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Divisorial domains

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Abstract. Let R be a domain with quotient field Q. R is divisorial if $R:(R:I) = I$ for every nonzero fractional ideal I of R. We prove that a local domain R, not a field, is divisorial if and only if Q/R has simple essential socle and R/rR is AB-5* for every nonzero $r \in R$. We give examples of non-divisorial and of non-finitely divisorial local domains such that Q/R has simple essential socle. If A is any R-submodule of Q with endomorphism ring R, we say that R is A-divisorial if $A:(A:X) = X$ for every nonzero submodule X of A. We prove that if a local noetherian domain R is A-divisorial for some A, then R is one-dimensional and A is finitely generated, i.e. A is isomorphic to a canonical ideal of R. If A is a fractional ideal of R we generalize the characterization of divisorial domains, namely we prove that R is A-divisorial if and only if Q/A has simple essential socle and R/rR is AB-5* for every nonzero $r \in R$.

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Introduction

Throughout this paper, R will denote an integral domain, not a field, and Q will be its field of quotients.

The notion of divisorial fractional ideal is a classical one and was introduced in the thirties: if F is a fractional ideal of R, we denote by F^{-1} the inverse of F, namely

$$F^{-1} = R : F = \{q \in Q \mid qF \leq R\}.$$

F is said to be divisorial if it coincides with its double inverse. Note that, since $R:F$ is canonically isomorphic to $\mathrm{Hom}_R(F,R)$, F^{-1} can be viewed as the R-dual of F. Thus we can say that F is divisorial if and only if it is R-reflexive. A domain R is said to be divisorial if every nonzero ideal I of R (equivalently, every nonzero fractional ideal) is divisorial. Divisorial domains have been studied by H. Bass, W. Heinzer and

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E. Matlis in the classical cases. More precisely, Bass in [4] and Matlis in [18] gave a characterization of the noetherian case, while Heinzer in [12] characterized the integrally closed divisorial domains.

In this paper we consider the general case and we give necessary and sufficient conditions for a domain to be divisorial. A first result in this direction appears in a paper by L. Salce and the author [6], where it is proved that a domain is divisorial if and only if it is h-local and every localization at a maximal ideal is divisorial. The necessity of this condition was already proved by Heinzer and Matlis, independently. This characterization is very useful, since it allows to reduce the problem to the local case. Another interesting result proved by Heinzer and Matlis states that, if a domain R is divisorial, then the R-module Q/R has essential socle. We note that, if a local domain R is either noetherian or integrally closed, then R is divisorial if and only if Q/R has simple essential socle or, differently stated, if and only if Q/R is cocyclic.

In Section 2 we show that the above characterization is valid also for other classes of local domains, but not always: we give an example of a local domain R such that Q/R has simple essential socle and nevertheless R is not divisorial (Example 2.11). Thus the problem is to find which condition has to be added to Q/R being cocyclic in order to obtain a characterization of local divisorial domains. The appropriate condition is the notion AB-5*, which is the dual of the Grothendieck condition AB-5 for abelian categories: we prove that a local domain R is divisorial if and only if Q/R is cocyclic and R/I is AB-5* for every nonzero ideal I of R (or, equivalently, for every $I = rR$, $0 \neq r \in R$). A consequence of the AB-5* condition on every proper homomorphic image of R is that the module $I/\boldsymbol{m}I$ ($\boldsymbol{m}$ the maximal ideal of R), is finitely generated, for every ideal I of R. But we exhibit an example showing that this more manageable condition (together with the condition Q/R cocyclic) is not sufficient to conclude that R is divisorial. A question raised by Heinzer was to decide whether the condition Q/R cocyclic is sufficient to guarantee that at least the finitely generated ideals of R are divisorial. We answer to the question negatively by exhibiting a counterexample (see Proposition 4.4, Section 4). This counterexample is constructed by making use of the notion of A-divisoriality, introduced in [6], which we now illustrate. If A is any R-submodule of Q and S is its endomorphism ring, then the A-inverse of any S-submodule X of Q is defined by

$$A : X = \{q \in Q \mid qX \leq A\}.$$

Thus the A-inverse of X is canonically isomorphic to the A-dual $\mathrm{Hom}_S(X, A)$ of X. A domain R is said to be A-divisorial if every nonzero S-submodule of A coincides with its A-double inverse, i.e. it is A-reflexive. As in the case of divisorial domains, the notion of A-divisoriality is a local one. In fact, in [6] it is shown that a domain R is A-divisorial for a proper nonzero submodule A of Q if and only if the endomorphism ring S of A is h-local and every localization at a maximal ideal $\boldsymbol{m}$ of S is $A_{\boldsymbol{m}}$-divisorial.

In Section 3 we characterize the local noetherian A-divisorial domains generalizing well known results on noetherian divisorial domains. More precisely, we consider a local noetherian domain R and an R-submodule A of Q with endomorphism ring R

and we show that, in case A is a fractional ideal of R (i.e. finitely generated), every characterization obtained by Bass and Matlis carries over simply by changing R with A. For instance, we prove that R is A-divisorial if and only if R has Krull dimension one and Q/A has simple socle, or equivalently, if and only if Q/A has simple essential socle. Moreover we show that, if a local noetherian domain R is A-divisorial, then A must be finitely generated and thus the above characterization applies. It is interesting to note that, if A is not finitely generated, then the condition on Q/A being cocyclic or even injective is not sufficient to conclude that R is A-divisorial.

At this point we have to remark that, after writing this paper, we happened to read a manuscript by W. Heinzer, J. Huckaba and I. Papick [13] which has a terminology similar to ours. They study domains R admitting an $\boldsymbol{m}$-canonical ideal which means, in our terminology, domains R which are A-divisorial for a fractional ideal A of R with endomorphism ring R. For instance in Proposition 4.3 they prove the same result we state in Theorem 3.2 (4), but with a different approach.

The investigation on A-divisorial noetherian domains, besides its intrinsic interest, is very useful in producing examples of local domains R such that Q/R is cocyclic: these are obtained in Section 4, by means of a generalization of the classical $D + \boldsymbol{m}$ construction introduced by Krull. More precisely, if R_0 is a local domain and A is an R_0-submodule of Q such that the endomorphism ring of A is R_0 and Q/A is cocyclic, then we consider the domain R defined by $R = R_0 + Ax + x^2Q[[x]]$, where x is an indeterminate. If $Q(R)$ is the quotient field of R, then $Q(R)/R$ is cocyclic, but depending on the choice of R_0 and A, we can make R to be divisorial or not. For instance, using a particular notion of independence for elements of the completion of a local noetherian domain considered in [15], we are able to choose a noetherian local domain R_0 and an A with endomorphism ring R_0, such that Q/A is cocyclic but not injective. This implies that the local domain R described above has non-divisorial finitely generated ideals, hence it answers negatively Heinzer's question mentioned at the beginning.

The problem of characterizing integrally closed A-divisorial domains is not completely solved. A question arising in this context is the following: assume that R is a local integrally closed A-divisorial domain for a fractional ideal A of R with endomorphism ring R. Does it follow that R is a valuation domain? (The same question is posed also in [13].) The above question is closely related to a difficult open problem posed by Heinzer in 1968, namely to decide whether the integral closure of a divisorial domain is a Prüfer domain.

In Section 5 we are able to give a generalization of our characterization of local divisorial domains. Namely we prove that a domain R is A-divisorial, for a fractional ideal A of R with endomorphism ring R, if and only if Q/A is cocyclic and R/rR is AB-5* for every nonzero element $r \in R$.

1 Preliminaries and notations

Throughout R will denote a commutative domain with identity and with quotient field Q. By an overring of R we will mean any ring between R and Q. A local domain is a domain with exactly one maximal ideal and it is not necessarily noetherian. A domain is h-local if every nonzero prime ideal is contained in a unique maximal ideal and every nonzero ideal is contained in finitely many maximal ideals.

If X and Y are R-submodules of Q, then $X : Y$ denotes the R-submodule of Q consisting of the elements $q \in Q$ such that $qY \leq X$. It is obvious that $X : Y$ is canonically isomorphic to $\mathrm{Hom}_R(Y, X)$.

We recall the notions of A-divisorial and A-reflexive modules as introduced in [6]. Given a fixed R-submodule A of Q, an R-module N is said to be *A-reflexive* (respectively *A-torsionless*) if the canonical homomorphism

$$\omega_N : N \to \mathrm{Hom}_R(\mathrm{Hom}_R(N, A), A)$$

defined by: $\omega_N(x)(f) = f(x)$ $(x \in N, f \in \mathrm{Hom}_R(N, A))$, is an isomorphism (respectively a monomorphism). Obviously A-reflexive modules are A-torsionless and A-torsionless modules are torsionfree. Recall that a fractional ideal I of the domain R is divisorial if $I = R : (R : I)$; thus, in our terminology, the divisorial fractional ideals are exactly the R-reflexive fractional ideals of R. If A is a fixed R-submodule of Q, we say that the domain R is *A-divisorial* (respectively *A-reflexive*), if every A-torsionless $\mathrm{End}_R(A)$-module of rank one (resp. of finite rank) is A-reflexive. Hence R is A-divisorial if and only if $A : (A : X) = X$ for every nonzero $\mathrm{End}_R(A)$-submodule X of A. To avoid trivial cases, we will always assume $0 \neq A \neq Q$. Following the terminology used in the literature, R-reflexive and R-divisorial domains will be simply called "reflexive" and "divisorial" respectively.

In Section 3, we will frequently use well known results on injective modules over noetherian rings; for instance, we will assume familiarity with the structure of indecomposable injective modules over such rings, with their endomorphism rings and also with Matlis duality on a complete local noetherian ring. For terminology and results on the subject, we refer to [16] or [23].

2 Local divisorial domains

As recalled in the Introduction, the investigation of divisorial domains can be reduced to the local case, since Proposition 5.4 in [6] states that a domain is divisorial if and only if it is h-local and every localization at a maximal ideal is divisorial. We first recall the characterization of divisorial local noetherian domains.

Theorem A (H. Bass Theorem 6.2, 6.3 [4]; E. Matlis, Theorem 3.8, [18]). *Let R be a local noetherian domain with maximal ideal $\mathbf{m}$. The following are equivalent:*

1. *R is divisorial.*
2. *Q/R is the injective envelope of the simple module $R/\mathbf{m}$.*
3. *Q/R is injective.*
4. *R has Krull dimension one and $(R : \mathbf{m})/R$ is simple.*
5. *R is reflexive.*

Definition 1. An R-module is said to be *cocyclic* if it has simple essential socle.

It is well known that, if a local domain R is divisorial, then Q/R is cocyclic. (See [18] or [12]). We can add the following characterization.

Theorem 2.1. *Let R be a local noetherian domain with maximal ideal $\boldsymbol{m}$. Then R is divisorial if and only if Q/R is cocyclic.*

Proof. The socle of Q/R is $(R : \boldsymbol{m})/R$, hence it is enough to show that if Q/R is cocyclic, then R has Krull dimension one. Assume that P is a nonzero prime ideal of R and let $0 \neq r \in P$. The annihilator of the element $r^{-1} + R \in Q/R$ is rR and by hypothesis, Q/R is embeddable in the injective envelope E of the simple R-module $R/\boldsymbol{m}$. Hence, using well known properties of E, a suitable power $\boldsymbol{m}^n$ of $\boldsymbol{m}$ is contained in $rR \subseteq P$ and thus P is the maximal ideal of R. □

The characterization of the integrally closed case given by Heinzer is the following.

Theorem B (W. Heinzer, Theorem 5.1, [12]). *Let R be a local integrally closed domain with maximal ideal $\boldsymbol{m}$. The following are equivalent:*

1. *R is divisorial.*
2. *R is a valuation domain with principal maximal ideal.*

As in the noetherian case, we show that the condition Q/R cocyclic is equivalent to the divisoriality. First we recall a Lemma.

Lemma 2.2 (Lemma 5.5, [6]). *Let R be a local domain with maximal ideal $\boldsymbol{m}$ such that Q/R is cocyclic. Then:*

1. *$\boldsymbol{m}$ is invertible if and only if R is a valuation divisorial domain.*
2. *If R is integrally closed, then R is a valuation divisorial domain.*

As a corollary of the preceding Lemma we obtain.

Theorem 2.3. *Let R be a local integrally closed domain with maximal ideal $\boldsymbol{m}$. Then R is divisorial if and only if Q/R is cocyclic.*

Remark 1. A consequence of Lemma 2.2 (1) is that, if R is a local domain with principal maximal ideal, then R is divisorial if and only if Q/R is cocyclic.

If R is a local domain with non-principal maximal ideal, then $R : \boldsymbol{m}$ coincides with $\boldsymbol{m} : \boldsymbol{m}$, hence it is the endomorphism ring of $\boldsymbol{m}$. Let us denote $\boldsymbol{m} : \boldsymbol{m}$ by R_1. Assume moreover that Q/R is cocyclic; then R_1 is two-generated over R and, in the terminology used in [9], R_1 is a unique minimal overring of R. The following Proposition, which illustrates the properties of R_1, is essentially Theorem. 5.7 in [6], but here we state it under weaker hypotheses. It is easy to see that the same proof applies with the obvious changes.

Proposition 2.4. *Let R be a local domain with non-principal maximal ideal $\mathbf{m}$ and such that Q/R is cocyclic. Let $R_1 = \mathbf{m} : \mathbf{m}$; then one and only one of the following cases can occur:*

1. *R_1 is local with maximal ideal $\mathbf{m}_1$ properly containing $\mathbf{m}$.*
   In this case $\mathbf{m}_1/\mathbf{m}$ is simple both as an R-module and as an R_1-module; moreover $\mathbf{m}_1^2 \subseteq \mathbf{m}$ and $R : \mathbf{m}_1 = \mathbf{m}_1$.
2. *R_1 has exactly two maximal ideals $\mathbf{m}_1$ and $\mathbf{m}_2$.*
   In this case $\mathbf{m} = \mathbf{m}_1 \cap \mathbf{m}_2$, $\mathbf{m}_i/\mathbf{m}$ is simple both as an R-module and as an R_1-module, for $i = 1, 2$; $R : \mathbf{m}_i = \mathbf{m}_j$, for $i \neq j$. Moreover R_1 is a Prüfer domain, intersection of two valuation domains V_1 and V_2 whose maximal ideals $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfy $\mathbf{m} = \mathbf{n}_1 \cap \mathbf{n}_2$.
3. *R_1 is local with maximal ideal $\mathbf{m}$.*
   In this case R_1 is a valuation domain.

The preceding Proposition allows us to consider only three different possibilities. We will give a complete description of the situation arising in case (2) and (3), while case (1) is more difficult, but also more interesting. The next two Lemmas are easy, but very useful.

Lemma 2.5. *Let R be a local domain with non-principal maximal ideal $\mathbf{m}$ and such that Q/R is cocyclic. Let $R_1 = \mathbf{m} : \mathbf{m}$; then an ideal I of R is an R_1-ideal if and only if I is not a principal ideal of R.*

Proof. Let I be a non-principal ideal of R and let $0 \neq c \in I$. There exists $d \in I$ such that $d \notin cR$. Consider $q = \frac{d}{c}$, then $qR + R$ properly contains R hence, by hypothesis, $qR + R \supseteq R_1$. Now for every $\alpha \in R_1$ we have $\alpha = qr + s$ for some $r, s \in R$; hence $c\alpha = dr + sc \in cR + dR \subseteq I$ and thus $R_1 I \subseteq I$. Conversely, it is clear that if $I = cR$, then $c\alpha \notin I$ for every $\alpha \in R_1 \backslash R$. □

Lemma 2.6 (Proposition 7.5, [6]). *Let V be a valuation domain with maximal ideal $\mathbf{m}$. Then V is $\mathbf{m}$-divisorial, i.e. $\mathbf{m} : (\mathbf{m} : I) = I$ for every nonzero ideal I of V.*

We settle now the case of a domain satisfying Proposition 2.4 (3).

Proposition 2.7. *Let R be a local domain with non-principal maximal ideal $\mathbf{m}$ such that $R_1 = \mathbf{m} : \mathbf{m}$ is local with maximal ideal $\mathbf{m}$. Then R is divisorial if and only if Q/R is cocyclic.*

Proof. As already observed, we have to prove only the sufficiency. Assume that I is an ideal of R which is principal or isomorphic to $\mathbf{m}$; then I is clearly divisorial.

Otherwise, by Lemma 2.5, I is an R_1 ideal and R_1 is a valuation domain by Proposition 2.4. Now if I is principal over R_1, say $I = \beta R_1$, then $R : (R : I) = \beta R : \boldsymbol{m} = I$. In the remaining case we have $R : I = \boldsymbol{m} : I = R_1 : I$, since I is non-principal neither over R nor over R_1. By Lemma 2.6, $\boldsymbol{m} : I$ cannot be principal over R, since otherwise $I \cong \boldsymbol{m}$. Thus $R : (\boldsymbol{m} : I) = \boldsymbol{m} : (\boldsymbol{m} : I) = I$ and R is divisorial. □

Example 2.8. Let V be a valuation domain of the form $K + \boldsymbol{m}$ where $\boldsymbol{m}$ is the maximal ideal of V and K is a field containing a subfield k_0 such that $[K : k_0] = 2$. Then the local domain $R = k_0 + \boldsymbol{m}$ is a divisorial domain satisfying Proposition 2.7.

We now consider case (2) of Proposition 2.4. We will use the well known fact that, if a family of ideals of a divisorial domain has nonzero intersection, then the inverse of the intersection is the sum of the inverses of the ideals in the family.

Proposition 2.9. *Let R be a local domain with non-principal maximal ideal $\boldsymbol{m}$ such that Q/R is cocyclic and $R_1 = \boldsymbol{m} : \boldsymbol{m}$ has exactly two maximal ideals $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Then R is divisorial if and only if R_1 is an h-local domain.*

Proof. Assuming that R is divisorial, we show that R_1 is h-local by imitating the proof of Lemmas 2.3 and 2.4 in [12] as follows. Let I be any nonzero ideal of R_1 contained in $\boldsymbol{m}_1 \cap \boldsymbol{m}_2$. Consider $\mathscr{B} = \{\mathscr{B}_\alpha\}$ the set of all the ideals B_α of R_1 such that $I \subseteq B_\alpha \subseteq \boldsymbol{m}_2$, but $B_\alpha \nsubseteq \boldsymbol{m}_1$. $\mathscr{B}$ is not empty, since $\boldsymbol{m}_2 \in \mathscr{B}$. Let $B_0 = \bigcap_\alpha B_\alpha$, then $B_0 \nsubseteq \boldsymbol{m}_1$. In fact, if $B_0 \subseteq \boldsymbol{m}_1$, then $R : B_0 = \sum_\alpha (R : B_\alpha) \supseteq R : \boldsymbol{m}_1 = \boldsymbol{m}_2$, by Proposition 2.4. Now $\boldsymbol{m}_2 = \boldsymbol{m} + \beta R$ for every $\beta \in \boldsymbol{m}_2 \backslash \boldsymbol{m}$; thus $\beta \in B_{\alpha_1}^{-1} + \cdots + B_{\alpha_n}^{-1}$ and, since $B_\alpha^{-1} \supseteq \boldsymbol{m}$ for every α, we conclude that $\boldsymbol{m}_2 \subseteq B_{\alpha_1}^{-1} + \cdots + B_{\alpha_n}^{-1}$. Hence $R : \boldsymbol{m}_2 = \boldsymbol{m}_1 \supseteq B_{\alpha_1} \cap \cdots \cap B_{\alpha_n}$, a contradiction. Now assume that the ideal I is a nonzero prime ideal of R_1; choosing $\beta \in B_0 \backslash \boldsymbol{m}_1$, we obtain $I + \beta^2 R_1 \nsubseteq \boldsymbol{m}_1$ and thus $B_0 \subseteq I + \beta^2 R_1$. Now $\beta \in I + \beta^2 R_1$ yields the contradiction I not prime. Hence any nonzero prime ideal of R_1 is contained in only one maximal ideal, i.e. R_1 is h-local. Conversely, let I be a nonzero ideal of R; to show that R is divisorial it is enough to assume that I is non-principal and non isomorphic to $\boldsymbol{m}$. Hence $R : I = \boldsymbol{m} : I$ and, by Lemma 2.5, I is an R_1 ideal. Moreover, since $\boldsymbol{m} : I$ is a fractional R_1-ideal, it cannot be principal over R, hence $R : (R : I) = \boldsymbol{m} : (\boldsymbol{m} : I)$. In the notations of Proposition 2.4 (2), $R_1 = V_1 \cap V_2$, where V_1 and V_2 are two valuation domains with maximal ideals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ respectively; moreover $(R_1)_{m_i} = V_i$, for $i = 1, 2$. Now, by hypothesis R_1 is h-local and using Lemma 2.3 [6] (which shows that over an h-local domain the operation":" commutes with localizations at maximal ideals), it is easy to see that $\boldsymbol{m} : (\boldsymbol{m} : I) = [\boldsymbol{n}_1 : (\boldsymbol{n}_1 : IV_1)] \cap [\boldsymbol{n}_2 : (\boldsymbol{n}_2 : IV_2)]$. Thus, by Lemma 2.6, $\boldsymbol{m} : (\boldsymbol{m} : I) = IV_1 \cap IV_2 = I$ and I is divisorial. □

Example 2.10. A noetherian domain satisfying the hypotheses of Proposition 2.9 is $R = (k[X, Y]_{(X,Y)}/(X^2 - Y^2 - X^3))$ where k is a field. Denoting by x, y the generators of the maximal ideal $\boldsymbol{m}$ of R we have $R_1 = R + ((x + y)/x)R$, $\boldsymbol{m}_1 = \boldsymbol{m} + ((x + y)/x)R$ and $\boldsymbol{m}_2 = \boldsymbol{m} + ((x - y)/x)R$.

A non-noetherian example is Example 4.3 in [9].

The preceding results describe many cases in which a local domain R is divisorial if and only if Q/R is cocyclic. We exhibit now an example showing that in general this characterization doesn't hold; the example uses the classical $D + \boldsymbol{m}$ construction.

Example 2.11. Let R_0 be a noetherian local divisorial domain with maximal ideal $\boldsymbol{m}_0$ and with quotient field Q. Let $V = Q[[x]]$ be the valuation domain of the power series ring with coefficients in Q and denote by $\boldsymbol{m}$ the maximal ideal of V. Consider $R = R_0 + \boldsymbol{m}$; R is local with maximal ideal $\boldsymbol{m}_0 + \boldsymbol{m}$ and quotient field $Q((x))$. It is straightforward to see that $R : (\boldsymbol{m}_0 + \boldsymbol{m}) = (R_0 : \boldsymbol{m}_0) + \boldsymbol{m}$ and using the fact that Q/R_0 is cocyclic as an R_0-module, we can show that $Q((x))/R$ is a cocyclic R-module. In fact, let $\xi = \sum_{i \geq -k} q_i x^i \in Q((x)) \backslash R$. Then, clearly $k \geq 0$; if $k = 0$, we must have $q_0 \notin R_0$. Thus there exists $0 \neq r_0 \in R_0$ such that $q_0 r_0 \in (R_0 : \boldsymbol{m}_0) \backslash R_0$, hence, $\xi r_0 \in (R : (\boldsymbol{m}_0 + \boldsymbol{m})) \backslash R$. If $k \geq 1$, then $qx^k \in R$ for every $q \in Q$; hence, if $\alpha \in (R_0 : \boldsymbol{m}_0) \backslash R_0$, we have $\xi q_{-k}^{-1} \alpha x^k \in (R : (\boldsymbol{m}_0 + \boldsymbol{m})) \backslash R$. By Corollary 4.4 in [5], R is divisorial if and only if every proper submodule of Q containing R_0 is a fractional ideal of R_0. Thus if we choose R_0 with non-finitely generated integral closure, then R is not divisorial.

Remark 2. It is well known that a local noetherian domain of dimension one has non-finitely generated integral closure if and only if it is analitically ramified (See Theorem 8 in [21] or Theorem 10.2 in [19]) and examples of analitically ramified local noetherian domains of dimension one are the famous examples by Nagata ((E3.1) in [20]). Hence the existence of a local noetherian divisorial domain with non-finitely generated integral closure is guaranteed for instance by Theorem 14.16 in [19].

Remark 3. It is easy to verify that the finitely generated ideals of the domain R constructed in Example 2.11 above are divisorial. We will show in Section 4 (Proposition 4.4) that the condition Q/R cocyclic is not sufficient to conclude that every finitely generated ideal is divisorial.

Our purpose is to find which condition has to be added to the hypothesis Q/R cocyclic in order to guarantee the divisoriality of the local domain R. The notion which allows us to obtain the wanted characterization is the AB-5* condition, which is the dual of the Grothendieck condition AB-5.

Definition 2. Let R be any ring. An R module M satisfies the *AB-5* condition* if

$$\bigcap_i (N_i + K) = \left(\bigcap_i N_i \right) + K$$

for every inverse system of submodules $\{N_i\}_{i \in I}$ of M and for every submodule K of M.

The class of modules satisfying the AB-5* condition contains the linearly compact modules, hence also the artinian modules. It is clearly closed under submodules and epimorphic images, but in general is not closed under finite direct sums. (See [1], [7].)

Proposition 2.12. *Let R be a local divisorial domain. Then R/rR is AB-5* for every nonzero $r \in R$.*

Proof. Let $0 \neq r \in R$ and let $\{J_i/rR\}_{i\in I}$ be an inverse system of submodules of R/rR; then $\{J_i\}_{i\in I}$ is a family of ideals of R with nonzero intersection. Hence $(\bigcap_i J_i)^{-1} = (\sum_i J_i^{-1})$ and $\{(J_i)^{-1}\}_{i\in I}$ is a direct system of fractional ideals of R. If K/rR is any submodule of R/rR, then the AB-5 condition implies:

$$\left(\sum_i J_i^{-1}\right) \cap K^{-1} = \sum_i (J_i^{-1} \cap K^{-1}).$$

But, by divisoriality, we also have $\sum_i (J_i^{-1} \cap K^{-1}) = [\bigcap_i (J_i + K)]^{-1}$, hence

$$\bigcap_i (J_i + K) = \left[\left(\sum_i J_i^{-1}\right) \cap K^{-1}\right]^{-1} = \left(\bigcap_i J_i\right) + K.$$

Passing to the homomorphic images in R/rR, we conclude that R/rR is AB-5*. □

We can now state the main result of this Section.

Theorem 2.13. *Let R be a local domain. Then R is divisorial if and only if Q/R is cocyclic and R/rR is AB-5* for every nonzero $r \in R$.*

Proof. In view of the preceding results we have to prove only the sufficiency. Let I be a nonzero ideal of R and consider $0 \neq r \in I$. By hypothesis, the ring R/rR is AB-5* and cocyclic, since it is isomorphic to $r^{-1}R/R \subseteq Q/R$. Hence, by Corollary 6 in [2], $\bar{R} = R/rR$ is a dual ring, i.e. every ideal of $\bar{R}$ coincides with its double annihilator in $\bar{R}$. Let $\mathrm{Ann}_{\bar{R}} N$ denote the annihilator in $\bar{R}$ of the module N; we have

$$\mathrm{Ann}_{\bar{R}}(I/rR) = \frac{(rR : I) \cap R}{rR} = \frac{rR : I}{rR},$$

where the second equality holds since $r \in I$ implies $r(R : I) \subseteq R$. We also have

$$\mathrm{Ann}_{\bar{R}}\left(\frac{rR : I}{rR}\right) = \frac{(rR : (rR : I)) \cap R}{rR} = \frac{R : (R : I)}{rR},$$

and again the second equality holds, since $rR : (rR : I) = R : (R : I) \subseteq R$. Hence we conclude that I is divisorial. □

Remark 4. Note that in the classical cases the AB-5* condition on every epimorphic image R/rR is automatically satisfied. In fact, in the noetherian case R has Krull dimension one, hence, for every nonzero element $r \in R$, R/rR is artinian and thus AB-5*. In the integrally closed case R is a valuation domain, hence R itself is AB-5*.

If Q/R is AB-5*, then clearly R/rR is AB-5*, for every nonzero $r \in R$. We don't know whether Q/R is necessarily AB-5* in case R is a divisorial domain. We will state this problem as Question 5.6 in a more general context.

We show that it is not possible to weaken the AB-5* condition.

Proposition 2.14. *Let R be local divisorial domain. Then Q/R is cocyclic and for every nonzero ideal I of R, $I/\boldsymbol{m}I$ is finitely generated.*

Proof. Let I be a nonzero ideal of R and take $0 \neq r \in \boldsymbol{m}I$. Then $I/\boldsymbol{m}I$ is an epimorphic image of I/rR which is an AB-5*-module, by Proposition 2.12, hence $I/\boldsymbol{m}I$ itself is AB-5*. It is well known that the socle of an AB-5*-module over a local (commutative) ring is finitely generated (see for instance Theorem 1.2 in [1]), hence $I/\boldsymbol{m}I$ is finitely generated as an R-module. □

Remark 5. The converse of the statement of the preceding Proposition doesn't hold, as the following Example shows.

Example 2.15. Let $R = R_0 + \boldsymbol{m}$ be as in Example 2.11. Assume moreover that R_0 is a totally divisorial local noetherian domain, i.e. a divisorial domain all of whose overrings are divisorial and such that the integral closure of R_0 is not finitely generated over R_0. (See Sections 6 and 7 in [6] and Example 3.5 in [14]). In Example 2.11, we have shown that, if $Q(R)$ denotes the quotient field of R, then $Q(R)/R$ is cocyclic, but R is not divisorial. Let $\boldsymbol{n}$ denote the maximal ideal of R. We show now that $I/\boldsymbol{n}I$ is finitely generated over R, for every nonzero ideal I of R. It is well known that every ideal of R is isomorphic to a fractional ideal of the form $J + \boldsymbol{m}$ where J is an R_0 submodule of Q, hence $I/\boldsymbol{n}I \cong J/\boldsymbol{m}_0 J$. Thus it is enough to show that $J/\boldsymbol{m}_0 J$ is finitely generated over R_0; this is obviously true if J is finitely generated or if $J = Q$. By results in [6] Section 6, the integral closure $\overline{R}_0$ of R_0 is a valuation domain with maximal ideal $a\overline{R}_0$, where a is an element in $\boldsymbol{m}_0$. Moreover, if J is non-finitely generated, then it is an $\overline{R}_0$-module, hence it is isomorphic to $a^k\overline{R}_0$, for some integer $k \in \mathbf{Z}$. This yields $J/\boldsymbol{m}_0 J \cong \overline{R}_0/a\overline{R}_0$ which is a simple R_0-module, by Lemma 6.4, [6].

3 A-divisorial noetherian domains

Recall that, if A is any proper R-submodule of Q with endomorphism ring S, R is A-divisorial if and only if $A : (A : X) = X$ for every nonzero S-submodule X of A. Thus a domain is A-divisorial if and only if the endomorphism ring of A is A-divisorial. Hence, to study the A-divisorial domains amounts to consider domains R which are A-divisorial for a module $A \subsetneq Q$ such that $A : A = R$. The first result to recall is that

A-divisoriality is a local property. In fact, Theorem 4.7 in [6] states that if A is an R-submodule of Q with $A : A = R$, then R is A-divisorial if and only if R is h-local and $R_{\boldsymbol{m}}$ is $A_{\boldsymbol{m}}$-divisorial for every maximal ideal $\boldsymbol{m}$ of R. Consequently we will reduce our investigation to the local case and thus, in this Section:

R will denote a noetherian local domain with maximal ideal $\boldsymbol{m}$.

If N is any R-module, $E_R(N)$ will denote the R-injective envelope of N.

Lemma 3.1. *Let R be a local noetherian A-divisorial domain, where A is an R-submodule of Q with $A : A = R$. Then:*

1. *Q/A is the injective envelope of the simple module $R/\boldsymbol{m}$.*
2. *For every nonzero R-submodule X of A such that A/X is cocyclic, there exists $0 \neq q_0 \in Q$ such that $X = q_0 A \cap A$.*

Proof. (1) By Lemma 4.3 and Proposition 4.6 in [6], Q/A is an essential extension of its socle $(A : \boldsymbol{m})/\boldsymbol{m}$; hence Q/A is embeddable in the injective envelope of the simple module $S = R/\boldsymbol{m}$. Let $E = E_R(S)$; we prove that $E = Q/A$ by showing that $\mathrm{Ann}_E\, \boldsymbol{m}^n = \mathrm{Ann}_{Q/A}\, \boldsymbol{m}^n$ for every $n \in \mathbf{N}$. Let $\bar{R} = R/\boldsymbol{m}^n$; then $\mathrm{Ann}_E\, \boldsymbol{m}^n = E_{\bar{R}}(S)$. Denote by Y the submodule $\mathrm{Ann}_{Q/A}\, \boldsymbol{m}^n$ of $E_{\bar{R}}(S)$. Then $Y = (A : \boldsymbol{m}^n)/A$ and it is easy to see that $\mathrm{Ann}_{\bar{R}}\, Y = \dfrac{A : (A : \boldsymbol{m}^n)}{\boldsymbol{m}^n}$, since $A : \boldsymbol{m}^n \supseteq A$ and $A : A = R$. By hypothesis, R is A-divisorial, thus we have $\mathrm{Ann}_{\bar{R}}\, Y = 0$. Hence, applying Matlis' duality valid over complete local noetherian rings, we conclude that $Y = \mathrm{Ann}_E\, \boldsymbol{m}^n = \mathrm{Ann}_{Q/A}\, \boldsymbol{m}^n$, for every $n \in \mathbf{N}$; consequently $E = Q/A$.

(2) By hypothesis $X = A : (A : X)$, hence $X = \bigcap_{X \subseteq qA} qA \cap A$. Since A/X has simple essential socle, there must exist $q_0 \in Q$ such that $X = q_0 A \cap A$. □

We will first assume that A is a fractional ideal of R, since in this case the characterization of A-divisorial noetherian domains closely resembles Theorems A and 2.1 stated in Section 2.

Theorem 3.2. *Let R be a local noetherian domain. Assume that A is a fractional ideal of R such that $A : A = R$. The following are equivalent:*

1. *R is A-divisorial.*
2. *Q/A is the injective envelope of the simple module $R/\boldsymbol{m}$.*
3. *Q/A is injective.*
4. *R has Krull dimension one and $(A : \boldsymbol{m})/\boldsymbol{m}$ is simple.*
5. *Q/A is cocyclic.*
6. *R is A-reflexive.*

Proof. (1) ⇒ (2). By Lemma 3.1.

(2) ⇒ (3). Obvious.

(3) $\Rightarrow$ (4). Our proof is similar to the proof of Theorem 5 in [17]. Assume that Q/A is injective and let Y be an indecomposable summand of Q/A. Then $Y = E_R(R/P)$ for a nonzero prime ideal P of R. By results in [16], a copy of the quotient field of R/P is contained in $Y \cap \mathrm{Ann}_{Q/A} P$. Now $\mathrm{Ann}_{Q/A} P = (A : P)/A$ is a finitely generated R/P-module, then the quotient field of R/P is finitely generated over R/P and thus P is the maximal ideal $\boldsymbol{m}$. We conclude that Q/A is a direct sum of copies of the module $E = E_R(R/\boldsymbol{m})$. If P' is any nonzero prime ideal of R, tensoring by $R_{P'}$ the exact sequence

$$0 \to A \to Q \to Q/A \to 0$$

we get $E \otimes R_{P'} \neq 0$, since A is finitely generated. Thus P' coincides with the maximal ideal of R, hence R has Krull dimension one. We can now apply Theorem. 15.5 in [19], to get that $(A : \boldsymbol{m})/\boldsymbol{m}$ is simple.

(4) $\Rightarrow$ (1). By Theorem 15.5 in [19].

(2) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (4). Without loss of generality we can assume that A is an ideal of R and that $Q/A \subseteq E = E_R(R/\boldsymbol{m})$. We have to prove that R has Krull dimension one. Let P be a nonzero prime ideal of R and let $0 \neq r \in P$ be such that $r^{-1} \notin A$. Then $\mathrm{Ann}_R(r^{-1} + A) = rA \subseteq P$ and, since $r^{-1} + A \in Q/A \subseteq E$, there exists $n \in \mathbf{N}$ such that $\boldsymbol{m}^n \subseteq P$, i.e. P is maximal.

(6) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (6). By Theorem 3.6 in [6] it is enough to show that $\mathrm{Ext}^1_R(I, A) = 0$, for every ideal I of R. Applying the functor $\mathrm{Hom}_R(I, -)$ to the exact sequence $0 \to A \to Q \to Q/A \to 0$, we obtain

$$0 \to \mathrm{Hom}_R(I, A) \to \mathrm{Hom}_R(I, Q) \xrightarrow{\phi} \mathrm{Hom}_R(I, Q/A) \to \mathrm{Ext}^1_R(I, A) \to 0.$$

We have already proved that, if condition (1) holds, then Q/A is injective, hence every element in $\mathrm{Hom}_R(I, Q/A)$ is the multiplication by an element $q + A \in Q/A$. Thus ϕ is surjective and $\mathrm{Ext}^1_R(I, A) = 0$. □

Remark 6. The preceding Theorem is a generalization of Theorem 15.5 in [19] which is stated in the hypothesis of Krull dimension one.

In [19], an ideal A satisfying the equivalent conditions of Theorem 3.2 is called a *canonical ideal* and it is proved that two canonical ideals are isomorphic.

Throughout the remaining of this Section $\hat{R}$ will denote the completion of R in the $\boldsymbol{m}$-adic topology and $Q(\hat{R})$ will denote the total ring of fractions of $\hat{R}$. We recall that every nonzero element of R is regular in $\hat{R}$, hence $Q\hat{R} \subseteq Q(\hat{R})$.

Lemma 3.3. *Let R be a local noetherian A-divisorial domain, where A is an R-submodule of Q with $A : A = R$. If E is the injective envelope of the simple module $R/\boldsymbol{m}$, then:*

1. $\mathrm{Hom}_R(A, E) \cong Q\hat{R}/\hat{R}$.
2. $\mathrm{Hom}_R(Q, E) \cong Q\hat{R}$.
3. $Q(\hat{R}) \cong Q\hat{R}$.

Proof. (1). By Lemma 3.1 (1) we can identify E with Q/A. Consider $f \in \mathrm{Hom}_R(A, E)$ and let $X = \mathrm{Ker}\, f$. By Lemma 3.1 (2), A/X is isomorphic to $(q_0^{-1}A + A)/A$ for some nonzero element $q_0 \in Q$. Since R is A-divisorial, we have that $q_0^{-1}A + A = A : I$ where $I = q_0 R \cap R$, hence $(q_0^{-1}A + A)/A = \mathrm{Ann}_E\, I$. Consider the situation described by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle g} \\ A/X & \xrightarrow[\phi]{\cong} & \mathrm{Ann}_E\, I \end{array}$$

where $\phi \circ \pi(a) = q_0^{-1}a + A$ and g is induced by f. Clearly g is injective and its image is contained in $\mathrm{Ann}_E\, I$. Since $\mathrm{Ann}_E\, I = E_{R/I}(R/\boldsymbol{m})$, it is well known that the endomorphism ring of $\mathrm{Ann}_E\, I$ is $\widehat{R/I} = \hat{R}/I\hat{R}$, (see [23]); thus g is the multiplication by an element $\eta \in \hat{R}$. This means that, for every $a \in A$, $f(a) = \eta(q_0^{-1}a + A)$.

Consider now the homomorphism

$$\psi : Q\hat{R} \to \mathrm{Hom}_R(A, E)$$

defined by $\psi(\eta r^{-1})(a) = \eta(r^{-1}a + A)$, for every $\eta \in \hat{R}$, $0 \neq r \in R$ and $a \in A$. ψ is well defined and the preceding argument shows that ψ is surjective. Assume that $\eta r^{-1} \in Q\hat{R}$ belongs to the kernel of ψ; then $\eta \in \mathrm{Ann}_{\hat{R}}(r^{-1}A)/A$. Now $(r^{-1}A)/A = \mathrm{Ann}_E\, rR$ coincides with the R/rR-injective envelope of the simple module $S \cong R/\boldsymbol{m}$, hence, by Matlis' duality, $\mathrm{Ann}_{\widehat{(R/rR)}}(r^{-1}A)/A = 0$. This means that $\mathrm{Ann}_{\hat{R}}(r^{-1}A)/A = r\hat{R}$. We have thus proved that $\eta r^{-1} \in \hat{R}$. Conversely it is clear that $\hat{R} \subseteq \mathrm{Ker}\, \psi$, thus we conclude that $\mathrm{Hom}_R(A, E) \cong Q\hat{R}/\hat{R}$.

(2). Applying the functor $\mathrm{Hom}_R(-, E)$ to the exact sequence

$$0 \to A \to Q \to Q/A \to 0$$

and recalling that we can identify E and Q/A, we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{R} & \xrightarrow{\alpha} & \mathrm{Hom}_R(Q, Q/A) & \xrightarrow{\sigma} & \mathrm{Hom}_R(A, Q/A) & \longrightarrow & 0 \\ & & {\scriptstyle id}\uparrow & & {\scriptstyle \chi}\uparrow & & {\scriptstyle \psi'}\uparrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \hat{R} & \longrightarrow & Q\hat{R} & \xrightarrow{\pi} & Q\hat{R}/\hat{R} & \longrightarrow & 0 \end{array}$$

where ψ' is the isomorphism defined in (1) and χ is defined as $\chi(\eta r^{-1})(q) = \eta(r^{-1}q + A)$. Thus the restriction of χ to $\hat{R}$ coincides with α and $\sigma \circ \chi = \psi' \circ \pi = \psi$, hence $\mathrm{Hom}_R(Q, Q/A) \cong Q\hat{R}$.

(3). Since E is an $\hat{R}$-module, we have the natural isomorphism

$$\mathrm{Hom}_R(Q, E) \cong \mathrm{Hom}_{\hat{R}}(Q\hat{R}, E).$$

Consider the embedding of $\hat{R}$-modules $0 \to Q\hat{R} \to Q(\hat{R})$. Applying the functor $\mathrm{Hom}_{\hat{R}}(-, E)$ and using (2) above, we obtain

$$\mathrm{Hom}_{\hat{R}}(Q(\hat{R}), E) \to \mathrm{Hom}_{\hat{R}}(Q\hat{R}, E) \cong Q\hat{R}.$$

$\mathrm{Hom}_{\hat{R}}(Q\hat{R}, E)$ is a divisible $\hat{R}$-module (where divisibility is clearly defined with respect to regular elements of $\hat{R}$), hence the same is true for $Q\hat{R}$. This means that for every regular element $\eta \in \hat{R}$, $\eta^{-1} \in Q\hat{R}$, i.e. $Q(\hat{R}) = Q\hat{R}$. □

We can now prove our main result on A-divisorial noetherian domains.

Theorem 3.4. *Let R be a local noetherian A-divisorial domain, where A is an R-submodule of Q with $A : A = R$. Then R has Krull dimension one and A is a fractional ideal of R.*

Proof. Denote by N the multiplicative system of the regular elements of $\hat{R}$. Then $Q(\hat{R}) = \hat{R}_N$ and by Lemma 3.3 (2) and (3), we have $\mathrm{Hom}_{\hat{R}}(\hat{R}_N, E) \cong \hat{R}_N$. Since E is an $\hat{R}$-injective module, it is well known that $\mathrm{Hom}_{\hat{R}}(\hat{R}_N, E)$ is an $\hat{R}_N$-injective module. This implies that the noetherian ring $\hat{R}_N$ is self-injective. By Theorem 24.5 in [8], $\hat{R}_N$ is also artinian, hence it has dimension 0. Since $N = \hat{R} \setminus Z(\hat{R})$, where $Z(\hat{R})$ denotes the set of the zero divisors of $\hat{R}$, we have that the prime ideals of $\hat{R}$ associated to zero are of height 0, hence they are minimal primes. Let $\{p_i\}_{i \le n}$ denote the set of the minimal primes of $\hat{R}$; then $\{p_i\hat{R}_N\}_{i \le n}$ is the set of the maximal ideals of $\hat{R}_N$ and it is routine to check that, in this case, $\hat{R}_N \cong \bigoplus_{i \le n} \hat{R}_{p_i}$. Thus, for every $i \le n$, $\hat{R}_{p_i}$ is a self-injective local artinian ring, since it is a localization and a summand of the self-injective ring $\hat{R}_N$. Moreover, being local, $\hat{R}_{p_i}$ is indecomposable, hence it is isomorphic to the injective envelope of its unique simple module $\hat{R}_{p_i}/p_i\hat{R}_{p_i}$. Now $\mathrm{Hom}_{\hat{R}}(\hat{R}_{p_i}, E)$ is an $\hat{R}_{p_i}$-injective module which is a summand of $\mathrm{Hom}_{\hat{R}}(\hat{R}_N, E) \cong \hat{R}_N \cong \bigoplus_{i \le n} E_{\hat{R}_N}(S_i)$ where $S_i = \hat{R}_N/p_i\hat{R}_N$ are the simple $\hat{R}_N$-modules. Hence

$$(a) \qquad \mathrm{Hom}_{\hat{R}}(\hat{R}_{p_i}, E) \cong \hat{R}_{p_i}.$$

Consider now the quotient field $Q(\hat{R}/p_i)$ of $\hat{R}/p_i$, we have

$$\mathrm{Hom}_{\hat{R}}(Q(\hat{R}/p_i), E) \cong \mathrm{Hom}_{\hat{R}}\left(\frac{\hat{R}_{p_i}}{p_i\hat{R}_{p_i}}, E\right).$$

By (a), $\operatorname{Hom}_{\hat{R}}(\hat{R}_{p_i}/p_i\hat{R}_{p_i}, E) \cong \operatorname{Ann}_{\hat{R}_{p_i}}(p_i\hat{R}_{p_i}) = \operatorname{Soc}(\hat{R}_{p_i})$. As noted above, $\operatorname{Soc}(\hat{R}_{p_i}) \cong \hat{R}_{p_i}/p_i\hat{R}_{p_i}$; moreover we have

$$\frac{\hat{R}_{p_i}}{p_i\hat{R}_{p_i}} \cong \operatorname{Hom}_{\hat{R}}(Q(\hat{R}/p_i), E) \cong \operatorname{Hom}_{\hat{R}/p_i}(Q(\hat{R}/p_i), \operatorname{Ann}_E p_i),$$

where $\operatorname{Ann}_E p_i$ is the $\hat{R}/p_i$-injective envelope of the simple module $\hat{R}/\boldsymbol{m}\hat{R}$. It follows that the local domain $R' = \hat{R}/p_i$ satisfies the condition $\operatorname{Hom}_{R'}(Q(R'), E') \cong Q(R')$, where E' is the R'-injective envelope of the R'-simple module. Proposition 5 in [17], implies then that $\hat{R}/p_i$ has Krull dimension one and thus the same is true for $\hat{R}$, since p_i is a minimal prime of $\hat{R}$. Since R has the same dimension of its completion $\hat{R}$, we finally conclude that R has dimension one. We can now apply Theorem 15.7 in [19]. In fact R has dimension one and, by Lemma 3.1, $E = Q/A$ is an epimorphic image of Q; hence R satisifes condition (6) of the above mentioned Theorem. In the proof of the implication (4) $\Rightarrow$ (6) in that Theorem, it is shown that A has to be finitely generated, or equivalently, that A is a fractional ideal of R. □

Remark 7. Theorem 3.4 implies that a noetherian domain of dimension greater than one cannot be A-divisorial for any $A \subseteq Q$ satisfying $A : A = R$.

If R has Krull dimension one, then any $A \subseteq Q$ such that $A : A = R$ and such that Q/A is cocyclic is necessarily finitely generated. In fact, using the correspondence between divisible submodules of $E_R(R/\boldsymbol{m})$ and strongly unramified extensions of R in Q proved by Theorem 6.8 and Corollary 15.3 in [19], Q/A must coincide with $E_R(R/\boldsymbol{m})$ and thus, arguing as in the last part of the proof of Theorem 3.4, A is finitely generated, hence A is isomorphic to a *canonical ideal*.

Remark 8. It should be pointed out that not every one-dimensional local noetherian domain R admits a canonical ideal. In fact in Theorem 15.7 of [19] it is proved that such an R has a canonical ideal if and only if $Q(\hat{R})$ is a Gorenstein ring of dimension 0; Ferrand and Raynaud in Proposition 3.1, [10] construct an example of a one-dimensional local noetherian domain R such that $Q(\hat{R})$ is not a Gorenstein ring.

We end this Section by showing that, if A is not a fractional ideal of R, then conditions (3) and (5) of Theorem 3.2 are no longer equivalent and they don't imply that R is A-divisorial. First we show that the condition (3) in Theorem 3.2 doesn't imply (1); namely we show that there exist noetherian domains R admitting a submodule A of Q such that $A : A = R$ and Q/A is injective, but R is not A-divisorial.

Example 3.5. Let R be any local complete noetherian domain with maximal ideal $\boldsymbol{m}$. Let A be any R-submodule of Q maximal with respect to the property $1 \notin A$. Then Q/A is cocyclic, hence $Q/A \subseteq E$, where $E = E_R(R/\boldsymbol{m})$. Now $\operatorname{Ann}_R Q/A = 0$, thus, by duality, $Q/A = E$. Let $q \in Q$ be an element of $A : A$, then the map

$$\bar{q} : Q/A \to Q/A$$

defined by $\bar{q}(x + A) = qx + A$, for every $x \in Q$, is an R-homomorphism of Q/A, hence $q \in R$ since $\mathrm{End}_R(Q/A) = \hat{R} = R$. If R has dimension greater than 1, then by Theorem 3.4, R is not A-divisorial.

We show now that, if $A \subseteq Q$ satisfies $A : A = R$, then the condition Q/A cocyclic does not always imply that Q/A is injective. To see this we use the notion of *primarily independence* introduced in [15]. We summarize here the results of [15] which will be needed.

($\cdot$) Theor. 3.9, [15]. Let $(S, \boldsymbol{m})$ be a countable noetherian local domain with maximal ideal $\boldsymbol{m}$; assume that S is excellent, normal and of dimension at least 2. (For instance, let $S = k[x, y]_{(x,y)}$ where k is a countable field). Let $\hat{S}$ denote the $\boldsymbol{m}$-adic completion of S; then there exists an element $\tau \in \boldsymbol{m}\hat{S}$ which is primarily independent over S.

($\cdot$) Proposition 3.4, [15]. Let $(S, \boldsymbol{m})$ be an excellent, normal local noetherian domain of dimension at least 2. Consider $R = S[\tau]_{(\boldsymbol{m},\tau)}$ where $\tau \in \boldsymbol{m}\hat{S}$ is primarily independent over S; then $Q(R) \cap \hat{S} = R$.

Proposition 3.6. *There exists a noetherian local domain R admitting a submodule A of Q with $A : A = R$, such that Q/A is cocyclic but not injective.*

Proof. Let $S = k[x, y]_{(x,y)}$, k a countable field and $\tau \in \boldsymbol{m}\hat{S}$ primarily independent over S. Consider $R = S[\tau]_{(\boldsymbol{m},\tau)}$ and denote by $\boldsymbol{n}$ the maximal ideal of R. Then the completion $\hat{R}$ of R in the $\boldsymbol{n}$-adic topology is isomorphic to $\hat{S}[[t]]$, where t is an indeterminate over $\hat{S}$. Let $\boldsymbol{a}$ be the prime ideal of $\hat{R}$ generated by $t - \tau$; then $\hat{R}/\boldsymbol{a} \cong \hat{S}$. Let $E = E_R(R/\boldsymbol{n})$ be the injective envelope of the simple module $R/\boldsymbol{n}$; it is well known that E is also the $\hat{R}$-injective envelope of $\hat{R}/\boldsymbol{n}\hat{R}$. Consider $D = \mathrm{Ann}_E\, \boldsymbol{a}$; we have that $D \subsetneq E_R(R/\boldsymbol{n})$, hence D is a non-injective cocyclic R-module. We claim that D is an epimorphic image of $Q(R)$. Clearly D is the $\hat{R}/\boldsymbol{a}$-injective envelope of $\hat{R}/\boldsymbol{n}\hat{R}$, hence it is isomorphic, as an $\hat{S}$-module, to $E_{\hat{S}}(S/\boldsymbol{m}) \cong E_S(S/\boldsymbol{m})$. Using the representation of $E_S(S/\boldsymbol{m})$ as a module of inverse polynomials, as described in [22], one gets that $E_S(S/\boldsymbol{m})$ is an S-epimorphic image of a submodule of the quotient field $Q(S)$ of S and thus, by injectivity, it is also an epimorphic image of $Q(S)$. Let $\phi : Q(S) \to E_S(S/\boldsymbol{m})$ be such an S-epimorphism; using the natural isomorphism

$$\mathrm{Hom}_S(Q(S), E_S(S/\boldsymbol{m})) \overset{\chi}{\cong} \mathrm{Hom}_{\hat{S}}(Q(S)\hat{S}, E_S(S/\boldsymbol{m}))$$

one gets an $\hat{S}$-surjection

$$\chi(\phi) : Q(S)\hat{S} \to E_S(S/\boldsymbol{m}).$$

Since $E_S(S/\boldsymbol{m})$ is $\hat{S}$-injective, we also have a surjection

$$\psi : Q(\hat{S}) \to E_S(S/\boldsymbol{m})$$

extending $\chi(\phi)$. Now, since $Q(S) \subseteq Q(R) \subseteq Q(\hat{S})$, the restriction of ψ to $Q(R)$ is a surjection and since $R \subseteq \hat{S}$, it is also an R-epimorphism of $Q(R)$ onto $D \cong E_S(S/\boldsymbol{m})$. Let now $A \subseteq Q(R)$ be such that $D \cong Q(R)/A$ and consider $q \in A : A$. The multiplication by q induces an endomorphism of $D \cong E_S(S/\boldsymbol{m})$, hence $q \in \hat{S} \cap Q(R)$, which coincides with R by Proposition 3.4 in [15]. □

4 Applications and examples

In this Section we give some examples by means of a generalization of the classical $D + \boldsymbol{m}$ construction. Our approach is inspired by the example appearing in [11]. The setting is illustrated by the following:

4.1. Let R_0 be a local domain with quotient field Q and maximal ideal $\boldsymbol{m}_0$; let A be an R_0-submodule of Q such that $A : A = R_0$. Consider $V = Q[[x]]$ and denote by $\boldsymbol{m}$ the maximal ideal of V; we are interested in studying the local domain R defined as follows:

$$R = R_0 + Ax + x^2 V.$$

Denote by $\boldsymbol{n}$ the maximal ideal of R. Then $\boldsymbol{n} = \boldsymbol{m}_0 + Ax + \boldsymbol{m}^2$ and the quotient field $Q(R)$ of R is $Q((x))$.

Lemma 4.2. *Let R_0, A and R be as in* 4.1. *Then $R : \boldsymbol{n} = R_0 + (A : \boldsymbol{m}_0)x + x^2 V$ and if Q/A is a cocyclic R_0-module, then $Q(R)/R$ is a cocyclic R-module.*

Proof. Let $\xi = \sum_{i \geq -k} q_i x^i \in Q((x))$. Then $\xi \in R : \boldsymbol{n}$ if and only if $i \geq 0$, $q_0 \boldsymbol{m}_0 \subseteq R_0$, $q_0 A \subseteq A$ and $q_1 \boldsymbol{m}_0 \subseteq A$; hence $\xi \in R_0 + (A : \boldsymbol{m}_0)x + x^2 V$. The converse is obvious. We have so shown that $Q(R)/R$ has simple socle and we show now that $Q(R)/R$ is an essential extension of its socle. Let $\xi = \sum_{i \geq -k} q_i x^i \in Q((x))\backslash R$. If $k \geq 1$, then $qx^{k+1} \in R$ for every $q \in Q$; hence, if $\alpha \in (A : \boldsymbol{m}_0)\backslash A$, we have that $0 \neq q_{-k}^{-1}\alpha x^{k+1} \in R$ and for some $v \in V$, $\xi q_{-k}^{-1}\alpha x^{k+1} = \alpha x + x^2 v$ is an element of $(R : \boldsymbol{n})\backslash R$. If $k = 0$, then we have either $q_0 \notin R_0$ or $q_0 \in R_0$ and $q_1 \notin A$. In the first case, since $A : A = R_0$, there exists $0 \neq a \in A$ such that $q_0 a \notin A$. Thus, since Q/A is a cocyclic R_0-module, there exists $0 \neq r_0 \in R_0$ such that $q_0 a r_0 \in (A : \boldsymbol{m}_0)\backslash A$. Hence, $\xi a r_0 x \in (R : \boldsymbol{n})\backslash R$. In the second case there is $0 \neq t_0 \in R_0$ such that $q_1 t_0 \in (A : \boldsymbol{m}_0)\backslash A$. Thus $\xi t_0 \in (R : \boldsymbol{n})\backslash R$. If $k \leq -1$, since $\xi \notin R$ it must be $k = -1$ and $q_1 \in Q\backslash A$. Thus there exists $0 \neq r_0 \in R_0$ such that $q_1 r_0 \in (A : \boldsymbol{m}_0)\backslash A$, hence $\xi r_0 \in (R : \boldsymbol{n})\backslash R$. □

We list now some properties of the fractional ideals of R.

Lemma 4.3. *Let R_0, A and R be as in* 4.1.

1. *Let T, T_1 be R_0-submodules of Q, such that $T_1 \supseteq TA$. Then $L = T + T_1 x + \boldsymbol{m}^2$ is a fractional ideal of R and we have:*

$$R : L = (A : T_1) + (A : T)x + \boldsymbol{m}^2,$$

$$R : (R : L) = A : (A : T) + (A : (A : T_1))x + \boldsymbol{m}^2.$$

2. *Write* $Q((x)) = \bigoplus_{i \in \mathbf{Z}} Qx^i$ *(as a* Q*-module) and let* π_i *be the corresponding projections onto* Q.

For every nonzero fractional ideal J *of* R *we can assume that* 0 *is the minimum index such that* $\pi_i(J) \neq 0$. *Hence* J *is either* $\pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$, *or*

$$J = \sum_{\alpha}(i_\alpha + n_\alpha x)R_0 + H_1 x + \boldsymbol{m}^2$$

where $\{i_\alpha\}_{\alpha \in \Lambda}$ *is a generating set for* $\pi_0(J)$*; for every* α, $n_\alpha \in \pi_1(J)$ *is chosen in such a way that* $i_\alpha + n_\alpha x \in J$ *and* $H_1 = \{q \in Q \mid qx \in J\}$.

Proof. (1). L is a fractional ideal of R since $x^2 L \subseteq R$. It is straight-forward to check that the inverse and the double inverse of L are as formulated.

(2) Let J be any nonzero fractional ideal of R. Multiplying by a suitable power of x, we can assume that the minimum value of the nonzero elements of J is exactly 0. This clearly amounts to $\pi_0(J) \neq 0$ and $\pi_i(J) = 0$, for every $i < 0$. If $0 \neq \xi \in J$ has value 0, then $\xi \boldsymbol{m}^2 = \boldsymbol{m}^2 \subseteq J$. Thus we have $J \subseteq \pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$. Assume that $J \subsetneq \pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$ and let $j \in J$; if $\pi_0(j) = 0$, then $j \in H_1 x + \boldsymbol{m}^2$. If $\pi_0(j) \neq 0$, then $\pi_0(j) = \sum_\alpha r_\alpha i_\alpha$ where the $r_\alpha \in R_0$ are almost all zero. Consider the element $j_0 = \sum_\alpha r_\alpha(i_\alpha + n_\alpha x)$, then $j_0 \in J$ and $j - j_0 \in H_1 x + \boldsymbol{m}^2$; hence the claim follows. □

We are now able to exhibit an example of a local domain R such that $Q(R)/R$ is cocyclic, but there exists a non-divisorial finitely generated ideal.

Proposition 4.4. *Let* R_0, A *and* R *be as in* 4.1. *Assume moreover that* R_0 *is noetherian and that* Q/A *is a non-injective cocyclic* R_0*-module. (Take for instance* R_0 *satisfying Proposition 3.6). Then* $Q(R)/R$ *is a cocyclic* R*-module and there exists a non-divisorial finitely generated ideal of* R.

Proof. $Q(R)/R$ is cocyclic by the preceding Lemma. Let E denote the injective envelope of the simple module $S = R_0/\boldsymbol{m}_0$. We show that the hypothesis Q/A non-injective implies the existence of an element $n \in \mathbf{N}$ such that $A : (A : \boldsymbol{m}_0^n) \supsetneq \boldsymbol{m}_0^n$. In fact, $\mathrm{Ann}_{Q/A}\, \boldsymbol{m}_0^n \subseteq \mathrm{Ann}_E\, \boldsymbol{m}_0^n$ and $\mathrm{Ann}_E\, \boldsymbol{m}_0^n$ is the $R_0/\boldsymbol{m}_0^n$-injective envelope of S. Assume that $A : (A : \boldsymbol{m}_0^n) = \boldsymbol{m}_0^n$, for every $n \in \mathbf{N}$; then the annihilator in $R_0/\boldsymbol{m}_0^n$ of $\mathrm{Ann}_{Q/A}\, \boldsymbol{m}_0^n$ is zero. Now $R_0/\boldsymbol{m}_0^n$ is a complete local noetherian ring, hence, by duality, $\mathrm{Ann}_{Q/A}\, \boldsymbol{m}_0^n$ coincides with the $R_0/\boldsymbol{m}_0^n$-injective envelope of S. Thus $\mathrm{Ann}_{Q/A}\, \boldsymbol{m}_0^n = \mathrm{Ann}_E\, \boldsymbol{m}_0^n$, for every $n \in \mathbf{N}$ and we conclude that $Q/A = E$, a contradiction. Let now $I = \boldsymbol{m}_0^n$, where $n \in \mathbf{N}$ is such that $A : (A : \boldsymbol{m}_0^n) \supsetneq \boldsymbol{m}_0^n$. Consider the R-ideal J generated by I, i.e. $J = I + IAx + x^2 V$; J is finitely generated. By Lemma 4.3 (1) $R : (R : J) \supsetneq J$, hence J is not divisorial. □

We illustrate some conditions under which R is divisorial.

Lemma 4.5. *Let R_0, A, R be as in* 4.1. *Assume that R_0 is A-divisorial and that every proper R_0-submodule of Q is embeddable in A. If moreover Q/A is injective, then R is divisorial.*

Proof. Let J be any nonzero fractional ideal of R; assume that $\pi_i(J) = 0$ for every $i < 0$ and $\pi_0(J) \neq 0$. If $J = \pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$ then, by Lemma 4.3 (1) and our hypotheses on R_0, we conclude that J is divisorial. Assume now that $J \subsetneq \pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$. Consider the representation of J as described in Lemma 4.3 (2). Using the injectivity of Q/A, we show that

$$\pi_0(R : J) = A : H_1.$$

In fact, an element of $R : J$ is of the form $q_0 + q_1x + m$ $(m \in \boldsymbol{m}^2)$ with q_0 and q_1 satisfying: $q_0\pi_0(J) \subseteq R_0$, $q_0H_1 \subseteq A$ and $q_0n_\alpha + q_1i_\alpha \in A$, for every α.

Since $A : A = R$ and $H_1 \supseteq \pi_0(J)A$, the first inclusion follows by $q_0H_1 \subseteq A$; thus $\pi_0(R : J) \subseteq A : H_1$. To prove the equality, let $q_0 \in A : H_1$ and consider the elements $z_\alpha = q_0n_\alpha + A \in Q/A$. Let $\phi : \pi_0(J) \to Q/A$ be the map defined by $\phi(i_\alpha) = z_\alpha$ for every $\alpha \in \Lambda$. ϕ is a well defined homomorphism; in fact, if $\sum_\alpha r_\alpha i_\alpha = 0$ with $r_\alpha \in R_0$, then $\sum_\alpha r_\alpha(i_\alpha + n_\alpha x) \in J$, implies that $\sum_\alpha r_\alpha n_\alpha \in H_1$. Hence $q_0(\sum_\alpha r_\alpha n_\alpha) \in A$ and thus $\sum_\alpha r_\alpha z_\alpha = \sum_\alpha r_\alpha(q_0n_\alpha) + A = 0$. By the injectivity of Q/A, there exists an element $q_1 \in Q$ such that $z_\alpha = i_\alpha q_1 + A$, i.e. $-i_\alpha q_1 + q_0n_\alpha \in A$; thus $q_0 - q_1x \in R : J$, hence $q_0 \in \pi_0(R : J)$.

Finally we show that $R : (R : J) = J$. By Lemma 4.3 (1), $R : (R : J) \subseteq \pi_0(J) + \pi_1(J)x + \boldsymbol{m}^2$. Let $d \in R : (R : J) \backslash J$; then $d = i_0 + i_1x + m$, where $i_0 \in \pi_0(J)$, $i_1 \in \pi_1(J)$ and $m \in \boldsymbol{m}^2$. We have $i_0 = \sum_\alpha r_\alpha i_\alpha$, $i_1 = \sum_\alpha t_\alpha n_\alpha + h_1$ $(r_\alpha, t_\alpha$ in R_0 and $h_1 \in H_1)$. Consider $j_0 = \sum_\alpha r_\alpha(i_\alpha + n_\alpha x) + h_1x + m$; then $j_0 \in J$ and $d - j_0 = \sum_\alpha(t_\alpha - r_\alpha)n_\alpha x$. Hence $\sum_\alpha(t_\alpha - r_\alpha)n_\alpha \notin H_1$, since $d - j_0 \notin J$. Recalling that $\pi_0(R : J) = A : H_1$ and that $A : (A : H_1) = H_1$ by hypothesis, we infer that there exists an element $q_0 + q_1x \in (R : J)$ such that $q_0(\sum_\alpha(t_\alpha - r_\alpha)n_\alpha) \notin A$. Since $d \in R : (R : J)$ we must have $d(q_0 + q_1x) \in R$; we also have $j_0(q_0 + q_1x) \in R$. Thus, since $d(q_0 + q_1x) = (d - j_0)(q_0 + q_1x) + j_0(q_0 + q_1x)$, we conclude that $(d - j_0)(q_0 + q_1x) = q_0\sum_\alpha(t_\alpha - r_\alpha)n_\alpha x + m'$ $(m' \in \boldsymbol{m}^2)$ is an element of R. Hence we get the contradiction $q_0\sum_\alpha(t_\alpha - r_\alpha)n_\alpha \in A$. □

In the next two Propositions we assume that R_0 is a noetherian or a valuation domain and we characterize the case in which the corresponding domain R is divisorial.

Proposition 4.6. *Let R_0 be a noetherian local domain, A and R be as in* 4.1. *Then R is a divisorial domain if and only if R_0 is analytically irreducible and A-divisorial.*

Proof. Assume that R is divisorial and let T be any R_0-submodule of Q. Let $L = T + \boldsymbol{m}$; by Lemma 4.3 (1), $R : (R : L) = A : (A : T) + \boldsymbol{m}$. Hence R_0 is A-divisorial and by the results proved in Section 3, we conclude that A is finitely generated. If moreover T is a proper submodule of Q containing R_0, then $A : T \neq 0$ and thus, T is finitely generated. By Theorem 7.1 and its Remark in [19], we conclude that R_0 is analytically irreducible. For the converse it is enough to apply Lemma 3.1 (1) and the preceding Lemma. □

Proposition 4.7. *Let R_0 be a valuation domain, A and R be as in* 4.1. *Then R is a divisorial domain if and only if R_0 is an almost maximal valuation domain and it is A-divisorial, namely $A \cong \boldsymbol{m}_0$.*

Proof. Let R be a divisorial domain. If T is any nonzero fractional ideal of R_0, then, by Lemma 4.3 (1), $A : (A : T) = T$, hence R_0 is A-divisorial. Since $A : A = R_0$, we conclude that $A \cong \boldsymbol{m}_0$ (by Proposition 7.5, [6]). Moreover, by Proposition 2.12 it follows that for every nonzero ideal J of R, R/J is an AB-5* module; hence, in particular, $R/\boldsymbol{m}^2$ is AB-5*. Now $R/\boldsymbol{m}^2$ is isomorphic to $R_0 + Ax$ has an $R[[x]]/(x^2)$-module; hence it is isomorphic, as an R-module, to the idealization $R_0 \ltimes A$ ($R_0 \ltimes A$ is the set of all pairs (r_0, a), $r_0 \in R_0$ and $a \in A$, with operation defined by $(r_0, a)(r_1, a_1) = (r_0 r_1, a r_1 + r_0 a_1)$.) Thus $R_0 \ltimes A$ is AB-5* as a ring and applying Corollary 3.6 in [3], we obtain that $\boldsymbol{m}_0/I\boldsymbol{m}_0$ is linearly compact discrete for every nonzero ideal I of R_0. It follows easily that also R_0/sR_0 is linearly compact discrete for every nonzero $s \in R_0$; hence R_0 is an almost maximal valuation domain. The converse is proved by applying Lemma 4.5, since every proper submodule of Q is clearly a fractional ideal of R_0 and it is well known that if R_0 is an almost maximal valuation domain, then Q/I is injective for every nonzero ideal I of R_0. □

Remark 9. Note that Propositions 4.6 and 4.7 imply that, if R is a local divisorial domain and P is a prime ideal of R, then R/P is not necessarily divisorial. In fact, we can consider R_0 satisfying the statements of the above mentioned Propositions and such that A is a non-principal fractional ideal of R_0. Then the corresponding domain R is divisorial, but letting $P = Ax + \boldsymbol{m}^2$, P is a prime ideal of R and $R/P \cong R_0$ is not a divisorial domain.

5 A-divisorial domains, A a fractional ideal

In this Section we give a characterization of A-divisorial local domains in case A is a fractional ideal. In fact, in this case we are able to generalize the characterization of divisorial domains given by Theorem 2.13. As noted in Section 3, we will consider R-submodules A of Q with endomorphism ring R.

Proposition 5.1. *Let R be a local A-divisorial domain, where A is a fractional ideal of R such that $A : A = R$. Then Q/A is cocyclic and A/X is AB-5* for every nonzero $X \subseteq A$.*

Proof. Q/A is cocyclic by Lemma 4.3 in [6]. Let $\{J_i/X\}_{i \in I}$ be an inverse system of submodules of A/X and let K/X be a submodule of A/X. Considering A-duals and arguing as in the proof of Proposition 2.12, we obtain

$$\bigcap_i (J_i + K) = \left(\bigcap_i J_i\right) + K$$

hence, passing to the images in A/X, we conclude that A/X is AB-5*. □

Lemma 5.2. *Let A be a fractional ideal of the domain R. Then the following are equivalent:*

1. *A/X is AB-5* for every nonzero $X \subseteq A$.*
2. *A/rA is AB-5* for every nonzero $r \in R$.*
3. *R/rR is AB-5* for every nonzero $r \in R$.*
4. *R/I is AB-5* for every nonzero $I \subseteq R$.*

Proof. Clearly we can assume that A is an ideal of R.

(1) ⇒ (2). Obvious

(2) ⇒ (3). Let $0 \neq r \in R$ be non-invertible and let $0 \neq s \in rA$; then $sr^{-1} + sA$ is a nonzero element of the AB-5* module A/sA whose annihilator in R is rA. Hence R/rA is AB-5* and thus its epimorphic image R/rR, is AB-5*, too.

(3) ⇒ (4). Obvious, since every R/I is an epimorphic image of some R/rR.

(4) ⇒ (1). Obvious. □

We can now generalize Theorem 2.13 as follows.

Theorem 5.3. *Let R be a local domain and A a fractional ideal of R such that $A : A = R$. Then R is A-divisorial if and only if Q/A is cocyclic and R/rR is AB-5* for every nonzero $r \in R$.*

Proof. In view of the preceding results it is enough to prove the sufficiency. By Lemma 5.2, R/rR and A/rA are AB-5* modules, for every nonzero $r \in R$; moreover A/rA is cocyclic, since it is isomorphic to $(r^{-1}A)/A \subseteq Q/A$. The hypothesis $A : A = R$ implies that A/rA is a faithful R/rR-module; hence the R/rR-module structure of A/rA gives rise to a non-degenerate bilinear product:

$$\mu : R/rR \times A/rA \to A/rA.$$

We can then apply Theorem 4 in [2], to conclude that there exixts an antiisomorphism between the lattice of submodules of R/rR and the lattice of submodules of A/rA which is induced by the annihilation of submodules via μ.

Let $\bar{R} = R/rR$ and let X/rA be a submodule of A/rA. Then

$$\mathrm{Ann}_{\bar{R}}(X/rA) = \frac{(rA : X) \cap R}{rA}$$

and $(rA : X) \cap R = rA : X$, since $rA \subseteq X$ implies $rA : X \subseteq rA : rA = R$. If $\bar{A} = A/rA$ and J/rR is a submodule of R/rR, we have

$$\mathrm{Ann}_{\bar{A}}(J/rR) = \frac{(rA : J) \cap A}{rA}.$$

Without loss of generality we may assume that A is an ideal of R. Let $0 \neq X \subseteq A$. Consider $0 \neq r \in X$; then $rA \subseteq X$. We have shown above that $(rA : (rA : X)) \cap A = X$. But $X \subseteq A$ implies $A : X \supseteq A : A = R$, hence $A : (A : X) \subseteq A : R = A$. Thus we conclude that $A : (A : X) = X$ for every nonzero submodule X of A, i.e. R is A-divisorial. □

Remark 10. In Section 3 we have characterized the (local) noetherian A-divisorial domains. The problem of characterizing the integrally closed A-divisorial domains seems to be much more difficult. For instance, we don't even have enough information about the integrally closed domains which are A-divisorial for a fractional ideal. In particular we formulate the following question.

Question 5.4. Assume that R is a local integrally closed A-divisorial domain for a fractional ideal A of R with endomorphism ring R. Does it follow that R is a valuation domain?

The same question is also asked in [13]. We believe that this question is closely related to a difficult open problem posed by W. Heinzer in 1968 in [12], namely the problem to decide whether the integral closure of a divisorial domain is a Prüfer domain.

Note that in the case of a local noetherian domain R, we have proved that R is A-divisorial (for a fractional ideal A of R) if and only if Q/A is cocyclic. We don't know whether the same statement holds for integrally closed domains, i.e. we ask the following.

Question 5.5. Assume that R is a local integrally closed domain and A is a fractional ideal of R with endomorphism ring R. Does the condition Q/A cocyclic imply that R is A-divisorial?

Another question was mentioned in Section 2 for divisorial domains. We formulate it now in the case of A-divisorial domains.

Question 5.6. Assume that R is an A-divisorial domain for a fractional ideal A of R with endomorphism ring R. Does it follow that Q/A is an AB-5* module?

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