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Contributions to Nonlinear PDEs arising in Conformal Geometry, Mean Field Games and Choquard models

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Abstract

The goal of this thesis is to analyze certain nonlinear elliptic Partial Differential Equations (briefly PDEs) that arise in Conformal Geometry, Mean Field Games theory, and Choquard models. Our primary focus is on the study of the existence and nonexistence of solutions, analyzing their asymptotic behavior, and examining the occurrence of concentrating phenomena. The contents of this manuscript have been organized into three distinct parts, each one focusing on a specific topic.

The first part of this work deals with some prescribed curvature problems. In particular, in Chapter 1 we address the problem of existence and compactness of entire solutions to the Gaussian curvature equation on $\mathbb{R}^2$ in the case of power-type and sign-changing prescribed curvature; while in Chapter 2 we deal with the corresponding prescribed Q-curvature problem in $\mathbb{R}^4$. While these two issues are undoubtedly interconnected, it is essential to recognize that each possesses distinct features that warrant in-depth analysis and attention. The content of Part I corresponds to the research papers [17, 18].

In the second part, we study second-order ergodic Mean-Field Games systems defined in the whole space $\mathbb{R}^N$ with a coercive potential V and aggregating nonlocal coupling, given in terms of a Riesz-type interaction kernel. In Chapter 3, we prove that the strength of the attractive term and the behavior of the diffusive part interact to produce three distinct regimes for the existence and nonexistence of classical solutions in our MFG system. On the other hand, in Chapter 4, exploiting a variational approach and a concentration-compactness argument, we show that in the vanishing viscosity limit, there is concentration of mass around minima of the potential V. This leads to proving existence of solutions to the potential-free system. The content of Part II corresponds to the research papers [19, 20].

The third part of this thesis focuses on boundary value problems for Choquard equations. More in detail, we investigate the existence of solutions when the domain is an annulus or an exterior domain, considering both Neumann and Dirichlet boundary conditions. The results of Chapter 5 are presented in the work [21].

Contents

Chapter 0

Introduction

This doctoral thesis is devoted to studying specific nonlinear elliptic Partial Differential Equations that arise in the context of Conformal Geometry, Mean Field Games theory, and Choquard models. More in detail, we use methods from Calculus of Variations, Regularity theory, and Optimal Control, to explore fundamental properties of nonlinear elliptic equations, such as the existence of solutions, their qualitative properties, and asymptotic decay. We also investigate concentration, blow-up, and singular phenomena.

The manuscript is split into three parts, each focusing on a specific and distinct topic. This introduction aims to clarify how our problems are closely connected, particularly in terms of the tools and techniques employed to prove our results.

Part I concerns a classical question in Geometric Analysis and more specifically in Conformal Geometry: the *problem of prescribing curvature*. In a nutshell: the task consists of finding whether a given function on a manifold M endowed with a metric g, can be the curvature of a conformal metric $g_u = e^{2u}g$. For a more thorough explanation, please refer to Section I.1. Our focus will be to investigate prescribed curvature problems defined in the whole Euclidean space $\mathbb{R}^2$ and $\mathbb{R}^4$ with the standard flat metric, assuming to have Gaussian curvature, and respectively Q-curvature, of the form $1 - |x|^p$. When dealing with prescribed curvature which is unbounded and changes sign, new challenges arise that cannot be tackled using the same methods as in previous works. A different approach is necessary.

Firstly, I studied existence and compactness of entire solutions to the following prescribed Gaussian curvature equation in $\mathbb{R}^2$

$$-\Delta u = (1 - |x|^p)e^{2u} \tag{1}$$

(refer to work [17]). If u satisfies (1), then the metric $e^{2u}|dx|^2$ has Gaussian curvature equal to $1 - |x|^p$. Using a variational approach together with a blow-up argument, I showed that (1) has solutions with prescribed total curvature equal to

$$\Lambda := \int_{\mathbb{R}^2} (1 - |x|^p)e^{2u}dx \in \mathbb{R},$$

if and only if

$$p \in (0, 2) \qquad \text{and} \qquad (2 + p)\pi \leq \Lambda < 4\pi$$

and I proved that such solutions remain compact as $\Lambda \to \bar{\Lambda} \in [(2 + p)\pi, 4\pi)$, while they produce a spherical blow-up as $\Lambda \uparrow 4\pi$.

As the Q-curvature is a higher-order equivalent of the Gaussian curvature, I have investigated also the related prescribed Q-curvature problem, that is

$$\begin{cases} \Delta^2 u = (1-|x|^p)e^{4u}, & \text{in } \mathbb{R}^4 \\ \Lambda := \int_{\mathbb{R}^4}(1-|x|^p)e^{4u}dx < \infty \end{cases} \tag{2}$$

(see paper [18]). I showed that for every polynomial P of degree 2 such that $\lim\limits_{|x|\to+\infty} P = -\infty$, and for every $\Lambda \in (0, 16\pi^2)$, there exists at least one solution to problem (2) which assume the form $u = w + P$, where w behaves logarithmically at infinity. Conversely, I proved that all solutions to (2) have the form $v + P$, where

$$v(x) = \frac{1}{8\pi^2}\int\limits_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy$$

and P is a polynomial of degree at most 2 bounded from above. Moreover, if u is a solution to (2), it has the following asymptotic behavior

$$u(x) = -\frac{\Lambda}{8\pi^2}\log|x| + P + o(\log|x|), \quad \text{as } |x| \to +\infty.$$

In the following two parts of the thesis, we shifted our attention to some nonlinear equations featuring nonlocal interactions. Over the last few years, the problem of modeling the collective behavior of a large number of interacting individuals has gained a lot of attention in the mathematical community. As one can easily guess, for simple systems it is possible to analyze each path separately and describe their behavior mathematically. However, as the number of individuals grows, the system becomes exceedingly complex, and tracking all the components becomes an impossible task. The key idea for addressing these challenges consists of using averaged information about the system. As a result, studying nonlinear equations that involve nonlocal interactions, has started to emerge as a new paradigm for modeling the collective behavior of many-body systems.

Part II focuses on a nonlocal problem arising from Mean Field Games. This recent theory describes Nash equilibria of differential games, with a very large number of identical and rational infinitesimal players aiming at minimizing a certain common cost by anticipating the distribution of the overall population. Please refer to Section II.1 for a general presentation of the theory of Mean-Field Games and its developments.

In particular, we consider second-order ergodic Mean-Field Games systems defined in the whole space $\mathbb{R}^N$, with a coercive potential V, and aggregating nonlocal coupling given in terms of a Riesz interaction kernel. Equilibria solve the following system of PDEs where a Hamilton-Jacobi-Bellman equation is combined with a Kolmogorov-Fokker-Planck equation for the mass distribution:

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - \int_{\mathbb{R}^N} \frac{m(y)}{|x-y|^{N-\alpha}}dy \\ -\Delta m - \operatorname{div}(m\nabla u(x)\,|\nabla u(x)|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{3}$$

In this setting, every player of the game is attracted to regions where the population is highly distributed, and the external potential V discourages agents from being far away from the origin. Due to the interplay between aggregating forces (described in terms of the Riesz attractive term and the coercive potential V) and dissipation (caused by the diffusive term in the system), we will obtain three different regimes for the existence and nonexistence of classical solutions to the MFG system (3). By means of a Pohozaev-type identity, we prove nonexistence of regular solutions to the MFG system without potential in the Hardy-Littlewood-Sobolev-supercritical regime. On the other hand, using a fixed point argument, we show the existence of classical solutions in the Hardy-Littlewood-Sobolev-subcritical regime at least for masses smaller than a given threshold value. In the mass-subcritical regime, we show that this threshold can be taken to be $+\infty$. Refer to work [20] for more details.

Moreover, considering the MFG system (3) with a small parameter $\varepsilon > 0$ in front of the Laplacian, I investigated concentration phenomena in the vanishing viscosity limit. More in-depth, assuming some restrictions on the strength of the attractive nonlocal term depending on the growth of the Hamiltonian, I studied the asymptotic behavior of solutions as $\varepsilon \to 0$. First, using a variational approach and a concentration-compactness argument, I obtained existence of classical solutions to potential-free MFG systems with Riesz-type coupling. Secondly, as expected when the diffusion becomes negligible, I proved concentration of mass around minima of the potential V. See paper [19].

Finally, in **Part III** of this dissertation we address boundary value problems for the Choquard equation. This nonlocal nonlinear PDE appears in physical models of multi-particle systems. At macro scales, the Choquard equation can be used to model boson stars and even the formation of black holes and dark matter galactic halos. At atomic scales, it was used by Prof. Roger Penrose to relate quantum mechanics and general relativity. Over the past decade, Choquard-type equations and other classes of nonlinear equations with nonlocal interactions, have emerged as a mathematical framework for modeling the behavior of self-interacting many-body systems at various scales: from atoms and molecules to flock patterns in animal swarms, up to the formation of stars and galaxies.

In [21], we consider the following nonlinear Choquard equation

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \qquad \text{in } \Omega \subset \mathbb{R}^N,$$

where $N \geq 2$, $p \in (1, +\infty)$, $V(x)$ is a continuous radial function such that $\inf_{x\in\Omega} V > 0$ and I_α is the Riesz potential of order $\alpha \in (0, N)$. Assuming Neumann or Dirichlet boundary conditions, we prove the existence of a positive radial solution to the corresponding boundary value problem when Ω is an annulus, or an exterior domain of the form $\mathbb{R}^N \setminus \overline{B_a(0)}$. We provide also a nonexistence result, that is if $p \geq \frac{N+\alpha}{N-2}$ the corresponding Dirichlet problem does not have any nontrivial regular solution in strictly star-shaped domains. Finally, when considering annular domains, letting $\alpha \to 0^+$ we obtain an existence result for the corresponding *local* problem with power-type nonlinearity.

0.1 Organization of the thesis

Each chapter of this thesis corresponds to a different paper as follows.
Part I:

- Bernardini C.; *Existence and Compactness of Conformal Metrics on the Plane with Unbounded and Sign-Changing Gaussian Curvature*, Vietnam J. Math. **51**, 463–487 (2023) doi:10.1007/s10013-021-00540-5.

- Bernardini C.; *Existence and asymptotic behavior of non-normal conformal metrics on $\mathbb{R}^4$ with sign-changing Q-curvature*, to appear in Commun. Contemp. Math. (2022) doi:10.1142/S0219199722500535.

Part II:

- Bernardini C., Cesaroni A.; *Ergodic Mean-Field Games with aggregation of Choquard-type,* J. Differential Equations **364**, 296-335 (2023) doi:10.1016/j.jde.2023.03.045.

- Bernardini C.; *Mass concentration for Ergodic Choquard Mean-Field Games,* (2022) submitted (preprint ArXiv: 2212.00132)

Part III:

- Bernardini C., Cesaroni A.: *Boundary value problems for Choquard equations*, (2023) submitted (preprint Arxiv 2305.09043)

Finally, during my Ph.D. studies, I had the opportunity to work on a slightly different topic concerning elliptic regularity. This issue is not contained in the present manuscript, we recall briefly the main result obtained, and we refer to the original paper for a complete treatment of the problem.

- Bernardini C., Vespri V., Zaccaron M.: *A note on Campanato's L^p-regularity with continuous coefficients*. Eurasian Math. Journal **13** (2022) no.4, 44–53.

We consider local weak solutions of elliptic equations in variational form with data in L^p. We refine the classical approach due to Campanato and Stampacchia and we prove the L^p-regularity for the solutions assuming the coefficients are merely continuous. This result shows that it is possible to prove the same sharp L^p-regularity results that can be proved via the classical singular kernel approach also with the variational regularity approach introduced by De Giorgi.

Part I

Prescribed curvature problems in Conformal Geometry

Introduction

In the first part of this thesis, we investigate some nonlinear Partial Differential Equations related to curvature invariants in Conformal Geometry. A model of such differential equations is the prescribed Gaussian curvature equation under conformal change of metrics, or analogously the prescribed Q-curvature equation. In what follows, we provide some preliminary notions and facts about the geometric quantities arising in the treatment. Afterwards, we give a brief introduction to our problems.

I.1 Prescribing curvature in conformal geometry

Let (M, g) be a 2-dimensional Riemannian manifold with Gaussian curvature K_g. We consider a metric g_u conformal to g, that is $g_u := e^{2u}g$ for some function $u \in C^\infty(M)$. If we denote by K_{g_u} the Gaussian curvature of (M, g_u), the *Gaussian Curvature Equation* reads as follows

$$-\Delta_g u + K_g = K_{g_u} e^{2u} \tag{I.1}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric g. Notice that for $n = 2$ it is defined as

$$\Delta_g := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_i} \right)$$

and it holds

$$\Delta_{g_u} \varphi = e^{-2u} \Delta_g \varphi, \qquad \text{for all } \varphi \in C^\infty(M),$$

hence Δ_g is a *conformally covariant operator*. Equation (I.1) provides a relation between K_g and K_{g_u}, so it shows how the Gaussian curvature behaves under conformal change of metrics. Moreover, identity (I.1) can be used to prove that the total Gaussian curvature is preserved under conformal change of metrics. More in detail, if M is closed (compact without boundary) and orientable, we have

$$\int_M K_g \, d\,\mathrm{vol}_g = \int_M K_{g_u} \, d\,\mathrm{vol}_{g_u}$$

namely the total Gaussian curvature is a *global conformal invariant*, and using the Gauss-Bonnet Theorem it holds

$$\int_M K_g \, d\,vol_g = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M. A classical problem in Conformal Geometry is the **problem of prescribing Gaussian curvature**: that is finding whether a given smooth function K on the manifold (M, g), can be the Gaussian curvature of a conformal metric g_u. Therefore, the question concerns the set of solutions u to

$$-\Delta_g u + K_g = K e^{2u}, \quad \text{in } M. \tag{I.2}$$

In other words, if u solves (I.2), then the conformal metric $g_u = e^{2u}g$ has Gaussian curvature equal to K.

The case when the surface M is the two-dimensional sphere $\mathbb{S}^2$ with the standard *round metric* $g_{\mathbb{S}^2}$, is the well-known Nirenberg problem (raised by Nirenberg in the years 1969-1970). Fundamental progress on the Nirenberg problem was made by Kazdan-Warner [118, 119], Aubin [7], Chang-Yang [49–51], Chang-Gursky-Yang [47], Chen-Ding [56] and many others. More broadly, in [119] Kazdan and Warner provided some necessary and sufficient conditions for a smooth function K on a given compact 2-manifold to be the Gaussian curvature of some metric, their results depended on the sign of the Euler characteristic of the consider manifold.

On the other hand, if $(M, g) = (\mathbb{R}^2, |dx|^2)$ we have $K_g = 0$ and Δ_g is the classical Laplace operator, so (I.2) reduces to

$$-\Delta u = Ke^{2u}, \quad \text{in } \mathbb{R}^2. \tag{I.3}$$

The first problem we investigate in this thesis deals with a specific instance of equation (I.3). Therefore, we briefly focus on the literature that pertains to this specific situation. The study of equation (I.3) started with the work of Liouville [138] who considered the case $K = 1$ (so (I.3) is called *Liouville equation* when $K = const. > 0$) and proved that any solution can be given by

$$u(\zeta, \xi) = \log\left(\frac{2|f'(z)|}{1 + |f(z)|^2}\right), \qquad z = \zeta + \xi i \in \mathbb{C}.$$

Later around the thirties Ahl'fors [4] studied solutions to equation (I.3) when K is a negative constant, proving that in this case (I.3) does not have any solution in the entire space $\mathbb{R}^2$ (see also the works of Wittich [183], Sattinger [174] and Oleinik [161]). The first existence result for solutions to equation (I.3) on the entire plane, when the function K is nonpositive and satisfies further growth conditions at infinity, was given by Ni [159] and it was then refined by McOwen [147] using a weighted Sobolev space approach. Later, a complete classification of all possible solutions to (I.3) in some important cases when K is nonpositive was obtained in [62–65]. If K is a positive constant Chen and Li [57] studied the following problem

$$\begin{cases} -\Delta u = Ke^{2u} & \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{2u} dx < \infty, \end{cases} \tag{I.4}$$

proving that every solution to (I.4) is radially symmetric with respect to some point in $\mathbb{R}^2$. In particular, all such solutions have the form

$$u(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2} - \log \sqrt{K}, \tag{I.5}$$

where $\lambda > 0$ and $x_0 \in \mathbb{R}^2$. On the other hand, if $K(x)$ is positive in some region, under suitable assumptions on the behavior of K at infinity, existence of solutions to equation (I.3) has been studied by McOwen [148], Aviles [8] Cheng and Lin [59, 60, 62] and many others. Finally, the study of compactness of solutions to (I.3) started with the seminal paper of Brézis and Merle [32], which led to a broader study, both in dimension 2 (see for instance the work of Li and Shafrir [129]), greater than 2 (using powers of the Laplacian, or GJMS-operators see e.g. [82, 83, 140, 143, 158]), or in dimension 1 (using the 1/2-Laplacian see e.g. [79, 80]).

Given a 4-dimensional Riemannian manifold M endowed with a metric g, the Q-curvature Q^4_g and the Paneitz operator P^4_g are defined as follows

$$Q^4_g := -\frac{1}{6}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right)$$

$$P^4_g := \Delta_g^2 + \mathrm{div}_g\left(\frac{2}{3}R_g g - 2\mathrm{Ric}_g\right) d,$$

where R_g denotes the scalar curvature, Ric_g the Ricci tensor, Δ_g the Laplace-Beltrami operator and d is the differential (acting on functions). These definitions have been introduced by Branson [25], Branson-Ørsted [28] and Paneitz [162], and then generalized to higher order Q-curvatures Q^{2m}_g and Paneitz operators P^{2m}_g on a $2m$-dimensional Riemannian manifold (M, g) (see [26, 27, 105] and also [87, 88]). Even if explicit formulas for P^{2m}_g and Q^{2m}_g in a general manifold do not exist when $2m > 4$, a formal definition is given in [105] (based on the work of Fefferman-Graham [86]).

The Paneitz operator and the Q-curvature are related by a generalized version of the Gauss identity, indeed they satisfy

$$P^{2m}_g u + Q^{2m}_g = Q^{2m}_{g_u} e^{2mu}$$

(compare with identity (I.1) when $m = 1$). Roughly speaking, the Paneitz operator can be seen as a higher-order Laplace-Beltrami operator and the Q-curvature is a sort of higher-order counterpart of the Gaussian curvature, this is also pointed out by the fact that in dimension 2 we have $P^2_g = -\Delta_g$ and $Q^2_g = K_g$.

The study of the Paneitz operator and Q-curvature has gained a lot of attention in conformal geometry due to their covariant properties. The Paneitz operator P^{2m}_g satisfies

$$P^{2m}_{g_u}(\varphi) = e^{-2mu} P^{2m}_g(\varphi) \quad \text{for all } \varphi \in C^\infty(M),$$

hence P^{2m}_g is *conformally covariant*. Moreover, the total Q-curvature is a *global conformal invariant*, namely if M is closed and $g_u = e^{2u} g$, we have

$$\int_M Q^{2m}_{g_u} d\,\mathrm{vol}_{g_u} = \int_M Q^{2m}_g d\,\mathrm{vol}_g$$

and this integral gives information on the topology of the manifold, indeed using the Gauss-Bonnet-Chern's Theorem [66] on a locally conformally flat closed manifold, we have

$$\int_M Q^{2m}_g d\,\mathrm{vol}_g = (2m-1)!\mathrm{vol}(\mathbb{S}^{2m})\frac{\chi(M)}{2}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. As above mentioned in the 2-dimensional case, also in the higher dimensional case we can investigate the **problem of prescribing Q-curvature**: given a smooth function K on (M, g), it corresponds to find solutions to the following equation

$$P^{2m}_g u + Q^{2m}_g = K e^{2mu}, \quad \text{in } M \tag{I.6}$$

where $K = Q^{2m}_{g_u}$ is the *prescribed* Q-curvature of the conformal metric $g_u = e^{2u} g$. When K is constant Chang and Yang [52] gave a partial affirmative answer to the question, assuming that the total Q-curvature of the metric g is less than $16\pi^2$ namely $\int_M Q^4_g d\mathrm{Vol}_g < 16\pi^2$ and the Paneitz operator is a positive operator whose kernel only consists of the constant functions. In view of a result of Gursky [109], the latter hypothesis is satisfied whenever $\int_M Q_g^4 d\mathrm{Vol}_g > 0$ and provided (M,g) is of positive Yamabe-type (this means that there is a conformal metric with positive constant scalar curvature). The same result in higher dimensions was later derived by Brendle [29] via a flow approach, again in the "subcritical" case, which precisely rules out the case when (M,g) is conformal to the standard sphere. The result of Chang-Yang has been extended recently by Djadli and Malchiodi [82] to the case where the kernel of P_g^4 only consists of the constant functions and $\int_M Q_g^4 d\mathrm{Vol}_g \neq 16k\pi^2$ for $i \in \mathbb{Z}^+$. Finally, Malchiodi and Struwe [140] studied a natural counterpart of the Nirenberg problem, namely to prescribe the Q-curvature of a conformal metric on the standard S^4 as a given function f. Their approach uses a geometric flow within the conformal class, which either leads to a solution to our problem as, in particular, in the case when $f \equiv const.$ or otherwise induces a blow-up of the metric near some point of S^4.

If we consider the Euclidean space $\mathbb{R}^{2m}$ endowed with the standard Euclidean metric $|dx|^2$ (flat metric) we have $P^{2m}_{|dx|^2} = (-\Delta)^m$ and $Q^{2m}_{|dx|^2} \equiv 0$, so equation (I.6) becomes

$$(-\Delta)^m u = Ke^{2mu}, \quad \text{in } \mathbb{R}^{2m}. \tag{I.7}$$

Due to its geometric meaning, the constant Q-curvature case for equation (I.7) has been extensively studied (see e.g. [46, 53, 57, 133, 142, 181] and the references therein). More in detail, if we consider the following equation

$$(-\Delta)^m u = (2m-1)!e^{2mu} \quad \text{in } \mathbb{R}^{2m} \tag{I.8}$$

and we assume that the volume is finite, that is $V := \int_{\mathbb{R}^{2m}} e^{2mu} dx < \infty$, it is well known that the function

$$u_{\lambda,x_0}(x) := \log\left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right) \tag{I.9}$$

solves (I.8) with $V = |\mathbb{S}^{2m}|$ for every $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$. Solutions of the form (I.9) are called *standard (or spherical) solutions* in the literature. Notice that they originate as pull-back under the stereographic projection of round metrics on $\mathbb{S}^{2m}$ (by *round* we mean that they are conformally diffeomorphic to the standard metric). There is no loss of generality in the particular choice of the constant K, we fixed $K \equiv (2m-1)!$ since it is the Q-curvature of the round sphere $(\mathbb{S}^{2m}, g)$. On this subject, we recall that Chang-Yang [53] showed that the round metrics are the only metrics on the $2m$-dimensional sphere having Q-curvature equal to $(2m-1)!$. As mentioned above, Chen and Li [57] classified all solutions to (I.8) when $m = 1$, proving that every solution is spherical. On the other hand, in higher-dimension non-spherical solutions do exist, Chang and Chen [46] proved that for any $m \geq 2$ and every $V \in (0, \mathrm{vol}(S^{2m}))$ there exists at least one non-spherical solution to (I.8), moreover Lin [133] in the case $m = 2$ and Martinazzi [142] for $m \geq 2$, classified all solutions to (I.8). We finally mention that the analogous problem in odd dimension has gained a lot of attention in recent years.

I.2 Our problem: sign-changing unbounded curvature.

The problem which we address in Chapter 1 deals with the following prescribed Gaussian curvature equation in $\mathbb{R}^2$

$$-\Delta u = (1 - |x|^p)e^{2u}, \tag{I.10}$$

where p is a real positive value.

First, we prove that equation (I.10) has solutions with prescribed total curvature equal to $\Lambda := \int_{\mathbb{R}^2}(1-|x|^p)e^{2u}dx \in \mathbb{R}$, if and only if

$$p \in (0,2) \qquad \text{and} \qquad (2+p)\pi \leq \Lambda < 4\pi.$$

More in detail, exploiting some estimates on the asymptotic behavior of solutions at infinity and a Pohozaev-type identity, we get a nonexistence result when $p > 0$ and $\Lambda < (2+p)\pi$ or $\Lambda \geq 4\pi$ and for every value of Λ when $p \geq 2$ (refer to Theorem 1.1.1 below). Then, we use a variational approach due to Chang and Chen (see Theorem 2.1 in [46], where, under suitable assumptions on the curvature K, they show existence of at least one solution to equation $(-\Delta)^{\frac{n}{2}}u = K(x)e^{nu}$ in $\mathbb{R}^n$) to prove existence of a radial solution u_λ to the perturbed problem

$$-\Delta u_\lambda = (\lambda - |x|^p)e^{-|x|^2}e^{2u_\lambda} \quad \text{on } \mathbb{R}^2,$$

such that

$$\Lambda = \int_{\mathbb{R}^2}(\lambda - |x|^p)e^{-|x|^2}e^{2u_\lambda}dx$$

(see Proposition 1.4.1 for further details). This result, together with a blow-up argument, will allow us to prove the existence part (see Theorem 1.1.2).

Furthermore, we show that such solutions remain compact as $\Lambda \to \bar{\Lambda} \in [(2+p)\pi, 4\pi)$, while they produce a spherical blow-up as $\Lambda \uparrow 4\pi$ (Theorem 1.1.3 and Theorem 1.1.4). The proof of the first fact is based on a blow-up analysis, the Kelvin transform and on quantization of the total curvature (using [144, Theorem 2]); while the second one relies on the fact that if $\Lambda \uparrow \Lambda_{\text{sph}}$, we could have only two cases: loss of curvature at infinity, or loss of compactness; we get that the second case occurs.

In Chapter 2, we consider the following prescribed Q-curvature problem

$$\begin{cases} \Delta^2 u = (1-|x|^p)e^{4u}, & \text{on } \mathbb{R}^4 \\ \Lambda := \int_{\mathbb{R}^4}(1-|x|^p)e^{4u}dx < \infty \end{cases} \tag{I.11}$$

where $p > 0$ is fixed. Taking advantage of the above-mentioned result by Chang and Chen, suitably adapted to this case, we extend to the non-normal case the existence results in [116]. More in detail, we show that for every polynomial P of degree 2 such that $\lim_{|x|\to+\infty} P = -\infty$, and for every $\Lambda \in (0, 16\pi^2)$, there exists at least one solution to problem (I.11) which assume the form $u = w + P$, where w behaves logarithmically at infinity (refer to Theorem 2.1.1).

Conversely, we prove that all solutions to (I.11) have the form $v + P$, where

$$v(x) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy$$

and P is a polynomial of degree at most 2 bounded from above. Moreover, if u is a solution to (I.11), it has the following asymptotic behavior

$$u(x) = -\frac{\Lambda}{8\pi^2}\log|x| + P + o(\log|x|), \quad \text{as } |x| \to +\infty$$

(see Theorem 2.1.2). The proof of this result is based some suitable upper and lower estimates for the function v, which allow us to obtain a Liouville-type theorem and then prove that the polynomial P is upper-bounded.

Chapter 1

Existence and compactness of conformal metrics on the plane with unbounded and sign-changing Gaussian curvature

1.1 Introduction to the problem and main results

We study existence and compactness of entire solutions of the equation

$$-\Delta u = Ke^{2u}, \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

where $K \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ is a given function. Equation (1.1) is the prescribed Gaussian curvature equation on $\mathbb{R}^2$. This means that if u satisfies (1.1), then the metric $e^{2u}|dx|^2$ has Gaussian curvature equal to K. Equations of this kind also appear in physics, see for example Bebernes and Ederly [14], Chanillo and Kiessling [54] and Kiessling [121].

In this chapter we will focus on a specific problem that arose recently from a work of Borer, Galimberti and Struwe [23] in the context of prescribing Gaussian curvature on 2-dimensional surfaces. More in detail, they studied the following sign-changing prescribed Gaussian curvature problem on a closed, connected Riemann surface (M, g) of genus greater than 1:

$$-\Delta_g u + K_g = (f_0 + \lambda)e^{2u} \qquad \text{on } M, \tag{1.2}$$

under the assumption that $\lambda > 0$ and $f_0 \le 0$ is a smooth non-constant function, which has only non-degenerate maxima ξ_0 with $f_0(\xi_0) = 0$. In the case of sign-changing curvature, the uniqueness of solutions may be lost (compare with [23, Theorem 1.1]), indeed, as already proven by Ding and Liu [81] there exists a value $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ the energy functional E_λ associated to (1.2) admits a local minimizer u_λ and a further critical point u^λ of mountain-pass type. In particular, through a mountain-pass technique and the *monotonicity trick*, Borer, Galimberti and Struwe [23] investigated the blow-up behavior of "large" solutions u^λ as $\lambda \downarrow 0$, and, upon rescaling, they obtained solutions to (1.1) with K either constant or $K(x) = 1 + (Ax, x)$ where $A = \frac{1}{2}Hess_{f_0}(\xi_\infty^{(i)})$ (see [23, Theorem 1.4]). More recently, Struwe [177] obtained a more precise characterization of this "bubbling" behavior: considering a closed surface of genus zero, he proved that all "bubbles" are spherical. This is achieved with the help of a Liouville-type result, in fact he proved that there are no solutions $u \in C^\infty(\mathbb{R}^2)$ to equation

$$-\Delta u = (1 + (Ax, x))e^{2u} \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

(where A is a negative definite and symmetric 2×2 matrix) with $u \le C$, such that the induced metric $e^{2u}|dx|^2$ has finite volume $e^{2u} \in L^1$ and $\int_{\mathbb{R}^2}(1 + (Ax, x))e^{2u}dx \in \mathbb{R}$. In this way, he proved that all blow-ups must be spherical also in the higher genus case.

If we do not assume that the prescribed curvature $f_0 + \lambda$ in the problem (1.2) is smooth, but we only require that $f_0 \in C^{0,\alpha}$ for some $\alpha \in (0, 1]$, then $Hess_{f_0}$ is no more defined and, upon rescaling, instead of (1.3), one might expect to find solutions to

$$-\Delta u = (1 - |x|^p)e^{2u} \quad \text{in} \quad \mathbb{R}^2, \qquad \Lambda := \int_{\mathbb{R}^2} (1 - |x|^p)e^{2u}dx < \infty, \tag{1.4}$$

for some $p > 0$. More precisely, for $p > 0$ we define

$$\Lambda_{\text{sph}} := 4\pi \qquad \text{and} \qquad \Lambda_{*,p} := (2 + p)\pi.$$

(the constant Λ_{sph} is the total curvature of the sphere S^2). Taking advantage of a Pohozaev-type identity, we are able to prove the following non-existence result:

Theorem 1.1.1. *Let $p > 0$ be fixed. For any $\Lambda \in (-\infty, \Lambda_{*,p}) \cup [\Lambda_{\text{sph}}, +\infty)$ problem (1.4) admits no solutions. In particular, for $p \ge 2$, problem (1.4) admits no solutions.*

Following an approach of Hyder and Martinazzi [116], and in sharp contrast with the nonexistence result of Struwe [177], we will show that problem (1.4) has solutions for every $p \in (0, 2)$ and for suitable values of Λ.

Theorem 1.1.2. *Let $p \in (0, 2)$ be fixed. Then for every $\Lambda \in (\Lambda_{*,p}, \Lambda_{\text{sph}})$ there exists a (radially symmetric) solution to problem (1.4). Such solutions have the following asymptotic behavior*

$$u(x) = -\frac{\Lambda}{2\pi} \log|x| + C + O(|x|^{-\alpha}), \quad \text{as } |x| \to \infty, \tag{1.5}$$

for every $\alpha \in [0, 1]$ such that $\alpha < \frac{\Lambda - \Lambda_{*,p}}{\pi}$, and

$$|\nabla u(x)| = O\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to \infty. \tag{1.6}$$

Observe that Theorem 1.1.1 and Theorem 1.1.2 do not cover the case $\Lambda = \Lambda_{*,p}$. In this case (see Proposition 1.3.4 and Lemma 1.3.6) relation (1.5) degenerates to

$$-\frac{\Lambda_{*,p} + o(1)}{2\pi} \log|x| \le u(x) \le -\frac{\Lambda_{*,p}}{2\pi} \log|x| + O(1), \quad \text{as } |x| \to +\infty,$$

which is compatible with the integrability of $(1 - |x|^p)e^{2u}$. We will study the case $\Lambda = \Lambda_{*,p}$ from the point of view of compactness, namely we will show that solutions to (1.4) are compact for Λ away from Λ_{sph}, and blow up spherically at the origin as $\Lambda \uparrow \Lambda_{\text{sph}}$.

Theorem 1.1.3. *Fix $p \in (0, 2)$, let $\{u_k \,|\, k \in \mathbb{N}\}$ be a sequence of solutions to (1.4) with $\Lambda = \Lambda_k \in [\Lambda_{*,p}, \Lambda_{\text{sph}})$ and $\Lambda_k \to \bar{\Lambda} \in [\Lambda_{*,p}, \Lambda_{\text{sph}})$. Then, up to subsequences, $u_k \to \bar{u}$ locally uniformly, where $\bar{u}$ is a solution to (1.4) with $\Lambda = \bar{\Lambda}$.*

Moreover, choosing $\Lambda_k \downarrow \Lambda_{*,p}$ and u_k given by Theorem 1.1.2, we obtain that (1.4) has a solution u also for $\Lambda = \Lambda_{*,p}$ and we have

$$u(x) \le -\frac{\Lambda_{*,p}}{2\pi} \log|x| - (1 + o(1)) \log\log|x|, \quad \text{as } |x| \to +\infty \tag{1.7}$$

and

$$|\nabla u(x)| = O\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to +\infty. \tag{1.8}$$

The proof of Theorem 1.1.3 relies on uniform controls of the integral of the Gaussian curvature at infinity. This is particularly subtle as $\Lambda_k \downarrow \Lambda_{*,p}$ since in this case Ke^{2u_k} is a priori no better than uniformly L^1 at infinity and we could have loss of negative curvature at infinity. This possibility is ruled out with an argument based on the Kelvin transform (see Lemma 1.5.4). Theorem 1.1.3 strongly uses the lack of scale invariance of equation (1.4).

Notice that for the constant curvature case (I.4), solutions given in (I.5) can blow up "spherically", we will show that this is also the case when $\Lambda \uparrow \Lambda_{\text{sph}}$.

Theorem 1.1.4. *Fix* $p \in (0, 2)$*, let* $\{u_k \,|\, k \in \mathbb{N}\}$ *be a sequence of radial solutions to* (1.4) *with* $\Lambda = \Lambda_k \uparrow \Lambda_{\text{sph}}$ *as* $k \to +\infty$*. Then*

$$(1 - |x|^p)e^{2u_k} \rightharpoonup \Lambda_{\text{sph}}\delta_0, \quad \text{as } k \to +\infty,$$

weakly in the sense of measures. Moreover, setting

$$\mu_k := 2e^{-u_k(0)}$$

and

$$\eta_k(x) := u_k(\mu_k x) - u_k(0) + \log 2,$$

we have

$$\eta_k(x) \xrightarrow[k\to\infty]{} \log \frac{2}{1 + |x|^2} \quad \textit{in } C^1_{loc}(\mathbb{R}^2).$$

Notice that all solutions to problem (1.4) are radially symmetric about the origin and monotone decreasing, this follows by Corollary 1 in the work of Naito [157] (to prove Corollary 1 Naito uses an approach based on the maximum principle in unbounded domains together with the method of moving planes).

The outline of this chapter is the following. In Section 1.2 we prove some regularity results for solution to (1.4). In Section 1.3, thanks to some estimates on the asymptotic behavior of solutions at infinity and to a Pohozaev-type identity, we prove Theorem 1.1.1. In Section 1.4, we use a variational approach due to Chang and Chen [46] to prove Proposition 1.4.1, which, together with a blow-up argument will allow us to prove the existence part of Theorem 1.1.2. Section 1.5 is devoted to the proof of Theorem 1.1.3, which is based on a blow-up analysis, the Kelvin transform, and on quantization of the total curvature (using [144, Theorem 2]). Finally, the proof of Theorem 1.1.4 in Section 1.6 is based on the fact that if $\Lambda \uparrow \Lambda_{\text{sph}}$, from Theorem 1.1.1, we could have only two cases: loss of curvature at infinity, or loss of compactness; from Lemma 1.6.1 we get that the second case occurs.

After the completion of the work the author learned that a result of Cheng and Lin (see [60, Theorem 1.1]) implies our Theorem 1.1.2. Their elegant proof is based on a Moser-Trudinger inequality in weighted Sobolev spaces. Our approach to Theorem 1.1.2, only based on ODE methods, is more elementary and will be the basis for the compactness Theorem 1.1.3, therefore we left the statement of Theorem 1.1.2 for completeness.

1.2 Regularity of solutions

Let $p > 0$ be fixed. First of all, we prove some regularity results for solutions to equation

$$-\Delta u = (1 - |x|^p)e^{2u} \quad \text{in } \mathbb{R}^2, \tag{1.9}$$

assuming that $u \in L^1_{loc}(\mathbb{R}^2)$ and $(1-|x|^p)e^{2u} \in L^1(\mathbb{R}^2)$.

In the following, if $\Omega \subseteq \mathbb{R}^2$ is an open set and $s \in \mathbb{R}$ $s \geq 0$, we will denote

$$C^s(\Omega) := \left\{ u \in C^{\lfloor s \rfloor}(\Omega) \;\middle|\; D^{\lfloor s \rfloor} u \in C^{0,s-\lfloor s \rfloor}(\Omega) \right\},$$

and

$$C^s_{loc}(\mathbb{R}^2) := \left\{ u \in C^0(\mathbb{R}^2) \;\middle|\; u|_\Omega \in C^s(\Omega) \text{ for every } \Omega \subset\subset \mathbb{R}^2 \right\}.$$

Proposition 1.2.1. *Let u be a solution to* (1.9)*. Then $u \in W^{2,r}_{loc}(\mathbb{R}^2)$ for $1 < r < \infty$.*

Proof. Following the proof of [115, Theorem 2.1], first we prove that $e^{2u} \in L^q_{loc}(\mathbb{R}^2)$ for any $q \geq 1$. Let $q \geq 1$ be fixed, take $\varepsilon = \varepsilon(q)$ such that $q < \frac{\pi}{2\varepsilon}$, we can find two functions f_1 and f_2 such that $(1-|x|^p)e^{2u} = f_1 + f_2$ and

$$f_1 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \qquad \|f_2\|_{\mathrm{L}^1(\mathbb{R}^2)} < \varepsilon.$$

We define the following

$$u_i(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right) f_i(y)dy, \quad i = 1, 2,$$

and

$$u_3 := u - u_1 - u_2.$$

In this way, u_3 is harmonic and hence $u_3 \in C^\infty(\mathbb{R}^2)$. Differentiating u_1 we obtain

$$\nabla u_1(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} f_1(y)dy$$

using the relation $|x||y|\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right| = |x-y|$ it's easy to prove that ∇u_1 is continuous, hence $u_1 \in C^1(\mathbb{R}^2)$. Concerning u_2 we have

$$\begin{aligned}
\int_{B_R} e^{8qu_2} dx &= \int_{B_R} \exp\left(\int_{\mathbb{R}^2} \frac{8q\|f_2\|}{2\pi} \log\left(\frac{|y|}{|x-y|}\right) \frac{f_2(y)}{\|f_2\|} dy\right) dx \\
&\leq \int_{B_R} \int_{\mathbb{R}^2} \exp\left(\frac{8q\|f_2\|}{2\pi} \log\left(\frac{|y|}{|x-y|}\right)\right) \frac{f_2(y)}{\|f_2\|} dy\, dx \\
&= \frac{1}{\|f_2\|} \int_{\mathbb{R}^2} f_2(y) \int_{B_R} \left(\frac{|y|}{|x-y|}\right)^{\frac{4q\|f_2\|}{\pi}} dx\, dy \leq C,
\end{aligned}$$

by Holder's inequality, we can conclude that $e^{2u} \in L^q_{loc}(\mathbb{R}^2)$ for any $q \geq 1$.

By assumption $p > 0$, so $(1-|x|^p) \in L^r_{loc}(\mathbb{R}^2)$ for every $1 \leq r \leq \infty$; it follows that $-\Delta u = (1-|x|^p)e^{2u} \in L^r_{loc}(\mathbb{R}^2)$ for each $1 \leq r \leq \infty$. By elliptic estimates (refer to [93, Theorem 9.11]), we have

$$u \in W^{2,r}_{loc}(\mathbb{R}^2), \quad \text{for every } r \in (1, \infty),$$

and by the Morrey-Sobolev embedding we get

$$u \in C^{1,\alpha}_{loc}(\mathbb{R}^2), \quad \text{for } \alpha \in (0, 1].$$

□

Proposition 1.2.2. *Let $p > 0$ be fixed and u be a solution of equation* (1.9)*. If $p \notin \mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \cap C^{p+2}_{loc}(\mathbb{R}^2)$, if $p-1 \in 2\mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \cap C^{s+2}_{loc}(\mathbb{R}^2)$ for $s < p$ and if $p \in 2\mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2)$.*

Proof. From Proposition 1.2.1 we have that $u \in W^{2,r}_{loc}(\mathbb{R}^2) \hookrightarrow C^{1,\alpha}_{loc}(\mathbb{R}^2)$ for $1 < r < \infty$ and $\alpha \in (0,1]$. Since $1-|x|^p \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ if $p \notin 2\mathbb{N}$, and belongs to $C^\infty(\mathbb{R}^2)$ if $p \in 2\mathbb{N}$, by bootstrapping regularity we get that $u \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ if $p \notin 2\mathbb{N}$ and $u \in C^\infty(\mathbb{R}^2)$ if $p \in 2\mathbb{N}$. Moreover, we can verify that for $p > 0$

$$1 - |x|^p \in \begin{cases} C^p(\mathbb{R}^2) & \text{if } p \notin \mathbb{N} \\ C^s_{loc}(\mathbb{R}^2) & \text{for } s < p, \text{ if } p-1 \in 2\mathbb{N} \\ C^\infty(\mathbb{R}^2) & \text{if } p \in 2\mathbb{N} \end{cases} .$$

Using Schauder estimates and bootstrapping regularity, we get that if $p \notin \mathbb{N}$ then $u \in C^{p+2}_{loc}(\mathbb{R}^2)$ and if $p-1 \in 2\mathbb{N}$ then $u \in C^{s+2}_{loc}(\mathbb{R}^2)$ for $s < p$. Hence we can conclude that if $p \notin \mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \cap C^{p+2}_{loc}(\mathbb{R}^2)$, if $p-1 \in 2\mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \cap C^{s+2}_{loc}(\mathbb{R}^2)$ for $s < p$ and if $p \in 2\mathbb{N}$ then $u \in C^\infty(\mathbb{R}^2)$. □

All solutions to (1.9) are in fact *normal solutions*, namely if u is a solution to (1.9) then u solves the integral equation

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy + c, \tag{1.10}$$

where $c \in \mathbb{R}$ (for a detailed proof of this fact see the proof of [59, Theorem 2.1]). Moreover, if we have more integrability (namely $\log(|\cdot|)(1-|y|^p)e^{2u} \in L^1(\mathbb{R}^2)$), equation (1.10) is equivalent to

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy + c', \tag{1.11}$$

where $c' \in \mathbb{R}$.

1.3 Nonexistence result

First of all, we can observe that the case $\Lambda < 0$ is not possible (this follows similar to the proof of [141, Theorem 1]).

Proposition 1.3.1. *Let $p > 0$ be fixed, if u is a solution to* (1.4) *then $\Lambda \geq 0$.*

Proof. Assume by contradiction that $\Lambda < 0$, then there exists $r_0 > 0$ such that

$$\int_{B_r} \Delta u \, dx \geq 0, \quad \forall r \geq r_0,$$

hence

$$\int_{\partial B_r} \frac{\partial u}{\partial \nu} d\sigma(x) \geq 0, \quad \forall r \geq r_0.$$

It follows that $\fint_{\partial B_r} u \, d\sigma$ is an increasing function for $r \geq r_0$, and consequently also $\exp\left(\fint_{\partial B_r} u \, d\sigma\right)$ is increasing for $r \geq r_0$. By Jensen inequality we get

$$\exp\left(2 \fint_{\partial B_r} u \, d\sigma\right) \leq \fint_{\partial B_r} e^{2u} d\sigma.$$

It follows that $\fint_{\partial B_r} e^{2u} d\sigma$ must be increasing for $r \geq r_0$, hence integrating $\int_{\mathbb{R}^2} e^{2u} = +\infty$. This leads to a contradiction since in this way Λ cannot be finite. □

In order to prove Theorem 1.1.1 we need the following Pohozaev-type identity and some asymptotic estimates at infinity.

Proposition 1.3.2. *Let $K(x) = 1 - |x|^p$ and u be a solution to the integral equation*

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right) K(y)e^{2u(y)}dy + c, \tag{1.12}$$

for some $c \in \mathbb{R}$, with $Ke^{2u} \in L^1(\mathbb{R}^2)$ and $|\cdot|^p e^{2u} \in L^1(\mathbb{R}^2)$. If

$$\lim_{R\to\infty} R^{2+p} \max_{|x|=R} e^{2u(x)} = 0 \tag{1.13}$$

then denoting by $\Lambda = \int_{\mathbb{R}^2} K(x)e^{2u(x)}dx$, we have

$$\frac{\Lambda}{\Lambda_{\text{sph}}}(\Lambda - \Lambda_{\text{sph}}) = -\frac{p}{2}\int\limits_{\mathbb{R}^2} |x|^p e^{2u(x)}dx. \tag{1.14}$$

Proof. In the spirit of the proof of [184, Theorem 2.1], differentiating equation (1.12) (by Proposition 1.2.2 u is sufficiently regular) and multiplying by x, we obtain

$$\langle x, \nabla u(x)\rangle = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\langle x, x-y\rangle}{|x-y|^2} K(y)e^{2u(y)}dy.$$

Multiplying both sides of the previous one by $K(x)e^{2u(x)}$ and integrating over the ball $B_R(0)$ for $R > 0$, we have

$$\begin{aligned}\int_{B_R} K(x)e^{2u(x)}\langle x, \nabla u(x)\rangle dx &= \\ &= \int_{B_R} K(x)e^{2u(x)}\left[-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\langle x, x-y\rangle}{|x-y|^2}K(y)e^{2u(y)}dy\right]dx.\end{aligned}$$

Integrating by parts the left-hand side:

$$\begin{aligned}\int_{B_R} K(x)e^{2u(x)}\langle x, \nabla u(x)\rangle dx &= \frac{1}{2}\int_{B_R} K(x)\langle x, \nabla e^{2u(x)}\rangle dx = \\ &= -\int_{B_R}\left(K(x) + \frac{1}{2}\langle x, \nabla K(x)\rangle\right)e^{2u(x)}dx + \frac{R}{2}\int_{\partial B_R} K(x)e^{2u(x)}d\sigma = \\ &= -\int_{B_R}\left(K(x) - \frac{p}{2}|x|^p\right)e^{2u(x)}dx + \frac{R}{2}\int_{\partial B_R}(1-|x|^p)e^{2u(x)}d\sigma.\end{aligned}$$

It's easy to see that

$$-\int_{B_R}\left(K(x) - \frac{p}{2}|x|^p\right)e^{2u(x)}dx \xrightarrow[R\to\infty]{} -\Lambda + \frac{p}{2}\int_{\mathbb{R}^2}|x|^p e^{2u(x)}dx$$

and concerning the boundary term, we have

$$\begin{aligned}\frac{R}{2}\int_{\partial B_R}(1-|x|^p)e^{2u(x)}d\sigma &\le \frac{R}{2}\max_{|x|=R} e^{2u(x)}\int_{\partial B_R}(1-|x|^p)d\sigma = \\ &= \pi R^2(1-R^p)\max_{|x|=R} e^{2u(x)} = \pi R^{p+2}\left(\frac{1}{R^p} - 1\right)\max_{|x|=R} e^{2u(x)}\end{aligned}$$

using (1.13) it goes to 0 if $R \to +\infty$. Regarding the right-hand side, we have

$$\begin{aligned}\int_{B_R} K(x)e^{2u(x)} \left[-\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\langle x, x-y\rangle}{|x-y|^2} K(y)e^{2u(y)}dy \right] dx = \\ = \frac{1}{2}\int_{B_R} K(x)e^{2u(x)} \left[-\frac{1}{2\pi}\int_{\mathbb{R}^2} K(y)e^{2u(y)}dy \right] dx \\ + \frac{1}{2}\int_{B_R} K(x)e^{2u(x)} \left[-\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\langle x+y, x-y\rangle}{|x-y|^2} K(y)e^{2u(y)}dy \right] dx.\end{aligned}$$

It follows immediately that

$$\frac{1}{2}\int_{B_R} K(x)e^{2u(x)} \left[-\frac{1}{2\pi}\int_{\mathbb{R}^2} K(y)e^{2u(y)}dy \right] dx \xrightarrow[R\to+\infty]{} -\frac{1}{4\pi}\Lambda^2$$

while, by dominated convergence, the second term goes to

$$-\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\langle x+y, x-y\rangle}{|x-y|^2} K(x)e^{2u(x)}K(y)e^{2u(y)}dy\, dx$$

which, changing variables, it's equal to 0. Finally combining we obtain (1.14). □

In order to study the asymptotic behavior of solution to (1.1), a useful trick is the Kelvin transform. We have the following result.

Proposition 1.3.3. *Let u be a solution to* (1.4) *and assume that* $(1-|x|^p)e^{2u} \in L^1(\mathbb{R}^2)$. *Then, the Kelvin transform of u, namely the function*

$$\tilde{u}(x) = u\left(\frac{x}{|x|^2}\right) - \alpha \log|x|, \qquad \textit{for} \quad x \neq 0, \tag{1.15}$$

where $\alpha := \frac{\Lambda}{2\pi}$, *satisfies*

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) K\left(\frac{y}{|y|^2}\right) \frac{e^{2\tilde{u}(y)}}{|y|^{4-2\alpha}}dy + c,$$

where $K(x) := 1 - |x|^p$, *hence* $\tilde{u}$ *is a solution to*

$$-\Delta\tilde{u}(x) = K\left(\frac{x}{|x|^2}\right)\frac{e^{2\tilde{u}}}{|x|^{4-2\alpha}}.$$

Proof. Following the proof of [116, Proposition 2.2], using (1.10) and then changing variables, we have

$$\begin{aligned}\tilde{u}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|y|}{\left|\frac{x}{|x|^2} - y\right|}\right) K(y)e^{2u(y)}dy - \frac{1}{2\pi}\log|x|\int_{\mathbb{R}^2} K(y)e^{2u(y)}dy + c \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x|\left|\frac{x}{|x|^2} - y\right|}\right) K(y)e^{2u(y)}dy + c \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x||y|\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|}\right) K\left(\frac{y}{|y|^2}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{4-2\alpha}}dy + c = \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) K\left(\frac{y}{|y|^2}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{4-2\alpha}}dy + c\end{aligned}$$

where in the last equality we have used that $|x||y|\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right| = |x-y|$. □

1.3.1 Asymptotic behavior of solutions

Proposition 1.3.4. *Let u be a solution to the integral equation*

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{2u(y)} dy + c,$$

where $c \in \mathbb{R}$, assume $K(y) \leq 0$ for $|y| \geq R_0 > 0$ and $Ke^{2u} \in L^1$. Then we have

$$u(x) \leq -\frac{\Lambda}{2\pi} \log|x| + O(1), \qquad \text{as} \quad |x| \to \infty \tag{1.16}$$

where $\Lambda = \int_{\mathbb{R}^2} Ke^{2u}$.

Proof. Following the proof of [116, Lemma 2.3] and adapting it to the two-dimensional case, we choose x such that $|x| \geq 2R_0$ (assuming $R_0 \geq 2$) and consider $\mathbb{R}^2 = A_1 \cup A_2 \cup A_3$ where

$$A_1 = B_{\frac{|x|}{2}}(x), \qquad A_2 = B_{R_0}(0), \qquad A_3 = \mathbb{R}^2 \setminus (A_1 \cup A_2).$$

It's easy to see that

$$\int_{A_1} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{2u(y)} dy \leq 0$$

because if $y \in A_1$ we have $K(y) \leq 0$ and $\log\left(\frac{|y|}{|x-y|}\right) \geq 0$. For $y \in A_2$ we have $\log\left(\frac{|y|}{|x-y|}\right) = -\log|x| + O(1)$ as $|x| \to +\infty$, hence

$$\int_{A_2} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{2u(y)} dy = -\log|x| \int_{A_2} Ke^{2u} dy + O(1), \quad \text{as } |x| \to +\infty.$$

If $y \in A_3$ we have $|x-y| \leq |x| + |y| \leq |x||y|$, hence $K(y) \log\left(\frac{|y|}{|x-y|}\right) \leq K(y) \log\left(\frac{1}{|x|}\right)$, so we obtain

$$\int_{A_3} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{2u(y)} dy \leq -\log|x| \int_{A_3} Ke^{2u} dy.$$

Finally we have

$$u(x) \leq \frac{1}{2\pi}\left(-\log|x| \int_{A_2 \cup A_3} Ke^{2u} dy\right) + O(1) \leq -\frac{\Lambda}{2\pi} \log|x| + O(1),$$

using the fact that $\int_{A_2 \cup A_3} Ke^{2u} \geq \int_{\mathbb{R}^2} Ke^{2u} = \Lambda$ since $K \leq 0$ in A_1. □

Corollary 1.3.5. *If $p \in (0,2)$, there exist no solutions to* (1.4) *for $\Lambda \geq 4\pi$.*

Proof. Assume that u solves (1.4) for some $\Lambda \geq 4\pi$. By Proposition 1.3.4 u satisfies (1.16), so hypothesis (1.13) in Proposition 1.3.2 is verified and therefore from (1.14) we must have $\Lambda < 4\pi$, a contradiction. □

Lemma 1.3.6. *Fix $p > 0$ and let u be a solution to* (1.4)*. Then, we have*

$$\Lambda \geq \Lambda_{*,p}$$

and

$$u(x) = -\frac{\Lambda + o(1)}{2\pi} \log|x|, \quad \text{as } |x| \to +\infty \tag{1.17}$$

Proof. We first prove the asymptotic estimate (1.17). Writing $u = u_1 + u_2$, where

$$u_2(x) = -\frac{1}{2\pi}\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)|y|^p e^{2u(y)}dy,$$

we obtain

$$u_1(x) = -\frac{\Lambda + o(1)}{2\pi}\log|x|, \quad \text{as } |x| \to +\infty, \tag{1.18}$$

(see the Appendix for a detailed proof of the estimate (1.18)). Let's consider $R \gg 1$ and $|x| \geq R+1$, define

$$h(R) := \frac{1}{2\pi}\int_{B_R^c} |y|^p e^{2u} dy,$$

it's easy to see that $h(R) \to 0$ as $R \to +\infty$, so we can write $h(R) = o_R(1)$. We have

$$-u_2(x) = \int_{B_R^c} h(R)\log\left(\frac{1}{|x-y|}\right)\chi_{|x-y|\leq 1}d\mu(y), \quad d\mu(y) = \frac{|y|^p e^{2u}}{\int_{B_R^c}|y|^p e^{2u}dy}dy.$$

By Jensen's inequality and Fubini's theorem we get

$$\int_{R+1<|x|<2R} e^{-2u_2}dx \leq \int_{B_R^c}\int_{R+1<|x|<2R}\left(1 + \frac{1}{|x-y|^{2h(R)}}\right)dx\, d\mu(y) \leq CR^2.$$

Hence using Holder's inequality and the previous one

$$R^2 \approx \int_{R+1<|x|<2R} e^{u_2}e^{-u_2}dx \leq CR\left(\int_{R+1<|x|<2R} e^{2u_2}dx\right)^{1/2}. \tag{1.19}$$

If we assume by contradiction that $\frac{\Lambda}{\pi} \leq p$, then $|y|^p e^{2u_1} \geq \frac{1}{|y|}$ for $|y|$ sufficiently large, therefore

$$o_R(1) = \int_{R+1<|x|<2R} |x|^p e^{2u_1}e^{2u_2}dx \gtrsim \frac{1}{R}\int_{R+1<|x|<2R} e^{2u_2}dx$$

which contradicts (1.19). Therefore we must have $\frac{\Lambda}{\pi} > p$, from this it follows that $|y|^p e^{2u_1} \leq C$ on $\mathbb{R}^2$, using the fact that $u_2 \leq 0$ we have $|y|^p e^{2u_1}e^{2u_2} < C$, and then

$$|u_2(x)| \leq C\int_{B_1(x)} \log\frac{1}{|x-y|}dy \leq C$$

this prove (1.17). Finally, if $\Lambda < \Lambda_{*,p}$ then $(1-|x|^p)e^{2u} \notin L^1(\mathbb{R}^2)$, hence it must be $\Lambda \geq \Lambda_{*,p}$. ☐

Lemma 1.3.7. *Let u be a solution to* (1.4) *with $\Lambda = \Lambda_{*,p} = (2+p)\pi$ and $\tilde{u}$ its Kelvin transform (as defined in* (1.15)*). Then*

$$\lim_{x\to 0}\tilde{u}(x) = -\infty. \tag{1.20}$$

and

$$\lim_{x\to 0}\Delta\tilde{u}(x) = +\infty. \tag{1.21}$$

Proof. In this case, we have

$$\tilde{u}(x) = u\left(\frac{x}{|x|^2}\right) - \left(1 + \frac{p}{2}\right)\log|x|, \quad \text{for } x \neq 0, \tag{1.22}$$

and $\tilde{u}$ satisfies

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right)\left(1 - \frac{1}{|y|^p}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{2-p}}dy + c, \quad \text{for } x \neq 0.$$

We follow the proof of [116, Lemma 2.7]. Using Proposition 1.3.4, which gives us a bound from above, we have $\sup_{B_1}\tilde{u} < +\infty$. Since u is continuous, using (1.22), we get

$$\left(1 - \frac{1}{|y|^p}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{2-p}} \leq \frac{C}{|y|^4} \qquad \text{on} \quad B_1^c. \tag{1.23}$$

For $x \neq 0$ we have

$$\begin{aligned}\tilde{u}(x) = &- \frac{1}{2\pi}\int\limits_{B_1} \log\left(\frac{1}{|x-y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy + \frac{1}{2\pi}\int\limits_{B_1}\log\left(\frac{1}{|x-y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{2-p}}dy + \\ &+ \frac{1}{2\pi}\int\limits_{B_1^c}\log\left(\frac{1}{|x-y|}\right)\left(1 - \frac{1}{|y|^p}\right)\frac{e^{2\tilde{u}(y)}}{|y|^{2-p}}dy + c.\end{aligned}$$

If $0 < |x| < 1$, using the fact that $\tilde{u} \leq C$ on B_1 and (1.23), the second and the third term of above are $O(1)$, therefore

$$\tilde{u}(x) = -\frac{1}{2\pi}\int\limits_{B_1}\log\left(\frac{1}{|x-y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy + O(1), \qquad \text{for } \ 0 < |x| < 1. \tag{1.24}$$

Assume by contradiction that $\tilde{u}(x_k) = O(1)$ for a sequence $x_k \to 0$, applying [116, Lemma 2.6] to (1.24) we have

$$\int_{B_1}\log\left(\frac{1}{|y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy < +\infty$$

and changing variables

$$\int_{B_1^c}\log(|y|)|y|^p e^{2u(y)}dy < +\infty. \tag{1.25}$$

Since $\Lambda = \Lambda_{*,p}$, as $|x| \to \infty$ we have

$$\begin{aligned}\int_{|y|\leq\sqrt{|x|}}(1-|y|^p)e^{2u(y)}dy &= \Lambda_{*,p} - \int_{|y|>\sqrt{|x|}}(1-|y|^p)e^{2u(y)}dy = \\ &= \Lambda_{*,p} + O\left(\frac{1}{\log|x|}\int_{|y|>\sqrt{|x|}}\log(|y|)(1-|y|^p)e^{2u(y)}dy\right) = \\ &= \Lambda_{*,p} + O\left(\frac{1}{\log|x|}\right) \qquad \text{as} \quad |x| \to +\infty\end{aligned}$$

where in the last equality we used (1.25). Using equation (1.11) for $|x| \gg 1$

$$u(x) = \frac{1}{2\pi}\left(\int_{|y|\leq\sqrt{|x|}} + \int_{\sqrt{|x|}\leq|y|\leq 2|x|} + \int_{|y|\geq 2|x|}\right)\log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy + c.$$

From the previous estimate, as $|x| \to \infty$,

$$\int_{|y|\leq\sqrt{|x|}} \log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy =$$
$$= (-\log|x| + O(1))\left(\Lambda_{*,p} + O\left(\frac{1}{\log|x|}\right)\right) = -\Lambda_{*,p}\log|x| + O(1).$$

Concerning the second integral, as $|x| \to \infty$ we get

$$\int_{\sqrt{|x|}\leq|y|\leq 2|x|} \log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy \geq \int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy$$
$$= -\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)\left|1-|y|^p\right|e^{2u}dy \geq -\frac{1}{|x|^2}\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)dy = O\left(\frac{1}{|x|^2}\right),$$

and for the third integral, using (1.25), we have

$$\frac{1}{2\pi}\int_{|y|\geq 2|x|}\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy = O(1).$$

Therefore, for $|x| \to \infty$

$$u(x) \geq -\frac{\Lambda_{*,p}}{2\pi}\log|x| + O(1)$$

but this means that $|\cdot|^p e^{2u} \notin L^1(\mathbb{R}^2)$, which is a contradiction. Hence (1.20) is proven.

In this case, $\tilde{u}$ is a solution to

$$-\Delta\tilde{u}(x) = \left(1 - \frac{1}{|x|^p}\right)\frac{e^{2\tilde{u}(x)}}{|x|^{2-p}},$$

using (1.22) and (1.17) we obtain that $\lim_{x\to 0}(-\Delta\tilde{u}(x)) = -\infty$, which proves (1.21). □

Proposition 1.3.8. *If $p \geq 2$, there exists no solution to* (1.4).

Proof. Assume by contradiction that for some $p \geq 2$ there exists a solution u of (1.4), then by Lemma 1.3.6 we must have $\Lambda \geq \Lambda_{*,p} \geq 4\pi$ since $p \geq 2$. If $\Lambda > \Lambda_{*,p}$, using Lemma 1.3.4 we have $u(x) \leq -\frac{\Lambda}{2\pi}\log|x| + c$ for $|x| \to +\infty$. In this way u satisfies (1.13) and hence by Proposition 1.3.2 follows that

$$\Lambda_{\text{sph}} \leq \Lambda_{*,p} < \Lambda < \Lambda_{\text{sph}},$$

which is a contradiction. If $\Lambda = \Lambda_{*,p}$, using (1.22) and (1.20) we observe that (1.13) is satisfied, and we proceed as in the previous case. □

Proof of Theorem 1.1.1. Theorem 1.1.1 is proven, we have just to combine Corollary 1.3.5, Lemma 1.3.6 and Proposition 1.3.8. □

If $\Lambda = \Lambda_{*,p}$ we obtain a sharper version of (1.20).

Lemma 1.3.9. *Fix $p \in (0,2)$, let u be a solution to* (1.4) *with $\Lambda = \Lambda_{*,p}$. Then we have*

$$\limsup_{|x|\to+\infty}\frac{u(x) + \left(1+\frac{p}{2}\right)\log|x|}{\log\log|x|} = -1.$$

Proof. Let $\tilde{u}$ be defined as in (1.22), u is a radial solution (this follows from [157, Corollary 1]) therefore also $\tilde{u}$ is radially symmetric. By Lemma 1.3.7 we have

$$\lim_{r\to 0}\tilde{u}(r) = -\infty, \qquad \lim_{r\to 0}\Delta\tilde{u}(r) = +\infty,$$

so there exists $\delta > 0$ such that $\tilde{u}$ is monotone increasing in $B_\delta(0)$. Using this and (1.24), we estimate for $|x| \to 0$ and get

$$\begin{aligned}
-\tilde{u}(x) &\geq \frac{1}{2\pi}\int_{2|x|\leq|y|<1}\log\left(\frac{1}{|x-y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy + O(1)\\
&= \frac{1}{2\pi}\int_{2|x|\leq|y|\leq\delta}\log\left(\frac{1}{|y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy + \frac{1}{2\pi}\int_{\delta\leq|y|<1}\log\left(\frac{1}{|y|}\right)\frac{e^{2\tilde{u}(y)}}{|y|^2}dy + O(1)\\
&\geq \frac{e^{2\tilde{u}(x)}}{2\pi}\int_{2|x|\leq|y|\leq\delta}\log\left(\frac{1}{|y|}\right)\frac{dy}{|y|^2} + O(1)\\
&= e^{2\tilde{u}(x)}\int_{2|x|}^{\delta}\frac{\log\frac{1}{\rho}}{\rho}d\rho + O(1), \quad \text{as } |x|\to 0.
\end{aligned}$$

Hence we have

$$-\tilde{u}(x) + O(1) \geq \frac{e^{2\tilde{u}(x)}}{2}\left(\log\left(\frac{1}{2|x|}\right)\right)^2, \quad \text{as } |x|\to 0.$$

Now taking the logarithm and rearranging, we get

$$\limsup_{x\to 0}\frac{\tilde{u}(x)}{\log\log\left(\frac{1}{|x|}\right)} \leq -1.$$

We prove now that the lim sup is equal to -1. Assume by contradiction, that the previous lim sup is less than -1, it must exist $\varepsilon > 0$ such that

$$\tilde{u}(x) \leq -\left(1+\frac{\varepsilon}{2}\right)\log\log\frac{1}{|x|}$$

for $|x|$ small. Therefore, recalling (1.24), for $|x|$ small we have

$$-\tilde{u}(x) \leq C\int_{B_1}\log\left(\frac{1}{|x-y|}\right)\frac{dy}{|y|^2|\log|y||^{2+\varepsilon}} + O(1).$$

We can split $\int_{B_1}\log\left(\frac{1}{|x-y|}\right)\frac{dy}{|y|^2|\log|y||^{2+\varepsilon}}$ into $I_1 + I_2 + I_3$ definig

$$I_i := \int_{A_i}\log\left(\frac{1}{|x-y|}\right)\frac{dy}{|y|^2|\log|y||^{2+\varepsilon}}$$

where

$$A_1 = B_{\frac{|x|}{2}}, \quad A_2 = B_{2|x|}\setminus B_{\frac{|x|}{2}} \text{ and } A_3 = B_1\setminus B_{2|x|}.$$

Concerning I_1, we observe that if $y \in B_{\frac{|x|}{2}}$ we have $\log\left(\frac{1}{|x-y|}\right) \sim \log\left(\frac{1}{|x|}\right)$ as $|x|\to 0$ and $\int_{B_{|x|/2}}\frac{1}{|y|^2|\log|y||^{2+\varepsilon}}dy = \left|\log\frac{|x|}{2}\right|^{-1-\varepsilon}$, hence

$$I_1 \leq \frac{C}{|\log|x||^{\varepsilon}}.$$

Regarding I_2 we have

$$I_2 \le \frac{C}{|x|^2 \big|\log|x|\big|^{2+\varepsilon}} \int_{B_{2|x|}} \log\left(\frac{1}{|y|}\right) dy = \frac{C}{\big|\log|x|\big|^{1+\varepsilon}}.$$

Finally it's easy to prove that $I_3 \le C$. We have obtained a contradiction to the fact that $-\tilde{u}(x) \to +\infty$ as $|x| \to 0$. $\square$

1.4 Existence result

The proof of the existence part of Theorem 1.1.2 will be based on a blow-up argument as done in [116], but first we need the following result, which we will prove using a variational approach due to Chang and Chen [46].

Proposition 1.4.1. *Let $p \in (0,2)$ be fixed. For every $\Lambda \in (0, \Lambda_{\rm sph})$ and $\lambda > 0$, there exists a radial solution u_λ to*

$$-\Delta u_\lambda = (\lambda - |x|^p) e^{-|x|^2} e^{2u_\lambda} \quad \text{on } \mathbb{R}^2, \tag{1.26}$$

such that

$$\Lambda = \int_{\mathbb{R}^2} (\lambda - |x|^p) e^{-|x|^2} e^{2u_\lambda} dx.$$

Moreover, such solution u_λ solves the integral equation

$$u_\lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) (\lambda - |y|^p) e^{-|y|^2} e^{2u_\lambda} dy + c_\lambda \tag{1.27}$$

for some constant $c_\lambda \in \mathbb{R}$.

Proof. Let's denote by $K_\lambda(x) = (\lambda - |x|^p)e^{-|x|^2}$. Following the proof of [46, Theorem 2.1], we identify each point in $\mathbb{R}^2$ with a point on S^2 through the stereographic projection $\Pi : S^2 \to \mathbb{R}^2$, we take $K(x) = K_\lambda(x)$ and $\mu = 1 - \frac{\Lambda}{\Lambda_{\rm sph}}$. A solution to

$$-\Delta u = K_\lambda(x) e^{2u} \quad \text{on } \mathbb{R}^2 \tag{1.28}$$

has the form

$$u_\lambda = w \circ \Pi^{-1} + (1-\mu)\eta_0,$$

where $\eta_0(x) = \log\left(\frac{2}{1+|x|^2}\right)$, $w = u + c$ such that u minimizes a certain functional defined on the set of functions

$$\left\{ v \in H^1_{\rm rad}(S^2) \,\middle|\, \int K_\lambda(x) e^{2v} dV > 0 \right\}$$

and c is a suitable constant such that $\int K_\lambda(x) e^{2w} dV = (1-\mu)\Lambda_{\rm sph}$. By construction it holds

$$\int K_\lambda(x) e^{2u_\lambda} dx = \int K_\lambda(x) e^{2w} dV = (1-\mu)\Lambda_{\rm sph} = \Lambda.$$

So there exists at least one radial solution to (1.26). In order to prove (1.27), consider

$$-\Delta_{g_0} w + (1-\mu) = (K_\lambda \circ \Pi) e^{-2\mu(\eta_0 \circ \Pi)} e^{2w}.$$

Since $(K_\lambda \circ \Pi)e^{-2\mu(\eta_0\circ\Pi)} \in L^\infty(S^2)$ and $e^{2w} \in L^q(S^2)$ for every $q \in [1,\infty)$, by elliptic estimates we get $w \in C^{1,\alpha}(S^2)$ for $\alpha \in (0,1)$. Therefore w is continuous in $S = (0,0,-1)$ and hence

$$u_\lambda(x) = (1-\mu)\eta_0(x) + w(S) + o(1) = \frac{\Lambda}{2\pi}\log|x| + C + o(1), \quad \text{as } |x| \to +\infty.$$

Defining

$$v_\lambda := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) K_\lambda(y) e^{2u_\lambda(y)} dy$$

and $h_\lambda := u_\lambda - v_\lambda$, we observe that

$$\Delta h_\lambda = 0, \qquad h_\lambda(x) = O(\log|x|) \quad \text{as } |x| \to +\infty$$

by Liouville's theorem h_λ must be constant. □

If w is a radial function belonging to $C^2(\mathbb{R}^2)$ and $0 \le r < R$, using the divergence theorem, we have the following identity

$$w(r) - w(R) = \int_r^R \frac{1}{2\pi t}\int_{B_t} -\Delta w \, dx \, dt. \tag{1.29}$$

In what follows, let u_λ be a radial solution to (1.26) given by Proposition 1.4.1.

Lemma 1.4.2. *For every $\lambda > 0$ we have $u_\lambda(x) \downarrow -\infty$ as $|x| \to +\infty$.*

Proof. Let us consider the function

$$r \longmapsto \int_{B_r} -\Delta u_\lambda(x) dx = \int_{B_r} (\lambda - |x|^p) e^{-|x|^2} e^{2u_\lambda(x)} dx$$

we observe that it is increasing on $[0, \lambda^{1/p}]$ and decreasing to $\Lambda > 0$ on $[\lambda^{1/p}, +\infty)$, so it follows that it is positive for every $r > 0$. Hence by (1.29) u_λ is a decreasing function of $|x|$ and using Proposition 1.3.4 we conclude that $u_\lambda \to -\infty$ as $|x| \to +\infty$. □

Lemma 1.4.3. *It holds that $\lambda e^{2u_\lambda(0)} \to +\infty$ as $\lambda \downarrow 0$.*

Proof. Assume that $\lambda e^{2u_\lambda(0)} \le C$ as $\lambda \downarrow 0$, then

$$\begin{aligned}\Lambda = \int_{\mathbb{R}^2} (\lambda - |x|^p) e^{-|x|^2} e^{2u_\lambda} dx &\le \int_{B_{\lambda^{1/p}}} (\lambda - |x|^p) e^{-|x|^2} e^{2u_\lambda} dx \\ &\le \int_{B_{\lambda^{1/p}}} \lambda e^{-|x|^2} e^{2u_\lambda(0)} dx \xrightarrow[\lambda \to 0]{} 0,\end{aligned}$$

which gives a contradiction. □

Now we define

$$\eta_\lambda(x) := u_\lambda(r_\lambda x) - u_\lambda(0)$$

where the values r_λ are non-negative and defined in such a way that

$$\lambda \, r_\lambda^2 \, e^{2u_\lambda(0)} = 1,$$

by Lemma 1.4.3 we have $r_\lambda \to 0$ as $\lambda \downarrow 0$, moreover $\eta_\lambda(0) = 0$ and $\eta_\lambda \leq 0$ (since u_λ is a radial decreasing function). A basic calculation shows that η_λ satisfies

$$-\Delta\eta_\lambda = \left(1 - \frac{r_\lambda^p |x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda}$$

and

$$\Lambda = \int_{\mathbb{R}^2} \left(1 - \frac{r_\lambda^p |x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx. \tag{1.30}$$

Observing that $\left(1 - \frac{r_\lambda^p|x|^p}{\lambda}\right) < 0$ in $B_{r_1}^c$ where $r_1 := \frac{\lambda^{1/p}}{r_\lambda}$, we have

$$0 < \Lambda < \int_{B_{r_1}} e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \leq \left|B_{r_1}\right|$$

where in the last inequality we employ the fact that since $\eta_\lambda \leq 0$ then $e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} \leq 1$. It follows that

$$\limsup_{\lambda \to 0} \frac{r_\lambda^p}{\lambda} < +\infty.$$

Since $\eta_\lambda(0) = 0$, using ODE theory, we have that, up to a subsequence,

$$\eta_\lambda \xrightarrow[\lambda \to 0]{} \eta \qquad \text{in } C^2_{loc}(\mathbb{R}^2) \tag{1.31}$$

and the function η satisfies

$$-\Delta\eta = (1 - \mu|x|^p)e^{2\eta}, \quad \text{in } \mathbb{R}^2$$

where $\mu := \lim_{\lambda \to 0} \frac{r_\lambda^p}{\lambda} \in [0, +\infty)$. Notice that at this stage we do not know if $\mu > 0$ and if $\int_{\mathbb{R}^2} (1 - |x|^p)e^{2\eta} = \Lambda$.

Lemma 1.4.4. *If $\mu = 0$, then $e^{2\eta} \in L^1(\mathbb{R}^2)$.*

Proof. From Lemma 1.4.2 and (1.31), we have that η is decreasing. Since η satisfies $-\Delta\eta = e^{2\eta}$, then $\Delta\eta \leq 0$ must be increasing. We have $\lim_{r\to\infty} \Delta\eta(r) =: c_0 \in [-\infty, 0]$, if $c_0 = 0$ then $\lim_{r\to\infty} e^{2\eta(r)} = 0$ and so $e^{2\eta} \in L^1(\mathbb{R}^2)$; if $c_0 < 0$ then $\eta(r) \lesssim -r^2$, and hence $e^{2\eta} \in L^1(\mathbb{R}^2)$. □

Lemma 1.4.5. *For every $\Lambda \in (\Lambda_{*,p}, \Lambda_{\text{sph}})$ we have $\mu > 0$.*

Proof. Assume by contradiction that $\mu = 0$, then η is a solution to

$$-\Delta\eta = e^{2\eta} \quad \text{in} \quad \mathbb{R}^2,$$

where $e^{2\eta} \in L^1(\mathbb{R}^2)$ from Lemma 1.4.4. Chen and Li in [57] proved that every solution to the previous equation with finite total Gaussian curvature, is a standard one, namely assumes the form

$$\eta(x) = \log\left(\frac{2\lambda}{1 + \lambda^2|x - x_0|^2}\right),$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^2$. Hence all solutions are radially symmetric with respect to some point $x_0 \in \mathbb{R}^2$. Since η is spherical, we have

$$\int_{\mathbb{R}^2} e^{2\eta} dx = |S^2| = \Lambda_{\text{sph}}.$$

Moreover, by assumption $\Lambda < \Lambda_{\rm sph}$, so we can fix $R_0 > 0$ such that

$$\Lambda < \int\limits_{B_{R_0}} e^{2\eta} dx \ .$$

Recalling that $r_\lambda \to 0$ and $\frac{r_\lambda^p}{\lambda} \to 0$ as $\lambda \downarrow 0$, one can find λ_0 (which depends on R_0) such that

$$\int\limits_{B_{R_0}} \left(1 - \frac{r_\lambda^p}{\lambda}|x|^p\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \geq \Lambda, \quad \text{for} \quad \lambda \in (0, \lambda_0). \tag{1.32}$$

Let us now define the function

$$\Gamma_\lambda(t) := \int\limits_{B_t} \left(1 - \frac{r_\lambda^p}{\lambda}|x|^p\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx,$$

we can observe that $\Gamma_\lambda(0) = 0$, Γ_λ is monotone increasing on the interval $\left[0, \frac{\lambda^{1/p}}{r_\lambda}\right]$ and it decreases to Λ on $\left[\frac{\lambda^{1/p}}{r_\lambda}, +\infty\right)$. From (1.32) we have

$$\Gamma_\lambda(t) \geq \Lambda, \quad \text{for } t \geq R_0 \text{ and } \lambda \in (0, \lambda_0). \tag{1.33}$$

Integrating from R_0 to $r \geq R_0$ we have

$$-\int_{R_0}^r \frac{\Gamma_\lambda(t)}{2\pi t} dt \leq -\int_{R_0}^r \frac{\Lambda}{2\pi t} dt = -\frac{\Lambda}{2\pi} \log \frac{r}{R_0} = -\left(1 + \frac{p}{2} + \delta\right) \log \frac{r}{R_0}$$

where $\delta > 0$ is such that $\Lambda_{*,p} + 2\delta\pi = \Lambda$, hence by (1.29) we have

$$\begin{aligned} \eta_\lambda(r) &\leq \eta_\lambda(R_0) - \left(1 + \frac{p}{2} + \delta\right) \log \frac{r}{R_0} = \\ &= C(R_0) - \left(1 + \frac{p}{2} + \delta\right) \log r, \qquad \forall r \geq R_0. \end{aligned}$$

This implies that

$$\lim_{R\to\infty} \lim_{\lambda\to 0} \int_{B_R^c} (1 + |x|^p) e^{2\eta_\lambda} dx = 0. \tag{1.34}$$

We can split

$$\Lambda = \int_{B_R} \left(1 - \frac{r_\lambda^p|x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx + \int_{B_R^c} \left(1 - \frac{r_\lambda^p|x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx$$

by uniform convergence the first term goes to $\int_{B_R} e^{2\eta} dx$ as $\lambda \to 0$, regarding the second term we have

$$\left| \int_{B_R^c} \left(1 - \frac{r_\lambda^p|x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \right| \leq \int_{B_R^c} (1 + |x|^p) e^{2\eta_\lambda} dx,$$

so taking the limit as $\lambda \to 0$ and using (1.34), we obtain

$$\Lambda = \int_{B_R} e^{2\eta} dx + o_R(1), \qquad \text{as } R \to +\infty.$$

It follows that

$$\Lambda = \int_{\mathbb{R}^2} \left(1 - \frac{r_\lambda^p|x|^p}{\lambda}\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \xrightarrow[\lambda\downarrow 0]{} \int_{\mathbb{R}^2} e^{2\eta} dx = \Lambda_{\rm sph}$$

which is absurd, hence we must have $\mu > 0$. $\square$

Proof of Theorem 1.1.2. Existence part. Since $\frac{r_\lambda^p}{\lambda} \xrightarrow[\lambda\to 0]{} \mu > 0$ by Lemma 1.4.5, choosing $R = \left(\frac{4}{\mu}\right)^{1/p}$ we have that for λ sufficiently small it holds

$$1 - \frac{r_\lambda^p}{\lambda}|x|^p \leq 1 - \frac{\mu}{2}|x|^p \leq -\frac{\mu}{4}|x|^p \qquad \text{for} \quad |x| \geq R.$$

Therefore, by (1.30) we obtain

$$\begin{aligned}\int_{B_R^c} \frac{\mu}{4}|x|^p e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx &\leq -\int_{B_R^c} \left(1 - \frac{r_\lambda^p}{\lambda}|x|^p\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx = \\ &= \int_{B_R} \left(1 - \frac{r_\lambda^p}{\lambda}|x|^p\right) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx - \Lambda \leq C\end{aligned} \tag{1.35}$$

where we used that $\eta_\lambda \leq 0$. So, from the fact that the integrand in B_R is uniformly bounded, it follows that

$$\int_{\mathbb{R}^2} (1 + |x|^p) e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \leq C.$$

From (1.35) we also have

$$\int_{B_R^c} e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx \leq \frac{C}{R^p} \xrightarrow[R\to+\infty]{} 0$$

uniformly in λ, and hence

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^2} e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx = \int_{\mathbb{R}^2} e^{2\eta} dx. \tag{1.36}$$

Using Fatou's lemma

$$\int_{\mathbb{R}^2} |x|^p e^{2\eta} dx \leq \lim_{\lambda\to 0} \int_{\mathbb{R}^2} |x|^p e^{-r_\lambda^2|x|^2} e^{2\eta_\lambda} dx. \tag{1.37}$$

From (1.36) and (1.37) we obtain

$$\int_{\mathbb{R}^2} (1 - \mu|x|^p) e^{2\eta} dx \geq \Lambda.$$

Now we are going to prove that the previous inequality is actually an equality. Since $\frac{r_\lambda^p}{\lambda} \to \mu > 0$ then

$$\frac{\lambda^{\frac{1}{p}}}{r_\lambda} \to \frac{1}{\mu^{\frac{1}{p}}} > 0,$$

proceeding as in the proof of Lemma 1.4.5, for $R_0 = 2\mu^{-\frac{1}{p}}$ and λ_0 sufficiently small, relation (1.33) holds and from it (1.34) follows. Finally, (1.34) implies that

$$\int_{\mathbb{R}^2} (1 - \mu|x|^p) e^{2\eta} dx = \Lambda.$$

Defining

$$u(x) := \eta(\rho x) + \log \rho, \qquad \rho := \mu^{-\frac{1}{p}}$$

we get the desired solution to (1.4).

Asymptotic behavior. Proof of estimate (1.5). Let u be a solution to (1.4) and $\tilde{u}$ its Kelvin transform as defined in (1.15). We have

$$-\Delta\tilde{u}(x) = \left(1 - \frac{1}{|x|^p}\right)\frac{e^{2\tilde{u}(x)}}{|x|^{4-\frac{\Lambda}{\pi}}} = O\left(\frac{1}{|x|^{4+p-\Lambda/\pi}}\right), \qquad \text{as } |x| \to 0.$$

Since for $\Lambda > \Lambda_{*,p}$

$$4 + p - \frac{\Lambda}{\pi} = 2 - \frac{\Lambda - \Lambda_{*,p}}{\pi} < 2$$

we obtain

$$-\Delta\tilde{u} \in L^q_{loc}(\mathbb{R}^2) \quad \text{for } 1 \leq q < \frac{1}{1 - \frac{\Lambda - \Lambda_{*,p}}{2\pi}},$$

hence by elliptic estimates we have

$$\tilde{u} \in \mathrm{W}^{2,q}_{\mathrm{loc}}(\mathbb{R}^2) \quad \text{for } 1 \leq q < \frac{1}{1 - \frac{\Lambda - \Lambda_{*,p}}{2\pi}}.$$

Now, by the Morrey-Sobolev embedding we obtain that

$$\tilde{u} \in C^{0,\alpha}_{loc}(\mathbb{R}^2)$$

for $\alpha \in [0,1]$ such that $\alpha < \frac{\Lambda - \Lambda_{*,p}}{\pi}$. From this (1.5) follows.
Proof of estimate (1.6). If u is a solution to (1.4), then u satisfies the integral equation (1.10) and differentiating under the integral sign, we obtain

$$|\nabla u(x)| = O\left(\int_{\mathbb{R}^2} \frac{1}{|x-y|}(1 + |y|^p)e^{2u(y)}dy\right)$$

since $\Lambda > \Lambda_{*,p}$, using (1.5) as $|x| \to \infty$ we get

$$(1 + |x|^p)e^{2u(x)} \leq \frac{C}{1 + |x|^{2+\delta}}$$

for $\delta > 0$. Hence, for $|x|$ large, we have

$$\begin{aligned}
|\nabla u(x)| &\leq C\left(\int_{B_{|x|/2}} + \int_{B_{2|x|}\setminus B_{|x|/2}} + \int_{B^c_{2|x|}}\right)\frac{1}{|x-y|}\frac{1}{1+|y|^{2+\delta}}dy \\
&\leq \frac{C}{|x|} + \frac{C}{|x|^{2+\delta}}\int_{B_{2|x|}\setminus B_{|x|/2}} \frac{1}{|x-y|}dy \leq \frac{C}{|x|}.
\end{aligned}$$

□

1.5 Compactness result

Let $p \in (0,2)$ be fixed and $\{u_k \,|\, k \in \mathbb{N}\}$ be a sequence of solutions to (1.4) with $\Lambda = \Lambda_k \in [\Lambda_{*,p}, \Lambda_{\mathrm{sph}})$, hence every solution u_k solves the integral equation

$$u_k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right)(1 - |y|^p)e^{2u_k(y)}dy + c_k. \tag{1.38}$$

Assuming that

$$\Lambda_k := \int_{\mathbb{R}^2}(1 - |x|^p)e^{2u_k(x)}dx \to \bar{\Lambda} \in [\Lambda_{*,p}, \Lambda_{\mathrm{sph}}), \tag{1.39}$$

we want to prove that $u_k \to \bar{u}$ (up to a subsequence) uniformly locally in $\mathbb{R}^2$, where $\bar{u}$ is a radial solution to (1.4) with $\Lambda = \bar{\Lambda}$.

With the same procedure used in the proof of Lemma 1.4.2, we can prove that u_k is radially decreasing.

Lemma 1.5.1. *We have $u_k(0) \geq C$ where C depends only on $\inf_k \Lambda_k$.*

Proof. Using the fact that u_k is radially decreasing, we have

$$\begin{aligned}\Lambda_k &= \int_{\mathbb{R}^2} (1-|x|^p)e^{2u_k(x)}dx \leq \int_{B_1} (1-|x|^p)e^{2u_k(x)}dx \\ &\leq \int_{B_1} e^{2u_k(x)}dx \leq \pi e^{2u_k(0)},\end{aligned}$$

therefore

$$u_k(0) \geq \frac{1}{2}\log\frac{\Lambda_k}{\pi} \geq \log\sqrt{p+2} > 0$$

since $\frac{\Lambda_k}{\pi} \in [2+p, 4)$. □

Since $\Lambda_k \in [\Lambda_{*,p}, \Lambda_{\text{sph}})$, using Proposition 1.3.2 (which can be applied thanks to Proposition 1.3.4 if $\Lambda_k \in (\Lambda_{*,p}, \Lambda_{\text{sph}})$ and thanks to Lemma 1.3.9 if $\Lambda_k = \Lambda_{*,p}$) we have the following Pohozaev identity:

$$\frac{\Lambda_k}{\Lambda_{\text{sph}}}(\Lambda_k - \Lambda_{\text{sph}}) = -\frac{p}{2}\int_{\mathbb{R}^2} |x|^p e^{2u_k(x)}dx. \tag{1.40}$$

Hence we have

$$\int_{\mathbb{R}^2} e^{2u_k}dx = \Lambda_k - \frac{2\Lambda_k}{p\Lambda_{\text{sph}}}(\Lambda_k - \Lambda_{\text{sph}}) \tag{1.41}$$

and taking the limit for $k \to +\infty$

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2} e^{2u_k}dx = \bar{\Lambda} - \frac{2\bar{\Lambda}}{p\Lambda_{\text{sph}}}(\bar{\Lambda} - \Lambda_{\text{sph}}). \tag{1.42}$$

Lemma 1.5.2.

$$\limsup_{k\to+\infty} u_k(0) < \infty$$

Proof. From (1.39), (1.40) and (1.41) we have

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^2} (1+|x|^p)e^{2u_k}dx < +\infty. \tag{1.43}$$

Differentiating (1.38), integrating over $B_1(0)$ and using Fubini's theorem, we get

$$\int_{B_1(0)} |\nabla u_k|dx \leq C\int_{\mathbb{R}^2}\left(\int_{B_1(0)}\frac{1}{|x-y|}dx\right)(1+|y|^p)e^{2u_k(y)}dy \leq C$$

where in the last inequality we used (1.43). Assume by contradiction that (up to a subsequence) $u_k(0) \to +\infty$ as $k \to +\infty$, by [144, Theorem 2] (the two-dimensional case was first studied by Brezis and Merle in [32] and by Li and Shafrir in [129]) we have that

$u_k \to -\infty$ locally uniformly in $B_1(0) \setminus \{0\}$. Moreover, we have quantization of the total curvature, namely

$$\lim_{k\to+\infty} \int_{B_r(0)} (1-|x|^p)e^{2u_k}dx = 4\pi,$$

where $r \in (0,1)$ is fixed. The blow-up at the origin is spherical, namely there exists a sequence of positive numbers

$$\mu_k := 2e^{-u_k(0)} \to 0, \quad \text{as } k \to +\infty,$$

such that, setting

$$\eta_k(x) := u_k(\mu_k x) - u_k(0) + \log 2,$$

we have

$$\eta_k(x) \xrightarrow[k\to\infty]{} \log \frac{2}{1+|x|^2} \qquad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2).$$

Since u_k is monotone decreasing, we have that $u_k \to -\infty$ locally uniformly in $\mathbb{R}^2 \setminus \{0\}$ and using (1.43) we get

$$\lim_{k\to+\infty} \int_{\mathbb{R}^2} e^{2u_k}dx = \Lambda_{\text{sph}}.$$

Comparing the previous one and (1.42), we have

$$1 = \frac{2\bar{\Lambda}}{p\Lambda_{\text{sph}}} \geq \frac{2\Lambda_{*,p}}{p\Lambda_{\text{sph}}} = \frac{2+p}{2p} > 1 \qquad \text{for} \quad p \in (0,2),$$

which is absurd. □

Lemma 1.5.3. *$u_k \to \bar{u}$ locally uniformly, where $\bar{u}$ is a solution to* (1.4) *with $\Lambda = \tilde{\Lambda} \geq \bar{\Lambda}$.*

Proof. Using Lemma 1.5.1 and Lemma 1.5.2 and the fact that u_k is radial decreasing, we have

$$u_k \leq u_k(0) = O(1),$$

therefore

$$-\Delta u_k = O_R(1) \quad \text{on } B_R.$$

Integrating (1.38) over B_R and using Fubini's theorem and (1.43), we obtain that

$$\int_{B_R} |u_k(x)|dx \leq C_R$$

therefore, by elliptic estimates we have

$$\|u_k\|_{\mathcal{C}^{1,\alpha}(B_{R/2})} \leq C_{R/2}$$

and hence, up to a subsequence $u_k \to \bar{u}$ in $\mathcal{C}^1_{loc}(\mathbb{R}^2)$. Finally by Fatou's lemma, we have that

$$\tilde{\Lambda} := \int_{\mathbb{R}^2} (1-|x|^p)e^{2\bar{u}}dx \geq \limsup_{k\to+\infty} \int_{\mathbb{R}^2} (1-|x|^p)e^{2u_k}dx = \bar{\Lambda}.$$

□

Lemma 1.5.4. *We have $\tilde{\Lambda} = \bar{\Lambda}$.*

Proof. Assume by contradiction that $\tilde{\Lambda} > \bar{\Lambda}$. Using the fact that $u_k \to \bar{u}$ in $\mathcal{C}^0_{loc}(\mathbb{R}^2)$, we get

$$\tilde{\Lambda} - \bar{\Lambda} = -\int_{B_R^c} (1 - |x|^p) e^{2u_k} dx + o_k(1) + o_R(1), \quad \text{as } k \to +\infty,\ R \to +\infty$$

hence

$$0 < \tilde{\Lambda} - \bar{\Lambda} = \lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_R^c} (|x|^p - 1) e^{2u_k} dx = \lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_R^c} |x|^p e^{2u_k} dx =: \rho. \tag{1.44}$$

We consider the Kelvin transform of u_k as defined in (1.15) for $x \neq 0$

$$\tilde{u}_k(x) = u_k\left(\frac{x}{|x|^2}\right) - \frac{\Lambda_k}{2\pi} \log|x|.$$

From Proposition 1.3.3 we have

$$\tilde{u}_k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) \left(1 - \frac{1}{|y|^p}\right) \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-p-\delta_k}} dy + c_k, \quad \delta_k = \frac{\Lambda_k - \Lambda_{*,p}}{\pi},$$

and with the same procedure for proving (1.24) we obtain

$$\tilde{u}_k(x) = -\frac{1}{2\pi} \int_{B_1} \log\left(\frac{1}{|x-y|}\right) \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy + O(1), \quad \text{if } 0 < |x| < 1. \tag{1.45}$$

If $\delta_k \not\to 0$, from (1.45) we have that $\tilde{u}_k = O(1)$ in B_1, but this contradicts the fact that $\rho > 0$, hence we must have $\delta_k \to 0$, namely $\Lambda_k \to \Lambda_{*,p}$. Let us define $\varepsilon_k > 0$ in such a way that

$$\int_{B_{\varepsilon_k}} \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy = \frac{\rho}{2}, \tag{1.46}$$

and we have that $\varepsilon_k \to 0$ as $k \to \infty$. We can observe that for $y \in B_{\varepsilon_k}$ and $x \in B^c_{2\varepsilon_k}$ $\log\left(\frac{1}{|x-y|}\right) = \log\left(\frac{1}{|x|}\right) + O(1)$, hence for $2\varepsilon_k \le |x| \le 1$ we get

$$\begin{aligned}\tilde{u}_k(x) &= -\frac{\log(1/|x|)}{2\pi} \int_{B_{\varepsilon_k}} \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy - \frac{1}{2\pi} \int_{B_1 \setminus B_{\varepsilon_k}} \log\left(\frac{1}{|x-y|}\right) \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy + O(1) \\ &\le -\frac{\rho}{4\pi} \log\left(\frac{1}{|x|}\right) + C.\end{aligned} \tag{1.47}$$

where in the last inequality, we used the fact that $\log\left(\frac{1}{|x-y|}\right)$ is lower bounded for $y \in B_1$ and $x \to 0$. From (1.47) we have that

$$\lim_{r\to 0} \lim_{k\to\infty} \sup_{B_r} \tilde{u}_k = -\infty \tag{1.48}$$

and moreover

$$\lim_{r\to 0} \lim_{k\to\infty} \int_{B_r \setminus B_{2\varepsilon_k}} \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy = 0,$$

which, using (1.44), implies

$$\lim_{k\to\infty} \int_{B_{2\varepsilon_k}} \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy = \rho. \tag{1.49}$$

Hence from (1.46), (1.48) and (1.49) we obtain

$$\frac{\rho}{2} = \lim_{k\to\infty} \int_{B_{2\varepsilon_k}\setminus B_{\varepsilon_k}} \frac{e^{2\tilde{u}_k(y)}}{|y|^{2-\delta_k}} dy = o(1) \int_{B_{2\varepsilon_k}\setminus B_{\varepsilon_k}} \frac{1}{|y|^2} dy = o(1), \quad \text{as } k \to \infty,$$

which is absurd, because we assumed $\rho > 0$. □

Proof of Theorem 1.1.3. Lemma 1.5.3 and Lemma 1.5.4 prove that, up to a subsequence, u_k converges to $\bar{u}$ locally uniformly in $\mathbb{R}^2$, where $\bar{u}$ is a solution to (1.4) with $\Lambda = \bar{\Lambda}$. Estimate (1.7) follows from Lemma 1.3.9, while for (1.8) we proceed in the same way as for proving (1.6). □

1.6 Spherical blow-up as $\Lambda_k \uparrow \Lambda_{\mathrm{sph}}$

Lemma 1.6.1. *Let* $\{u_k \,|\, k \in \mathbb{N}\}$ *be a sequence of solutions to* (1.4) *with* $\Lambda_k \uparrow \Lambda_{\mathrm{sph}}$*. Then*

$$u_k(0) \to +\infty, \quad \text{as } k \to +\infty.$$

Proof. By Lemma 1.5.1 we have that $u_k(0) \geq C$. Assume by contradiction that, up to a subsequence, $u_k(0) \to s \in \mathbb{R}$ as $k \to +\infty$. Proceeding as in the proof of Lemma 1.5.3, we get that $u_k \to \bar{u}$, where $\bar{u}$ is a solution to (1.4) for $\Lambda \geq 4\pi$, which contradicts Theorem 1.1.1 □

Proof of Theorem 1.1.4. By Lemma 1.6.1 the sequence $\{u_k \,|\, k \in \mathbb{N}\}$ blows-up at the origin. Moreover, we have

$$\int_{B_1(0)} |\nabla u_k| dx \leq C \int_{\mathbb{R}^2} \left(\int_{B_1(0)} \frac{1}{|x-y|} dx \right) (1 + |y|^p) e^{2u_k(y)} dy \leq C.$$

We can conclude using [144, Theorem 2]. □

1.7 Appendix

Proof of the estimate (1.18). Since $u_1 = u - u_2$ one has

$$\begin{aligned} u_1(x) = {} & \frac{1}{2\pi} \int_{\mathbb{R}^2\setminus B_1(x)} \log\left(\frac{|y|}{|x-y|}\right)(1 - |y|^p) e^{2u(y)} dy + \\ & + \frac{1}{2\pi} \int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right) e^{2u(y)} dy + \frac{1}{2\pi} \int_{B_1(x)} \log|y|(1-|y|^p) e^{2u(y)} dy + c = \\ = {} & \frac{1}{2\pi}(I_1 + I_2 + I_3) + c. \end{aligned}$$

First of all we estimate I_1 using the split of $\mathbb{R}^2$ introduced in the proof of Proposition 1.3.4

$$I_1 = \int_{B_{R_0}(0)} (\cdot) dy + \int_{B_{\frac{|x|}{2}}\setminus B_1(x)} (\cdot) dy + \int_{\mathbb{R}^2\setminus(B_{|x|/2}\cup B_{R_0})} (\cdot) dy.$$

If $y \in B_{R_0}$ we have $\log\left(\frac{|y|}{|x-y|}\right) = -\log|x| + O(1)$ as $|x| \to \infty$, hence

$$\int_{B_{R_0}} \log\left(\frac{|y|}{|x-y|}\right)(1 - |y|^p) e^{2u(y)} dy = (-\log|x| + O(1))(\Lambda + o_{R_0}(1)).$$

For $y \in B_{|x|/2}(x) \setminus B_1(x)$ we have $\log\left(\frac{|y|}{|x-y|}\right) = O(\log|x|)$ as $|x| \to \infty$, hence

$$\int\limits_{B_{|x|/2}(x)\setminus B_1(x)} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy = o_{R_0}(1)\,O(\log|x|).$$

For $y \in \mathbb{R}^2 \setminus (B_{|x|/2}(x) \cup B_{R_0})$ we can observe that $\log\left(\frac{|y|}{|x-y|}\right) = O(1)$ as $|x| \to \infty$, so

$$\int\limits_{\mathbb{R}^2\setminus(B_{|x|/2}(x)\cup B_{R_0})} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{2u(y)}dy = O(1)\,o_{R_0}(1).$$

Concerning I_2, using (1.16) as $|x| \to \infty$ we have

$$I_2 = \int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)e^{2u(y)}dy \le C\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)|y|^{-\frac{\Lambda}{\pi}}dy$$

from Proposition 1.3.1 $\Lambda \ge 0$, hence if $\Lambda = 0$ we have $I_2 \le O(1)$ and if $\Lambda > 0$ we obtain $I_2 = o(1)$ as $|x| \to \infty$. Let's consider I_3, as $|x| \to \infty$ we get

$$I_3 = O\left(\log|x|\int_{B_1(x)}(1-|y|^p)e^{2u(y)}dy\right) = o(\log|x|)$$

where, since $(1-|y|^p)e^{2u(y)}\chi_{B_1(x)} \to 0$ a.e. as $|x| \to \infty$, using dominated convergence we have $\int_{B_1(x)}(1-|y|^p)e^{2u(y)}dy = \int_{\mathbb{R}^2}(1-|y|^p)e^{2u(y)}\chi_{B_1(x)}dy \to 0$ as $|x| \to \infty$. Summing up, we can conclude that

$$u_1(x) = -\frac{\Lambda + o(1)}{2\pi}\log|x|, \quad \text{as } |x| \to +\infty.$$

□

Chapter 2

Existence and asymptotic behavior of non-normal conformal metrics on $\mathbb{R}^4$ with sign-changing Q-curvature

2.1 Introduction to the problem and main results

Let $p > 0$ be fixed, we consider the prescribed Q-curvature equation

$$\Delta^2 u = (1 - |x|^p) e^{4u} \quad \text{in } \mathbb{R}^4, \tag{2.1}$$

under the assumption

$$\Lambda := \int_{\mathbb{R}^4} (1 - |x|^p) e^{4u} dx < \infty. \tag{2.2}$$

Geometrically, this means that if u is a solution to (2.1)-(2.2), then the metric $g_u := e^{2u}|dx|^2$, which is conformal to the Euclidean metric on $\mathbb{R}^4$, has Q-curvature equal to $1 - |x|^p$ and finite total Q-curvature Λ. In what follows, we will always assume that $(1 - |x|^p)e^{4u} \in L^1(\mathbb{R}^4)$. Let u be a solution to (2.1)-(2.2), we define v as

$$v(x) := \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) (1 - |y|^p) e^{4u} dy. \tag{2.3}$$

Definition 2.1.1 (Normal and non-normal solutions). *We call a solution u to* (2.1)-(2.2) *normal if there exists a constant $c \in \mathbb{R}$ such that u solves the following integral equation*

$$u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) (1 - |y|^p) e^{4u} dy + c.$$

All other solutions to problem (2.1)-(2.2) *are called non-normal.*

For what concerns normal solutions to (2.1)-(2.2), recently, A. Hyder and L. Martinazzi [116] proved some existence and non-existence results. In particular, among other things, they showed that problem (2.1)-(2.2) admits normal solutions if and only if $p \in (0, 4)$ and $\Lambda_{*,p} \le \Lambda < \Lambda_{\mathrm{sph}}$ where $\Lambda_{*,p} := \left(1 + \frac{p}{4}\right) 8\pi^2$ and $\Lambda_{\mathrm{sph}} := 16\pi^2$. Moreover, every normal solution has the following asymptotic behavior

$$u(x) = -\frac{\Lambda}{8\pi^2} \log|x| + C + O(|x|^{-\alpha}), \quad \text{as } |x| \to \infty$$

for every $\alpha \in [0,1]$ such that $\alpha < \frac{\Lambda - \Lambda_{*,p}}{2\pi^2}$.

Motivated by the above results, we study the properties of more general solutions (not necessarily normal) to problem (2.1)-(2.2). Although for $p \geq 4$, problem (2.1)-(2.2) admits no normal solutions, we prove that non-normal solutions do exist. To this end, we consider all polynomials P of degree 2 such that

$$\lim_{|x|\to\infty} P(x) = -\infty$$

and define the set

$$\mathcal{P}_2 := \{P \text{ polynomial in } \mathbb{R}^4 \,|\, \deg P = 2 \quad \text{and} \quad \lim_{|x|\to\infty} P(x) = -\infty\}.$$

By means of a result of S.-Y. A. Chang and W. Chen (see Theorem 2.1 in [46] where, under suitable assumptions on the curvature K, using a variational approach, they prove existence of at least one solution to equation $(-\Delta)^{\frac{n}{2}} u = K(x)e^{nu}$ in $\mathbb{R}^n$) we prove the following theorem, which extends to the non-normal case the existence results in [116].

Theorem 2.1.1. *Let $p > 0$ be fixed. Then for any $P \in \mathcal{P}_2$ and for every $\Lambda \in (0, \Lambda_{\text{sph}})$, there exists at least one solution to problem* (2.1)-(2.2) *of the form*

$$u = w + P$$

where

$$w(x) = -\frac{\Lambda}{8\pi^2} \log|x| + C + o(1), \quad \text{as } |x| \to +\infty.$$

We shall prove the following classification result.

Theorem 2.1.2. *Let u be a solution to* (2.1)-(2.2) *such that $(1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4)$ and v as defined in* (2.3)*. Then there exists an upper-bounded polynomial P of degree at most 2 such that*

$$u = v + P.$$

Moreover, u has the following asymptotic behavior

$$u(x) = -\frac{\Lambda}{8\pi^2} \log|x| + P(x) + o(\log|x|), \quad \text{as } |x| \to \infty.$$

Note that P being upper bounded means that P has even degree, and since P has degree at most 2, this implies that P could only have degree 2 or be constant. For this reason, we can rephrase Theorem 2.1.2 by saying that all solutions to problem (2.1)-(2.2) have the form $v + P$, where v behaves logarithmically at infinity and P is an upper bounded polynomial of degree 2 if the solution is non-normal, whereas P is constant if the solution is normal.

Remark 1. We can observe that the function w of Theorem 2.1.1 and the function v of Theorem 2.1.2 differ by a constant.

In order to prove Theorem 2.1.2, we obtain some suitable upper and lower estimates for the function v (see Lemma 2.3.1 and Proposition 2.5.3 below). First, we prove the lower estimate (2.6). To this end, we need to overcome some difficulties compared to previous works, due to the fact that the Q-curvature is not constant and it changes sign (compare e.g. to the proof of Lemma 2.4 in [133]). The lower estimate (2.6) will be crucial to obtain a Liouville-type theorem (see Theorem 2.4.2), also in this case the proof is quite delicate because the estimate for $-v$ contains the singular integral

$$A(x) := \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.$$

Then we show that the polynomial P is upper-bounded (see Proposition 2.5.2). To do this, we take advantage of a useful result of Gorin and the fact that $|A(x)| \leq C$. In order to estimate the singular integral $A(x)$ and prove Proposition 2.5.3, we take some ideas from the proof of Lemma 13 in [142] and the one of Lemma 2.4 in [133], but our case is more challenging. This is due to the fact that the singular integral $A(x)$ is over $B_1(x)$ and we need a radius $\tau \in (0,1)$ which can be fixed later. In addition, we lack a good estimate for $\int_{\mathbb{R}^4 \setminus B_R} v^+ \, dx$, which indeed is necessary in [142], and we do not know a priori the sign of Δu, which is fundamental to apply a Harnack inequality as in [133].

Open problem. Can we find non-normal solutions to (2.1)-(2.2) with arbitrarily large but finite total Q-curvature Λ? In the constant Q-curvature case C.-S. Lin [133] (see also [46]) proved that this is not the case, indeed he showed that all solutions to

$$\begin{cases} \Delta^2 u = 6e^{4u} & \text{in } \mathbb{R}^4 \\ e^{4u} \in L^1(\mathbb{R}^4) \end{cases}$$

satisfy $V \in (0, \mathrm{vol}(\mathbb{S}^4)]$. Unfortunately, his approach is no longer applicable to our case, in fact when $u = v + P$ with P polynomial of degree 2, we need $Q(x) = (1-|x|^p)e^{4P}$ to be radially decreasing, which is not true in general. Since $x \cdot \nabla Q(x)$ does not have a fixed sign, using methods from [145], it would be interesting to see whether there exist solutions to (2.1)-(2.2) with total Q-curvature $\Lambda \geq \Lambda_{\mathrm{sph}}$.

2.2 Existence of solutions

In this section, we take advantage of a result of A. Chang and W. Chen (see [46, Theorem 2.1]). Using a variational approach in a Sobolev space defined on a conical singular manifold, they prove existence of at least one solution to equation

$$(-\Delta)^{\frac{n}{2}} w = K(x)e^{nw} \quad \text{in } \mathbb{R}^n,$$

in even dimensions, assuming that K is positive somewhere and for some $s > 0$, $K(x) = O\left(\frac{1}{|x|^s}\right)$ near infinity.

Proof of Theorem 2.1.1. Let us fix $P \in \mathcal{P}_2$ and $\Lambda \in (0, \Lambda_{\mathrm{sph}})$. By [46, Theorem 2.1] and its proof (refer also to Section 7 in [116]) setting $K(x) := (1-|x|^p)e^{4P}$ and $\mu := 1 - \frac{\Lambda}{\Lambda_{\mathrm{sph}}} \in (0,1)$, one can find at least one solution w to equation

$$\Delta^2 w = K(x)\, e^{4w}, \quad \text{in } \mathbb{R}^4$$

such that

$$\int_{\mathbb{R}^4} K(x)\, e^{4w}\, dx = (1-\mu)\Lambda_{\mathrm{sph}} = \Lambda.$$

It follows immediately that

$$u := w + P$$

is the desired solution to problem (2.1)-(2.2). More precisely, w is of the form

$$w = \tilde{w} \circ \Pi^{-1} + (1-\mu)w_0$$

where $w_0 = \log\left(\frac{2}{1+|x|^2}\right)$, $\Pi : \mathbb{S}^4 \to \mathbb{R}^4$ denotes the stereographic projection, $\tilde{w} = \bar{w} + C$, where $\bar{w}$ minimize a certain functional which takes values in a Sobolev space defined on a conical singular manifold, and C is a suitable constant such that

$$\int K(x)e^{4\tilde{w}}\, dV = (1-\mu)\Lambda_{\text{sph}}$$

where the corresponding volume element is $dV = e^{4(1-\mu)w_0}dx$. We have

$$P^4_{g_0}\tilde{w} + 6(1-\mu) = (K \circ \Pi)e^{-4\mu(w_0\circ\Pi)}e^{4\tilde{w}},$$

from this identity, with the same argument as in Section 7 of [116], we obtain $\tilde{w} \in C^{3,\alpha}(\mathbb{S}^4)$ for $\alpha \in (0,1)$. In particular, $\tilde{w}$ in continuous at the South pole $S = (0,0,0,0,-1)$, which implies

$$w(x) = (1-\mu)w_0(x) + \tilde{w}(S) + o(1) = -\frac{\Lambda}{8\pi^2}\log|x| + C + o(1), \quad \text{as } |x| \to \infty.$$

□

Remark 2. If $P \in \mathcal{P}_2$ is a radially symmetric polynomial, there exists at least one non-normal radial solution to problem (2.1)-(2.2) of the form $u = w + P$. This follows from the fact that we can minimize the previous functional over radial functions and obtain $\bar{w}$ radially symmetric.

2.3 Asymptotic behavior

In all this section, let u be a solution to problem (2.1)-(2.2), we define

$$v(x) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy. \tag{2.4}$$

Obviously, we have $\Delta^2 v(x) = (1-|x|^p)e^{4u}$ in $\mathbb{R}^4$.

Lemma 2.3.1. *For $|x| \geq 4$, there exists a constant C such that*

$$v(x) \leq -\frac{\Lambda}{8\pi^2}\log|x| + C. \tag{2.5}$$

Proof. For $|x| \geq 4$, we decompose $\mathbb{R}^4 = B_1(0) \cup A_1 \cup A_2 \cup A_3$ where $A_1 = \{y \,|\, |y-x| \leq |x|/2\}$, $A_2 = \{y \,|\, 1 \leq |y| \leq 2\}$ and $A_3 = \mathbb{R}^4 \setminus (A_1 \cup A_2 \cup B_1)$. For $y \in B_1$ we have

$$\log\left(\frac{|y|}{|x-y|}\right) \leq -\log|x| + C$$

hence

$$\int_{B_1} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \leq (-\log|x| + C)\int_{B_1}(1-|y|^p)e^{4u}dy.$$

For $y \in A_1$ we have $\log\left(\frac{|y|}{|x-y|}\right) \geq 0$ hence the integral over A_1 is negative. For what concern A_2 we have

$$\int_{A_2} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy$$
$$= \int_{A_2} \log(|y|)\,(1-|y|^p)e^{4u}dy - \int_{A_2} \log(|x-y|)(1-|y|^p)e^{4u}dy$$

using the fact that the first integral is non positive, we get

$$\leq -\int_{A_2} \log(|x-y|)(1-|y|^p)e^{4u}dy \leq -\log|x| \int_{A_2}(1-|y|^p)e^{4u}dy + C$$

where in the last inequality we used that for $y \in A_2$ $\log|x-y| \leq \log|x| + C$. For $y \in A_3$ since $|x-y| \leq |x| + |y| \leq |x|\,|y|$ we have

$$\log\left(\frac{|y|}{|x-y|}\right) \geq -\log|x|$$

and since in this case $1-|y|^p < 0$ we get

$$\log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p) \leq -\log(|x|)\,(1-|y|^p)$$

and hence

$$\int_{A_3} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \leq -\log(|x|)\int_{A_3}(1-|y|^p)e^{4u}dy.$$

Summing up, we finally obtain

$$v(x) \leq -\frac{1}{8\pi^2}\log(|x|)\int_{A_1^C}(1-|y|^p)e^{4u}dy + C,$$

since $\int_{A_1^C}(1-|y|^p)e^{4u}dy \geq \Lambda$ we have

$$v(x) \leq -\frac{\Lambda}{8\pi^2}\log|x| + C.$$

□

Lemma 2.3.2. *For any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for $|x| \geq R$*

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 5\varepsilon)\log|x| - \frac{1}{8\pi^2}\int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy. \tag{2.6}$$

Proof. We can choose $R_0 = R_0(\varepsilon) > 1$ such that

$$\int_{B_{R_0}}(1-|y|^p)e^{4u}dy \leq \Lambda + \varepsilon.$$

Let us take $R > 2R_0$ and assume that $|x| \geq R$, we can decompose

$$\mathbb{R}^4 = B_{R_0}(0) \cup A_1 \cup A_2$$

where

$$A_1 := \{y \in \mathbb{R}^4 \,:\, |y-x| \le |x|/2\}$$
$$A_2 := \{y \in \mathbb{R}^4 \,:\, |y-x| > |x|/2,\ |y| \ge R_0\}.$$

For $|x| \ge R$ and $|y| \le R_0$ we have

$$\log\left(\frac{|y|}{|x-y|}\right) \le -\log|x| + C < 0$$

hence, we get

$$\begin{aligned}
\int_{B_{R_0}\setminus B_1} & \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \\
&\ge (-\log|x|+C)\int_{B_{R_0}\setminus B_1}(1-|y|^p)e^{4u}dy \\
&\ge (-\log|x|+C)\int_{B_{R_0}}(1-|y|^p)e^{4u}dy \\
&\ge (-\log|x|+C)(\Lambda+\varepsilon) \ge -(\Lambda+\varepsilon)\log|x|
\end{aligned}$$

where we used the fact that $\int_{B_{R_0}\setminus B_1}(1-|y|^p)e^{4u}dy \le \int_{B_{R_0}}(1-|y|^p)e^{4u}dy$. Concerning the integral over B_1 we have

$$\begin{aligned}
\int_{B_1}\log\left(\frac{|x-y|}{|y|}\right)(1-|y|^p)e^{4u}dy &\le \int_{B_1}\log\left(\frac{|x-y|}{|y|}\right)e^{4u}dy \\
&= \int_{B_1}\log\left(\frac{1}{|y|}\right)e^{4u}dy + \int_{B_1}\log(|x-y|)e^{4u}dy \le C
\end{aligned}$$

using Holder's inequality. Therefore, we obtain

$$\int_{B_{R_0}}\log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \ge -(\Lambda+\varepsilon)\log|x| - C. \tag{2.7}$$

We observe that $\log|x-y| \ge 0$ for $y \notin B_1(x)$, $\log|y| \le \log(2|x|)$ for $y \in A_1$, $\int_{A_1}(1-|y|^p)e^{4u}dy \ge -\varepsilon$ and $\log(2|x|) \le 2\log|x|$ for $|x| \ge R$, hence we get

$$\begin{aligned}
\int_{A_1} & \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \\
&= \int_{A_1}\log(|y|)\,(1-|y|^p)e^{4u}dy - \int_{A_1}\log(|x-y|)(1-|y|^p)e^{4u}dy \\
&\ge \log(2|x|)\int_{A_1}(1-|y|^p)e^{4u}dy - \int_{B_1(x)}\log(|x-y|)(1-|y|^p)e^{4u}dy \\
&\ge 2\log(|x|)\int_{A_1}(1-|y|^p)e^{4u}dy - \int_{B_1(x)}\log(|x-y|)(1-|y|^p)e^{4u}dy \\
&\ge -2\varepsilon\log|x| - \int_{B_1(x)}\log(|x-y|)(1-|y|^p)e^{4u}dy. \qquad (2.8)
\end{aligned}$$

For $y \in A_2$, in the case $|y| \le 2|x|$ we have $\frac{|y|}{|x-y|} \le 4$, while in the case $|y| \ge 2|x|$ we get $\frac{|y|}{|x-y|} \le 2$, so when $y \in A_2$ we have the estimate

$$\log\left(\frac{|y|}{|x-y|}\right) \le \log 4,$$

hence using that $\int_{A_2}(1-|y|^p)e^{4u}dy \geq -\varepsilon$, we obtain

$$\int_{A_2} \log\left(\frac{|y|}{|x-y|}\right)(1-|y|^p)e^{4u}dy \geq \log(4)\int_{A_2}(1-|y|^p)e^{4u}dy \geq -\varepsilon\log 4. \tag{2.9}$$

Putting together (2.7), (2.8) and (2.9), possibly taking R larger, we get

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda+5\varepsilon)\log|x| + \frac{1}{8\pi^2}\int_{B_1(x)}\log\left(\frac{1}{|x-y|}\right)(1-|y|^p)e^{4u}dy$$

□

From (2.6) changing signs, it follows that for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \geq R$

$$-v(x) \leq \frac{1}{8\pi^2}(\Lambda+5\varepsilon)\log|x| + \frac{1}{8\pi^2}\int_{B_1(x)}\log(|x-y|)(1-|y|^p)e^{4u}dy. \tag{2.10}$$

2.4 A Liouville-type theorem

To prove a Liouville-type theorem (see Theorem 2.4.2 below) we will need the following useful result.

Lemma 2.4.1. *Let u be a measurable function such that $(1-|y|^p)e^{4u} \in L^1(\mathbb{R}^4)$. Then for any $x \in \mathbb{R}^4$ it holds*

$$\fint_{B_r(x)} u^+ dy \to 0, \quad \textit{as } r \to +\infty.$$

Proof. Let $x \in \mathbb{R}^4$ be fixed, using the fact that $4u^+ \leq e^{4u}$ we get

$$4\fint_{B_r(x)} u^+dy \leq \fint_{B_r(x)} e^{4u}dy = \frac{C}{r^4}\int_{B_r(x)}\frac{1}{1-|y|^p}(1-|y|^p)e^{4u}dy. \tag{2.11}$$

Observing that for $y \in B_r(x)$ we have $|y| \leq r + |x|$ and $|y| \geq -r + |x|$, we obtain the following inequalities

$$\frac{1}{1-|y|^p} \leq \frac{1}{1-(|x|+r)^p},$$

and

$$\frac{1}{1-|y|^p} \geq \frac{1}{1-(|x|-r)^p},$$

by means of them we get

$$4\fint_{B_r(x)} u^+dy \leq \frac{C}{r^4}\left[\frac{1}{1-(|x|+r)^p}\int\limits_{B_r(x)\cap B_1}(1-|y|^p)e^{4u}dy \right.$$
$$\left. + \frac{1}{1-(|x|-r)^p}\int\limits_{B_r(x)\cap B_1^c}(1-|y|^p)e^{4u}dy\right]$$

since $\int_{B_r(x)}(1-|y|^p)\,e^{4u}dy < \infty$, we can estimate (2.11) with $O(r^{-p-4})$ as $r \to \infty$. The claim follows as $r \to +\infty$ since by assumption $p > 0$. □

We are now in position to prove the following Liouville-type theorem, which will be crucial to prove that $u - v$ is a polynomial of degree at most 2.

Theorem 2.4.2. *Consider $h : \mathbb{R}^4 \to \mathbb{R}$ such that*

$$\Delta^2 h = 0 \qquad \text{and} \qquad h \leq u - v.$$

Assume that $(1-|y|^p)e^{4u} \in L^1(\mathbb{R}^4)$, $v \in L^1_{loc}(\mathbb{R}^4)$ and further that (2.10) holds. Then, h is a polynomial of degree at most 2.

Proof. We take some ideas from the proof of [142, Theorem 6], but this proof is more delicate since our estimate for $-v$ contains the singular integral

$$A(x) := \frac{1}{8\pi^2}\int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.$$

By elliptic estimates for biharmonic functions (see [142, Proposition 4]) for any $x \in \mathbb{R}^4$ we have

$$|D^3 h(x)| \leq \frac{C}{r^3}\fint_{B_r(x)} |h(y)|dy = -\frac{C}{r^3}\fint_{B_r(x)} h(y)dy + \frac{2C}{r^3}\fint_{B_r(x)} h^+ dy. \tag{2.12}$$

From Pizzetti's formula (refer e.g. to [142]) we have

$$\fint_{B_r(x)} h(y)dy = O(r^2), \quad \text{as } r \to \infty. \tag{2.13}$$

In order to estimate the term $\frac{2C}{r^3}\fint_{B_r(x)} h^+ dy$, we observe that

$$\fint_{B_r(x)} h^+ dy \leq \fint_{B_r(x)} u^+ dy + C\fint_{B_r(x)} (-v)^+ dy,$$

thanks to Lemma 2.4.1 the term $\fint_{B_r(x)} u^+ dy \to 0$. Using Tonelli's theorem, we can prove that $A \in L^1(\mathbb{R}^4)$ as follows

$$\begin{aligned}
\int_{\mathbb{R}^4} |A(x)|\, dx &= \int_{\mathbb{R}^4} \left| \frac{1}{8\pi^2}\int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy \right| dx \\
&\leq \int_{\mathbb{R}^4} \frac{1}{8\pi^2}\int_{B_1(x)} \big|\log|x-y|\big| \big|1-|y|^p\big| e^{4u} dy\, dx \\
&= C\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \chi_{|x-y|<1} \big|\log|x-y|\big| \, \big|1-|y|^p\big| e^{4u} dy\, dx \\
&= C\int_{\mathbb{R}^4} \big|1-|y|^p\big| e^{4u} \int_{B_1(y)} \log\left(\frac{1}{|x-y|}\right) dx\, dy \\
&= C\int_{\mathbb{R}^4} \big|1-|y|^p\big| e^{4u} dy < \infty.
\end{aligned}$$

Since $\frac{1}{8\pi^2}(\Lambda + 5\varepsilon)\log|x| + A(x) \geq 0$ for $|x| \geq R > 2$, we get

$$(-v)^+ \leq \frac{1}{8\pi^2}(\Lambda + 5\varepsilon)\log|x| + A(x)$$

for $x \in \mathbb{R}^4 \setminus B_R$. Taking into account that $A(x) \in L^1$ we obtain

$$\fint_{B_r(x)} (-v)^+ dy \leq C\fint_{B_r(x)} \log(|y|+1)\, dy + \fint_{B_r(x)} A(y)dy \leq C\log r + \frac{C}{r^4}$$

(if $y \in B_R(0)$ the previous estimate for $(-v)^+$ does not hold, we can overcome this problem since $v \in L^1_{loc}(\mathbb{R}^4)$). Hence,

$$\fint_{B_r(x)} h^+ dy \leq \fint_{B_r(x)} u^+ dy + \frac{C}{r^4} + C \log r. \tag{2.14}$$

From (2.13) and (2.14), we get that all terms in (2.12) go to 0 as $r \to \infty$, hence we obtain $D^3 h \equiv 0$. ☐

2.5 Proof of the classification result

In order to prove that all solutions to problem (2.1)-(2.2) have the form $v + P$, where v behaves logarithmically at infinity and P is an upper-bounded polynomial of degree at most 2, we proceed by steps.

Theorem 2.5.1. *Let u be a solution to problem* (2.1)-(2.2) *such that $(1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4)$ and v as in* (2.4). *Then*

$$u = v + P$$

where P is a polynomial of degree at most 2. Moreover, $\Delta u(x)$ can be represented by

$$\Delta u(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2}(1 - |y|^p)e^{4u} dy + C, \tag{2.15}$$

where C is a constant.

Proof. Consider $P = u - v$, we have $\Delta^2 P = 0$. From (2.10) using Theorem 2.4.2 we can conclude that $u = v + P$ where P is a polynomial of degree at most 2. Hence $\Delta u = \Delta v + \Delta P$, it follows immediately that $\Delta P = C$ where C is a constant and from (2.4) we have that $\Delta v = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2}(1 - |y|^p)e^{4u} dy$. ☐

Let us prove that the polynomial P is upper-bounded.

Proposition 2.5.2. *Let P be the polynomial of Theorem 2.5.1. Then*

$$\sup_{x \in \mathbb{R}^4} P(x) < +\infty.$$

Proof. **Step 1. Estimate of the term $|A(x)|$.** We take some ideas from the proof of [142, Lemma 13] and the one of [133, Lemma 2.4], but our case is more challenging. In what follows, C denotes a generic constant which may change from line to line and also within the same line. We observe that

$$\begin{aligned}
|A(x)| &= \left| \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1 - |y|^p)e^{4u} dy \right| \\
&\leq \frac{1}{8\pi^2} \int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right) |1 - |y|^p| \, e^{4u} dy \\
&= \frac{1}{8\pi^2} \left(\int_{B_1(x) \setminus B_\tau(x)} + \int_{B_\tau(x)} \right) \log\left(\frac{1}{|x-y|}\right) |1 - |y|^p| \, e^{4u} dy
\end{aligned}$$

where $\tau \in (0,1)$ will be fixed later. Since $\log\left(\frac{1}{|x-y|}\right) \in (0, -\log \tau)$ for $y \in B_1(x) \setminus B_\tau(x)$ and by assumption $(1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4)$, we have

$$\int_{B_1(x) \setminus B_\tau(x)} \log\left(\frac{1}{|x-y|}\right) |1 - |y|^p| \, e^{4u} dy < C.$$

In order to estimate the integral over $B_\tau(x)$ we proceed as follows. By Holder's inequality we get

$$\begin{aligned}
&\int_{B_\tau(x)} \log\left(\frac{1}{|x-y|}\right) |1-|y|^p| \, e^{4u} dy \\
&\qquad \le \left(\int_{B_\tau(x)} \log\left(\frac{1}{|x-y|}\right)^2 dy\right)^{1/2} \left(\int_{B_\tau(x)} |1-|y|^p|^2 \, e^{8u} dy\right)^{1/2} \\
&\qquad \le \left(\int_{B_\tau(x)} \log\left(\frac{1}{|x-y|}\right)^2 dy\right)^{1/2} \left(\int_{B_\tau(x)} |1-|y|^p|^4 \, dy\right)^{1/4} \left(\int_{B_\tau(x)} e^{16u} dy\right)^{1/4}.
\end{aligned} \tag{2.16}$$

Fix $0 < \varepsilon_0 < 1$, we can choose $R_0 > 6$ sufficiently large such that

$$\int_{B_4(x)} |1-|y|^p| \, e^{4u} dy \le \varepsilon_0 \tag{2.17}$$

for $|x| \ge R_0$. Let h be the solution of

$$\begin{cases} \Delta^2 h = f & \text{on } B_4(x) \\ h = \Delta h = 0 & \text{on } \partial B_4(x) \end{cases}$$

where $f(y) = (1-|y|^p)e^{4u(y)}$, then by [142, Theorem 7] (refer also to [133, Lemma 2.3]) for any $k \in \left(0, \frac{8\pi^2}{\|f\|_{L^1(B_4(x))}}\right)$, we have $e^{4k|h|} \in L^1(B_4(x))$, and

$$\int_{B_4(x)} e^{4k|h|} dy \le C \tag{2.18}$$

where C is a constant which depends on k but is independent from x. For $y \in B_4(x)$ define $q(y) := u(y) - h(y)$, then q satisfies

$$\begin{cases} \Delta^2 q = 0 & \text{on } B_4(x) \\ \Delta q = \Delta u \quad \text{and} \quad q = u & \text{on } \partial B_4(x) \end{cases}.$$

Integrating equation $\Delta^2 u = (1-|y|^p)e^{4u}$ on $B_\rho(x)$ we get

$$\int_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta u) d\sigma = \int_{B_\rho(x)} (1-|y|^p)e^{4u} dy.$$

Dividing by $\omega_4 \rho^3$ and integrating with respect to ρ from 0 to R (we assume $R < 5$), using Fubini's theorem, we obtain

$$\int_0^R \frac{1}{\omega_4 \rho^3} \int_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta u) d\sigma \, d\rho = \fint_{\partial B_R(x)} \Delta u \, d\sigma - \Delta u(x)$$

and similarly

$$\int_0^R \frac{1}{\omega_4 \rho^3} \int_{B_\rho(x)} (1-|y|^p)e^{4u} dy \, d\rho = \frac{1}{4\pi^2} \int_{B_R(x)} (1-|y|^p)e^{4u} \left[\frac{1}{|x-y|^2} - \frac{1}{R^2}\right] dy.$$

Hence

$$\fint_{\partial B_R(x)} \Delta u \, d\sigma = \Delta u(x) + \frac{1}{4\pi^2} \int_{B_R(x)} (1-|y|^p) e^{4u} \left[\frac{1}{|x-y|^2} - \frac{1}{R^2} \right] dy$$

by means of identity (2.15) we get

$$-\fint_{\partial B_R(x)} \Delta u \, d\sigma = \frac{1}{4\pi^2} \int_{|x-y|\geq R} \frac{1-|y|^p}{|x-y|^2} e^{4u} dy + \frac{1}{4\pi^2 R^2} \int_{B_R(x)} (1-|y|^p) e^{4u} dy - C_1.$$

If we take $R = 4$ we have

$$-\fint_{\partial B_4(x)} \Delta u \, d\sigma \leq C. \tag{2.19}$$

Let G be the Green's function for the operator Δ on $B_4(x)$, namely

$$\Delta G = \delta_x \qquad \text{and} \qquad G = 0 \;\text{ on }\; \partial B_4(x),$$

we have

$$-\Delta q(x) = -\int_{\partial B_4(x)} \frac{\partial G}{\partial n} \Delta u \, d\sigma = -\int_{\partial B_4(x)} c_0 \, \Delta u \, d\sigma \leq C$$

where by [142, Lemma 12] $c_0 > 0$ and in the last inequality we used (2.19). Since c_0 is a positive constant there exist some $\tau \in (0,1)$ such that if $\xi \in B_{4\tau}(x)$ and G_ξ is the Green's function defined by

$$\Delta G_\xi = \delta_\xi, \quad G_\xi = 0 \;\text{ on }\; \partial B_4(x),$$

then

$$0 \leq \frac{\partial G_\xi(\eta)}{\partial r} \leq C, \quad \text{for } \eta \in \partial B_4(x), \quad r := \frac{\eta - x}{4}$$

and as before we get

$$-\Delta q(y) \leq C \quad \text{on } B_{4\tau}(x). \tag{2.20}$$

Define $\tilde{q}(y) := -\Delta q(y)$, obviously q satisfies

$$\begin{cases} \Delta q(y) = -\tilde{q}(y) & \text{in } B_4(x) \\ q = u & \text{on } \partial B_4(x) \end{cases}$$

hence by elliptic estimates (refer to [93, Theorem 8.17]) for any $\ell > 1$ and $\sigma > 2$

$$\sup_{B_\tau(x)} q \leq c(\ell, \sigma) \left(\|q^+\|_{L^\ell(B_{4\tau}(x))} + \|\tilde{q}\|_{L^\sigma(B_{4\tau}(x))} \right).$$

From (2.20) we get $\|\tilde{q}\|_{L^\sigma(B_{4\tau}(x))} \leq C$. Since $q = u - h$, it follows that $q^+(y) \leq u^+(y) + |h(y)|$ for $y \in B_4(x)$ and hence

$$\int_{B_{4\tau}(x)} (q^+)^2 \leq C \int_{B_{4\tau}(x)} e^{2q^+} \leq C \left(\int_{B_{4\tau}(x)} e^{4u^+} \right)^{1/2} \left(\int_{B_{4\tau}(x)} e^{4|h|} \right)^{1/2}.$$

Note that

$$e^{4u^+} \leq 1 + e^{4u} \leq 1 + |1 - |y|^p| \, e^{4u}, \quad \text{for } |y| \geq 2^{1/p}.$$

Since $B_{4\tau}(x) \subset B^c_{2^{1/p}}$ (eventually choosing R_0 greater) from (2.17) we get

$$\int_{B_{4\tau}(x)} e^{4u^+} dy \leq C$$

finally, together with (2.18), we obtain

$$\|q^+\|_{L^2(B_{4\tau}(x))} < C.$$

In this way we have shown that

$$\sup_{B_\tau(x)} q \leq C,$$

hence for $y \in B_\tau(x)$ we have

$$u(y) = q(y) + h(y) \leq C + |h(y)|,$$

from which we get

$$\int_{B_\tau(x)} e^{16u} dy \leq C \int_{B_\tau(x)} e^{16|h|} dy < C. \tag{2.21}$$

where in the last inequality we used (2.18). From (2.16) using (2.21) we get that

$$\int_{B_\tau(x)} \log\left(\frac{1}{|x-y|}\right) \big|1 - |y|^p\big| e^{4u} dy \leq C$$

hence $|A(x)| \leq C$.
Step 2. Let us define

$$f(r) := \sup_{x \in \partial B_r} P(x)$$

and assume by contradiction that $\sup_{\mathbb{R}^4} P = +\infty$. From [104, Theorem 3.1] it must exist $s > 0$ such that

$$\lim_{r \to +\infty} \frac{f(r)}{r^s} = +\infty.$$

Moreover, P is a polynomial of degree at most 2, hence $|\nabla P(x)| \leq c|x|$ for $|x|$ large. From Lemma 2.3.2 using the fact that $|A(x)| \leq C$, we get that there is R (sufficiently large) such that for every $r \geq R$ we can find x_r with $|x_r| = r$ such that

$$u(y) = v(y) + P(y) \geq r^s, \quad \text{for } |y - x_r| \leq \frac{1}{r}.$$

We consider

$$\begin{aligned}
-\int_{B_1^c} (1 - |y|^p) e^{4u} dy &= \int_{B_1^c} (|y|^p - 1) e^{4u} dy \geq \int_{B_R^c} (|y|^p - 1) e^{4u} dy \\
&\geq \int_{B_R^c} e^{4u} dy \geq C \int_R^{+\infty} \int_{\partial B_r \cap B_{1/r}(x_r)} e^{4u} d\sigma\, dr \\
&\geq C \int_R^{+\infty} \int_{\partial B_r \cap B_{1/r}(x_r)} e^{4r^s} d\sigma\, dr \geq C \int_R^{+\infty} \frac{e^{4r^s}}{r^3} dr = +\infty.
\end{aligned}$$

which is absurd since $(1 - |y|^p) e^{4u} \in L^1(\mathbb{R}^4)$. □

The fact that P is upper-bounded implies that P has even degree, hence or P has degree 2 or is constant. We are now in the position to prove a more precise estimate from below for the function v.

Proposition 2.5.3. *Let u be a solution to* (2.1)-(2.2) *such that* $(1-|y|^p)e^{4u} \in L^1(\mathbb{R}^4)$ *and v as defined in* (2.4). *Then, given any $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that for $|x| \geq R$ it holds*

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 6\varepsilon)\log|x|. \tag{2.22}$$

Moreover, we have

$$\lim_{|x|\to+\infty} \Delta v(x) = 0. \tag{2.23}$$

Proof. First we prove (2.22). From Lemma 2.3.2 for any $\varepsilon > 0$ there exists $R > 0$ such that for $|x| \geq R$

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 5\varepsilon)\log|x| - \frac{1}{8\pi^2}\int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.$$

Notice that a priori the second term on the right-hand side may be very little, but in the proof of Proposition 2.5.2 we have proven that $|A(x)| \leq C$, so (2.22) follows at once.

Now we prove (2.23). Differentiating we have

$$\Delta v(x) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{1}{|x-y|^2}(1-|y|^p)e^{4u}dy.$$

For any $\sigma > 0$, by dominated convergence

$$\int\limits_{\mathbb{R}^4\setminus B_\sigma(x)} \frac{(1-|y|^p)e^{4u}}{|x-y|^2}dy \to 0, \quad \text{as } |x| \to +\infty.$$

By Holder's inequality we get

$$\int\limits_{B_\sigma(x)} \frac{(1-|y|^p)e^{4u}}{|x-y|^2}dy \leq \left(\int\limits_{B_\sigma(x)} \frac{(1-|y|^p)^k}{|x-y|^{2k}}dy\right)^{1/k} \left(\int\limits_{B_\sigma(x)} e^{4k'u}dy\right)^{1/k'}$$

if σ is small enough, by (2.21) we can conclude. □

Proof of Theorem 2.1.2. It follows from Lemma 2.3.1, Theorem 2.5.1, Proposition 2.5.2 and Proposition 2.5.3. □

Corollary 2.5.4. *Any solution u to* (2.1)-(2.2) *is bounded from above.*

Proof. The solution u is continuous and $u = v + P$. Moreover from (2.5) we have that $v(x) \leq C$ on B_4^c and from Proposition 2.5.2 we have $\sup_{\mathbb{R}^4} P(x) < +\infty$. □

Part II

Stationary Mean Field Games with Riesz-type aggregation.

Mean Field Games theory

The theory of Mean Field Games (MFG in short) has been introduced around 2006 in a series of seminal papers by Lasry and Lions [123–125] in order to model Nash equilibria of differential games with infinitely many interacting agents. Independently at about the same time, Caines, Huang and Malhamé [112–114] developed the analogous concept of "Nash centainly equivalence principle". Since then, the study of MFG rapidly grew up, also encouraged by its powerful applications in a wide range of disciplines: equations of this kind arise in Economics, Finance, models of social systems and crowd motions. For a complete presentation of the theory and its applications, we refer the reader to P.-L. Lions series of lectures at Collège de France [137], the lectures by Gueant, Lasry and Lions [108] and also [1, 34, 42, 43, 102] among many others.

In the Mean Field Games theory, players are assumed to be *indistinguishable* and *"rational"*, that is each of them optimizes his/her behavior by taking into account the behavior of the other players, in this sense each individual strategy is influenced by some averages of quantities depending on the states of the other agents. In other words, each agent chooses his/her optimal strategy in view of global information that are available to him/her and that result from the action of all other players, which is described through the distribution law of the dynamical states. Moreover, agents are *infinitesimal*, namely they are small compared to the collection of all other controllers and hence individually have a negligible influence on the game. In this setting, the key idea underlying the theory comes from Statistical Mechanics, and consists in a mean-field approach to describe equilibria in a system of many interacting identical particles (see for instance the notes by Sznitman [178]). The other central concept is the notion of Nash equilibrium, which describes how agents play in an optimal way by taking into account the others' strategies. In particular, there is a Nash equilibrium when no controller has interest to deviate unilaterally from the planned control.

II.1 Mean Field Games: an overview

Let us briefly describe from a PDE viewpoint, the heuristic derivation of the MFG system in the simplest case where the state space is $\mathbb{R}^N$ and the time horizon is finite (we mainly follow the approach of [1, Chapter 1]). We stress that, even if our problem deals with the stationary case, we decided to introduce MFG system in the time-dependent case to be as generic as possible. On the other hand, it is not our aim to provide all details and proof, since a rigorous treatment on the topic can be found in the above mentioned references.

Let us assume to have a differential game with infinitely many players, each agent controls his/her own dynamics, which is described by the following stochastic differential equation (SDE) with values in $\mathbb{R}^N$

$$X_s = x + \int_t^s b(X_r, v_r, m(r))dr + \sqrt{2\varepsilon}\, B_s$$

where $x \in \mathbb{R}^N$, the control v belongs to the set $\mathcal{A}$ of admissible control processes, $m \in \mathcal{P}(\mathbb{R}^N)$ is the distribution of all players, the drift $b : \mathbb{R}^N \times \mathcal{A} \times \mathcal{P}(\mathbb{R}^N) \to \mathbb{R}^N$ is sufficiently smooth, ε is a non-negative parameter and B_s is a N-dimensional standard Brownian motion starting at 0. Notice that at this stage we assume that m is given, from the point of view of the model we could think to it as the anticipation made by agents of the future distribution of players. Each player controls the state X through the control v in order to minimize a certain common cost. In a typical MFG model, the cost each agent wants to minimize is

$$J(t, x, v) = \mathbb{E}\left[\int_t^T L(X_s, v_s, m(s))ds + G(X_T, m(T))\right]$$

where $T > 0$ is the finite horizon of the problem and L and G are continuous maps. We define the value function of this optimal control problem as the map $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$u(t, x) = \inf_{v \in \mathcal{A}} J(t, x, v).$$

Taking advantage of some basic notions of optimal control, dynamic programming and Itô's formula, one can verify that at least formally, the value function u solves the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(x, Du, m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(T, x) = G(x, m(T)) \end{cases}$$

where (slight abuse of notation)

$$H(x, p, m) := \sup_{v \in \mathcal{A}} \big[- L(x, v, m) - b(x, v, m) \cdot p \big]$$

is the Hamiltonian of our problem. Finally, if $v^*(t, x)$ is defined as the maximum point in the definition of H when $p = Du(t, x)$, that is

$$H(x, Du, m) = -L(x, v^*, m) - b(x, v^*, m) \cdot p,$$

by standard arguments in control theory, one can verify that $v^*(t, x)$ is the optimal feedback, i.e. the optimal strategy to implement at time t in the state x. Moreover, under suitable assumptions, the drift is of the form

$$b(x, v^*, m) = -D_p H(x, Du, m).$$

Using the control v^* agents play in optimal way and the dynamic of each of them is described by

$$dX_s^* = b(X_s^*, v^*(s, X_s^*), m(s)) + \sqrt{2\varepsilon}\, dB_s.$$

Since we assume that all the agents behave in this way, and moreover that the initial position and the noise in their dynamics are independent (there is no common noise), studying the mean field limit, we obtain that at time s the distribution of all agents $\tilde{m}(s)$

is described by the law of X_s^* with initial distribution $m_0 \in \mathcal{P}(\mathbb{R}^N)$. Hence, the function $\tilde{m}$ satisfies in the sense of distributions the following Kolmogorov equation

$$\begin{cases} \partial_t \tilde{m}(t,x) - \varepsilon \Delta \tilde{m}(t,x) + \operatorname{div}\big(\tilde{m}(t,x) b(x, v^*(t,x), m(t))\big) = 0 \quad \text{in } (0,T)\times\mathbb{R}^N \\ \tilde{m}(0) = m_0 \quad \text{in } \mathbb{R}^N \end{cases}.$$

If all agents play optimally, in an equilibrium regime for the game, we expect that the anticipation m is correct, namely $m(t) = \tilde{m}(t)$. This leads to the MFG system

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(x, Du, m) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^N \\ \partial_t m - \varepsilon \Delta m - \operatorname{div}\big(m\, D_p H(x, Du, m)\big) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^N \\ m(0) = m_0, \quad u(x,T) = G(x, m(T)) \end{cases}. \tag{II.1}$$

In particular if the cost of control is separate from the mean-field dependent one, the MFG system assumes the following form:

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(x, Du) = F(x,m) \qquad \text{in } (0,T)\times\mathbb{R}^N \\ \partial_t m - \varepsilon \Delta m - \operatorname{div}\big(m\, D_p H(x, Du)\big) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^N \\ m(0) = m_0, \quad u(T) = G(x, m(T)) \end{cases}. \tag{II.2}$$

The above systems consist of a backward Hamilton-Jacobi equation coupled with a forward Kolmogorov-Fokker-Planck equation. In particular, the Hamilton-Jacobi equation is related to the optimal control problem of a typical small agent and it is solved by the value function of each player; while the Kolmogorov equation is related to the distribution law of the individual states and provides the evolution of the population density. The MFG system (II.1) describes, from a PDE viewpoint, Nash equilibria of differential games with infinitely many players. Indeed, the deviation of a single agent does not change the population dynamics, this is due to the fact that each agent is assumed to be "small" compared to the whole population. It follows that in the individual optimization the behavior of the other players, and hence their distribution $m(t)$, can be taken as given. In short, all agents play an optimal strategy while freezing the others' choices.

Conditions under which the MFG system admits a solution, and also its uniqueness and stability, can be found already in the first works by Lasry and Lions [123–125] in the case when F and G are monotone *smoothing operators* on the space of probability measures (i.e. the behavior of the couplings depends on the global behavior of m). Uniqueness of the solution is in general not expected, if Lasry-Lions' monotonicity condition fails (see e.g. [10, 33, 42, 43]), while it is proved to hold for short time horizon. On the other hand, existence of smooth solutions in the case of *local couplings*, namely when the functions F and G depend on the (Lebesgue) density of the measure m, is much more involved. We refer the reader to [100, 101, 103] for existence of smooth solutions under either growth limitations on the Hamiltonian or growth limitations on the coupling (see also [40]). In this context, starting from [125] where existence of weak solutions has been proved, a complete theory was then developed in [169, 170] proving uniqueness and stability of weak solutions in quite general setting (see also [35, 36]). We refer the reader to [1, Chapter 1] and the references therein, for a complete overview on this topic.

We are now in the position to heuristically introduce **second order stationary MFG systems**, this class of systems will be the main object of study of Part II. Similarly

to before, we assume that the dynamics of a typical player is described by the following controlled SDE
$$dX_t = -v_t dt + \sqrt{2}\, dB_t, \quad X_0 = x \in \mathbb{R}^N,$$
where v_t is the controlled velocity and B_t is a standard N-dimensional Brownian motion. Each agent wants to minimize the following long-time average cost
$$J(x,v) := \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T L(v_t) + F(X_t, m(X_t))\, dt\right],$$
where the Lagrangian L represents the cost of moving with velocity v_t, while the function $F(X_t, m(X_t))$ represents the cost of being at position X_t, modeling the coupling between the individual and the overall population. Following the same arguments as before (refer also to [16, 111]), we can associate to this optimal control problem a *stationary* Hamilton-Jacobi-Bellman equation
$$-\Delta u(x) + H\big(\nabla u(x)\big) + \lambda = F\big(x, m(x)\big) \quad \text{in } \mathbb{R}^N$$
where the Hamiltonian H is the Legendre transform of L, that is $H(p) := \sup_{a\in\mathbb{R}^N}[-L(a) + p\cdot a]$. It follows that the optimal velocity of a typical player is given in feedback form by $-\nabla H(\nabla u(\cdot))$. Since the drift is autonomous, if all agents play in an optimal way then, the law of X_t becomes stable in the long-time regime, namely as $t\to+\infty$ it converges to the measure $\tilde{m}$ which solves a *stationary* Kolmogorov-Fokker-Planck equation with drift $b(x) = -\nabla H(\nabla u(x))$. In an equilibrium regime, the invariant measure $\tilde{m}$ (independent of the initial position) coincides with the population density m. Equilibria of the game are encoded in the following *stationary* MFG system
$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = F(x,m) & \text{in } \mathbb{R}^N \\ -\Delta m - \operatorname{div}\big(m\nabla H(\nabla u)\big) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = 1 \end{cases} \tag{II.3}$$
where the unknowns are the function u and m, and the ergodic constant $\lambda\in\mathbb{R}$. Notice that λ can be interpreted as the value of the game, indeed if there exists an admissible control v^* such that $v^*(x) \in \operatorname{argmax}_{a\in\mathbb{R}^N}[a\cdot\nabla u - L(a)]$ then, $\lambda = J(x, v^*)$.

As already pointed out in [123], system (II.3) can also be defined as the limit, when the number of players tends to infinity, of Nash equilibria of ergodic differential games (refer to [34] for the analogous result for time-dependent problems).

A natural problem that arises in the literature is the study of the long-time behavior of the system (II.2) in connection with the corresponding ergodic control problem for mean field games (see [137]). The convergence of the MFG system in $(0,T)$, as the time horizon T tends to infinity, towards a stationary ergodic system of the form (II.3), has been proved under some special assumptions but it remains still open in the general case. More in detail, the long-time behavior is completely described for purely quadratic Hamiltonians [39, 40], in the case of smoothing couplings and uniformly convex Hamiltonians [41] and for local couplings and globally Lipschitz Hamiltonian [171] (see also [37, 146]). This is also related to the study of the long time stability of solutions, it can be proved that solutions become nearly stationary for most of the time, showing a so-called *turnpike pattern* (see [171]).

Finally, we point out that in order to study existence of solutions for MFG systems, the main arguments in the literature are based on variational methods, convex optimization, and elliptic regularity (refer e.g. to [15, 35, 38, 44, 97, 98, 102, 149] and to

Chapter 4). We can also construct solutions to MFG systems by exploiting a fixed point argument and an approximation procedure (see [9, 70, 102, 123] and also [69, 76] for multi-population systems), we will use this strategy in Chapter 3. Notice that, introducing sequences of regular couplings that converge to the original ones and studying the corresponding approximating problem, is a very common tool in such kind of problems (refer to [44, 70, 137]), indeed using the stability of the equation and some a priori estimates, one can prove that approximate solutions converge to a solution of the initial problem.

II.2 Our problem: a Riesz-type coupling

In Chapter 3 and Chapter 4, we study second-order ergodic Mean Field Games systems defined in the whole space $\mathbb{R}^N$, with attractive nonlocal coupling given in terms of a Riesz interaction kernel. Our analysis takes into account both the model in presence of an external coercive potential V and the potential-free case. More in detail, given $M > 0$, we consider elliptic systems of the form

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - \int_{\mathbb{R}^N} \frac{m(x)}{|x-y|^{N-\alpha}} dy \\ -\Delta m - \operatorname{div}(m \nabla u(x)\, |\nabla u(x)|^{\gamma - 2}) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \quad \text{in } \mathbb{R}^N \tag{II.4}$$

where $\gamma > 1$ and $\alpha \in (0, N)$ are fixed. Note that the unknowns in the previous system are the functions u, m and the constant $\lambda \in \mathbb{R}$, which can be interpreted as a Lagrange multiplier related to the mass constraint $\int_{\mathbb{R}^N} m = M$, or as the value of the game.
In Chapter 3, we will assume that the potential V is a locally Hölder continuous coercive function, that is there exist b and C_V positive constants such that

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \leq V(x) \leq C_V(1 + |x|)^b, \quad \forall x \in \mathbb{R}^N. \tag{II.5}$$

Notice that the assumption on V to be non-negative is not restrictive, we can assume more generally that V is bounded from below and shift appropriately λ. On the other hand, in Chapter 4 we will study the corresponding potential-free MFG system, that is we assume that V is identically 0.
The coupling in the system is given through the interaction term $-K_\alpha * m$, where $K_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$, defined for every $x \in \mathbb{R}^N \setminus \{0\}$ as

$$K_\alpha(x) = \frac{1}{|x|^{N-\alpha}}.$$

Finally, we assume, for sake of simplicity, that the Hamiltonian H in the system (II.4) has the form $H(p) = \frac{1}{\gamma}|p|^\gamma$. Actually, the results we obtain in Chapter 3 and Chapter 4 also hold under more general assumptions on the Hamiltonian, namely assuming that $H : \mathbb{R}^N \to \mathbb{R}$ is strictly convex, $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and there exist C_H, $K > 0$ and $\gamma > 1$, such that $\forall p \in \mathbb{R}^N$ the following conditions hold

$$C_H|p|^\gamma - K \leq H(p) \leq C_H|p|^\gamma$$

$$\nabla H(p) \cdot p - H(p) \geq K^{-1}|p|^\gamma - K$$

$$|\nabla H(p)| \leq K|p|^{\gamma-1}.$$

Our aim is to study existence and nonexistence of classical solutions to the MFG system (II.4) in the two above-mentioned cases. We recall that by *classical solution* we mean a triple $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$ for every $p \in (1, +\infty)$ solving the system.

Before discussing our results more in detail, let us take a look at some considerations regarding our model and the current state-of-the-art.

In our setting, each agent wants to minimize the following long-time average cost

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{|v_t|^{\gamma'}}{\gamma'} + V(X_t) - K_\alpha * m(X_t)dt\right]$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ and $m(x)$ is the density of population at $x \in \mathbb{R}^N$. We observe that the potential cost has two components: the term V and the term $-K_\alpha * m$. The coercive potential V describes spatial preferences of agents and hence (if present) discourages them from being far away from the origin. The Riesz-type interaction potential $-K_\alpha * m$, represents the coupling between the individual and the overall population, due to it every player of the game is attracted towards regions where the population is highly concentrated. In addition, the dynamics of each player is subject to a Brownian noise which induces a dissipation effect. So, existence results for classical solutions to the MFG system (II.4) will depend on balancing between dissipation and aggregation. Indeed, if aggregating forces are too strong, the mass m tends to concentrate and hence to develop singularities, while if the diffusion dominates, we might have loss of mass at infinity, in both cases we expect nonexistence of classical solutions.

In the particular case when $\gamma = \gamma' = 2$, using the Hopf-Cole transformation $v(x) := e^{-u(x)/2}$ (see [123, 125]), we can reduce our MFG system to a single PDE. More in detail, exploiting the previous change of variable and setting $m(x) = v^2(x)$, the MFG system (II.4) proves to be equivalent to the following normalized Choquard equation

$$\begin{cases} -2\Delta v + (V(x) - \lambda)v = (K_\alpha * v^2)v \\ \int_{\mathbb{R}^N} v^2(x)dx = M, \qquad v > 0 \end{cases} \qquad \text{in } \mathbb{R}^N, \tag{II.6}$$

with associated energy

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} 2|\nabla v|^2 + V(x)v^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)\, v^2(y)}{|x-y|^{N-\alpha}} dx\, dy.$$

Notice that the relation between MFG systems and normalized nonlinear elliptic equations has recently been exploited in [164] in the case of nonlinear Schrödinger systems. Choquard-type equations have been intensively studied during last decades since they appear in the context of various mean-field type physical models. The peculiarity of the Choquard equation lies in the attractive interaction potential, which is given in terms of a Riesz interaction kernel and therefore it is weaker and with longer range than the usual power-type potential in the classical Nonlinear Schrödinger equation. Existence of solutions to the normalized Choquard equation (II.6) in the case $V \equiv 0$, was first investigated using variational methods by Lieb [130]. In particular, he proved existence and uniqueness (up to translations) of solutions when $N = 3$ and $\alpha = 2$ by using symmetric decreasing rearrangement inequalities. Then, P.-L Lions [134] proved that there exists a minimum of the energy associated to (II.6) (with $V \equiv 0$) when we restrict the infimum to functions with spherical symmetry, while more recently Li and Ye [128] studied existence of positive solutions by using a minimax procedure and the concentration-compactness principle. We refer the reader to [135, 136, 152, 156] and references therein for a complete overview of the topic. Finally, as we shall see later, the existence result in Chapter 4 (Theorem 4.1.1 below) provides a more general result for the range of values α such that the normalized Choquard equation (II.6) with $V \equiv 0$ has a solution, but it leaves open the problem of symmetry of solutions.

Our model is defined in the whole Euclidean space $\mathbb{R}^N$, while usually, Mean Field Games systems are defined on bounded domains, with Neumann or periodic boundary conditions, in order to avoid non-compactness issues (see e.g. [70, 169]). We recall some works in the non-compact setting: in particular [11] in the linear-quadratic framework, [170] in the time-dependent case, [99] for regularity results, and finally [44] where a system analogous to (II.4) but with power-type coupling, has been considered. The unbounded setting brings new difficulties. Indeed, in order to avoid loss of mass, the diffusion induced by the Brownian motion has to be compensated by the optimal velocity, which is a priori unknown and depends on the distribution m itself and on the confining potential V (if present).

The second distinctive feature of our Mean Field Game is the nonlocal attractive coupling, we will refer to such kind of models as *focusing* MFG systems, namely models with coupling which encourages aggregation. Stationary *focusing* MFG systems with different assumptions on the coupling have been studied e.g. in [44, 45, 70, 71, 97]. The problem of existence of solutions to focusing MFG systems requires different approaches than the ones that have been developed in the literature to study *defocusing* MFG systems, namely models where individuals avoid areas with high density of population. Indeed, an increasing coupling is essential if one seeks for uniqueness of equilibria, and it is in general crucial in many existence and regularity arguments (see [102]).

Finally, a similar MFG system but with local decreasing coupling defined in terms of a power-type function, has been studied in [44]. We point out that, unlike it, in our setting the nonlocal attractive coupling models a long-range attractive force between players, moreover, in order to deal with the Riesz-term we need different techniques compared to the ones used in [44].

In **Chapter 3**, we show how the interplay between dissipation (induced by the diffusive term in the system) and aggregating forces (resulting from the Riesz-type attractive coupling and the action of the coercive potential V), could affect existence of classical solutions to the MFG system (II.4). In particular, we observe that the strength of the Riesz potential is related to the parameter α, which has two critical thresholds: the *Hardy-Littlewood-Sobolev-critical value* $\alpha = N - 2\gamma'$ and the *mass-critical value* $\alpha = N - \gamma'$. It follows that the MFG system (II.4) exhibits three different regimes which correspond to $\alpha \in (0, N - 2\gamma'$, $\alpha \in (N - 2\gamma', N - \gamma']$ and $\alpha \in (N - \gamma', N)$.

In the *Hardy-Littlewood-Sobolev-supercritical regime* $0 < \alpha < N - 2\gamma'$, assuming $V \equiv 0$, we get that "regular" solutions, namely satisfying some integrability and boundary conditions at infinity, do not exist. In order to prove this nonexistence result (refer to Theorem 3.1.1) we take advantage of a suitable Pohozaev-type identity and we argue by contradiction. Notice that, starting from the celebrated identity due to Pohozaev [168] and the well-known extension by Pucci and Serrin [172], Pohozaev-type identities have been used to prove nonexistence results for various kinds of nonlinear PDEs (see also

Section 1.3 and Section 5.4).

On the other hand, in the *Hardy-Littlewood-Sobolev-subcritical regime* $N-2\gamma'<\alpha<N$, taking advantage of the fixed-point structure naturally associated to the MFG system, we obtain existence of classical solutions to (II.4) (refer to Theorem 3.1.2). More in detail, we consider a regularized version of problem (II.4), obtained by convolving the Riesz-interaction term with a sequence of standard symmetric mollifiers approximating the unit. In our case, it is crucial to obtain a uniform L^∞-estimate on the Riesz-coupling, this bound can not be obtained without the above mentioned "approximating" procedure (see Lemma 3.4.5). Then, exploiting an extension of the classical Schauder Fixed Point Theorem, we show that solutions to the "regularised" version of our MFG system do exist. Actually, in the case when $N-2\gamma'\le\alpha\le N-\gamma'$ we obtain existence of solutions assuming that the total mass M is smaller than a threshold value M_0, while if $N-\gamma'<\alpha<N$ solutions exist for every total mass $M>0$. Finally, we show some a priori uniform (not depending on the mollifiers) estimates on the solutions of the regularized problem, using them we can pass to the limit and obtain a classical solution of the MFG system (II.4). We refer the reader to Chapter 3 and the work [20] for further details.

The previous results can also be interpreted from a variational point of view, this perspective provides a better understanding of our model's behavior and of the deep analogy with normalized Choquard-type equations. As Lasry and Lions first pointed out in [125], taking into account the variational nature of the MFG system, solutions to (II.4) are related to critical points of the following energy functional (assuming $\varepsilon=1$)

$$\mathcal{E}(m,w):=\begin{cases}\int\limits_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)+V(x)\,m\,dx-\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy & \text{if } (m,w)\in\mathcal{K}_{\varepsilon,M}\\ +\infty & \text{otherwise}\end{cases} \tag{II.7}$$

where

$$L\left(-\frac{w}{m}\right):=\begin{cases}\frac{1}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} & \text{if } m>0\\ 0 & \text{if } m=0,\ w=0\\ +\infty & \text{otherwise}\end{cases}$$

and the constrained set is defined as

$$\mathcal{K}_{\varepsilon,M}:=\Big\{(m,w)\in(L^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N))\times L^1(\mathbb{R}^N)\quad\text{s.t.}\quad\int_{\mathbb{R}^N} m\,dx=M,\quad m\ge 0\ \text{a.e.}$$
$$\varepsilon\int_{\mathbb{R}^N} m(-\Delta\varphi)\,dx=\int_{\mathbb{R}^N} w\cdot\nabla\varphi\,dx\quad\forall\varphi\in C_0^\infty(\mathbb{R}^N)\Big\} \tag{II.8}$$

with

$$q:=\begin{cases}\frac{N}{N-\gamma'+1} & \text{if } \gamma'<N\\ \gamma' & \text{if } \gamma'\ge N\end{cases}.$$

If $N-\gamma'<\alpha<N$, so in the *mass-subcritical regime*, the energy $\mathcal{E}$ is bounded from below. More precisely, using elliptic regularity results for the Kolmogorov equation (see Proposition 3.2.4 below), the Hardy-Littlewood-Sobolev inequality and the fact that $V\ge 0$, we get

$$\mathcal{E}(m,w)\ge C_1\|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}}-C_2\|m\|^2_{L^\beta(\mathbb{R}^N)}$$

where $\beta=\frac{2N}{N+\alpha}$, and in this regime we have $\frac{2\gamma'}{N-\alpha}>2$. Hence, $\inf_{(m,w)\in\mathcal{K}_{1,M}}\mathcal{E}(m,w)$ is well-defined and by means of classical direct methods and compactness arguments, it is possible to construct global minimizers. Then, a linearization argument and a convex duality theorem allow us to show that minimizers (m, w) of $\mathcal{E}$ correspond to solutions to the MFG system (II.4) (for more details we refer to Chapter 4 and also [19, 20, 44, 45]). In the *mass-critical regime*, namely, for $\alpha = N - \gamma'$, the energy is bounded from below just for sufficiently small masses M, and we may construct in this range global minimizers (refer to Section 4.6.1). Finally, in the *mass-supercritical regime*, namely for $0 < \alpha < N - \gamma'$, the energy is not bounded from below in general, so no global minimum can be found. Nonetheless, some compactness of sequences with finite energy is still available in the *Hardy-Littlewood-Sobolev-subcritical regime* $N - 2\gamma' < \alpha < N - \gamma'$. More in-depth, we consider a minimization problem adding a smallness constraint on the $L^{\frac{2N}{N+\alpha}}$-norm of m and we show that if the total mass is sufficiently small, then constrained minimizers are actually local free minimizers of the problem (see Section 4.6.2). This argument provides solutions to our Mean-Field Game, but at this stage, we do not know if they coincide with solutions we obtained by using the Schauder fixed-point approach. A similar procedure dealing with local minimizers has recently been developed in [72] for MFG on bounded domains with Neumann boundary conditions and local attractive interaction potential of polynomial type. Moreover, since the energy becomes more and more negative as the $L^{\frac{2N}{N+\alpha}}$-norm of m increases (as it can be observed by a simple rescaling argument), then we expect that with a nontrivial adaptation of the mountain-pass theorem, it should be possible to construct in the Hardy-Littlewood-Sobolev-subcritical regime $N - 2\gamma' < \alpha < N - \gamma'$ also solutions to the MFG with a min-max procedure (analogously to what is done in the case of normalized Choquard equation, see [128]). We plan to investigate this issue in the future.

In **Chapter 4** we investigate the problem of existence of classical solutions to the MFG system (II.4) for $\alpha \in (N - \gamma', N)$ when $V \equiv 0$. The strategy of the proof is the following: we consider an auxiliary MFG system where the Brownian motion depends on a parameter $\varepsilon > 0$ and there is an external confining potential V. Using the above-mentioned variational approach (recall that in this regime the energy $\mathcal{E}$ is bounded from below) we are able to find solutions to the auxiliary system. Then, we define a suitable rescaling of solutions and we study their asymptotic behavior in the vanishing viscosity limit, namely as $\varepsilon \to 0$. Taking advantage of an appropriate adaptation of the concentration-compactness Lions theorem, we prove that no loss of mass occurs in the limit, this implies existence of solutions to the potential-free MFG in the mass subcritical regime $\alpha \in (N - \gamma', N)$. Moreover, solutions are global minimizers of the energy $\mathcal{E}$ with $V \equiv 0$, among competitors which satisfy an appropriate integrability condition (see Theorem 4.1.1). Finally, we prove that in the vanishing viscosity setting, there is concentration of mass around minima of the potential V. We left open the problem of existence of solutions to the potential-free MFG system when $\alpha \in (N - 2\gamma', N - \gamma']$. We refer the reader to Chapter 4 and the research paper [19] for additional details.

Open Problems. An interesting open question concerns the analysis of the corresponding time-dependent MFG system, that is a system of the form

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{\gamma}|\nabla u|^\gamma = -\int_{\mathbb{R}^N} \frac{m(x,t)}{|x-y|^{N-\alpha}} dx & \text{in } \mathbb{R}^N \times (0,T) \\ \partial_t m - \Delta m - \operatorname{div}(m \nabla u(x)\, |\nabla u(x)|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ m(x,0) = m_0(x), \;\; u(x,T) = u_T(x) & \text{on } \mathbb{R}^N \end{cases}$$

where where $\int_{\mathbb{R}^N} m_0 \, dx = M$, $m_0 \geq 0$ a.e. and $m_0, u_T \in C^2(\mathbb{R}^N)$. Finite-horizon MFG systems in the case of local coupling as been investigated for instance in [36, 75, 102].

We expect that a similar approach as the one exploited in our works for the case of stationary equations, should work also in the parabolic setting. Hence, the next step could be to study existence of solutions to the previous evolutionary MFG system, and their long-time stability. On this point, we recall that convergence as $T \to +\infty$ is almost completely open in the case of non-monotone coupling (see [74] in the periodic setting for mildly non-monotone MFG).

Chapter 3

Ergodic Mean-Field Games with aggregation of Choquard-type

3.1 Introduction to the problem and main results

In this chapter, we study ergodic Mean-Field Games systems defined in the whole space $\mathbb{R}^N$ with a coercive potential V and attractive nonlocal coupling, defined in terms of a Riesz interaction kernel. More in details, given $M > 0$, we consider elliptic systems of the form

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m(x) \\ -\Delta m - \operatorname{div}(m\nabla u(x)\,|\nabla u(x)|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{3.1}$$

where $\gamma > 1$ and $\alpha \in (0, N)$ are fixed. Note that the unknowns in the system (3.1) are the functions u, m and the constant $\lambda \in \mathbb{R}$. We will assume that the potential V is a locally Hölder continuous coercive function, that is there exist b and C_V positive constants such that

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \leq V(x) \leq C_V(1 + |x|)^b, \quad \forall x \in \mathbb{R}^N. \tag{3.2}$$

The assumption on V to be non-negative is not restrictive, we can assume more generally that V is bounded from below and shift appropriately λ.

We provide existence and nonexistence results of classical solutions solving the MFG system (3.1), where by *classical solution* we will mean a triple $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$ for every $p \in (1, +\infty)$, solving the system. Our focus will be to obtain classical solutions which satisfy some integrability conditions and boundary conditions at ∞, which will be meaningful from the point of view of the game. In particular, we will require some integrability properties of the optimal speed with respect to m and of the confining potential V with respect to m, namely

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad |\nabla m||\nabla u| \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad Vm \in L^1(\mathbb{R}^N). \tag{3.3}$$

Indeed, if one looks at the Kolmogorov equation, such integrability properties are important to ensure some minimal regularity of m and uniqueness of the invariant distribution itself (see [110, 150]). Regularity and boundedness of m is quite crucial in our setting: indeed, due to the aggregating forces, m has an intrinsic tendency to concentrate and hence to develop singularities. Moreover, the Lagrange multiplier λ will be uniquely defined as the generalized principal eigenvalue (see for details [12, 68, 117]): if $m \in L^1(\mathbb{R}^N)$ is fixed and such that $K_\alpha * m \in C^{0,\theta}(\mathbb{R}^N)$ for some $\theta \in (0,1)$, we define λ as

$$\lambda := \sup\left\{c \in \mathbb{R} \,\middle|\, \exists v \in C^2(\mathbb{R}^N) \text{ solving } \Delta v + \frac{1}{\gamma}|\nabla v|^\gamma + c = V - K_\alpha * m\right\}.$$

Once we know this value exists, it is possible to show that there exists $u \in C^2(\mathbb{R}^N)$ solving the HJB equation with such value λ, and that such solution u is coercive i.e.

$$u(x) \to +\infty \qquad \text{as } |x| \to +\infty \tag{3.4}$$

and moreover its gradient has polynomial growth (see Section 3.2 and the references [12, 68, 117]). Note that (3.4) is a quite natural "boundary" condition for ergodic HJB equations on the whole space, indeed the optimal speed would give rise to an ergodic process, at least heuristically, if $-\nabla u \cdot x < 0$ for $|x| \to +\infty$ (refer to [110] and references therein, for more details about ergodic problems on the whole space and their characterization in terms of Lyapunov functions).

Existence results for such classical solutions will depend on the interplay between the dissipation (i.e. by the diffusive term in the system) and the aggregating forces (described in terms of the Riesz potential K_α and the coercive potential V). So, denoting by γ' the conjugates exponent of γ, that is $\gamma' = \frac{\gamma}{\gamma-1}$, we get that the MFG system (3.1) shows three different regimes which correspond to $\alpha \in (0, N-2\gamma')$, $\alpha \in (N-2\gamma', N-\gamma']$ and $\alpha \in (N-\gamma', N)$. We will refer to $\alpha = N-2\gamma'$ as the *Hardy-Littlewood-Sobolev-critical exponent* and to $\alpha = N-\gamma'$ as the *mass-critical* (or L^2-critical) *exponent*, in analogy with the regimes appearing in the study of the Choquard equation (II.6) when $\gamma' = 2$. Obviously if $\gamma' \geq N$, there exists just one regime, which will be the mass-subcritical regime $\alpha \in [0, N)$, whereas if $\frac{N}{2} \leq \gamma' < N$ there will be just 2 regimes.

First of all, we observe that for classical solutions to (3.1) with $V \equiv 0$ and which satisfy (3.3), a Pohozaev-type identity holds (see Proposition 3.3.2):

$$(2-N)\int_{\mathbb{R}^N} \nabla u \cdot \nabla m\,dx + \left(1 - \frac{N}{\gamma}\right)\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda N M + \frac{\alpha+N}{2}\int_{\mathbb{R}^{2N}} \frac{m(x)m(y)}{|x-y|^{N-\alpha}}dxdy.$$

Notice that, also in the case when the potential V does not vanish identically, assuming the integrability condition $m\nabla V \cdot x \in L^1(\mathbb{R}^N)$, a similar identity holds. For MFG in the periodic setting with polynomial interaction potential an analogous Pohozaev identity has been proved in [70], while for the case of the Choquard equation we refer to [156] and references therein.

In the *Hardy-Littlewood-Sobolev-supercritical regime* $0 < \alpha < N-2\gamma'$, the Pohozaev identity, together with the fact that $\lambda \leq 0$ (see Lemma 3.2.12), implies that solutions to the MFG system (3.1) do not exist. More precisely, we obtain the following nonexistence result.

Theorem 3.1.1. *Assume that* $\alpha \in (0, N-2\gamma')$ *and* $V \equiv 0$. *Then, the MFG system* (3.1) *has no solutions* $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N) \times \mathbb{R}$ *which satisfy* (3.3) *and* (3.4).

In the case when $N-2\gamma' < \alpha < N$ we obtain existence of classical solutions to the MFG system (3.1) by means of a Schauder fixed point argument (refer to [9] and see also [70]). More in detail, we consider a regularized version of problem (3.1), obtained by convolving the Riesz-interaction term with a sequence of standard symmetric mollifiers (see (3.42) below). Taking advantage of the fixed-point structure associated to the MFG system and exploiting the Schauder Fixed Point Theorem, we show that solutions to the "regularized" version of the MFG system do exist. Then, we provide a priori uniform estimates on the solutions to the regularized problem, which allow us to pass to the limit and obtain a classical solution of the MFG system (3.1).

Theorem 3.1.2. *Assume that the potential V is locally Hölder continuous and satisfies* (3.2)*. We have the following results:*

i. if $N-\gamma' < \alpha < N$ then, for every $M>0$ the MFG system (3.1) *admits a classical solution (u,m,λ);*

ii. if $N-2\gamma' < \alpha \leq N-\gamma'$ then, there exists a positive real value $M_0 = M_0(N,\alpha,\gamma,C_V,b)$ such that if $M \in (0,M_0)$ the MFG system (3.1) *admits a classical solution (u,m,λ).*

Moreover in both cases there exists a constant $C>0$ such that

$$|\nabla u(x)| \leq C(1+|x|)^{\frac{b}{\gamma}} \qquad u(x) \geq C|x|^{\frac{b}{\gamma}+1} - C^{-1},$$

where $C = C(C_V,b,\gamma,N,\lambda,\alpha)$, $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$ and it holds

$$m|\nabla u|^{\gamma} \in L^1(\mathbb{R}^N), \qquad mV \in L^1(\mathbb{R}^N), \qquad |\nabla u|\,|\nabla m| \in L^1(\mathbb{R}^N).$$

The Hardy-Littlewood-Sobolev critical exponent is not covered by our analysis. Indeed it is possible to prove existence of solutions to the regularized problem also in this case, for sufficiently small masses (see Theorem 3.4.7). Nevertheless, in order to pass to the limit in the regularization, we need to obtain a priori L^{∞} bounds on solutions m_k to the regularized problem, starting from uniform bounds in $L^{\frac{2N}{N+\alpha}} \cap L^1$. This is not possible at the critical level $\alpha = N-2\gamma'$, due to critical rescaling properties of the Sobolev critical exponent. A priori uniform L^{∞} bounds on m_k only hold in the range where we have a uniform bound in L^q for $q > \frac{N}{\gamma'+\alpha}$ (see Theorem 3.2.13) and $\frac{2N}{N+\alpha} > \frac{N}{\gamma'+\alpha}$ only in the Hardy-Littlewood-Sobolev-subcritical regime. One way to circumvent this difficulty could be to obtain at the critical level $\alpha = N-2\gamma'$, by using regularity estimates on the viscous Hamilton-Jacobi equation and on the Fokker-Planck equation and a smallness condition on $\|m\|_{\frac{N}{N-\gamma'}}$, a priori uniform bounds on $\|m\|_q$ for some $q > \frac{N}{N-\gamma'}$, in order to be able to apply Theorem 3.2.13. This kind of result has recently been obtained recently in [72] for MFG on bounded domains with Neumann boundary conditions and nonlinear Schrödinger-type potential. This problem is related to the maximal regularity of solutions to viscous Hamilton-Jacobi equations $-\Delta u + |\nabla u|^{\gamma} = f(x)$ (see [73, 77, 96]).

Note that in the mass-subcritical regime, solutions to the MFG exist for every total mass $M>0$, while in the mass-supercritical and mass-critical regime (namely for $\alpha \in (N-2\gamma', N-\gamma']$) we provide existence just for sufficiently small total masses, below some threshold value M_0. This different behavior is due to the fact that when $N-2\gamma' < \alpha \leq N-\gamma'$ the interaction attractive potential is stronger than the diffusive part, so if the total mass M is too large, the mass m tends to concentrate and hence to develop singularities.

We recall that solutions to (3.1) correspond to critical points of the energy $\mathcal{E}$ over the constrained set $\mathcal{K}_{1,M}$ (as defined in (II.7) and (II.8) respectively). So, we can analogously find solutions to (3.1) using a variational approach. We refer the reader to Chapter 4 for more details.

This chapter is organized as follows. Section 3.2 contains some preliminary results. In particular, we recall regularizing properties of the Riesz interaction kernel, some a priori elliptic estimates for solutions to the Kolmogorov equation, a priori gradient estimates for solutions to the Hamilton-Jacobi-Bellman equation and finally uniform L^∞ bounds for m, solution to (3.1). In Section 3.3 we provide the Pohozaev identity and the proof of the nonexistence result Theorem 3.1.1, while Section 3.4 contains the proof of the existence result Theorem 3.1.2.

In what follows, $C, C_1, C_2, K_1, \dots$ denote generic positive constants which may change from line to line and also within the same line.

3.2 Preliminaries

In this section we introduce some preliminary results which will be useful in this chapter and in Chapter 4. We refer the reader to [20, §2], [19] and the references [32, 68, 117] for more details.

3.2.1 Regularity results for the Kolmogorov equation

Lemma 3.2.1. *Let $u \in C^{2,\theta}(\mathbb{R}^N)$ and $m \in W^{1,2}(\mathbb{R}^N)$ be a solution (in the distributional sense) to*

$$-\Delta m(x) - \operatorname{div}\left(m(x)\,\nabla u(x)\,|\nabla u|^{\gamma-2}\right) = 0 \quad \text{in } \mathbb{R}^N, \tag{3.5}$$

where $\gamma > 1$ is fixed. Then, $m \in C^{2,\theta}(\mathbb{R}^N)$. Moreover, if $m \geq 0$ and $m \not\equiv 0$, then $m(x) > 0$ for any $x \in \mathbb{R}^N$.

Proof. If $\gamma \geq 2$, then m solves

$$-\Delta m - b(x) \cdot \nabla m(x) - m(x) \operatorname{div} b(x) = 0$$

where $b(x) := |\nabla u|^{\gamma-2}\nabla u(x) \in C^{1,\theta}(\mathbb{R}^N)$ and $\operatorname{div} b(x) \in C^{0,\theta}(\mathbb{R}^N)$. By elliptic regularity (see e.g. [93, Theorem 8.24]) we get that $m \in C^{0,\alpha}$ for a certain $\alpha \in (0,1]$. Denoting by $f := m\nabla u|\nabla u|^{\gamma-2}$ we have $-\Delta m = \operatorname{div} f$ where $f \in C^{0,\alpha}$, then by [93, Theorem 4.15] we get that $m \in C^{1,\alpha}$ and hence

$$-\Delta m = \operatorname{div}\left(m\nabla u|\nabla u|^{\gamma-2}\right) \in C^{0,\min\{\alpha,\theta\}}$$

so $m \in C^{2,\min\{\alpha,\theta\}}$. Iterating we finally obtain that $m \in C^{2,\theta}$. If $1 < \gamma < 2$, $b(x)$ is just an Hölder continuous function, hence m is a weak solution of equation (3.5). In this case, we can replace $b(x)$ with $b_\varepsilon(x) := \nabla u(x)(\varepsilon + |\nabla u|^2)^{\frac{\gamma}{2}-1}$ and m_ε is a-posteriori a classical solution to the approximate equation

$$-\Delta m - \operatorname{div}\left(m(x)\, b_\varepsilon(x)\right) = 0.$$

We can conclude letting $\varepsilon \to 0$. If $m \geq 0$ on $\mathbb{R}^N$, we also have that m satisfy

$$-\Delta m - b(x) \cdot \nabla m(x) - \left(\operatorname{div} b(x)\right)^+ m(x) \leq 0,$$

since $\int_{\mathbb{R}^N} m\, dx = M > 0$, the Strong Minimum Principle (refer e.g to [93, Theorem 8.19]) implies that $m > 0$ in $\mathbb{R}^N$ (indeed m can not be equal to 0, unless it is constant, which is impossible). ◻

We will use the following result (proved in [44, Proposition 2.4]) which takes advantage of some classical elliptic regularity results of Agmon [3].

Proposition 3.2.2. *Let $m \in L^p(\mathbb{R}^N)$ for $p > 1$ and assume that for some $K > 0$*

$$\left| \int_{\mathbb{R}^N} m\Delta\varphi \, dx \right| \leq K \|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then, $m \in W^{1,p}(\mathbb{R}^N)$ and there exists a constant $C > 0$ depending only on p such that

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \leq C\, K.$$

We prove now some a priori estimates for solutions to the Kolmogorov equation. Let us fix $p \in (1, +\infty)$ and $M > 0$.

Proposition 3.2.3. *Let us consider a couple $(m, w) \in (L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} m(-\Delta\varphi)dx = \int_{\mathbb{R}^N} w \cdot \nabla\varphi \, dx, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Assume also that $\int_{\mathbb{R}^N} m(x)\, dx = M$, $m \geq 0$ a.e. and

$$E := \int_{\mathbb{R}^N} m \left| \frac{w}{m} \right|^{\gamma'} dx < +\infty.$$

Then, we have that

$$m \in W^{1,r}(\mathbb{R}^N)$$

for r such that $\frac{1}{r} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p} + \frac{1}{\gamma'}$ (i.e. $r = \frac{p\gamma'}{\gamma'+p-1}$) and there exists a constant C, depending on r, such that

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \leq C(E + M)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{3.6}$$

Proof. Using Hölder inequality (since $\frac{1}{r} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p} + \frac{1}{\gamma'}$, it holds $\frac{1}{p\gamma} + \frac{1}{\gamma'} + \frac{1}{r'} = 1$) we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} m\, \Delta\varphi \, dx \right| &\leq \int_{\mathbb{R}^N} |w|\, |\nabla\varphi| dx = \int_{\mathbb{R}^N} \left(\left| \frac{w}{m} \right|^{\gamma'} m \right)^{\frac{1}{\gamma'}} m^{\frac{1}{\gamma}} |\nabla\varphi| \, dx \\
&\leq \left(\int_{\mathbb{R}^N} \left| \frac{w}{m} \right|^{\gamma'} m \, dx \right)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla\varphi\|_{L^{r'}(\mathbb{R}^N)}
\end{aligned}$$

and hence

$$\left| \int_{\mathbb{R}^N} m\, \Delta\varphi \, dx \right| \leq E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla\varphi\|_{L^{r'}(\mathbb{R}^N)}.$$

Since $\|m\|_{L^1(\mathbb{R}^N)} = M$ and $m \in L^p(\mathbb{R}^N)$, by interpolation we get

$$\|m\|_{L^r(\mathbb{R}^N)} \leq \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} M^{\frac{1}{\gamma'}} \tag{3.7}$$

therefore $m \in L^r(\mathbb{R}^N)$. From Proposition 3.2.2 with $K = E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}$, we obtain that $m \in W^{1,r}(\mathbb{R}^N)$ and there exists a constant $C > 0$, depending on r, such that

$$\|\nabla m\|_{L^r(\mathbb{R}^N)} \leq C\, E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{3.8}$$

By (3.7) and (3.8), we can conclude that

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \leq \left(M^{\frac{1}{\gamma'}} + CE^{\frac{1}{\gamma'}}\right) \|m\|_{L^p}^{\frac{1}{\gamma}} \leq C(E+M)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}.$$

□

Proposition 3.2.4. *Under the assumption of Proposition 3.2.3, we have the following results:*

i) if $1 < p < 1 + \frac{\gamma'}{N}$ *then, there exists* $\delta_1 = \frac{1}{p-1}\left(\frac{\gamma'}{N} + 1 - p\right)$ *such that*

$$\|m\|_{L^p(\mathbb{R}^N)}^{(1+\delta_1)p} \leq C\, M^{(1+\delta_1)p-1}\, E \tag{3.9}$$

where C *is a constant depending on* N, γ *and* p;

ii) if $\gamma' < N$ *and* $1 < p \leq \frac{N}{N-\gamma'}$ *then, there exists* $\delta_2 = \frac{1}{p-1}\frac{\gamma'}{N}$ *and a constant* C *depending on* N, γ *and* p *such that*

$$\|m\|_{L^p(\mathbb{R}^N)}^{p\delta_2} \leq C(E+M)M^{p\delta_2-1}. \tag{3.10}$$

Proof. i) The proof of (3.9) follows from [44, Lemma 2.8]. *ii)* As before let $\frac{1}{r} = \frac{1}{p}\left(1 - \frac{1}{\gamma'}\right) + \frac{1}{\gamma'}$, if $\gamma' < N$ then $r < \gamma' < N$, so by Gagliardo-Niremberg inequality and (3.6) we get

$$\|m\|_{L^{r^*}(\mathbb{R}^N)} \leq C\|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}(E+M)^{\frac{1}{\gamma'}} \tag{3.11}$$

where $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$ and C is a constant depending on N, p and γ'. One can observe that $\frac{1}{r^*} - \frac{1}{p} = \frac{pN-N-p\gamma'}{p\gamma' N} \leq 0$, that is $r^* \geq p$, so by interpolation there exists $\theta \in (0,1]$ such that

$$\|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\theta}} \leq M^{\frac{1-\theta}{\theta}} \|m\|_{L^{r^*}(\mathbb{R}^N)}$$

and from (3.11) we get that

$$\|m\|_{L^p(\mathbb{R}^N)}^{\left(\frac{1}{\theta}-\frac{1}{\gamma}\right)\gamma'} \leq C(E+M)M^{\frac{1-\theta}{\theta}\gamma'}.$$

By simple computations we have that

$$\left(\frac{1}{\theta} - \frac{1}{\gamma}\right)\gamma' = \frac{\gamma'}{N}\frac{p}{p-1}$$

and

$$\left(\frac{1}{\theta} - 1\right)\gamma' = \frac{\gamma'}{N}\frac{p}{p-1} - 1$$

denoting by δ_2 the quantity $\frac{1}{p-1}\frac{\gamma'}{N}$, we finally obtain (3.10). □

Remark 3. In the following we will use (3.9) and (3.10) in the case when $p = \frac{2N}{N+\alpha}$. It will be useful to observe that if $\gamma' \geq N$ then $1 < \frac{2N}{N+\alpha} < 2 \leq 1 + \frac{\gamma'}{N}$, hence estimate (3.9) holds. In the case when $\gamma' < N$, if $N - \gamma' \leq \alpha < N$ then, $1 < \frac{2N}{N+\alpha} < 1 + \frac{\gamma'}{N}$ and hence from (3.9) we get that

$$\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} \leq CM^{\frac{2\gamma'}{N-\alpha}-1}E; \tag{3.12}$$

whereas if $N - 2\gamma' \leq \alpha < N - \gamma'$, we may use estimate (3.10), which gives us

$$\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} \leq C(E+M)M^{\frac{2\gamma'}{N-\alpha}-1}. \tag{3.13}$$

Finally, we recall the following a priori elliptic regularity result (see [44, Proposition 2.8, Corollary 2.9]).

Proposition 3.2.5. *Let*

$$q := \begin{cases} \frac{N}{N-\gamma'+1} & \text{if } \gamma' < N \\ \gamma' & \text{if } \gamma' \geq N \end{cases},$$

the couple $(m, w) \in (L^q(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ *be such that*

$$\int_{\mathbb{R}^N} m(-\Delta\varphi)dx = \int_{\mathbb{R}^N} w \cdot \nabla\varphi \, dx, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N)$$

with $\int_{\mathbb{R}^N} m(x)\, dx = M$, $m \geq 0$ *a.e. and*

$$E := \int_{\mathbb{R}^N} m \left| \frac{w}{m} \right|^{\gamma'} dx < +\infty.$$

Then, it holds:

i)

$$m \in L^\beta(\mathbb{R}^N), \quad \forall \beta \in \left[1, \frac{N}{N-\gamma'}\right) \qquad (\forall \beta \in [1, +\infty), \text{ if } \gamma' \geq N)$$

and there exists a constant C *depending on* N, β *and* γ' *such that*

$$\|m\|_{L^\beta(\mathbb{R}^N)} \leq C(E + M);$$

ii)

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < q$$

and there exists a constant C *depending on* N, ℓ *and* γ' *such that*

$$\|m\|_{W^{1,\ell}(\mathbb{R}^N)} \leq C(E + M);$$

iii) if $\gamma' > N$*, we have also*

$$m \in C^{0,\theta}(\mathbb{R}^N), \quad \forall \theta \in \left(0, 1 - \frac{N}{\gamma'}\right)$$

and there exists a constant C *depending on* N, θ *and* γ' *such that*

$$\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \leq C(E + M).$$

Proof. From Proposition 3.2.3 we have

$$m \in W^{1,r_0}(\mathbb{R}^N) \quad \text{for} \quad \frac{1}{r_0} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{q} + \frac{1}{\gamma'}.$$

Case $\gamma' < N$. Since $1 < r_0 < \gamma' < N$, by Sobolev embedding theorem and interpolation, we get that

$$m \in L^\beta(\mathbb{R}^N) \quad \forall \beta \leq q_1 \tag{3.14}$$

where q_1 is the Sobolev critical exponent, i.e.

$$q_1 := \frac{Nr_0}{N - r_0} = \frac{qN\gamma'}{N\gamma' - N + q(N - \gamma')},$$

(notice that $q_1 > q$ since $q < \frac{N}{N-\gamma'}$). From (3.14), using Proposition 3.2.3 again, we have

$$m \in W^{1,\ell}(\mathbb{R}^N) \qquad \forall \ell \le r_1 = \frac{q_1\gamma'}{\gamma'-1+q_1}.$$

As before, by Sobolev embedding theorem and interpolation, we have that

$$m \in L^{\beta}(\mathbb{R}^N) \qquad \forall \beta \le q_2 = \frac{q_1 N\gamma'}{N\gamma' - N + q_1(N-\gamma')}.$$

Iterating the previous argument, we observe that $q_{j+1} = f(q_j)$ where $f(s) := \frac{sN\gamma'}{N\gamma'-N+s(N-\gamma')}$. Since f is an increasing function if $s < \frac{N}{N-\gamma'}$ and it has a fixed point for $\bar{s} = \frac{N}{N-\gamma'}$, we obtain that

$$m \in L^{\beta}(\mathbb{R}^N), \quad \forall \beta < \frac{N}{N-\gamma'}$$

and

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < \frac{N}{N-\gamma'+1}.$$

Moreover, for any fixed $\beta < \frac{N}{N-\gamma'}$, taking $r = r(\beta)$ such that $\frac{1}{r} = \left(1-\frac{1}{\gamma'}\right)\frac{1}{\beta} + \frac{1}{\gamma'}$, from estimate (3.6) and the Sobolev embedding theorem (notice that $r^* > \beta$) we get that there exists a constant C depending on N and r such that

$$\|m\|_{L^{\beta}(\mathbb{R}^N)} \le C(E+M)^{\frac{1}{\gamma'}} \|m\|^{\frac{1}{\gamma}}_{L^{\beta}(\mathbb{R}^N)}.$$

and hence

$$\|m\|_{L^{\beta}(\mathbb{R}^N)} \le C_1(E+M).$$

Finally, from (3.6) we obtain

$$\|m\|_{W^{1,\ell}(\mathbb{R}^N)} \le C_2(E+M).$$

Case $\gamma' = N$. Since $r_0 < \gamma' = N$, we can apply the Sobolev embedding theorem and with the same argument as before we obtain

$$q_{j+1} = \frac{N}{N-1} q_j.$$

Obviously $q_{j+1} > q_j$, by iteration we get that

$$m \in L^{\beta}(\mathbb{R}^N), \quad \forall \beta < +\infty$$

and

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < \gamma'.$$

The estimates on the norms follow in the same way as the previous case.

Case $\gamma' > N$. Since $m \in L^{\gamma'}(\mathbb{R}^N)$, by interpolation $m \in L^N(\mathbb{R}^N)$ and we can go back to the previous case. In particular, we have that $m \in W^{1,\ell}(\mathbb{R}^N)$ for $N < \ell < \gamma'$, hence by Morrey's embedding

$$m \in C^{0,\theta}(\mathbb{R}^N), \quad \text{for } 0 < \theta < 1 - \frac{N}{\gamma'}$$

and there exists a constant C, depending on θ, N and γ', such that

$$\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \le C(E+M).$$

□

3.2.2 Some properties of the Riesz potential

We recall here some properties of the Riesz potential, which will be useful in the following in order to deal with the Riesz-type interaction term.

Definition 3.2.1. *Given $\alpha \in (0, N)$ and $f \in L^1_{loc}(\mathbb{R}^N)$, we define the Riesz potential of order α of the function f as*

$$K_\alpha * f(x) := \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} dy, \quad x \in \mathbb{R}^N.$$

The Riesz potential K_α is well-defined as an operator on the whole space $L^r(\mathbb{R}^N)$ if and only if $r \in \left[1, \frac{N}{\alpha}\right)$. We state now the following well-known theorems (for which refer e.g. to [132, Theorem 4.3] and [182, Theorem 14.37]).

Theorem 3.2.6 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < N$ and $1 < r < \frac{N}{\alpha}$. Then, for any $f \in L^r(\mathbb{R}^N)$*

$$\|K_\alpha * f\|_{L^{\frac{Nr}{N-\alpha r}}(\mathbb{R}^N)} \leq C\|f\|_{L^r(\mathbb{R}^N)}$$

where C is a constant depending only on N, α and r.

Theorem 3.2.7. *Let $0 < \lambda < N$ and $p, r > 1$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C(N, \lambda, p)$ (independent of f and g) such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)\, g(y)}{|x-y|^\lambda} dx\, dy \right| \leq C\|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^r(\mathbb{R}^N)}. \tag{3.15}$$

Remark 4. If $0 < \alpha < N$ and $f \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, then there exists a sharp constant C, depending only on N and α, such that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)\, f(y)}{|x-y|^{N-\alpha}} dx\, dy \right| \leq C\|f\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}. \tag{3.16}$$

As shown in [131], in this case the constant C can be computed explicitly and there exist explicit optimizers for (3.16) (while neither the constant nor the optimizers are known for $p \neq r$, although do exist).

Regarding the L^∞-norm and the Hölder continuity of the Riesz potential, we recall here the following results.

Theorem 3.2.8. *Let $0 < \alpha < N$, $1 < r \leq +\infty$ be such that $r > \frac{N}{\alpha}$ and $s \in [1, \frac{N}{\alpha})$. Then, for every $f \in L^s(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ we have that*

$$\|K_\alpha * f\|_{L^\infty(\mathbb{R}^N)} \leq C_1\|f\|_{L^r(\mathbb{R}^N)} + C_2\|f\|_{L^s(\mathbb{R}^N)} \tag{3.17}$$

where $C_1 = C_1(N, \alpha, r)$ and $C_2 = C_2(N, \alpha, s)$.

Proof. We observe that

$$\frac{1}{|x|^{N-\alpha}} \in L^p(B_1), \quad \forall p \in \left[1, \frac{N}{N-\alpha}\right)$$

and it is well-known that $\int_{B_1(0)} \frac{1}{|x|^{(N-\alpha)p}} dx = \frac{\omega_N}{N-(N-\alpha)p}$. By Hölder inequality we get

$$\int_{B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq \left(\int_{B_1} |f(x-y)|^r \, dy \right)^{\frac{1}{r}} \left(\int_{B_1} \frac{1}{|y|^{(N-\alpha)r'}} \, dy \right)^{\frac{1}{r'}}$$
$$\leq \|f\|_{L^r(\mathbb{R}^N)} \left(\frac{\omega_N}{N-(N-\alpha)r'} \right)^{\frac{1}{r'}} \leq C_1 \|f\|_{L^r(\mathbb{R}^N)}$$

using the fact that $r' < \frac{N}{N-\alpha}$, since by assumption $r > \frac{N}{\alpha}$. On the other hand

$$\frac{1}{|x|^{N-\alpha}} \in L^p(B_1^c), \quad \forall p \in \left(\frac{N}{N-\alpha}, +\infty \right]$$

hence

$$\int_{\mathbb{R}^N \setminus B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq \left(\int_{\mathbb{R}^N \setminus B_1} |f(x-y)|^s dy \right)^{\frac{1}{s}} \left\| \frac{1}{|y|^{N-\alpha}} dy \right\|_{L^{s'}(B_1^c)} \leq C \|f\|_{L^s(\mathbb{R}^N)},$$

since $(N-\alpha)s' > N$. We can conclude that

$$|K_\alpha * f(x)| \leq \int_{B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy + \int_{\mathbb{R}^N \setminus B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq C_1 \|f\|_{L^r(\mathbb{R}^N)} + C_2 \|f\|_{L^s(\mathbb{R}^N)},$$

where $C_1 = C_1(N, \alpha, r)$ and $C_2 = C_2(N, \alpha, s)$. □

Remark 5. It follows immediately that if we take $r = +\infty$ and $s = 1$ we have

$$\int_{B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq \frac{\omega_N}{\alpha} \|f\|_{L^\infty(B_1)}$$

and hence

$$\|K_\alpha * f\|_{L^\infty(\mathbb{R}^N)} \leq C_{\alpha,N} \|f\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^1(\mathbb{R}^N)}.$$

Theorem 3.2.9. *Let $1 < r < +\infty$ and $0 < \alpha < N$ be such that $0 < \alpha - \frac{N}{r} < 1$. Then, for every $f \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ we have that*

$$K_\alpha * f \in C^{0,\alpha - \frac{N}{r}}(\mathbb{R}^N)$$

and there exists a constant C, depending on r, α and N, such that

$$\frac{\left| K_\alpha * f(x) - K_\alpha * f(y) \right|}{\|x-y\|^{\alpha - \frac{N}{r}}} \leq C \|f\|_{L^r(\mathbb{R}^N)}.$$

Proof. Concerning Hölder regularity results for the Riesz potential, one may refer to [151, Theorem 2.2, p.155] and [84, Theorem 2]. □

3.2.3 Some results on the Hamilton-Jacobi-Bellman equation

By a straightforward adaptation of [44, Theorem 2.5 and Theorem 2.6], we obtain a priori regularity estimates for solutions to some Hamilton-Jacobi-Bellman equations defined on the whole euclidean space $\mathbb{R}^N$. The following propositions are stated under slightly more general assumptions than ones of our problem.

Proposition 3.2.10. *Assume that $K_\alpha * m \in L^\infty(\mathbb{R}^N)$ and that V satisfies* (3.2) *with $b \geq 0$. Let $(u, c) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ be a classical solution to the HJB equation*

$$-\Delta u + \frac{1}{\gamma}|\nabla u(x)|^\gamma + c = V(x) - K_\alpha * m(x) \quad \text{in } \mathbb{R}^N, \tag{3.18}$$

for $\gamma > 1$ fixed. Then

i. there exists a constant $C_1 > 0$, depending on $C_V, b, \gamma, N, c, \|K_\alpha * m\|_\infty$, such that

$$|\nabla u(x)| \leq C_1(1 + |x|)^{\frac{b}{\gamma}};$$

ii. if u is bounded from below and $b \neq 0$, then there exist a constant $C_2 > 0$ such that

$$u(x) \geq C_2|x|^{\frac{b}{\gamma}+1} - C_2^{-1}, \quad \forall x \in \mathbb{R}^N.$$

The same result holds also in the case when $b = 0$, but we have to require in addition that there exists $\delta > 0$ such that $V(x) - K_\alpha * m(x) - c > \delta > 0$ for $|x|$ sufficiently large.

Proof. The thesis follows applying [44, Theorem 2.5 and Theorem 2.6]. □

Let us define

$$\lambda := \sup\{c \in \mathbb{R} \,|\, (3.18) \text{ has a solution } u \in C^2(\mathbb{R}^N)\} \tag{3.19}$$

Proposition 3.2.11. *Besides the hypothesis of Proposition 3.2.10, let us assume also that $V - K_\alpha * m$ is locally Hölder continuous. Then*

i) $\lambda < +\infty$ and there exists $u \in C^2(\mathbb{R}^N)$ such that the pair (u, λ) solves (3.18).

ii) if $b \neq 0$ in (3.2)*, u is unique up to additive constants (namely if $(v, \lambda) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ solves* (3.18) *then there exists $k \in \mathbb{R}$ such that $u = v + k$) and there exists a constant $K > 0$ such that*

$$u(x) \geq K|x|^{\frac{b}{\gamma}+1} - K^{-1}, \quad \forall x \in \mathbb{R}^N.$$

Proof. It follows by [44, Theorem 2.7]. We may observe also that

$$\lambda = \sup\{c \in \mathbb{R} \,|\, (3.18) \text{ has a subsolution } u \in C^2(\mathbb{R}^N)\}.$$

□

Finally, we conclude with an estimate on the Lagrange multiplier λ defined in (3.19).

Lemma 3.2.12. *Let $(u, \lambda) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ be a solution to the HJB equation* (3.18).

i If $V \equiv 0$, then $\lambda \leq 0$;

ii if V satisfies (3.2) *then $\lambda \leq C$ for some constant depending on b, C_V, γ, N.*

Proof. The proof is based on the same argument of [70, Lemma 3.3]. Let us consider the function $\mu_\delta(x) := \left(\frac{\delta}{2\pi}\right)^{N/2} e^{\frac{-\delta|x|^2}{2}}$ for $x \in \mathbb{R}^N$ and $\delta > 0$. Obviously $\int_{\mathbb{R}^N} \mu_\delta(x)\,dx = 1$. From the definition of Legendre transform we get that

$$\frac{1}{\gamma}|\nabla u|^\gamma = \sup_{\alpha \in \mathbb{R}^N}\left(\nabla u \cdot \alpha - \frac{|\alpha|^{\gamma'}}{\gamma'}\right) \geq \nabla u \cdot (\delta x) - \frac{|\delta x|^{\gamma'}}{\gamma'}$$

hence

$$-\Delta u(x) + \nabla u \cdot (\delta x) - \frac{1}{\gamma'}|\delta x|^{\gamma'} + \lambda \le V(x) - m * K_\alpha(x).$$

Multiplying the previous inequality by μ_δ and integrating over B_R we obtain

$$-\int_{B_R} \Delta u(x)\mu_\delta + \int_{B_R} \nabla u \cdot (\delta x)\mu_\delta - \int_{B_R} \frac{1}{\gamma'}|\delta x|^{\gamma'}\mu_\delta + \lambda \int_{B_R} \mu_\delta \le \int_{B_R} (V(x) - m * K_\alpha)\mu_\delta.$$

Integrating by parts (notice that $\int_{B_R} \nabla u \cdot \nabla \mu_\delta = -\int_{B_R} \nabla u \cdot (\delta x)\mu_\delta$) we get

$$-\int_{\partial B_R} \mu_\delta \nabla u \cdot \nu \, d\sigma - \frac{1}{\gamma'}\int_{B_R} |\delta x|^{\gamma'}\mu_\delta \, dx + \lambda \int_{B_R} \mu_\delta \, dx \le \int_{B_R} (V(x) - m * K_\alpha)\mu_\delta \, dx$$

and since $\int_{B_R} m * K_\alpha(x)\mu_\delta(x)\, dx \ge 0$, we have

$$\lambda \int_{B_R} \mu_\delta \, dx \le \int_{\partial B_R} \mu_\delta \nabla u \cdot \nu \, d\sigma + \frac{1}{\gamma'}\int_{B_R} |\delta x|^{\gamma'}\mu_\delta \, dx + \int_{B_R} V(x)\mu_\delta \, dx. \tag{3.20}$$

For $\delta > 0$ fixed, the first integral in the RHS of (3.20) can be estimated as follows

$$\left| \int_{\partial B_R} \mu_\delta \nabla u \cdot \nu \, d\sigma \right| \le C\delta^{\frac{N}{2}} e^{-\frac{\delta R^2}{2}} \|\nabla u\|_{L^\infty(\partial B_R)} |\partial B_R| \to 0, \qquad \text{as} \quad R \to +\infty$$

taking advantage of the gradient estimates on ∇u proved in Proposition 3.2.10. So, sending $R \to +\infty$ in (3.20) and using (3.2) we get

$$\lambda \le \frac{1}{(2\pi)^{\frac{N}{2}}} \frac{\delta^{\frac{\gamma'}{2}}}{\gamma'} \int_{\mathbb{R}^N} |y|^{\gamma'} e^{-\frac{|y|^2}{2}} \, dy + \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} V\left(\frac{y}{\sqrt{\delta}}\right) e^{-\frac{|y|^2}{2}} \, dy.$$

If $V \equiv 0$, then sending $\delta \to 0$ in the previous inequality, we conclude immediately $\lambda \le 0$. If $V \not\equiv 0$, we may choose $\delta = 1$ in the previous inequality and conclude recalling (3.2). □

3.2.4 Uniform a priori L^∞-bounds on m

We state now the following result, which provides uniform a priori L^∞ bounds on m.

Theorem 3.2.13. *We consider a sequence of classical solutions (u_k, m_k, λ_k) to the following MFG system*

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = W_k(x) - G_{k,\alpha}[m](x) \\ -\Delta m - \operatorname{div}\left(m \nabla u |\nabla u|^{\gamma-2}\right) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \ge 0 \end{cases}$$

where $W_k : \mathbb{R}^N \to \mathbb{R}$ satisfies assumption (3.2) with constant C_V, b independent of k and $G_{k,\alpha} : L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ is such that $G_{k,\alpha}[m] \ge 0$ for all $m \in L^1$ with $m \ge 0$. We assume also that there exists $\alpha \in (0, N)$ such that for all $s \in \left[1, \frac{N}{\alpha}\right)$ and $r \in \left(\frac{N}{\alpha}, +\infty\right]$ taking $m \in L^s(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ there holds

$$\|G_{k,\alpha}[m]\|_{L^\infty(\mathbb{R}^N)} \le C_1 \|m\|_{L^r(\mathbb{R}^N)} + C_2 \|m\|_{L^s(\mathbb{R}^N)} \tag{3.21}$$

where $C_1 = C_1(N, \alpha, r)$ and $C_2 = C_2(N, \alpha, s)$.
If u_k are bounded from below and satisfy (3.4), and $m_k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|m_k\|_{L^q} \le C_q$ for some $q > \frac{N}{\alpha + \gamma'}$ then, there exists a positive constant C not depending on k such that

$$\|m_k\|_{L^\infty(\mathbb{R}^N)} \le C, \quad \forall k \in \mathbb{N}.$$

Proof. We follow the argument of the proof of [44, Theorem 4.1] (refer also to [70] for the analogous result on $\mathbb{T}^N$) but we have to define a different rescaling in this case.

Up to addition of constants we may assume $\inf u_k(x) = 0$. We assume by contradiction that

$$\sup_{\mathbb{R}^N} m_k = L_k \to +\infty$$

and we define

$$\delta_k := \begin{cases} L_k^{-\beta}, & \text{if } \ \gamma' \le N \ \text{ and } \ q \le \frac{N}{\gamma'} \\ L_k^{-\frac{1}{\gamma'}}, & \text{if either } \ \gamma' > N \ \text{ or } \ \gamma' \le N, \ q > \frac{N}{\gamma'} \end{cases}$$

where $\beta > 0$ (so $\delta_k \to 0$) has to be chosen in the following way. We fix

$$r \in \left(\frac{N}{\alpha}, \frac{Nq}{N - q\gamma'} \right) \tag{3.22}$$

(note that since $q > \frac{N}{\gamma'+\alpha}$ the interval is not empty) and if $q = \frac{N}{\gamma'}$ we fix $r \in \left(\frac{N}{\alpha}, +\infty\right)$. Then we choose β such that

$$\frac{1}{\gamma'}\left(1 - \frac{q}{r}\right) \le \beta < \frac{q}{N}.$$

We rescale (u_k, m_k, λ_k) as follows:

$$v_k(x) := \delta_k^{\frac{2-\gamma}{\gamma-1}} u_k(\delta_k x) + 1, \qquad n_k(x) := L_k^{-1} m_k(\delta_k x), \qquad \tilde{\lambda}_k := \delta_k^{\gamma'} \lambda_k.$$

Observe that $0 \le n_k(x) \le 1$ and $\sup n_k = 1$ and moreover that $v_k(x) \ge 1$ for all x. So we obtain that $(v_k, n_k, \tilde{\lambda}_k)$ is a solution to

$$\begin{cases} -\Delta v_k + \frac{1}{\gamma}|\nabla v_k|^\gamma + \tilde{\lambda}_k = V_k(x) - \tilde{g}_k(x) \\ -\Delta n_k - \operatorname{div}(n_k \nabla v_k |\nabla v_k|^{\gamma-2}) = 0 \end{cases}$$

where

$$V_k(x) := \delta_k^{\gamma'} W_k(\delta_k x) \qquad \text{and} \qquad \tilde{g}_k(x) := \delta_k^{\gamma'} G_{k,\alpha}[m_k](\delta_k x).$$

Observe that by assumption (3.2) there holds

$$C_V^{-1} \delta_k^{\gamma'} (\max\{|\delta_k x| - C_V, 0\})^b \le V_k(x) \le C_V(1 + \delta_k^{\gamma'+b}|x|)^b, \quad \forall x \in \mathbb{R}^N.$$

Computing the equation in a minimum point of u_k we obtain $\lambda_k \ge -\|G_{k,\alpha}[m_k]\|_\infty$ and reasoning as in Lemma 3.2.12, we get that $\lambda_k \le C$, for some C just depending on γ, C_V, b, so we get

$$-\|\tilde{g}_k\|_\infty = -\delta_k^{\gamma'} \|G_{k,\alpha}[m_k]\|_\infty \le \tilde{\lambda}_k \le \delta_k^{\gamma'} C.$$

If $\gamma' > N$ or $\gamma' \le N$ and $q > \frac{N}{\gamma'}$ we apply (3.21) with $r = +\infty$ and $s = 1$ and we get

$$\|\tilde{g}_k\|_\infty \le \delta_k^{\gamma'} (C_1 L_k + C_2 M) = L_k^{-1}(C_1 L_k + C_2 M) \le C$$

which in turns gives also that $|\tilde{\lambda}_k| \le C$. If $\gamma' > N$ there holds

$$\|n_k\|_{L^1} = \int_{\mathbb{R}^N} n_k(x)dx = \delta_k^{\gamma'-N} \|m_k\|_{L^1} = \delta_k^{\gamma'-N} M \to 0 \quad \text{and} \quad 0 \le n_k \le 1 = \sup n_k,$$

while if $\gamma' \le N$ and $q > \frac{N}{\gamma'}$ we have that

$$\|n_k\|_{L^q} = L_k^{-1}\delta_k^{-\frac{N}{q}}\|m_k\|_{L^q} \le L_k^{\frac{N}{q\gamma'}-1}C_q \to 0 \quad \text{and} \quad 0 \le n_k \le 1 = \sup n_k.$$

If $\gamma' \le N$ and $q \le \frac{N}{\gamma'}$ first of all we observe that, since $\beta < \frac{q}{N}$,

$$\|n_k\|_{L^q} = L_k^{-1}\delta_k^{-\frac{N}{q}}\|m_k\|_{L^q} \le L_k^{\beta\frac{N}{q}-1}C_q \to 0 \qquad \text{and} \quad 0 \le n_k \le 1 = \sup n_k.$$

We apply (3.21) with r as in (3.22) and $s = 1$ and we get, using interpolation between L^q and L^∞ to estimate the norm $\|m_k\|_{L^r}$, that there holds

$$\|\tilde{g}_k\|_\infty \le \delta_k^{\gamma'}(C_1\|m_k\|_{L^{\frac{N}{N-2\gamma'}}} + C_2C_q) \le L_k^{-\beta\gamma'}(CL_k^{1-\frac{q}{r}} + C_2C_q) \le CL_k^{1-\beta\gamma'-\frac{q}{r}} \le C$$

since $\beta\gamma' > 1 - \frac{q}{r}$. This in turns implies that $|\tilde{\lambda}_k| \le C$. The remaining part of the proof follows exactly the same lines of the proof of [44, Theorem 4.1], since we have uniform bounds on $\tilde{\lambda}_k$ and on $\|\tilde{g}_k\|_\infty$, either the L^1 or the L^q norm of n_k vanishing as $k \to +\infty$, whereas $\|n_k\|_\infty = 1$. In particular one shows that if x_k is an approximated maximum point of n_k (that is $n_k(x_k) = 1 - \delta$), then necessarily $\delta_k^{\gamma'+b}|x_k|^b \to +\infty$. If it is not the case, using a priori gradient estimates on v_k as in Proposition 3.2.10, we get that n_k is uniformly (in k) Hölder continuous in the ball $B_1(x_k)$, contradicting the fact that $n_k \ge 0$ and either $\|n_k\|_{L^q} \to 0$ or $\|n_k\|_{L^1} \to 0$. On the other hand, if $\delta_k^{\gamma'+b}|x_k|^b \to +\infty$, we may construct a Lyapunov function for the system, which allows for some integral estimates on n_k showing again a uniform (in k) Hölder bound for n_k in $B_{1/2}(x_k)$ and again getting a contradiction. Therefore one concludes that $L_k \to +\infty$ is not possible. □

3.3 Pohozaev identity and nonexistence of solutions

In this section, we study the MFG system (3.1) in the case $V \equiv 0$, i.e.

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u(x)|^\gamma + \lambda = -K_\alpha * m(x) \\ -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \quad \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} m = M, \quad m \ge 0 \end{cases} \tag{3.23}$$

The following Lemma (see Lemma 3.2 in [70]) will be useful in order to control the behavior of m, $\nabla u, \nabla m$ at infinity.

Lemma 3.3.1. *Let $h \in L^1(\mathbb{R}^N)$. Then, there exists a sequence $R_n \to \infty$ such that*

$$R_n \int_{\partial B_{R_n}} |h(x)|dx \to 0, \quad \text{as } n \to \infty.$$

In order to prove nonexistence of solutions to the MFG system (3.23) in the *super-critical regime* $0 < \alpha < N - 2\gamma'$, we need a Pohozaev-type identity.

Proposition 3.3.2 (Pohozaev identity). *Let $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N) \times \mathbb{R}$ be a solution to (3.23) such that*

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad |\nabla m||\nabla u| \in L^1(\mathbb{R}^N).$$

Then, the following equality holds

$$(2-N)\int_{\mathbb{R}^N} \nabla u \cdot \nabla m \, dx + \left(1 - \frac{N}{\gamma}\right)\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda N M + \frac{\alpha + N}{2}\int_{\mathbb{R}^N} m(x)K_\alpha * m(x)\, dx. \tag{3.24}$$

Proof. From Lemma 3.2.1, we get that m is twice differentiable, so the following computations are justified. Consider the first equation in (3.23), multiplying each term by $\nabla m \cdot x$ and integrating over $B_R(0)$ for $R > 0$, we get

$$-\int_{B_R} \Delta u \, \nabla m \cdot x \, dx + \frac{1}{\gamma} \int_{B_R} |\nabla u(x)|^\gamma \nabla m \cdot x \, dx + \lambda \int_{B_R} \nabla m \cdot x \, dx = -\int_{B_R} K_\alpha * m(x) \nabla m \cdot x \, dx. \tag{3.25}$$

We take into account each term of (3.25) separately. Integrating by parts the first term, we have

$$-\int_{B_R} \Delta u \, \nabla m \cdot x \, dx = \int_{B_R} \nabla u \cdot \nabla(\nabla m \cdot x) dx - \int_{\partial B_R} (\nabla u \cdot \nu)(\nabla m \cdot x) \, d\sigma, \tag{3.26}$$

we observe that

$$\int_{B_R} \nabla u \cdot \nabla(\nabla m \cdot x) dx = \int_{B_R} \sum_{i=1}^N u_{x_i} (\nabla m \cdot x)_{x_i} = \int_{B_R} \nabla u \cdot \nabla m + \int_{B_R} \sum_{i,j} u_{x_i} m_{x_i x_j} x_j$$

and integrating by parts the last term of the previous one we get

$$\begin{aligned}\int_{B_R} \sum_{i,j} (u_{x_i} x_j) m_{x_i x_j} &= \int_{\partial B_R} \sum_{i,j} u_{x_i} m_{x_i} x_j \cdot \frac{x_j}{R} - \int_{B_R} \sum_{i,j} m_{x_i} u_{x_i x_j} x_j - N \int_{B_R} \sum_i u_{x_i} m_{x_i} \\ &= \int_{\partial B_R} (\nabla u \cdot \nabla m) x \cdot \nu d\sigma - \int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) + (1-N) \int_{B_R} \nabla u \cdot \nabla m.\end{aligned}$$

Note that $x \cdot \nu = R$ on ∂B_R. Coming back to (3.26) we obtain

$$\begin{aligned}-\int_{B_R} \Delta u \, \nabla m \cdot x \, dx = &-\int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) dx + (2-N) \int_{B_R} \nabla u \cdot \nabla m \, dx \\ &+ \int_{\partial B_R} (\nabla u \cdot \nabla m)(x \cdot \nu) d\sigma - \int_{\partial B_R} (\nabla u \cdot \nu)(\nabla m \cdot x) \, d\sigma.\end{aligned} \tag{3.27}$$

Concerning the second and the third term in (3.25), we get that

$$\begin{aligned}\frac{1}{\gamma} \int_{B_R} |\nabla u(x)|^\gamma \nabla m \cdot x \, dx &= \frac{1}{\gamma} \int_{\partial B_R} |\nabla u|^\gamma m \, x \cdot \nu \, d\sigma - \frac{1}{\gamma} \int_{B_R} m \, \mathrm{div}(|\nabla u|^\gamma x) dx \\ &= \frac{1}{\gamma} \int_{\partial B_R} |\nabla u|^\gamma m \, x \cdot \nu \, d\sigma - \frac{1}{\gamma} \int_{B_R} m \, \nabla(|\nabla u|^\gamma) \cdot x \, dx - \frac{N}{\gamma} \int_{B_R} m |\nabla u(x)|^\gamma dx\end{aligned} \tag{3.28}$$

and

$$\lambda \int_{B_R} \nabla m \cdot x \, dx = \lambda \int_{\partial B_R} m \, x \cdot \nu d\sigma - \lambda N \int_{B_R} m \, dx. \tag{3.29}$$

Similarly, multiplying the second equation in (3.23) by $\nabla u \cdot x$ and integrating over $B_R(0)$

we get

$$\begin{aligned}\int_{B_R} \Delta m \nabla u \cdot x\, dx &= -\int_{B_R} \operatorname{div}(m|\nabla u|^{\gamma-2}\nabla u)\nabla u \cdot x\, dx \\ &= \int_{B_R} \nabla(\nabla u \cdot x)\cdot (m\,|\nabla u|^{\gamma-2}\nabla u)\, dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2}\nabla u \cdot \nu\, d\sigma \\ &= \int_{B_R} \frac{1}{\gamma} m \nabla(|\nabla u|^{\gamma})\cdot x\, dx + \int_{B_R} m|\nabla u|^{\gamma} dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2}\nabla u \cdot \nu\, d\sigma\end{aligned} \tag{3.30}$$

where we have integrated by parts and then used the following identity

$$\frac{1}{\gamma}\nabla(|\nabla u|^{\gamma})\cdot x = |\nabla u|^{\gamma-2}\nabla u \cdot \nabla(\nabla u \cdot x) - |\nabla u|^{\gamma}.$$

Integrating by parts the LHS of (3.30) we get

$$\begin{aligned}&\int_{\partial B_R} (\nabla m \cdot \nu)(\nabla u \cdot x) d\sigma - \int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) dx \\ &\quad= \int_{B_R} \frac{m}{\gamma}\nabla(|\nabla u|^{\gamma})\cdot x\, dx + \int_{B_R} m|\nabla u|^{\gamma} dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2}\nabla u \cdot \nu\, d\sigma\end{aligned}$$

and then isolating the first term in the second line

$$\begin{aligned}-\frac{1}{\gamma}\int_{B_R} m\nabla(|\nabla u|^{\gamma})\cdot x\, dx = &\int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) dx - \int_{\partial B_R} (\nabla m \cdot \nu)(\nabla u \cdot x)\, d\sigma \\ &+ \int_{B_R} m|\nabla u|^{\gamma} dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2}\nabla u \cdot \nu\, d\sigma\end{aligned} \tag{3.31}$$

plugging (3.31) in (3.28) we obtain

$$\begin{aligned}\frac{1}{\gamma}\int_{B_R} |\nabla u(x)|^{\gamma}\nabla m \cdot x\, dx &= \frac{1}{\gamma}\int_{\partial B_R} m|\nabla u|^{\gamma} x \cdot \nu\, d\sigma \\ &\quad+ \int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x)\, dx - \int_{\partial B_R} (\nabla m \cdot \nu)(\nabla u \cdot x)\, d\sigma \\ &\quad+ \left(1 - \frac{N}{\gamma}\right)\int_{B_R} m|\nabla u|^{\gamma} dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2}\nabla u \cdot \nu\, d\sigma.\end{aligned} \tag{3.32}$$

For what concern the Riesz's potential term, since $m \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ from Theorem 3.2.6 it follows that $K_\alpha * m \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, hence by Hölder inequality

$$\left|\int_{B_R} K_\alpha * m(x)\nabla m \cdot x\, dx\right| \le R\int_{B_R} |K_\alpha * m|\, |\nabla m|\, dx \le R\, \|K_\alpha * m\|_{L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)} \|\nabla m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)},$$

this proves that the term $-\int_{B_R} K_\alpha * m(x)\nabla m \cdot x\, dx$ is finite. We get

$$\begin{aligned}\int_{B_R} K_\alpha * m(x)\nabla m \cdot x\, dx &= \int_{B_R}\int_{\mathbb{R}^N} \frac{m(y)\nabla m(x)\cdot x}{|x-y|^{N-\alpha}} dy\, dx = \int_{\mathbb{R}^N}\int_{B_R} \frac{m(y)\nabla m(x)\cdot x}{|x-y|^{N-\alpha}} dx\, dy \\ &= \int_{\mathbb{R}^N}\int_{\partial B_R} \frac{m(x)\, m(y)}{|x-y|^{N-\alpha}}(x\cdot\nu)\, d\sigma(x)\, dy - \int_{\mathbb{R}^N}\int_{B_R} m(x)\operatorname{div}_x\left(\frac{m(y)}{|x-y|^{N-\alpha}} x\right) dx\, dy\end{aligned} \tag{3.33}$$

and furthermore

$$\int_{\mathbb{R}^N}\int_{B_R} m(x)\,\mathrm{div}_x\left(\frac{m(y)}{|x-y|^{N-\alpha}}x\right)dx\,dy =$$
$$= (\alpha-N)\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x-y)\cdot x}{|x-y|^2}dx\,dy + N\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy =$$
$$= \frac{\alpha+N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy + \frac{\alpha-N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy \tag{3.34}$$

where we used that $\frac{x\cdot(x-y)}{|x-y|^2}=\frac{1}{2}+\frac{(x+y)\cdot(x-y)}{2|x-y|^2}$. Summing up (3.27), (3.29), (3.32), (3.33) and (3.34) we get the following identity

$$(2-N)\int_{B_R}\nabla u\cdot\nabla m\,dx + \left(1-\frac{N}{\gamma}\right)\int_{B_R} m|\nabla u|^\gamma dx - \frac{\alpha+N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy$$
$$-\lambda N\int_{B_R} m(x)\,dx - \frac{\alpha-N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy = I_{\partial B_R} \tag{3.35}$$

where

$$I_{\partial B_R} = \int_{\partial B_R}\left[-\nabla u\cdot\nabla m - \frac{m}{\gamma}|\nabla u|^\gamma - \lambda\, m\right](x\cdot\nu)d\sigma - \int_{\mathbb{R}^N}\int_{\partial B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}(x\cdot\nu)\,d\sigma(x)\,dy$$
$$+\int_{\partial B_R}(\nabla u\cdot\nu)(\nabla m\cdot x) + (\nabla m\cdot\nu)(\nabla u\cdot x) + m|\nabla u|^{\gamma-2}(\nabla u\cdot x)(\nabla u\cdot\nu)\,d\sigma.$$

Now, we let R go to infinity in (3.35). We observe that (changing variables x and y)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy = 0.$$

Moreover

$$|I_{\partial B_R}| \le R\int_{\partial B_R}(3|\nabla u|\,|\nabla m| + 2m\,|\nabla u|^\gamma + |\lambda| m)\,d\sigma + \int_{\mathbb{R}^N} R\int_{\partial B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}d\sigma(x)\,dy$$

since by assumption $|\nabla u|\,|\nabla m|$, $m|\nabla u|^\gamma$ and $m\in L^1(\mathbb{R}^N)$, by Lemma 3.3.1, we get that for some sequence $R_n\to+\infty$

$$R_n\int_{\partial B_{R_n}}\Big(3|\nabla u|\,|\nabla m| + 2m\,|\nabla u|^\gamma + |\lambda| m\Big)d\sigma \to 0, \quad \text{as } n\to+\infty.$$

By means of the same argument, since $m\in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ implies that

$$G(x) := \int_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dy \in L^1(\mathbb{R}^N)$$

(by Theorem 3.2.7), we get that there exists a sequence $R_n\to+\infty$ such that

$$R_n\int_{\partial B_{R_n}} G(x)dx \to 0, \quad \text{as } n\to+\infty,$$

which conclude the proof. □

We are now in position to prove nonexistence of classical solutions with prescribed integrability and boundary conditions at ∞.

Proof of Theorem 3.1.1. We argue by contradiction. Assume to have a triple $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N) \times \mathbb{R}$ solution to (3.23) such that $u \to +\infty$ as $|x| \to +\infty$ and it holds

$$m|\nabla u|^\gamma, \quad |\nabla m||\nabla u| \quad \in L^1(\mathbb{R}^N).$$

From Proposition 3.3.2 we have the following Pohozaev-type identity

$$(2-N)\int_{\mathbb{R}^N} \nabla u \cdot \nabla m \, dx + \left(1 - \frac{N}{\gamma}\right)\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda N M + \frac{\alpha + N}{2}\int_{\mathbb{R}^N} m(K_\alpha * m)\, dx. \tag{3.36}$$

Moreover, we obtain the following identities

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla m \, dx = -\frac{1}{\gamma}\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx - \lambda M - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx\, dy \tag{3.37}$$

and

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla m \, dx = -\int_{\mathbb{R}^N} m|\nabla u|^\gamma \, dx. \tag{3.38}$$

Proof of (3.37). Multiplying the first equation in (3.23) by m and integrating over B_R we obtain

$$\int_{B_R} \nabla u \cdot \nabla m \, dx - \int_{\partial B_R} m \nabla u \cdot \nu \, d\sigma + \frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma dx + \lambda \int_{B_R} m \, dx = -\int_{B_R}\int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dy\, dx. \tag{3.39}$$

By Holder's inequality and using the fact that $m|\nabla u|^\gamma \in L^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |\nabla u| m \, dx \le \left(\int_{\mathbb{R}^N} |\nabla u|^\gamma m \, dx\right)^{\frac{1}{\gamma}} M^{\frac{1}{\gamma'}} < +\infty,$$

hence $|\nabla u| m \in L^1(\mathbb{R}^N)$ and by Lemma 3.3.1, we get that for some sequence $R_n \to +\infty$

$$\int_{\partial B_{R_n}} m \nabla u \cdot \nu \, d\sigma \to 0, \quad \text{as } n \to +\infty.$$

Equality (3.37) follows letting $R \to \infty$ in (3.39).
Proof of (3.38). For any $s > 0$ let us define the set

$$X_s := \{x \in \mathbb{R}^N \mid u(x) \le s\}$$

and the function

$$v_s(x) := u(x) - s, \quad \forall x \in \mathbb{R}^N.$$

After a translation we may assume $u(0) = 0$. In this way, $\cup_{s>0} X_s = \mathbb{R}^N$, every X_s is non-empty and bounded since $u(x) \to +\infty$ as $|x| \to +\infty$. Multiplying the second equation in (3.23) by v_s and integrating by parts, we get

$$\int_{X_s} \nabla v_s \cdot \nabla m \, dx = -\int_{X_s} m\, |\nabla u|^{\gamma-2} \nabla u \cdot \nabla v_s \, dx,$$

since $\nabla v_s = \nabla u$, we obtain (3.38) letting $s \to +\infty$.

Plugging (3.38) in (3.37) we get

$$\left(1-\frac{1}{\gamma}\right)\int\limits_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda M + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy$$

and hence

$$\int\limits_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda\gamma' M + \gamma' \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy. \tag{3.40}$$

Using (3.38) in (3.36), we have

$$\left(\frac{N}{\gamma'}-1\right)\int\limits_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda N\, M + \frac{\alpha+N}{2}\int\limits_{\mathbb{R}^N} m(K_\alpha * m)dx$$

and finally from (3.40) we obtain

$$\left(\frac{N-2\gamma'-\alpha}{2}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy = \gamma'\lambda\, M.$$

Recall that by Lemma 3.2.12, we have that $\lambda \le 0$ and by assumption $N - 2\gamma' - \alpha > 0$, so we get a contradiction. □

Remark 6. One could observe that the the previous proof (with slight changes) holds also in the case when $u \to -\infty$ as $|x| \to +\infty$, hence one may ask why we do not consider this possibility. This is due to the fact that the property of ergodicity for the process is strictly related to the existence of a Lyapunov function (refer to [110]). More in detail, a sufficient condition to have an ergodic process is

$$\nabla u \cdot x > 0, \quad \text{for } x \text{ large}$$

(see also [68] and references therein). As a consequence, the case $u \to -\infty$ as $|x| \to +\infty$ is not relevant.

Remark 7. A Pohozaev identity similar to (3.24) can be obtained also in the case where V does not vanishes identically (requiring in addition that $m\nabla V \cdot x \in L^1(\mathbb{R}^N)$ and $mV \in L^1(\mathbb{R}^N)$). Nevertheless, the nonexistence result of Theorem 3.1.1 can not be extended to the MFG system (3.1) when V satisfies (3.2). This is due to the fact that in this case we could not have that $\lambda \le 0$ (refer to Lemma 3.2.12).

3.4 Existence of classical solutions to the MFG system

First of all we consider a regularised version of problem (3.1), namely

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m * \varphi_k(x) \\ -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \qquad \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \ge 0 \end{cases} \tag{3.41}$$

where $(\varphi_k)_k$ is a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$ (i.e. a sequence of symmetric functions on $\mathbb{R}^N$ such that $\varphi_k \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\varphi_k \subset \overline{B_{1/k}(0)}$, $\int \varphi_k = 1$ and $\varphi_k \ge 0$). For every k fixed, using the Schauder Fixed Point Theorem, we will prove existence of (u_k, m_k, λ_k) solution to (3.41), and then, exploiting a priori uniform estimates on these solutions, we will show that we may pass to the limit as $k \to +\infty$ and get a solution of the MFG system (3.1).

3.4.1 Solution of the regularized problem

We consider (3.41) with k fixed:

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m * \varphi(x) \\ -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \quad \text{in } \mathbb{R}^N. \tag{3.42}$$

We are going to construct solution to (3.42) by using the following version of the well-known Schauder Fixed Point Theorem. Construction of solutions to MFG systems by exploiting fixed point arguments is quite classical in the literature, see [9, 70, 102, 123].

Theorem 3.4.1 (Corollary 11.2 in [93])**.** *Let A be a closed and convex set in a Banach space X and let $\mathcal{F}$ be a continuous map from A into itself such that the image $\mathcal{F}(A)$ is precompact. Then, $\mathcal{F}$ has a fixed point.*

Let $\xi, C > 0$ (which will be chosen later), $M > 0$ and $\bar{p} > \frac{N}{\alpha}$, we define the set

$$A_{\xi,M,C} := \left\{ \mu \in L^{\bar{p}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \,\middle|\, \|\mu\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi,\ \int_{\mathbb{R}^N} \mu\, dx = M, \right. \\ \left. \mu \geq 0,\ \int_{\mathbb{R}^N} \mu V(x)\, dx \leq C \right\}. \tag{3.43}$$

Lemma 3.4.2. *For any choice of $\xi, M, C > 0$, the set $A_{\xi,M,C} \subset L^{\bar{p}}(\mathbb{R}^N)$ is closed and convex.*

Proof. The set $A_{\xi,M,C}$ is convex since it is intersection of convex sets.

Let now $(\mu_n)_n$ be a sequence in $A_{\xi,M,C}$ which converges in $L^{\bar{p}}$ to $\bar{\mu}$. Obviously $\bar{\mu} \geq 0$ and since $\mu_n \rightharpoonup \bar{\mu}$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ by weak lower semicontinuity of the norm we have that

$$\|\bar{\mu}\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \liminf \|\mu_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi.$$

From Fatou's Lemma we get also that $\int_{\mathbb{R}^N} \bar{\mu} V(x) \leq \liminf \int_{\mathbb{R}^N} \mu_n V(x) \leq C$. Note that due to the fact that $0 \leq \int_{\mathbb{R}^N} \mu_n V(x) \leq C$, and that V is coercive, see (3.2), μ_n are uniformly integrable, since for every $R >> 1$, $0 \leq \int_{|x|\geq R} \mu_n dx \leq \frac{C_V}{R^b} \int_{\mathbb{R}^N} V(x)\mu_n dx \leq \frac{CC_V}{R^b}$. Due to the fact that $\mu_n \to \bar{\mu}$ in $L^{\bar{p}}$, we have also that they have uniformly absolutely continuous integrals, so we may apply the Vitali convergence theorem and obtain that $\mu_n \to \bar{\mu}$ in $L^1(\mathbb{R}^N)$ and hence $\int_{\mathbb{R}^N} \bar{\mu}\, dx = M$. This proves that $\bar{\mu} \in A_{\xi,M,C}$, and hence that $A_{\xi,M,C}$ is closed. □

We define the map $F : A_{\xi,M,C} \to C^2(\mathbb{R}^N) \times \mathbb{R}$ which to every element $\mu \in A_{\xi,M,C}$ associates a solution $(u, \bar{\lambda}) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \mu * \varphi(x), \quad \text{in } \mathbb{R}^N \tag{3.44}$$

where $\bar{\lambda}$ is defined as in (3.19) (refer to [12]); and the map G which to the couple $(u, \bar{\lambda})$ associates the function m which solves (weakly)

$$\begin{cases} -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases}. \tag{3.45}$$

We look for a fixed point of the map $\mathcal{F} : \mu \mapsto m$ defined as the composition of F and G, namely $\mathcal{F}(\mu) := G(F(\mu))$.

We are going to show that, once we have fixed M (in an appropriate range), it is possible to choose appropriately ξ and C in such a way that the map $\mathcal{F}$ defined on $A_{\xi,M,C}$ satisfies the assumptions of the Schauder Fixed Point Theorem 3.4.1. As we will see, the regularization with φ in the system (3.42) is necessary in order to get precompactness of the image of $\mathcal{F}$. We start with some preliminary results.

Proposition 3.4.3. *Let us consider $\mu \in A_{\xi,M,C}$, $(u,\bar\lambda) = F(\mu)$ and $m = G(u,\bar\lambda) = \mathcal{F}(\mu)$. Then,*

i) there exists a positive constant C depending on $C_V, b, \gamma, N, \bar\lambda, \|K_\alpha * \mu * \varphi\|_\infty$ such that

$$|\nabla u(x)| \le C(1+|x|)^{\frac{b}{\gamma}}. \tag{3.46}$$

ii) the function u is unique up to addition of constants and there exists $C > 0$ such that

$$u(x) \ge C|x|^{\frac{b}{\gamma}+1} - C^{-1}. \tag{3.47}$$

iii) it holds

$$-K_1 \le \bar\lambda \le K_2 \tag{3.48}$$

where K_1 and K_2 are positive constants depending respectively on $\|K_\alpha * \mu * \varphi\|_\infty$ and on C_V, b, γ, N.

iv) the function m is unique, $m \in (W^{1,1}\cap L^\infty)(\mathbb{R}^N)$, $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$, $m \in W^{1,p}(\mathbb{R}^N)$ $\forall p > 1$ and it holds

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \le C\|m^{\frac{1}{p}}|\nabla u|^{\gamma-1}\|_{L^p(\mathbb{R}^N)}\|m^{1-\frac{1}{p}}\|_{L^\infty(\mathbb{R}^N)}. \tag{3.49}$$

Moreover, the following integrability properties are verified

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N), \qquad mV \in L^1(\mathbb{R}^N), \qquad |\nabla u|\,|\nabla m| \in L^1(\mathbb{R}^N). \tag{3.50}$$

Proof. *i)* Since $\mu * \varphi \in L^1(\mathbb{R}^N) \cap L^{\bar p}(\mathbb{R}^N)$ with $\bar p > \frac{N}{\alpha}$, by Theorem 3.2.8 and Theorem 3.2.9 we obtain $K_\alpha * (\mu * \varphi) \in C^{0,\theta}(\mathbb{R}^N)$ for some $\theta \in (0,1)$ and $\|K_\alpha * \mu * \varphi\|_\infty \le C_{N,\alpha,\bar p}\|\mu * \varphi\|_{L^{\bar p}} + \|\mu * \varphi\|_{L^1} \le C_{N,\alpha,\bar p}\|\mu\|_{L^{\bar p}} + M$. Therefore we can apply Proposition 3.2.10, which gives us the following estimate

$$|\nabla u(x)| \le C(1+|x|)^{\frac{b}{\gamma}}$$

where C is a constant depending on $C_V, b, \gamma, N, \bar\lambda, \|K_\alpha * (\mu * \varphi)\|_\infty$. This proves (3.46).

ii) Since, by construction, u is a solution to (3.44) with $\lambda = \bar\lambda$ then by Proposition 3.2.11 *ii)* it follows uniqueness up to additive constants and (3.47).

iii) The fact that $\bar\lambda \le K_2$ is a direct consequence of Lemma 3.2.12. Furthermore, if $\bar x$ is a minimum point of u, evaluating (3.44) at $\bar x$ we have that

$$\bar\lambda \ge V(\bar x) - K_\alpha * \mu * \varphi(\bar x) \ge -\|K_\alpha * \mu * \varphi\|_\infty \ge -K_1$$

since $V(x) \geq 0$ in $\mathbb{R}^N$.

iv) For $r > 1$, let us consider the function $h(x) := u(x)^r$, one can observe that

$$
\begin{aligned}
-\Delta h + |\nabla u|^{\gamma-2}\nabla u \cdot \nabla h &= r u^{r-1}\left(-(r-1)\frac{|\nabla u|^2}{u} - \Delta u + |\nabla u|^\gamma\right) \\
&= r u^{r-1}\left(-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma\right) \\
&= r u^{r-1}\left(-\bar{\lambda} + V - K_\alpha * \mu * \varphi - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma\right),
\end{aligned}
$$

where in the last equality we used the fact that u solves (3.44). Denoting by

$$
H(x) := -\bar{\lambda} + V(x) - K_\alpha * \mu * \varphi(x) - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma,
$$

from (3.48), (3.2) and the fact that $K_\alpha * \mu * \varphi \in L^\infty$, we get

$$
H(x) \geq (r-1)|\nabla u|^\gamma\left(\frac{1}{\gamma'(r-1)} - \frac{|\nabla u|^{2-\gamma}}{u}\right) + C_V^{-1}|x|^b - C \geq 1, \quad \text{for } |x| > R
$$

taking R sufficiently large. Hence, for $|x| > R$

$$
-\Delta h + |\nabla u|^{\gamma-2}\nabla u \cdot \nabla h \geq C|x|^{(\frac{b}{\gamma}+1)(r-1)} > 0
$$

this means that h is a Lyapunov function for the stochastic process with drift $|\nabla u|^{\gamma-2}\nabla u$, and since m solves (3.45), it is the density of the invariant measure associated to this process. So, from [150, Proposition 2.3] we get that

$$
m|x|^{(\frac{b}{\gamma}+1)(r-1)} \in L^1(\mathbb{R}^N)
$$

and more in general, since the value of $r > 1$ can be chosen arbitrarily, we have that for any $q > 0$

$$
m|x|^q \in L^1(\mathbb{R}^N).
$$

In particular $m|x|^b \in L^1(\mathbb{R}^N)$, so taking into account estimates (3.46) and (3.2) we obtain that

$$
m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad mV \in L^1(\mathbb{R}^N).
$$

With the same argument (since $|\nabla u|^{p(\gamma-1)}$ has polynomial growth) it follows that

$$
m|\nabla u|^{p(\gamma-1)} \in L^p(\mathbb{R}^N), \quad \forall p > 1
$$

hence from [150, Corollary 3.2 and Theorem 3.5] we get that

$$
m \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
$$

Moreover, using the fact that m is a weak solution to the Kolmogorov equation and Hölder inequality, we obtain that for any $\phi \in C_0^\infty(\mathbb{R}^N)$ we have

$$
\left|\int_{\mathbb{R}^N} m\Delta\phi\, dx\right| \leq \int_{\mathbb{R}^N} m|\nabla u|^{\gamma-1}|\,|\nabla\phi|\, dx \leq \|m^{\frac{1}{p}}|\nabla u|^{\gamma-1}\|_p \|m^{1-\frac{1}{p}}\|_\infty \|\nabla\phi\|_{p'}.
$$

Since $m^{\frac{1}{p}}|\nabla u|^{\gamma-1} \in L^p(\mathbb{R}^N)$ and $m^{1-\frac{1}{p}} \in L^\infty(\mathbb{R}^N)$, by Proposition 3.2.2 we get that

$$
m \in W^{1,p}(\mathbb{R}^N), \ \ \forall p > 1
$$

and estimate (3.49) holds. Finally, from [150, Theorem 3.1] we have that $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \frac{|\nabla m|^2}{m} < +\infty$, so using Hölder inequality we obtain

$$\int_{\mathbb{R}^N} |\nabla u|\,|\nabla m| \leq \left\| \,|\nabla u|\sqrt{m}\, \right\|_2 \left\| \frac{|\nabla m|}{\sqrt{m}} \right\|_2 < +\infty.$$

Since the function u is unique up to additive constants, ∇u is fixed and hence, by existence of a Lyapunov function, it follows immediately uniqueness of m solution to the Kolmogorov equation. □

We show now that once we fix the mass M (in $(0, +\infty)$ in the mass-subcritical case, or below a certain threshold in the mass-supercritical and mass-critical regime), then we may choose the constants ξ and C in the definition (3.43) of the set $A_{\xi,M,C}$ such that the map $\mathcal{F}$ maps $A_{\xi,M,C}$ into itself.

Lemma 3.4.4. *We have the following results:*

i) if $N - \gamma' < \alpha < N$, then for any $M > 0$, there exist $\xi, C > 0$ such that $\mathcal{F}$ maps $A_{\xi,M,C}$ into itself;

ii) if $N-2\gamma' \leq \alpha \leq N-\gamma'$, then there exists a positive real value $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ such that if $M \in (0, M_0)$ there exist $\xi, C > 0$ such that $\mathcal{F}$ maps the set $A_{\xi,M,C}$ into itself.

Proof. Let $\mu \in A_{\xi,M,C}$, $m = \mathcal{F}(\mu)$ and $(u, \bar\lambda) = F(\mu)$ as above. Since by Proposition 3.4.3 *iv)* $m \in L^\infty(\mathbb{R}^N)$, by interpolation it follows that $m \in L^{\bar p}(\mathbb{R}^N)$. Multiplying (3.44) by m and integrating over B_R, we obtain

$$-\int_{B_R} m\Delta u\,dx + \frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma dx + \bar\lambda \int_{B_R} m\,dx = \int_{B_R} V(x)m\,dx - \int_{B_R} m(K_\alpha * \mu * \varphi)\,dx$$

and integrating by parts the first term

$$\begin{aligned}\int_{B_R} \nabla m\cdot\nabla u\,dx - \int_{\partial B_R} m\nabla u\cdot\nu\,d\sigma + \frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma dx + \bar\lambda\int_{B_R} m\,dx& \\ = \int_{B_R} V(x)m\,dx - \int_{B_R} m(K_\alpha * \mu * \varphi)\,dx.&\end{aligned} \tag{3.51}$$

From the fact that $\int_{\mathbb{R}^N} m = M$ and $m|\nabla u|^\gamma \in L^1(\mathbb{R}^N)$, by Hölder inequality we get that $m|\nabla u| \in L^1(\mathbb{R}^N)$, hence by Lemma 3.3.1 for some sequence $R_n \to +\infty$ we have that $\int_{\partial B_{R_n}} m\nabla u\cdot\nu\,d\sigma \to 0$. Since $m(K_\alpha * \mu * \varphi) \in L^1(\mathbb{R}^N)$ and (3.50) holds, letting R go to $+\infty$ in (3.51) we obtain that

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx = -\frac{1}{\gamma}\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx - \bar\lambda M + \int_{\mathbb{R}^N} V(x)m\,dx - \int_{\mathbb{R}^N} m(K_\alpha * \mu * \varphi)\,dx. \tag{3.52}$$

Moreover, from the fact that m solves (weakly) the Kolmogorov equation in (3.45), following the proof of identity (3.38), we have that

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx = -\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx. \tag{3.53}$$

Putting together (3.52) and (3.53) we get that

$$\frac{1}{\gamma'}\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx + \int_{\mathbb{R}^N} mV\,dx = \bar\lambda M + \int_{\mathbb{R}^N} m(K_\alpha * \mu * \varphi)\,dx. \tag{3.54}$$

From (3.54), using the fact that $\bar{\lambda} \leq K_2$ (from (3.48)) and (3.15), we have

$$\begin{aligned}\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx &\leq C_1 M + C_2 \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|\mu * \varphi\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\leq C_1 M + C_2 \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|\mu\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\leq C_1 M + C_2\, \xi \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \end{aligned} \tag{3.55}$$

where $C_1 = C_1(\gamma, C_V, b, N)$ and $C_2 = C_2(\alpha, N, \gamma)$.
Choice of ξ. First of all we show that we may choose ξ in such a way that if $\mu \in A_{\xi,M,C}$ then $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = \|\mathcal{F}(\mu)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi$.
Let us fix $a := \frac{2\gamma'}{N-\alpha}$. Notice that $a > 2$ if $\alpha > N - \gamma'$, $a = 2$ if $\alpha = N - \gamma'$, $a \in (1,2)$ if $N - 2\gamma' < \alpha < N - \gamma'$ and $a = 1$ when $\alpha = N - 2\gamma'$.
In the case when $N - \gamma' \leq \alpha < N$, using estimate (3.12), we get

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C M^{a-1} \int_{\mathbb{R}^N} m|\nabla u|^\gamma dx \tag{3.56}$$

where C is a constant depending on N, α and γ; whereas if $N - 2\gamma' \leq \alpha < N - \gamma'$ using estimate (3.13), we get

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C M^{a-1} \left(\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx + M \right) \tag{3.57}$$

where C is a constant depending on N, α and γ. From (3.55) and either (3.56) or (3.57) we obtain that

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C_1 M^a + C_2 M^{a-1} \xi \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}. \tag{3.58}$$

We define the function

$$f(t) := t^a - C_2 M^{a-1} \xi t - C_1 M^a$$

and observe that (3.58) is equivalent to

$$f\left(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \right) \leq 0.$$

When $a > 1$, $f(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}) \leq 0$ is equivalent to $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq t_0$, where t_0 is the unique zero of f. So, in order to conclude that $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi$ it is sufficient to choose ξ such that $f(\xi) \geq 0$.

Case $N - \gamma' < \alpha < N$. In this case since $a > 2$ and $f(\xi) = \xi^a - C_2 M^{a-1} \xi^2 - C_1 M^a$ then for every fixed $M > 0$, there exists ξ_M such that $f(\xi) \geq 0$ for every $\xi \geq \xi_M$ and we have done.

Case $\alpha = N - \gamma'$. In this case $a = 2$, so arguing as before, and recalling that $f(\xi) = \xi^2 - C_2 M \xi^2 - C_1 M^2$, we get that whenever $M < \frac{1}{C_2} := M_0$ there exists ξ_M such that $f(\xi) \geq 0$ for every $\xi \geq \xi_M$.

Case $N - 2\gamma' < \alpha < N - \gamma'$. In this case $a \in (1,2)$. Denote $g(t) := t^a - C_2 M^{a-1} t^2 - C_1 M^a$. We aim to find ξ such that $g(\xi) \geq 0$. This is possible if and only if $g(t_{max}) \geq 0$, where $t_{max} = \left(\frac{a}{2 C_2 M^{a-1}} \right)^{\frac{1}{2-a}}$ is the maximum point of g. Evaluating g in this point we get

$$g(t_{max}) = \left(\frac{2-a}{2} \right) C_2^{-\frac{a}{2-a}} \left(\frac{a}{2} \right)^{\frac{a}{2-a}} M^{\frac{a(1-a)}{2-a}} - C_1 M^a.$$

Since $a \in (1,2)$ we have that $\frac{2-a}{2} > 0$ and $\frac{a(1-a)}{2-a} < 0$, hence

$$g(t_{max}) \geq 0$$

provided that M is sufficiently small. We may choose $\xi = t_{max}$ (or more generally ξ in the range of values t such that $g(t) \geq 0$).

Case $\alpha = N - 2\gamma'$. Since $a = 1$, the function f reads $f(t) = t - C_2\xi t - C_1 M$ and since $f(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}) \leq 0$ we have, if $\xi < \frac{1}{C_2}$,

$$\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \frac{C_1 M}{1 - C_2\xi}.$$

We look for some condition on M under which we may choose ξ such that $\frac{C_1 M}{1-C_2\xi} \leq \xi$. Observe that this is equivalent to $C_2\xi^2 - \xi + C_1 M \leq 0$. If $M \leq \frac{1}{4C_1C_2}$, then it is sufficient to choose ξ in the range $\left[\frac{1-\sqrt{1-4C_1C_2M}}{2C_2}, \frac{1+\sqrt{1-4C_1C_2M}}{2C_2}\right] \cap \left(0, \frac{1}{C_2}\right)$.

Choice of C. Notice that in each of the previous cases, from (3.54), using (3.48), (3.15) and the fact that $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi$, we get

$$\int_{\mathbb{R}^N} mV dx \leq C_1 M + C_2 \xi^2.$$

So it is sufficient to choose C greater or equal to $C_1 M + C_2\xi^2$.
We can conclude that $\mathcal{F}$ maps the set $A_{\xi,M,C}$ into itself. □

We show now that the image of $\mathcal{F}$ is precompact, that is relatively compact. Here is the main point in which the regularization with the mollifier φ comes into play.

Lemma 3.4.5. *Let M and ξ, C as given by Lemma 3.4.4. Then the image $\mathcal{F}(A_{\xi,M,C})$ is precompact.*

Proof. Let us consider a sequence $(m_n)_n \subset \mathcal{F}(A_{\xi,M,C})$, in order to prove that $\mathcal{F}(A_{\xi,M,C})$ is precompact in $A_{\xi,M,C}$, we have to show that that $(m_n)_n$ admits a subsequence converging in $L^{\bar{p}}$-norm to a point belonging to $A_{\xi,M,C}$. There exists a sequence $(\mu_n)_n \subset A_{\xi,M,C}$ such that $\mathcal{F}(\mu_n) = m_n$ for every $n \in \mathbb{N}$, considering also $(u_n, \bar{\lambda}_n) = F(\mu_n)$, we have that for every $n \in \mathbb{N}$ the triple $(u_n, m_n, \bar{\lambda}_n)$ is such that

$$\begin{cases} -\Delta u_n + \frac{1}{\gamma}|\nabla u_n|^\gamma + \bar{\lambda}_n = V(x) - K_\alpha * \mu_n * \varphi(x) \\ -\Delta m_n - \operatorname{div}(m_n \nabla u_n |\nabla u_n|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} m_n = M \quad m_n \geq 0. \end{cases}.$$

Note that by Young's convolution inequality $\|\mu_n * \varphi\|_{L^q(\mathbb{R}^N)} \leq \|\mu_n\|_{L^1(\mathbb{R}^N)}\|\varphi\|_{L^q(\mathbb{R}^N)} = M\|\varphi\|_{L^q(\mathbb{R}^N)}$ for every q. Therefore by Theorem 3.2.8 and Theorem 3.2.9 we get that $K_\alpha * \mu_n * \varphi \in L^\infty \cap C^{0,\theta}$ for some $\theta \in (0,1)$ uniformly in n, and in particular $\|K_\alpha * \mu_n * \varphi\|_{L^\infty(\mathbb{R}^N)} \leq C$, for some C independent of n. By Proposition 3.4.3 we have that u_n are bounded from below, that $m_n \in L^\infty$ and that $\bar{\lambda}_n$ are equibounded in n, so applying Theorem 3.2.13 (actually a simpler version, with $W_n(x) = V(x) - K_\alpha * \mu_n * \varphi$ and $G_{k,\alpha} \equiv 0$) we obtain that there exists a positive constant C not depending on n such that

$$\|m_n\|_{L^\infty(\mathbb{R}^N)} \leq C, \qquad \forall n \in \mathbb{N}. \tag{3.59}$$

Now we use Proposition 3.2.5 *ii*), since $m_n \in L^q(\mathbb{R}^N)$ (where q is defined as in Proposition 3.2.5) and $E_n \leq C_1 M + C_2 \xi^2$ we get that

$$\|m_n\|_{W^{1,\ell}(\mathbb{R}^N)} \leq C, \quad \forall \ell < q$$

where the constant C does not depend on n. Hence, by Sobolev compact embeddings, $m_n \to \bar{m}$ strongly in $L^s(K)$ for $1 \leq s < q^*$ and for every $K \subset\subset \mathbb{R}^N$. Moreover, using the fact that $\int_{\mathbb{R}^N} m_n V\,dx \leq C$ uniformly in n and (3.2) we get that for $R > 1$

$$C \geq \int_{\mathbb{R}^N} m_n V\,dx \geq \int_{|x|\geq R} m_n V\,dx \geq C R^b \int_{|x|\geq R} m_n(x)\,dx$$

that is

$$\int_{|x|\geq R} m_n(x)dx \to 0, \text{ as } R \to +\infty.$$

Using also the uniform estimate (3.59), from the Vitali Convergence Theorem we obtain that up to sub-sequences

$$m_n \to \bar{m} \quad \text{in } L^1(\mathbb{R}^N) \tag{3.60}$$

and as a consequence $\int_{\mathbb{R}^N} \bar{m}(x)dx = M$. Finally, from (3.59) and (3.60), we deduce that $m_n \to \bar{m}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$. Since $A_{\xi,M,C}$ is closed and by Lemma 3.4.4 we have that $\mathcal{F}(A_{\xi,M,C}) \subset A_{\xi,M,C}$, we may conclude that $\bar{m} \in A_{\xi,M,C}$. □

Finally we show that $\mathcal{F}$ is continuous.

Lemma 3.4.6. *Let ξ, M and C as given by Lemma 3.4.4. Then, the map $\mathcal{F}$ is continuous.*

Proof. Let $(\mu_n)_n$ be a sequence in $A_{\xi,M,C}$ such that $\mu_n \to \tilde{\mu} \in A_{\xi,M,C}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$. In order to prove that the map $\mathcal{F}$ is continuous, we have to show that $\mathcal{F}(\mu_n) \to \mathcal{F}(\tilde{\mu})$ with respect to the $L^{\bar{p}}$-norm, that is $m_n \to \tilde{m}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$.
We consider the sequence made by the couples $(u_n, \bar{\lambda}_n) \in C^2(\mathbb{R}^N) \times \mathbb{R}$, where $(u_n, \bar{\lambda}_n) = F(\mu_n)$ $\forall n \in \mathbb{N}$, as previously defined. As observed in Lemma 3.4.5, $K_\alpha * \mu_n * \varphi$ is uniformly bounded in L^∞. So by Proposition 3.4.3 we have that $\bar{\lambda}_n$ are uniformly bounded, that

$$|\nabla u_n(x)| \leq C(1 + |x|^{\frac{b}{\gamma}}) \qquad \text{uniformly in } n$$

and then consequently

$$|\Delta u_n| \leq C(1 + |x|^b) \qquad \text{uniformly in } n. \tag{3.61}$$

Up to extracting a subsequence we assume that $\bar{\lambda}_n \to \lambda^{(1)}$. Since u_n is a classical solution to the HJB equation, by classical elliptic regularity estimates applied to $v_n(x) := u_n(x) - u_n(0)$ (refer e.g to [93, Theorem 8.32]) for any $\theta \in (0, 1]$ and $K \subset\subset \mathbb{R}^N$ we get

$$\|v_n\|_{C^{1,\theta}_{loc}(K)} \leq C \qquad \text{uniformly in } n$$

(notice that the previous estimate holds for $\theta = 1$ thanks to (3.61)). By Arzelà-Ascoli Theorem, up to extracting a subsequence, we get that

$$v_n \to u^{(1)} \qquad \text{locally uniformly in } C^{1,\theta}$$

and hence

$$\nabla u_n \to \nabla u^{(1)} \qquad \text{locally uniformly in } C^{0,\theta}.$$

Since $\|(\mu_n - \tilde{\mu}) * \varphi\|_{L^{\bar{p}}(\mathbb{R}^N)} \leq 2M\|\varphi\|_{L^{\bar{p}}(\mathbb{R}^N)}$, by Theorem 3.2.9 we get that

$$\|K_\alpha * \varphi * \mu_n\|_{C^{0,\alpha-N/\bar{p}}} \leq C, \qquad \text{uniformly in } n$$

and from Theorem 3.2.8

$$\|K_\alpha * \varphi * \mu_n - K_\alpha * \varphi * \tilde{\mu}\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,\alpha,\bar{p}}\|\mu_n - \tilde{\mu}\|_{L^{\bar{p}}(\mathbb{R}^N)} + \|\mu_n - \tilde{\mu}\|_{L^1(\mathbb{R}^N)}.$$

Since $\mu_n \to \tilde{\mu}$ in $L^1(\mathbb{R}^N) \cap L^{\bar{p}}(\mathbb{R}^N)$, then up to subsequences

$$K_\alpha * \varphi * \mu_n \longrightarrow K_\alpha * \varphi * \tilde{\mu} \quad \text{locally uniformly.}$$

By stability with respect to locally uniform convergence, we get that $(u^{(1)}, \lambda^{(1)})$ is a solution (in the viscosity sense) to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \varphi * \tilde{\mu}(x), \quad \text{on } \mathbb{R}^N.$$

Let $(\tilde{u}, \tilde{\lambda}) = F(\tilde{\mu})$, we want to show that $\tilde{\lambda} = \lambda^{(1)}$. Assume by contradiction that $\tilde{\lambda} \neq \lambda^{(1)}$, without loss of generality we can assume that $\lambda^{(1)} < \tilde{\lambda} - 2\varepsilon$ for a certain $\varepsilon > 0$. Then, for n sufficiently large $\bar{\lambda}_n < \tilde{\lambda} - \varepsilon$ and, possibly enlarging n, we have also $\|K_\alpha * \varphi * \mu_n - K_\alpha * \varphi * \tilde{\mu}\|_\infty \leq \varepsilon$. One can observe that

$$-\Delta\tilde{u} + \frac{1}{\gamma}|\nabla\tilde{u}|^\gamma + \tilde{\lambda} - \varepsilon - V(x) + K_\alpha * \varphi * \mu_n(x) \leq 0,$$

i.e. $\tilde{u}$ is a subsolution to the equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \tilde{\lambda} - \varepsilon = V(x) - K_\alpha * \varphi * \mu_n(x).$$

Since by definition (see [44, Theorem 2.7 (i)])

$$\bar{\lambda}_n := \sup\left\{\lambda \in \mathbb{R} \,\middle|\, -\Delta u + \frac{|\nabla u|^\gamma}{\gamma} + \lambda = V - K_\alpha * \mu_n * \varphi \quad \text{has a subsolution in } C^2(\mathbb{R}^N)\right\}$$

it must be $\bar{\lambda}_n \geq \tilde{\lambda} - \varepsilon$, which yields a contradiction. Therefore $\tilde{\lambda} = \lambda^{(1)}$. By Proposition 3.4.3 *ii*), $\tilde{u}$ is unique up to addition of constants, namely there exists $c \in \mathbb{R}$ such that $\tilde{u} = u^{(1)} + c$, it follows that $\nabla\tilde{u} = \nabla u^{(1)}$. Once we have the sequence of function u_n, we construct the sequence $(m_n)_n \subset \mathcal{F}(A_{\xi,M,C})$ such that for every $n \in \mathbb{N}$ fixed, it holds

$$\begin{cases} -\Delta m_n - \operatorname{div}(m_n \nabla u_n |\nabla u_n|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} m_n = M, \quad m_n \geq 0 \end{cases}.$$

From Lemma 3.4.5, up to extracting a subsequence

$$m_n \to m^{(1)} \text{ in } L^{\bar{p}}(\mathbb{R}^N)$$

where $m^{(1)} \in A_{\xi,M,C}$. Since $\nabla u_n|\nabla u_n|^{\gamma-2} \to \nabla\tilde{u}|\nabla\tilde{u}|^{\gamma-2}$ locally uniformly in $\mathbb{R}^N$, we get that $m^{(1)}$ is a weak solution to

$$-\Delta m - \operatorname{div}(m\nabla\tilde{u}|\nabla\tilde{u}|^{\gamma-2}) = 0$$

that has $\tilde{m} = \mathcal{F}(\tilde{\mu})$ as unique solution. This proves that $m_n \to \tilde{m}$ in $L^{\bar{p}}(\mathbb{R}^N)$. □

We are ready to prove the following result on existence of solutions to the regularised MFG system (3.42).

Theorem 3.4.7. *We get the following results:*

i. if $N-\gamma' < \alpha < N$ *then, for every* $M > 0$ *the MFG system* (3.42) *admits a classical solution;*

ii. if $N-2\gamma' \leq \alpha \leq N-\gamma'$ *then, there exists a positive real value* $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ *such that if* $M \in (0, M_0)$ *the MFG system* (3.42) *admits a classical solution.*

Proof. From Lemma 3.4.2, Lemma 3.4.4, Lemma 3.4.5 and Lemma 3.4.6 assumptions of Theorem 3.4.1 are verified, hence the map $\mathcal{F}$ has a fixed point m_φ. The fixed point m_φ together with the couple $(u_\varphi, \bar{\lambda}_\varphi) = F(m_\varphi)$ obtained solving the Hamilton-Jacobi-Bellman equation with Riesz potential term equal to $K_\alpha * m_\varphi * \varphi$, is a solution to the MFG system (3.42). □

3.4.2 Limiting procedure

Let $(\varphi_k)_k$ be a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$. For every $k \in \mathbb{N}$ (under the additional assumption that the constrained mass M is sufficiently small in the case when $N-2\gamma' < \alpha \leq N-\gamma'$) from Theorem 3.4.7 we can construct a classical solution $(u_k, m_k, \bar{\lambda}_k)$ to the corresponding regularised MFG system (3.41). Our aim now is passing to the limit as $k \to +\infty$ and prove that $(u_k, m_k, \bar{\lambda}_k)$ converges to a solution of the MFG system (3.1).

We need some preliminary apriori estimates.

Lemma 3.4.8. *Let* $\alpha \in (N - 2\gamma', N)$ *and* $(u_k, m_k, \bar{\lambda}_k)$ *be a solution to the regularized MFG system* (3.41) *as constructed in Theorem 3.4.7. Then, there exist* C_1, C_2, C_3 *positive constants independent of* k *such that*

$$\|m_k\|_{L^\infty(\mathbb{R}^N)} \leq C_1,$$

$$|\bar{\lambda}_k| \leq C_2$$

and

$$|\nabla u_k| \leq C_3(1 + |x|^{\frac{b}{\gamma}}) \qquad |\Delta u_k| \leq C_3(1 + |x|^b). \tag{3.62}$$

Proof. Note that if $m \in L^r(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ for any $r \in \left(\frac{N}{\alpha}, +\infty\right]$ and $s \in \left[1, \frac{N}{\alpha}\right)$, by Theorem 3.2.8 we have that

$$\begin{aligned}\|K_\alpha * \varphi_k * m\|_{L^\infty(\mathbb{R}^N)} &\leq C_{N,\alpha,r,s}(\|\varphi_k * m\|_{L^r(\mathbb{R}^N)} + \|\varphi_k * m\|_{L^s(\mathbb{R}^N)}) \\ &\leq C_1\|m\|_{L^r(\mathbb{R}^N)} + C_2\|m\|_{L^s(\mathbb{R}^N)}\end{aligned}$$

So, since $m_k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|m_k\|_{\frac{2N}{N+\alpha}} \leq \xi$, u_k are bounded from below and satisfy (3.4), we may apply Theorem 3.2.13 with $W_k \equiv V$, $G_{k,\alpha}[m] = K_\alpha * \varphi_k * m$ and $q = \frac{2N}{N+\alpha} > \frac{N}{\alpha+\gamma'}$ and conclude the uniform L^∞ bounds on m_k.
Now, by Proposition 3.4.3, we get that $\bar{\lambda}_k$ are equibounded in k and that

$$|\nabla u_k(x)| \leq C(1 + |x|^{\frac{b}{\gamma}}) \qquad |\Delta u_k| \leq C(1 + |x|^b)$$

where C is independent of k. □

Proof of Theorem 3.1.2. Since for any $k \in \mathbb{N}$, u_k is a classical solution to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^{\gamma} + \bar{\lambda}_k = V(x) - K_\alpha * \varphi_k * m_k$$

by Lemma 3.4.8 and elliptic estimates (refer to [93, Theorem 8.32]) applied to $v_k(x) := u_k(x) - u_k(0)$, we obtain that for every $K \subset\subset \mathbb{R}^N$ and $\theta \in (0,1]$

$$\|v_k\|_{C^{1,\theta}_{loc}(K)} \le C \qquad \text{uniformly with respect to } k$$

hence up to extracting a subsequence

$$v_k \to \bar{u} \qquad \text{locally uniformly in } C^1 \text{ on compact sets.}$$

Similarly, since m_k weak solution to $-\Delta m - \operatorname{div}(m|\nabla u_k|^{\gamma-2}\nabla u_k) = 0$, for every $\varphi \in C_0^\infty(K)$ it holds

$$\left| \int_K m_k \Delta\varphi \, dx \right| \le \|\nabla\varphi\|_{L^1(K)} \|m_k |\nabla u_k|^{\gamma-1}\|_{L^\infty(K)}.$$

Using the uniform L^∞ estimates on m_k and the estimates (3.62), by Proposition 3.2.2 and Sobolev embedding, we get that for every $\beta \in (0,1)$

$$\|m_k\|_{C^{0,\beta}(K)} \le C \qquad \text{uniformly with respect to } k$$

so up to extracting a subsequence

$$m_k \to \bar{m} \qquad \text{locally uniformly.}$$

Since the values of $\bar{\lambda}_k$ are equibounded with respect to k, we have that $\bar{\lambda}_k \to \bar{\lambda}$ up to a subsequence.Again recalling that $\int V(x) m_k \le C$ uniformly in k, we conclude by Vitali Convergence Theorem that $m_k \to \bar{m}$ in $L^1(\mathbb{R}^N)$ and hence $\int_{\mathbb{R}^N} \bar{m} = M$. From the strong convergence in $L^1(\mathbb{R}^N)$ and the uniform L^∞ estimates, we obtain also that

$$m_k \to \bar{m} \quad \text{in } L^p(\mathbb{R}^N)$$

for every $p \in [1, +\infty)$. We finally have that

$$K_\alpha * \varphi_k * m_k \to K_\alpha * m \quad \text{locally uniformly.}$$

We can pass to the limit and obtain that $(\bar{u}, \bar{m}, \bar{\lambda})$ is a solution to the MFG system (3.1). $\square$

Chapter 4

Concentration for Ergodic Choquard Mean-Field Games

4.1 Introduction to the problem and main results

In the present chapter, we keep on the study, initiated in Chapter 3, of stationary Mean-Field Games systems defined in the whole space $\mathbb{R}^N$ with aggregating nonlocal coupling of Riesz-type. In particular, given $M > 0$, we consider systems of the form

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^{\gamma} + \lambda = -K_{\alpha} * m(x) \\ -\Delta m - \operatorname{div}(m\nabla u|\nabla u|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{4.1}$$

where $\gamma > 1$ is fixed and $K_{\alpha} : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$.

Focusing on the *mass-subcritical regime* $N - \gamma' < \alpha < N$ (where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ), we provide existence of classical solutions to the MFG system (4.1). Notice that by *classical solution* we mean a triple $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$ for every $p \in (1, +\infty)$, solving the system. More precisely, we obtain the following existence result.

Theorem 4.1.1. *Let $N - \gamma' < \alpha < N$. Then, for every $M > 0$ there exists $(\bar{u}, \bar{m}, \bar{\lambda})$ classical solution to the MFG system* (4.1). *Moreover, there exist C_1, C_2, C_3 and C_4 positive constants such that*

$$\bar{u}(x) \geq C_1|x| - C_1^{-1}, \qquad |\nabla \bar{u}| \leq C_2$$

and

$$0 < \bar{m}(x) \leq C_3 e^{-C_4|x|}.$$

Remark 8. Solutions to the MFG system (4.1) are invariant by translation, namely if $(\bar{u}(x), \bar{m}(x), \bar{\lambda})$ is a classical solution to (4.1) then for every $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$, also $\big(\bar{u}(x + x_0) + c, \bar{m}(x + x_0), \bar{\lambda}\big)$ is a classical solution to (4.1). Therefore, the constants C_1 and C_4 appearing in the previous theorem, depend on the choice of the solution.

Theorem 4.1.1 partially completes the study of existence of solutions to the potential-free MFG system (4.1) started in Chapter 3. In particular, in Theorem 3.1.1, using a Pohozaev-type identity, one proves that if $0 < \alpha < N - 2\gamma'$, "regular" solutions to the MFG system (4.1) (namely satisfying some quite natural integrability conditions and boundary conditions at infinity) do not exist. It remains still open the problem of existence of solutions to (4.1) when $\alpha \in [N-2\gamma', N-\gamma']$.

On the other hand, the main result in Chapter 3 deals with the study of MFG systems with Riesz-type coupling and external confining potential V. More in detail, exploiting a Schauder fixed point argument, one proves that the MFG system admits a classical solution for every total mass $M>0$ if $\alpha \in (N-\gamma', N)$, and for sufficiently small masses $M<M_0$ if $\alpha \in (N-2\gamma', N-\gamma']$ (see Theorem 3.1.2 for more details). Notice that, using a fixed point approach, the presence of the coercive potential V adds compactness to the problem and proves to be essential to conclude, so the existence result in this chapter does not cover the case when the potential V is identically 0. In order to deal with the potential-free system we take advantage a variational argument. This approach allows us to obtain some uniform (namely not depending on the viscosity parameter) estimates on the solutions, which will be crucial in the vanishing viscosity setting and which can not be obtained by means of a fixed point technique.

Remark 9. From Theorem 4.1.1 we obtain a more general result for the range of values α such that the normalized Choquard equation (II.6) with $V\equiv 0$ has a solution, but we left open the problem of symmetry of solutions.

Finally, if the coupling is given by $K_\alpha * m$, dissipating forces dominate and we expect nonexistence of solutions in the noncompact setting $\mathbb{R}^N$; while if we consider a bounded domain with periodic or Neumann boundary conditions a similar approach to the case of power-like coupling should work (see e.g. [70, 98]).

In order to study system (4.1), we consider an ergodic MFG system defined in the whole space $\mathbb{R}^N$ with an external confining potential V and Brownian noise which depends on $\varepsilon>0$. Specifically, we take into account systems of the form

$$\begin{cases} -\varepsilon\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m(x) \\ -\varepsilon\Delta m - \operatorname{div}(m\nabla u|\nabla u|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m\geq 0 \end{cases} \tag{4.2}$$

where we assume that the potential V is locally Hölder continuous and there exist two positive constants b and C_V such that

$$C_V^{-1}(\max\{|x|-C_V,0\})^b \leq V(x) \leq C_V(1+|x|)^b, \quad \forall x\in\mathbb{R}^N. \tag{4.3}$$

Studying the asymptotic behavior of rescaled solutions to the MFG system (4.2) in the vanishing viscosity limit, we are able to prove existence of classical solutions to the MFG system (4.1) (without the potential term V). As a matter of fact, letting $\varepsilon\to 0$, the dynamic of each player is no subject anymore to the dissipation effect induced by the Brownian motion, so we expect aggregation of players. In particular, in the vanishing viscosity limit the mass m tends to concentrate, the introduction of the coercive potential V, which represents spatial preferences of agents, rules out this possibility and leads to concentration of mass around minima of the potential V.

Let us summarize the main tools to prove our results. As above mentioned, taking into account the variational nature of the MFG system, solutions to (4.2) are related to critical points of the energy functional $\mathcal{E}$ (as defined in (II.7)) over the constrained set $K_{\varepsilon,M}$ (see (II.8)). Using some regularity results for the Kolmogorov equation (refer to Proposition 3.2.4), the Hardy-Littlewood-Sobolev inequality and the fact that V is non-negative, we show that the energy $\mathcal{E}$ is bounded from below when $N-\gamma'<\alpha<N$.

By classical direct methods and compactness arguments, we obtain minimizers $(m_\varepsilon, w_\varepsilon)$ of $\mathcal{E}$. Finally, passing to another functional with linearized Riesz-term and using convex duality arguments (see for instance [33, 35, 36, 44, 45]), we are able to construct the associated solutions $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ of the MFG system (4.2). Then, in order to investigate the behavior of the system in the vanishing viscosity limit, we define a suitable rescaling of u_ε, m_ε and λ_ε. We also translate the reference system by y_ε, where y_ε is a point of minimum for the value function u_ε, in such a way around y_ε the mass remains positive and we can rule out vanishing of the total mass in the limit. We obtain a triple $(\bar{m}_\varepsilon, \bar{u}_\varepsilon, \tilde{\lambda}_\varepsilon)$ which solves the following MFG system

$$\begin{cases} -\Delta \bar{u}_\varepsilon + \frac{1}{\gamma}|\nabla \bar{u}_\varepsilon|^\gamma + \tilde{\lambda}_\varepsilon = \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} V\left(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}(y+y_\varepsilon)\right) - K_\alpha * \bar{m}_\varepsilon(y) \\ -\Delta \bar{m}_\varepsilon - \operatorname{div}(\bar{m}_\varepsilon \nabla \bar{u}_\varepsilon |\nabla \bar{u}_\varepsilon|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} \bar{m}_\varepsilon = M \end{cases}$$

(see Subsection 4.4.1 for more details). Exploiting a concentration-compactness argument (refer to the seminal work of P.-L. Lions [136]) as done in [44], we are able to prove that there is no loss of mass when passing to the limit as $\varepsilon \to 0$. We show that in the vanishing viscosity limit, the rescaled solutions converge (up to sub-sequences) to $(\bar{u}, \bar{m}, \bar{\lambda})$ *classical* solution to the MFG system (4.1). Moreover, solutions to (4.1) are related to minimum points of the following energy

$$\mathcal{E}_0(m, w) := \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx\, dy$$

over the constrained set

$$\mathcal{B} := \left\{(m, w) \in \mathcal{K}_{1,M} \;\middle|\; m(1+|x|^b) \in L^1(\mathbb{R}^N)\right\}.$$

We obtain the following theorem, which states existence of solutions to (4.2) and concentration of mass.

Theorem 4.1.2. *Let $N - \gamma' < \alpha < N$. Assume that the potential V is locally Hölder continuous and satisfies* (4.3)*. Then, for every $\varepsilon, M > 0$ there exists $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ classical solution to* (4.2)*, such that $(m_\varepsilon, -m_\varepsilon \nabla u_\varepsilon |\nabla u_\varepsilon|^{\gamma-2})$ is a minimum of the energy $\mathcal{E}$.*

Moreover, there exists a sequence $\varepsilon \to 0$ and a sequence of points $x_\varepsilon = \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y_\varepsilon$ around which there is concentration of mass, namely for every $\eta > 0$ there exist $R, \varepsilon_0 > 0$ such that

$$\int_{B(x_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} R)} m_\varepsilon(x)\, dx \geq M - \eta$$

for all $\varepsilon < \varepsilon_0$ and

$$x_\varepsilon \to \bar{x}, \quad \text{as } \varepsilon \to 0$$

where $\bar{x}$ is a minimum point of the potential V and $V(\bar{x}) = 0$.

Finally, if $0 < \alpha < N - \gamma'$ the energy $\mathcal{E}$ is not bounded from below, so global minima do not exist. Despite this, since in the *mass-supercritical and HLS-subcritical regime* $N - 2\gamma' < \alpha < N - \gamma'$, we still get some a priori estimates for the $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$-norm of m (see (3.13)), we can look for critical points, that are no global minimizers (called *bound states* in the NLS literature). Hence, assuming for sake of simplicity $\varepsilon = 1$ (same results hold for $\varepsilon > 0$ fixed), we study the previous minimization problem requiring a smallness constraint on the $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$-norm of m. More in detail, we minimize $\mathcal{E}$ over the constraint set

$$\mathcal{K}_{1,M,\xi} := \left\{ (m,w) \in \mathcal{K}_{1,M} \,\middle|\, \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le \xi \right\},$$

where ξ is the value obtained from Lemma 3.4.4. Taking advantage of some results obtained in Chapter 3 and adapting the previous minimization procedure, we prove that if the total mass M is sufficiently small, local *free* minimizers of $\mathcal{E}$ exist. This procedure allows us to construct classical solutions to the MFG system (4.2), which a priori are not the same as those obtained in Chapter 3 by using a fixed point argument. More precisely, we get the following result.

Theorem 4.1.3. *Let $N - 2\gamma' < \alpha \le N - \gamma'$ and $\varepsilon = 1$. Assume that the potential V is locally Hölder continuous and satisfies* (4.3)*. Then, there exists a positive real value $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ such that if $M \in (0, M_0)$ the MFG system* (4.2) *admits a classical solution $(\tilde{u}, \tilde{m}, \tilde{\lambda})$. Moreover, $\tilde{m}$ is bounded in $L^\infty(\mathbb{R}^N)$ and the function $\tilde{u}$ satisfies for some constant $C > 0$ the following estimates*

$$\tilde{u}(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1}$$

$$|\nabla \tilde{u}(x)| \le C(1 + |x|^{\frac{b}{\gamma}}).$$

Unfortunately, due to the absence of adequate estimates for the value $\tilde{\lambda}$ in this regime, we are unable to obtain the corresponding outcome of Theorem 4.1.1 when the potential is identically zero.

The outline of this chapter is the following. In Section 4.2 we provide some preliminary results. In particular, we provide some integrability, Hölder continuity and compactness results for the Riesz-potential term when considering couples $(m, w) \in K_{\varepsilon,M}$ with finite kinetic energy E. We state also a more general version of Theorem 3.2.13 for $\alpha \in (N-\gamma', N)$, which gives uniform L^∞ bounds on m. In Section 4.3, using a variational approach, we prove that minimizers of the energy do exist, and from them, we obtain associated solutions to the MFG system. We are in the position to analyze the asymptotic behavior of solutions in the vanishing viscosity limit, in particular, in Section 4.4 we prove existence of groundstates to MFG system defined in the whole space $\mathbb{R}^N$ with Riesz-type coupling and without the confining potential V. Finally, in Section 4.5, we show concentration of the mass towards minima of V. We conclude this Chapter with the study of the energy associated to the MFG system (4.2) when $\alpha \in (N-2\gamma', N-\gamma']$ in Section 4.6.

In what follows, $C, C_1, C_2, K_1, \ldots$ denote generic positive constants which may change from line to line and also within the same line.

4.2 Preliminaries

We state here some preliminary results for couples $(m, w) \in \mathcal{K}_{\varepsilon,M}$ with finite kinetic energy E. For further details we refer the reader to [19, 20, 44]. In what follows we always assume that $\varepsilon, M > 0$ are fixed.

First of all, we have the following compactness result.

Proposition 4.2.1. *Let us consider a sequence of couples* $(m_n, w_n) \in \mathcal{K}_{\varepsilon,M}$ *such that* $E_n < C$ *uniformly in* n*. Assume also that there exists a couple* $(\bar{m}, \bar{w}) \in \mathcal{K}_{\varepsilon,M}$ *such that* $\bar{E} < +\infty$ *and* $m_n \to \bar{m}$ *in* $L^1(\mathbb{R}^N)$ *as* $n \to +\infty$*. Then*

$$m_n \to \bar{m} \text{ in } L^s(\mathbb{R}^N), \quad \forall s \in \left[1, \frac{N}{N-\gamma'}\right)$$

(the previous holds $\forall s \in [1, +\infty)$ *if* $\gamma' \geq N$*).*

Proof. If $\gamma' < N$, from Proposition 3.2.5 we have that $m_n, \bar{m} \in L^\beta(\mathbb{R}^N)$ $\forall \beta < \frac{N}{N-\gamma'}$ and $\|m\|_{L^\beta(\mathbb{R}^N)} \leq C(E+M)$. Let us pick $1 \leq s < \frac{N}{N-\gamma'}$ and $s_1 \in \left(s, \frac{N}{N-\gamma'}\right)$, by interpolation we get there exists $\theta \in (0,1)$ (depending on s and s_1) such that

$$\|\bar{m} - m_n\|_{L^s(\mathbb{R}^N)} \leq \|\bar{m} - m_n\|^\theta_{L^1(\mathbb{R}^N)} \|\bar{m} - m_n\|^{1-\theta}_{L^{s_1}(\mathbb{R}^N)}.$$

We observe that

$$\|\bar{m} - m_n\|_{L^{s_1}(\mathbb{R}^N)} \leq \|\bar{m}\|_{L^{s_1}} + \|m_n\|_{L^{s_1}} \leq C(\bar{E} + M) + C(E_n + M) \leq C_1.$$

Hence $\|\bar{m} - m_n\|_{L^{s_1}(\mathbb{R}^N)}$ is bounded, since $\|\bar{m} - m_n\|_{L^1(\mathbb{R}^N)} \to 0$ we can conclude. The same argument holds in the case when $\gamma' \geq N$. □

We are able to prove more precise integrability results for the Riesz term $K_\alpha * m$ and also a compactness result.

Corollary 4.2.2. *Assume that* $(m, w) \in \mathcal{K}_{\varepsilon,M}$ *and* $E < +\infty$*.*

i) If $\gamma' \geq N$*, then*

$$K_\alpha * m \in L^\beta(\mathbb{R}^N), \quad \forall \beta \in \left(\frac{N}{N-\alpha}, +\infty\right).$$

ii) In the case $\gamma' < N$*, if* $N - \gamma' \leq \alpha < N$*, then*

$$K_\alpha * m \in L^\beta(\mathbb{R}^N), \quad \forall \beta \in \left(\frac{N}{N-\alpha}, +\infty\right);$$

whereas if $0 < \alpha < N - \gamma'$*, then*

$$K_\alpha * m \in L^\beta(\mathbb{R}^N), \quad \forall \beta \in \left(\frac{N}{N-\alpha}, \frac{N}{N-\gamma'-\alpha}\right).$$

Moreover, there exists a constant C *depending on* N*,* α*,* γ' *and* β *such that*

$$\|K_\alpha * m\|_{L^\beta(\mathbb{R}^N)} \leq C(E + M).$$

Proof. Case $\gamma' \geq N$*.* From Proposition 3.2.5 i) we have in particular that $m \in L^\beta(\mathbb{R}^N)$ $\forall \beta \in (1, \frac{N}{\alpha})$, hence by Theorem 3.2.6 it follows claim *i)*. *Case* $\gamma' < N$*.* From Proposition 3.2.5 i) it holds that $m \in L^\beta(\mathbb{R}^N)$ $\forall \beta < \frac{N}{N-\gamma'}$. In the case when $N - \gamma' \leq \alpha < N$, we have that $m \in L^\beta(\mathbb{R}^N)$ $\forall \beta \in (1, \frac{N}{\alpha})$ and we can conclude as before. Whereas in the case when $0 < \alpha < N - \gamma'$, $m \in L^\beta(\mathbb{R}^N)$ $\forall \beta \in (1, \frac{N}{N-\gamma'})$, by Theorem 3.2.6 we can conclude the proof of claim *ii)*. □

Corollary 4.2.3. *Under the assumptions of Proposition 4.2.1, we have the following:*

i) if $\gamma' < N$ and $\alpha > N - 2\gamma'$, it holds

$$(K_\alpha * m_n)m_n \xrightarrow[n\to+\infty]{} (K_\alpha * \bar{m})\bar{m}, \quad \text{in } L^1(\mathbb{R}^N);$$

ii) if $\gamma' \geq N$, then for every $\alpha \in (0, N)$

$$(K_\alpha * m_n)m_n \xrightarrow[n\to+\infty]{} (K_\alpha * \bar{m})\bar{m}, \quad \text{in } L^1(\mathbb{R}^N).$$

Proof. Let us consider $\bar{r} := \frac{Nr}{N-\alpha r}$ and $(\bar{r})'$ its conjugate exponent, namely $(\bar{r})' = \frac{Nr}{Nr-N+\alpha r}$. If $\gamma' < N$, from Proposition 4.2.1, we observe that in order to have $m_n \to \bar{m}$ in $(L^r \cap L^{(\bar{r})'})(\mathbb{R}^N)$ for a certain $r \in \left(1, \frac{N}{N-\gamma'}\right)$, it is sufficient to require that $(\bar{r})' < \frac{N}{N-\gamma'}$, that is

$$\frac{N}{\alpha + \gamma'} < r < \frac{N}{N - \gamma'}$$

and hence

$$\alpha > N - 2\gamma'.$$

In particular $m_n \to \bar{m}$ in $L^r(\mathbb{R}^N)$, so from Theorem 3.2.6 it follows that

$$K_\alpha * m_n \to K_\alpha * \bar{m}, \quad \text{in } L^{\bar{r}}(\mathbb{R}^N)$$

and since $m_n \to \bar{m}$ in $L^{(\bar{r})'}(\mathbb{R}^N)$

$$(K_\alpha * m_n)m_n \to (K_\alpha * \bar{m})\bar{m}, \quad \text{in } L^1(\mathbb{R}^N).$$

The case $\gamma' \geq N$ is analogous. □

Taking advantage of the integrability results in Proposition 3.2.5, we are able to prove Hölder continuity of the term $K_\alpha * m$, for couples $(m, w) \in \mathcal{K}_{\varepsilon,M}$ with finite kinetic energy E.

Corollary 4.2.4. *Assume that $(m, w) \in \mathcal{K}_{\varepsilon,M}$ and $E < +\infty$.*

i) If $\gamma' \geq N$, then

$$K_\alpha * m \in C^{0,\theta}(\mathbb{R}^N), \quad \forall \theta \in (0, \min\{1, \alpha\}).$$

ii) If $\gamma' < N$ and $\alpha > N - \gamma'$, then

$$K_\alpha * m \in C^{0,\theta}(\mathbb{R}^N), \quad \forall \theta \in (0, \min\{1, \alpha - (N - \gamma')\}).$$

Proof. The thesis follows from Theorem 3.2.9 and Proposition 3.2.5. □

We recall here a Brezis-Lieb-type lemma for the Riesz potential (refer to [31, Theorem 1] for the classical Brezis-Lieb Lemma). It will be a key tool in Section 4.4.

Lemma 4.2.5 (Lemma 2.4 in [152])**.** *Let $0 < \alpha < N$, $p \in \left[1, \frac{2N}{N+\alpha}\right)$ and $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. If $f_n \to f$ almost everywhere in $\mathbb{R}^N$ as $n \to +\infty$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (K_\alpha * |f_n|^p)|f_n|^p - \int_{\mathbb{R}^N} (K_\alpha * |f_n - f|^p)|f_n - f|^p = \int_{\mathbb{R}^N} (K_\alpha * |f|^p)|f|^p.$$

4.2.1 Uniform L^∞-bounds on m in the mass-subcritical regime

Finally, we state a simplified version of Theorem 3.2.13, indeed in the mass-subcritical regime $N-\gamma' < \alpha < N$, uniform a priori L^∞-bounds on m hold under more general assumptions.

Theorem 4.2.6. *Let $\alpha \in (N-\gamma', N)$ and $(s_k)_k$, $(t_k)_k$ be bounded positive real sequences. We consider a sequence of classical solutions (u_k, m_k, λ_k) to the following MFG system*

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = s_k V(t_k x) - K_\alpha * m(x) \\ -\Delta m - \operatorname{div}\left(m \nabla u |\nabla u|^{\gamma-2}\right) = 0 & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases}$$

where the potential V satisfies assumption (4.3) with constant C_V, b independent of k. We assume that for every k, $m_k \in L^\infty(\mathbb{R}^N)$ and u_k are bounded from below. Then, there exists a positive constant C not depending on k such that

$$\|m_k\|_{L^\infty(\mathbb{R}^N)} \leq C, \quad \forall k \in \mathbb{N}.$$

Proof. We follow the argument of the proof of Theorem 3.2.13 (refer to [20, Theorem 2.12] and see also [44, Theorem 4.1]), but since we are in the mass-subcritical regime, some assumptions we require in the previous one can be weakened.
We assume by contradiction that

$$\sup_{\mathbb{R}^N} m_k = L_k \to +\infty$$

and we define

$$\delta_k := \begin{cases} L_k^{-\frac{1}{\alpha+\gamma'}} & \text{if } \gamma' \leq N \\ L_k^{-\frac{1}{\gamma'}} & \text{if } \gamma' > N \end{cases}.$$

We rescale (u_k, m_k, λ_k) as follows:

$$v_k(x) := \delta_k^{\frac{2-\gamma}{\gamma-1}} u_k(\delta_k x) + 1, \qquad n_k(x) := L_k^{-1} m_k(\delta_k x), \qquad \tilde{\lambda}_k := \delta_k^{\gamma'} \lambda_k.$$

In this way $0 \leq n_k(x) \leq 1$, $\sup n_k = 1$ and $\int_{\mathbb{R}^N} n_k(x)dx = \delta_k^a M \to 0$ since $a = \alpha + \gamma' - N > 0$ if $\gamma' \leq N$ and $a = \gamma' - N > 0$ if $\gamma' \geq N$. Moreover, since up to addition of constants we may assume $\inf u_k(x) = 0$, we have $v_k(x) \geq 1$ for all $x \in \mathbb{R}^N$. We get that $(v_k, n_k, \tilde{\lambda}_k)$ is a solution to

$$\begin{cases} -\Delta v_k + \frac{1}{\gamma}|\nabla v_k|^\gamma + \tilde{\lambda}_k = V_k(x) - K_\alpha * n_k(x) \\ -\Delta n_k - \operatorname{div}(n_k \nabla v_k |\nabla v_k|^{\gamma-2}) = 0 \end{cases} \tag{4.4}$$

where

$$V_k(x) := \delta_k^{\gamma'} s_k V(t_k \delta_k x).$$

Notice that from (4.3), denoting by $\sigma_k = \delta_k^{\gamma'+b} s_k t_k^b$ we have

$$\delta_k^{\gamma'} s_k C_V^{-1} (\max\{t_k \delta_k |x| - C_V, 0\})^b \leq V_k(x) \leq C_V(1 + \sigma_k |x|^b), \quad \forall x \in \mathbb{R}^N$$

and by Theorem 3.2.8 we get that

$$|K_\alpha * n_k(x)| \leq C_{N,\alpha} \|n_k\|_\infty + \|n_k\|_1 \leq C + \delta_k^a M \leq 2C, \quad \text{uniformly in } k.$$

Moreover, by computing the HJB equation of (4.4) in a minimum point of v_k we have that

$$\tilde{\lambda}_k \geq -K_\alpha * n_k(\bar{x}) \geq -C$$

while with the same computations as in Lemma 3.2.12 ([20, Lemma 2.11]) we obtain that $\lambda_k \leq C$ where C depends on γ, C_V, b, N, this gives $-C \leq \tilde{\lambda}_k \leq \delta_k^{\gamma'} C$ hence $|\tilde{\lambda}_k| \leq C$ uniformly.

We can conclude following the proof of [44, Theorem 4.1]. More in detail, if x_k is an approximated maximum point of n_k (that is $n_k(x_k) = 1 - \delta$), then either $\sigma_k |x_k|^b \to +\infty$ up to subsequences or $\sigma_k |x_k|^b \leq C$ for some $C > 0$. We suppose that the second possibility occurs, using a priori gradient estimates on v_k, we get that n_k is uniformly (in k) Hölder continuous in the ball $B_1(x_k)$, which contradicts the fact that $n_k \geq 0$ and $\|n_k\|_{L^1} \to 0$. Then $\sigma_k |x_k|^b \to +\infty$, in this case we may construct a Lyapunov function for the system and hence some integral estimates on n_k, this allows us to obtain a uniform (in k) Hölder bound for n_k, which yields an absurd. This proves that $L_k \to +\infty$ is not possible. □

4.3 Existence of ground states for $\varepsilon > 0$

In this section, we provide existence of classical solutions to the MFG system (4.2) using a minimization procedure. Notice that, even if this result partially covers the existence result obtained in Chapter 3, the variational approach proves to be essential to obtain some suitable estimates that will be necessary in the vanishing viscosity setting and hence to prove concentration phenomena as $\varepsilon \to 0$.

If $\gamma' < N$, condition

$$N - \gamma' \leq \alpha < N \tag{4.5}$$

is *necessary* for the problem

$$\min_{(m,w) \in \mathcal{K}_{\varepsilon,M}} \mathcal{E}(m, w)$$

to be well-posed. Indeed, let us consider $m = ce^{-|x|}$ such that $\int_{\mathbb{R}^N} m(x)dx = M$ and $w = \varepsilon \nabla m$, in this way $(m, w) \in \mathcal{K}_{\varepsilon,M}$. For $\sigma > 0$ define

$$m_\sigma(x) := \frac{m(\sigma^{-1}x)}{\sigma^N} \qquad \text{and} \qquad w_\sigma(x) := \frac{w(\sigma^{-1}x)}{\sigma^{N+1}},$$

we get that $(m_\sigma, w_\sigma) \in \mathcal{K}_{\varepsilon,M}$ and

$$\mathcal{E}(m_\sigma, w_\sigma) = \frac{1}{\sigma^{\gamma'}} \left[\int_{\mathbb{R}^N} \frac{m}{\gamma'} \left| \frac{w}{m} \right|^{\gamma'} + \sigma^{\gamma'} V m \, dX - \frac{\sigma^{\gamma' - N + \alpha}}{2} \int_{\mathbb{R}^N} m(X)(K_\alpha * m)(X) \, dX \right].$$

From (4.3) we have that $\int_{\mathbb{R}^N} \frac{m}{\gamma'} |\frac{w}{m}|^{\gamma'} + \sigma^{\gamma'} V m \, dX \leq C$, so if $\alpha < N - \gamma'$ and $\sigma \to 0$ the Riesz term in the energy dominates and

$$\mathcal{E}(m_\sigma, w_\sigma) \to -\infty, \quad \text{as } \sigma \to 0.$$

Actually, condition (4.5) is also *sufficient*, in fact we prove that if $N - \gamma' < \alpha < N$ the energy $\mathcal{E}$ is bounded from below; and in the case when $\alpha = N - \gamma'$, requiring in addition that the constraint mass M is sufficiently small, the energy $\mathcal{E}$ is non-negative (see Section 4.6.1). Hence, the minimum problem is well defined and by means of classical direct methods we are able to obtain minimizers. Notice that in the case when $\gamma' \geq N$, the above condition (4.5) reduces to $0 < \alpha < N$.

In what follows we address the case $N - \gamma' < \alpha < N$. Without loss of generality, we may assume $\varepsilon \in (0, 1]$ fixed. Let us define

$$e_\varepsilon(M) := \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m, w).$$

Lemma 4.3.1. *Assume that $N - \gamma' < \alpha < N$ and let $(m, w) \in \mathcal{K}_{\varepsilon,M}$. Then, there exist $C_1 = C_1(N, \gamma, \alpha, M)$, $C_2 = C_2(N, \gamma, \alpha, M)$ and $K = K(M, C_V, b, N, \alpha, \gamma)$ positive constants such that*

$$-C_1\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \le e_\varepsilon(M) \le -C_2\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K. \tag{4.6}$$

Proof. Let us fix $\beta := \frac{2N}{N+\alpha}$, since $1 < \beta < 1 + \frac{\gamma'}{N}$ by (3.12), (3.15) and the fact that $V \ge 0$, we get

$$\mathcal{E}(m, w) \ge c_1\varepsilon^{\gamma'}\|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} - c_2\|m\|^2_{L^\beta(\mathbb{R}^N)} \tag{4.7}$$

where c_1 is a constant depending on N, α, γ, M and c_2 is a constant which depends on N and α. Minimizing the RHS of (4.7), we obtain that

$$c_1\varepsilon^{\gamma'}\|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} - c_2\|m\|^2_{L^\beta(\mathbb{R}^N)} \ge (c_3 - c_4)\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}$$

hence, there exists a constant $C_1 > 0$ depending on N, γ, α, M such that

$$\mathcal{E}(m, w) \ge -C_1\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}.$$

In order to prove the estimate from above it is enough to show that for a suitable couple $(\tilde{m}, \tilde{w}) \in \mathcal{K}_{\varepsilon,M}$ it holds

$$\mathcal{E}(\tilde{m}, \tilde{w}) \le -C_2\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K.$$

Let us consider a smooth function $\varphi : [0, +\infty) \to \mathbb{R}$ defined as $\varphi(r) = e^{-r}$. We define

$$\tilde{m}(x) := M\tau^N I_1\varphi(\tau|x|)$$

$$\tilde{w}(x) := \varepsilon\nabla\tilde{m}(x)$$

where τ is a positive constant to be fixed and $I_1^{-1} := \int_{\mathbb{R}^N} e^{-|y|}dy$, obviously $(\tilde{m}, \tilde{w}) \in \mathcal{K}_{\varepsilon,M}$. We get that

$$\int_{\mathbb{R}^N} \tilde{m}\left|\frac{\tilde{w}}{\tilde{m}}\right|^{\gamma'} dx = M\varepsilon^{\gamma'}\tau^{\gamma'},$$

$$\int_{\mathbb{R}^N} V(x)\,\tilde{m}\,dx \le MC_V + MC_V I_1 I_2\frac{1}{\tau^b}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\tilde{m}(x)\tilde{m}(y)}{|x-y|^{N-\alpha}}dx\,dy = M^2I_1^2I_3\tau^{N-\alpha}$$

where $I_2 := \int_{\mathbb{R}^N} |y|^b\varphi(|y|)dy$ and $I_3 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi(|x|)\varphi(|y|)}{|x-y|^{N-\alpha}}dx\,dy$. Now, coming back to the energy, we obtain

$$\mathcal{E}(\tilde{m}, \tilde{w}) \le M(\varepsilon\tau)^{\gamma'} + MC_V + MC_V I_1I_2\frac{1}{\tau^b} - \frac{1}{2}M^2I_1^2I_3\,\tau^{N-\alpha}$$

finally taking $\tau = \frac{1}{A}\varepsilon^{-\frac{\gamma'}{\gamma'-N+\alpha}}$ we get

$$\mathcal{E}(\tilde{m},\tilde{w}) \le \left(\frac{M}{A^{\gamma'}} - \frac{M^2 I_1^2 I_3}{2A^{N-\alpha}}\right) \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + MC_V + MC_V I_1 I_2 \frac{1}{\tau^b} \le -C_2 \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K$$

choosing A large enough. □

In particular, from Lemma 4.3.1, it follows that for $N-\gamma' < \alpha < N$, $e_\varepsilon(M)$ is finite. We have also the following a priori bounds.

Proposition 4.3.2. *Let $(m,w) \in \mathcal{K}_{\varepsilon,M}$ such that $e_\varepsilon(M) \ge \mathcal{E}(m,w) - \eta$ for some positive η. Then*

$$\|m\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \tag{4.8}$$

$$\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx \le C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K \tag{4.9}$$

and

$$\int_{\mathbb{R}^N} V(x)\, m\, dx \le C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K \tag{4.10}$$

where C and K are positive constants depending on $M, N, \alpha, C_V, b, \gamma$ and η.

Proof. Let us denote $\beta := \frac{2N}{N+\alpha}$, if $(m,w) \in \mathcal{K}_{\varepsilon,M}$ and $e_\varepsilon(M) \ge \mathcal{E}(m,w) - \eta$ for some $\eta > 0$, we have

$$\begin{aligned} c + \eta \ge e_\varepsilon(M) + \eta \ge \mathcal{E}(m,w) &\ge \frac{1}{\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\,m(y)}{|x-y|^{N-\alpha}} dx\,dy \\ &\ge C_1 \varepsilon^{\gamma'} M^{1-\frac{2\gamma'}{N-\alpha}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} - C_2\|m\|^2_{L^\beta(\mathbb{R}^N)}, \end{aligned} \tag{4.11}$$

where in the first inequality, we used that by Lemma 4.3.1 there exists a positive constant c depending on $M, C_V, b, \gamma, N, \alpha$ such that $e_\varepsilon(M) \le c$, while in the last inequality we exploit estimates (3.12) and (3.15). Since $\frac{\gamma'}{N-\alpha} > 1$, choosing C sufficiently large (not depending on ε) we have

$$C_1 \varepsilon^{\gamma'} M^{1-\frac{2\gamma'}{N-\alpha}} \left(\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} C\right)^{\frac{\gamma'}{N-\alpha}} - C_2\left(\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} C\right) \ge c + \eta,$$

hence we must have

$$\|m\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} C.$$

From (4.11) we get that

$$\begin{aligned} \frac{1}{\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx &\le c + \eta + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\,m(y)}{|x-y|^{N-\alpha}} dx\,dy \\ &\le c + \eta + C\|m\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K \end{aligned}$$

which proves (4.9). Finally, since

$$\frac{1}{\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \int_{\mathbb{R}^N} V(x)\, m\, dx = \mathcal{E}(m,w) + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}},$$

using (3.15) and (4.8), we obtain

$$\int_{\mathbb{R}^N} V(x)m\,dx \leq \frac{1}{\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \int_{\mathbb{R}^N} V(x)\,m\,dx \leq \mathcal{E}(m,w) + C\|m\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$
$$\leq e_\varepsilon(M) + \eta + C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \quad (4.12)$$

which gives estimate (4.10). □

By means of classical direct methods, we prove that for every $\varepsilon, M > 0$ there exists a minimizer $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$ of the energy $\mathcal{E}$.

Proposition 4.3.3. *For every $\varepsilon > 0$ and $M > 0$, there exists a minimizer $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$ of the energy $\mathcal{E}$, namely*

$$\mathcal{E}(m_\varepsilon, w_\varepsilon) = \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w).$$

For every minimizer $(m_\varepsilon, w_\varepsilon)$ of $\mathcal{E}$, we have that

$$\|m_\varepsilon\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \quad (4.13)$$

$$\int_{\mathbb{R}^N} m_\varepsilon\left|\frac{w_\varepsilon}{m_\varepsilon}\right|^{\gamma'} dx \leq C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K \quad (4.14)$$

and

$$\int_{\mathbb{R}^N} V(x)m_\varepsilon\,dx \leq C\varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} + K, \quad (4.15)$$

where $C > 0$ and K are two constants not depending on ε. Moreover, it holds

$$m_\varepsilon(1+|x|)^b \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad w_\varepsilon(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N). \quad (4.16)$$

Proof. Let us consider a minimizing sequence $(m_n, w_n) \in \mathcal{K}_{\varepsilon,M}$, namely $\mathcal{E}(m_n, w_n) \to e_\varepsilon(M)$ as $n \to +\infty$. For n sufficiently large $e_\varepsilon(M) \geq \mathcal{E}(m_n, w_n) - 1$, so estimates (4.8), (4.9) and (4.10) hold. By Proposition 3.2.5, using (4.9), we get that

$$\|m_n\|_{W^{1,r}(\mathbb{R}^N)} \leq C, \quad \forall r < q$$

where C does not depend on n, hence by Sobolev compact embedding, up to subsequences $m_n \to m_\varepsilon$ in $L^s(K)$ for $1 \leq s < q^* := \frac{qN}{N-q}$ and $K \subset\subset \mathbb{R}^N$. We observe that if $A \subset \mathbb{R}^N$ it holds

$$\int_A m_n(x)\,dx = \int_{\mathbb{R}^N} m_n(x)\chi_A(x)\,dx \leq \|m_n\|_{\frac{2N}{N+\alpha}}\|\chi_A\|_{\frac{2N}{N-\alpha}};$$

hence using (4.8) we get that for every $\mu > 0$ there exists $\delta_\mu > 0$ such that

$$\int_A m_n(x)\,dx \leq \mu$$

for every n and for any $A \subset \mathbb{R}^N$ such that $meas(A) < \delta_\mu$. Using estimate (4.10) and (4.3) we obtain that for $R > 1$

$$C \geq \int_{\mathbb{R}^N} m_n V\,dx \geq \int_{B_R^c} m_n V\,dx \geq CR^b \int_{B_R^c} m_n(x)\,dx$$

namely, for every $\eta > 0$ there exists $R > 1$ such that $\int_{|x|>R} m_n(x)\,dx \leq \eta$ for any n (more precisely, for every n greater than a certain value n_0). Thus, using the Vitali Convergence Theorem, up to extracting a subsequence, we have that

$$m_n \to m_\varepsilon \quad \text{in } L^1(\mathbb{R}^N)$$

and consequently

$$\int_{\mathbb{R}^N} m_\varepsilon(x)\,dx = M.$$

Moreover, since by Sobolev embedding, m_n are bounded in $L^s(\mathbb{R}^N)$ for every $s \in [1, q^*)$, we have also that $m_n \to \bar{m}$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. If we assume $\beta = \frac{2N}{N+\alpha}$, by Hölder inequality we get also

$$\int_{\mathbb{R}^N} |w_n|^{\frac{\gamma'\beta}{\gamma'-1+\beta}} dx \leq \left(\int_{\mathbb{R}^N} m_n^{1-\gamma'} |w_n|^{\gamma'} dx \right)^{\frac{\beta}{\gamma'-1+\beta}} \|m_n\|_{L^\beta(\mathbb{R}^N)}^{\frac{\beta(\gamma'-1)}{\gamma'-1+\beta}}$$

hence using (4.8) and (4.9)

$$w_n \rightharpoonup w_\varepsilon \quad \text{in } L^{\frac{\gamma'\beta}{\gamma'-1+\beta}}(\mathbb{R}^N).$$

From (4.8), (4.9) and (4.10) passing to the limit as $n \to +\infty$ and using Fatou's Lemma we obtain estimates (4.13), (4.14) and (4.15).

We can infer that $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$. Since the functional $\int_{\mathbb{R}^N} m|\frac{w}{m}|^{\gamma'} + V(x)m\,dx$ is sequentially lower semi-continuous with respect to the weak convergence and by Corollary 4.2.3 we have that $(K_\alpha * m_n)m_n \to (K_\alpha * m_\varepsilon)m_\varepsilon$ in $L^1(\mathbb{R}^N)$, then $(m_\varepsilon, w_\varepsilon)$ is a minimum of the energy $\mathcal{E}$.

Finally the fact that $m_\varepsilon(1+|x|)^b \in L^1(\mathbb{R}^N)$ follows from (4.15) and (4.3); whereas, by Hölder inequality

$$\int_{\mathbb{R}^N} |w_\varepsilon|(1+|x|)^{b/\gamma} dx \leq \left(\int_{\mathbb{R}^N} m_\varepsilon^{-\frac{\gamma'}{\gamma}} |w_\varepsilon|^{\gamma'} dx \right)^{\frac{1}{\gamma'}} \left(\int_{\mathbb{R}^N} m_\varepsilon(1+|x|)^b dx \right)^{\frac{1}{\gamma}}$$

since $\int_{\mathbb{R}^N} m_\varepsilon^{-\frac{\gamma'}{\gamma}} |w_\varepsilon|^{\gamma'} dx = \int_{\mathbb{R}^N} m_\varepsilon \left|\frac{w_\varepsilon}{m_\varepsilon}\right|^{\gamma'} dx$, using (4.14) and the fact that $m_\varepsilon(1+|x|)^b \in L^1$, we obtain that $w_\varepsilon(1+|x|)^{b/\gamma} \in L^1(\mathbb{R}^N)$. □

Once we have obtained minimizers $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$ of the energy $\mathcal{E}$, we construct the associated solutions $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ of the MFG system (4.2).

Proposition 4.3.4. *Let $N - \gamma' < \alpha < N$. Assume that the potential V is locally Hölder continuous and satisfies* (4.3). *Then, for every $\varepsilon, M > 0$ there exists a classical solution $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ to the MFG system* (4.2) *such that*

$$u_\varepsilon(x) \geq C_\varepsilon |x|^{1+\frac{b}{\gamma}} - C_\varepsilon^{-1}$$

$$|\nabla u_\varepsilon(x)| \leq C_\varepsilon(1 + |x|^{\frac{b}{\gamma}})$$

where C_ε positive constant. Moreover, u_ε is unique up to additive constants, $m_\varepsilon \in L^\infty(\mathbb{R}^N)$ and there exist C_1, C_2, K positive constants not depending on ε such that

$$-C_1 \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \leq \lambda_\varepsilon \leq K - C_2 \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}. \tag{4.17}$$

Proof. Following the proof of [44, Proposition 3.4], let us consider the set of test functions

$$\mathcal{A} := \left\{ \psi \in C^2(\mathbb{R}^N) \;\middle|\; \limsup_{|x|\to+\infty} \frac{|\nabla\psi(x)|}{|x|^{b/\gamma}} < +\infty, \; \limsup_{|x|\to+\infty} \frac{|\Delta\psi(x)|}{|x|^{b}} < +\infty \right\}.$$

From Proposition 4.3.3 there exists at least one minimizer $(m_\varepsilon, w_\varepsilon)$ of the energy $\mathcal{E}$, and one can verify (using (4.16) and integrating by parts) that

$$-\varepsilon \int_{\mathbb{R}^N} m_\varepsilon \Delta\psi \, dx = \int_{\mathbb{R}^N} w_\varepsilon \cdot \nabla\psi \, dx, \quad \forall \psi \in \mathcal{A} \tag{4.18}$$

(see (3.18) in [44], for details). Since every minimizer satisfies (4.16) and (4.18), minimizing $\mathcal{E}$ on $\mathcal{K}_{\varepsilon,M}$ is equivalent to minimize $\mathcal{E}$ on the following constraint set

$$\mathcal{K} := \Bigg\{ (m,w) \in (L^1 \cap W^{1,r}) \times L^{\frac{\gamma'\beta}{\gamma'+\beta-1}}(\mathbb{R}^N) \;\Bigg|\; (m,w) \text{ satisfies } (4.16),\ (4.18),\ \int_{\mathbb{R}^N} m = M,\ m \geq 0 \Bigg\}$$

where $r < q$. Now we prove that if $(m_\varepsilon, w_\varepsilon)$ is a minimizer of $\mathcal{E}$ on $\mathcal{K}$, then $(m_\varepsilon, w_\varepsilon)$ is also a minimizer of the functional

$$\tilde{\mathcal{E}}(m,w) := \frac{1}{\gamma'} \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx + \int_{\mathbb{R}^N} V(x) m \, dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\, m_\varepsilon(y)}{|x-y|^{N-\alpha}} dx\, dy$$

on $\mathcal{K}$. Define

$$\Phi(m,w) := \begin{cases} \frac{1}{\gamma'} \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx & \text{if } (m,w) \in \mathcal{K} \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\Psi(m) := \int_{\mathbb{R}^N} V(x) m \, dx - \frac{1}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\, m(y)}{|x-y|^{N-\alpha}} dx\, dy,$$

we have

$$\mathcal{E}(m,w) = \Phi(m,w) + \Psi(m).$$

For any $(m,w) \in \mathcal{K}$ and $\lambda \in (0,1)$ we define $m_\lambda := (1-\lambda)m_\varepsilon + \lambda m$ and $w_\lambda := (1-\lambda)w_\varepsilon + \lambda w$, by minimality of $(m_\varepsilon, w_\varepsilon)$ it holds

$$\Phi(m_\lambda, w_\lambda) - \Phi(m_\varepsilon, w_\varepsilon) \geq \Psi(m_\varepsilon) - \Psi(m_\lambda) \tag{4.19}$$

and by convexity of Φ

$$\lambda \left(\Phi(m,w) - \Phi(m_\varepsilon, w_\varepsilon)\right) \geq \Phi(m_\lambda, w_\lambda) - \Phi(m_\varepsilon, w_\varepsilon). \tag{4.20}$$

From (4.19) and (4.20) we obtain that

$$\lambda \left(\Phi(m,w) - \Phi(m_\varepsilon, w_\varepsilon)\right) \geq -\lambda \int_{\mathbb{R}^N} V(m - m_\varepsilon) dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m_\varepsilon(y)\,(m - m_\varepsilon)(x)}{|x-y|^{N-\alpha}} dy\, dx + o(\lambda),$$

dividing by λ and letting λ go to 0, we get

$$-\int_{\mathbb{R}^N} V(m - m_\varepsilon) dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m_\varepsilon(y)\,(m(x) - m_\varepsilon(x))}{|x-y|^{N-\alpha}} dy\, dx \leq \Phi(m,w) - \Phi(m_\varepsilon, w_\varepsilon)$$

for any $(m, w) \in \mathcal{K}$. Hence, the couple $(m_\varepsilon, w_\varepsilon)$ minimizes $\tilde{\mathcal{E}}$ on $\mathcal{K}$.
Let us consider the following functional

$$\mathcal{L}(m, w, \lambda, \psi) := \tilde{\mathcal{E}}(m, w) + \int_{\mathbb{R}^N} \varepsilon m \Delta\psi + w\nabla\psi - \lambda m \, dx + \lambda M.$$

One can easily verify that

$$\min_{(m,w)\in\mathcal{K}} \tilde{\mathcal{E}}(m, w) = \min_{(m,w)\in E} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \mathcal{L}(m, w, \lambda, \psi)$$

where

$$E := \left\{ (m, w) \in (L^1 \cap W^{1,r})(\mathbb{R}^N) \times L^{\frac{\gamma'\beta}{\gamma'+\beta-1}}(\mathbb{R}^N) \,\middle|\, (m, w) \text{ satisfies } (4.16) \right\}.$$

Proceeding as in [44, Proposition 3.4], by means of the Fan's minimax theorem (refer to Theorem 2.3.7 in [24]) and the Rockafellar interchange theorem (see [173, Theorem 3A]), we get that

$$\begin{aligned}
&\min_{(m,w)\in E} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \mathcal{L}(m, w, \lambda, \psi) \\
&= \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \min_{(m,w)\in E} \int_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} + Vm - (K_\alpha * m_\varepsilon)m + \varepsilon m\Delta\psi + w\nabla\psi - \lambda m \, dx + \lambda M \\
&= \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \int_{\mathbb{R}^N} \min_{(m,w)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^N} m \left(\frac{1}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} + V - K_\alpha * m_\varepsilon + \varepsilon\Delta\psi + \frac{w}{m}\nabla\psi - \lambda \right) dx + \lambda M.
\end{aligned}$$

We observe now that

$$\min_{(m,w)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^N} m \left(\frac{1}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} + V - K_\alpha * m_\varepsilon + \varepsilon\Delta\psi + \frac{w}{m}\nabla\psi - \lambda \right) = 0$$

if $\varepsilon\Delta\psi - \frac{1}{\gamma}|\nabla\psi|^\gamma - \lambda + V - K_\alpha * m_\varepsilon \geq 0$, while it is $-\infty$ otherwise. This proves that

$$\min_{(m,w)\in\mathcal{K}} \tilde{\mathcal{E}}(m, w) = \sup_{(\lambda,\psi)\in B} \lambda M$$

where

$$B := \left\{ (\lambda, \psi) \in \mathbb{R} \times \mathcal{A} \,\middle|\, -\varepsilon\Delta\psi + \frac{1}{\gamma}|\nabla\psi|^\gamma + \lambda \leq V - K_\alpha * m_\varepsilon \text{ on } \mathbb{R}^N \right\}.$$

From Corollary 4.2.2 and Corollary 4.2.4 we have that $K_\alpha * m_\varepsilon \in L^\beta(\mathbb{R}^N) \cap C^{0,\theta}(\mathbb{R}^N)$, hence by Proposition 3.2.11 ([20, Proposition 2.10]) there exists a couple $(u_\varepsilon, \lambda_\varepsilon) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ such that

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\gamma}|\nabla u_\varepsilon|^\gamma + \lambda_\varepsilon = V(x) - K_\alpha * m_\varepsilon(x), \quad \text{on } \mathbb{R}^N \tag{4.21}$$

where

$$\lambda_\varepsilon := \sup \left\{ \lambda \in \mathbb{R} \,\middle|\, -\varepsilon\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V - K_\alpha * m_\varepsilon \text{ has a subsolution } u_\varepsilon \in C^2(\mathbb{R}^N) \right\}$$

and u_ε is unique up to additive constants. Using also Proposition 3.2.10 ([20, Proposition 2.9]) we have

$$u_\varepsilon(x) \geq C_\varepsilon |x|^{\frac{b}{\gamma}+1} - C_\varepsilon^{-1}, \qquad |\nabla u_\varepsilon(x)| \leq C_\varepsilon (1+|x|)^{\frac{b}{\gamma}}.$$

Since

$$\varepsilon|\Delta u_\varepsilon| \leq \frac{1}{\gamma}|\nabla u_\varepsilon|^\gamma + |\lambda_\varepsilon| + V(x) + K_\alpha * m_\varepsilon(x) \leq C(1+|x|)^b,$$

it follows that $\limsup_{|x|\to+\infty} \frac{|\Delta u_\varepsilon(x)|}{|x|^b} < +\infty$, hence $u_\varepsilon \in \mathcal{A}$. This proves that $\sup_{(\lambda,\psi)\in B} \lambda M = \lambda_\varepsilon M$ and consequently

$$\lambda_\varepsilon M = \tilde{\mathcal{E}}(m_\varepsilon, w_\varepsilon) = \mathcal{E}(m_\varepsilon, w_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m_\varepsilon(x)m_\varepsilon(y)}{|x-y|^{N-\alpha}}dx\,dy. \tag{4.22}$$

Now, by (4.22), (4.21) and (4.18) (since $u_\varepsilon \in \mathcal{A}$) we get that

$$0 = \int\limits_{\mathbb{R}^N} \left(\frac{1}{\gamma'}\left|\frac{w_\varepsilon}{m_\varepsilon}\right|^{\gamma'} + V - K_\alpha * m_\varepsilon - \lambda_\varepsilon\right) m_\varepsilon dx = \int\limits_{\mathbb{R}^N} \left(\frac{1}{\gamma'}\left|\frac{w_\varepsilon}{m_\varepsilon}\right|^{\gamma'} - \varepsilon\Delta u_\varepsilon + \frac{|\nabla u_\varepsilon|^\gamma}{\gamma}\right) m_\varepsilon dx$$
$$= \int\limits_{\mathbb{R}^N} \left(\frac{1}{\gamma'}\left|\frac{w_\varepsilon}{m_\varepsilon}\right|^{\gamma'} + \frac{w_\varepsilon}{m_\varepsilon}\cdot\nabla u_\varepsilon + \frac{1}{\gamma}|\nabla u_\varepsilon|^\gamma\right) m_\varepsilon\, dx$$

and we must have

$$\frac{w_\varepsilon}{m_\varepsilon} = -\nabla u_\varepsilon |\nabla u_\varepsilon|^{\gamma-2}, \quad \text{on the set } \{m_\varepsilon > 0\}.$$

We can conclude that $\varepsilon\Delta m_\varepsilon + \operatorname{div}(m_\varepsilon \nabla u_\varepsilon |\nabla u_\varepsilon|^{\gamma-2}) = 0$ in weak sense.

Proof of estimate (4.17). From (4.22) we get that

$$\lambda_\varepsilon = \frac{1}{M}\mathcal{E}(m_\varepsilon, w_\varepsilon) - \frac{1}{2M}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m_\varepsilon(x)m_\varepsilon(y)}{|x-y|^{N-\alpha}}dx\,dy \tag{4.23}$$

hence, by (4.6) and (3.16) we get that

$$\lambda_\varepsilon \geq -c_1 \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} - c_2 \|m_\varepsilon\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \geq -C_1 \varepsilon^{-\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}} \tag{4.24}$$

using (4.8) in the last inequality. Moreover, from (4.23) we have

$$\lambda_\varepsilon \leq \frac{1}{M}\mathcal{E}(m_\varepsilon, w_\varepsilon) = \frac{1}{M}\inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w)$$

and using (4.6), we conclude the proof of estimate (4.17).

Finally, the function $m_\varepsilon \in L^\infty(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ for very $p \geq 1$, this follows proving that the function u_ε^r for $r > 1$ is a Lyapunov function for the stochastic process with drift $\nabla u_\varepsilon |\nabla u_\varepsilon|^{\gamma-2}$ and m_ε density of the invariant measure associated to the process, then using some results in [150] and Proposition 3.2.2 (we refer the reader to the proof of [20, Proposition 4.3 *iv*)] for more details). This proves that the triple $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ is a classical solution to the MFG system (4.2). □

4.4 Asymptotic analysis of solutions

In this section, assuming $\alpha \in (N - \gamma', N)$, we want to study the behavior of agents when the Brownian noise vanishes. To this end, we analyze the asymptotic behavior of a solution $(m_\varepsilon, u_\varepsilon, \lambda_\varepsilon)$ to the MFG system (4.2) as $\varepsilon \to 0$.

4.4.1 The rescaled problem and some a priori estimates

For $\varepsilon > 0$, let us define a suitable rescaling for m, u and λ, which preserves the mass of m:

$$\tilde{m}(y) := \varepsilon^{\frac{N\gamma'}{\gamma'-N+\alpha}} m\left(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y\right),$$

$$\tilde{u}(y) := \varepsilon^{\frac{\gamma'(N-\alpha)-\gamma'-N+\alpha}{\gamma'-N+\alpha}} u\left(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y\right),$$

$$\tilde{\lambda} := \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} \lambda$$

and a rescaled potential

$$V_\varepsilon(y) := \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} V\left(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y\right). \tag{4.25}$$

Notice that V_ε vanishes locally as $\varepsilon \to 0$, hence, passing to the limit, we can not take advantage of the coercivity of V in order to prove that there is no loss of mass (compare with the proof of Proposition 4.3.3) indeed we will use a concentration-compactness argument. From assumptions (4.3) on the potential V, we get the corresponding assumptions on V_ε:

$$C_V^{-1}\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\left(\max\left\{\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}|y| - C_V, 0\right\}\right)^b \le V_\varepsilon(y) \le C_V \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\left(1 + \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}|y|\right)^b. \tag{4.26}$$

The rescaled Riesz-type interaction term is

$$K_\alpha * \tilde{m}(y) = \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} K_\alpha * m\left(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y\right).$$

Hence, if the triple $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ is a classical solution to the MFG system (4.2) (from Proposition 4.3.4 there exists at least one solution to (4.2)), one can verify that

$$\begin{aligned}
&-\Delta\tilde{u}_\varepsilon(y) + \frac{1}{\gamma}|\nabla\tilde{u}_\varepsilon(y)|^\gamma + \tilde{\lambda}_\varepsilon \\
&\quad = -\varepsilon^{\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}+1}\Delta u_\varepsilon(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y) + \varepsilon^{\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}\frac{1}{\gamma}\big|\nabla u_\varepsilon(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y)\big|^\gamma + \varepsilon^{\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}\lambda_\varepsilon \\
&\quad = \varepsilon^{\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}\left[-\varepsilon\Delta u_\varepsilon + \frac{1}{\gamma}|\nabla u_\varepsilon|^\gamma + \lambda_\varepsilon\right] = \varepsilon^{\frac{\gamma'(N-\alpha)}{\gamma'-N+\alpha}}\left[V(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y) - K_\alpha * m(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y)\right] \\
&\quad = V_\varepsilon(y) - K_\alpha * \tilde{m}(y)
\end{aligned}$$

and also

$$\begin{aligned}
&-\Delta\tilde{m}_\varepsilon(y) - \operatorname{div}\big(\tilde{m}_\varepsilon(y)\nabla\tilde{u}_\varepsilon(y)|\nabla\tilde{u}_\varepsilon(y)|^{\gamma-2}\big) \\
&\quad = -\varepsilon^{\frac{N\gamma'+2\gamma'}{\gamma'-N+\alpha}}\Delta m_\varepsilon(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y) - \operatorname{div}\left(\varepsilon^{\frac{N\gamma'+N-\alpha}{\gamma'-N+\alpha}} m_\varepsilon\nabla u_\varepsilon|\nabla u_\varepsilon|^{\gamma-2}(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y)\right) \\
&\quad = \varepsilon^{\frac{N\gamma'+N-\alpha+\gamma'}{\gamma'-N+\alpha}}[-\varepsilon\Delta m_\varepsilon - \operatorname{div}(m_\varepsilon\nabla u_\varepsilon|\nabla u_\varepsilon|^{\gamma-2})] = 0.
\end{aligned}$$

Hence

$$\begin{cases} -\Delta\tilde{u}_\varepsilon + \frac{1}{\gamma}|\nabla\tilde{u}_\varepsilon|^\gamma + \tilde{\lambda}_\varepsilon = V_\varepsilon(y) - K_\alpha * \tilde{m}_\varepsilon(y) \\ -\Delta\tilde{m}_\varepsilon - \operatorname{div}(\tilde{m}_\varepsilon\nabla\tilde{u}_\varepsilon|\nabla\tilde{u}_\varepsilon|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N}\tilde{m}_\varepsilon = M \end{cases} \tag{4.27}$$

In order to prove that there is no loss of mass when passing to the limit as $\varepsilon \to 0$, we translate the reference system at a point around which the mass $\tilde{m}_\varepsilon$ remains positive. By Proposition 4.3.4, $u_\varepsilon \in C^2(\mathbb{R}^N)$ is coercive, hence there exists a point $y_\varepsilon \in \mathbb{R}^N$ such that

$$\tilde{u}_\varepsilon(y_\varepsilon) = \min_{\mathbb{R}^N}\tilde{u}_\varepsilon(y).$$

Let us define

$$\bar{u}_\varepsilon(y) := \tilde{u}_\varepsilon(y + y_\varepsilon) - \tilde{u}_\varepsilon(y_\varepsilon)$$

$$\bar{m}_\varepsilon(y) := \tilde{m}_\varepsilon(y + y_\varepsilon)$$

in this way we have $\bar{u}_\varepsilon(0) = 0 = \min_{\mathbb{R}^N}\bar{u}_\varepsilon$. One can immediately verify that $(\bar{m}_\varepsilon, \bar{u}_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a classical solution to

$$\begin{cases} -\Delta\bar{u}_\varepsilon + \frac{1}{\gamma}|\nabla\bar{u}_\varepsilon|^\gamma + \tilde{\lambda}_\varepsilon = V_\varepsilon(y + y_\varepsilon) - K_\alpha * \bar{m}_\varepsilon(y) \\ -\Delta\bar{m}_\varepsilon - \operatorname{div}(\bar{m}_\varepsilon\nabla\bar{u}_\varepsilon|\nabla\bar{u}_\varepsilon|^{\gamma-2}) = 0 & . \\ \int_{\mathbb{R}^N}\bar{m}_\varepsilon = M \end{cases} \tag{4.28}$$

We define also the rescaled energy as

$$\mathcal{E}_\varepsilon(m, w) := \int_{\mathbb{R}^N}\frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + V_\varepsilon(y + y_\varepsilon)m\,dy - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy,$$

for couples $(m, w) \in \mathcal{K}_{\varepsilon,M}$ with $m > 0$. Notice that $\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) = \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\mathcal{E}(m_\varepsilon, w_\varepsilon)$, hence if $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$ is a minimizer of $\mathcal{E}$, then $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ is a minimizer of $\mathcal{E}_\varepsilon$ on $\mathcal{K}_{1,M}$ (where $\bar{w}_\varepsilon = -\bar{m}_\varepsilon\nabla\bar{u}_\varepsilon|\nabla\bar{u}_\varepsilon|^{\gamma-2}$). We will denote

$$\tilde{e}_\varepsilon(M) := \min_{(m,w)\in\mathcal{K}_{1,M}}\mathcal{E}_\varepsilon(m, w).$$

From (4.6) by rescaling, we get

$$-C_1 \le \tilde{e}_\varepsilon(M) \le -C_2 + K\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} \tag{4.29}$$

where C_1, C_2, K are positive constants not depending on ε.

First of all, we prove the following a priori estimates.

Lemma 4.4.1. *Let $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ be a classical solution to* (4.28). *Then, there exist C_1, C_2, C positive constants not depending on ε such that*

$$-C_1 \le \tilde{\lambda}_\varepsilon \le -C_2 \tag{4.30}$$

$$\int_{\mathbb{R}^N}\bar{m}_\varepsilon|\nabla\bar{u}_\varepsilon|^\gamma dx \le C \tag{4.31}$$

$$\|\bar{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C \tag{4.32}$$

$$\varepsilon^{\frac{\gamma'(N-\alpha+b)}{\gamma'-N+\alpha}}|y_\varepsilon|^b \leq C \tag{4.33}$$

$$0 \leq V_\varepsilon(y+y_\varepsilon) \leq C\left(1+\varepsilon^{\frac{\gamma'(N-\alpha+b)}{\gamma'-N+\alpha}}|y|^b\right) \tag{4.34}$$

$$|\nabla \bar{u}_\varepsilon(y)| \leq C(1+|y|)^{\frac{b}{\gamma}} \qquad \textit{and} \qquad \bar{u}_\varepsilon(y) \geq C|y|^{1+\frac{b}{\gamma}} - C^{-1}. \tag{4.35}$$

Moreover, for R sufficiently large we have

$$\int_{B_R(0)} \bar{m}_\varepsilon(y)dy \geq C. \tag{4.36}$$

Proof. Estimates (4.30) and (4.31) follow, by rescaling, from (4.17) and (4.14) respectively. From Proposition 4.3.4 we have that for every ε, $\bar{u}_\varepsilon$ are bounded from below and $\bar{m}_\varepsilon \in L^\infty(\mathbb{R}^N)$, so by Theorem 4.2.6 with $s_\varepsilon = \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}$ and $t_\varepsilon = \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}$, we obtain the uniform L^∞-bound (4.32). Evaluating the first equation of (4.28) in $y=0$, we get

$$\tilde{\lambda}_\varepsilon \geq V_\varepsilon(y_\varepsilon) - K_\alpha * \bar{m}_\varepsilon(0) \tag{4.37}$$

from estimates (4.30), (4.32) and (4.26) we get (4.33); using it and (4.26) again, we obtain (4.34).
Since (4.30), (4.32) and (4.34) hold and $\bar{u}_\varepsilon$ is bounded from below, from Proposition 3.2.10 (see [20, Proposition 2.9]) we get estimates (4.35) (which are uniform with respect to ε since $\|K_\alpha * \bar{m}_\varepsilon\|_\infty \leq C_{N,\alpha}\|\bar{m}_\varepsilon\|_\infty + M \leq C$ uniformly in ε).
From (4.37), using the fact that $V_\varepsilon \geq 0$ and (4.30), we get that there exists a positive constant C not depending on ε such that

$$K_\alpha * \bar{m}_\varepsilon(0) \geq C > 0,$$

hence

$$C \leq \int_{B_R} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy + \int_{\mathbb{R}^N\setminus B_R} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy \leq \int_{B_R} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy + \frac{M}{R^{N-\alpha}}.$$

This implies that for $R>0$ sufficiently large

$$\int_{B_R} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy \geq C_1 > 0.$$

Moreover, if $r<R$ we have

$$C_1 \leq \int_{B_R\setminus B_r} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy + \int_{B_r} \frac{\bar{m}_\varepsilon(y)}{|y|^{N-\alpha}}dy \leq \frac{1}{r^{N-\alpha}}\int_{B_R\setminus B_r} \bar{m}_\varepsilon(y)dy + \|\bar{m}_\varepsilon\|_\infty \int_{B_r} \frac{dy}{|y|^{N-\alpha}}.$$

Keeping in mind that $\int_{B_r} \frac{dy}{|y|^{N-\alpha}} = cr^\alpha$ and (4.32), we can infer that choosing r sufficiently small

$$\frac{1}{r^{N-\alpha}}\int_{B_R\setminus B_r} \bar{m}_\varepsilon(y)dy \geq C_2 > 0$$

and consequently

$$\int_{B_R} \bar{m}_\varepsilon(y)dy \geq C_3 > 0.$$

□

4.4.2 Convergence of solutions

At this stage we are able to prove a convergence result, which however, do not ensure conservation of mass in the limit.

Proposition 4.4.2. *If $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a classical solution to (4.28), then as $\varepsilon \to 0$ up to extracting a subsequence we have that*

$$\tilde{\lambda}_\varepsilon \to \bar{\lambda}$$

and

$$\bar{u}_\varepsilon \to \bar{u}, \qquad \nabla \bar{u}_\varepsilon \to \nabla \bar{u}, \qquad \bar{m}_\varepsilon \to \bar{m}, \qquad \text{locally uniformly.}$$

The triple $(\bar{u}, \bar{m}, \bar{\lambda})$ is a classical solution to

$$\begin{cases} -\Delta \bar{u} + \frac{1}{\gamma}|\nabla \bar{u}|^\gamma + \bar{\lambda} = g(x) - K_\alpha * \bar{m}(x) \\ -\Delta \bar{m} - \operatorname{div}(\bar{m}\nabla \bar{u}|\nabla \bar{u}|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} \bar{m}\, dx = a \end{cases} \tag{4.38}$$

where g is a continuous function such that, up to subsequence, $V_\varepsilon(x+y_\varepsilon) \xrightarrow[\varepsilon \to 0]{} g(x)$ locally uniformly and $a \in (0, M]$. Moreover, there exist C_1, C_2, c_1, c_2 positive constants such that

$$\bar{u}(y) \geq C_1|y| - C_1{}^{-1} \qquad \text{and} \qquad |\nabla \bar{u}| \leq C_2 \tag{4.39}$$

and finally

$$\bar{m}(x) \leq c_1 e^{-c_2|x|}, \quad \text{on } \mathbb{R}^N. \tag{4.40}$$

Proof. By means of the previous uniform estimates and the fact $\bar{u}_\varepsilon$ is a classical solution to the HJB equation, using [93, Theorem 8.32], we get that for any compact set K in $\mathbb{R}^N$ and for any $\theta \in (0,1]$

$$\|\bar{u}_\varepsilon\|_{C^{1,\theta}(K)} \leq C \qquad \text{locally uniformly in } \varepsilon.$$

Using (4.32) and (4.35), by Proposition 3.2.2 and Sobolev embedding, we get that for every $\mu \in (0,1)$

$$\|\bar{m}_\varepsilon\|_{C^{0,\mu}(K)} \leq C \qquad \text{locally uniformly in } \varepsilon.$$

Hence, up to subsequences, we have that as $\varepsilon \to 0$

$$\bar{u}_\varepsilon \to \bar{u}, \qquad \text{locally uniformly in } C^1 \text{ on compact sets}$$

and

$$\bar{m}_\varepsilon \to \bar{m} \qquad \text{locally uniformly on compact sets}$$

and weakly in $W^{1,p}(B_R)$ $\forall p > 1$ and $R > 0$. Moreover, from (4.30) it follows that up to extracting a subsequence $\tilde{\lambda}_\varepsilon \to \bar{\lambda}$. From (4.32) and the fact that $\|\bar{m}_\varepsilon\|_{L^1(\mathbb{R}^N)} = M$, by interpolation we get that for every $p \in (1, +\infty)$ it holds $\|\bar{m}_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq C$ uniformly in ε. Using Theorem 3.2.8 and Theorem 3.2.9 we have that $\|K_\alpha * \bar{m}_\varepsilon\|_{C^{0,\alpha-\frac{N}{r}}(\mathbb{R}^N)} \leq C$ uniformly in ε and up to extracting a subsequence

$$K_\alpha * \bar{m}_\varepsilon \xrightarrow[\varepsilon \to 0]{} K_\alpha * \bar{m}, \quad \text{locally uniformly.}$$

Moreover, from estimate (4.34) we have that up to subsequences $V_\varepsilon(x+y_\varepsilon) \to g(x)$ locally uniformly, where g is a continuous function such that $0 \leq g(x) \leq C$. Finally, from (4.36) it follows that $\int_{\mathbb{R}^N} \bar{m}(x)\,dx = a \in (0, M]$. By stability with respect to uniform convergence, $\bar{u}$ solves in the viscosity sense

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \bar{\lambda} = g(x) - K_\alpha * \bar{m}(x)$$

and using the regularity of the HJB equation we get that $\bar{u} \in C^2$. Finally, by the strong convergence $\nabla u_\varepsilon \to \nabla \bar{u}$, we get that $\bar{m}$ solves

$$\Delta m - \operatorname{div}(m\nabla\bar{u}|\nabla\bar{u}|^{\gamma-2}) = 0$$

and with the same procedure as Proposition 3.4.3 *iv*), we get that $\bar{m} \in W^{1,p}(\mathbb{R}^N)$ for every $p \in (1, +\infty)$. Hence, $(\bar{u}, \bar{m}, \bar{\lambda})$ is a classical solution to (4.38).

In order to prove (4.39) we use Proposition 3.2.10. Notice that, if f is a non-negative Hölder continuous function such that $\int_{\mathbb{R}^N} f^\beta\,dx < +\infty$ for a certain $\beta > 1$, then $f(x) \to 0$ as $|x| \to +\infty$ (see [44, Lemma 2.2] which is stated in the case $\beta = 1$ but it can be easily generalised to $\beta > 1$). Since $K_\alpha * \bar{m} \in C^{0,\theta}(\mathbb{R}^N) \cap L^\beta(\mathbb{R}^N)$ and it is non-negative, we get that

$$K_\alpha * \bar{m}(x) \to 0, \quad \text{as } |x| \to +\infty$$

and hence

$$\liminf_{|x|\to+\infty} \left(g(x) - K_\alpha * \bar{m}(x) - \bar{\lambda}\right) \geq -\bar{\lambda} > 0.$$

From Proposition 3.2.10 we get (4.39).

Since we can choose $k > 0$ such that the function $\varphi(x) := e^{k\bar{u}(x)}$ is a Lyapunov function for the process, from [150, Proposition 2.6] we get that

$$e^{k\bar{u}} \in L^1(\bar{m})$$

and finally from [150, Theorem 6.1] it follows (4.40). □

4.4.3 No loss of mass when passing to the limit

First, we prove that the energy functional $\mathcal{E}_\varepsilon(m, w)$ holds a sort of sub-additive property. Then, we assume by contradiction to have loss of mass, namely that $\int_{\mathbb{R}^N} \bar{m}\,dx = a \in (0, M)$, by means of a concentration-compactness argument we prove that this leads to an absurd. Hence $\bar{m}$ has still L^1-norm equal to M.

Lemma 4.4.3. *For all $a \in (0, M)$, there exists a positive constant C depending on a, M and the other constants of the problem (but not on ε) such that*

$$\tilde{e}_\varepsilon(M) < \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M - a) - C.$$

Proof. Let us assume that $a \geq \frac{M}{2}$ and fix $c > 1$ and $B > 0$. If $(m, w) \in \mathcal{K}_{1,B}$ we get

$$\begin{aligned}\tilde{e}_\varepsilon(cB) &\leq \mathcal{E}_\varepsilon(cm, cw) = \int_{\mathbb{R}^N} \frac{cm}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + cV_\varepsilon(x + y_\varepsilon)m\,dx - \frac{c^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\,m(y)}{|x - y|^{N-\alpha}}dx\,dy \\ &= c\mathcal{E}_\varepsilon(m, w) - \frac{c(c-1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\,m(y)}{|x - y|^{N-\alpha}}dx\,dy \qquad (4.41)\end{aligned}$$

If $(m, w) \in \mathcal{K}_{1,B}$ is a minimizer of $\mathcal{E}_\varepsilon$, we have

$$-C_2(B) + K\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} \geq \tilde{e}_\varepsilon(B) \geq -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{m(x)\,m(y)}{|x - y|^{N-\alpha}}dx\,dy$$

notice that the constant C_2 is the one that appears in (4.29) and depends on B and on the others variables of the problem. Taking ε sufficiently small, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy \geq \frac{C_2(B)}{2} > 0. \tag{4.42}$$

and using (4.42) in (4.41) we get

$$\tilde{e}_\varepsilon(cB) < c\tilde{e}_\varepsilon(B) - c(c-1)\frac{C_2(B)}{2}. \tag{4.43}$$

Taking $B = a$ and $c = M/a$ in (4.43) we have

$$\tilde{e}_\varepsilon(M) < \frac{M}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left(\frac{M}{a}-1\right)\frac{C_2(a)}{2} = \tilde{e}_\varepsilon(a) + \frac{M-a}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left(\frac{M}{a}-1\right)\frac{C_2(a)}{2}$$

if $a = M/2$ we have done, whereas if $a > M/2$ we take $B = M-a$ and $c = \frac{a}{M-a}$ in (4.43) and multiplying by $\frac{M-a}{a}$, we get

$$\frac{M-a}{a}\,\tilde{e}_\varepsilon(a) < \tilde{e}_\varepsilon(M-a) - \left(\frac{a}{M-a}-1\right)\frac{C_2(M-a)}{2} \leq \tilde{e}_\varepsilon(M-a).$$

We can conclude that

$$\tilde{e}_\varepsilon(M) < \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - \frac{M}{a}\left(\frac{M}{a}-1\right)\frac{C_2(a)}{2}.$$

In the case when $a < M/2$ we replace a with $M-a$. □

From estimate (4.40), it follows that there exists a positive constant $\bar{c}$ such that $\bar{m} \leq \bar{c}e^{-|x|}$. For $R > 0$ (which will be fixed later) let us define

$$\nu_R(x) := \begin{cases} \bar{c}e^{-R} & \text{if } |x| \leq R \\ \bar{c}e^{-|x|} & \text{if } |x| > R \end{cases}.$$

We have the following splitting of the energy $\mathcal{E}_\varepsilon$.

Lemma 4.4.4. *Let $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ be a minimizer of $\mathcal{E}_\varepsilon$, $\bar{m}$ and $\bar{u}$ obtained from Proposition 4.4.2 and $\bar{w}_\varepsilon \to \bar{w} = -\bar{m}\nabla\bar{u}|\nabla\bar{u}|^{\gamma-2}$ locally uniformly. If $\int_{\mathbb{R}^N}\bar{m}dx = a \in (0, M)$, then*

$$\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \geq \mathcal{E}_\varepsilon(\bar{m}, \bar{w}) + \mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) + o_\varepsilon(1) - CR^{b+N}e^{-R} \tag{4.44}$$

as $\varepsilon \to 0$.

Proof. Following the arguments of the proof of [44, Theorem 5.6], we recall some facts that we will need. By definition, $\bar{m}(x) \leq \nu_R(x)$ for $|x| > R$ and

$$\int_{\mathbb{R}^N}\nu_R(x)dx = \omega_N R^N\bar{c}e^{-R} + \int_{\mathbb{R}^N\setminus B_R}\bar{c}e^{-|x|} \leq Ce^{-R}R^N \to 0, \quad \text{as } R \to +\infty. \tag{4.45}$$

Since $\bar{m}_\varepsilon \to \bar{m}$ and $\nabla\bar{u}_\varepsilon \to \nabla\bar{u}$ locally uniformly as $\varepsilon \to 0$, there exists ε_0, which depends on R, such that $\forall\varepsilon \leq \varepsilon_0$

$$|\bar{m}_\varepsilon - \bar{m}| + \left||\nabla\bar{u}_\varepsilon|^{\gamma-2}\nabla\bar{u}_\varepsilon - |\nabla\bar{u}|^{\gamma-2}\nabla\bar{u}\right| \leq \bar{c}e^{-R}, \quad \text{for } |x| \leq R.$$

Moreover, $\forall \varepsilon \leq \varepsilon_0$

$$\bar{m}_\varepsilon - \bar{m} + 2\nu_R \geq \nu_R, \quad \forall x \in \mathbb{R}^N$$

and hence

$$|\bar{m}_\varepsilon - \bar{m}| \leq \bar{m}_\varepsilon - \bar{m} + 2\nu_R. \tag{4.46}$$

We estimate each term of the energy $\mathcal{E}_\varepsilon$ separately. Concerning the kinetic term, notice that the function $(m,w) \mapsto \frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'}$ is convex and in particular

$$\partial_m\left(\frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'}\right) = -\frac{1}{\gamma}\left|\frac{w}{m}\right|^{\gamma'} \qquad \text{and} \qquad \nabla_w\left(\frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'}\right) = \frac{w}{m}\left|\frac{w}{m}\right|^{\gamma'-2}.$$

By convexity, we estimate separately the integral over B_R and the integral over $\mathbb{R}^N \setminus B_R$, obtaining

$$\int_{B_R} \frac{\bar{m}_\varepsilon}{\gamma'}\left|\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\right|^{\gamma'} dx \geq \int_{B_R} \frac{\bar{m}}{\gamma'}\left|\frac{\bar{w}}{\bar{m}}\right|^{\gamma'} dx + \int_{B_R} \frac{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}{\gamma'}\left|\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\right|^{\gamma'} dx - CR^N e^{-R}, \tag{4.47}$$

$$\int_{\mathbb{R}^N\setminus B_R} \frac{\bar{m}_\varepsilon}{\gamma'}\left|\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\right|^{\gamma'} dx \geq \int_{\mathbb{R}^N\setminus B_R} \frac{\bar{m}}{\gamma'}\left|\frac{\bar{w}}{\bar{m}}\right|^{\gamma'} dx + \int_{\mathbb{R}^N\setminus B_R} \frac{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}{\gamma'}\left|\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\right|^{\gamma'} dx - CR^{N+b}e^{-R}. \tag{4.48}$$

we also have (refer to estimate (5.44) in [44])

$$\int_{\mathbb{R}^N} V_\varepsilon(x+y_\varepsilon)\bar{m}_\varepsilon dx \geq \int_{\mathbb{R}^N} V_\varepsilon(x+y_\varepsilon)\bar{m} dx + \int_{\mathbb{R}^N} V_\varepsilon(x+y_\varepsilon)(\bar{m}_\varepsilon - \bar{m} + 2\nu_R) dx - CR^{b+N}e^{-R}. \tag{4.49}$$

Regarding the Riesz term in the energy $\mathcal{E}_\varepsilon$, since by Proposition 4.4.2 $\bar{m}_\varepsilon(x) \to \bar{m}(x)$ a.e. as $\varepsilon \to 0$ and $(\bar{m}_\varepsilon)_\varepsilon$ is a bounded sequence in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ (it follows by interpolation using the uniform estimate (4.32)), applying Lemma 4.2.5 we get that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{m}_\varepsilon(x)\bar{m}_\varepsilon(y)}{|x-y|^{N-\alpha}} dx\,dy \\
&\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{m}(x)\bar{m}(y)}{|x-y|^{N-\alpha}} dx\,dy + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{m}_\varepsilon(x) - \bar{m}(x)|\,|\bar{m}_\varepsilon(y) - \bar{m}(y)|}{|x-y|^{N-\alpha}} dx\,dy \\
&\quad \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{m}(x)\bar{m}(y)}{|x-y|^{N-\alpha}} dx\,dy + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\bar{m}_\varepsilon - \bar{m} + 2\nu_R)(x)\,(\bar{m}_\varepsilon - \bar{m} + 2\nu_R)(y)}{|x-y|^{N-\alpha}} dx\,dy
\end{aligned}$$

where in the last inequality we used (4.46). Hence

$$\begin{aligned}
&\int_{\mathbb{R}^N} (K_\alpha * \bar{m}_\varepsilon)\bar{m}_\varepsilon \\
&\quad \leq \int_{\mathbb{R}^N} (K_\alpha * \bar{m})\bar{m} + \int_{\mathbb{R}^N} \big(K_\alpha * (\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\big)(\bar{m}_\varepsilon - \bar{m} + 2\nu_R) + o_\varepsilon(1).
\end{aligned} \tag{4.50}$$

Finally, putting together estimates (4.47), (4.48), (4.49) and (4.50), we obtain (4.44). □

We are now in position to prove that there is no loss of mass passing to the limit.

Theorem 4.4.5. *Let $N-\gamma'<\alpha<N$, assume that $(\bar m_\varepsilon,\bar w_\varepsilon)$ is a minimizer of $\mathcal{E}_\varepsilon$, $\bar m$ and $\bar u$ obtained from Proposition 4.4.2. Then,*

$$\int_{\mathbb{R}^N}\bar m\,dx=M$$

and hence $\bar m_\varepsilon\to\bar m$ in $L^1(\mathbb{R}^N)$. Moreover, for every $\eta>0$, there exist $R,\varepsilon_0>0$ such that for all $\varepsilon\le\varepsilon_0$

$$\int_{B(0,R)}\bar m_\varepsilon(x)\,dx\ge M-\eta, \tag{4.51}$$

namely

$$\int_{B(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}y_\varepsilon,\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}R)} m_\varepsilon(x)\,dx\ge M-\eta. \tag{4.52}$$

Proof. From (4.45) we get

$$\int_{\mathbb{R}^N}(\bar m_\varepsilon-\bar m+2\nu_R)dx=M-a+2\int_{\mathbb{R}^N}\nu_R dx\to M-a,\quad\text{as } R\to+\infty.$$

Let us define

$$C_R:=\frac{M-a}{M-a+2\int_{\mathbb{R}^N}\nu_R},$$

we observe that $0<C_R<1$ and $C_R\to1$ as $R\to+\infty$, moreover the couple $\big(C_R(\bar m_\varepsilon-\bar m+2\nu_R),C_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R)\big)\in\mathcal{K}_{M-a}$ and it follows that

$$\begin{aligned}C_R\mathcal{E}_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R,\bar w_\varepsilon-\bar w+2\nabla\nu_R)&=\mathcal{E}_\varepsilon\Big(C_R(\bar m_\varepsilon-\bar m+2\nu_R),C_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R)\Big)\\&+\frac{C_R^2-C_R}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\bar m_\varepsilon-\bar m+2\nu_R)(x)\,(\bar m_\varepsilon-\bar m+2\nu_R)(y)}{|x-y|^{N-\alpha}}dx\,dy.\end{aligned}$$

Notice that $C_R^2-C_R<0$ and from (4.32) and the fact that $\|\bar m_\varepsilon\|_{L^1(\mathbb{R}^N)}=M$, by interpolation we get that $\|\bar m_\varepsilon\|_{L^p(\mathbb{R}^N)}\le C$ uniformly in ε for every $p\in(1,+\infty)$. By (3.15) we get that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\bar m_\varepsilon-\bar m+2\nu_R)(x)\,(\bar m_\varepsilon-\bar m+2\nu_R)(y)}{|x-y|^{N-\alpha}}dx\,dy\right|&\le C\|\bar m_\varepsilon-\bar m+2\nu_R\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\\&\le C\left(\|\bar m_\varepsilon\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}+\|\bar m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}+2\|\nu_R\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\right)^2\le C\end{aligned}$$

where the constant C is independent of ε. Hence

$$\begin{aligned}C_R\mathcal{E}_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R,&\bar w_\varepsilon-\bar w+2\nabla\nu_R)\\&\ge\mathcal{E}_\varepsilon\Big(C_R(\bar m_\varepsilon-\bar m+2\nu_R),C_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R)\Big)+C\frac{C_R^2-C_R}{2}\\&\ge\tilde e_\varepsilon(M-a)+C\frac{C_R^2-C_R}{2}.\end{aligned}$$

Using this in (4.44) we have that

$$\begin{aligned}\tilde e_\varepsilon(M)\ge\tilde e_\varepsilon(a)+\tilde e_\varepsilon(M-a)&+(1-C_R)\mathcal{E}_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R,\bar w_\varepsilon-\bar w+2\nabla\nu_R)\\&+o_\varepsilon(1)-CR^{b+N}e^{-R}+C(C_R^2-C_R)\end{aligned}$$

by (4.29) we have $\mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) \geq -K$, hence

$$\tilde{e}_\varepsilon(M) \geq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - (1-C_R)K + o_\varepsilon(1) - CR^{b+N}e^{-R} + C(C_R^2 - C_R)$$

finally, from Lemma 4.4.3 we get

$$0 > -C > -(1-C_R)K + o_\varepsilon(1) - CR^{b+N}e^{-R} + C(C_R^2 - C_R)$$

letting $R \to +\infty$ this yields a contradiction. We can conclude following the proof of [44, Corollary 5.7]. □

4.4.4 Proof of Theorem 4.1.1

We are ready to prove that the triple $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ converges to $(\bar{u}, \bar{m}, \bar{\lambda})$ solution to the MFG system (4.1).

Proof of Theorem 4.1.1. Let $(\bar{u}, \bar{m}, \bar{\lambda})$ be the triple obtained from Proposition 4.4.2 and set $\bar{w} := -\bar{m}\nabla\bar{u}|\nabla\bar{u}|^{\gamma-2}$. We have that $(\bar{m}, \bar{w}) \in \mathcal{B}$, indeed from Proposition 4.4.2 and Theorem 4.4.5 we get that $(\bar{m}, \bar{w}) \in \mathcal{K}_{1,M}$, and using estimate (4.40) it follows

$$\int_{\mathbb{R}^N} \bar{m}(1+|x|^b)dx \leq \int_{\mathbb{R}^N} c_1 e^{-c_2|x|}(1+|x|)^b dx < +\infty.$$

Since $\bar{m}_\varepsilon \to \bar{m}$ in $L^1(\mathbb{R}^N)$, $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$, $(\bar{m}, \bar{w}) \in K_{1,M}$ and $\bar{E}_\varepsilon, \bar{E} < +\infty$, using Corollary 4.2.3 we get that $(K_\alpha * \bar{m}_\varepsilon)\bar{m}_\varepsilon \to (K_\alpha * \bar{m})\bar{m}$ in $L^1(\mathbb{R}^N)$. Moreover, $\bar{w}_\varepsilon \to \bar{w}$ locally uniformly and weakly in $L^{\frac{\gamma'\beta}{\gamma'-1+\beta}}(\mathbb{R}^N)$.
It follows that the energy $\mathcal{E}_0$ is lower semi-continuous and it holds

$$\begin{aligned}
\mathcal{E}_0(\bar{m}, \bar{w}) &= \int_{\mathbb{R}^N} \frac{\bar{m}}{\gamma'} \left|\frac{\bar{w}}{\bar{m}}\right|^{\gamma'} dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{m}(x)\,\bar{m}(y)}{|x-y|^{N-\alpha}} dx\,dy \\
&\leq \liminf_\varepsilon \left(\int_{\mathbb{R}^N} \frac{\bar{m}_\varepsilon}{\gamma'} \left|\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\right|^{\gamma'} dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar{m}_\varepsilon(x)\,\bar{m}_\varepsilon(y)}{|x-y|^{N-\alpha}} dx\,dy \right) \\
&\leq \liminf_\varepsilon \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \qquad (4.53)
\end{aligned}$$

where in the last inequality we used the fact that $V \geq 0$. Moreover, if $(m, w) \in \mathcal{B}$, using (4.26), we have

$$0 \leq \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} m(y+y_\varepsilon)V_\varepsilon(y+y_\varepsilon)dy \leq \lim_{\varepsilon\to 0} C_V \varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}} \int_{\mathbb{R}^N} (1+|y|)^b m(y)dy = 0$$

from which it follows that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(m(\cdot+y_\varepsilon), w(\cdot+y_\varepsilon)\big) = \mathcal{E}_0(m, w).$$

Using the fact that $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ is a minimizer for $\mathcal{E}_\varepsilon$ and then (4.53), we finally get

$$\mathcal{E}_0(m, w) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(m(\cdot+y_\varepsilon), w(\cdot+y_\varepsilon)\big) \geq \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \geq \mathcal{E}_0(\bar{m}, \bar{w}),$$

this proves that

$$\mathcal{E}_0(\bar{m}, \bar{w}) = \min_{(m,w)\in\mathcal{B}} \mathcal{E}_0(m, w).$$

Since $(\bar m_\varepsilon, \bar w_\varepsilon)$ and $(\bar m, \bar w)$ are minimizers of $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0$ respectively, we obtain that

$$\begin{aligned}\mathcal{E}_\varepsilon(\bar m_\varepsilon, \bar w_\varepsilon) &\le \int_{\mathbb{R}^N} \left(\frac{\bar m}{\gamma'}\left|\frac{\bar w}{\bar m}\right|^{\gamma'} + V_\varepsilon\,\bar m - \bar m(K_\alpha * \bar m)\right)(y+y_\varepsilon)dy \\ &= \mathcal{E}_0(\bar m, \bar w) + \int_{\mathbb{R}^N} V_\varepsilon(y+y_\varepsilon)\,\bar m(y+y_\varepsilon)dy \le \mathcal{E}_0(\bar m_\varepsilon, \bar w_\varepsilon) + C\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\end{aligned}$$

where in the last inequality we used also (4.26) and the fact that $(\bar m, \bar w) \in \mathcal{B}$. It follow immediately that

$$\int_{B(0,R)} \bar m_\varepsilon(y)\,V_\varepsilon(y+y_\varepsilon)dy \le C\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}.$$

From (4.26) and (4.51) we get that there exists a positive constant C such that

$$\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}|y_\varepsilon| \le C \tag{4.54}$$

and hence using (4.26) again we obtain that

$$0 \le V_\varepsilon(y+y_\varepsilon) \le C_V\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\left(1+\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}|y+y_\varepsilon|\right)^b \le C\varepsilon^{\frac{(N-\alpha)\gamma'}{\gamma'-N+\alpha}}\left(1+\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}|y|\right)^b$$

which allows us to conclude that $V_\varepsilon(y+y_\varepsilon) \to 0$ locally uniformly as $\varepsilon \to 0$. This prove that the function g defined in Theorem 4.4.2 is actually zero. □

4.5 Concentration of the mass

The following result allows us to localize the points where the mass concentrates.

Proposition 4.5.1. *As $\varepsilon \to 0$, the sequence $\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y_\varepsilon$ converges, up to subsequences, to a point $\bar x \in \mathbb{R}^N$ such that $V(\bar x) = 0$.*

Proof. Let $z \in \mathbb{R}^N$ (to be fixed later), by minimality of $(\bar m_\varepsilon, \bar w_\varepsilon)$ and $(\bar m(\cdot+z), \bar w(\cdot+z))$ we get

$$\begin{aligned}\mathcal{E}_\varepsilon(\bar m_\varepsilon, \bar w_\varepsilon) &\le \mathcal{E}_\varepsilon\big(\bar m(\cdot+z), \bar w(\cdot+z)\big) = \mathcal{E}_0(\bar m, \bar w) + \int_{\mathbb{R}^N} \bar m(y+z)\,V_\varepsilon(y+y_\varepsilon)dy \\ &\le \int_{\mathbb{R}^N} \frac{\bar m_\varepsilon}{\gamma'}\left|\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\right|^{\gamma'} dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\bar m_\varepsilon(x)\,\bar m_\varepsilon(y)}{|x-y|^{N-\alpha}}dx\,dy + \int_{\mathbb{R}^N} \bar m(y+z)\,V_\varepsilon(y+y_\varepsilon)dy\end{aligned}$$

hence

$$\int_{\mathbb{R}^N} \bar m_\varepsilon(y)\,V_\varepsilon(y+y_\varepsilon)dy \le \int_{\mathbb{R}^N} \bar m(y+z)\,V_\varepsilon(y+y_\varepsilon)dy = \int_{\mathbb{R}^N} \bar m(y)\,V_\varepsilon(y+y_\varepsilon-z)dy$$

and using (4.25) and the fact that $\bar m_\varepsilon(y) = \varepsilon^{\frac{N\gamma'}{\gamma'-N+\alpha}} m_\varepsilon\big(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}y + \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}y_\varepsilon\big)$ we get

$$\int_{\mathbb{R}^N} m_\varepsilon(y)\,V(y)dy \le \int_{\mathbb{R}^N} \bar m(y)\,V\big(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}(y+y_\varepsilon-z)\big)dy.$$

By assumption the potential V is a locally Hölder continuous coercive function, so it has a global minimum at a point $\bar z \in \mathbb{R}^N$, and by a shift of λ we may assume that $V(\bar z) = 0$. Let us fix $z = y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N+\alpha}}\bar z$, it holds

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N} \bar m(y)\,V\big(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}y+\bar z\big)dy \le \limsup_{\varepsilon\to 0} c_1\int_{\mathbb{R}^N} e^{-c_2|y|}V\big(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}}y+\bar z\big)dy = 0. \tag{4.55}$$

Moreover, by (4.54), we get that (up to subsequences)

$$\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y_\varepsilon \to \bar{x} \in \mathbb{R}^N$$

and by (4.52) denoting by $B := B\big(\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} R\big)$

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} m_\varepsilon(y) V(y)\,dy &\geq \liminf_{\varepsilon\to 0} \int_B m_\varepsilon(y)V(y)\,dy \\ &\geq \liminf_{\varepsilon\to 0} \min_{x\in B} V(x) \int_B m_\varepsilon(y)\,dy \geq \liminf_{\varepsilon\to 0} \min_{x\in B} V(x)(M-\eta) \\ &\geq (M-\eta)V(\bar{x}). \end{aligned} \tag{4.56}$$

From (4.55) and (4.56) we obtain that $V(\bar{x}) = 0$. □

Proof of Theorem 4.1.2. It follows from Proposition 4.5.1 and Theorem 4.4.5. □

Remark 10. Arguing as in [44] (refer to Proposition 5.13 and its proof) one can prove that if V has a finite number of minima $x_i \in \mathbb{R}^N$ for $i = 1, \dots, n$ and can be written as

$$V(x) = h(x) \prod_{i=1}^{n} |x - x_i|^{b_i}$$

for a certain function $C_V^{-1} \leq h(x) \leq C_V$ on $\mathbb{R}^N$ and $b_i > 0$ such that $\sum_{i=1}^n b_i = b$, then the sequence $\varepsilon^{\frac{\gamma'}{\gamma'-N+\alpha}} y_\varepsilon$, as $\varepsilon \to 0$, converges (up to subsequences) to the more stable minimum of V (namely the point x_j such that $b_j = \max_{i=1\dots n} b_i$).

4.6 Variational approach in the regime $N - 2\gamma' < \alpha \leq N - \gamma'$

In this last section, we show how the variational approach developed in Section 4.3, can be used to construct solutions to the MFG system (4.2) in the *mass-supercritical and HLS-subcritical regime* and in the *mass-critical case*. Despite that, we are not able to obtain suitable estimates for the value of the game λ_ε analogous to (4.17), which indeed proves to be essential in the vanishing viscosity argument. Hence, using the same technique as in Section 4.4, we are unable to obtain existence of classical solutions to the MFG system (4.1) when $N - 2\gamma' < \alpha \leq N - \gamma'$.

More precisely, when $\alpha = N - \gamma'$ we show that if the total mass M is sufficiently small, then the energy is bounded from below, and by direct methods, we can construct global minimizers, which corresponds to solutions of the MFG system (4.2).

On the other hand, if $N - 2\gamma' < \alpha < N - \gamma'$ the energy is not bounded from below, in this case, we prove that for sufficiently small total masses M, it is possible to construct *local* minimizers of the energy. The key idea consists of considering a constrained variational problem, where the constraint is on the $L^{\frac{2N}{N+\alpha}}$-norm of m, constructing minimizers of the constrained problem and finally showing that actually such minimizers are local free minimizers of the energy. In this way, we can construct solutions to the MFG system arising from local minima of the energy. Moreover in this regime, as the $L^{\frac{2N}{N+\alpha}}$-norm of m is becoming larger, the energy is decreasing to $-\infty$, so the problem should present a sort of mountain pass geometry and we expect existence of saddle-type critical points of the energy. Currently, we are not able to construct such saddle-type critical points, but this open question is in our opinion quite interesting and it will be the subject of future investigation. Moreover, in the vanishing viscosity limit, where up to rescaling the coercive potential is disappearing, we expect that local minimizers are subject to vanishing (following the terminology of the concentration-compactness argument), that is, roughly speaking, they lose their mass "at infinity", while the saddle-type critical points are going to concentrate, converging in the limit to a solution of the potential-free MFG system.

In order to deal with the Riesz-interaction term which is not Hölder continuous a priori for $\alpha \in (N-2\gamma', N-\gamma']$ (refer to Corollary 4.2.4), we first regularize the problem convolving the Riesz term with a standard symmetric mollifier. We can assume, for sake of simplicity, $\varepsilon = 1$. More precisely, we consider the following approximation of the MFG system (4.2)

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \varphi_k * m(x) \\ -\Delta m - \operatorname{div}(m\nabla u(x)\,|\nabla u(x)|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{4.57}$$

where (φ_k) is a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$. We associate to (4.57) the following energy

$$\mathcal{E}_k(m,w) := \begin{cases} \int\limits_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V\,m\,dx - \frac{1}{2}\int\limits_{\mathbb{R}^N} m(x)K_\alpha * \varphi_k * m(x)dx & \text{if } (m,w) \in \mathcal{K}_{1,M} \\ +\infty & \text{otherwise} \end{cases}$$

where

$$L\left(-\frac{w}{m}\right) := \begin{cases} \frac{1}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} & \text{if } m > 0 \\ 0 & \text{if } m = 0,\ w = 0 \\ +\infty & \text{otherwise} \end{cases}.$$

4.6.1 Existence of minima in the critical case $\alpha = N - \gamma'$

In the critical case $\alpha = N - \gamma'$, if the total mass M is sufficiently small then, the energy $\mathcal{E}_k$ is bounded from below.

Lemma 4.6.1. *Let $\alpha = N-\gamma'$ and $(m,w) \in \mathcal{K}_{1,M}$. Then, there exists $M_0 > 0$ (depending on N and γ) such that for any $M \in (0, M_0]$*

$$\mathcal{E}_k(m,w) \geq 0. \tag{4.58}$$

Hence, $\inf_{(m,w)\in\mathcal{K}_{1,M}} \mathcal{E}_k(m,w)$ is finite.

Proof. Similarly to the proof of Lemma 4.3.1, let us fix $\beta := \frac{2N}{2N-\gamma'}$, we get

$$\mathcal{E}_k(m,w) \geq \left(C_1\frac{1}{M} - C_2\right)\|m\|^2_{L^\beta(\mathbb{R}^N)} \tag{4.59}$$

where C_1 and C_2 are constants depending on N and γ. If $\frac{C_1}{M} - C_2 > 0$, that is $M \leq M_0$ where $M_0 := \frac{C_1}{C_2}$, we have that

$$\mathcal{E}_k(m,w) \geq 0.$$

□

As before, by classical direct methods we prove that for every $M \in (0, M_0]$ there exists a global minimizer $(m_k, w_k) \in \mathcal{K}_{1,M}$ of the regularised energy $\mathcal{E}_k$, this allows us to construct the corresponding associated solutions (u_k, m_k, λ_k) of the regularised problem.

With the same arguments of Subsection 3.4.2, since we have uniform L^∞-bounds on m_k, we can finally pass to the limit as $k \to +\infty$ in the MFG system and obtain a solution to the initial problem (4.2) for $\alpha = N - \gamma'$.

4.6.2 Existence of local minima for $N - 2\gamma' < \alpha < N - \gamma'$

In this subsection, assuming $\gamma' < N$, we study the *mass-supercritical and HLS-subcritical regime* $N - 2\gamma' < \alpha < N - \gamma'$. We consider the MFG system (4.57) with k fixed

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \varphi * m(x) \\ -\Delta m - \operatorname{div}(m\nabla u(x)\, |\nabla u(x)|^{\gamma-2}) = 0 & \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{4.60}$$

From Theorem 3.4.7 (which takes advantage of a fixed point argument) there exists a positive real value $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ such that if $M \in (0, M_0)$ the MFG system (4.60) admits a classical solution $(u_\varphi, m_\varphi, \lambda_\varphi)$. Moreover, m_φ belongs to the set

$$A_{\xi,M,C} := \left\{ \mu \in (L^{\bar{p}} \cap L^1)(\mathbb{R}^N) \,\middle|\, \|\mu\|_{\frac{2N}{N+\alpha}} \leq \xi,\ \|\mu\|_1 = M,\ \mu \geq 0,\ \int_{\mathbb{R}^N} \mu V(x)\, dx \leq C \right\}$$

where $\bar{p} > \frac{N}{\alpha}$ and once fixed the mass $M \in (0, M_0)$, the values of ξ and C are given by Lemma 3.4.4. We associate to the system (4.60) the following energy

$$\mathcal{E}(m, w) := \begin{cases} \int\limits_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} + V(x) m dx - \frac{1}{2} \int\limits_{\mathbb{R}^N} m(x) K_\alpha * \varphi * m(x) dx & \text{if } (m, w) \in \mathcal{K}_{1,M} \\ +\infty & \text{otherwise} \end{cases}.$$

In the case where $N - 2\gamma' < \alpha < N - \gamma'$, the energy $\mathcal{E}$ is no a priori bounded from below, hence we have to look for local minimizers over the constraint set

$$\mathcal{K}_{1,M,\xi} := \left\{ (m, w) \in \mathcal{K}_{1,M} \,\middle|\, \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi \right\}.$$

Lemma 4.6.2. *Let us assume that* $N - 2\gamma' < \alpha < N - \gamma'$, $M \in (0, M_0)$ *and* $(m, w) \in \mathcal{K}_{1,M,\xi}$. *Then,*

$$\mathcal{E}(m, w) \geq -\frac{M_0}{\gamma'} - \frac{C_2}{2}\xi^2$$

where C_2 *is a constant depending on* N *and* α.

Proof. Let us fix $\beta := \frac{2N}{N+\alpha}$, by (3.13), (3.15) and the fact that $V \geq 0$, we get

$$\mathcal{E}(m, w) \geq \frac{1}{\gamma' C} M^{1 - \frac{2\gamma'}{N-\alpha}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} - \frac{M}{\gamma'} - \frac{C_2}{2}\|m\|^2_{L^\beta(\mathbb{R}^N)} \geq -\frac{M_0}{\gamma'} - \frac{C_2}{2}\xi^2$$

where C_2 is a constant which depends on N and α. □

Hence, for every $M \in (0, M_0)$ the **local minimum problem** is well-defined over the constraint set $\mathcal{K}_{1,M,\xi}$, we define

$$e(M,\xi) := \inf_{(m,w)\in\mathcal{K}_{1,M,\xi}} \mathcal{E}(m,w).$$

Notice that the couple $(m_\varphi, m_\varphi\nabla u_\varphi|\nabla u_\varphi|^{\gamma-2})$ obtained in Section 3.4 taking advantage of a fixed point argument, belongs to $\in \mathcal{K}_{1,M,\xi}$ and it holds

$$\begin{aligned}\mathcal{E}(m_\varphi, w_\varphi) &= \int_{\mathbb{R}^N} \frac{m_\varphi}{\gamma'}|\nabla u_\varphi|^\gamma + V(x)m_\varphi\,dx - \frac{1}{2}\int_{\mathbb{R}^N} m_\varphi\,(K_\alpha * \varphi * m_\varphi)\,dx \\ &= \lambda_\varphi M + \frac{1}{2}\int_{\mathbb{R}^N} m_\varphi\,(K_\alpha * \varphi * m_\varphi)\,dx \leq C_\lambda M + \frac{C_2}{2}\xi^2\end{aligned}$$

where we used that $\int_{\mathbb{R}^N} \frac{m_\varphi}{\gamma'}|\nabla u_\varphi|^\gamma + V m_\varphi\,dx = \lambda_\varphi M + \int_{\mathbb{R}^N} m_\varphi\,(K_\alpha * \varphi * m_\varphi)\,dx$ (refer to identity (3.54)). It follows immediately that

$$e(M,\xi) \leq C_\lambda M + \frac{C_2}{2}\xi^2. \tag{4.61}$$

We prove now that the values of the energy $\mathcal{E}$ for the couples $(m,w) \in \mathcal{K}_{1,M,\xi}$ such that $\|m\|_{\frac{2N}{N+\alpha}} = \xi$ are strictly greater than the infimum $e(M,\xi)$.

Lemma 4.6.3. *We have that*

$$\inf_{\substack{(m,w)\in\mathcal{K}_{1,M,\xi}\\ \|m\|_{\frac{2N}{N+\alpha}}=\xi}} \mathcal{E}(m,w) > \inf_{(m,w)\in\mathcal{K}_{1,M,\xi}} \mathcal{E}(m,w).$$

Proof. Let us consider a couple $(\hat{m},\hat{w}) \in \mathcal{K}_{1,M,\xi}$ such that $\|\hat{m}\|_{L^{\frac{2N}{N+\alpha}}} = \xi$. Then, denoting by $a := \frac{2\gamma'}{N-\alpha}$ and with g the function $g(t) = t^a - C\,C_2\gamma' M^{a-1}t^2 - CM^a - \gamma' C C_\lambda M^a$, we have

$$\begin{aligned}\mathcal{E}(\hat{m},\hat{w}) &\geq \frac{1}{\gamma' C}M^{1-a}\xi^a - \frac{M}{\gamma'} - \frac{C_2}{2}\xi^2 = \frac{1}{\gamma' C M^{a-1}}\Big(\xi^a - CM^a - C\,C_2\gamma' M^{a-1}\xi^2\Big) + \frac{C_2}{2}\xi^2 \\ &= \frac{1}{\gamma' C M^{a-1}}\big(g(\xi) + \gamma' C C_\lambda M^a\big) + \frac{C_2}{2}\xi^2.\end{aligned}$$

Since from Lemma 3.4.4 we have chosen ξ in such a way that $g(\xi) > 0$, we get that

$$\mathcal{E}(\hat{m},\hat{w}) > C_\lambda M + \frac{C_2}{2}\xi^2 \geq \mathcal{E}(m_\varphi, w_\varphi).$$

The claim follows. □

Remark 11. If $(\bar{m},\bar{w})$ minimizes the energy $\mathcal{E}$ over the set $\mathcal{K}_{1,M,\xi}$, then there must exists a positive real value η ($\eta \leq \xi$) such that

$$\|m_k\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi - \eta < \xi$$

We have the following a priori bounds.

Proposition 4.6.4. *Let $(m,w) \in \mathcal{K}_{1,M,\xi}$ such that $e(M,\xi) \geq \mathcal{E}(m,w) - c$ for some positive c. Then*

$$\int_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} dx \leq \frac{C_2}{2}\xi^2 + K \tag{4.62}$$

and

$$\int_{\mathbb{R}^N} V(x)\, m\, dx \leq \frac{C_2}{2}\xi^2 + K$$

where C_2 and K are positive constants depending respectively on N, α and on M, α, b, γ, N.

Proof. From (4.61) there exists a positive constant C_e depending on $M, C_V, b, \gamma, N, \alpha$ such that

$$e(M,\xi) \leq C_e.$$

Let us denote with $\beta := \frac{2N}{N+\alpha}$, we have

$$C_e + c \geq e(M,\xi) + c \geq \mathcal{E}(m,w) \geq \int_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} dx - \frac{1}{2}\int_{\mathbb{R}^N} m\,(K_\alpha * \varphi * m) dx.$$

From the previous we get

$$\begin{aligned}\frac{1}{\gamma'}\int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx &\leq C_e + c + \frac{1}{2}\int_{\mathbb{R}^N} m(x)(K_\alpha * \varphi * m)(x) dx \\ &\leq C_e + c + \frac{C_2}{2}\|m\|^2_{L^\beta(\mathbb{R}^N)} \leq K + \frac{C_2}{2}\xi^2\end{aligned}$$

which proves (4.62). Finally, since it holds $\frac{1}{\gamma'}\int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx + \int_{\mathbb{R}^N} V(x)\, m\, dx = \mathcal{E}(m,w) + \frac{1}{2}\int_{\mathbb{R}^N} m(K_\alpha * \varphi * m)$, using (3.15), we obtain

$$\int_{\mathbb{R}^N} V(x) m\, dx \leq \mathcal{E}(m,w) + \frac{C_2}{2}\|m\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq e(M,\xi) + c + \frac{C_2}{2}\xi^2 \leq K + \frac{C_2}{2}\xi^2.$$

□

By means of classical direct methods, we prove that for every $M \in (0, M_0)$ there exists a local minimizer $(\bar{m}, \bar{w}) \in \mathcal{K}_{1,M,\xi}$ of the energy $\mathcal{E}$.

Proposition 4.6.5. *For every $M \in (0, M_0)$ there exists $(\bar{m}, \bar{w})$ local minimizer of the energy $\mathcal{E}$ over the set $\mathcal{K}_{1,M,\xi}$, namely*

$$\mathcal{E}(\bar{m}, \bar{w}) = \inf_{(m,w)\in\mathcal{K}_{1,M,\xi}} \mathcal{E}(m,w).$$

For every local minimizer $(\bar{m}, \bar{w})$ of $\mathcal{E}$ we have that

$$\int_{\mathbb{R}^N} \frac{\bar{m}}{\gamma'} \left|\frac{\bar{w}}{\bar{m}}\right|^{\gamma'} dx \leq \frac{C_2}{2}\xi^2 + K$$

and

$$\int_{\mathbb{R}^N} V(x)\bar{m}\, dx \leq \frac{C_2}{2}\xi^2 + K.$$

Moreover, it holds

$$\bar{m}(1+|x|)^b \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad \bar{w}(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N). \tag{4.63}$$

Proof. It follows as the proof of Proposition 4.3.3. □

Once we have obtained local minimizers $(\bar m, \bar w) \in \mathcal{K}_{1,M,\xi}$ of the energy $\mathcal{E}$, we construct the associated solutions $(\bar u, \bar m, \bar\lambda)$ of the MFG system (4.60).

Proposition 4.6.6. *If $N - 2\gamma' < \alpha < N - \gamma'$, then for any $M \in (0, M_0)$ there exists a classical solution $(\bar u, \bar m, \bar\lambda)$ to the MFG system* (4.60). *Moreover, the following estimates hold*

$$\bar u(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1} \tag{4.64}$$

$$|\nabla \bar u(x)| \le C(1 + |x|^{\frac{b}{\gamma}}). \tag{4.65}$$

Proof. This proof is very similar to the one of Proposition 4.3.4, so we shall omit many details, we recall only the different facts. From Proposition 4.6.5 there exists at least one local minimizer $(\bar m, \bar w)$ of the energy $\mathcal{E}$, and one can verify that

$$-\int_{\mathbb{R}^N} \bar m \Delta\psi \, dx = \int_{\mathbb{R}^N} \bar w \cdot \nabla\psi \, dx, \quad \forall \psi \in \mathcal{A} \tag{4.66}$$

(refer to the proof of identity (3.18) in [44], for more details). Since every local minimizer satisfies (4.63) and (4.66), minimizing $\mathcal{E}$ on $\mathcal{K}_{1,M,\xi}$ is equivalent to minimize $\mathcal{E}$ on the following constraint set

$$\mathcal{K} := \left\{ (m,w) \in (L^1 \cap W^{1,r}) \times L^{\frac{\gamma'\beta}{\gamma'+\beta-1}}(\mathbb{R}^N) \;\middle|\; \begin{array}{c} (m,w) \text{ satisfies (4.63), (4.66), } \int_{\mathbb{R}^N} m = M \\ m \ge 0 \quad \text{and} \quad \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le \xi \end{array} \right\}$$

where $r < q$. Let us consider in addition the following set

$$\tilde{\mathcal{K}} := \left\{ (m,w) \in (L^1 \cap W^{1,r}) \times L^{\frac{\gamma'\beta}{\gamma'+\beta-1}}(\mathbb{R}^N) \;\middle|\; \begin{array}{c} (m,w) \text{ satisfies (4.63), (4.66)} \\ \int_{\mathbb{R}^N} m = M \qquad m \ge 0 \end{array} \right\}$$

and define

$$\Phi(m,w) := \begin{cases} \int_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} dx & \text{if } (m,w) \in \tilde{\mathcal{K}} \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\Psi(m) := \int_{\mathbb{R}^N} V(x) m \, dx - \frac{1}{2} \int_{\mathbb{R}^N} m(K_\alpha * \varphi * m) dx.$$

In this way $\mathcal{E}(m,w) = \Phi(m,w) + \Psi(m)$. For every couple $(m,w) \in \tilde{\mathcal{K}}$ and for every $\lambda \in (0, \Lambda)$ (where Λ will be fixed later) we define $m_\lambda := (1-\lambda)\bar m + \lambda m$ and $w_\lambda := (1-\lambda)\bar w + \lambda w$. One can verify that $(m_\lambda, w_\lambda) \in \mathcal{K}$, indeed we have

$$\|m_\lambda\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le (1-\lambda)\|\bar m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} + \lambda\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le (1-\lambda)(\xi-\eta) + \lambda\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$

(see Remark 11). Taking $\Lambda = \min\left\{1, \frac{\eta}{\|m\|_{\frac{2N}{N+\alpha}}}\right\}$, it follows that

$$\|m_\lambda\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} < (1-\lambda)\xi - (1-\lambda)\eta + \eta = \xi - \lambda(\xi - \eta) \le \xi$$

since $\xi-\eta \geq 0$. The other conditions follow immediately. Since $(\bar{m}, \bar{w})$ is a local minimum of $\mathcal{E}$ over $\mathcal{K}$ and $(m_\lambda, w_\lambda) \in \mathcal{K}$, we get that

$$\mathcal{E}(m_\lambda, w_\lambda) \geq \mathcal{E}(\bar{m}, \bar{w})$$

hence

$$\Phi(m_\lambda, w_\lambda) - \Phi(\bar{m}, \bar{w}) \geq \Psi(\bar{m}) - \Psi(m_\lambda) \tag{4.67}$$

and by convexity of Φ

$$\lambda\left(\Phi(m, w) - \Phi(\bar{m}, \bar{w})\right) \geq \Phi(m_\lambda, w_\lambda) - \Phi(\bar{m}, \bar{w}). \tag{4.68}$$

From (4.67) and (4.68), dividing by λ and letting λ go to 0, we finally obtain that

$$\Phi(m, w) - \Phi(\bar{m}, \bar{w}) \geq -\int_{\mathbb{R}^N} V(x)(m - \bar{m})dx + \int_{\mathbb{R}^N} (m - \bar{m})K_\alpha * \varphi * \bar{m}$$

Hence, the couple $(\bar{m}, \bar{w})$ minimizes also the following convex functional on $\tilde{\mathcal{K}}$

$$\tilde{\mathcal{E}}(m, w) := \int_{\mathbb{R}^N} \frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} dx + \int_{\mathbb{R}^N} V(x)m\, dx - \int_{\mathbb{R}^N} m(x)(K_\alpha * \varphi * \bar{m})(x)dx.$$

The remaining part of the proof is analogous to the one of Proposition 4.3.4 and can be summarized as follows. We include the differential constraint in the functional by adding a supremum, so that using the Fan's min-max theorem, we may interchange the min and the sup. In this way, we can define the minimum of $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{K}}$ as the supremum of the values of λM for which the HJB equation has a regular subsolution. Finally, since $\varphi * \bar{m} \in L^r(\mathbb{R}^N)$ for a certain suitable r by Theorem 3.2.9 we get that $K_\alpha * \varphi * \bar{m} \in C^{0,\alpha-\frac{N}{r}}(\mathbb{R}^N)$, hence we can use Proposition 3.2.11 and obtain the couple $(\bar{u}, \bar{\lambda}) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ and the estimates (4.64)-(4.65). □

Finally, the function $\bar{m}$ is bounded from above in $\mathbb{R}^N$.

Proposition 4.6.7. *Let $(\bar{u}, \bar{m}, \bar{\lambda})$ be a solution to the MFG system* (4.60)*. Then, $\bar{m} \in L^\infty(\mathbb{R}^N)$.*

Proof. The proof is analogous to the one of Proposition 3.4.3 *iv*). □

Theorem 4.6.8. *Let $N - 2\gamma' < \alpha < N - \gamma'$. Assume that the potential V is locally Hölder continuous and satisfies* (4.3)*. Then, for every $M \in (0, M_0)$ there exists $(\bar{u}, \bar{m}, \bar{\lambda})$ classical solution to the MFG system* (4.60)*. Moreover, $\bar{m}$ is bounded in $L^\infty(\mathbb{R}^N)$ and the function $\bar{u}$ satisfies for some constant $C > 0$ the following estimates*

$$\bar{u}(x) \geq C|x|^{1+\frac{b}{\gamma}} - C^{-1}$$

$$|\nabla \bar{u}(x)| \leq C(1 + |x|^{\frac{b}{\gamma}}).$$

Proof. It follows from Proposition 4.6.6 and Proposition 4.6.7. □

Proof of Theorem 4.1.3. If $(\varphi_k)_k$ is a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$, from Theorem 4.6.8 we get that for every $k \in \mathbb{N}$ and $M \in (0, M_0)$ there exists a classical solution $(\bar{u}_k, \bar{m}_k, \bar{\lambda}_k)$ to the regularised MFG system (4.57). Proceeding as in Lemma 3.4.8, we get uniform L^∞ bounds on $\bar{m}_k$, that the values $\bar{\lambda}_k$ are equibounded in k and

$$|\nabla \bar{u}_k| \leq C_1(1 + |x|^{\frac{b}{\gamma}}) \qquad |\Delta \bar{u}_k| \leq C_2(1 + |x|^b)$$

where C_1 and C_2 are positive constants not depending on k.

We are now in the position to pass to the limit as $k \to +\infty$ (for more details we refer the reader to the proof of Theorem 3.1.2) and prove that $(\bar{u}_k, \bar{m}_k, \bar{\lambda}_k)$ converges to a triple $(\tilde{u}, \tilde{m}, \tilde{\lambda})$ solution to the MFG system (4.2). $\square$

Part III

Choquard equation.

Chapter 5

Boundary value problems for Choquard equations

5.1 Introduction to the problem and main results

In the present chapter, we study the following nonlinear Choquard equation

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{5.1}$$

with Dirichlet or Neumann boundary conditions. We assume that $N \in \mathbb{N}$, $N \geq 2$, the exponent in the nonlinearity is a real value $p > 1$ and the potential $V : \Omega \to \mathbb{R}$ is a continuous radial function such that $\inf_{x \in \Omega} V > 0$. Here, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$, which is defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{C_{N,\alpha}}{|x|^{N-\alpha}}, \quad \text{where} \quad C_{N,\alpha} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha}, \tag{5.2}$$

recall that by $I_\alpha * f$ in a domain $\Omega \subset \mathbb{R}^N$, we mean the convolution $I_\alpha * (\chi_\Omega f)$ in $\mathbb{R}^N$. We consider both the case when the domain Ω is an annulus centered at the origin, namely there exist two real values $0 < a < b < +\infty$ such that

$$\Omega = A_{a,b} := \{x \in \mathbb{R}^N \mid a < |x| < b\}$$

and the case of the exterior domain

$$\Omega = \mathbb{R}^N \setminus \overline{B_a(0)} = \{x \in \mathbb{R}^N \mid |x| > a > 0\}.$$

The Choquard equation has been extensively studied over the last decades, since it arises in the modeling of several mean-field physical phenomena. More in detail, the Choquard-Pekar equation

$$-\Delta u + u = (I_2 * |u|^2)u \qquad \text{in } \mathbb{R}^3, \tag{5.3}$$

has been first introduced by S. Pekar [163] in 1954 to model the quantum mechanics of a polaron at rest, then P. Choquard used it to describe an electron trapped in its own hole. A further application was proposed by R. Penrose [165–167], who used it to model self-gravitating matter. Existence of solutions to equation (5.3) in the normalized framework, namely imposing that $\|u\|_{L^2(\mathbb{R}^3)} = \mu$, has been first investigated using variational methods by E.H. Lieb [130] and in more general cases by P.-L. Lions [134] (refer also to [19, 20] for further results about existence of solutions to more general Choquard-type systems). In particular, using symmetric decreasing rearrangement inequalities, E.H. Lieb proved that there exists a minimizing solution, which is radial and unique up to translations, while more recently, L. Ma and L. Zhao [139] classified all positive solutions to (5.3) (see also [126, 180]).

On the other hand, the Choquard equation on $\mathbb{R}^N$ with a more general nonlocal nonlinearity depending on a parameter $p > 1$, that is the following semilinear elliptic equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^N,$$

admits a nontrivial solution $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ with $\nabla u \in H^1_{loc}(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}_{loc}(\mathbb{R}^N)$ if and only if $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. We bring up e.g. [13, 67, 152, 154, 156] for a complete overview on the topic and also [78, 90, 92] for existence of sign-changing solutions. The situation changes when adding an external potential V in the equation, that is considering

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \qquad \text{in } \mathbb{R}^N, \tag{5.4}$$

since the presence of the (possibly) variable potential V influences particles and hence it could affect existence of solutions (see for instance [155, 179] and [156, §4], for further discussion and references).

As we mentioned above, Choquard equations on the whole space $\mathbb{R}^N$ have been extensively studied in past decades, while there are few results about Choquard equations on other types of domain $\Omega \subset \mathbb{R}^N$. Notice that the notion of solution to the Choquard equation is *nonlocal*, that is if $u \geq 0$ (weakly) solves (5.4), than u is only a supersolution to the same equation in $\Omega \subset \mathbb{R}^N$. We refer, among others, to [89] for a Brezis-Nirenberg type critical problem of the Choquard equation on bounded domains, to [91] who proved that for slightly subcritical Choquard problems the number of positive solutions depends on the topology of the domain, and to [94, 95] for existence results at the HLS critical level for problems on non-contractible domains which contain a sufficiently large annulus. On the other hand, for what concerns exterior domains, we mention the work of V. Moroz and J. Van Schaftingen [153] regarding sharp Liouville-type nonexistence results for supersolutions in some suitable range of the parameter p, and optimal decay rates for solutions (see also [55, 78] and the references therein).

In this chapter, we prove that problem (5.1) with Neumann or Dirichlet boundary conditions, admits a positive radial solution in annular domains $\Omega = A_{a,b}$ for every $p \geq 1$, while in exterior domains $\Omega = \mathbb{R}^N \setminus \bar{B}_a$ for every $p > \frac{N+\alpha}{N}$. Our result generalizes to the case of nonlocal nonlinear equations a classical result by Kadzan and Warner [120] about existence of solutions to nonlinear equations with power-like nonlinearities in annular domains.

Problem (5.1) has a variational structure: weak solutions are formally critical points (with $u > 0$ on Ω) of the action functional $\mathcal{A}$ defined for a function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{A}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + V(x)u^2\right) - \frac{1}{2p}\int_\Omega \left(I_\alpha * |u|^p\right)|u|^p.$$

Notice that due to the classical Sobolev Embedding Theorem $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [2, 2^*]$, the above energy is well-defined and sufficiently differentiable on the Sobolev space $H^1(\mathbb{R}^N)$ if $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$. Setting the problem in an annular domain or in the exterior of a ball, and looking for radial solutions, we are able to enlarge the range of parameters p for which the energy is well-defined. We will consider the Sobolev space of radial functions

$$H^1_{rad}(\Omega) := \{u \in H^1(\Omega) \mid u(x) = u(|x|)\}$$

with the usual norm

$$\|u\|_{H^1} := \left(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right)^{\frac{1}{2}} = \left(\int_\Omega |u|^2 + |\nabla u|^2\right)^{\frac{1}{2}}.$$

We take advantage of the fact that the embedding

$$H^1_{rad}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for every $q \geq 1$ if $\Omega = A_{a,b}$ and for every $q > 2$ if $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$ (see Theorem 5.2.1 and Theorem 5.2.2 below). Analogous results hold also for $H^1_{0,rad}(\Omega)$ (see Corollary 5.2.3 and Corollary 5.2.4 below). Then, in order to find solutions to (5.1) with homogeneous Neumann or Dirichlet boundary conditions we consider a constrained variational problem (see also [155] and [179]). For every $\alpha \in (0, N)$ fixed, we look for minimizers of the energy functional

$$Q(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + Vu^2 \tag{5.5}$$

over the constrained set

$$M_\alpha := \left\{u \in H^1_{rad}(\Omega) \;\middle|\; \int_\Omega (I_\alpha * u^p)\, u^p = 1\right\}. \tag{5.6}$$

Notice that the minimizers u_α can be chosen non-negative (possibly taking $|u_\alpha|$ since it holds $\int_\Omega \big|\nabla|u_\alpha|\big|^2 dx = \int_\Omega |\nabla u_\alpha|^2 dx$). Hence, if

$$Q(u_\alpha) = J_\alpha := \inf\{Q(u) \mid u \in M_\alpha\}, \tag{5.7}$$

up to multiplication by a constant, the function u_α is a positive groundstate of (5.1) with Neumann homogeneous boundary conditions.

Our main results are the following.

Theorem 5.1.1. *Let $N \geq 2$, $\alpha \in (0, N)$, $\Omega = A_{a,b}$ and $V(|x|)$ be a continuous radial function on Ω such that $\inf_{x\in\Omega} V(x) > 0$. Then, for every $p \in [1, +\infty)$ there exists $u \in H^1_{rad}(\Omega)$ which solves*

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}. \tag{5.8}$$

On the other hand, if $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$ the boundary value problem (5.8) admits a solution for every $p \in \left(\frac{N+\alpha}{N}, +\infty\right)$.

The analogous result holds in the case of Dirichlet boundary conditions. Notice that if $N \geq 3$ and Ω is an annular domain, we can weaken the assumptions on the potential V, covering in this way also the case of the Choquard equation with the unperturbed Laplacian, that is when $V \equiv 0$.

Theorem 5.1.2. *Let $N \geq 2$, $\alpha \in (0, N)$, $\Omega = A_{a,b}$ and $V(|x|)$ be a continuous radial function on Ω such that $\inf_{x\in\Omega} V(x) > 0$ if $N = 2$ and $V(|x|) \geq 0$ if $N \geq 3$. Then, for every $p \in [1, +\infty)$ there exists $u \in H^1_{0,rad}(\Omega)$ which solves*

$$\begin{cases} -\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} . \tag{5.9}$$

If $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$ the boundary value problem (5.9) admits a solution when $\inf_{x\in\Omega} V(x) > 0$ for every $p \in \left(\frac{N+\alpha}{N}, +\infty\right)$.

On the other hand, when Ω is a smooth domain in $\mathbb{R}^N$ and the solution u is sufficiently regular, we are able to obtain a Pohozaev-type identity, which in turns implies the triviality of u, when the domain Ω is strictly star-shaped.

Theorem 5.1.3. *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a smooth domain strictly star-shaped with respect to the origin, $p \geq \frac{N+\alpha}{N-2}$ and $V \in C^1(\Omega)$. Let $u \in H^1_0(\Omega) \cap H^2(\Omega) \cap W^{1,\frac{2pN}{N+\alpha}}(\Omega)$ be a solution to*

$$\begin{cases} -\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} . \tag{5.10}$$

Assume that

$$u^2V, \;\; u^2\nabla V \cdot x \;\in L^1(\Omega)$$

and

$$\int_\Omega u^2 \nabla V \cdot x \geq 0.$$

If $p = \frac{N+\alpha}{N-2}$, we assume that $V > 0$ in Ω, whereas if $p > \frac{N+\alpha}{N-2}$ it is sufficient that $V \geq 0$. Then, $u \equiv 0$.

Finally, in the case $\Omega = A_{a,b}$, we prove a Γ-convergence type result as $\alpha \to 0^+$, relating the minimization problem (5.7) with

$$J_0 = \inf\{Q(u) \mid u \in M_0\}$$

where

$$M_0 := \left\{ u \in H^1_{rad}(\Omega) \;\middle|\; \|u\|_{L^{2p}(\Omega)} = 1, \quad u \geq 0 \right\}.$$

This argument allows us to recover the existence result of Kadzan and Warner [120] for solutions to the corresponding local problem

$$\begin{cases} -\Delta u + Vu = u^{2p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.11}$$

and to extend it also to the case V is nonconstant. Notice that, the analogous result holds also with Dirichlet boundary conditions, in this context we may refer among others to [106, 160] for some related results.

Theorem 5.1.4. *Let us assume that* $\Omega = A_{a,b}$*,* $N \geq 2$*,* $V(|x|)$ *be a continuous radial function on* Ω *such that* $\inf_{x\in\Omega} V(x) > 0$ *and* p *is a fixed real value in* $[1, +\infty)$*. Then,*

$$\lim_{\alpha\to 0^+} J_\alpha = J_0$$

and if $\{u_\alpha\}$ *is a sequence of minimizers for* J_α*, there exists* $u_0 \in M_0$ *such that as* $\alpha \to 0^+$

$$u_\alpha \to u_0 \ \text{ in } H^1_{rad}(\Omega)$$

and $J_0 = Q(u_0)$*. Moreover,* u_0 *is a solution to* (5.11) *up to multiplication by a constant.*

Theorem 5.1.5. *Under the assumptions of Theorem 5.1.2, the same result as Theorem 5.1.4 holds in the case of Dirichlet boundary condition on* $\Omega = A_{a,b}$*.*

This chapter is organized as follows. In Section 5.2, we provide some preliminary results about Sobolev embeddings for radial functions defined in annular domains or in exterior domains. Using a variational approach, in Section 5.3, we prove existence of a positive radial solution to our problem. In Section 5.4, we obtain a suitable Pohozaev-type identity which allows us to prove the nonexistence result of Theorem 1.3. Finally, Section 5.5 is devoted to the study of the limiting problem as $\alpha \to 0^+$.

5.2 Sobolev embeddings for radial functions

We state here some results on Sobolev embeddings for $H^1_{rad}(\Omega)$ and $H^1_{0,rad}(\Omega)$, both in the cases when the domain $\Omega \subset \mathbb{R}^N$ is an annulus and when $\Omega = \{x \in \mathbb{R}^N \,|\, |x| > a > 0\}$. Since we have not found a detailed proof in the literature, we recall it here by completeness.

Theorem 5.2.1. *Let* $N \geq 2$ *and* $\Omega = A_{a,b}$*. For every* $p \in [1, +\infty)$*, the following immersion*

$$H^1_{rad}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact. Moreover, for every $p \geq 1$ *there exists a positive constant* $C = C(N, p, a, b)$ *such that*

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{H^1_{rad}(\Omega)}. \tag{5.12}$$

Remark 12. If $a = 0$ namely $\Omega = A_{0,b} = B_b(0)$, from the classical Rellich-Kondrachov Theorem (refer to [2, Theorem 6.2]) we have that the previous result holds for every $p \in [1, 2^*)$ if $N \geq 3$, and for every $p \in [1, +\infty)$ if $N = 2$.

Proof. If $N = 2$, since Ω is bounded and smooth, we have that for all $p \in [1, +\infty)$ the embedding

$$H^1(\Omega) \to L^p(\Omega)$$

is compact (see e.g. [2, Theorem 6.2]).

If $N \geq 3$, every function $u \in H^1_{rad}(A_{a,b})$ can be extended to a function $\bar{u} \in H^1_{rad}(\mathbb{R}^N)$ such that $\bar{u}|_{A_{a,b}} = u$ and $\bar{u}(x) = 0$ for $|x|$ sufficiently large with (by construction we have that $\bar{u} \equiv 0$ for $|x| \geq 2b - a$)

$$\|\bar{u}\|_{L^2(\mathbb{R}^N)} \leq C\|u\|_{L^2(A_{a,b})} \qquad \text{and} \qquad \|\bar{u}\|_{H^1(\mathbb{R}^N)} \leq C\|u\|_{H^1(A_{a,b})}$$

where C depends only on $|b-a|$, see [30, Theorem 8.6]. We recall also that every function $\bar{u} \in H^1(\mathbb{R})$ is represented by a continuous function on $\bar{\mathbb{R}}$, which we denote again by $\bar{u}$, and such that

$$\bar{u}(x) - \bar{u}(y) = \int_y^x \bar{u}'(r)\,dr, \qquad \forall x, y \in [-\infty, +\infty]$$

(refer to [30, Theorem 8.2]). Hence following [175], for any $u \in H^1_{rad}(A_{a,b})$ and $x \in A_{a,b}$ we have that

$$
\begin{aligned}
|u(x)|^2 = |\bar{u}(|x|)|^2 &= \left| \int_{|x|}^{+\infty} \frac{d}{dr} |\bar{u}(r)|^2 dr \right| \leq \int_{|x|}^{+\infty} \left| \frac{d}{dr} |\bar{u}(r)|^2 \right| dr \\
&\leq 2 \left(\int_{|x|}^{+\infty} |\bar{u}(r)|^{\frac{2N}{N-2}} r^{N-1} dr \right)^{\frac{N-2}{2N}} \left(\int_{|x|}^{+\infty} \left| \frac{d}{dr} \bar{u}(r) \right|^2 r^{N-1} dr \right)^{\frac{1}{2}} \left(\int_{|x|}^{+\infty} r^{(2-N)N-1} dr \right)^{\frac{1}{N}} \\
&\leq C|x|^{2-N} \|\bar{u}\|_{L^{2^*}(\mathbb{R}^N)} \|\nabla \bar{u}\|_{L^2(\mathbb{R}^N)} \leq C|x|^{2-N} \|\nabla \bar{u}\|^2_{L^2(\mathbb{R}^N)} \\
&\leq C|x|^{2-N} \|u\|^2_{H^1(A_{a,b})}
\end{aligned}
$$

where $C = C(N, a, b)$ and we used the Hölder's inequality and the classical Sobolev Embedding Theorem (refer to [30, Theorem 9.9] or [127, Corollary 11.9]). Hence for every $p \in [1, +\infty)$ we get

$$
\int_{A_{a,b}} |u(x)|^p dx = C \int_a^b r^{N-1} |u(r)|^p dr \leq C \|u\|^p_{H^1(A_{a,b})} \int_a^b r^{N-1+p-\frac{Np}{2}} dr
$$

which proves estimate (5.12). In order to prove compactness, let $\{u_n \,|\, n \in \mathbb{N}\} \subset H^1_{rad}(\Omega)$ be a bounded sequence, then there exists a constant $M > 0$ such that $\|u_n\|_{H^1(\Omega)} \leq M$ $\forall n \in \mathbb{N}$. Using (5.12), we get that the family $\{u_n \,|\, n \in \mathbb{N}\}$ is bounded in $L^p(\Omega)$ for any $p \in [1, +\infty)$. Let us show that $\{u_n \,|\, n \in \mathbb{N}\}$ is uniformly equi-continuous in $L^p(\Omega)$. For every $k \in \mathbb{N}_*$ we define

$$
\Omega_k := \left\{ x \in \Omega \,\middle|\, \operatorname{dist}(x, \partial\Omega) > \frac{1}{k} \right\}.
$$

Using the Hölder's inequality and the Sobolev embedding, we can prove that $\forall k \in \mathbb{N}_*$ and $\forall h \in \mathbb{R}^N$ it holds

$$
\|\tau_h u - u\|_{L^1(\Omega \setminus \Omega_k)} \leq C \|u\|_{H^1(\Omega)} \mathcal{L}(\Omega \setminus \Omega_k)^{\frac{1}{2}}, \tag{5.13}
$$

where $C = C(N, a, b)$ and $\tau_h u(x) := u(x+h)$ with u extended with 0 outside Ω. Moreover $\forall k \in \mathbb{N}_*$ and $\forall h \in \mathbb{R}^N$ such that $|h| \leq \frac{1}{k}$ we have

$$
\|\tau_h u - u\|_{L^1(\Omega_k)} \leq |h| \, \|\nabla u\|_{L^2(\Omega)} \mathcal{L}(\Omega)^{\frac{1}{2}}. \tag{5.14}
$$

Hence from (5.13) and (5.14) we get that $\forall k \in \mathbb{N}_*$, $\forall h \in \mathbb{R}^N$ such that $|h| \leq \frac{1}{k}$ and $\forall n \in \mathbb{N}$ we have

$$
\begin{aligned}
\|\tau_h u_n - u_n\|_{L^1(\Omega)} &= \|\tau_h u_n - u_n\|_{L^1(\Omega \setminus \Omega_k)} + \|\tau_h u_n - u_n\|_{L^1(\Omega_k)} \\
&\leq M \left(C \mathcal{L}(\Omega \setminus \Omega_k)^{1/2} + |h| \, \mathcal{L}(\Omega)^{\frac{1}{2}} \right).
\end{aligned}
$$

For every fixed $\varepsilon > 0$ we can choose k_0 such that if $|h| \leq \frac{1}{k_0}$ then

$$
|h| \, \mathcal{L}(\Omega)^{\frac{1}{2}} \leq \frac{\varepsilon}{M} \qquad \text{and} \qquad \mathcal{L}(\Omega \setminus \Omega_k)^{1/2} \leq \frac{\varepsilon}{CM},
$$

hence we get that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall h \in \mathbb{R}^N$ with $|h| \leq \delta$ and for any $n \in \mathbb{N}$ it holds

$$
\|\tau_h u_n - u_n\|_{L^1(\Omega)} \leq \varepsilon.
$$

Finally for every $p \in (1, +\infty)$ fixed, by interpolation there exist $q \in (p, +\infty)$ and $\theta \in (0, 1)$ such that

$$\begin{aligned}\|\tau_h u_n - u_n\|_{L^p(\Omega)} &\le \|\tau_h u_n - u_n\|_{L^1(\Omega)}^{\theta} \|\tau_h u_n - u_n\|_{L^q(\Omega)}^{1-\theta} \\ &\le \|\tau_h u_n - u_n\|_{L^1(\Omega)}^{\theta} \left[\|\tau_h u_n\|_{L^q(\Omega)} + \|u_n\|_{L^q(\Omega)}\right]^{1-\theta} \\ &\le \|\tau_h u_n - u_n\|_{L^1(\Omega)}^{\theta} \left[2\|u_n\|_{L^q(\Omega)}\right]^{1-\theta} \\ &\le \|\tau_h u_n - u_n\|_{L^1(\Omega)}^{\theta} (2CM)^{1-\theta}\end{aligned}$$

where in the last inequality we used (5.12). This proves equicontinuity in $L^p(\Omega)$, namely $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall h \in \mathbb{R}^N$ with $|h| \le \delta$ and for any $n \in \mathbb{N}$ it holds

$$\|\tau_h u_n - u_n\|_{L^p(\Omega)} \le \varepsilon.$$

Then by the compactness criterion in L^p, we can conclude that the sequence $\{u_n \,|\, n \in \mathbb{N}\}$ converges (up to subsequences) in $L^p(\Omega)$. □

Theorem 5.2.2. *Let $N \ge 2$ and $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$. For every $p \in [2, +\infty)$ we have the following continuous immersion*

$$H^1_{rad}(\Omega) \hookrightarrow L^p(\Omega);$$

which is also compact for $p \in (2, +\infty)$.

Proof. Case $N \ge 3$. Let $u \in H^1_{rad}(\Omega)$, since it is radial we can identify it with a function in $H^1\big((a, +\infty)\big)$ which we still denote by u. Since $u \in H^1((a, +\infty))$ then $\lim\limits_{|x| \to +\infty} u(x) = 0$, by the same argument as in the proof of Theorem 5.2.1, we get that

$$|u(x)| \le C|x|^{1-\frac{N}{2}} \|u\|_{H^1(\Omega)} \tag{5.15}$$

where $C = C(N, a)$. If $p > 2^*$, $N + p - \frac{Np}{2} < 0$ and so

$$\int_\Omega |u(x)|^p dx \le C_1 \|u\|^p_{H^1(\Omega)} \int_a^{+\infty} r^{N-1+p-\frac{Np}{2}} dr = C_2 \|u\|^p_{H^1(\Omega)}. \tag{5.16}$$

From (5.16) and the classical Sobolev embedding (see [2, Theorem 5.4]) which holds for $p \in [2, 2^*]$, we can conclude that if $u \in H^1_{rad}(\Omega)$

$$\|u\|_{L^p(\Omega)} \le C_{N,p,a} \|u\|_{H^1(\Omega)}, \qquad \forall p \in [2, +\infty)$$

and hence

$$H^1_{rad}(\Omega) \hookrightarrow L^p(\Omega), \qquad \forall p \in [2, +\infty).$$

Now in order to prove compactness, we proceed as in [5, Theorem 11.2] (see also [175]). Let $\{u_n \,|\, n \in \mathbb{N}\} \subset H^1_{rad}(\Omega)$ be a bounded sequence, without loss of generality we can assume that $u_n \rightharpoonup 0$ in $H^1_{rad}(\Omega)$. From (5.15) it follows that

$$|u_n(x)| \le C_1 |x|^{1-\frac{N}{2}}, \qquad \text{for every } n \in \mathbb{N}$$

so if $p > 2$, given $\varepsilon > 0$ there exist $C_2, R > 0$ (we can always assume $R > a$) such that

$$|u_n(x)|^{p-2} \le C_1 |x|^{\left(1-\frac{N}{2}\right)(p-2)} \le C_2 \varepsilon, \qquad \text{for } |x| \ge R.$$

Using this we get that

$$\int_{|x|\geq R} |u_n(x)|^p dx \leq C_2\varepsilon \int_{|x|\geq R} |u_n(x)|^2 dx \leq C_2\varepsilon \|u_n\|^2_{H^1(\Omega)} \leq C_3\varepsilon. \tag{5.17}$$

We consider now the annulus $A := \{x \in \mathbb{R}^N \mid a < |x| < R\}$, since the sequence $\{u_n \mid n \in \mathbb{N}\}$ is bounded in $H^1_{rad}(A)$, recalling the compact embedding $H^1_{rad}(A) \subset\subset L^p(A)$ for any $p \in [1, +\infty)$ (see Theorem 5.2.1) we get that $u_n \to 0$ strongly in $L^p(A)$. It follows that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\int_{a<|x|<R} |u_n(x)|^p dx \leq \varepsilon. \tag{5.18}$$

Combining (5.17) and (5.18), we get that for every $n \geq n_0$

$$\int_\Omega |u_n(x)|^p dx \leq C_4\varepsilon$$

which proves that $u_n \to 0$ in $L^p(\Omega)$ $\forall p \in (2, +\infty)$.

Case $N = 2$. The continuous immersion $H^1(\Omega) \hookrightarrow L^p(\Omega)$ holds for every $p \in [2, +\infty)$ (see [2, Theorem 5.4 Case B]). For what concern compactness, we proceed as in the previous case using that for any $u \in H^1_{rad}(\Omega)$ it holds

$$|u(x)| \leq C|x|^{-\frac{1}{4}} \|u\|_{H^1(\Omega)}.$$

Actually, if $N = 2$ the embedding is compact for every $p \in [2, +\infty)$. □

Remark 13. Notice that if $N \geq 3$ the embedding of $H^1_{rad}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ is compact for all $p \in (2, 2^*)$ (see [5, Theorem 11.2] and also [175]).

Regarding the Sobolev space $H^1_{0,rad}(\Omega)$ we get the following results.

Corollary 5.2.3. *Let $N \geq 2$ and $\Omega = A_{a,b}$. For every $p \in [1, +\infty)$, the following compact immersion holds*

$$H^1_{0,rad}(\Omega) \hookrightarrow L^p(\Omega).$$

Moreover, if $N \geq 3$ there exists a positive constant $C = C(N, p, a, b)$ such that $\forall p \geq 1$

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \tag{5.19}$$

Proof. It follows similarly to the proof of Theorem 5.2.1. If $N = 2$, since $A_{a,b}$ is bounded, we have that for all $p \in [1, +\infty)$ the embedding

$$H^1_0(\Omega) \hookrightarrow L^p(\Omega)$$

is compact (see e.g. [2, Theorem 6.2] which holds also in $H^1_0(\Omega)$). If $N \geq 3$ and $U \subset \mathbb{R}^N$ is an arbitrary open set, then for every $u \in H^1_0(U)$ it holds

$$\|u\|_{L^{2^*}(U)} \leq C_N \|\nabla u\|_{L^2(U)} \tag{5.20}$$

(classical Gagliardo-Nirenberg-Sobolev inequality for $H^1_0(U)$). Since we are working with functions in $H^1_{0,rad}(\Omega)$, extending u by 0 outside Ω and proceeding as before we obtain the following estimate

$$|u(x)|^2 \leq C|x|^{2-N} \|u\|_{L^{2^*}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C|x|^{2-N} \|\nabla u\|^2_{L^2(\Omega)}$$

where in the last inequality we used (5.20). Estimate (5.19) follows immediately; for what concerns compactness we can replicate the arguments before. □

Remark 14. Notice that differently from Theorem 5.2.1, if $N \geq 3$ only the gradient of u appears in the right-hand-side of inequality (5.19). This fact will be useful in the next section, in order to relax the assumptions on the potential V for problems defined on annular domains.

Corollary 5.2.4. *Let $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$. For every $p \in [2, +\infty)$ we have the following continuous immersion*

$$H^1_{0,rad}(\Omega) \hookrightarrow L^p(\Omega).$$

Hence for every $p \geq 2$ there exists a positive constant $C = C(N, p, a)$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{H^1_{rad}(\Omega)}$$

moreover, if $N \geq 3$ for any $p \geq 2^*$ we have

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \tag{5.21}$$

The previous immersion is compact for every $p \in (2, +\infty)$.

Proof. Case $N \geq 3$. Let $u \in H^1_{0,rad}(\Omega)$, by the same argument as before and using estimate (5.20), we get that

$$|u(x)| \leq C|x|^{1-\frac{N}{2}}\|\nabla u\|_{L^2(\Omega)}$$

where $C = C(N, a)$. If $p > 2^*$, $N + p - \frac{Np}{2} < 0$ and so

$$\int_\Omega |u(x)|^p dx \leq C_1\|\nabla u\|^p_{L^2(\Omega)} \int_a^{+\infty} r^{N-1+p-\frac{Np}{2}} dr = C_2\|\nabla u\|^p_{L^2(\Omega)},$$

which proves estimate (5.21). By the previous inequality and the classical Sobolev embedding (see [2, Theorem 5.4]) which holds for $p \in [2, 2^*]$, we can conclude that if $u \in H^1_{0,rad}(\Omega)$

$$\|u\|_{L^p(\Omega)} \leq C_{N,p,a}\|u\|_{H^1(\Omega)}, \qquad \forall p \in [2, +\infty)$$

and hence the continuous embedding follows.

Case $N = 2$. The continuous immersion $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ holds for every $p \in [2, +\infty)$ (see [2, Theorem 5.4 Case B]).

In both cases, the proof of compactness follows the same arguments as in the proof of Theorem 5.2.2. □

5.3 Existence of a constrained minimizer

We construct a radial solution to the Neumann boundary value problem (5.8) as minimizer of $Q(u)$ on the constrained set M_α (as defined in (5.5) and (5.6)).

Proposition 5.3.1. *Let $N \geq 2$, $\alpha \in (0, N)$ fixed, $\Omega = A_{a,b}$ and $V(|x|)$ be a continuous radial function on Ω such that $\inf_{x\in\Omega} V(x) > 0$. Then, for every $p \in [1, +\infty)$, there exists $u_\alpha \in M_\alpha$ non-negative function such that*

$$Q(u_\alpha) = J_\alpha := \inf_{u\in M_\alpha} Q(u).$$

Proof. Let $J_\alpha := \inf_{u\in M_\alpha} Q(u) \geq 0$ and $\{u_n \,|\, n\in\mathbb{N}\} \in M_\alpha$ be a minimizing sequence for J_α. We can assume that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$Q(u_n) < J_\alpha + 1$$

from which, using that $\inf_{x\in\Omega} V(x) > 0$, we deduce that

$$\int_\Omega |\nabla u_n|^2 dx < J_\alpha + 1, \qquad \int_\Omega u_n^2\, dx \leq \frac{Q(u_n)}{\inf V} < \frac{J_\alpha+1}{\inf V}.$$

This proves that the sequence $\{u_n \,|\, n\in\mathbb{N}\}$ is bounded in $H^1_{rad}(\Omega)$, hence there exists $u_\alpha \in H^1_{rad}(\Omega)$ such that $u_n \rightharpoonup u_\alpha$ weakly in $H^1_{rad}(\Omega)$ (up to subsequences) as $n\to\infty$ and almost everywhere in Ω. Notice that the function u_α can be chosen non-negative, possibly taking $|u_\alpha|$. Now, in order to conclude, we have to verify that $u_\alpha \in M_\alpha$. Using the Hardy-Littlewood-Sobolev inequality (3.15) we get that

$$\begin{aligned}
&\left|\int_\Omega (I_\alpha * |u_n|^p)|u_n|^p - \int_\Omega (I_\alpha * |u_\alpha|^p)|u_\alpha|^p\right| \\
&= C_{N,\alpha}\left|\int_\Omega\int_\Omega \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}dx\,dy - \int_\Omega\int_\Omega \frac{|u_\alpha(x)|^p|u_\alpha(y)|^p}{|x-y|^{N-\alpha}}dx\,dy\right| \\
&= C_{N,\alpha}\left|\int_\Omega\int_\Omega \frac{(|u_n(x)|^p - |u_\alpha(x)|^p)(|u_n(y)|^p + |u_\alpha(y)|^p)}{|x-y|^{N-\alpha}}dx\,dy\right| \\
&\leq C\Big\|\,|u_n|^p - |u_\alpha|^p\Big\|_{L^{\frac{2N}{N+\alpha}}(\Omega)}\Big\|\,|u_n|^p + |u_\alpha|^p\Big\|_{L^{\frac{2N}{N+\alpha}}(\Omega)} \\
&\leq C\Big\||u_n|^p - |u_\alpha|^p\Big\|_{L^{\frac{2N}{N+\alpha}}(\Omega)}\left(\|u_n\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)} + \|u_\alpha\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)}\right).
\end{aligned}$$

Using estimate (5.12) and the fact that $\|u_n\|_{H^1(\Omega)} \leq C$, we obtain that for every $p \in [1,+\infty)$

$$\|u_n\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)} + \|u_\alpha\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)} \leq C$$

uniformly in n. If $p=1$ we are done, since using Theorem 5.2.1 we have that $\|u_n - u_\alpha\|_{\frac{2N}{N+\alpha}} \to 0$. On the other hand if $p>1$, in order to deal with the term $\||u_n|^p - |u_\alpha|^p\|_{L^{\frac{2N}{N+\alpha}}(\Omega)}$, we take advantage of the following estimate. Using convexity of the function x^p, for $x>0$ and $p>1$, we get that

$$\Big||u_n|^p - |u_\alpha|^p\Big| \leq p\Big||u_n| - |u_\alpha|\Big|\left(|u_n|^{p-1} + |u_\alpha|^{p-1}\right)$$

and so using Hölder's inequality

$$\begin{aligned}
&\Big\||u_n|^p - |u_\alpha|^p\Big\|_{L^{\frac{2N}{N+\alpha}}(\Omega)} \\
&\leq C\Big\||u_n| - |u_\alpha|\Big\|_{L^{\frac{2Nr}{N+\alpha}}(\Omega)}\Big\||u_n|^{p-1} + |u_\alpha|^{p-1}\Big\|_{L^{\frac{2Nr'}{N+\alpha}}(\Omega)} \\
&\leq C\Big\||u_n| - |u_\alpha|\Big\|_{L^{\frac{2Nr}{N+\alpha}}(\Omega)}\left(\|u_n\|^{p-1}_{L^{\frac{2Nr'(p-1)}{N+\alpha}}(\Omega)} + \|u_\alpha\|^{p-1}_{L^{\frac{2Nr'(p-1)}{N+\alpha}}(\Omega)}\right) \qquad (5.22)
\end{aligned}$$

where r and r' are conjugate exponents. Choosing r such that

$$\frac{2Nr'(p-1)}{N+\alpha} \geq 1,$$

we can use estimate (5.12) again, and since $\{u_n\}$ is bounded in $H^1_{rad}(\Omega)$, we get that

$$\|u_n\|^{p-1}_{L^{\frac{2Nr'(p-1)}{N+\alpha}}(\Omega)} + \|u_\alpha\|^{p-1}_{L^{\frac{2Nr'(p-1)}{N+\alpha}}(\Omega)} \leq C$$

uniformly in n. Finally, since the embedding is compact (refer to Theorem 5.2.1) we have that up to subsequences

$$\|u_n - u_\alpha\|_{L^{\frac{2Nr}{N+\alpha}}(\Omega)} \to 0.$$

This proves that

$$\Big\| |u_n|^p - |u_\alpha|^p \Big\|_{L^{\frac{2N}{N+\alpha}}(\Omega)} \to 0$$

and consequently that

$$\int_\Omega (I_\alpha * |u_\alpha|^p)|u_\alpha|^p dx = 1.$$

We can conclude that $u_\alpha \in M_\alpha$. □

Proposition 5.3.2. *Under the assumptions of Proposition 5.3.1, let* $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$. *Then, for every* $p \in \left(\frac{N+\alpha}{N}, +\infty\right)$ *there exists* $u_\alpha \in M_\alpha$ *non-negative function such that*

$$Q(u_\alpha) = J_\alpha := \inf_{u \in M_\alpha} Q(u).$$

Proof. We proceed in the same way as the previous proof, but since $\Omega = \{x \in \mathbb{R}^N \mid |x| > a > 0\}$, we use the following Sobolev Embedding (see Theorem 5.2.2): for every $q \in [2, +\infty)$ we have the continuous immersion

$$H^1_{rad}(\Omega) \hookrightarrow L^q(\Omega)$$

which is also compact for $q \in (2, +\infty)$. Since $\{u_n\}$ is bounded in $H^1_{rad}(\Omega)$, it follows that for every $p \in \left(\frac{N+\alpha}{N}, +\infty\right)$

$$\|u_n\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)} + \|u_\alpha\|^p_{L^{\frac{2Np}{N+\alpha}}(\Omega)} \leq C$$

uniformly in n. Recalling estimate (5.22), in this case we have to require that

$$\frac{2Nr'(p-1)}{N+\alpha} \geq 2 \qquad \text{and} \qquad \frac{2Nr}{N+\alpha} > 2.$$

If $p \geq 2$, taking $r = r' = 2$, the two previous conditions are satisfied. If $\frac{N+\alpha}{N} < p < 2$, we set $r' = \frac{N+\alpha}{N(p-1)}$ and consequently $r = \frac{N+\alpha}{2N+\alpha-Np}$, in this way

$$\frac{2Nr'(p-1)}{N+\alpha} = 2 \qquad \text{and} \qquad \frac{2Nr}{N+\alpha} = \frac{2N}{2N+\alpha-Np} > 2$$

which concludes the proof. □

Proof of Theorem 5.1.1. Let $u_\alpha \in M_\alpha$ be a minimizer for J_α (see Proposition 5.3.1 and Proposition 5.3.2). By classical arguments, u_α solves

$$\begin{cases} -\Delta u_\alpha + V(x)u_\alpha = \mu_\alpha (I_\alpha * |u_\alpha|^p)|u_\alpha|^{p-2}u_\alpha & \text{in } \Omega \\ u_\alpha > 0 & \text{in } \Omega \\ \frac{\partial u_\alpha}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where μ_α is a Lagrange multiplier, and $u_\alpha > 0$ by the Strong Maximum Principle. Therefore $|\mu_\alpha|^{\frac{1}{2p-2}} u_\alpha$ provides a solution to (5.8). □

Proof of Theorem 5.1.2. When dealing with homogeneous Dirichlet boundary conditions we will investigate existence of minima of the functional Q over the constrained set

$$M_{\alpha,0} := \left\{ u \in H^1_{0,rad}(\Omega) \;\middle|\; \int_\Omega (I_\alpha * |u|^p)\,|u|^p = 1 \right\}.$$

Exploiting Corollary 5.2.3 and Corollary 5.2.4, one can prove that there exists $u_{\alpha,0}$ non-negative function which achieves the infimum

$$J_{\alpha,0} := \inf\{Q(u) \,|\, u \in M_{\alpha,0}\}.$$

Hence, the results of Proposition 5.3.1 and Proposition 5.3.2 hold also for Dirichlet boundary conditions, and we can conclude by rescaling. Let us assume now that $V \geq 0$, $N \geq 3$ and $\Omega = A_{a,b}$. If $\{u_n\}$ is a minimizing sequence for $J_{\alpha,0}$, exploiting estimate (5.19) we get that

$$\int_{A_{a,b}} u_n^2\,dx \leq C \int_{A_{a,b}} |\nabla u_n|^2 dx \leq C\,Q(u_n) < C(J_{\alpha,0}+1).$$

Hence, if $N \geq 3$ we obtain existence of solutions to the problem (5.9) on $\Omega = A_{a,b}$ also in the more general case $V(|x|) \geq 0$. □

5.4 Nonexistence result

First, we prove a suitable version of the celebrated Pohozaev identity (refer to the seminal papers [168, 172] and also to [6, 89] for the Choquard case).

Theorem 5.4.1. *Let Ω be a smooth domain in $\mathbb{R}^N$. Assume that $u \in H^1_0(\Omega) \cap W^{2,2}(\Omega) \cap W^{1,\frac{2Np}{N+\alpha}}(\Omega)$ is a solution to*

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)u^{p-2}u, \quad \textit{in } \Omega \tag{5.23}$$

such that $u^2 V \in L^1(\Omega)$ and $u^2 \nabla V \cdot x \in L^1(\Omega)$. Then the following Pohozaev-type identity holds

$$\left(2 - N + \frac{\alpha+N}{p}\right)\int_\Omega |\nabla u|^2 dx - \left(N - \frac{\alpha+N}{p}\right)\int_\Omega V u^2\,dx - \int_\Omega u^2 \nabla V \cdot x\,dx = I_{\partial\Omega}, \tag{5.24}$$

where ν is the exterior unit normal at $\partial\Omega$ and $u_\nu := \frac{\partial u}{\partial \nu}$ and $I_{\partial\Omega} = \int_{\partial\Omega} u_\nu^2\,(x\cdot\nu)\,d\sigma$.

Proof. Since u solves (5.23), multiplying each term by $\nabla u \cdot x$ and integrating over Ω, we get

$$-\int_\Omega \Delta u\, \nabla u \cdot x\,dx + \int_\Omega V u\, \nabla u \cdot x\,dx = \int_\Omega (I_\alpha * |u|^p)|u|^{p-2}u\, \nabla u \cdot x\,dx.$$

We consider each term of the previous equality separately. Integrating by parts the first term, we have

$$-\int_\Omega \Delta u\, \nabla u\cdot x\, dx = \int_\Omega \nabla u\cdot\nabla(\nabla u\cdot x)dx - \int_{\partial\Omega} (\nabla u\cdot \nu)(\nabla u\cdot x)\, d\sigma$$
$$= \int_\Omega |\nabla u|^2 + \int_\Omega \sum_{i,j} u_{x_i} u_{x_i\, x_j} x_j - \int_{\partial\Omega} (\nabla u\cdot\nu)(\nabla u\cdot x)\, d\sigma. \quad (5.25)$$

Notice that integrating by parts

$$\int_\Omega \sum_{i,j} (u_{x_i} x_j) u_{x_i x_j} = \int_{\partial\Omega} |\nabla u|^2\, x\cdot\nu\, d\sigma - \int_\Omega \nabla u\cdot\nabla(\nabla u\cdot x) + (1-N)\int_\Omega |\nabla u|^2. \quad (5.26)$$

Putting (5.26) into (5.25), and integrating by parts again, we find

$$-\int_\Omega \Delta u\, \nabla u\cdot x\, dx = (2-N)\int_\Omega |\nabla u|^2\, dx + \int_\Omega \Delta u\, \nabla u\cdot x\, dx +$$
$$+\int_{\partial\Omega} |\nabla u|^2\, x\cdot\nu\, d\sigma - 2\int_{\partial\Omega} (\nabla u\cdot\nu)(\nabla u\cdot x)\, d\sigma.$$

Since $u=0$ on $\partial\Omega$, one has that $\nabla u(x) = u_\nu \nu$ where $u_\nu = \frac{\partial u}{\partial \nu}$, hence we get

$$-\int_\Omega \Delta u\, \nabla u\cdot x\, dx = \frac{2-N}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2}\int_{\partial\Omega} u_\nu^2\, x\cdot\nu\, d\sigma. \quad (5.27)$$

Concerning the second term, integrating by parts we have

$$\int_\Omega V u\, \nabla u\cdot x\, dx = \int_{\partial\Omega} V u^2\, x\cdot\nu\, d\sigma - \int_\Omega V u\, \nabla u\cdot x\, dx - N\int_\Omega V u^2\, dx - \int_\Omega u^2 \nabla V\cdot x\, dx$$

and hence

$$\int_\Omega V u\, \nabla u\cdot x\, dx = \frac{1}{2}\int_{\partial\Omega} V u^2\, x\cdot\nu\, d\sigma - \frac{N}{2}\int_\Omega V u^2\, dx - \frac{1}{2}\int_\Omega u^2\nabla V\cdot x\, dx. \quad (5.28)$$

As for the Riesz term we get

$$\int_\Omega (I_\alpha * |u|^p)|u|^{p-2} u\, \nabla u\cdot x\, dx = c\int_\Omega\int_\Omega \frac{|u(y)|^p\, |u(x)|^{p-2} u(x)\nabla u\cdot x}{|x-y|^{N-\alpha}} dy\, dx =$$
$$= c\int_\Omega\int_{\partial\Omega} \frac{|u(y)|^p\, |u(x)|^p}{|x-y|^{N-\alpha}}(x\cdot\nu)\, d\sigma(x)\, dy - c\int_\Omega\int_\Omega |u(y)|^p\, u(x)\, \mathrm{div}_x\left(\frac{u(x)|u(x)|^{p-2} x}{|x-y|^{N-\alpha}}\right) dx\, dy, \quad (5.29)$$

where $c = c(N,\alpha)$ is the constant in the definition of the Riesz potential. We have that

$$
\begin{aligned}
&\int_\Omega\int_\Omega |u(y)|^p u(x)\,\mathrm{div}_x\left(\frac{u(x)|u(x)|^{p-2}}{|x-y|^{N-\alpha}}x\right)dx\,dy\\
&= \int_\Omega\int_\Omega \frac{|u(y)|^p\,u(x)|u(x)|^{p-2}\nabla u\cdot x}{|x-y|^{N-\alpha}}dx\,dy + (p-2)\int_\Omega\int_\Omega \frac{|u(y)|^p\,u(x)|u(x)|^{p-2}\nabla u\cdot x}{|x-y|^{N-\alpha}}dx\,dy\\
&\quad + (\alpha-N)\int_\Omega\int_\Omega \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}\frac{(x-y)\cdot x}{|x-y|^2}dx\,dy + N\int_\Omega\int_\Omega \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}dx\,dy\\
&= (p-1)\int_\Omega\int_\Omega \frac{|u(y)|^p\,u(x)|u(x)|^{p-2}\nabla u\cdot x}{|x-y|^{N-\alpha}}dx\,dy + \frac{\alpha+N}{2}\int_\Omega\int_\Omega \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}dx\,dy\\
&\quad + \frac{\alpha-N}{2}\int_\Omega\int_\Omega \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy
\end{aligned}
\tag{5.30}
$$

where we used that $\frac{x\cdot(x-y)}{|x-y|^2} = \frac{1}{2} + \frac{(x+y)\cdot(x-y)}{2|x-y|^2}$ and moreover we observe that by symmetry

$$
\int_\Omega\int_\Omega \frac{|u(x)|^p\,|u(y)|^p}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy = 0.
$$

Using (5.30) in (5.29) we finally get

$$
\begin{aligned}
&\int_\Omega (I_\alpha * |u|^p)|u|^{p-2}u\,\nabla u\cdot x\,dx\\
&\quad = \frac{c}{p}\int_\Omega\int_{\partial\Omega} \frac{|u(y)|^p\,|u(x)|^p}{|x-y|^{N-\alpha}}(x\cdot\nu)\,d\sigma(x)\,dy - \frac{\alpha+N}{2p}\int_\Omega (I_\alpha * |u|^p)|u|^p dx.
\end{aligned}
\tag{5.31}
$$

Summing up (5.27), (5.28) and (5.31), and using the fact that $u = 0$ on $\partial\Omega$ we obtain the following identity

$$
\begin{aligned}
\frac{2-N}{2}\int_\Omega |\nabla u|^2\,dx - \frac{N}{2}\int_\Omega Vu^2\,dx - \frac{1}{2}\int_\Omega u^2\nabla V\cdot x\,dx&\\
+\frac{\alpha+N}{2p}\int_\Omega (I_\alpha * |u|^p)|u|^p\,dx = \frac{1}{2}\int_{\partial\Omega} u_\nu^2\,x\cdot\nu\,d\sigma.&
\end{aligned}
\tag{5.32}
$$

Finally, testing equation (5.23) with u, we infer that

$$
\int_\Omega (I_\alpha * |u|^p)|u|^p\,dx = \int_\Omega |\nabla u|^2 dx + \int_\Omega Vu^2\,dx
$$

using this relation in (5.32) we conclude the proof of the Pohozaev-type identity (5.24). □

Remark 15. Notice that the previous integrability assumptions on u can be weakened for suitable values of the nonlinearity p, indeed one can adapt the proof of regularity for solutions to the Choquard equation in the whole space $\mathbb{R}^N$ to other kinds of domains (see [152, Proposition 4.1] and also [154, Theorem 2]).

As a consequence, using the Pohozaev identity we can prove that if the domain Ω is strictly star-shaped, the value $p = \frac{N+\alpha}{N-2}$ is critical from the point of view of existence of non-trivial solution for the Dirichlet problem (5.10). Notice that in the literature also $p = \frac{\alpha+N}{N}$ is a critical value for the Choquard equation in the whole space $\mathbb{R}^N$ and for Choquard boundary value problems defined in exterior domains of the form $\mathbb{R}^N \setminus \overline{B_a(0)}$ (refer to Theorem 5.1.1 and Theorem 5.1.2).

Proof of Theorem 5.1.3. Since $u \in H_0^1(\Omega) \cap W^{2,2}(\Omega) \cap W^{1,\frac{2Np}{N+\alpha}}(\Omega)$ solves (5.10) and we assumed that $u^2V,\ u^2\nabla V \cdot x \in L^1(\Omega)$, we have the following Pohozaev identity

$$\left(N-2-\frac{\alpha+N}{p}\right)\int_\Omega |\nabla u|^2 dx + \left(N-\frac{\alpha+N}{p}\right)\int_\Omega V u^2\, dx + \int_\Omega u^2 \nabla V \cdot x\, dx + \int_{\partial\Omega} u_\nu^2\,(x\cdot\nu)\, d\sigma = 0$$

Since Ω is strictly star-shaped with respect to $0 \in \mathbb{R}^N$, we have that $x \cdot \nu > 0$ on $\partial\Omega$, moreover by assumptions $\int_\Omega u^2 \nabla V \cdot x\, dx \geq 0$. So if $V \geq 0$ and $p > \frac{N+\alpha}{N-2}$, u must be identically equal to 0, and in the case $p = \frac{N+\alpha}{N-2}$ the same holds if $V > 0$. □

5.5 Limiting problem

In this section we will always assume that $\Omega = A_{a,b}$, $N \geq 2$, p is a fixed real value in $[1, +\infty)$ and $V(|x|)$ is a continuous radial function on Ω such that $\inf_{x\in\Omega} V(x) > 0$. Notice that if $0 \leq \alpha_2 < \alpha_1 < N$ for every couple of points $x, y \in B_b(0)$ it holds

$$\frac{1}{|x-y|^{N-\alpha_1}} \leq (\max\{1, 2b\})^{\alpha_1-\alpha_2} \frac{1}{|x-y|^{N-\alpha_2}},$$

from which using that $C_{N,\alpha_1} < C_{N,\alpha_2}$ (as defined in (5.2)) it follows that

$$\int_\Omega (I_{\alpha_1} * |f|)|f|\, dx \leq C \int_\Omega (I_{\alpha_2} * |f|)|f|\, dx \tag{5.33}$$

where $C = (\max\{1, 2b\})^{\alpha_1}$. From Proposition 5.3.1 we have that for every fixed $\alpha \in (0, N)$, there exists a non-negative function $u_\alpha \in M_\alpha$ which minimizes $Q(u)$, that is $J_\alpha = Q(u_\alpha)$.

We recall the following classical result (see [122, Theorem D]).

Theorem 5.5.1 (Riesz kernel as an approximation of the identity)**.** *Let $f \in L^p(\mathbb{R}^N)$ for $p \in [1, +\infty)$. If the Riesz potential $I_\alpha * f$ is well-defined, we have that*

$$I_\alpha * f(x) \to f(x), \quad \text{as } \alpha \to 0^+$$

at each Lebesgue point of f.

Following some arguments in [107], we prove weak convergence of minimizers when $\alpha \to 0^+$.

Proposition 5.5.2. *Let $\{\alpha_n\}$ be a sequence of real values in $(0, N)$ such that $\alpha_n \to 0^+$ as $n \to +\infty$ and $u_{\alpha_n} \in M_{\alpha_n}$ be a sequence of minimizers to J_{α_n} respectively. Then, there exists $u_0 \in H^1_{rad}(\Omega)$ such that, up to subsequences*

$$u_{\alpha_n} \rightharpoonup u_0 \quad \text{in } H^1_{rad}(\Omega) \qquad u_{\alpha_n} \to u_0 \quad \text{in } L^q(\Omega), \quad \forall q \in [1, +\infty)$$

and

$$\int_\Omega u_0^{2p}(x)\,dx = 1.$$

Proof. First we prove that the sequence $\{u_{\alpha_n} \,|\, n \in \mathbb{N}\}$ is bounded in $H^1_{rad}(\Omega)$. Let us consider a non-negative test function $\eta \in M_{\alpha_1}$ and for every $\alpha \in (0, \alpha_1)$ let us define

$$\eta_\alpha := \frac{\eta}{\left(\int_\Omega (I_\alpha * \eta^p)\eta^p\,dx\right)^{\frac{1}{2p}}}.$$

Denoting by $a = \left(\int_\Omega (I_\alpha * \eta^p)\eta^p\,dx\right)^{\frac{1}{2p}}$ we observe that

$$\int_\Omega (I_\alpha * \eta_\alpha^p)\eta_\alpha^p\,dx = \frac{1}{a^{2p}}\int_\Omega (I_\alpha * \eta^p)\eta^p\,dx = 1$$

hence $\eta_\alpha \in M_\alpha$. For every $\alpha \in (0, \alpha_1)$, using (5.33), we get

$$Q(\eta_\alpha) = \frac{Q(\eta)}{a^2} = \frac{Q(\eta)}{\left(\int_\Omega (I_\alpha * \eta^p)\eta^p\,dx\right)^{\frac{1}{p}}} \leq \frac{Q(\eta)}{\left(\frac{1}{C}\int_\Omega (I_{\alpha_1} * \eta^p)\eta^p\,dx\right)^{\frac{1}{p}}} = C_1\,Q(\eta)$$

where $C_1 = C_1(b, \alpha_1, p)$ and in the last equality we exploited the fact that $\eta \in M_{\alpha_1}$. Since $\eta_\alpha \in M_\alpha$, it follows that $J_\alpha \leq C_1\,Q(\eta)$, therefore $J_\alpha \leq C_1$ for every $\alpha \in (0, \alpha_1)$. Hence if u_{α_n} is a minimizer associated to α_n, we have that

$$\int_\Omega |\nabla u_{\alpha_n}|^2 \leq C_1 \qquad \text{and} \qquad \int_\Omega |u_{\alpha_n}|^2 \leq C_1$$

uniformly in n. This proves that the sequence $\{u_{\alpha_n} \,|\, n \in \mathbb{N}\}$ is bounded in $H^1_{rad}(\Omega)$ as $\alpha_n \to 0^+$, so there exists $u_0 \in H^1_{rad}(\Omega)$ such that, up to subsequences $u_{\alpha_n} \rightharpoonup u_0$ in $H^1_{rad}(\Omega)$. We can conclude using the compact Sobolev Embedding for radial functions (refer to Theorem 5.2.1).
Now we prove that $\int_\Omega u_0^{2p}(x)\,dx \geq 1$. We observe that if $u_{\alpha_1} \in M_{\alpha_1}$, using (5.33) it holds

$$1 = \int_\Omega (I_{\alpha_1} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p\,dx \leq \lim_{\alpha_2 \to 0^+} C\int_\Omega (I_{\alpha_2} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p\,dx. \tag{5.34}$$

Passing to the limit as $\alpha_1 \to 0^+$ in (5.34), we get that

$$1 = \lim_{\alpha_1 \to 0^+}\int_\Omega (I_{\alpha_1} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p\,dx \leq \lim_{\alpha_1 \to 0^+}\lim_{\alpha_2 \to 0^+} C\int_\Omega (I_{\alpha_2} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p\,dx,$$

and using monotone convergence and Theorem 5.5.1, we obtain that

$$1 \leq \lim_{\alpha_1 \to 0^+} C\int_\Omega \lim_{\alpha_2 \to 0^+}(I_{\alpha_2} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p\,dx = \lim_{\alpha_1 \to 0^+} C\int_\Omega |u_{\alpha_1}(x)|^{2p}\,dx.$$

Since $u_{\alpha_n} \to u_0$ in $L^q(\Omega)$ for every $q \in [1, +\infty)$ and $C = (\max\{1, 2b\})^{\alpha_1} \to 1$ as $\alpha_1 \to 0$, we can conclude that

$$1 \leq \int_\Omega u_0(x)^{2p}\,dx.$$

As for the inverse inequality, we observe that again by (5.33), for $\alpha_2 > \alpha_1$

$$\lim_{\alpha_1 \to 0^+} \int_\Omega (I_{\alpha_2} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p dx \le \lim_{\alpha_1 \to 0^+} C \int_\Omega (I_{\alpha_1} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p dx = C$$

where $C = (\max\{1, 2b\})^{\alpha_2}$. Passing to the limit as $\alpha_2 \to 0^+$ we get

$$1 \ge \lim_{\alpha_2 \to 0^+} \lim_{\alpha_1 \to 0^+} \int_\Omega (I_{\alpha_2} * |u_{\alpha_1}|^p)|u_{\alpha_1}|^p dx = \lim_{\alpha_2 \to 0^+} \int_\Omega (I_{\alpha_2} * |u_0|^p)|u_0|^p dx$$

where to get the last equality, we proceed as in the proof of Proposition 5.3.1. Finally, using again the Monotone Convergence Theorem and Theorem 5.5.1, we obtain that

$$1 \ge \int_\Omega u_0(x)^{2p}\, dx$$

which concludes the proof. □

Now we prove that every non-negative function belonging to M_0 can be seen as the limit of a suitable approximating sequence.

Proposition 5.5.3. *Let $u \in M_0$ be a non-negative function. Then for every $\alpha \in (0, N)$ there exists $w_\alpha \in M_\alpha$ such that $w_\alpha \to u$ in $H^1_{rad}(\Omega)$ as $\alpha \to 0^+$.*

Proof. For every $\alpha \in (0, N)$, let us consider the function

$$w_{\sigma_\alpha} := \sigma_\alpha u, \qquad \text{where} \quad \sigma_\alpha = \left(\int_\Omega (I_\alpha * |u|^p)|u|^p \right)^{-\frac{1}{2p}}.$$

It follows immediately that

$$\int_\Omega (I_\alpha * |w_{\sigma_\alpha}|^p)|w_{\sigma_\alpha}|^p dx = \sigma_\alpha^{2p} \int_\Omega (I_\alpha * |u|^p)|u|^p dx = 1,$$

hence w_{σ_α} belongs to M_α. In order to conclude we observe that

$$\lim_{\alpha \to 0^+} \sigma_\alpha = \lim_{\alpha \to 0^+} \left(\int_\Omega (I_\alpha * |u|^p)|u|^p dx \right)^{-\frac{1}{2p}} = \left(\int_\Omega |u(x)|^{2p}\, dx \right)^{-\frac{1}{2p}} = 1$$

using in the last equality that $u \in M_0$. □

In order to conclude we have to show that the constrained variational problems J_α converges to the limit problem J_0 as $\alpha \to 0^+$.

Proposition 5.5.4. *Let us consider a sequence of real values $\alpha \in (0, N)$ such that $\alpha \to 0^+$ and let $\{u_\alpha\}$ be a sequence of minimizers of $Q(u)$ in M_α respectively. Then, up to subsequences, we have that*

$$u_\alpha \to u_0 \quad \text{in } H^1_{rad}(\Omega)$$

and

$$J_\alpha = \inf_{u \in M_\alpha} Q(u) = Q(u_\alpha) \xrightarrow[\alpha \to 0^+]{} J_0 = \inf_{u \in M_0} Q(u) = Q(u_0).$$

Proof. From Proposition 5.5.2 there exists $u_0 \in M_0$ such that $u_\alpha \rightharpoonup u_0$ in $H^1_{rad}(\Omega)$. So, by lower semicontinuity with respect to weak convergence, we get that

$$J_0 \le Q(u_0) \le \liminf_{\alpha\to 0^+} Q(u_\alpha) = \liminf_{\alpha\to 0^+} J_\alpha \tag{5.35}$$

where we used also that $J_0 := \inf_{u\in M_0} Q(u)$ and that every u_α is a minimizer for J_α. Conversely, let $u \in M_0$ be a non-negative function, by Proposition 5.5.3 there exists a sequence of function $w_\alpha \in M_\alpha$ such that $w_\alpha \to u$ in $H^1_{rad}(\Omega)$ as $\alpha \to 0^+$, so it holds

$$Q(u) = \lim_{\alpha\to 0^+} Q(w_\alpha) \ge \limsup_{\alpha\to 0^+} J_\alpha. \tag{5.36}$$

Notice that in the minimization problem $J_0 = \inf_{u\in M_0} Q(u)$, we can equivalently minimize over the constrained set $\{u \in M_0 \mid u \ge 0\}$ (possibly taking $|u|$ instead of u). Passing to the infimum in (5.36), we get that

$$J_0 = \inf_{\{u\in M_0 | u\ge 0\}} Q(u) \ge \limsup_{\alpha\to 0^+} J_\alpha$$

and hence from (5.35) we finally obtain

$$J_0 \ge \limsup_{\alpha\to 0^+} J_\alpha \ge \liminf_{\alpha\to 0^+} J_\alpha \ge J_0,$$

which concludes the proof. □

Proof of Theorem 5.1.4. Since u_0 is a minimizer for J_0, we have that

$$\begin{cases} -\Delta u_0 + V(x)u_0 = \mu_0 u_0^{2p-1} & \text{in } \Omega \\ u_0 > 0 & \text{in } \Omega \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where μ_0 is a Lagrange multiplier. A suitable multiple of u_0 (namely $|\mu_0|^{\frac{1}{2p-2}} u_0$) provides a solution to (5.11). □

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