fully support accessibility features, such as Talkback and VoiceOver, the Asset UI Accessibility Plugin (UAP) was used.

The application was designed to enable the representation of lines and parabolas. This choice was made to assess the effectiveness of the multimodal approach on a small sample before including more complex functions. The application is available in Italian and English language, and it was tested with Italian participants from the University of Padua.

A. USER INTERFACE

GraficiAccessibili offers two interaction modalities: *Disegna*, i.e., draw, and *Studia*, i.e., study. The *Studia* modality (see Figure 1) allows to depict only one type of function, currently limited to lines and parabolas, to study it in detail. It displays the curve on the cartesian plane and "plays" the sounds associated with the points of the curve. Moreover, the user can select the "Trova Equazione" mode, I. e., find the function, that encourages tactile interaction with the selected function, by exploring the cartesian plane with the finger. The user must find five points of the curve, helped by sound and haptic cues.

Disegna allows users to define and graphically represent up to three functions simultaneously, chosen between lines and parabolas. This modality is thought for users with prior knowledge about different functions that they represent in the cartesian plane. For each function, it is possible to listen again sounds associated with it or remove the function from the cartesian plane.

Functions are added by users through specific text areas. The *Studia* mode offers a text input area for adding lines in either their explicit or implicit form. The same applies to parabolas, for which the app provides a distinct input text area based on its axis of symmetry. In *Disegna* mode, the same input text fields are offered, but they are presented by dialog boxes. This design decision is in line with our goal of enabling users to autonomously represent various types of functions on the cartesian plane unregarding their abilities. It underscores the wide range of options available to users while simultaneously highlighting the unique characteristics of each selected function. The use of a dedicated text area, accompanied by a label describing the function to be added, helps mitigate errors made by many users due to a potential lack of prior knowledge about the function to draw. Additionally, this choice enhances the overall accessibility of the application by explicitly specifying the elements of the selected function. This setup is particularly well-suited for being used with a screen reader, as it provides clear and descriptive information about coefficients and parameters.

The app calculates the points of the chosen functions and plots the curve of the function on the cartesian plane and plays its specific sound. Functions currently represented on the cartesian plane, with a small segment of the same color to make a visual connection, and features available for users to interact with, are listed below the input area. Users can press the play button to listen again the functions, and the trash button to delete it. Inside *Studia* mode, users can also press the magnifier lens button to start the *Trova Equazione* mode.

The points of the depicted function are instantiated as GameObjects. In particular, we've utilized Unity's feature *Prefab* to handle multiple objects of the same type. Each point is an instance of the *CapsuleSegment* prefab. It consists of a cylinder with a half-sphere at each end, serving as the fundamental building block for creating curves. The prefab includes an Audio Source component that assigns a sound cue to a point. Additionally, it has an invisible button that acts as a listener to trigger playback and haptic feedback events.

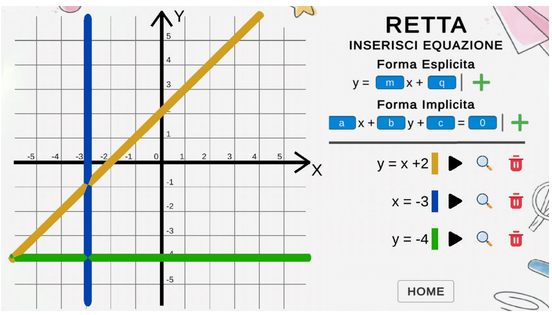

FIGURE 1. **Snapshot of "Study Line."**

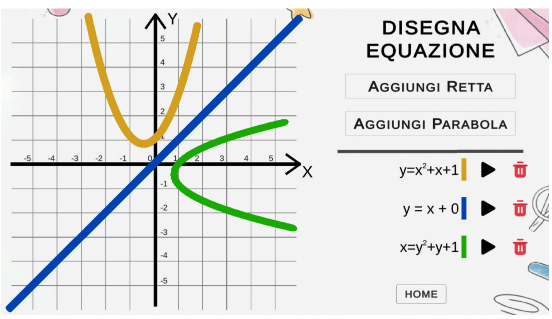

FIGURE 2. **Snapshot of "Draw Mode."**

B. SOUND AND SONIFICATION PROCESS

We have designed a specific way of sonifying functions to make them recognizable. In mathematical terms, a function represents a mathematical expression that describes the relationship between the coordinates of points within a coordinate system. Each point is defined by an ordered pair, wherein the primary and secondary coordinates are respectively termed the abscissa (X) and the ordinate (Y). In order to achieve our goal, coordinates must necessarily be distinguishable solely on the basis of sound. To recognize the abscissa of each point, we used the beats of a metronome. The beats are categorized into two types: the first, a stronger beat, represents the points $[-5, 0, +5]$ on the x-axis, while the second, a lighter beat, is utilized for the remaining points $[-4, -3, -2, -1, +1, +2, +3, +4]$. To recognize the ordinate of a point, we employ musical notes instead. The choice of musical notes required various experiments, consisting of testing different timbres, sounds that changed according to a pitch manipulation based on values along the Y-axis, and the employment of several sounds simultaneously for each point of a function. Through these experiments, we have ascertained that some choices to are the best for representing certain information, while other choices have turned out to be subjective. Based on the result of previous research and our experiments, we set the sounds of points with positive abscissa higher and the sounds of points with negative abscissa lower. About the choice of sound to use, we ascertained that there is no universal framework to easily sonify functions, so each research work we have consulted showed a different setting of sounds. In our case, we used the sounds of the *Major Pentatonic* scale. Our major pentatonic scale consists of the following notes C, D, E, G, A generated by a virtual synthesizer. The notes were mapped following these criteria:

- The musical note A is used in the first quadrant, it has as frequency 220 Hz (La3);
- The musical note G is used in the second quadrant, it has as frequency 196 Hz (Sol3);
- The musical note E is used in the third quadrant, it has as frequency 164,81 Hz (Mi3);
- The musical note D is used in the fourth quadrant, it has as frequency 146,83 Hz (Re3);
- The musical note C is used to sonify functions that stands for lines parallel to the x axes. It has as frequency 130,81 Hz (Do3). Since this function is constant, we have used this note as a particular case.

Moreover, each note was scaled to the next semi-note of the pentatonic scale when the points were in the upper half of the quadrant. This method of scaling allows for the distinction of a total of eight sounds along the Y axis.

We use the sound of a bell to indicate the intersection of a function with the x or y-axis. We choose the pentatonic scale for specific reasons. Firstly, it is a scale that is very popular and its sound results to be familiar to everyone. Secondly, the pentatonic scale harmonizes well with various chords and musical contexts. This versatility makes it suitable for layering over different harmonies, providing a smoother and more pleasant musical experience which is also adapt for improvisation. Additionally, the pentatonic scale reduces the likelihood of playing incorrect notes during improvisation by avoiding notes that could introduce significant dissonance, such as the 4th and 7th degrees in the major scale. The sonification of functions such as parabolas which can lie on all four quadrants of the cartesian plane, implies the adoption of sounds that are flexible and easily recognizable by everyone.

The sonification process for functions operates as follows: the app traverses values along the x-axis, producing corresponding beats for each point. When a point of a function aligns with a specific x-value, the app reproduces the sound corresponding to the point along the y-axis. Consequently, the number of beats indicates the position of the point along the x-axis, while the sound conveys its placement along the y-axis. Given the use of two distinct sounds, users might be required to listen to the function multiple times to count the beats and to recognize the quarters where the function lies according to the listened note. This approach generates situations where users can hear only the beats of the metronome for a while, indicating that the chosen function is not placed in that area of the cartesian plane.

Function recognition relies significantly on the variations in the tones of the sounds. If we consider lines, the transition occurs in a singular direction, either from low to high or vice versa, depending on whether the line is increasing or decreasing. In the case of parabolas, the shift in musical notes occurs twice. For positive parabolas with the axis of symmetry parallel to the y-axis, there is an initial shift from high to low leading up to the vertex, followed by a subsequent shift in the opposite direction. In the case of a parabola with the axis of symmetry parallel to the x-axis, users might hear two different musical notes sequentially. This happens because such functions often exhibit two branches situated in different quadrants, resulting in distinct sounds for points with the same x value but located in different quarters. In this case, the closer are the two branches of the parabolas, the faster the reproduction of the sounds of the points is. The other remarkable case is the line parallel to the y-axis. In this case, all the points of this line have the same abscissa, so users listen in one second all at once the sound of the function. The last important case is the line parallel to the x-axis. This line is constant, so we have associated only one musical note with all its points, although they are on different quarters.

C. TOUCH

Haptic Feedback plays a crucial role in exploring the cartesian plane by touch. Our idea consists of integrating haptic feedback on the points that are the most relevant to perceive. Firstly, we implemented haptic feedback along x and y-axis. Haptic feedback comes together with a speech file which explicitly says also the coordinate of that point. In this way, users can perceive points by a multimodal approach. The same procedure holds for the points of function. The vibration is implemented at intervals of every 4 segments depicted for the function as a compromise to avoid high battery consumption and distracting the user from listening to the sound. We consider this choice as the best compromise for users to perceive vibration of both lines and parabolas at a reasonable frequency. Figure 3 shows the segments of the line $y = x$ which offer haptic feedback.

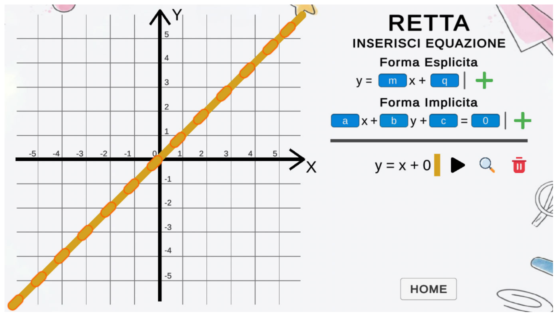

FIGURE 3. Buttons featuring haptic vibration.

D. TROVA EQUAZIONE

The *Trova Equazione* feature was specifically designed for individuals with visual impairments to facilitate their study of functions. These users require time to familiarize themselves with the application and comprehend the spatial proportions of the cartesian plane on tablet devices. This feature was developed with their needs in mind, aiming to assist them in the learning process. The method consists of highlighting the points of interest for a selected function and encouraging users to explore the cartesian plane through touch to find these points. During this process, all functions already depicted, except the chosen one, are hidden. The points of interest are six, equidistant from each other. They are red colored to distinguish them from the rest of the curve and to provide high contrast against the background. This color choice also takes into account the needs of users with color blindness. We developed this feature to lead users to explore the entire cartesian plane and to study the function with the finger, starting from the points at the extremes of the function. Users can exit this modality only touching all the six points of interest. The goal is to use sounds and haptic feedback to follow the direction of the line or curve.

Haptic feedback plays a crucial role because it helps to indicate the right direction to follow. When a point of interest is touched, it triggers the reproduction of a speech file which explicitly says how many points were currently found. This information is also displayed on the right side of the interface with a label. This interface section also displays the formula of the selected function and a *Play* button that allows users to replay the associated sound.

The goal of *Trova Equazione* is to assist people with visual impairments in learning spatial information about the placement of elements on the interface, such as the Cartesian Plane and the functions plotted on it. Figure 4 shows the current feature related to the study of an increasing line.

IV. EXPERIMENTS AND RESULTS

We conducted different experiments to assess the effectiveness of the multimodal approach implemented in our

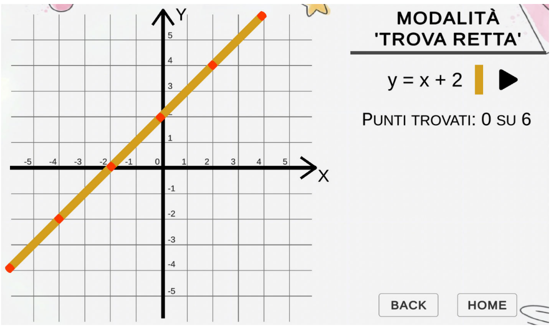

FIGURE 4. Snapshot of "Trova Equazione."

project. The testing process was conducted in two stages. Initially, a group of 31 sighted users tested the application, providing us with their insights about the usability of our system and testing the sounds implemented to distinguish functions. Then, the system was tested by people with visual impairments, providing valuable insight into its effectiveness with the target user group.

The primary objective of these tests was to verify the accuracy of the application in sonifying the functions, ensuring that the generated sounds accurately represent the corresponding mathematical functions. Feedback from these tests has been useful in understanding the validity of features implemented in the application and whether we are providing a valuable tool to people with visual impairments. In addition, we have submitted a System Usability Scale (SUS) questionnaire to assess the perceived usability of our app. The SUS is a widely recognized tool for measuring usability and consists of a 10-item questionnaire with responses ranging from Strongly agree to Strongly disagree [45].

We created a test version of the app for our experiment. It differs from the final version in a few ways. We remove the study mode to focus on testing the effectiveness of sound representations. We added a Setup-Test scene to collect user information before the test. We also added a new scene for each test question.

A. EXPERIMENT WITH SIGHTED USERS

In the first test phase, a group of 31 students was asked to identify a function based only on auditory cues and to choose the answer among four distinct options shown through images. We conducted this test with individuals who do not have any disabilities to measure the application's accuracy. The experiments were conducted following the protocol of the standard **HCI**. Firstly, the tests were conducted by interview to gain insights into the user's experience. Interviews can give an in-depth understanding of the users' values, perceptions, and experiences.

The experiments start by explaining to each user the reasons for the development of the application. Then, we described the process of sonification of functions and showed some videos to explain in detail how the beats of a metronome were implemented and how lines and parabolas sound. Later, each user had 5 minutes to use the *Disegna* mode to become familiar with the app. During this time, they were asked to try functions of various types to understand how they were sonified. During this phase, users were free to ask us any question about the application. Finally, users underwent the test, which was anonymous and without any time constraints.

1) INFORMATION ABOUT THE USERS

All participants were students of the Bachelor's degree in Psychology and volunteered for the study, ensuring no workers were involved. The first experiment engaged 31 students aged between 18 and 28 years, in particular 15 are males and 16 are females. The equal gender distribution helps to ensure that the results of this experiment are not biased towards a particular gender. Most participants were under 22 years old and in the first year of their bachelor's degree. The average age of participants is 21.09, so very similar to the age at which students typically complete high school. We targeted this group because they had recent exposure to math concepts like the graph of linear or quadratic functions of one real variable. These concepts will hence forth be referred to as line and parabola. We gathered data on the high school attended by participants to determine whether there is a correlation between their mathematical background and their ability to discern functions using sound. The majority of participants attended the *Liceo Scientifico*, which is a type of high school in Italy that dedicates the highest number of hours per week to the study of Mathematics.

2) DATA ANALYSIS

Each user was assigned a code that was not related to their actual identity to keep the test anonymous. The experiment consisted of asking users to listen to a sound and recognize which type of function corresponds to it among the four options presented.

During this phase of the tests, we stopped talking to users and let them answer questions alone. The system informed the user about the outcome of their answer, which could be correct, partially correct, or incorrect. An answer was considered partially correct if the user understood the type of the function but made an error in recognizing its concavity or if it is an increasing or decreasing line. Each correct answer scored 1 point, partially correct 0.5, and incorrect 0.0. Users could play more times the sound of the function to improve their understanding. The number of playbacks was considered indicative of the number of attempts that a participant needed before responding. The answer of each user was saved as a *JSON* fil with the following structure:

- playerName: the code associated to the user;
- gender;
- age;
- scores: a list containing the score for each question;
- totalPlayerScore: the sum of the collected scores;

- times: a list containing the time spent by the user to answer to each question;
- totalTime: total times spent by the user to answer to each question;
- attempts: a list containing the number of playbacks for each question;
- answerGiven: a list containing the answer to each question;
- correctAnswer: a list containing the correct answer for each question.

We decided to have a total of 10 questions to keep the duration of the test reasonable. Each question corresponds to a different type of sound that can be encountered in our application. The chosen functions are described in Appendix V-B. The user can listen to the sound of a question multiple times without any limit. The answer can only be provided once and cannot be modified. Once the user has provided an answer to a question, they receive feedback on the accuracy of their response and step to the following question. As the test progresses, the difficulty of the questions increases. The initial six questions pertained to the various types of lines and parabolas, which could be identified. The subsequent questions were specific instances of these functions, which could be answered correctly only if the process of sonification had been correctly understood. Figure 5 shows the scene used for the experiments. The scene provides no information about the cumulative points collected by users during the test. All data is collected anonymously to avoid placing undue pressure on participants. The showing of current scores or the time spent for the tests can cause users to feel frustrated and can result in inaccurate responses to the question. In step with this premise, the scene contains only essential information, such as the number of remaining questions, and the play button to listen again to the sound cue of the function to guess.

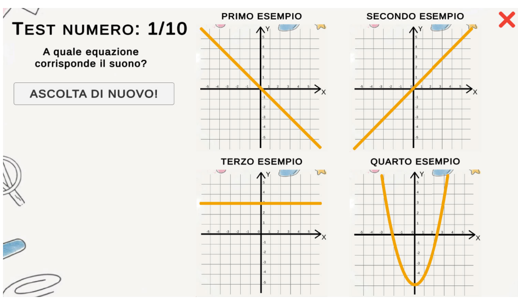

FIGURE 5. First question of the experiment.

Lastly, we asked users to respond to a System Usability Scale (SUS) questionnaire. The total time spent for each interview is approximately 20 minutes per user.

3) QUALITATIVE AND QUANTITATIVE ANALYSIS

The tests performed showed important results. At least half of the participants managed to answer all the questions correctly. This percentage increases to two-thirds of the users when we take into account responses that were partially correct. Figure 6 shows the number of correct answers for each question. The chart shows that users properly distinguished the majority of test functions. For the evaluation of the correctness of our tests, we have used the average correctness computed for each question as the scores of correct questions divided by the total number of questions. The average correctness of the test is 70%, with a total of 218 correct answers out of 310. When considering partially correct answers, the overall correctness increases to 75%, with a total of 253 correct and partial answers out of 310. The cumulative score, which accounts for both fully and partially correct responses, is 235.5 on the maximum score of 310.

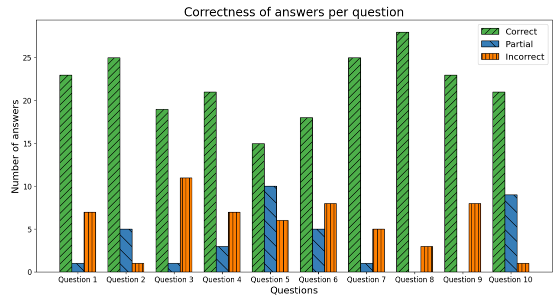

FIGURE 6. Correctness of answers per question.

As expected, the most recognizable functions were the lines, mainly because of the change in sound from low to high or vice versa. Questions about lines (questions 1, 2, 8, and 9) reported an average correctness of 79%, which increased to 82% when partially correct answers were included. Of the 124 answers, 99 were completely correct and 105 partially correct. Questions concerning parabolas (questions 3, 4, 7, 10) with a symmetry axis parallel to the y axis obtained an average correctness of 69% when considering only fully corrected answers, with 86 fully corrected answers out of 124, and 75% when including partially corrected ones, with 93 points out of 124. Similar results were obtained for questions related to parabolas with the axis of symmetry parallel to the x-axis. The users provided 43 fully corrected answers over 62. When considering partial correctness, this value increases to 49, with a total score of 46 points. Based on these data, the average correctness for this type of question is 69%, which rises to 74% when including partially correct answers.

The analysis of Figure 6 reveals the tendency of users to learn from their mistakes during the execution of tests. The first question pertains to an increasing line and yields an average correctness of 74% for fully correct answers, with 23 participants correctly answering out of 31. The second question pertains to a decreasing line and saw 25 participants correctly answer out of 31, with the average correctness increasing to 80% for fully correct answers. The same pattern holds for both types of parabolas. Of the 31 participants in the third question, 19 answered correctly, with an average of 61%. This increased to 67% in the fourth question, which pertains to the same type of function but with a negative sign. The seventh question still refers to a parabola with branches pointing up, and the average score recorded in this case is even higher, with a value of 80% for fully corrected answers and 83% when including partially corrected ones. 25 participants correctly answered the question, and one partially answered.

The sound of a parabola with a parallel x-axis is introduced in the fifth question. For this type of function, a single x-value corresponds to two y-values, excluding the vertex. It is important to remember that parabolas with a symmetry axis parallel to the x-axis are not functions, but algebraic curves. This creates difficulties in sonification (since each value of the independent variable must correspond to two sounds, one sound at the vertex, or no sound). Furthermore, it creates problems for understanding, as undergraduate students do not study these types of curves in detail. More in detail, this means that users might hear sounds from two different quadrants. The question had the lowest average correctness, with 48%. Only 15 out of 31 participants correctly answered and 10 out of 31 participants gave a partially correct answer. The sixth question refers to this type of function but with opposite concavity and got a higher average correctness, which was 58%. In this case, 18 of the 31 students correctly answered. Interestingly, only 5 participants partially answered, resulting in an average correctness of 74%. This data is surprising because users had less time to memorize this sound.

The highest average of correctness is recorded in question eight, with a percentage of 90%, and a total of 28 fully corrected answers out of 31. Interestingly, no one provided a partially corrected answer for this question. This question corresponds to a line parallel to the x-axis, demonstrating that the use of a constant sound was effective.

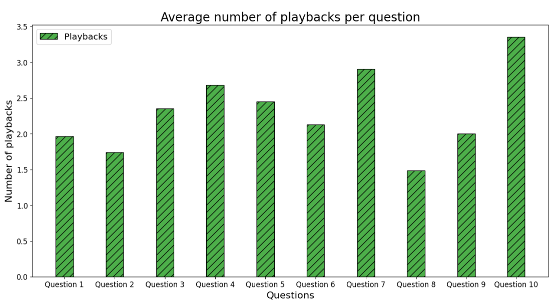

FIGURE 7. Average number of playbacks per question.

The last question was tricky and required users to accurately distinguish sounds and beats to answer correctly (see Figure 8). The result is that 21 of the 31 students correctly answered the question, with an average correctness of 67%. Despite this, a high number of partially corrected answers were given, 9 out of 31. This shows that the process of sonification was correctly understood and that the students were able to discriminate between the types of function. The error committed is about the recognition of the position of the parabola along the x-axis. Taking into account partially corrected answers, the average correctness increases to 96%.

The correctness of answers is influenced by various factors. For instance, recognizing the abscissa and ordinate of a point through beats and musical notes requires multiple listens to sounds associated with functions. Taking into account this information, we consider the number of playbacks to be the attempt of the users to answer a question. Figure 7 shows the average number of playbacks for each question.

This plot shows each question of the test on the x-axis and along the y-axis the number of playbacks. From this plot, it is possible to highlight some evidence: the lowest average number of registered attempts is 1.5. This information suggests that most participants needed to listen to the sounds associated with each question multiple times. As we expected, question 8, which pertains to a line parallel to the x-axis, recorded the lowest number of playbacks, due to the absence of any change in the reproduced sound.

The tenth question demanded the highest number of playbacks. It required identifying the sound produced by a parabola that intersects the x-axis at two points and accurately determining its location on the x-axis. The question provides as possible answers two options of the same type of parabola but translated on the x-axis by 4 units. Therefore, the elevated number of playbacks aligns with our expectations. The tenth question is shown in Figure 8.

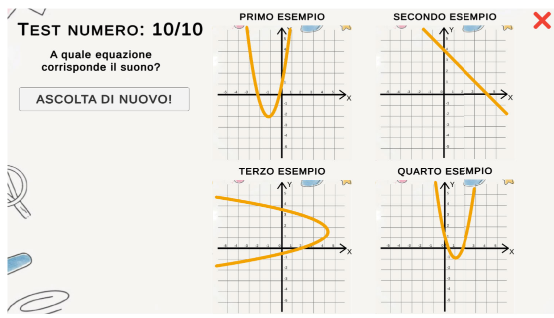

FIGURE 8. The tenth question the test.

The time spent by the users for each question was analyzed as an additional element to understand the effectiveness of the sonification process. As expressed in the previous section, the auditory channel is known to convey concepts more immediately compared to tactile tools. In this experiment, the sound listening of each question was fixed to 10 second as a design choice. Therefore, taking into account this information, Figure 9 shows the average time spent on each question.

Consistent with the data on the number of playbacks per question, the chart shows that users spent at least 20 seconds on almost every question. This information reaffirms the necessity to listen to the sound cues at least twice for the users. The question that demanded the most time (question 7) does not align with the question that had the highest number of playbacks of the previous plot, but they both

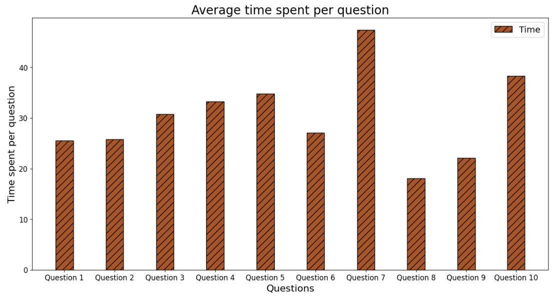

FIGURE 9. **Average time spent per question.**

depict 3 parabolas and a line. The seventh and tenth question reproduce the sounds of a convex parabola with its axis of symmetry parallel to the y-axis and pose a question to discriminate it with a similar case but translated in the direction of the x-axis. Therefore, the difficulty relies on properly recognizing not only the trend of the functions but also the moments in which sound cues are played. In both cases, the correct answer could only be provided if the concept of beats had been correctly assimilated. Consistent with the lower number of playback required, question eight also turned out to be the one where users spent the least time in providing an answer. This situation is also coherent, with question 8 being the most recognized sound. Finally, on average, parabolas required more time than lines, likely due to the presence of a more complex sound structure.

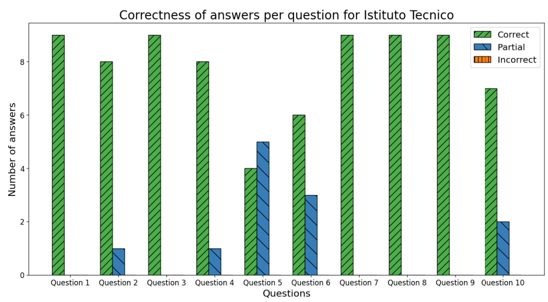

FIGURE 10. **Correctness of answer per question for Istituto Tecnico.**

The correctness of the answers can also depend on the background of users. In our experiments, we involved participants who attend different types of high school, to assess the process of sonification according to any type of background which students could have. 15 students out of 31 attended *Liceo Scientifico*, which dedicates more hours to the study of STEM subjects. For this reason, we expected that these participants were the ones with the best responses to the test. The total number of correct questions provided by the students of *Liceo Scientifico* is 105 out of 150. Based on these data, the average correctness of this participant category is 70%. This value is in line with the general average correctness computed for the chart in Figure 6. Taking into account the partially corrected answers, we obtain a result of 121 correct answers from 150. In this case, the average correctness increases to 75%, consistent with the value obtained for the general case.

The second macro-category of participants includes students who attended *Istituto Tecnico*. This category includes 9 out of 31 participants in the tests.

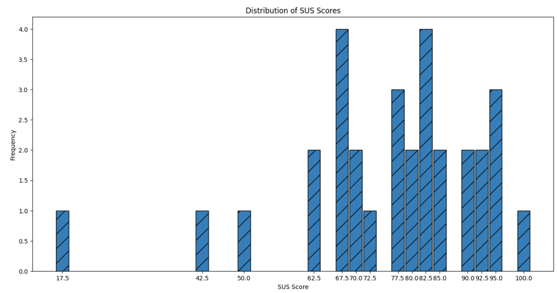

FIGURE 11. **SUS score distribution.**

Figure 10 shows that participants who attended the *Istituto Tecnico* achieved the best results in the this phase of the experiments. Out of 90 total answers, only 12 were partially correct, 0 were incorrect, and 78 were correct. In particular, no student from the technical institute gave an incorrect or partially correct answer to questions 1, 3, 7, 8, and 9. As a result, the average correctness of the answers per question is 86%, considering only the correct answers. Since there were no wrong answers, the average correctness rate is 100%, when including partially correct answers. The performance of the students of *Istituto Tecnico* aligns with the data provided by the Italian Ministry of Education on the study of mathematics in high school. Every year, the Italian Ministry of Education provides national tests, known as *Invalsi*, to evaluate student skills levels in Italian, Mathematics, and English. In 2022, students from technical high schools achieved the highest scores in Mathematics [44].

Lastly, there were 7 participants who come from different type of high schools. The number of participants of this category was considered to be too low to be statistical relevant.

4) SYSTEM USABILITY SCALE

The System Usability Scale (SUS) is a reliable and straightforward tool to measure the usability of a system or application. Created by John Brooke in 1986, it has become an industry standard for evaluating various products and services, from hardware to websites. The SUS consists of a 10-item questionnaire with five response options, ranging from *Strongly agree* to *Strongly disagree*. The SUS is easy to implement, provides reliable results with small samples, and effectively distinguishes between usable and unusable systems. However, the interpretation of scores can be complex. The SUS score ranges from 0 to 100, and is computed by adjusting the responses to a 10-item questionnaire and multiplying the total by 2.5. A score above 68 is considered above average [45]. Scores can be converted into percentile ranks to facilitate comparison with a database [46].

The SUS questionnaire for this project contains the following questions:

1) I think I would like to use this application frequently;
2) I found the application complex without needing it;
3) I found the application very easy to use;
4) I think I would need the support of someone who is already able to use the application;
5) I found the various features of the application well integrated;
6) I found inconsistencies among the various features of the application;
7) I think most people could learn to use the application easily;
8) I found the application very complex to use;
9) I became very familiar with the application during its use;
10) I needed to learn many processes before being able to use the application effectively.

The final SUS score obtained for our application is 76.12, i. e., it surpasses the average score, signifying that our application exhibits better usability compared to many other systems.

Figure 11 shows the distribution of the SUS score obtained by our participants. The chart shows that only 5 people out of 31 gave answers that generate a SUS score below the average, 12 out of 31 a score in the range $67 - 81$, and 14 out of 31 a score that is above 82, which is excellent according to SUS standard. This distribution underscores the high usability of our application. On average, only a small portion of the sample found the application not usable. On the other hand, the majority of participants rated it as excellent. Therefore this data confirms the high usability of our system and validates the design decisions made during the development phase. This is a strong indication that our application is user-friendly and effectively meets the needs of our users.

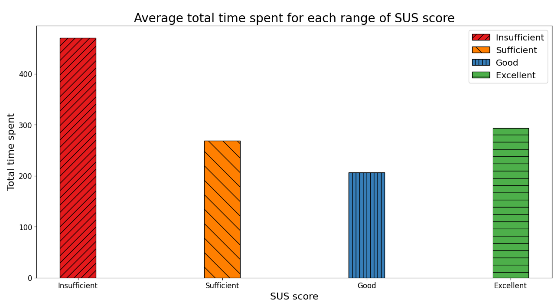

FIGURE 12. Average time spent for each range of SUS score.

The distribution of the SUS score also shows some outliers. An outlier is a data point that differs significantly from other observations and can be due to variability in the measurement, or it may be the result of an experimental error. The data more in contrast to other observations is the SUS score of 17.5, obtained from a single participant. Excluding it from the computation of the final SUS score, we get a value of 78.03.

In addition, we examined the correlation between the total time users spent on the test and the SUS score they generated. We expected that people with lower SUS scores would require more time to complete the test, in relation to the low usability of the application according to their needs. Therefore, we had categorized the scores in four different categories as follow:

1) Insufficient: SUS score below 67
2) Sufficient: SUS score between 68 and 74
3) Good: SUS score between 75 and 81
4) Excellent: SUS score above 82

and analyze the average time spent for each. Figure 12 shows the chart of this analysis.

As expected, the average of the total time spent by users is the highest for those who generated a lower SUS score. In this case, the average total time spent by these users is approximately 450 seconds, which is the time taken to complete the test. These data align with their assessment of the low usability of the application. However, it is important to note that there are only 5 users in the insufficient range. A further examination of the data reveals that the remaining 26 participants spent between 200 and 300 seconds on average to complete the test. It is notable that the 'excellent' category represents the second-highest average time spent. This suggests that, in addition to the usability of the product, the time taken may also reflect the effort participants put into answering the questions correctly.

5) USERS FEEDBACK

The feedback provided by users during the tests was of fundamental importance. Many participants reported that, despite not fully recalling all mathematical concepts, they successfully answered at least 6 out of the test questions. Several users requested longer demo sessions to have more time to associate better each note with its corresponding quadrant. By the end of the tests, many students expressed confidence that regular use of the application would enable them to recognize all types of equations and to avoid replicating the errors made during the tests. A notable observation was the different strategies adopted by each user to distinguish equations. Common approaches included focusing on beats and the bell sound marking intersections with the x or y-axis, interpreting the ascending or descending trend of the sound, and using beats to pinpoint the equation's position along the x-axis.

B. EXPERIMENTS WITH PEOPLE WITH VISUAL IMPAIRMENTS

After the excellent results obtained during the first testing phase, we decided to involve people with visual impairments. We tested two users with total blindness, one user with partial blindness, and two users with low vision. The participants were recruited through personal contacts to start the snowball sampling procedure. They were volunteers who showed their interest in joining the tests of our application. 3 out of 5 were students, one was a worker, and the last participant was a musician and expert in accessibility issues. The tests in question were conducted by interview, with the difference that people with visual impairments had more time to familiarize themselves with the application than sighted users. In particular, each visual disability can require specific needs for which applying the same protocol did not seem to be the right strategy in this case.

1) USERS WITH BLIND USERS

The test conducted with users with total blindness produced excellent results.

The first experiment involved a user who is totally blind since birth. He is more than 40 years old, and a musician with a perfect pitch. Initially, the application was introduced to the participant with an explanation of its development and the features it offers. The test focused on drawing lines within the *Studia* mode. It lasted five minutes due to the participant's availability.

Firstly, we showed an example of increasing and decreasing lines in order to let the participant familiarize with the sonification process. The participant showed an understanding of the correlation between the shift in sound and the trend of lines and correctly identified each note employed, given his background as a musician.

Then, the participant tested the *Trova Equation* mode to look for the bisector of the first and third quadrants. The participant required an initial phase where he explored the interface in order to gather spatial information about the distribution of elements. The implementation of speech files indicating the coordinates along the axes of the plane turned out to be crucial for him to recognize where he was. Then, the user relied on haptic feedback to properly locate functions. The participant interpreted the vibration feedback as key points that indicate the direction to follow. In a short time, he properly located the function on the cartesian plane.

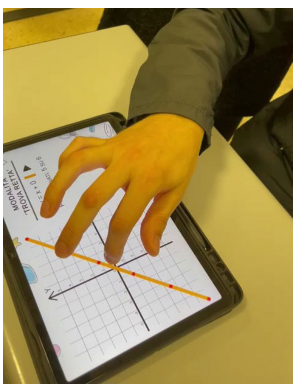

FIGURE 13. First test conducted with a participant experiencing total blindness.

Finally, the subject encouraged us to continue the development of the application, trying to integrate more different types of functions and showing enthusiasm for the adopted multimodal approach. The participant also appreciated the support of Talkback, noting its importance for accessibility. Figure 13 shows a moment from the conducted experiment.

The second experiment involved a 30 years old woman, a master's student in Italian Literature who showed a propensity for Mathematics during high school. The test lasted about 50 minutes. We followed the same protocol adopted for sighted users; therefore, the experiment starts by explaining to the participants the reasons for this research and the features of the application. Then, we explain our sonification process, with a focus on how to properly distinguish the values along the x-axis, the notes involved for the y-axis and examples of lines and parabolas. The participant never studied Mathematics by auditory channels and the first impression recorded was of strangeness. For this reason, it was necessary to play the sounds several times, so that the sonification process could be understood and assimilated.

FIGURE 14. Person with total blindness that tests our application.

The participant correctly understood where a function starts and ends by counting the beat of the metronome to distinguish the values along the x-axis. The sounds adopted for the values along the y-axis did not guarantee the same precision but gave only an idea where the points could be. Although this was the case, the participant considered the change in tone to be an effective method to transmit information about the function trends. Regarding the tactile aspect, the participant initially studies the distribution of elements on the interface. In this case, speech files were also considered crucial for recognizing exactly the position of her finger along the cartesian plane. In this way, the user was able to properly find functions and move along the x-axis and the y-axis to identify coordinates. Nevertheless, she indicates a desire for more explicit instructions during the exploration phase, particularly in instances where the finger is positioned at a distance from the x-axis and the y-axis and the desired functions. She proposes the use of speech files, such as "right" or "left", to enhance user guidance and facilitate greater autonomy. Figure 14 shows the participant while exploring the cartesian plane.

The final phase of the experiment replicates the same test conducted by sighted users, with the key distinction being that the user was also asked to describe the function without providing the typical four possible answers.

TABLE 1. Results of test with a totally blind person.

Age	30	Gender	Female
Question number	**Score**	**Time**	**# playbacks**
1	1	25.9	1
2	1	18.5	1
3	1	52.1	1
4	1	44.0	3
5	1	21.6	1
6	1	22.9	1
7	1	16.1	1
8	1	19.7	1
9	1	27.6	2
10	1	43.5	2
Total	10	292.4	14

Table 1 shows that the participant demonstrated proficiency in distinguishing all functions and needed on average less time and fewer replays than sighted people. During the test, the participant used her fingers to show us the trend of the function she was listening to, making estimations about the starting point and the quadrants in which the function was located. In the case of parabolas, the participant also attempted to estimate the position of the vertices. All this information was corrected. After the conclusion of the test, the participant expressed enthusiasm for the application. In her opinion, tools such as tactile graphs are still more precise and suitable for the study of functions. However, the participant recognized the potential of the application in providing autonomy for people with visual impairments since tactile graphs must be prepared by someone else. In particular, the compatibility of the application with TalkBack or Voice Over and its effective multimodal approach made the participant feel more independent compared to traditional tools.

The third experiment was conducted with a partially blind young woman. The participant, a 23-year-old worker, does not have a perfect pitch. The participant was no longer familiar with mathematical concepts such as lines and parabolas and studied Mathematics only using tactile tools. Unlike the previous participant, the sonification process required more time to be assimilated, and her precision in distinguishing points along the x-axis was lower, possibly due to varying mathematical predispositions. The implemented sounds were considered pleasant, and the shift in tone was indicated as effective in indicating coordinate trends along the y-axis.

The exploration with haptic feedback was comparable to that of the previous participants. Similarly, the subject required an initial phase to gather spatial information about the distribution of elements on the interface. The speech files were still considered useful and valuable for indicating the current position of users. However, the subject also requested the inclusion of vibrating divider or auditory cues to aid in spatial orientation. Furthermore, it was recommended that directional speech indication or the position of the number on a clock must be implemented for navigation assistance. Table 2 shows the results of her test.

TABLE 2. Results of test with a partially blind person.

Age	23	Gender	Female
Question number	**Scores**	**Times**	**# playbacks**
1	1	33.3	1
2	1	18.2	1
3	1	48.0	2
4	1	13.8	3
5	1	156.2	1
6	1	15.8	4
7	1	150.9	1
8	1	31.9	1
9	1	153.1	3
10	0.5	79.3	3
Total	9.5	700.922	20

The test was performed in the same way as it was asked to the previous participant. All types of function were correctly guessed, which confirms the effectiveness of the sonification process. However, the test took more time than the average of sighted people and of the previous participant. During its execution, we noticed more uncertainty regarding the recognition of the position of the function in the quadrants of the plane. As expressed before, this might be related to a less inclination to Mathematics of the user. In general, the feedback expressed by this participant is that the application requires a training phase to familiarize the user with the sounds and spatial information of the interface. In her opinion, the application cannot be given as a tool to be used immediately autonomously, and in the initial stages, it also requires the assistance of an external person. The difficulty met in the recognition of specific functions such as horizontal parabolas was attributed by her to the low time of use of the app: she thought that a more prolonged use of the application would guarantee the guessing of all types of functions. The participant confirmed that the main advantage of this application is the feeling of autonomy which can give people with visual impairment to draw functions, despite that tactile graph still guarantees greater precision.

The results of these experiments are very good: all users were on target with the goal of this app and responded appropriately to the features of our application. The tests were correctly executed and confirmed the correct implementation of sound cues. The elements considered critical instead could be used for future works.

2) USERS WITH LOW VISION

Two tests were performed on individuals with low vision who provided valuable feedback on the suitability of our user interface for people with partial use of vision. Their insights covered various aspects, including the use of colors, font size, elements of the cartesian plane, and the contrasts of these elements with the background. In addition, their educational background provided valuable insight into improving our application features. The tests carried out following the same protocol adopted for sighted users.

The first test involved a woman 29 years old with low vision and who does not have perfect pitch. Firstly, she provided feedback on color contrast and font size: the color contrast between the elements of the UI was considered excellent, meanwhile, she suggested enlarging the font size for values along the x and y axes. Subsequently, the participant proposed that the significant points of a function could be highlighted within the *Trova Equazione* feature. These points can include intersections with the x-axis and the y-axis, as well as minimum or maximum points. In addition, the participant suggested that the coordinates of these points could be explicitly stated, to help users to learn the trend of the function.

Finally, the user performed the same tests conducted by sighted users. Table 3 shows the results.

TABLE 3. Results of the first test with person with low vision.

Age	29	Gender	Female
Question number	**Scores**	**Times**	**# playbacks**
1	1	12.0	1
2	1	7.3	1
3	1	10.7	1
4	1	9.3	1
5	0.5	21.8	2
6	1	10.2	1
7	1	9.5	1
8	1	7.0	1
9	1	8.5	1
10	0.5	10.9	1
Total	9	107.646	11

The outcome of this experiment was a score of 9 out of 10, with two partially correct answers. The time taken for this test was relatively low, even compared to the tests conducted with sighted individuals. Generally, the participant was able to accurately distinguish the functions solely on the basis of sound. A notable aspect of this experiment is that the participant only needed to replay the sounds of a function once during the test, but still achieved a high score.

The second experiment involved a 27 year old man, with low vision and who does not have a perfect pitch. This participant showed different needs compared to the previous one. The first feedback provided is to select a different color for the points of interest in the *Trova Equazione* mode according to the color of the functions. This advice would improve the contrast and visibility of the points of interest and would help people who experience color blindness. The second suggestion is to make the sound more uniform to better highlight the position of points along the y-axis. Our implementation of musical notes was perceived as corrected to indicate an increasing or a decreasing sound, but there is room for improvement to achieve greater precision. The third advice regards the colors and font size of elements on the *Studia* mode. It was suggested to increase the contrast of the input fields and to use a sharper font for the text of functions to enhance readability. Other elements of the interface, such as the start menu, and the cartesian plane, were instead evaluated positively. Table 4 shows the results of his tests.

TABLE 4. Results of the second test with person with low vision.

Age	27	Gender	Male
Question number	**Scores**	**Times**	**# playbacks**
1	1	12.5	1
2	1	11.4	1
3	0.0	10.5	1
4	1	9.7	1
5	1	9.0	1
6	1	9.6	1
7	1	10.8	1
8	1	5.9	1
9	0.0	5.5	1
10	1	13.5	1
Total	9	98.859	10

The outcome had a score of 8 out of 10, and no responses were partially corrected. The participant was never required to replay the sound of a function during the test. The time recorded for the test is notable for being short, as the listening time for each question is 10 seconds. This suggests that the participant guessed some answers before the sound of the function ended up playing. Despite this, the participant was still able to accurately distinguish the trends of the functions. The participant expressed confidence that these errors could be mitigated with regular use of the application.

Both low vision users promoted most of the elements of the interfaces and provided important insights about the improvements in the UI of our application. Both participants confirmed the opinions on the pleasantness of sounds, but the lack of precision for identifying the ordinates. The metronome beats were confirmed to be effective in identifying the abscissas.

V. CONCLUSION

Our experiments show that this app contributes to making the study of mathematical functions accessible to everyone, since the developed application was tailored for people with visual impairments and allows them to study lines and parabolas through a multimodal approach that utilizes both auditory and tactile means to perceive functions. The tests confirmed the efficacy of our sonification process and assessed the usability of our product also with respect to a general audience. Feedback gained during the test helped us to understand the limitations of our product and how it could be improved.

A. LIMITATION AND FUTURE WORKS

The application showed positive results on the tests, but there are some limitations. First of all, the application allows for the representation of only two types of function which are lines and parabolas. This decision was made to focus development efforts on perfecting the representation of these two function types before expanding to other functions. The app currently available in the Google Play Store[1] also

[1]https://www.math.unipd.it/ gaggi/graficiAccessibili/

implements hyperbolas, and we plan to add other types of function.

The app works better on Android tablets than on iPads due to the lack of haptic feedback on the latter. The counter-version app for the iPhone works properly instead, but the smaller screen size limits the user experience.

By collecting user insights during the testing phase and analyzing the limitations of this project, we have identified potential extensions and improvements of this work. Given the education context of the app, a possible new feature of this project could include the possibility of sharing a function between two users, to allow teachers to send functions to study to students.

After the test phase, we add a tutorial to assist users in becoming familiar with the app's features. It is crucial to recognize that not all users possess prior knowledge of mathematical functions and the required skills to use the app. Consequently, a tutorial is essential to explain which functions can be studied and how they are sonified.

The feature *Trova Equazione* can be improved by changing the criteria for finding the points of interest of functions. Points of interest could include intersections with the cartesian plane axes and coordinates featuring integer abscissas and ordinates. Moreover, these points could be displayed on the interface, presenting their coordinates to provide students with insight into the slope and trends of functions. Furthermore, each point could be associated with a speech audio file that indicates its coordinates.

Another feature that we have added after the test phase is the improvement of the sonification process with a wider range of notes to better discern the points along the y-axis. Moreover, since the pleasantness of the sound cue is subjective, the app allows users to choose the timbre of the sound involved in the sonification process.

Regarding the exploration of the cartesian plane by touch, it is necessary to implement a feature which indicates the direction to follow. This feature could rely on speech indication for the direction to follow or on using the position of the numbers on a clock to suggest the direction to follow.

Lastly, further development of the application could include customization options to cater to the specific needs of each user.

B. CLOSING REMARKS

The main goals of this research were achieved and the testing phases confirmed the right path taken during the development of the application. Despite some limitations, the results of our tests confirmed the effectiveness of this app. This application has the potential to provide a concrete tool for helping students with visual impairments in the study of mathematics.

The prevalence of electronic devices in today's society has not been adequate for people with visual impairments. The tests conducted with these individuals highlighted the challenges they face in studying mathematics and, more broadly, STEM subjects. These challenges underscore the pressing need for more inclusive educational tools.

Although this application is not yet complete, it represents a significant step forward. Currently, the developed application can be used in conjunction with standard tools to provide autonomy to users with visual impairments. The enthusiasm shown by the participants during the tests was inspiring for us and validates the efforts invested in developing this project.

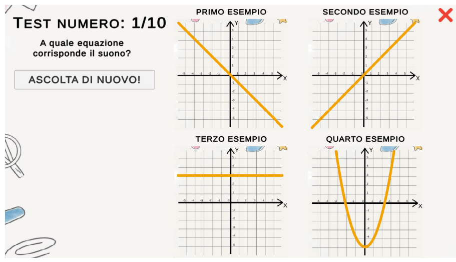

FIGURE 15. First question of the test.

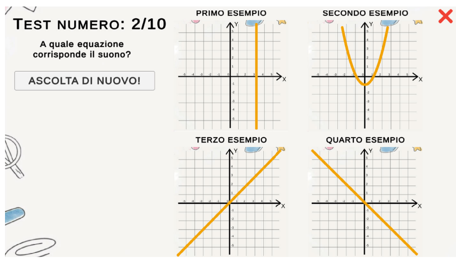

FIGURE 16. Second question of the test.

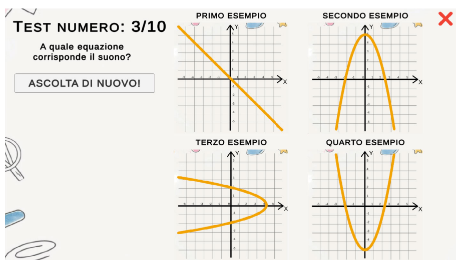

FIGURE 17. Third question of the test.

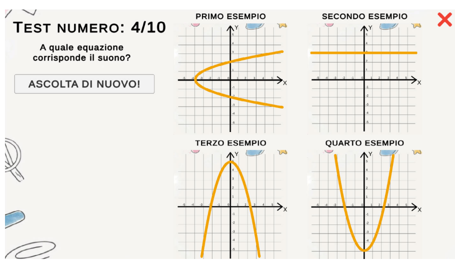

FIGURE 18. Fourth question of the test.

APPENDIX
TESTS

The functions chosen for the questions of the test are the following:

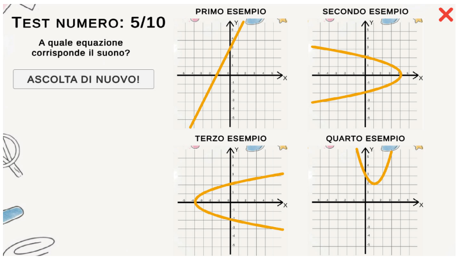

FIGURE 19. Fifth question of the test.

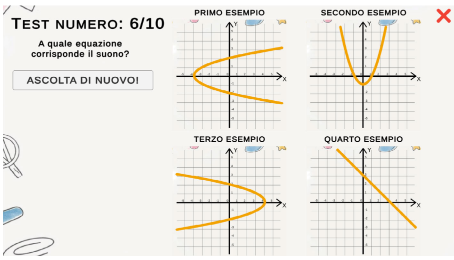

FIGURE 20. Sixth question of the test.

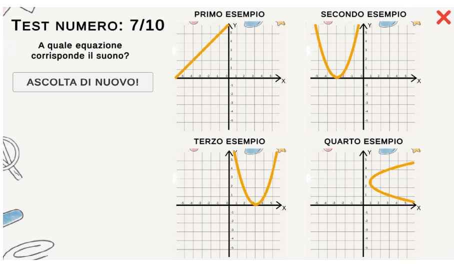

FIGURE 21. Seventh question of the test.

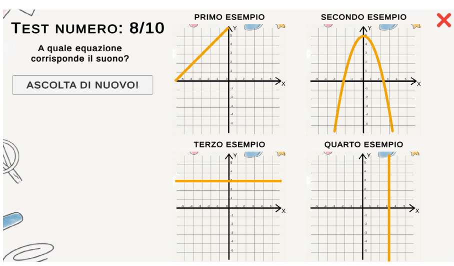

FIGURE 22. Eighth question of the test.

1) $y = x$. It is reproduced the sound of an increasing line. Users had to discern the sound among the following options: decreasing line (0.5 points), increasing lines (1.0 point), line parallel to the x-axis (0.5 points), positive parabolas with axis of symmetry parallel to the y-axis (0.0 points) (See Figure 15).
2) $y = -x$. It is reproduced the sound of a decreasing line. Users had to distinguish the sound of the equation among the following options: line parallel to the y-axis (0.5 points), positive parabola with vertical axis of symmetry (0.0 points), increasing line (0.5 points), decreasing line (1.0 point) (See Figure 16).

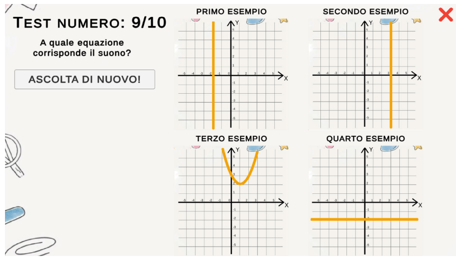

FIGURE 23. Ninth question of the test.

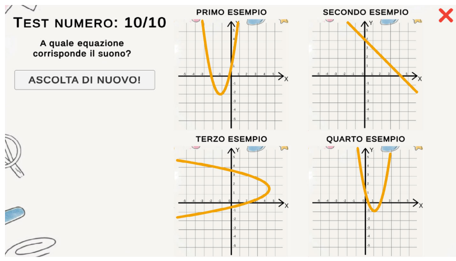

FIGURE 24. Tenth question of the test.

3) $y = x^2 - 5$. It is reproduced the sound of a positive parabola with vertical axis of symmetry. Users had to distinguish the sound of the equation among the following options: decreasing line (0.0 points), negative parabola with vertical axis of symmetry (0.5 points), negative parabola with horizontal axis of symmetry (0.0 points), positive parabola with vertical axis of symmetry (1.0 point)(See Figure 17).
4) $y = -x^2 + 5$. It is reproduced the sound of a negative parabola with vertical axis of symmetry. Users had to distinguish the sound of the equation among the following options: positive parabola with horizontal axis of symmetry (0.0 points), line parallel to the x-axis (0.0 points), negative parabola with vertical axis of symmetry (1.0 point), positive parabola with vertical axis of symmetry (0.5 points) (See Figure 18).
5) $x = -y^2 + 5$. It is reproduced the sound of a positive parabola with horizontal axis of symmetry. Users had to distinguish the sound of the equation among the following options: increasing line (0.0 points), negative parabola with horizontal axis of symmetry (0.5 points), positive parabola with horizontal axis of symmetry (1.0 point), positive parabola with vertical axis of symmetry (0.0 points)(See Figure 19).
6) $x = y^2 - 4$. It is reproduced the sound of a negative parabola with horizontal axis of symmetry. Users had to distinguish the sound of the equation among the following options: positive parabola with horizontal axis of symmetry (0.5 points),positive parabola with vertical axis of symmetry (0.0 points),negative parabola with horizontal axis of symmetry (1.0 point), decreasing line (0.0 points)(See Figure 20).
7) $y = x^2 - 6x + 9$. It is reproduced the sound of a positive parabola with vertical axis of symmetry and lies only on the first quadrant. Users had to distinguish

the sound of the equation among the following options: increasing line (0.0 points),positive parabola with vertical axis of symmetry which lies only on the second quadrant (0.5 points),positive parabola with vertical axis of symmetry which lies only on the first quadrant (1.0 point), positive parabola with horizontal axis of symmetry which lies only on the first quadrant (0.0 points) (See Figure 21).

8) $y = 3$. It is reproduced the sound of a line parallel to the x-axis. Users had to distinguish the sound of the equation among the following options: increasing line (0.0 points),negative parabola with vertical axis of symmetry (0.0 points),line parallel to the x-axis (1.0 point), line parallel to the y-axis (0.0 points) (See Figure 22).
9) $x = -2$. It is reproduced the sound of a line parallel to the y-axis and has as abscissa -3. Users had to distinguish the sound of the equation among the following options: line parallel to the y-axis and has as abscissa -3 (1.0 point), line parallel to the y-axis and has as abscissa $+3$ (0.0 points),positive parabola with vertical axis of symmetry (0.0 points), line parallel to the x-axis (0.0 points) (See Figure 23).
10) $y = 2x^2 - 4x + 1$. It is reproduced the sound of a positive parabola with vertical axis of symmetry and its vertex has coordinate $(0, -1)$. Users had to distinguish the sound of the equation among the following options: positive parabola with vertical axis of symmetry and its vertex has coordinate $(-1, -2)$, with a score of 0.5 points, decreasing line (0.0 points), negative parabola with axis of symmetry parallel to the x-axis (0.0 points), positive parabola with vertical axis of symmetry and its vertex has coordinate $(0, -1)$, with a score of 1.0 point (See Figure 24)

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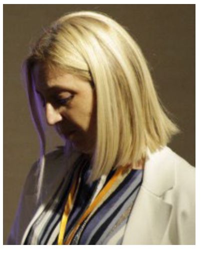

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