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Research article

Spectral stability of the curl curl operator via uniform Gaffney inequalities on perturbed electromagnetic cavities[†]

Pier Domenico Lamberti[1,*] **and Michele Zaccaron**[2,*]

[1] Dipartimento di Tecnica e Gestione dei Sistemi Industriali (DTG), University of Padova, Stradella S. Nicola 3, 36100 Vicenza, Italy

[2] Dipartimento di Matematica 'Tullio Levi-Civita', University of Padova, Via Trieste 63, 35121 Padova, Italy

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* **Correspondence:** Email: lamberti@math.unipd.it, zaccaron@math.unipd.it.

Abstract: We prove spectral stability results for the *curlcurl* operator subject to electric boundary conditions on a cavity upon boundary perturbations. The cavities are assumed to be sufficiently smooth but we impose weak restrictions on the strength of the perturbations. The methods are of variational type and are based on two main ingredients: the construction of suitable Piola-type transformations between domains and the proof of uniform Gaffney inequalities obtained by means of uniform a priori H^2-estimates for the Poisson problem of the Dirichlet Laplacian. The uniform a priori estimates are proved by using the results of V. Maz'ya and T. Shaposhnikova based on Sobolev multipliers. Connections to boundary homogenization problems are also indicated.

Keywords: Maxwell's equations; spectral stability; cavities; shape sensitivity; boundary homogenization

1. Introduction

In this paper we study the spectral stability of the *curlcurl* operator on an electromagnetic cavity Ω in $\mathbb{R}^3$ upon perturbation of the shape of Ω. The cavity Ω is a bounded connected open set (shortly, a bounded domain), the boundary of which is enough regular to guarantee the validity of the celebrated

Gaffney inequality, that is

$$\|u\|_{H^1(\Omega)^3} \le C\left(\|u\|_{L^2(\Omega)^3} + \left\|\operatorname{curl} u\right\|_{L^2(\Omega)^3} + \left\|\operatorname{div} u\right\|_{L^2(\Omega)}\right) \tag{1.1}$$

for all vector fields $u \in L^2(\Omega)^3$ with distributional $\operatorname{curl} u \in L^2(\Omega)^3$ and $\operatorname{div} u \in L^2(\Omega)$, and satisfying the so-called electric boundary conditions

$$\nu \times u = 0, \text{ on } \partial\Omega .$$

Here ν denotes the unit outer normal to $\partial\Omega$ and $H^1(\Omega)$ is the standard Sobolev space of functions in $L^2(\Omega)$ with first order weak derivatives in $L^2(\Omega)$. It is classical that the Gaffney inequality holds for domains Ω with boundaries of class C^2, but the regularity can be relaxed in order to include boundaries of class $C^{1,\beta}$ with $\beta > 1/2$, see [22, 38].

The eigenvalue problem under consideration is

$$\begin{cases} \operatorname{curl}\,\operatorname{curl} u = \lambda u, & \text{in } \Omega, \\ \nu \times u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and is immediately derived from the time-harmonic Maxwell's equations

$$\operatorname{curl} E - \mathrm{i}\,\omega\, H = 0\,, \ \operatorname{curl} H + \mathrm{i}\,\omega\, E = 0\,, \tag{1.3}$$

where E, H denote the spatial parts of the electric and the magnetic field respectively and $\omega > 0$ is the angular frequency. Indeed, taking the curl in the first equation of (1.3) and setting $\lambda = \omega^2$, one immediately obtains problem (1.2). Note that here the medium filling Ω is homogeneous and isotropic and for simplicity the corresponding electric permittivity ε and magnetic permeability μ have been normalized by setting $\varepsilon = \mu = 1$. The boundary conditions are those of a perfect conductor, namely $\nu \times E = 0$ and $H \cdot \nu = 0$. Thus, the vector field u in (1.2) plays the role of the electric field E (similarly, the magnetic field would satisfy the same equation but with the other boundary conditions $u \cdot \nu = 0$ and $\nu \times \operatorname{curl} u = 0$).

We observe that the study of electromagnetic cavities is quite important in applications, for example in designing cavity resonators or shielding structures for electronic circuits, see e.g., [24, Chp. 10]. We also refer to [12, 17, 30, 35, 36, 39] for details and references concerning the mathematical theory of electromagnetism. See also [13–16, 31, 32, 37, 41–44].

The spectrum of problem (1.2) is discrete and consists of a divergent sequence of positive eigenvalues $\lambda_n[\Omega]$ of finite multiplicity.

In this paper, we study the dependence of $\lambda_n[\Omega]$ and the corresponding eigenfunctions upon variation of Ω. It seems to us that very little is known in the literature. The case of domain perturbations of the form $\Phi(\Omega)$ where Φ is a regular diffeomorphism from Ω to $\Phi(\Omega)$ are considered in [29] and [33] where differentiability results and Hadamard type formulas for shape derivatives are proved. We also quote the pioneering work [28] where the Hadamard formula was found on the base of heuristic computations. We note that shape derivatives are used in inverse electromagnetic scattering in [25–27].

The aim of the present paper is to prove spectral stability results under less stringent assumptions on the families of domain perturbations. For this purpose, we adopt the approach of [3] further developed in [2, 18–20].

Given a fixed domain Ω, we consider a family of domains Ω_ϵ, $\epsilon > 0$ converging to Ω as $\epsilon \to 0$. The convergence of Ω_ϵ to Ω will be described by means of a fixed atlas $\mathcal{A}$, that is a finite collection of rotated parallelepipeds V_j, $j = 1, \dots, s$ covering the domains under consideration and such that if V_j touches the boundaries of the domains then $\Omega \cap V_j$ and $\Omega_\epsilon \cap V_j$ are given by the subgraphs of two functions $g_j, g_{\epsilon,j}$ in two variables, say $\bar{x} = (x_1, x_2)$. Thus the convergence of Ω_ϵ to Ω is understood in terms of the convergence of $g_{\epsilon,j}$ to g_j as $\epsilon \to 0$.

It is not surprising that if $g_{\epsilon,j}$ converges uniformly to g_j together with its first and second derivatives as $\epsilon \to 0$ (in which case one talks of C^2-convergence) then we have spectral stability of the *curlcurl* operator, which means that the eigenvalues and eigenfunctions of the problem in Ω_ϵ converge to those in Ω as $\epsilon \to 0$. It is also not surprising that if $g_{\epsilon,j}$ converges uniformly to g_j together with its first derivatives and

$$\sup_{\epsilon>0} \sup_{\bar{x}\in\mathbb{R}^2} |D^2 g_{\epsilon,j}(\bar{x})| \neq \infty \tag{1.4}$$

then we have spectral stability again. (These results are also immediate consequences of the results of the present paper.) The main question here is whether it is possible to relax condition (1.4). For example, if we assume that $g_{\epsilon,j}$ is of the form

$$g_{\epsilon,j} = \epsilon^\alpha b_j(\bar{x}/\epsilon) \tag{1.5}$$

where $\alpha > 0$ and b_j is a fixed $C^{1,1}$ function, condition (1.4) is encoded by the inequality $\alpha \geq 2$. In this model case, the question is whether one can get spectral stability for $\alpha < 2$. Note that a profile of the form (1.5) is typical in the study of boundary homogenization problems and thin domains, see for example [2–4, 9, 11, 19–21].

This problem was solved for the biharmonic operator with intermediate boundary conditions (modelling an elastic hinged plate) in [3] where condition (1.4) is relaxed by introducing a suitable notion of weighted convergence which allows to prove spectral stability for $\alpha > 3/2$ in the model problem above. That condition is described here in (3.14). It is remarkable that the threshold $3/2$ is sharp since for $\alpha \leq 3/2$ spectral stability does not occur for the problem discussed in [3] (in particular, it is proved in [3] that for $\alpha < 3/2$ a degeneration phenomenon occurs and for $\alpha = 3/2$ a strange term in the limit appears, as in many homogenization problems). An analogous trichotomy is found in [20] for the biharmonic operator subject to certain Steklov type boundary conditions.

In this paper, we prove that the relaxed convergence (3.14) guarantees the spectral stability of the *curlcurl* operator. Our result requires that the Gaffney inequality (1.1) holds for all domains Ω_ϵ with a constant C independent of ϵ. Again, if one does not assume the validity of the uniform bound (1.4), then proving that a uniform Gaffney inequality holds is highly non-trivial. Here we manage to do this, by exploting the approach of [34, Ch. 14] based on the use of Sobolev multipliers and the notion of domains of class $M_2^{3/2}(\delta)$. In particular, if we assume that

$$|\nabla g_{\epsilon,j}(\bar{x}) - \nabla g_{\epsilon,j}(\bar{y})| \leq M|\bar{x} - \bar{y}|^\beta \tag{1.6}$$

for all $\bar{x}, \bar{y} \in \mathbb{R}^2$, with $\beta \in]1/2, 1]$ and M independent of ϵ, and we also assume that the sup-norms of functions $|\nabla g_{\epsilon,j}|$ are sufficienlty small, then our domains belong to the class $M_2^{3/2}(\delta)$ with δ small enough. This allows to apply [34, Thm. 14.5.1] which guarantees the validity of a uniform H^2- a priori estimate for the Dirichlet Laplacian which, in turn, is equivalent to the uniform Gaffney inequality.

In conclusion, the convergence of the domains Ω_ϵ to Ω in the sense of (3.14) combined with the validity of (1.6) and the smallness of the gradients of the profile functions $g_{\epsilon,j}$ guarantees the spectral stability of the *curlcurl* operator. Note that, in principle, since Ω is of class C^1 one may think of choosing from the very beginning an atlas which guarantees that the gradients of the profile functions are as small as required (indeed, it is enough to adapt the atlas to the tangent planes of a sufficiently big number of boundary points of Ω). Then the convergence in the sense of (3.14) would imply the smallness of the gradients of the profile functions of Ω_ϵ as well.

By setting $\beta = \alpha - 1$, we deduce that a uniform Gaffney inequality holds for the example provided by (1.5) if $\alpha > 3/2$. Moreover, if $\alpha > 3/2$ then spectral stability occurs for the same example since in this case also the convergence (3.14) occurs.

The case $\alpha \leq 3/2$ is more involved and we plan to address it in a forthcoming paper, see Remark 4. We note that if $\alpha < 3/2$ one cannot expect the validity of uniform Gaffney inequalities, in particular because the regularity assumptions $C^{1,\beta}$ for $\beta > 1/2$ is optimal for the validity of the Gaffney inequality itself, see [22, 38].

One of the main tools used in this paper is a Piola-type transform which allows to pull back functions from Ω to Ω_ϵ preserving the boundary conditions. In particular the transformation depends on ϵ and is constructed in such a way that for any fixed compact set K contained in $\Omega \cap \Omega_\epsilon$, it does not modify the values of the vector fields on K for ϵ sufficiently small. Our Piola transform is constructed by pasting together local Piola transforms defined in each local chart of the atlas and for this reason it is called here *Atlas Piola transform*. We believe that our construction has its own interest.

This paper is organized as follows. Section 2 is devoted to preliminaries and notation concerning the atlas classes, the functions spaces and the weak formulations of our problems. Section 3 is devoted to the construction of the Atlas Piola transform and to the proof of its main properties, see Theorem 2. In Section 4 we prove our main stability theorem, namely Theorem 4. Section 5 is devoted to the proof of uniform a priori estimates and uniform Gaffney inequalities - see Corollaries 1, 2 - and contains the corresponding applications to the spectral stability problems, see Theorems 8, 9.

2. Preliminaries and notation

2.1. *Classes of open sets*

In this paper we consider open sets Ω in $\mathbb{R}^N$, in particular in $\mathbb{R}^3$, with sufficiently regular boundaries. This means that Ω can be described in a neighborhood of any point of the boundary as the subgraph of a sufficiently regular function g defined in a local system of orthogonal coordinates. The regularity of Ω depends on the regularity of the functions g. Since we aim at studying domain perturbation problems, following [8] and [20], we find convenient to use the notion of *atlas*, that is a collection $\mathcal{A}$ of rotated parallelepipeds V_j, $j = 1, \dots, s$, which cover Ω and such that if V_j touches the boundary of Ω then $\Omega \cap V_j$ is a subgraph of a function g_j. The parallelepipeds will also be called *local charts*. More precisely, in Definition 1 below the atlas $\mathcal{A}$ is defined as $(\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ where s is the total number of cuboids used to cover Ω, s' is the number of cuboids touching the boundary of Ω, V_j are the cuboids, r_j are the rotations used to change variables in the representations of the local charts, and ρ is a parameter controlling the minima and maxima of the functions g_j. Note that in this paper the atlas $\mathcal{A}$ will be often fixed, while the functions g_j, hence Ω, will be perturbed.

Given a set $V \subset \mathbb{R}^N$ and a parameter $\rho > 0$, we write $V_\rho := \{x \in V : d(x, \partial V) > \rho\}$.

Definition 1. *Let $\rho > 0, s, s' \in \mathbb{N}, s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids (i.e., rotations of rectangle parallelepipeds in $\mathbb{R}^N$) and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.*

A bounded domain $\Omega \subset \mathbb{R}^N$ is said to be of class $C_M^{k,\gamma}(\mathcal{A})$ with $k \in \mathbb{N} \cup \{0\}$, $\gamma \in [0,1]$ and $M > 0$ if it satisfies the following conditions:

(i) $\Omega \subset \bigcup_{j=1}^s (V_j)_\rho$ *and* $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ *for* $j = 1, \dots, s'$ *and* $V_j \cap \partial\Omega = \emptyset$ *for* $s' + 1 \leq j \leq s$;

(iii) *for* $j = 1, \dots, s$ *we have*

$$r_j(V_j) = \left\{x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \dots, N\right\},$$

for $j = 1, \dots, s'$ *we have*

$$r_j(V_j \cap \Omega) = \left\{x = (\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W_j, a_{Nj} < x_N < g_j(\bar{x})\right\},$$

where $\bar{x} = (x_1, x_2)$,

$$W_j = \left\{\bar{x} \in \mathbb{R}^{N-1}, a_{ij} < x_i < b_{ij}, i = 1, \dots, N-1\right\}$$

and the functions $g_j \in C^{k,\gamma}(\overline{W_j})$ *for any* $j = 1, \dots, s'$. *Moreover, for* $j = 1, \dots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho$$

for all $\bar{x} \in \overline{W_j}$.

(iv)

$$\sup_{|\alpha| \leq k} \|D^\alpha g_j\|_{L^\infty(W_j)} + \sup_{|\alpha| = k} \sup_{\substack{\bar{x}, \bar{y} \in W_j \\ \bar{x} \neq \bar{y}}} \frac{|D^\alpha g_j(\bar{x}) - D^\alpha g_j(\bar{y})|}{|\bar{x} - \bar{y}|^\gamma} \leq M$$

for $j = 1, \dots, s'$.

We say that Ω is of class $C^{k,\gamma}(\mathcal{A})$ if it is of class $C_M^{k,\gamma}(\mathcal{A})$ for some $M > 0$; we say that Ω is of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}(\mathcal{A})$ for some atlas $\mathcal{A}$.

2.2. Function spaces

In this section, we recall basic facts and notation for the function spaces that will be used in the following. We refer e.g., to [23, Ch. 2] for more details.

Here by Ω we denote a bounded domain - that is, a bounded connected open set - in $\mathbb{R}^3$. Since the differential problems under consideration are associated with self-adjoint operators, the space $L^2(\Omega)$ is understood here as a space of real-valued functions and is endowed with the scalar product $\int_\Omega u \cdot v \, dx$ defined for all vector fields $u, v \in L^2(\Omega)^3$. The space of vector fields $u \in L^2(\Omega)^3$ with distributional curl in $L^2(\Omega)^3$ is denoted by $H(\mathrm{curl}, \Omega)$ and is endowed with the norm defined by

$$\|u\|_{H(\mathrm{curl},\Omega)} = \left(\|u\|^2_{L^2(\Omega)^3} + \|\mathrm{curl}\, u\|^2_{L^2(\Omega)^3}\right)^{1/2}$$

for all $u \in H(\operatorname{curl}, \Omega)$. The closure in $H(\operatorname{curl}, \Omega)$ of the space of C^∞-functions with compact support in Ω is denoted by $H_0(\operatorname{curl}, \Omega)$. The following lemma characterizes the space $H_0(\operatorname{curl}, \Omega)$ and is analogous to the well-known characterization of the Sobolev space $H_0^1(\Omega)$ (see e.g., [6]). We include a short proof. Here by v^0 we denote the extension-by-zero of a vector field v, that is

$$v^0 = \begin{cases} v & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \Omega\,. \end{cases}$$

Lemma 1. *Let Ω be a bounded open set of class $C^{0,1}$ and $u \in H(\operatorname{curl}, \Omega)$. Then $u \in H_0(\operatorname{curl}, \Omega)$ if and only if $u^0 \in H(\operatorname{curl}, \mathbb{R}^3)$, in which case $\operatorname{curl}(u^0) = (\operatorname{curl} u)^0$.*

Proof. Suppose that u^0 belongs to $H(\operatorname{curl}, \mathbb{R}^3)$. Thus, there exists $v \in (L^2(\mathbb{R}^3))^3$ such that

$$\int_\Omega u \cdot \operatorname{curl} \varphi \, dx = \int_{\mathbb{R}^3} v \cdot \varphi \, dx \qquad \text{for all } \varphi \in (C_c^\infty(\mathbb{R}^3))^3. \tag{2.1}$$

Since it holds in particular for all test functions $\varphi \in (C_c^\infty(\Omega))^3$, then necessarily $v = \operatorname{curl} u$ on Ω. On the other hand, since we can take any $\varphi \in (C_c^\infty(\mathbb{R}^3 \setminus \bar{\Omega}))^3$, we see that $v = 0$ outside Ω. Hence we can rewrite (2.1) as follows

$$\int_\Omega u \cdot \operatorname{curl} \varphi \, dx = \int_\Omega \operatorname{curl} u \cdot \varphi \, dx \qquad \text{for all } \varphi \in (C_c^\infty(\mathbb{R}^3))^3. \tag{2.2}$$

By [23, Lemma 2.4] it follows that $u \in H_0(\operatorname{curl}, \Omega)$. The converse implication is straighforward. □

We note that if Ω is sufficiently regular, say of class $C^{0,1}$, the space $H_0(\operatorname{curl}, \Omega)$ coincides with the set of square integrable vector fields in Ω whose curl is also square integrable, and such that their tangential trace at the boundary $\partial\Omega$ is zero (see [23, Thm. 2.12]). In particular, we have that

$$H_0(\operatorname{curl}, \Omega) \cap (C^\infty(\bar{\Omega}))^3 = \left\{ u \in (C^\infty(\bar{\Omega}))^3 : \nu \times u|_{\partial\Omega} = 0 \right\},$$

where $C^\infty(\bar{\Omega})$ denotes smooth compactly supported functions of $\mathbb{R}^3$ restricted to $\bar{\Omega}$.

We denote by $H(\operatorname{div}, \Omega)$ the space of vector fields $u \in L^2(\Omega)^3$ with distributional divergence in $L^2(\Omega)^3$, endowed with the norm defined by

$$\|u\|_{H(\operatorname{div},\Omega)} = \left(\|u\|^2_{L^2(\Omega)^3} + \|\operatorname{div} u\|^2_{L^2(\Omega)} \right)^{1/2}$$

for all $u \in H(\operatorname{div}, \Omega)$. Finally, we set

$$\|u\|_{X(\Omega)} = \left(\|u\|^2_{L^2(\Omega)^3} + \|\operatorname{curl} u\|^2_{L^2(\Omega)^3} + \|\operatorname{div} u\|^2_{L^2(\Omega)} \right)^{1/2},$$

and we consider the space

$$X_{\mathrm{N}}(\Omega) := H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$$

endowed with the norm defined above, that is $\|u\|_{X_{\mathrm{N}}(\Omega)} = \|u\|_{X(\Omega)}$ for all $u \in X_{\mathrm{N}}(\Omega)$. We also set $X_{\mathrm{N}}(\operatorname{div} 0, \Omega) := \{u \in X_{\mathrm{N}}(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}$.

Recall that $H^1(\Omega)$ is the standard Sobolev space of functions in $L^2(\Omega)$ with first order weak derivatives in $L^2(\Omega)$. The celebrated Gaffney inequality allows to prove that the space $X_{\mathrm{N}}(\Omega)$ is continuously embedded into the space $H^1(\Omega)^3$ provided Ω is sufficiently regular. Namely, we have the following result, see e.g., [23, Theorem 3.7].

Theorem 1. *Let Ω is a bounded open set in $\mathbb{R}^3$ of class $C^{1,1}$. Then $X_{\mathrm{N}}(\Omega)$ is continuously embedded into $H^1(\Omega)^3$, and there exists $C > 0$ such that the Gaffney inequality*

$$\|u\|_{H^1(\Omega)^3} \le C\|u\|_{X_{\mathrm{N}}(\Omega)}, \tag{2.3}$$

holds for all $u \in X_{\mathrm{N}}(\Omega)$.

By the previous theorem it immediately follows that the space $X_{\mathrm{N}}(\Omega)$ is compactly embedded into $L^2(\Omega)^3$, since this is true for the space $H^1(\Omega)^3$.

As we shall see, the regularity assumptions on Ω in Theorem 1 can be relaxed since the inequality holds for domains of class $C^{1,\beta}$ with $\beta \in]1/2, 1]$, but some care is required, see Section 5.

2.3. Weak formulations and resolvent operators

Since for our purposes we prefer to work in the space $X_{\mathrm{N}}(\Omega)$ rather than in the space $X_{\mathrm{N}}(\mathrm{div}\, 0, \Omega)$, following [14, 15], we introduce a penalty term in the equation and we replace problem (1.2) by the problem

$$\begin{cases} \mathrm{curl}\, \mathrm{curl}\, u - \tau\nabla\, \mathrm{div}\, u = \lambda u, & \text{in } \Omega, \\ \mathrm{div}\, u = 0, & \text{on } \partial\Omega, \\ \nu \times u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where τ is any fixed positive real number.

It is easy to see that problem (2.4) can be formulated in the weak sense as follows

$$\int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, \varphi\, dx + \tau \int_\Omega \mathrm{div}\, u\ \mathrm{div}\, \varphi\, dx = \lambda \int_\Omega u \cdot \varphi\, dx, \quad \text{for all } \varphi \in X_{\mathrm{N}}(\Omega), \tag{2.5}$$

in the unknowns $u \in X_{\mathrm{N}}(\Omega)$ and $\lambda \in \mathbb{R}$. Is obvious that the solutions of (1.2) are exactly the divergence free solutions of (2.5). (Moreover, the weak formulation of (1.2) can be obtained simply by replacing $X_{\mathrm{N}}(\Omega)$ by $X_{\mathrm{N}}(\mathrm{div}\, 0, \Omega)$ in (2.5).)

Problem (2.5) admits also solutions which are not divergence free and which are given by the gradients of the solutions to the Helmohltz equation with Dirichlet boundary conditions. Namely, $u = \nabla f$ where f solves the following problem

$$\begin{cases} -\Delta f = \Lambda f, & \text{in } \Omega, \\ f = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

with $\Lambda = \frac{\lambda}{\tau}$. In fact, we have the following result from [15].

Lemma 2. *If Ω is a bounded domain in $\mathbb{R}^3$ of class $C^{0,1}$, then the set of all eigenpairs (λ, u) of problem* (2.4) *is the union of the set of all eigenpairs (λ, u) of problem* (1.2) *and the set of all eigenpairs of the form $(\tau\Lambda, \nabla f)$ where (Λ, f) is an eigenpair of problem* (2.6).

Thus, we can directly study problem (2.5) rather than the original problem (1.2): this will always be understood in the following. In fact, studying the spectral stability of problem (2.5) is equivalent to studying the spectral stability of problem (1.2) because the spurious eigenpairs introduced by the penalty term are given by the eigenpairs of the Dirichlet Laplacian which are stable for our class of domain perturbations (see [3]).

In order to study spectral stability problems, it is also convenient to recast the eigenvalue problems under consideration in the form of eigenvalue problems for compact self-adjoint operators and this can be done by passing to the analysis of the corresponding resolvent operators. A direct way of doing so, consists in defining the operator T from $X_{\mathrm{N}}(\Omega)$ to its dual $(X_{\mathrm{N}}(\Omega))'$ by setting

$$< Tu, \varphi >= \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, \varphi\, dx + \tau \int_\Omega \mathrm{div}\, u\ \mathrm{div}\, \varphi\, dx, \tag{2.7}$$

for all $u, \varphi \in X_{\mathrm{N}}(\Omega)$, and considering the map J from $L^2(\Omega)^3$ to $(X_{\mathrm{N}}(\Omega))'$ defined by

$$< Ju, \varphi >= \int_\Omega u \cdot \varphi\, dx,$$

for all $u \in L^2(\Omega)^3$ and $\varphi \in X_{\mathrm{N}}(\Omega)$. By restricting J to $X_{\mathrm{N}}(\Omega)$ (and denoting the restriction by the same symbol J), and using the Riesz Theorem it turns out that the operator $T + J$ is a homeomorphism from $X_{\mathrm{N}}(\Omega)$ to its dual. The inverse operator $(T + J)^{-1}$ will serve for our purposes as discussed above. In fact, the following theorem holds.

Lemma 3. *If Ω is a bounded domain in $\mathbb{R}^3$ such that the embedding ι of $X_{\mathrm{N}}(\Omega)$ into $L^2(\Omega)^3$ is compact, then the operator S_Ω from $L^2(\Omega)^3$ to itself defined by*

$$S_\Omega u = \iota \circ (T + J)^{-1} \circ J$$

is a non-negative compact self-adjoint operator in $L^2(\Omega)^3$ whose eigenvalues μ are related to the eigenvalues λ of problem (2.5) *by the equality $\mu = (\lambda + 1)^{-1}$.*

By the previous lemma and standard spectral theory it follows that the spectrum $\sigma(S_\Omega)$ of S_Ω can be represented as $\sigma(S_\Omega) = \{0\} \cup \{\mu_n(\Omega)\}_{n\in\mathbb{N}}$, where $\mu_n(\Omega)$, $n \in \mathbb{N}$ is a decreasing sequence of positive eigenvalues of finite multiplicity, which converges to zero. Consequently, the eigenvalues of problem (2.5) can be represented by the sequence $\lambda_n(\Omega)$, $n \in \mathbb{N}$ defined by $\lambda_n(\Omega) = \mu_n^{-1}(\Omega) - 1$. Moreover, the classical Min-Max Principle yields the following variational representation

$$\lambda_n(\Omega) = \min_{\substack{V\subset X_{\mathrm{N}}(\Omega)\\ \dim V = n}} \max_{u\in V\setminus\{0\}} \frac{\int_\Omega |\,\mathrm{curl}\, u|^2 dx + \tau \int_\Omega |\,\mathrm{div}\, u|^2 dx}{\int_\Omega |u|^2\, dx}. \tag{2.8}$$

3. A Piola-type approximation of the identity

Given two domains Ω and $\tilde{\Omega}$ in $\mathbb{R}^3$ and a diffeomorphism $\Phi : \tilde{\Omega} \to \Omega$ of class $C^{1,1}$, the standard way to pull-back vector fields from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ consists in using the (covariant) Piola transform defined by

$$u(x) = ((v \circ \Phi)\, \mathrm{D}\,\Phi\,)\,(x), \quad \text{for all } x \in \tilde{\Omega}, \tag{3.1}$$

for all $v \in X_{\mathrm{N}}(\Omega)$, see e.g., [35]. In fact, it turns out that $v \in H_0(\mathrm{curl}, \Omega)$ if and only if $u \in H_0(\mathrm{curl}, \tilde{\Omega})$, in which case we have

$$(\mathrm{curl}\, v) \circ \Phi = \frac{\mathrm{curl}\, u\, (\mathrm{D}\,\Phi)^T}{\det{(\mathrm{D}\,\Phi)}}\,. \tag{3.2}$$

Note that for functions u, v in H^1 we also have

$$(\operatorname{div} v) \circ \Phi = \frac{\operatorname{div}\left[u(\mathrm{D}\,\Phi)^{-1}(\mathrm{D}\,\Phi)^{-T}\det(\mathrm{D}\,\Phi)\right]}{\det(\mathrm{D}\,\Phi)}, \tag{3.3}$$

and in this case $v \in X_{\mathrm{N}}(\Omega) \cap H^1(\Omega)^3$ if and only if $u \in X_{\mathrm{N}}(\tilde{\Omega}) \cap H^1(\tilde{\Omega})^3$. See [33] for more details. Unfortunately, given two domains Ω and $\tilde{\Omega}$, in general it is not possible to define explicitly a diffeomorphism between Ω and $\tilde{\Omega}$ (even if it is known a priori that the two domains are diffeomorphic). Nevertheless, it is important for our purposes to define an operator which allows to pass from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ as the Piola transform does. This can be done by assuming that Ω and $\tilde{\Omega}$ belong to the same atlas class and using a partition of unity in order to paste together Piola transforms defined locally, as described in the following. Note that the specific choice of local Piola transforms reflects our need for a transformation close to the identity.

Let $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^3$ and let $\Omega, \tilde{\Omega}$ be two domains of class $C^{1,1}(\mathcal{A})$. Let $g_j, \tilde{g}_j$ be the profile functions of Ω and $\tilde{\Omega}$ as in Definition 1. Assume that $k \in]0, +\infty[$ is such that

$$k > \max_{j=1,\dots,s'} \|\tilde{g}_j - g_j\|_\infty, \text{ and } \tilde{g}_j - k > a_{3,j} + \rho, \ \forall j = 1, \dots, s'. \tag{3.4}$$

For any $j = 1, \dots, s'$ we set

$$\hat{g}_j := \tilde{g}_j - k \tag{3.5}$$

and we define the map $h_j : r_j(\overline{\tilde{\Omega} \cap V_j}) \to \mathbb{R}$

$$h_j(\bar{x}, x_3) := \begin{cases} 0, & \text{if } a_{3j} \le x_3 \le \hat{g}_j(\bar{x}), \\ (\tilde{g}_j(\bar{x}) - g_j(\bar{x}))\left(\frac{x_3 - \hat{g}_j(\bar{x})}{\tilde{g}_j(\bar{x}) - \hat{g}_j(\bar{x})}\right)^3, & \text{if } \hat{g}_j(\bar{x}) < x_3 \le \tilde{g}_j(\bar{x}), \end{cases} \tag{3.6}$$

and the map

$$\Phi_j : r_j(\overline{\tilde{\Omega} \cap V_j}) \to r_j(\overline{\Omega \cap V_j}), \qquad \Phi_j(\bar{x}, x_3) := (\bar{x}, x_3 - h_j(\bar{x}, x_3)). \tag{3.7}$$

Note that Φ_j coincides with the identity map on the set

$$K_j := \left\{(\bar{x}, x_3) \in W_j \times]a_{3j}, b_{3j}[\, : a_{3j} < x_3 < \hat{g}_j(\bar{x})\right\}. \tag{3.8}$$

Finally, if $s' + 1 \le j \le s$ we define $\Phi_j : r_j(\overline{V_j}) \to r_j(\overline{V_j})$ to be the identity map.

Observe that since $h_j \in C^{1,1}(r_j(\overline{\tilde{\Omega} \cap V_j}))$, then Φ_j is of class $C^{1,1}$, and so is the following map

$$\Psi_j : \overline{\tilde{\Omega} \cap V_j} \to \overline{\Omega \cap V_j}, \qquad \Psi_j := r_j^{-1} \circ \Phi_j \circ r_j. \tag{3.9}$$

An easy computation shows that if

$$k > \frac{3}{\alpha} \max_{j=1,\dots,s'} \|\tilde{g}_j - g_j\|_\infty \tag{3.10}$$

for some constant $\alpha \in]0, 1[$ then

$$0 < 1 - \alpha \le \det(\mathrm{D}\,\Psi_j(x)) \le 1 + \alpha \quad \text{for any } x \in \tilde{\Omega} \cap V_j. \tag{3.11}$$

Let $\{\psi_j\}_{j=1}^s$ be a C^∞-partition of unity associated with the open cover $\{V_j\}_{j=1}^s$ of the compact set $\overline{\cup_{j=1}^s (V_j)_\rho}$ that is $0 \le \psi_j \le 1$, $\mathrm{supp}(\psi_j) \subset V_j$ for all $j = 1, \dots, s$, and $\sum_{j=1}^s \psi_j \equiv 1$ in $\overline{\cup_{j=1}^s (V_j)_\rho}$, in particular also in $\overline{\Omega \cup \tilde{\Omega}}$. Note that this is a partition of unity is independent of $\Omega, \tilde{\Omega}$ in the atlas class under consideration.

Since for any $\varphi \in X_{\mathrm{N}}(\Omega)$ we have $\varphi = \sum_{j=1}^s \varphi_j$ where $\varphi_j = \psi_j \varphi$, then it is natural to give the following definition (note that here we consider open sets of class $C^{1,1}$ hence the spaces X_{N} are embedded into H^1).

Definition 2. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^3$ and $\Omega, \tilde{\Omega}$ be two domains of class $C^{1,1}(\mathcal{A})$. Assume that $k > 0$ satisfies* (3.4)*, and $\{\psi_j\}_{j=1}^s$ is a partition of unity as above. The* Atlas Piola transform *from Ω to $\tilde{\Omega}$, with parameters $\mathcal{A}$, k, and $\{\psi_j\}_{j=1}^s$, is the map from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ defined by*

$$\mathcal{P}\varphi := \sum_{j=1}^{s'} \tilde{\varphi}_j + \sum_{j=s'+1}^{s} \varphi_j \tag{3.12}$$

for all $\varphi \in X_{\mathrm{N}}(\Omega)$, where

$$\tilde{\varphi}_j(x) := \begin{cases} (\varphi_j \circ \Psi_j(x))\, \mathrm{D}\,\Psi_j(x), & \text{if } x \in \tilde{\Omega} \cap V_j, \\ 0, & \text{if } x \in \tilde{\Omega} \setminus V_j, \end{cases} \tag{3.13}$$

for any $j = 1, \dots, s'$.

Note that $\mathcal{P}\varphi \in X_{\mathrm{N}}(\tilde{\Omega})$ because $(\varphi_j \circ \Psi_j)\, \mathrm{D}\,\Psi_j \in X_{\mathrm{N}}(\tilde{\Omega} \cap V_j)$ (observe that the support of φ_j is compact in V_j), hence $\tilde{\varphi}_j \in X_{\mathrm{N}}(\tilde{\Omega})$.

This Atlas Piola transform will be used in this paper for a family $\Omega_\epsilon, \epsilon > 0$ of domains of class $C^{1,1}(\mathcal{A})$, converging in some sense to a domain Ω of class $C^{1,1}(\mathcal{A})$. In this case, Ω_ϵ will play the role of the domain $\tilde{\Omega}$ and the corresponding transformation will allow us to pass from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\Omega_\epsilon)$.

Given a family of domains Ω_ϵ, $\epsilon > 0$, and a fixed domain Ω, all of class $C^{1,1}(\mathcal{A})$, we shall denote by $g_{\epsilon,j}$ and g_j the corresponding profile functions (defined on W_j) of Ω_ϵ and Ω respectively, as in Definition 1. Following [3, 20], we use a notion of convergence for the open sets Ω_ϵ to Ω, which is expressed in terms of convergence of the the profile functions $g_{\epsilon,j}$ to g_j. Namely, we assume that for any $\epsilon > 0$ there exists $\kappa_\epsilon > 0$ such that for any $j \in \{1, \dots, s'\}$

$$\begin{aligned} &(i) \;\; \kappa_\epsilon > \max_{j=1,\dots,s'} \left\| g_{\epsilon,j} - g_j \right\|_{L^\infty(W_j)}; \\ &(ii) \;\; \lim_{\epsilon \to 0} \kappa_\epsilon = 0; \\ &(iii) \;\; \lim_{\epsilon \to 0} \frac{\max_{j=1,\dots,s'} \left\| D^\beta (g_{\epsilon,j} - g_j) \right\|_{L^\infty(W_j)}}{\kappa_\epsilon^{3/2-|\beta|}} = 0 \quad \text{for all } \beta \in \mathbb{N}^3 \text{ with } |\beta| \le 2. \end{aligned} \tag{3.14}$$

Note that if every function $g_{\epsilon,j}$ converges to g_j uniformly together with the first order derivatives and condition (1.4) is satisfied (in particular, if the second order derivatives of $g_{\epsilon,j}$ converge uniformly to those of g_j) then conditions (3.14) are fulfilled, see [3]. Note also that the exponent $3/2$ in (3.14) turns out to be optimal in the analysis of [3] and plays a crucial role for instance in proving inequality (3.41).

We now fix a partition of unity $\{\psi_j\}_{j=1}^s$ associated with the covering of cuboids of the atlas $\mathcal{A}$ as above, and independent of Ω_ϵ and Ω. We also choose $k = 6\kappa_\epsilon$ and we denote by $\mathcal{P}_\epsilon$ the Atlas Piola transform from Ω to Ω_ϵ (with parameters $\mathcal{A}$, k, $\{\psi_j\}_{j=1}^{s}$). Note that conditions (3.4), (3.10) (3.11) are satisfied with $\alpha = 1/2$ if ϵ is sufficiently small.

In the following, we shall denote by $\hat{g}_{\epsilon,j}$, $h_{\epsilon,j}$, $\Phi_{\epsilon,j}$, $K_{\epsilon,j}$, $\Psi_{\epsilon,j}$, $\tilde{\varphi}_{\epsilon,j}$ all quantities defined in (3.5), (3.6), (3.7), (3.8), (3.9), (3.13) respectively, with $\tilde{\Omega} = \Omega_\epsilon$ and $k = 6\kappa_\epsilon$.

Then we can prove the following theorem. We note that in the proof, some technical issues related to pasting together functions defined in different charts are treated in the spirit of the arguments used in [20] for the Sobolev spaces $H^2(\Omega)$.

Theorem 2. *Let $\Omega_\epsilon, \epsilon > 0$, and Ω be bounded domains of class $C^{1,1}(\mathcal{A})$. Assume that Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense of* (3.14). *Let $\mathcal{P}_\epsilon$ be the Atlas Piola transform from Ω to Ω_ϵ defined for ϵ sufficiently small as above. Then the following statements hold:*

(i) *for any $\epsilon > 0$ the function $\mathcal{P}_\epsilon$ maps $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\Omega_\epsilon)$ with continuity;*

(ii) *for any compact set $\mathcal{K}$ contained in Ω there exists $\epsilon_{\mathcal{K}} > 0$ such that*

$$(\mathcal{P}_\epsilon\varphi)(x) = \varphi(x), \quad \forall x \in \mathcal{K} \tag{3.15}$$

for all $\epsilon \in]0, \epsilon_{\mathcal{K}}[$ and $\varphi \in X_{\mathrm{N}}(\Omega)$;

(iii) *the limit*

$$\left\|\mathcal{P}_\epsilon\varphi\right\|_{X_{\mathrm{N}}(\Omega_\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} \left\|\varphi\right\|_{X_{\mathrm{N}}(\Omega)}, \tag{3.16}$$

holds for all $\varphi \in X_{\mathrm{N}}(\Omega)$;

(iv) *the limit*

$$\left\|\mathcal{P}_\epsilon\varphi - \varphi\right\|_{X(\Omega_\epsilon\cap\Omega)} \underset{\epsilon\to 0}{\longrightarrow} 0, \tag{3.17}$$

holds for all $\varphi \in X_{\mathrm{N}}(\Omega)$.

Proof. Let $\varphi \in X_{\mathrm{N}}(\Omega)$ be fixed. Note that Ω is of class $C^{1,1}$ hence the Gaffney inequality holds and $\varphi \in H^1(\Omega)^3$. Moreover, $\varphi_j \in X_{\mathrm{N}}(\Omega)$ for all $j = 1, \dots, s'$ hence $\tilde{\varphi}_{\epsilon,j}$ belongs to $X_{\mathrm{N}}(\Omega_\epsilon)$ for all $j = 1, \dots, s'$. It follows that $\mathcal{P}_\epsilon\varphi \in X_{\mathrm{N}}(\Omega_\epsilon)$. The continuity of the operator follows by standard calculus, the Gaffney inequality and formulas (3.2), (3.3). Thus, statement (i) holds.

For any fixed compact set $\mathcal{K}$ contained in Ω, since $\hat{g}_{\epsilon,j}$ converges uniformly to g_j, we have

$$\mathcal{K} \cap V_j \subset r_j^{-1}(K_{\epsilon,j})$$

for all $j = 1, \dots, s'$ and ϵ sufficiently small; this, combined with the fact that $\Phi_{\epsilon,j}$ coincides with the identity on $K_{\epsilon,j}$, it follows that $\tilde{\varphi}_{\epsilon,j} = \varphi_j$ on $\mathcal{K}$ for all ϵ sufficiently small and (3.15) follows.

We now prove statement (iii). We have to prove the following limiting relations:

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\mathcal{P}_\epsilon\varphi\right|^2 = \int_{\Omega}\left|\varphi\right|^2, \tag{3.18}$$

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\operatorname{curl}\mathcal{P}_\epsilon\varphi\right|^2 = \int_{\Omega}\left|\operatorname{curl}\varphi\right|^2, \tag{3.19}$$

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\operatorname{div}\mathcal{P}_\epsilon\varphi\right|^2 = \int_{\Omega}\left|\operatorname{div}\varphi\right|^2. \tag{3.20}$$

We begin by proving (3.18). To see this, it just suffices to show that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}=\int_{\Omega}\varphi_j\cdot\varphi_h, \tag{3.21}$$

and

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\tilde{\varphi}_{\epsilon,j}\cdot\varphi_i=\int_{\Omega}\varphi_j\cdot\varphi_i \tag{3.22}$$

for any $j,h\in\{1,\dots,s'\}$ and $i\in\{s'+1,\dots,s\}$. We will only show (3.21), since the computations to prove (3.22) are similar. We will first see that

$$\lim_{\epsilon\to 0}\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2=0. \tag{3.23}$$

Notice that for any $j\in\{1,\dots,s'\}$ we have $\left|(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})\right|\to 0$ as ϵ goes to 0. Moreover, if $w\in\mathbb{R}^3$ is a vector, then $\left|w\,\mathrm{D}\,\Psi_{\epsilon,j}\right|=\left|w\,\mathrm{D}\,\Phi_{\epsilon,j}\right|\le C|w|$, since

$$\mathrm{D}\,\Phi_{\epsilon,j}=\begin{pmatrix}1&0&0\\0&1&0\\-\frac{\partial h_{\epsilon,j}}{\partial x_1}&-\frac{\partial h_{\epsilon,j}}{\partial x_2}&1-\frac{\partial h_{\epsilon,j}}{\partial x_3}\end{pmatrix}$$

and the first derivatives of $h_{\epsilon,j}$ are all bounded due to the hypothesis on the functions $g_{\epsilon,j}$ (see also (3.37)). Note that here and in what follows, by c we denote a constant independent of ϵ which may vary from line to line. Then by using also (3.11), we have

$$\begin{aligned}\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2dy&=\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|(\varphi_j\circ\Psi_{\epsilon,j})\,\mathrm{D}\,\Psi_{\epsilon,j}\right|^2dy\\&\le c\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\varphi_j\circ\Psi_{\epsilon,j}\right|^2dy\\&=c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\frac{\left|\varphi_j\right|^2}{\left|\det(\mathrm{D}\,\Psi_{\epsilon,j})\circ\Psi_{\epsilon,j}^{(-1)}\right|}dx\\&\le c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\varphi_j\right|^2dx\xrightarrow[\epsilon\to 0]{}0.\end{aligned}$$

By (3.23) we deduce that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\tilde{\varphi}_{\epsilon,j}\right|^2=\int_{\Omega}\left|\varphi_j\right|^2. \tag{3.24}$$

Indeed, since $\Psi_{\epsilon,j}$ is the identity on $r_j^{-1}(K_{\epsilon,j})\subset\Omega\cap\Omega_\epsilon$, using (3.23) yields

$$\int_{\Omega_\epsilon}\left|\tilde{\varphi}_{\epsilon,j}\right|^2=\int_{r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2+\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2\xrightarrow[\epsilon\to 0]{}\int_{\Omega\cap V_j}\left|\varphi_j\right|^2=\int_{\Omega}\left|\varphi_j\right|^2.$$

Observe now that

$$\begin{aligned}\int_{\Omega_\epsilon} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} &= \int_{\Omega_\epsilon\cap V_j\cap V_h} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}\\ &= \int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} + \int_{(\Omega_\epsilon\cap V_j\cap V_h)\setminus(r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h}))} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}.\end{aligned} \tag{3.25}$$

It is obvious that

$$\lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} = \lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})} \varphi_j\cdot\varphi_h = \int_\Omega \varphi_j\cdot\varphi_h. \tag{3.26}$$

Here and in the following we will make use of the identity

$$\begin{aligned}&(\Omega_\epsilon\cap V_j\cap V_h)\setminus(r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})) =\\ &\qquad [(\Omega_\epsilon\cap V_j\cap V_h)\setminus r_j^{-1}(K_{\epsilon,j})]\cup[(r_j^{-1}(K_{\epsilon,j})\setminus r_h^{-1}(K_{\epsilon,h}))\cap V_h].\end{aligned} \tag{3.27}$$

Observe that by (3.23), (3.24) we get

$$\left|\int_{(\Omega_\epsilon\cap V_j\cap V_h)\setminus r_j^{-1}(K_{\epsilon,j})} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}\right| \le \left(\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2\right)^{\frac12}\left(\int_{\Omega_\epsilon\cap V_h}\left|\tilde{\varphi}_{\epsilon,h}\right|^2\right)^{\frac12} \xrightarrow[\epsilon\to 0]{} 0, \tag{3.28}$$

and

$$\left|\int_{(r_j^{-1}(K_{\epsilon,j})\setminus r_h^{-1}(K_{\epsilon,h}))\cap V_h} \tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}\right| \le \left(\int_{r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2\right)^{\frac12}\left(\int_{(\Omega_\epsilon\cap V_h)\setminus r_h^{-1}(K_{\epsilon,h})}\left|\tilde{\varphi}_{\epsilon,h}\right|^2\right)^{\frac12} \xrightarrow[\epsilon\to 0]{} 0. \tag{3.29}$$

Hence, by formula (3.27), we see that the second term of the sum in (3.25) vanishes as ϵ goes to zero, hence we deduce the validity of (3.21) from (3.25) and (3.26).

We now prove (3.19). Again, we need to check that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \operatorname{curl}\tilde{\varphi}_{\epsilon,j}\cdot\operatorname{curl}\tilde{\varphi}_{\epsilon,h} = \int_\Omega \operatorname{curl}\varphi_j\cdot\operatorname{curl}\varphi_h \tag{3.30}$$

for any $j,h\in\{1,\dots,s'\}$. Note that

$$\mathrm{D}\,\Phi_{\epsilon,j}^{(-1)} = \frac{1}{\det(\mathrm{D}\,\Phi_{\epsilon,j})}\begin{pmatrix} 1-\frac{\partial h_{\epsilon,j}}{\partial x_3} & 0 & 0\\ 0 & 1-\frac{\partial h_{\epsilon,j}}{\partial x_3} & 0\\ \frac{\partial h_{\epsilon,j}}{\partial x_1} & \frac{\partial h_{\epsilon,j}}{\partial x_2} & 1\end{pmatrix}\circ\Phi_{\epsilon,j}^{(-1)},$$

and recall that $\Psi_{\epsilon,j} = r_j^{-1}\circ\Phi_{\epsilon,j}\circ r_j$. Moreover, by (3.2) we have

$$\operatorname{curl}\tilde{\varphi}_{\epsilon,j} = \left(\operatorname{curl}\varphi_j\circ\Psi_{\epsilon,j}\right)(\mathrm{D}\,\Psi_{\epsilon,j})^{-T}\det\mathrm{D}(\Psi_{\epsilon,j}) \ \text{ on } \Omega_\epsilon\cap V_j$$

so that

$$\left|\operatorname{curl}\tilde{\varphi}_{\epsilon,j}\right| \le c\left|\operatorname{curl}\varphi_j\circ\Psi_{\epsilon,j}\right|.$$

Then, with computations analogous to those performed above, we get

$$\lim_{\epsilon\to 0}\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\operatorname{curl}\tilde{\varphi}_{\epsilon,j}\right|^2 = 0. \tag{3.31}$$

It is also obvious that

$$\lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})}\left|\operatorname{curl}\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega\left|\operatorname{curl}\varphi_j\right|^2 \tag{3.32}$$

and thus

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\operatorname{curl}\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega\left|\operatorname{curl}\varphi_j\right|^2. \tag{3.33}$$

By using the same argument above, formula (3.27) together with the new identities (3.31)–(3.33), we obtain (3.30) .

Finally, we prove (3.20). To do so, we need to prove that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\operatorname{div}\tilde{\varphi}_{\epsilon,j}\operatorname{div}\tilde{\varphi}_{\epsilon,h} = \int_\Omega\operatorname{div}\varphi_j\operatorname{div}\varphi_h \tag{3.34}$$

for any $j,h\in\{1,\dots,s'\}$. Here and in the rest of the proof, the vectors under consideration will be represented as follows: $\varphi_j=(\varphi_j^1,\varphi_j^2,\varphi_j^3)$ and $\Psi_{\epsilon,j}=(\Psi^1_{\epsilon,j},\Psi^2_{\epsilon,j},\Psi^3_{\epsilon,j})$.

Since $\varphi\in X_{\mathrm{N}}(\Omega)$ and the Gaffney inequality holds on Ω, it follows that $\varphi\in H^1(\Omega)^3$. Thus, recalling that $\tilde{\varphi}_{\epsilon,j}(x)=(\varphi_j\circ\Psi_{\epsilon,j}(x))\,\mathrm{D}\,\Psi_{\epsilon,j}(x)$ for all $x\in\Omega_\epsilon\cap V_j$, it is possible to apply the chain rule and obtain that

$$\operatorname{div}(\tilde{\varphi}_{\epsilon,j}) = \sum_{m,n,i=1}^3\underbrace{\left(\frac{\partial\varphi_j^m}{\partial x_n}(\Psi_{\epsilon,j})\frac{\partial\Psi^n_{\epsilon,j}}{\partial x_i}\frac{\partial\Psi^m_{\epsilon,j}}{\partial x_i}\right)}_{\text{type A}}+\sum_{m,i=1}^3\underbrace{\varphi_j^m(\Psi_{\epsilon,j})\frac{\partial^2\Psi^m_{\epsilon,j}}{\partial x_i^2}}_{\text{type B}}\quad\text{in }\Omega_\epsilon\cap V_j, \tag{3.35}$$

where the terms in the first sum are called of type A and the others are called terms of type B.

Recall that $h_{\epsilon,j}$ are the functions in (3.6) used to define the diffeomorphisms $\Phi_{\epsilon,j}$. We observe that by the Leibniz rule we have

$$D^\alpha h_{\epsilon,j}(x) = \sum_{0\le\gamma\le\alpha}\binom{\alpha}{\gamma}D^\gamma\left(g_{\epsilon,j}(\bar{x})-g_j(\bar{x})\right)D^{\alpha-\gamma}\left(\frac{x_3-\hat{g}_{\epsilon,j}(\bar{x})}{g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})}\right)^3$$

hence by standard calculus (note that the denominator in the previous formula is the constant $k=6\kappa_\epsilon$) and (3.14) we get

$$\left|D^{\alpha-\gamma}\left(\frac{x_3-\hat{g}_{\epsilon,j}(\bar{x})}{g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})}\right)^3\right| \le \frac{c}{\left|g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})\right|^{|\alpha|-|\gamma|}} \le \frac{c}{\kappa_\epsilon^{|\alpha|-|\gamma|}}. \tag{3.36}$$

Therefore

$$\left\|D^\alpha h_{\epsilon,j}\right\|_\infty \le c\sum_{0\le\gamma\le\alpha}\frac{\left\|D^\gamma(g_{\epsilon,j}-g_j)\right\|_\infty}{\kappa_\epsilon^{|\alpha|-|\gamma|}} \tag{3.37}$$

for all $\epsilon > 0$ sufficiently small. It follows by the definitions of $\Psi_{\epsilon,j}, \Phi_{\epsilon,j}$, by (3.37) and part (*iii*) of condition (3.14), that for all $m, i = 1, 2, 3$

$$\left\| \frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2} \right\|_{L^\infty(\Omega_\epsilon \cap V_j)} = o(\kappa_\epsilon^{-1/2}), \quad \text{as } \epsilon \to 0. \tag{3.38}$$

We claim that

$$\lim_{\epsilon \to 0} \int_{(\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \operatorname{div} \tilde{\varphi}_{\epsilon,j} \right|^2 = 0. \tag{3.39}$$

To prove that, we analyse first the terms of type A in (3.35). By changing variables in integrals we get:

$$\begin{aligned}
&\int_{(\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \frac{\partial \varphi_j^m}{\partial x_n} \circ \Psi_{\epsilon,j} \right|^2 \left| \frac{\partial \Psi^n_{\epsilon,j}}{\partial x_i} \right|^2 \left| \frac{\partial \Psi^m_{\epsilon,j}}{\partial x_i} \right|^2 dy \\
&\leq c \int_{(\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \frac{\partial \varphi_j^m}{\partial x_n} \circ \Psi_{\epsilon,j} \right|^2 dy \\
&= c \int_{(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \frac{\partial \varphi_j^m}{\partial x_n} \right|^2 \frac{1}{\left| \det(\mathrm{D}\, \Psi_{\epsilon,j}) \circ \Psi^{(-1)}_{\epsilon,j} \right|} dx \\
&\leq c \int_{(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \frac{\partial \varphi_j^m}{\partial x_n} \right|^2 dx \xrightarrow[\epsilon \to 0]{} 0.
\end{aligned} \tag{3.40}$$

We now consider the terms of type B. By setting $\eta_j(z) := \varphi_j(r_j^{-1}(z))$ and recalling (3.38) we have that

$$\begin{aligned}
&\int_{(\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j^m(\Psi_{\epsilon,j}) \frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2} \right|^2 dy \\
&\quad \leq \left\| \frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2} \right\|^2_{L^\infty(\Omega_\epsilon \cap V_j)} \int_{(\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j(\Psi_{\epsilon,j}) \right|^2 dy \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j \right|^2 \frac{1}{\left| \det(\mathrm{D}\, \Psi_{\epsilon,j}) \circ \Psi^{(-1)}_{\epsilon,j} \right|} dx \\
&\quad \leq o(\kappa_\epsilon^{-1}) \int_{(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j(x) \right|^2 dx \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{r_j(\Omega \cap V_j) \setminus K_{\epsilon,j}} \left| \eta_j(z) \right|^2 dz \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{W_j} \left(\int_{\hat{g}_{\epsilon,j}(\bar{z})}^{g_j(\bar{z})} \left| \eta_j(\bar{z}, z_3) \right|^2 dz_3 \right) d\bar{z} \\
&\quad \leq o(\kappa_\epsilon^{-1}) \int_{W_j} \left| g_j(\bar{z}) - \hat{g}_{\epsilon,j}(\bar{z}) \right| \left\| \eta_j(\bar{z}, \cdot) \right\|^2_{L^\infty(a_{3j}, g_j(\bar{z}))^3} d\bar{z} \\
&\quad \leq o(\kappa_\epsilon^{-1}) \left\| g_j - \hat{g}_{\epsilon,j} \right\|_{L^\infty(W_j)} \int_{W_j} \left\| \eta_j(\bar{z}, \cdot) \right\|^2_{H^1(a_{3j}, g_j(\bar{z}))^3} d\bar{z} \\
&\quad \leq o(\kappa_\epsilon^{-1})\, \kappa_\epsilon \left\| \eta_j \right\|^2_{H^1(r_j(\Omega \cap V_j))^3} \xrightarrow[\epsilon \to 0]{} 0.
\end{aligned} \tag{3.41}$$

Here we have used the following one dimensional embedding estimate for Sobolev functions (see e.g., Burenkov [7]):

$$\|f\|_{L^\infty(a,b)} \le c\|f\|_{H^1(a,b)}$$

for all $f \in H^1(a,b)$, where the constant $c = c(d)$ is uniformly bounded for $|b-a| > d$. We conclude that (3.39) holds.

Using (3.39), the fact that $\Psi_{\epsilon,j}$ in $r_j^{-1}(K_{\epsilon,j})$ coincides with the identity and that $\left|(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})\right| \to 0$ as ϵ goes to 0, we deduce that

$$\lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})}\left|\operatorname{div}\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega \left|\operatorname{div}\varphi_j\right|^2, \tag{3.42}$$

and

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\operatorname{div}\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega \left|\operatorname{div}\varphi_j\right|^2. \tag{3.43}$$

With (3.39), (3.42) and (3.43) in mind, in order to prove (3.34), it suffices to reproduce the same argument used before starting from (3.25) combined with formula (3.27). We omit the details. Thus statement (iii) is proved.

The proof of statement (iv) follows by the same considerations above. First of all, for any $j = 1, \dots, s'$ the function $\tilde{\varphi}_{\epsilon,j}$ coincides with φ_j on $r_j^{-1}(K_{\epsilon,j})$. Thus $\mathcal{P}_\epsilon\varphi = \varphi$ on $(\cup_{j=1,\dots,s'} r_j^{-1}(K_{\epsilon,j})) \cup (\cup_{j=s'+1,\dots,s} V_j)$. It follows that

$$\|\mathcal{P}_\epsilon\varphi - \varphi\|_{X(\Omega_\epsilon\cap\Omega)} \le \|\mathcal{P}_\epsilon\varphi - \varphi\|_{X(\cup_{j=1,\dots,s'}(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j}))}$$

This combined with by the limiting relations (3.23), (3.31) and (3.39) yields the validity of statement (iv). □

4. Spectral stability

Let Ω_ϵ, $\epsilon > 0$, and Ω be bounded domains in $\mathbb{R}^3$ of class $C^{1,1}(\mathcal{A})$. For simplicity, it is convenient to set $\Omega_0 = \Omega$. In this section, we prove that if Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense of (3.14), and a uniform Gaffney inequality holds on the domains Ω_ϵ then we have spectral stability for the curl curl operator defined on the domains Ω_ϵ with respect to the reference domain Ω. By uniform Gaffney inequality, we mean that the spaces $X_{\mathrm{N}}(\Omega_\epsilon)$ are embedded into $H^1(\Omega_\epsilon)^3$ and there exists a positive constant C independent of ϵ such that

$$\|u\|_{H^1(\Omega_\epsilon)^3} \le C\|u\|_{X_{\mathrm{N}}(\Omega_\epsilon)}, \tag{4.1}$$

for all $u \in X_{\mathrm{N}}(\Omega_\epsilon)$ and $\epsilon > 0$. (Note that by Theorem 1, for every $\epsilon > 0$ there exists a positive constant C_ϵ, possibly depending on ϵ, such that (4.1) holds, but here we need a constant independent of ϵ).

To do so, for any $\epsilon \ge 0$, we denote by S_ϵ the operator S_{Ω_ϵ} from $L^2(\Omega_\epsilon)$ to itself defined in Lemma 3. Recall that if $f_\epsilon \in L^2(\Omega_\epsilon)^3$ is the datum of the following Poisson problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon - \tau\nabla\operatorname{div} v_\epsilon + v_\epsilon = f_\epsilon, & \text{in } \Omega_\epsilon, \\ \operatorname{div} v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \\ v_\epsilon \times \nu = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{4.2}$$

then the unique solution $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon)$ is precisely $S_\epsilon f_\epsilon$, that is $v_\epsilon = S_\epsilon f_\epsilon$. Recall that τ is a fixed positive constant (one could normalise it by setting $\tau = 1$ but we prefer to keep it as it is also with reference to other papers where it is important to have the possibility to use different values of τ, see for example [33, Remark 2.13]). In this section we prove that S_ϵ compactly converges to S_0 as $\epsilon \to 0$, and this implies spectra stability. This has to be understood in the following sense.

We denote by $E = \{E_\epsilon\}_{\epsilon>0}$ the family of the extension-by-zero operators $E_\epsilon : L^2(\Omega)^3 \to L^2(\Omega_\epsilon)^3$ defined by

$$E_\epsilon\varphi = \varphi^0 = \begin{cases} \varphi, & \text{if } x \in \Omega_\epsilon \cap \Omega, \\ 0, & \text{if } x \in \Omega_\epsilon \setminus \Omega, \end{cases} \tag{4.3}$$

for all $\varphi \in L^2(\Omega)^3$. Note that under our assumptions we have that for all $\varphi \in L^2(\Omega)^3$

$$\lim_{\epsilon\to 0}\left\|E_\epsilon\varphi\right\|_{L^2(\Omega_\epsilon)^3} = \left\|\varphi\right\|_{L^2(\Omega)^3},$$

since $\left|\Omega \setminus (\Omega_\epsilon \cap \Omega)\right| \to 0$ as ϵ goes to 0. We recall the following definition from [40], see also [1] and [10].

Definition 3. *Let $u_\epsilon \in L^2(\Omega_\epsilon)$, for $\epsilon > 0$, be a family of functions. We say that u_ϵ E-converges to $u_0 \in L^2(\Omega)$ as $\epsilon \to 0$ and we write $u_\epsilon \xrightarrow{E} u_0$ if*

$$\|u_\epsilon - E_\epsilon u_0\|_{L^2(\Omega_\epsilon)} \to 0, \ \text{ as } \ \epsilon \to 0\,.$$

We also say that S_ϵ E-converges to S_0 as $\epsilon \to 0$ and we write $S_\epsilon \xrightarrow{EE} S_0$ if for any family of functions $f_\epsilon \in L^2(\Omega_\epsilon)$, we have

$$f_\epsilon \xrightarrow{E} f_0 \implies S_\epsilon f_\epsilon \xrightarrow{E} S_0 f_0\,.$$

Finally, we say that S_ϵ E-compact converges to S_0 as $\epsilon \to 0$ and we write $S_\epsilon \xrightarrow{C} S_0$ if $S_\epsilon \xrightarrow{EE} S_0$ and for any family of functions $f_\epsilon \in L^2(\Omega_\epsilon)$, with $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ and any sequence of positive numbers ϵ_n with $\epsilon_n \to 0$, there exists a subsequence ϵ_{n_k} and $u \in L^2(\Omega)$ such that $S_{\epsilon_{n_k}} f_{n_k} \xrightarrow{E} u$.

The following theorem from [40, Thm. 6.3] holds.

Theorem 3. *If $S_\epsilon \xrightarrow{C} S_0$ then the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

If we denote by $\mu_n(\epsilon)$, $n \in \mathbb{N}$ the sequence of eigenvalues of S_ϵ and by $u_n(\epsilon)$, $n \in \mathbb{N}$ a corresponding orthonormal sequence of eigenfunctions, then the stability of eigenvalues and eigenfunctions stated above has to be interpreted in the following sense:

(i) $\mu_n(\epsilon) \to \mu_n(0)$ as $\epsilon \to 0$.

(ii) For any sequence ϵ_k, $k \in \mathbb{N}$, converging to zero there exists an orthonormal sequence of eigenfunctions $u_n(0)$, $n \in \mathbb{N}$ in $L^2(\Omega)^3$ such that, possibly passing to a subsequence of ϵ_k, $u_n(\epsilon_k) \xrightarrow{E} u_n(0)$.

(iii) Given m eigenvalues $\mu_n(0), \dots, \mu_{n+m-1}(0)$ with $\mu_n(0) \neq \mu_{n-1}(0)$ and $\mu_{n+m-1}(0) \neq \mu_{n+m}(0)$ and corresponding orthonormal eigenfunctions $u_n(0), \dots, u_{n+m-1}(0)$, there exist m orthonormal generalized eigenfunctions (i.e., linear combinations of eigenfunctions) $v_n(\epsilon), \dots, v_{n+m-1}(\epsilon)$ associated with $\mu_n(\epsilon), \dots, \mu_{n+m-1}(\epsilon)$ such that $v_{n+i}(\epsilon) \xrightarrow{E} u_{n+i}(0)$ for all $i = 0, 1, \dots, m-1$.

Recall that μ is an eigenvalue of S_ϵ if and only if $\lambda = \mu^{-1}$ is an eigenvalue of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon - \tau\nabla \operatorname{div} v_\epsilon + v_\epsilon = \lambda v_\epsilon, & \text{in } \Omega_\epsilon, \\ \operatorname{div} v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \\ v_\epsilon \times \nu = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{4.4}$$

and that the corresponding eigenfunctions are the same. Note that the eigenvalues of (4.4) differ from those of (2.4) just by a translation. Thus, studying the stability of eigenvalues and eigenfunctions of the problem (4.4) or (2.4), is equivalent to studying the spectral stability of the family of operators S_ϵ. To do so, we recall that the weak formulation of problem (4.2) reads as follows: find $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon)$ such that

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \eta \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \eta \, dx + \tau \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \eta \, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \eta \, dx \tag{4.5}$$

for all $\eta \in X_{\mathrm{N}}(\Omega_\epsilon)$.

Suppose that for some $C > 0$ we have that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \leq C$ for every $\epsilon > 0$. Then, setting $\eta = v_\epsilon$ in (4.5) and observing that $\int_{\Omega_\epsilon} f_\epsilon \cdot v_\epsilon \, dx \leq \frac{1}{2}\int_{\Omega_\epsilon} |f_\epsilon|^2 \, dx + \frac{1}{2}\int_{\Omega_\epsilon} |v_\epsilon|^2 \, dx$, we get

$$\frac{1}{2}\int_{\Omega_\epsilon} |v_\epsilon|^2 \, dx + \int_{\Omega_\epsilon} |\operatorname{curl} v_\epsilon|^2 \, dx + \tau \int_{\Omega_\epsilon} |\operatorname{div} v_\epsilon|^2 \, dx \leq \frac{1}{2}\int_{\Omega_\epsilon} |f_\epsilon|^2 \, dx.$$

This in turn implies that for all $\epsilon > 0$

$$\|v_\epsilon\|_{X_{\mathrm{N}}(\Omega_\epsilon)} = \left(\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)^3} + \|\operatorname{curl} v_\epsilon\|^2_{L^2(\Omega_\epsilon)^3} + \|\operatorname{div} v_\epsilon\|^2_{L^2(\Omega_\epsilon)}\right)^{1/2} \leq c\|f_\epsilon\|_{L^2(\Omega_\epsilon)} = O(1). \tag{4.6}$$

In order to prove the E-convergence of the operators S_ϵ, it is necessary to consider the limit of functions v_ϵ. We note that if $\Omega \subset \Omega_\epsilon$ for all $\epsilon > 0$ then it would suffice to consider the restriction of v_ϵ to Ω and pass to the weak limit in Ω. Otherwise, it is convenient to extend functions v_ϵ to the whole of $\mathbb{R}^3$. To do so, we observe that by the uniform Gaffney inequality combined with inequality (4.6) it follows that $\|v_\epsilon\|_{H^1(\Omega_\epsilon)^3}$ is uniformly bounded. Moreover, the domains Ω_ϵ belong to the same Lipschitz class $C^{0,1}_M(\mathcal{A})$ for some $M > 0$ hence the functions v_ϵ can be extended to the whole of $\mathbb{R}^3$ with a uniformly bounded norm, see e.g., [7]. Thus, in the sequel we shall directly make the following assumption:

$$v_\epsilon \in H^1(\mathbb{R}^3)^3 \cap X_{\mathrm{N}}(\Omega_\epsilon), \quad \sup_{\epsilon>0} \|v_\epsilon\|_{\in H^1(\mathbb{R}^3)^3} \neq \infty\,. \tag{4.7}$$

Thus the family $\{v_\epsilon|_\Omega\}_{\epsilon>0}$ is bounded in $H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)$ and we can extract a sequence $\{v_{\epsilon_n}|_\Omega\}_{n\in\mathbb{N}}$, with $\epsilon_n \to 0$ as n goes to ∞, such that

$$v_{\epsilon_n}|_\Omega \underset{n\to\infty}{\rightharpoonup} v \qquad \text{weakly in } H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega) \tag{4.8}$$

for some $v \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)$. It turns out that v preserves the boundary conditions as the following lemma clarifies.

Lemma 4. *Assume that for some $\epsilon_n > 0$ with $\epsilon_n \to 0$, there exists $v \in H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ such that $\{v_{\epsilon_n}|_\Omega\}_{n\in\mathbb{N}}$ weakly converges to v in $H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$. Then $v \in X_{\mathrm{N}}(\Omega)$.*

Proof. To prove that $v \in X_{\mathrm{N}}(\Omega)$ we just need to make sure that $v \in H_0(\mathrm{curl}, \Omega)$. Since $v_{\epsilon_n} \in H_0(\mathrm{curl}, \Omega_{\epsilon_n})$ for all $n \in \mathbb{N}$, by Lemma 1 we know that the extension-by-zero $v^0_{\epsilon_n}$ of v_{ϵ_n} belongs to $H(\mathrm{curl}, \mathbb{R}^3)$ for all $n \in \mathbb{N}$. By the reflexivity of $H(\mathrm{curl}, \mathbb{R}^3)$ and the boundedness of the sequence $\left\{v^0_{\epsilon_n}\right\}_{n\in\mathbb{N}}$, we deduce that possibly passing to a subsequence, there exists a function $\tilde{v} \in H(\mathrm{curl}, \mathbb{R}^3)$ such that $v^0_{\epsilon_n} \rightharpoonup \tilde{v}$ weakly in $H(\mathrm{curl}, \mathbb{R}^3)$ as n goes to ∞. It suffices to show that $\tilde{v} = v^0$. Since $v^0_{\epsilon_n}$ is equal to zero outside of Ω_{ϵ_n}, it is clear that $\tilde{v} = 0$ a.e. in $\mathbb{R}^3 \setminus \Omega$. Moreover, since $v_{\epsilon_n}|_\Omega$ weakly converges to both $v, \tilde{v}$ in $H(\mathrm{curl}, \Omega)$, we have that $v = \tilde{v}$ a.e. in Ω. Thus the extension by zero of v to the whole of $\mathbb{R}^3$ is precisely $\tilde{v}$ and belongs to $H(\mathrm{curl}, \mathbb{R}^3)$. Using Lemma 1 again, we see that $v \in H_0(\mathrm{curl}, \Omega)$. □

Lemma 5. *Assume that condition* (3.14) *and the uniform Gaffney inequality* (4.1) *hold. For any $\epsilon > 0$ let $f_\epsilon \in L^2(\Omega_\epsilon)^3$. Suppose that $\sup_{\epsilon>0}\left\|f_\epsilon\right\|_{L^2(\Omega_\epsilon)^3} \neq \infty$ and that the extension-by-zero of the functions f_ϵ converge weakly in $L^2(\Omega)^3$ to some function $f \in L^2(\Omega)^3$ as $\epsilon \to 0$. For all $\epsilon > 0$, let $v_\epsilon := S_\epsilon f_\epsilon$ the (unique) weak solution in $X_{\mathrm{N}}(\Omega_\epsilon)$ of* (4.5) *with datum f_ϵ. Assume* (4.7) *and suppose that $v_\epsilon \rightharpoonup v$ weakly in $H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ to some $v \in H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$. Then $v = S_0 f$.*

Proof. First of all, we note that by Lemma 4, $v \in X_{\mathrm{N}}(\Omega)$. Define for $u, w \in H(\mathrm{curl}, \Omega_\epsilon) \cap H(\mathrm{div}, \Omega_\epsilon)$

$$Q_{\Omega_\epsilon}(u, w) := \int_{\Omega_\epsilon} u \cdot w \, dx + \int_{\Omega_\epsilon} \mathrm{curl}\, u \cdot \mathrm{curl}\, w \, dx + \tau \int_{\Omega_\epsilon} \mathrm{div}\, u \cdot \mathrm{div}\, w \, dx,$$

which is equivalent to the scalar product for the space $H(\mathrm{curl}, \Omega_\epsilon) \cap H(\mathrm{div}, \Omega_\epsilon)$. The square of the induced norm will be denoted with $Q_{\Omega_\epsilon}(\cdot)$. Note that since v_ϵ is the solution with datum f_ϵ, then we have that

$$Q_{\Omega_\epsilon}(v_\epsilon, \eta) = \int_{\Omega_\epsilon} f_\epsilon \cdot \eta$$

for all $\eta \in X_{\mathrm{N}}(\Omega_\epsilon)$.

Let φ be any function in $X_{\mathrm{N}}(\Omega)$ and let $\mathcal{P}_\epsilon\varphi$ the Atlas Piola trasform of φ. Since $\mathcal{P}_\epsilon\varphi \in X_{\mathrm{N}}(\Omega_\epsilon)$, we deduce tha

$$Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) = \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon\varphi \tag{4.9}$$

for all $\epsilon > 0$. We now show that

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon\varphi = \int_\Omega f \cdot \varphi \tag{4.10}$$

and

$$\lim_{\epsilon\to 0} Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) = Q_\Omega(v, \varphi) \tag{4.11}$$

In order to prove the first limit, it suffices to prove that

$$\int_{\Omega_\epsilon \cap V_j} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j} \underset{\epsilon\to 0}{\longrightarrow} \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{4.12}$$

for any $j = 1, \ldots, s'$, where $\tilde{\varphi}_{\epsilon,j}$ is defined in (3.13) (with $\tilde{\Omega}$ replaced by Ω_ϵ), since it is obvious that

$$\int_{\Omega_\epsilon \cap V_j} f_\epsilon \cdot \varphi_j \underset{\epsilon\to 0}{\longrightarrow} \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{4.13}$$

for any $j = s' + 1, \ldots, s$. We have that

$$\int_{\Omega_\epsilon \cap V_j} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j} = \int_{r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \varphi_j + \int_{(\Omega_\epsilon \cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j}. \tag{4.14}$$

Obviously

$$\lim_{\epsilon\to 0} \int_{r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \varphi_j = \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{4.15}$$

since the extension-by-zero of f_ϵ weakly converge to f in $L^2(\Omega)^3$, $\sup_{\epsilon>0}\left\|f_\epsilon\right\|_{L^2(\Omega_\epsilon)^3} < \infty$ and $\left|(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})\right|$ goes to 0 as $\epsilon \to 0$. Meanwhile

$$\left|\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j}\right| \leq \left\|f_\epsilon\right\|_{L^2(\Omega_\epsilon)^3} \left(\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2\right)^{\frac{1}{2}} \xrightarrow[\epsilon\to 0]{} 0 \tag{4.16}$$

by (3.23) and the hypothesis that $\sup_{\epsilon>0}\left\|f_\epsilon\right\|_{L^2(\Omega_\epsilon)^3} \neq \infty$. From (4.14), (4.15) and (4.16) we immediately deduce (4.12). Hence we have proved (4.10).

Let us now focus on (4.11). We write

$$\begin{aligned} Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) &= Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \\ &= Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi - \varphi) + Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \\ &= Q_\Omega(v_\epsilon, \varphi) - Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon, \varphi) + Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi - \varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \end{aligned} \tag{4.17}$$

By the weak convergence of v_ϵ to v we have that

$$Q_\Omega(v_\epsilon, \varphi) \to Q_\Omega(v, \varphi), \ \text{ as } \epsilon \to 0. \tag{4.18}$$

By the Cauchy-Schwarz inequality and Theorem 2, (iv) we get that

$$Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi - \varphi) \leq (Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon))^{1/2}(Q_{\Omega_\epsilon\cap\Omega}(\mathcal{P}_\epsilon\varphi - \varphi))^{1/2} \to 0, \ \text{ as } \epsilon \to 0 \tag{4.19}$$

Similarly,

$$Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon, \varphi) \leq (Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon))^{1/2}(Q_{\Omega\setminus\Omega_\epsilon}(\varphi))^{1/2} \to 0 \ \text{ as } \epsilon \to 0. \tag{4.20}$$

Finally,

$$Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \leq (Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon))^{1/2}(Q_{\Omega_\epsilon\setminus\Omega}(\mathcal{P}_\epsilon\varphi))^{1/2} \to 0 \ \text{ as } \epsilon \to 0 \tag{4.21}$$

since by (3.23), (3.31) and (3.39) it follows that $Q_{\Omega_\epsilon\setminus\Omega}(\mathcal{P}_\epsilon\varphi) \to 0$ as $\epsilon \to 0$. By combining (4.18)–(4.21), we deduce that limit (4.11) holds.

In conclusion, by using the limiting relations (4.10) and (4.11) in equation (4.9) we conclude that

$$Q_\Omega(v, \varphi) = \int_\Omega f \cdot \varphi$$

which means that v is the solution in $X_{\mathrm{N}}(\Omega)$ of the given problem with datum $f \in L^2(\Omega)$, as required.

□

Remark 1. *A careful inspection of the proof of Lemma 5 reveals that the uniform Gaffney inequality has been used only to prove the limiting relations* (4.18)–(4.20) *since the functions v_ϵ are required here to be defined on Ω and to have uniformly bounded norms. This problem does not occur if $\Omega \subset \Omega_\epsilon$ in which case only the Gaffney inequality in Ω is necessary. However, the uniform Gaffney inequality will be used in an essential way in the following statements also in the particular case $\Omega \subset \Omega_\epsilon$*

In the next lemma we prove that S_ϵ E-converges to S_0 as $\epsilon \to 0$.

Lemma 6. *Assume that condition* (3.14) *and the uniform Gaffney inequality* (4.1) *hold. Let $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ be such that $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f \in L^2(\Omega)^3$ for some function $f \in L^2(\Omega)^3$. Set $v_\epsilon := S_\epsilon f_\epsilon$ and $v := S_0 f$. Then $v_\epsilon \xrightarrow[\epsilon\to 0]{E} v$, hence $S_\epsilon \xrightarrow[\epsilon\to 0]{EE} S_0$.*

Proof. Since $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f$, then $\left\|f_\epsilon\right\|_{L^2(\Omega_\epsilon)^3} \le C$ for all $\epsilon > 0$ sufficiently small and consequently $\|v_\epsilon\|_{X_{\mathrm{N}}(\Omega_\epsilon)}$ is uniformly bounded with respect to ϵ, as shown in (4.6). By the uniform Gaffney inequality it follows that also $\|v_\epsilon\|_{H^1(\Omega_\epsilon)^3}$ is uniformly bounded. In particular

$$\lim_{\epsilon\to 0}\|v_\epsilon\|_{L^2(\Omega_\epsilon\setminus\Omega)^3} = 0$$

because $\left|\Omega_\epsilon \setminus \Omega\right| \to 0$ as ϵ goes to 0. This can be proved using the same argument used for (3.41) as follows:

$$\begin{aligned}
\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|v_\epsilon(x)\right|^2 dx &= \int_{r_j(\Omega_\epsilon\cap V_j)\setminus K_{\epsilon,j}} \left|v_\epsilon \circ r_j^{-1}(z)\right|^2 dz \\
&= \int_{W_j}\left(\int_{\hat{g}_{\epsilon,j}(\bar{z})}^{g_{\epsilon,j}(\bar{z})} \left|v_\epsilon \circ r_j^{-1}(\bar{z}, z_3)\right|^2 dz_3\right) d\bar{z} \\
&\le \int_{W_j} \left|g_{\epsilon,j}(\bar{z}) - \hat{g}_{\epsilon,j}(\bar{z})\right| \left\|v_\epsilon \circ r_j^{-1}(\bar{z},\cdot)\right\|^2_{L^\infty(a_{3j}, g_{\epsilon,j}(\bar{z}))^3} d\bar{z} \\
&\le \left\|g_{\epsilon,j} - \hat{g}_{\epsilon,j}\right\|_{L^\infty(W_j)} \int_{W_j} \left\|v_\epsilon \circ r_j^{-1}(\bar{z},\cdot)\right\|^2_{H^1(a_{3j}, g_{\epsilon,j}(\bar{z}))^3} d\bar{z} \\
&\le \kappa_\epsilon \left\|v_\epsilon \circ r_j^{-1}\right\|^2_{H^1(r_j(\Omega_\epsilon\cap V_j))^3} \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}$$

Hence to prove that $v_\epsilon \xrightarrow[\epsilon\to 0]{E} v$ we just have to show that

$$\lim_{\epsilon\to 0}\left\|v_\epsilon|_\Omega - v\right\|_{L^2(\Omega)^3} = 0. \tag{4.22}$$

Recall that $\{v_\epsilon|_\Omega\} \subset H^1(\Omega)^3$ is bounded in H^1-norm. Select now a sequence $\{v_{\epsilon_n}\}_{n\in\mathbb{N}}$ from the family. By the compact embedding of $H^1(\Omega)^3$ into $L^2(\Omega)^3$ we have that, up to choosing a subsequence, $v_{\epsilon_n}|_\Omega \to v^*$ strongly in $L^2(\Omega)^3$ and $v_{\epsilon_n}|_\Omega \rightharpoonup v^*$ weakly in $H^1(\Omega)^3$ for some $v^* \in H^1(\Omega)^3$. By Lemma 5 we have that $v^* = S_0 f = v \in X_{\mathrm{N}}(\Omega)$. This shows that for any extracted sequence of the family $\{v_\epsilon|_\Omega - v\}_{\epsilon>0}$, there exist a subsequence such that $\left\|v_{\epsilon_{n_k}}|_\Omega - v\right\|_{L^2(\Omega)^3} \xrightarrow[k\to\infty]{} 0$. Thus we can conclude that $\left\|v_\epsilon|_\Omega - v\right\|_{L^2(\Omega)^3} \xrightarrow[\epsilon\to 0]{} 0$, which is exactly (4.22). □

Remark 2. *The hypothesis of Lemma 6 can be weakened to only require that* $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3}$ *are uniformly bounded and that the extenstion-by-zero of* f_ϵ *(restricted to* Ω*) weakly converges to* f *in* $L^2(\Omega)^3$ *as* ϵ *goes to 0, which is a weaker assumption than* $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f$*.*

Finally we can state and prove the main theorem

Theorem 4. *Let* $\mathcal{A}$ *be an atlas in* $\mathbb{R}^3$ *and* $\{\Omega_\epsilon\}_{\epsilon>0}$ *be a family of bounded domains of class* $C^{1,1}(\mathcal{A})$ *converging to a bounded domain* Ω *of class* $C^{1,1}(\mathcal{A})$ *as* $\epsilon \to 0$*, in the sense that condition* (3.14) *holds. Suppose that the uniform Gaffney inequality* (4.1) *holds. Then* $S_\epsilon \xrightarrow{C} S_0$ *as* $\epsilon \to 0$*. In particular, spectral stability occurs: the eigenvalues of the operator* S_ϵ *converge to the eigenvalues of the operator* S_0*, and the eigenfunctions of the operator* S_ϵ *E-converge to the eigenfunctions of the operator* S_0 *as* $\epsilon \to 0$*.*

Proof. By Lemma 6 we have that $S_\epsilon \xrightarrow[\epsilon\to 0]{EE} S$. Now, suppose that we are given a family of data $\{f_\epsilon\}_{\epsilon>0}$ such that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \le 1$ for all $\epsilon > 0$, and extract a sequence $\{f_{\epsilon_n}\}_{n\in\mathbb{N}}$ from it. We have to show that we can always find a subsequence $\epsilon_{n_k} \to 0$ and a function $v \in L^2(\Omega)^3$ such that

$$S_{\epsilon_{n_k}} f_{\epsilon_{n_k}} \xrightarrow[k\to\infty]{E} v. \tag{4.23}$$

Possibly passing to a subsequence, we can find a function f to which the restriction to Ω of the extension-by-zero of $\{f_{\epsilon_n}\}_{n\in\mathbb{N}}$ weakly converge in $L^2(\Omega)^3$. Setting $v := S_0 f$, we can apply Lemma 6 and Remark 2 to find out that (4.23) holds.

Finally, the spectral stability is a consequence of the compact convergence of compact operators as stated in Theorem 3. □

5. Uniform Gaffney Inequalities and applications to families of oscillating boundaries

In this section we give sufficient conditions that guarantee the validity of a uniform Gaffney inequality of the type (4.1) for a family of Lipschitz domains Ω_ϵ, $\epsilon > 0$, belonging to the same class $C^{0,1}_M(\mathcal{A})$. To do so, we exploit a known relation between Gaffney inequalities and a priori estimates for the Dirichlet Laplacian that we formulate in our setting. We note that one of the two implications (namely, the validity of the Gaffney inequality implies the validity of the a priori estimate) is quite standard. The other one is a bit more involved, hence, for the sake of completeness, we include a proof.

Theorem 5. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *of class* $C^{0,1}_M(\mathcal{A})$*. Then the Gaffney inequality* (2.3) *holds for all* $u \in X_{\mathrm{N}}(\Omega)$ *and a constant* $C > 0$ *independent of* u *if and only if (the weak, variational) solutions* $\varphi \in H^1_0(\Omega)$ *to the Poisson problem*

$$\begin{cases} -\Delta\varphi = f, & \text{in } \Omega,\\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

satisfy the a priori estimate

$$\|\varphi\|_{H^2(\Omega)} \le \tilde{C}\|f\|_{L^2(\Omega)} \tag{5.2}$$

for all $f \in L^2(\Omega)$ *and a constant* $\tilde{C} > 0$ *independent of* f*. In particular, the constants* C *and* $\tilde{C}$ *depend on each other,* M *and* $\mathcal{A}$*.*

Proof. Assume that the a priori estimate (5.2) holds. We set $H^1_{\rm N}(\Omega) := X_{\rm N}(\Omega) \cap H^1(\Omega)^3$ and $E(\Omega) = \{\nabla\varphi : \varphi \in H^1_0(\Omega),\ \Delta\varphi \in L^2(\Omega)\}$. By [5, Thm. 4.1] there exists two linear operators $P : X_{\rm N}(\Omega) \to H^1_{\rm N}(\Omega)$ and $Q : X_{\rm N}(\Omega) \to E(\Omega)$ such that $u = Pu + Qu$ for all $u \in X_{\rm N}(\Omega)$ and such that

$$\|Pu\|_{H^1(\Omega)^3} + \|Qu\|_{L^2(\Omega)^3} + \|\operatorname{div} Qu\|_{L^2(\Omega)^3} \le C_{BS}\|u\|_{X_{\rm N}(\Omega)}$$

for all $u \in X_{\rm N}(\Omega)$, for some positive constant C_{BS}. A careful inspection of the proof of [5, Thm. 4.1] reveals that C_{BS} depends only on M, $\mathcal{A}$. By definition, $Qu = \nabla\varphi$ with $\varphi \in H^1_0(\Omega)$ and $\Delta\varphi \in L^2(\Omega)$. Since we have assumed that (5.2) holds, then

$$\|\varphi\|_{H^2(\Omega)} \le \tilde{C}\|\Delta\varphi\|_{L^2(\Omega)} = \tilde{C}\|\operatorname{div} Qu\|_{L^2(\Omega)} \le \tilde{C}C_{BS}\|u\|_{X_{\rm N}(\Omega)}$$

Thus, since $\|Qu\|_{H^1(\Omega)^3}$ is obviously controlled by $\|\varphi\|_{H^2(\Omega)}$ we deduce that

$$\|u\|_{H^1(\Omega)^3} \le \|Pu\|_{H^1(\Omega)^3} + \|Qu\|_{H^1(\Omega)^3} \le C\|u\|_{X_{\rm N}(\Omega)},$$

for all $u \in X_{\rm N}(\Omega)$, and (2.3) is proved.

Viceversa, assume that (2.3) holds and let φ be a solution to (5.1). Since $\nabla\varphi \in X_{\rm N}(\Omega)$, by (2.3) it follows that

$$\begin{aligned}
\|\nabla\varphi\|_{H^1(\Omega)^3} &\le C(\|\nabla\varphi\|_{L^2(\Omega)^3} + \|\operatorname{curl}\nabla\varphi\|_{L^2(\Omega)^3} + \|\operatorname{div}\nabla\varphi\|_{L^2(\Omega)^3}) \\
&= C(\|\nabla\varphi\|_{L^2(\Omega)^3} + \|\Delta\varphi\|_{L^2(\Omega)}) \\
&\le C(c_{\mathcal{P}}\|\Delta\varphi\|_{L^2(\Omega)} + \|\Delta\varphi\|_{L^2(\Omega)}) \\
&\le C(c_{\mathcal{P}} + 1)\|f\|_{L^2(\Omega)},
\end{aligned}$$

where we have used [20, Lemma 1] and $c_{\mathcal{P}}$ denotes the usual Poincaré constant. This, combined with the Poincaré's inequality, immediately implies (5.2). □

Example 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 such that around a boundary point (identified here with the origin) is described by the subgraph $x_N < g(\bar{x})$ of the C^1 function defined by*

$$g(x_1, \dots, x_{N-1}) = |x_1|/\log|x_1|$$

It is proved in [34, 14.6.1] that for this domain the a priori estimate (5.2) *does not hold. Thus, by Theorem 5 it follows that not even the Gaffney inequality holds for this domain for $N = 3$.*

Theorem 5 highlights the importance of proving the a priori estimate (5.2) and getting information on the constant $\tilde{C}$. We do this by following the approach of Maz'ya and Shaposhnikova [34] and using the notion of domains Ω with boundaries $\partial\Omega$ of class $\mathcal{M}_2^{3/2}(\delta)$. We re-formulate the definition in Maz'ya and Shaposhnikova [34, § 14.3.1] by using the atlas classes. Here we can treat the general case of domains in $\mathbb{R}^N$ with $N \ge 2$.

Note that in this section, following [34] we find it convenient to assume directly that the functions g_j describing the boundary of Ω as in Definition 1 are extended to the whole of $\mathbb{R}^{N-1}$ and belong to the corresponding function spaces defined on $\mathbb{R}^{N-1}$.

Definition 4. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and $\delta > 0$. We say that a bounded domain Ω in $\mathbb{R}^N$ is of class $\mathcal{M}_2^{3/2}(\delta, \mathcal{A})$ if Ω is of class $C^{0,1}(\mathcal{A})$ and the gradients ∇g_j of the functions g_j describing the boundary of Ω as in Definition 1 belong to the space $MW_2^{1/2}(\mathbb{R}^{N-1})$ of Sobolev multipliers with*

$$\|\nabla g_j\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} \leq \delta \tag{5.3}$$

for all $j = 1, \dots s'$. We say that a bounded domain Ω in $\mathbb{R}^N$ is of class $\mathcal{M}_2^{3/2}(\delta)$ if it is of class $\mathcal{M}_2^{3/2}(\delta, \mathcal{A})$ for some atlas $\mathcal{A}$.

Recall that $MW_2^{1/2}(\mathbb{R}^{N-1}) = \{f \in W_{2,loc}^{1/2}(\mathbb{R}^{N-1}) : \ f\varphi \in W_2^{1/2}(\mathbb{R}^{N-1})$ for all $\varphi \in W_2^{1/2}(\mathbb{R}^{N-1})\}$ and that $\|f\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} = \sup\{\|f\varphi\|_{W_2^{1/2}(\mathbb{R}^{N-1})} : \|\varphi\|_{W_2^{1/2}(\mathbb{R}^{N-1})} = 1\}$, where $W_2^{1/2}(\mathbb{R}^{N-1})$ denotes the standard Sobolev space with fractional order of smoothness $1/2$ and index of summability 2 (for simplicity, in (5.3) we use the the same symbol for the norm of a vector field).

Remark 3. *Note that by [34, Thm. 4.1.1] there exists $c > 0$ depending only on N such that the functions g_j in Definition 4 satisfy the estimate $\|\nabla g_j\|_{L^\infty(\mathbb{R}^{N-1})} \leq c\|\nabla g_j\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} \leq c\delta$, see also [34, Thm. 14.6.4]. Thus, if Ω is of class $\mathcal{M}_2^{3/2}(\delta, \mathcal{A})$ then it is also of class $C_M^{0,1}(\mathcal{A})$ with $M = c\delta$.*

The following theorem is a reformulation of [34, Thm. 14.5.1]

Theorem 6. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. If Ω is a bounded domain of class $\mathcal{M}_2^{3/2}(\delta, \mathcal{A})$ for some δ sufficiently small (depending only on N) then the a priori estimate* (5.2) *holds for some constant $\tilde{C}$ depending only on N and $\mathcal{A}$.*

By [34, Corollaries. 14.6.1, 14.6.2] it is possible to prove the following theorem based on the condition (5.5) from [34, (14.6.9)] . Here, given an atlas $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$, by a refinement of $\mathcal{A}$ we mean an atlas of the type $\widetilde{\mathcal{A}} = (\tilde{\rho}, \tilde{s}, \tilde{s}', \{\tilde{V}_j\}_{j=1}^{\tilde{s}}, \{\tilde{r}_j\}_{j=1}^{\tilde{s}})$ where $\tilde{\rho} \leq \rho$, $s \leq \tilde{s}$, $s' \leq \tilde{s}'$, $\cup_{j=1}^{\tilde{s}} \tilde{V}_j = \cup_{j=1}^{s} V_j$, $\{\tilde{r}_j\}_{j=1}^{\tilde{s}} \subset \{r_j\}_{j=1}^{s}$, which can be thought as an atlas constructed from $\mathcal{A}$ by replacing each cuboid $V_j = r_j(W_j \times]a_{N,j}, b_{N,j}[)$ by a finite number of cuboids of the form $\widetilde{V}_{j,l} = r_j(\widetilde{W}_{j,l} \times]a_{N,j}, b_{N,j}[)$, $l = 1, \dots m_j$, where $W_j = \cup_{l=1}^{m_j} \widetilde{W}_{j,l}$.

Theorem 7. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let Ω be a bounded domain of class $C_M^{0,1}(\mathcal{A})$. Let ω be a (non-decreasing) modulus of continuity for the gradients ∇g_j of the functions g_j describing the boundary of Ω, that is*

$$|\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})| \leq \omega(|\bar{x} - \bar{y}|) \tag{5.4}$$

*for all $\bar{x}, \bar{y} \in \mathbb{R}^{N-1}$. Assume that there exists $D > 0$ such that the function ω satisfies the following condition**

$$\int_0^\infty \left(\frac{\omega(t)}{t}\right)^2 dt \leq D\,. \tag{5.5}$$

Then there exists $C > 0$ depending only on N, $\mathcal{A}$, D such that if $M \leq C\delta$ then, possibly replacing the atlas $\mathcal{A}$ with a refinement of $\mathcal{A}$, Ω is of class $\mathcal{M}_2^{3/2}(\delta, \mathcal{A})$.

Proof. We begin with the case $N \geq 3$. By [34, Cor. 14.6.1] there exists $c > 0$ depending only on N such that if $x = (\bar{x}, g_j(\bar{x})) \in \partial\Omega$ is any point of the boundary represented in local charts by a profile

*Here only the integrability at zero really matters and one could consider integrals defined in a neigborhhood of zero.

function g_j and the following inequality

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho(\bar{x})}\frac{\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}}+\|\nabla g_j\|_{L^\infty(B_\rho(\bar{x}))}\right)\le c\delta \tag{5.6}$$

is satisfied, then, possibly replacing the atlas $\mathcal{A}$ with a refinement of its, Ω is of class $\mathcal{M}_2^{3/2}(\delta,\mathcal{A})$. Here $|E|$ denotes the $N-1$-dimensional Lebesgue measure of E,

$$D_{3/2}(g_j,B_\rho)(\bar{x})=\left(\int_{B_\rho(\bar{x})}|\nabla g_j(\bar{x})-\nabla g_j(\bar{y})|^2|\bar{x}-\bar{y}|^{-N}d\bar{y}\right)^{1/2},$$

and $B_\rho(\bar{x})$ the ball in $\mathbb{R}^{N-1}$ of radius ρ and centre $\bar{x}$. We refer to [34, §14.7.2] for the local characterization of the boundaries of domains of class $\mathcal{M}_2^{3/2}(\delta,\mathcal{A})$.

We have

$$\begin{aligned}\int_E\int_{B_\rho(\bar{x})}|\nabla g_j(\bar{x})-\nabla g_j(\bar{y})|^2|\bar{x}-\bar{y}|^{-N}d\bar{y}d\bar{x}&\le\int_E\int_{B_\rho(\bar{x})}\frac{\omega^2(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|^N}d\bar{y}d\bar{x}\\&=\int_E\int_{B_\rho(0)}\frac{\omega^2(|\bar{h}|)}{|\bar{h}|^N}d\bar{h}d\bar{x}=\sigma_{N-2}|E|\int_0^\rho\left|\frac{\omega(t)}{t}\right|^2dt\le\sigma_{N-2}D|E|\,.\end{aligned}\tag{5.7}$$

Here σ_m denotes the m-dimensional measure of the m-dimensional unit sphere. Thus

$$\frac{\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}}\le(\sigma_{N-2}D)^{1/2}|E|^{\frac{1}{2(N-1)}}=O(\rho^{1/2}),$$

hence

$$\frac{\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}}\le c\delta, \tag{5.8}$$

provided ρ is sufficiently small. Thus, inequality (5.6) follows if we assume directly that $\|\nabla g_j\|_{L^\infty(B_\rho)}\le c\delta$.

In the case $N=2$, by [34, Cor. 14.6.1] it suffices to replace (5.6) by the following inequality

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho(\bar{x})}\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}|\log|E||^{1/2}+\|\nabla g_j\|_{L^\infty(B_\rho(\bar{x}))}\right)\le c\delta \tag{5.9}$$

and use the same argument as above. □

By combining Theorems 6 and 7, we deduce the validity of the following result

Corollary 1. *Under the same assumptions of Theorem 7, there exists $\tilde{C}>0$ depending only on N, $\mathcal{A}$, D such that if $M<\tilde{C}^{-1}$ then the a priori estimate* (5.2) *holds.*

Finally, by Theorem 5 and Corollary 1 we deduce the following result ensuring the validity of uniform Gaffney inequality that can be used in our spectral stability results.

Corollary 2. *Under the same assumptions of Theorem 7 with $N=3$, there exists $C>0$ depending only on $\mathcal{A}$ and D such that if $M<C^{-1}$ then the Gaffney inequality* (2.3).

5.1. Applications to families of oscillating boundaries

It is clear that in order to apply Theorem 7 and Corollaries 1, 2, it suffices to assume that the gradients ∇g_j of the functions g_j describing the boundary of a domain Ω as in Definition 1 are of class $C^{0,\beta}$ with $\beta \in]1/2, 1]$, that is

$$|\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})| \leq K|\bar{x} - \bar{y}|^{\beta} \tag{5.10}$$

for some positive constant K and all $\bar{x}, \bar{y} \in W_j$, and that the functions g_j have sufficiently small Lipschitz constants. As we have already mentioned, in principle, the second condition is not a big obstruction to the application of these results, since for a domain of class C^1 one can find a sufficiently refined atlas, adapted to the tangent planes of a finite number of boundary points, such that the C^1 norms, hence the Lipschitz constants, of the profile functions g_j are arbitrarily close to zero. Thus, we can apply our results to uniform classes of domains of class $C^{1,\beta}$ since condition (5.5) would be satisfied exactly because $\beta > 1/2$ (as we have said, here what matters is the behaviour of the modulus of continuity $\omega(t)$ for t close to zero and one can assume directly that $\omega(t)$ is constant for t big enough).

Thus, we can prove the following result. Note that here the domains Ω_ϵ are assumed to be of class $C^{1,1}$ and that they belong to the uniform class $C^{1,\beta}_K(\mathcal{A})$ with $K > 0$ fixed, which in particular implies the validity of (5.10) for all functions $g_{\epsilon,j}$ and all $\epsilon > 0$. (Recall that the operators S_ϵ are defined in the beginning of Section 4.)

Theorem 8. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^3$ and $\{\Omega_\epsilon\}_{\epsilon>0}$ be a family of bounded domains of class $C^{1,1}(\mathcal{A})$ converging to a bounded domain Ω of class $C^{1,1}(\mathcal{A})$ as $\epsilon \to 0$, in the sense that condition* (3.14) *holds. Suppose that Ω is of class $C^{0,1}_M(\mathcal{A})$ with M small enough as in Corollary 2. Suppose also that all domains Ω_ϵ are of class $C^{1,\beta}_K(\mathcal{A})$ with the same parameters $\beta \in]1/2, 1]$ and $K > 0$. Then the uniform Gaffney inequality* (4.1) *holds provided ϵ is small enough. Moreover, $S_\epsilon \xrightarrow{C} S$ as $\epsilon \to 0$. In particular, spectral stability occurs: the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

Proof. Since Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense that condition (3.14) holds, it follows that the gradients of the functions $g_{\epsilon,j}$ describing the boundary of Ω_ϵ converge uniformly to the gradients of the functions g_j describing the boundary of Ω. Thus, Ω_ϵ is of class $C^{0,1}_M(\mathcal{A})$ provided ϵ is small enough. By the discussion above, Corollary 2 is applicable and the uniform Gaffney inequality (4.1) holds provided ϵ is small enough. Then the last part of the statement follows by Theorem 4. □

A prototype for the classes of domains under discussion is given by domains designed by profile functions often used in homogenization theory, in particular in the study of thin domains. Namely, assume that one of the profile functions $g_{\epsilon,j}$, call it g_ϵ, is of the form

$$g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon) \tag{5.11}$$

for some function b of class $C^{1,1}(\mathbb{R}^2)$ and $\alpha > 0$, and assume that the gradient of b is bounded. If $\omega_{\nabla b}$ is a (non-decreasing) modulus of continuity of ∇b, then we have

$$|\nabla g_\epsilon(\bar{x}) - \nabla g_\epsilon(\bar{y})| = \epsilon^{\alpha-1}|\nabla b(\bar{x}/\epsilon) - \nabla b(\bar{y}/\epsilon)| \leq \epsilon^{\alpha-1}\omega_{\nabla b}\left(\frac{\bar{x}-\bar{y}}{\epsilon}\right),$$

hence the function ω to be considered in (5.4) is given by $\omega(t) = \epsilon^{\alpha-1}\omega_{\nabla b}(t/\epsilon)$. Observe that

$$\int_0^\infty \left(\frac{\omega(t)}{t}\right)^2 dt = \epsilon^{2\alpha-2}\int_0^\infty \left(\frac{\omega_{\nabla b}(t/\epsilon)}{t}\right)^2 dt = \epsilon^{2\alpha-3}\int_0^\infty \left(\frac{\omega_{\nabla b}(s)}{s}\right)^2 ds\,. \tag{5.12}$$

Moreover, since b is assumed to be of class $C^{1,1}$, we have that $\omega_{\nabla b}(t) \le ct$ for t in a neighborhhood of zero. Thus, if $\alpha \ge 3/2$ and ϵ_0 is any fixed positive constant, it follows that that

$$\sup_{\epsilon\in]0,\epsilon_0]} \epsilon^{2\alpha-3}\int_0^\infty \left(\frac{\omega_{\nabla b}(s)}{s}\right)^2 ds \ne \infty\,. \tag{5.13}$$

Since the gradient of g_ϵ is arbitrarily close to zero for ϵ sufficiently small, we have that Theorem 7 and Corollaries 1, 2 are applicable and the Gaffney inequality (4.1) holds for all ϵ sufficiently small, with a constant $C > 0$ independent of ϵ. The same arguments can be applied to families of profile functions of the type

$$g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})$$

where b is as above and ψ is a fixed $C^{1,1}$ function with bounded gradient. Thus, we can state the following stability result concerning a local perturbation of a domain Ω.

Theorem 9. *Let W be a bounded open rectangle in $\mathbb{R}^2$, $b \in C^{1,1}(\mathbb{R}^2)$ with bounded gradient, $b \ge 0$, and $\psi \in C^{1,1}_c(W)$, $\alpha > 3/2$. Assume that Ω and Ω_ϵ, $\epsilon > 0$ are domains of class $C^{1,1}$ in $\mathbb{R}^3$ satisfying the following condition:*

(i) $\Omega \cap (W\times]-1,1[) = \{(\bar{x}, x_3) \in \mathbb{R}^3 :\ \bar{x} \in W,\ -1 < x_3 < 0\}$;

(ii) $\Omega_\epsilon \cap (W\times]-1,1[) = \{(\bar{x}, x_3) \in \mathbb{R}^3 :\ \bar{x} \in W,\ -1 < x_3 < \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})\}$ *where* $b \in C^{1,1}(\mathbb{R}^2)$ *has bounded gradient, and* $\psi \in C^{1,1}_c(W)$;

(iii) $\Omega \setminus (W\times]-1,1[) = \Omega_\epsilon \setminus (W\times]-1,1[)$;

Then the family $\{\Omega_\epsilon\}_{\epsilon>0}$ converges to Ω in the sense that condition (3.14) *holds. Moreover, the uniform Gaffney inequality* (4.1) *holds and $S_\epsilon \xrightarrow{C} S$ as $\epsilon \to 0$. In particular, spectral stability occurs: the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

Proof. By assumptions, the domains Ω and Ω_ϵ belong to the same atlas class $C^{1,1}(\mathcal{A})$ for a suitable atlas $\mathcal{A}$, and $W\times]-1,1[$ is one of the local charts of $\mathcal{A}$. In particular, the profile functions describing the boundaries of Ω and Ω_ϵ in that chart are given by $g(\bar{x}) = 0$ and $g_\epsilon = \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})$ for all $\bar{x} \in W$.

As in the proof of [3, Thm. 7.4], if $\tilde{\alpha} \in]3/2, \alpha[$ is fixed then one can easily check that conditions (3.14) are satisfied with $k_\epsilon = \epsilon^{2\tilde{\alpha}/3}$. By (5.2) and the discussion above, it follows that the Gaffney inequality (2.3) holds with a constant C independent of ϵ, provided ϵ is sufficiently small. To complete the proof it suffices to apply Theorem 4. □

Remark 4. *It is clear that condition* (5.13) *is satisfied also in the case $\alpha = 3/2$. Thus the uniform Gaffney inequality* (4.1) *holds also in the case $\alpha = 3/2$ in Theorem 9. However, in this case the convergence of Ω_ϵ to Ω in the sense of* (3.14) *is not guaranteed hence we cannot directly deduce that we have spectral stability. Thus, another method has to be used in the analysis of the stability problem* *for $\alpha = 3/2$. For example, in the case of non-constant periodic functions b one could use the unfolding method as in [11], adopted also in [2, 3, 19, 20]: in those papers, for $\alpha = 3/2$ we have spectral instability in the sense that the limiting problem differs from the given problem in Ω by a strange term appearing in the boundary conditions (as often happens in homogenization problems). We plan to discuss the details of this problem for the curlcurl operator in a forthcoming paper, but we can already mention that a preliminary formal analysis would indicate that no strange limit appears in the limiting problem for $\alpha = 3/2$. On the other hand, at the moment we are not able to formulate any conjecture for the case $\alpha < 3/2$ although, on the base of the results of [11] concerning the Navier-Stokes system, a degeneration phenomenon (to Dirichlet boundary conditions) could not be excluded.*

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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