Conditioned Tandem Networks

Bernardo D'Auria*

1 Introduction

Most real systems, especially the ones publicly accessible and requiring sharing resources, inevitably generate waiting lines. Recently we have especially tested this kind of circumstances when the pandemia, generated by COVID-19, has worldwide produced many chaotic situations such as shortages of Intensive Care Units or long lines behinds small shops to avoid agglomerations in closed environments.

In some cases, the citizens, here referred to as customers, had at their disposal many alternatives, such as other options about when and where to get what they needed. From a modeling viewpoint, this means that the incoming flow to a particular observed service may be highly affected by the resulting efficiency of the way the service is delivered.

Customers generally make their decisions by observing the system from *outside*, such as seeing the actual queueing line in front of a pharmacy or observing how quick an on-line booking time-window closed for sold out. However, their perception of the system is incomplete and may often lead to unexpected results due to the partial information they have access to. Indeed, the type of partial information may deeply affect the way we estimate the system behavior especially in those cases where the system is made of multiple stages. This analysis is the main subject of this short note.

Behavioral queueing has recently become an increasing subject of investigation, where classical models have been analyzed by using game-theoretical techniques, we refer to the book of Hassin and Haviv [2] for a general background. However, most of the results commonly refer to single queueing systems and just very few results (e.g. [3]) are known about more general systems such as queueing networks.

To be pragmatic, here we focus on the simplest example of queueing network, the two stages Jackson tandem network, and, as we show in the sequel, interesting questions are still open and similar ones can be asked for more general scenarios.

As a preliminary study on this subject we refer to [1], where the two-node Jackson tandem network has been analyzed under different levels of information (none, full and partial) and equilibrium strategies have been investigated for each of them.

2 Problem statement

The system consists of two nodes in series, each of type $\cdot/M/1$, that is, they have one server, an infinite capacity and service times exponentially distributed, see Figure 1. The arrival process is an independent homogeneous Poisson process with rate λ.

A representation of the state at time t is given by the vector $(Q_1(t), Q_2(t))$, where $Q_l(t)$ denotes the number of customers in queue $l \in \{1,2\}$. By defining $S_l(n,m)$ as the conditional sojourn time spent in queue $l \in \{1,2\}$ by a tagged customer accepted to enter in the system while this being in the state $(n-1,m)$, and

*Department of Economics,G. d'Annuzio University of Chieti-Pescara, Italy. bernardo.dauria@unich.it

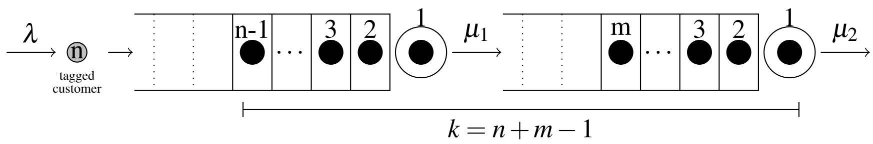

Figure 1: The two-node tandem network

by $T_l(n,m) = E[S_l(n,m)]$ the corresponding expectation, we have that $T_1(n,m) = n/\mu_1$ and $T_2(n,m)$ satisfies the following recursive formula, see [1], and set $c_\mu = \mu_2/(\mu_1+\mu_2)$,

$$T_2(n,m) = c_\mu^m T_2(n-1,1) + \frac{\mu_1}{\mu_2}\sum_{k=1}^{m} c_\mu^k T_2(n-1,m+2-k)\ . \tag{1}$$

An arriving tagged customer decides to join or bulk the system according to a profit function $P(\mathscr{I})$, whose value decreases with the system waiting time. The amount of information $\mathscr{I}$, revealed to the customer, determines the different scenarios and for each of them we may search for an optimal strategy telling when to join. In particular, the *unobservable* case assumes that no information at all is revealed. The *fully*-observable case occurs when s/he knows the current state, i.e. (n,m). The intermediate case, the *partially*-observable, may be subdivided in *total*, *first-queue* and *second-queue* subcases, where the revealed information is given by $n+m$, n, and m respectively.

While the unobservable, the fully and the total scenarios are completely understood, see [1], much less is known about the first-queue and the second-queue situations, in the following jointly referred as the *one-queue* scenarios. We use the subscript o to denote the index of the observed queue and u for the unobserved one.

Assuming the one-queue scenario, we define the threshold strategy $X \in \mathbb{R}^+$ as the strategy to join when the number of customers in the observed queue is less than $\lfloor X \rfloor$. If X is not integer (mixed-strategy), whenever in the observed queue there are X customers the tagged customer will join with probability $X - \lfloor X \rfloor$.

In the one-queue scenario, study with respect to the X strategy and the system parameters, the stochastic monotonicity of the stationary r.v.'s $(\tilde{Q}_u^X | \tilde{Q}_o^X = x)$.

In the one-queue scenario, prove the existence of an equilibrium symmetric strategy X, for the profit function $P^X(x) = R - C_1\,T_1^X(x) - C_2\,T_2^X(x)$, as well as determine the corresponding socially optimal threshold strategy.

3 Discussion

Both scenarios, the first- and the second-queue cases, can be analyzed by using a matrix analytical approach, or equivalently implementing recursive formulas as in Equation (1), helped by tools developed in [4] and in [5], respectively.

Looking at the values in Table 1, it is easy to conclude that, in general, the stochastic increasing monotonicity in terms of n (for the first-queue case) does not hold. Indeed, the computations show that the conditional distribution can be a decreasing function of n. Similar results are valid for the second-queue case. It would be interesting to understand how this property depends on the parameters of the system.

The point of Problem 2 is that it is still not clear if the increment of the incoming flow, due to an increment in the strategy threshold X, implies that one system looks stochastically more congested than another at a given level of the observation x.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$X=3$	0.0515	0.0511	0.0472	0.00226	
$X=4$	0.0515	0.0515	0.0511	0.0472	0.00226

Table 1: First-queue case, values of $\mathbb{E}[\tilde{Q}_2^X|\tilde{Q}_1^X = n]$ for $\lambda = 0.049$, $\lambda/\mu_1 = 0.98$ and $\mu_1/\mu_2 = 0.05$.

As for the Problem 2, while for the total-queue case it has been found in [1] that a dominant threshold strategy exists, the same reasoning does not work for the one-queue scenario. It can be expected that the first-queue case would be easier to handle as the application of an X-strategy implies that the number of customers in the first queue is bounded above by the same quantity. This is not the case for the second-queue scenario, where the X-strategy does not rule out the possibility that in the second queue more than X customers may be waiting in line.

For the region of the parameters where the increasing monotonicity holds, Problem 2 should be easier to solve. Therefore, it is interesting to characterize this region, in terms of μ_1 and μ_2, as well as to understand what happens outside it.

References

[1] B. D'Auria and S. Kanta. Pure threshold strategies for a two-node tandem network under partial information. *Operations Research Letters*, 43(5):467–470, Sept. 2015.

[2] R. Hassin and M. Haviv. *To Queue or Not to Queue: Equilibrium Behavior in Queueing Systems*. Springer US, Boston, MA, 2003.

[3] B. Kim and J. Kim. Equilibrium strategies for a tandem network under partial information. *Operations Research Letters*, 44(4):532–534, July 2016.

[4] D. P. Kroese, W. R. W. Scheinhardt, and P. G. Taylor. Spectral properties of the tandem Jackson network, seen as a quasi-birth-and-death process. *The Annals of Applied Probability*, 14(4):2057–2089, 2004.

[5] L. Leskelä and J. Resing. A tandem queueing network with feedback admission control. In *Proceedings of the 1st EuroFGI International Conference on Network Control and Optimization*, NET-COOP'07, page 129–137, Berlin, Heidelberg, 2007. Springer-Verlag.