

**FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS
OVER FINITE SIMPLE GROUPS OF LIE TYPE VII.
SEMISIMPLE CLASSES IN $\mathbf{PSL}_n(q)$ AND $\mathbf{PSp}_{2n}(q)$**

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ABSTRACT. We show that the Nichols algebra of a simple Yetter-Drinfeld module over a projective special linear group over a finite field whose support is a semisimple orbit has infinite dimension, provided that the elements of the orbit are reducible; we obtain a similar result for all semisimple orbits in a finite symplectic group except in low rank. We prove that orbits of irreducible elements in the projective special linear groups of odd prime degree could not be treated with our methods. We conclude that any finite-dimensional pointed Hopf algebra H with group of group-like elements isomorphic to $\mathbf{PSL}_n(q)$ ($n \geq 4$), $\mathbf{PSL}_3(q)$ ($q > 2$), or $\mathbf{PSp}_{2n}(q)$ ($n \geq 3$), is isomorphic to a group algebra.

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1. INTRODUCTION

1.1. **The problem.** Let G be a finite group. The conjugacy class of $x \in G$ is denoted by \mathcal{O}_x^G or \mathcal{O}_x . The subgroup of G generated by $I \subset G$ is denoted by $\langle I \rangle$. Consider the following properties of a conjugacy class \mathcal{O} of G :

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- (C) There are $H < G$ and $r, s \in H \cap \mathcal{O}$ such that $rs \neq sr$, $H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle$, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$ and $\min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2$ or $\max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4$.
- (D) There exist $r, s \in \mathcal{O}$ such that $\mathcal{O}_r^{(r,s)} \neq \mathcal{O}_s^{(r,s)}$ and $(rs)^2 \neq (sr)^2$.
- (F) There are $r_a \in \mathcal{O}$, $a \in \mathbb{I}_4 = \{1, 2, 3, 4\}$, such that $\mathcal{O}_{r_a}^{(r_a:a \in \mathbb{I}_4)} \neq \mathcal{O}_{r_b}^{(r_b:a \in \mathbb{I}_4)}$ and $r_a r_b \neq r_b r_a$ for $a \neq b \in \mathbb{I}_4$.

We say that \mathcal{O} is of type C, D, F when the corresponding property holds. As explained in the Introduction to [5], to which we refer the reader, the next question has profound implications for the classification of finite-dimensional Nichols algebras, and therefore, of finite-dimensional pointed Hopf algebras.

Question 1. *Determine which conjugacy classes of a given finite (non-abelian) group G are of type C, D or F.*

Indeed, if a conjugacy class \mathcal{O} is type C, D or F, then the Nichols algebra associated with any Yetter-Drinfeld module of group type with support isomorphic to \mathcal{O} has infinite dimension; for brevity we say in this case that \mathcal{O} *collapses*. For the purposes of this paper further precision on Nichols algebras is not needed.

If \mathcal{O} is neither of type C, D nor F then we say that it is *kthulhu*. It follows at once from the previous definitions that if $\mathcal{O} \cap H$ consists of commuting elements or is a single conjugacy class of H for any $H \leq G$, then \mathcal{O} is *kthulhu*.

Intuitively, checking whether a class satisfies the criteria of types C, D or F can be best carried out by means of inductive arguments that are more flexible in the language of racks, see Section 2. Conjugacy classes are the prototypical examples of racks. We will say that a rack is *abelian* if it consists of commuting elements. In this situation, none of the three criteria applies but the Hopf algebraic problem can be attacked with methods stemming from the abelian group case. However, not all *kthulhu* racks are abelian, and abelian group techniques are not always effective. We say that a rack is *sober* if every subrack is either abelian or indecomposable [1, §1.5]; and *austere* if every subrack generated by two elements is either abelian or indecomposable [3, 2.11]. Clearly, sober implies austere and austere implies *kthulhu*. Subsidiary to Question 1, we propose:

Question 2. *Determine which conjugacy classes of a given finite (non-abelian) group G are sober or austere, or *kthulhu*.*

1.2. Simple groups of Lie type. It is natural, and convenient, to start addressing Questions 1 and 2 by assuming that G is simple non-abelian, see [9] for the importance of this reduction. When G is alternating or sporadic, this was addressed in [7, 8, 14, 15]. The series of papers [1, 2, 3, 4, 5, 12] treat the case when G is simple of Lie type. In the first five papers, Questions 1 and 2 were answered for non-semisimple conjugacy classes in Chevalley or Steinberg groups. The sixth is devoted to Suzuki and Ree groups.

In the present paper we deal with Questions 1 and 2 for semisimple conjugacy classes in the classical Chevalley groups. The main new difficulty is due to the deeper influence of arithmetics, as opposed to the unipotent classes and the mixed classes, which can be reduced in most of the cases to a unipotent one in a smaller group. We summarize our main results and then discuss the proofs and the main new challenges in the present paper.

Theorem I. *Let $\mathcal{O} \neq \{e\}$ be a semisimple conjugacy class in a group G .*

- (i) *If $G = \mathbf{PSL}_n(q)$, then any \mathcal{O} not listed in Table 1 collapses.*
- (ii) *If $G = \mathbf{PSp}_n(q)$, then any \mathcal{O} collapses with the possible exception of the orbit of non-trivial involutions if $n = 2$ and $q \in \{3, 5, 7\}$.*

Part (i) is proved in Theorem 5.4; see Remark 5.3 for the cases with small q excluded in the statement. Part (ii) is proved in Theorem 7.1. For other Chevalley groups, there is substantial information in Theorems 4.1 and 6.2.

Roughly the proofs of these results go as follows: pick a simple group \mathbf{G} , a surjective morphism of groups $\pi : G \rightarrow \mathbf{G}$ and conjugacy classes \mathfrak{D} and \mathcal{O} in G and \mathbf{G} respectively, such that $\pi(\mathfrak{D}) = \mathcal{O}$. Then look at the subgroups H intersecting \mathfrak{D} . If $H \cap \mathfrak{D}$ splits as more than one conjugacy class of H for one $H \leq G$, then work out the details to have that \mathfrak{D} is of type C or D and that this is preserved by the projection $\pi : \mathfrak{D} \rightarrow \mathcal{O}$. When G is of Lie type, the subgroup H is usually found by looking in one way or another at the structure of the algebraic group behind G .

But if $\pi(H) \cap \mathcal{O}$ is either abelian or one conjugacy class of $\pi(H)$ for every $H \leq G$, then \mathcal{O} is kthulhu. In the previous papers in the series, which dealt with unipotent and mixed conjugacy classes, when this happened, either \mathbf{G} was ‘small’ and had few subgroups, or the class \mathcal{O} was unipotent of small size and fitted into some well-behaved infinite family. In this case, the field of definition of the group played a marginal role.

In the present paper, devoted to semisimple classes, we discovered a new collection of kthulhu classes with a different behaviour: any conjugacy class \mathcal{O} of an irreducible element in $\mathbf{PSL}_n(q)$ where n is an odd prime is kthulhu. To show this we used the main result of [17] to get the list of the $H \leq G$ intersecting \mathfrak{D} and use some arithmetic manipulations. This outcome, and the techniques used, differ significantly with those of the previous of the series and underline the connection of semisimple classes with arithmetical aspects.

1.3. Applications to Hopf algebras. As in previous papers, we say that a finite group G collapses if every finite-dimensional pointed Hopf algebra H with $G(H) \simeq G$ is necessarily isomorphic to $\mathbb{C}G$. As a corollary of our main Theorem and results from previous papers in the series, we obtain new families of groups that collapse, see Theorem III, extending [3, Theorem 1.2]. For this, we first draw the complete list of kthulhu classes in the simple

TABLE 1. Kthulhu semisimple classes in $\mathbf{PSL}_n(q)$.

| n | q | class | Remark |
|-----------|--|--------------------------|---------|
| 2 | 2, $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$ | (3) | abelian |
| | 3, $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$ | (2^2) | abelian |
| | 4, $\mathbf{PSL}_2(4) \simeq \mathbb{A}_5$ | (5) | sober |
| | 5, $\mathbf{PSL}_2(5) \simeq \mathbb{A}_5$ | ($1, 2^2$) | sober |
| | 9, $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$ | (1, 5) | kthulhu |
| | even and not a square | irreducible, order 3 | sober |
| | all | irreducible, order > 3 | sober |
| odd prime | all | irreducible | kthulhu |

groups $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$ for $n \geq 2$, any q . This information combines the main result of this paper with a corrected version of [1, Table 1], [3, Table 3] and [5, Theorem 1.1] and a careful analysis of the cases of the groups $\mathbf{PSL}_2(q)$ for $q = 2, 3, 4, 5, 7, 9$, where some exceptions occur. We point out that the classes labeled $(1^{r_1}, 2)$ in $\mathbf{Sp}_{2n}(9)$, occurring in [3, Tables 3,5] are in fact not kthulhu: they are of type C, as each of them includes a non-trivial unipotent class of type (2) in $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$ which is of type C, cf. Example 2.7. Also, the classes of involutions in $\mathbf{PSL}_2(7)$ in [3, Table 1] are not kthulhu: since $\mathbf{PSL}_3(2) \simeq \mathbf{PSL}_2(7)$, they are of type C by [3, Lemma 2.12].

Theorem II. *Let G be either $\mathbf{PSL}_n(q)$ or $\mathbf{PSp}_{2n}(q)$, $n \geq 2$ and let \mathcal{O} be a non-trivial conjugacy class in G different from the class of a split involution in $\mathbf{PSp}_4(7)$. Then \mathcal{O} is kthulhu if and only if it occurs in Table 2.*

It remains open to determine whether the conjugacy class of split involutions in $\mathbf{PSp}_4(7)$ is kthulhu.

The next result combines [1, Theorem 1.4], [3, Theorem 1.2] and [5, Theorem 1.2] with Theorem I.

Theorem III. *The groups $\mathbf{PSL}_n(q)$ with $n \geq 4$, $\mathbf{PSL}_3(q)$ with $q > 2$, and $\mathbf{PSp}_{2n}(q)$, $n \geq 3$, collapse. \square*

In the group $\mathbf{PSL}_3(2)$, there is just one class that could not be treated, namely the regular unipotent class \mathcal{O} , which is sober. Actually $\mathbf{PSL}_3(2) \simeq \mathbf{PSL}_2(7)$ and for the latter group, \mathcal{O} is semisimple.

1.4. Conventions. If $a \leq b \in \mathbb{N}_0$, then $\mathbb{I}_{a,b}$ denotes $\{a, a+1, \dots, b\}$; also $\mathbb{I}_a = \mathbb{I}_{1,a}$ for simplicity.

Let G be a group. The centraliser of $X \subset G$ is denoted by $C_G(X)$. If $x, y \in G$, then $x \triangleright y := xyx^{-1}$. We write $X \geq Y$, or $Y \leq X$, to express that Y is a subrack of X (or a subgroup, or more generally a subobject in a given category). The normality of a subgroup is expressed by $N \triangleleft G$. If $x \in G$ and $H \leq G$, then \mathcal{O}_x^H will indicate the orbit of x under the conjugation action.

TABLE 2. Kthulhu classes, $G = \mathbf{PSL}_n(q)$ or $\mathbf{PSp}_{2n}(q)$, $n \geq 2$.

| G | n | q | type of class | description/label |
|------------------------|-----------|---|---------------|------------------------|
| $\mathbf{PSL}_n(q)$ | 2 | even or else odd and not a square | unipotent | (2) |
| | | 5 | semisimple | involution |
| | | all | semisimple | irreducible, $ x > 3$ |
| | | even and not a square | semisimple | irreducible $ x = 3$ |
| | 3 | 2 | unipotent | (3) |
| | odd prime | all | semisimple | irreducible |
| $\mathbf{PSp}_{2n}(q)$ | ≥ 2 | even | unipotent | $W(1)^a \oplus V(2)$ |
| | | odd and not a square | unipotent | $(1^{r_1}, 2)$ |
| | 2 | even | unipotent | $W(2)$ |
| | | 3,5 | semisimple | split involution |

Let $q = p^m$, p a prime and $m \in \mathbb{N}$. Let \mathbb{F}_q be the field with q elements and \mathbb{k} the algebraic closure of \mathbb{F}_q . We denote by $\mathbb{G}_n(\mathbb{k})$ the group of n -th roots of unity in a field \mathbb{k} .

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2. RACKS

2.1. Racks. As in previous papers we use the language of racks; see [9] for more information. A rack is a pair $(\mathcal{O}, \triangleright)$ where \mathcal{O} is a non-empty set and $\triangleright : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ is a self distributive operation such that $\varphi_x := x \triangleright _$ is bijective for any $x \in \mathcal{O}$. A subset $\mathcal{O}' \subset \mathcal{O}$ is a subrack if $\mathcal{O}' \triangleright \mathcal{O}' \subset \mathcal{O}'$. Let $\text{Inn } \mathcal{O}$ be the group generated by the image of the map $\varphi : \mathcal{O} \rightarrow \mathbb{S}_{\mathcal{O}}$. The main examples of racks considered in this paper are (unions of) conjugacy classes

of a finite group with \triangleright being the conjugation. A rack $(\mathcal{O}, \triangleright)$ is abelian if $x \triangleright y = y$ for any $x, y \in \mathcal{O}$. Also, a rack is indecomposable if it can not be presented as the disjoint union of two subracks and decomposable otherwise.

The following observation will be useful, especially when dealing with orthogonal groups.

Remark 2.1. Let G be a finite group, $N \triangleleft G$, $g \in G - N$. Then the orbit \mathcal{O}_g^N of g under the conjugation action of N is a subrack of \mathcal{O}_g^G .

This is a special case of [13, Remark 3.2] that can be verified directly. Notice that if $N \leq G$ is not normal, then \mathcal{O}_g^N may fail to be a subrack of \mathcal{O}_g^G . For instance, take $G = \mathbb{S}_4$, $g = (123)$ and $N = \langle (12)(34) \rangle$; then $\mathcal{O}_g^N = \{(123), (142)\}$ is not closed under the rack operation.

2.2. Racks of type C. The following notion was introduced in [3, Definition 2.3] motivated by [18, Theorem 2.1].

Definition 2.2. A rack X is of type C if there are a decomposable subrack $Y = R \amalg S \leq X$, $r \in R$ and $s \in S$ such that $r \triangleright s \neq s$,

$$R = \mathcal{O}_r^{\text{Inn}Y}, \quad S = \mathcal{O}_s^{\text{Inn}Y}, \quad \min\{|R|, |S|\} > 2 \text{ or } \max\{|R|, |S|\} > 4.$$

The group-theoretical reformulation (C) of the definition of type C is [3, Lemma 2.8]. We need a variation of [3, Lemma 2.8] in order to encompass the situation in Remark 2.1. The proof can be repeated verbatim: we recall it here for completeness.

Lemma 2.3. *Let G be a finite group, $g \in G$ and $N \triangleleft G$. The orbit $\mathcal{O} = \mathcal{O}_g^N$ is of type C if and only if there are $H \leq \langle \mathcal{O} \rangle$, $r, s \in \mathcal{O} \cap H$ such that*

$$(2.1) \quad \mathcal{O}_r^H \neq \mathcal{O}_s^H;$$

$$(2.2) \quad rs \neq sr;$$

$$(2.3) \quad H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle;$$

$$(2.4) \quad \min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2 \quad \text{or} \quad \max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4.$$

Proof. Assume that r, s and H are as above and set $R := \mathcal{O}_r^H$ and $S := \mathcal{O}_s^H$. If $r' = h \triangleright r \in \mathcal{O}_r^H = R$ for some $h \in H$, then there exist $x_1, \dots, x_k \in \mathcal{O}$ such that $h = x_1 \cdots x_k$. Hence

$$r' = x_1 \triangleright (x_2 \triangleright (\cdots (x_k \triangleright r))) \in \mathcal{O}(\triangleright \mathcal{O}(\triangleright \cdots \mathcal{O} \triangleright \mathcal{O})) \subset \mathcal{O},$$

so $R \subset \mathcal{O} \cap H$ and similarly $S \subset \mathcal{O} \cap H$. By (2.1) the subset $Y := R \amalg S \subset \mathcal{O}$ is a decomposable subrack, and $r \triangleright s \neq s$ is (2.2). In addition,

$$\mathcal{O}_r^{\text{Inn}Y} = \mathcal{O}_r^{\langle R, S \rangle} = \mathcal{O}_r^H = R,$$

where the second equality follows from (2.3), and similarly, $\mathcal{O}_s^{\text{Inn } Y} = S$. The estimate on R and S is (2.4). Hence \mathcal{O} is of type C.

Conversely, let $X = \mathcal{O}$ and r, s, R, S, Y be as in Definition 2.2. Setting $H := \langle R, S \rangle$, we immediately have (2.3), $H \leq \langle \mathcal{O} \rangle$, $R = \mathcal{O}_r^H$, $S = \mathcal{O}_s^H$ and $r, s \in \mathcal{O} \cap H$. Finally (2.2), (2.1) and (2.4) are straightforward. \square

Remark 2.4. Let G, N and \mathcal{O} be as in Lemma 2.3.

(a) If there exist $r, s \in \mathcal{O}$ satisfying (2.2), $\mathcal{O}_r^{\langle r, s \rangle} \neq \mathcal{O}_s^{\langle r, s \rangle}$ and:

$$\min\{|\mathcal{O}_r^{\langle r, s \rangle}|, |\mathcal{O}_s^{\langle r, s \rangle}|\} > 2 \quad \text{or} \quad \max\{|\mathcal{O}_r^{\langle r, s \rangle}|, |\mathcal{O}_s^{\langle r, s \rangle}|\} > 4,$$

then \mathcal{O} is of type C by Lemma 2.3 applied to $H := \langle r, s \rangle$.

(b) If $|g|$ is odd and $r, s \in \mathcal{O}$, then for any $H \leq G$ containing r and s the estimate (2.4) follows from (2.2), since then $|s| = |r| = |g|$ is odd, whence $|\mathcal{O}_r^H| \geq |\mathcal{O}_r^{\langle s \rangle}| \geq 3$, and similarly for \mathcal{O}_s^H . This generalizes [3, Lemma 2.7 (b)] to the situation of Remark 2.1.

Example 2.5. Let $n \geq 5$ be odd. Then the conjugacy class \mathcal{O} of n -cycles in \mathbb{S}_n is of type C. Indeed, \mathcal{O} splits into two classes \mathcal{O}' and \mathcal{O}'' in \mathbb{A}_n and $|\mathcal{O}'| = |\mathcal{O}''| = \frac{(n-1)!}{2} > n$ elements. Therefore, if $r \in \mathcal{O}'$, there exists $s \in \mathcal{O}''$ such that $s \notin C_{\mathbb{A}_n}(r) = \langle r \rangle$ and the result follows from Remark 2.4.

Example 2.6. The class \mathcal{O} corresponding to the partition $(1^2, 2^2)$ in \mathbb{A}_6 is of type C. Indeed, let $H := C_{\mathbb{A}_6}(56) \simeq \mathbb{S}_4$. Then, $\mathcal{O}' := H \cap \mathcal{O}$ contains all involutions of the form $(ab)(cd)$ for $a, b, c, d \notin \{5, 6\}$ and those of the form $(ab)(56)$ for $a, b \notin \{5, 6\}$. Therefore, $|\mathcal{O}'| = 9$ and \mathcal{O}' contains all involutions in H . Now, the involutions in \mathbb{S}_4 are parted into two classes of size 6 and 3, respectively, and \mathbb{S}_4 contains non-commuting non-conjugate involutions. Hence, we can find $r, s \in H \cap \mathcal{O}$ such that $r \triangleright s \neq s$ and $\mathcal{O}_r^H \neq \mathcal{O}_s^H$, with $|\mathcal{O}_r^H| = 3$ and $|\mathcal{O}_s^H| = 6$. Finally, $\langle \mathcal{O}_s^H, \mathcal{O}_r^H \rangle = H$ because \mathbb{S}_4 is generated by its involutions. We conclude by Lemma 2.3.

Example 2.7. The class of 3-cycles in $G = \mathbb{A}_3$ or \mathbb{A}_4 is kthulhu because its intersection with any subgroup of G is either abelian or a conjugacy class.

The class \mathcal{O} of 3-cycles in \mathbb{A}_n for $n \geq 5$ and the class \mathcal{O}' labeled (3^2) in \mathbb{A}_6 are of type C. Indeed, $\mathcal{O} \cap \mathbb{A}_4$ splits into the classes $\mathcal{O}_{(123)}$ and $\mathcal{O}_{(124)}$. Since the representatives do not commute, \mathcal{O} is of type C by Remark 2.4. Any non-inner automorphism of \mathbb{S}_6 interchanges \mathcal{O} and \mathcal{O}' in \mathbb{A}_6 , so \mathcal{O}' is of type C as well.

Here is an easy but useful application of the previous Lemma.

Lemma 2.8. *Let G be a finite group, $H \leq G$, $x \in H$. Assume that*

$$(2.5) \quad H \text{ is not abelian;}$$

$$(2.6) \quad H = \langle \mathcal{O}_x^H \rangle;$$

$$(2.7) \quad \text{there exists } s \in \mathcal{O}_x^G \cap H : s \notin \mathcal{O}_x^H, |\mathcal{O}_s^H| > 2.$$

Then \mathcal{O}_x^G is of type C.

Proof. There is $r \in \mathcal{O}_x^H$ such that $rs \neq sr$; otherwise $s \in Z(H)$ by (2.6), hence $|\mathcal{O}_s^H| = 1$. Thus (2.2) holds and (2.1) and (2.3) are clear by construction. Finally, $|\mathcal{O}_r^H| > 2$ by (2.5), thus (2.4) holds. \square

Here is another way to detect racks of type C.

Lemma 2.9. *Let G_1 and G_2 be finite groups, $a_1 \neq b_1 \in G_1$, $a_2, b_2 \in G_2$. Set $r = (a_1, a_2), s = (b_1, b_2) \in G := G_1 \times G_2$. Assume that*

$$(2.8) \quad a_1 b_1 = b_1 a_1, \quad a_2 b_2 \neq b_2 a_2;$$

$$(2.9) \quad \mathcal{O}_{a_1}^{G_1} = \mathcal{O}_{b_1}^{G_1}, \quad \mathcal{O}_{a_2}^{G_2} = \mathcal{O}_{b_2}^{G_2};$$

$$(2.10) \quad G_2 = \langle \mathcal{O}_{a_2}^{G_2} \rangle.$$

Then $\mathcal{O}_r^G = \mathcal{O}_{a_1}^{G_1} \times \mathcal{O}_{a_2}^{G_2}$ is of type C.

Proof. Let $H = \langle \{a_1\} \times \mathcal{O}_{a_2}^{G_2}, \{b_1\} \times \mathcal{O}_{a_2}^{G_2} \rangle \ni r, s$; (2.2) is evident. We claim that $\mathcal{O}_r^H = \{a_1\} \times \mathcal{O}_{a_2}^{G_2}$. Indeed, \subseteq follows from (2.8), and \supseteq from (2.10):

$$\begin{aligned} y \in G_2 &\implies \exists x_1, \dots, x_t \in \mathcal{O}_{a_2}^{G_2} : y = x_1 \dots x_t \\ &\implies (a_1, y \triangleright a_2) = (a_1, x_1) \triangleright ((a_1, x_2) \triangleright \dots (a_1, a_2)) \in \mathcal{O}_r^H. \end{aligned}$$

Similarly, $\mathcal{O}_s^H = \{b_1\} \times \mathcal{O}_{a_2}^{G_2}$. Hence (2.1) and (2.3) follow. Finally, if $|\mathcal{O}_r^H| = |\mathcal{O}_s^H| = |\mathcal{O}_{a_2}^{G_2}| \leq 2$, then a_2 and b_2 commute, contradicting (2.8). \square

Remark 2.10. Let G_1 and G_2 be finite groups, $a_1 \neq b_1 \in G_1$. The hypotheses of Lemma 2.9 on G_2 hold when $G_2/Z(G_2)$ is a non-abelian simple group and $a_2 \in G_2$ is not central. Namely, $\langle \mathcal{O}_{a_2}^{G_2} \rangle \triangleleft G_2$, hence it is all of G_2 giving (2.10). Furthermore there is $b_2 \in \mathcal{O}_{a_2}^{G_2}$ that does not commute with a_2 , because G_2 is not abelian, as needed in (2.8).

2.3. Racks of type D. A rack X is of type D if it has a decomposable subrack $Y = R \amalg S$ with elements $r \in R, s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$ [7, Definition 3.5]. If \mathcal{O} is a conjugacy class in a finite group G , then the rack \mathcal{O} is of type D if and only if (D) holds, see [7].

3. ALGEBRAIC GROUPS

Let \mathbb{G} be a connected reductive algebraic group defined over $\mathbb{k} = \overline{\mathbb{F}_q}$ and let $F : \mathbb{G} \rightarrow \mathbb{G}$ be a Frobenius map, that is a \mathbb{F}_q -split Steinberg endomorphism [22, Chapter 21]. Thus there exists an F -stable torus \mathbb{T} such that $F(t) = t^q$ for $t \in \mathbb{T}$, and $\mathbb{G}^F = \mathbb{G}(\mathbb{F}_q)$ is the finite group of \mathbb{F}_q -points. We make more precise assumptions on \mathbb{G} in each Subsection below. The main objectives of the paper are encompassed in the following situations:

- ◊ The group \mathbb{G} is either $\mathbf{SL}_n(\mathbb{k})$ or $\mathbf{Sp}_{2n}(\mathbb{k})$ ($n \geq 2$) and

$$\mathbf{G} := \mathbb{G}^F / Z(\mathbb{G}^F) = [\mathbb{G}^F, \mathbb{G}^F] / Z([\mathbb{G}^F, \mathbb{G}^F])$$

is a finite simple group.

- ◊ The group \mathbb{G} is either $\mathbf{SO}_{2n+1}(\mathbb{k})$ ($n \geq 2$, p is odd) or $\mathbf{SO}_{2n}(\mathbb{k})$ ($n \geq 4$) and $\mathbf{G} := [\mathbb{G}^F, \mathbb{G}^F] / Z(\mathbb{G}^F)$ is a finite simple group.

In both situations we say that \mathbf{G} is a (classical) Chevalley group. We allow $\mathbf{SO}_5(\mathbb{k})$ whose simply connected cover is $\mathbf{Sp}_6(\mathbb{k})$, and $\mathbf{SL}_2(3)$, $\mathbf{SL}_2(4)$, $\mathbf{SL}_2(5)$, $\mathbf{SL}_2(9)$ for flexibility in some recursive arguments.

Let Φ be the root system of \mathbb{G} and fix a subset Δ of simple roots. Let Q , respectively Λ , be the root, respectively weight, lattice. Then $\Lambda = \bigoplus_{i \in \mathbb{I}} \mathbb{Z}\omega_i$; here θ is the rank of \mathbb{G} , $\mathbb{I} = \mathbb{I}_\theta$ and $(\omega_i)_{i \in \mathbb{I}}$ are the fundamental weights. Let W be the Weyl group of Φ and let $s_\alpha \in W$ be the reflection corresponding to $\alpha \in \Phi$. Also, $\alpha_i^\vee : \mathbb{k}^\times \rightarrow \mathbb{T}$, $i \in \mathbb{I}$, are the simple coroots. Then

$$\omega_i(\alpha_j^\vee(\xi)) = \xi^{\delta_{ij}}, \quad \xi \in \mathbb{k}^\times, \quad i, j \in \mathbb{I}.$$

If $\alpha \in \Phi$, then there is a monomorphism $x_\alpha : \mathbb{k} \rightarrow \mathbb{G}$ of abelian groups; let $\mathbb{U}_\alpha = \text{Im } x_\alpha$ (a root subgroup) and let \mathbb{U} , respectively \mathbb{U}^- , be the subgroup of \mathbb{G} generated by the \mathbb{U}_α 's with $\alpha \in \Delta$, respectively $-\alpha \in \Delta$.

3.1. The classical groups. In this section we fix notation for the classical groups we will deal with. For $d \geq 1$ we set $J_d = \begin{pmatrix} & & & 1 \\ & & & \\ & & \cdot & \\ & & & \\ 1 & & & \end{pmatrix}$. We denote by Fr_q the Frobenius map $\mathbf{GL}_d(\mathbb{k}) \rightarrow \mathbf{GL}_d(\mathbb{k})$ given by $(a_{ij}) \mapsto (a_{ij}^q)$, and similarly the restriction to any suitable subgroup.

We will often use the automorphism $\phi : \mathbf{GL}_d(\mathbb{k}) \rightarrow \mathbf{GL}_d(\mathbb{k})$ given by:

$$(3.1) \quad \phi(A) := J_d {}^t A^{-1} J_d.$$

The symplectic group $\mathbf{Sp}_{2n}(\mathbb{k})$. The symplectic group $\mathbf{Sp}_{2n}(\mathbb{k})$ is the subgroup of $\mathbf{GL}_{2n}(\mathbb{k})$ leaving invariant the bilinear form $\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$. Thus $\mathbf{Sp}_{2n}(\mathbb{k})$ consists of the invertible matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

$$(3.2) \quad {}^t C J_n A = {}^t A J_n C, \quad {}^t B J_n D = {}^t D J_n B, \quad -{}^t C J_n B + {}^t A J_n D = J_n.$$

In this case, $F = \text{Fr}_q$ and $\mathbf{G} = \mathbf{Sp}_{2n}(q) / Z(\mathbf{Sp}_{2n}(q)) = \mathbf{PSp}_{2n}(q)$.

The orthogonal group $\mathbf{SO}_{2n+1}(\mathbb{k})$. Let p be odd. The orthogonal group $\mathbf{SO}_{2n+1}(\mathbb{k})$ is the subgroup of $\mathbf{SL}_{2n+1}(\mathbb{k})$ leaving invariant the bilinear form J_{2n+1} . Thus $\mathbf{SO}_{2n+1}(\mathbb{k})$ consists of the invertible matrices

$$X = \begin{pmatrix} A & e & B \\ f & k & g \\ C & h & D \end{pmatrix}, \quad A, B, C, D \in \mathbb{k}^{n \times n}, \quad e, {}^t f, {}^t g, h \in \mathbb{k}^{n \times 1}, \quad k \in \mathbb{k}$$

such that $\det X = 1$ and

$$(3.3) \quad \begin{aligned} {}^t C J_n A + {}^t f f + {}^t A J_n C &= 0, & {}^t C J_n e + {}^t f k + {}^t A J_n h &= 0, \\ {}^t C J_n B + {}^t f g + {}^t A J_n D &= J_n, & {}^t h J_n e + k^2 + {}^t e J_n h &= 1, \\ {}^t h J_n B + k g + {}^t e J_n D &= 0, & {}^t D J_n B + {}^t g g + {}^t B J_n D &= 0. \end{aligned}$$

In this case $F = \text{Fr}_q$ and $\mathbf{G} = [\mathbf{SO}_{2n+1}(q), \mathbf{SO}_{2n+1}(q)] = \mathbf{P}\Omega_{2n+1}(q)$.

The orthogonal group $\mathbf{SO}_{2n}(\mathbb{k})$. Let $n \geq 4$. The orthogonal group $\mathbf{SO}_{2n}(\mathbb{k})$ is the subgroup of matrices in $\mathbf{SL}_{2n}(\mathbb{k})$ preserving the quadratic form $\sum_{i=1}^n x_i x_{2n-i+1}$. If p is odd, such elements automatically preserve the bilinear form with associated matrix J_{2n} . Thus $\mathbf{SO}_{2n}(\mathbb{k})$ consists of those matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{SL}_{2n}(\mathbb{k})$ with $A, B, C, D \in \mathbb{k}^{n \times n}$, such that

$$(3.4) \quad {}^t C J_n A + {}^t A J_n C = 0, \quad {}^t C J_n B + {}^t A J_n D = J_n, \quad {}^t D J_n B + {}^t B J_n D = 0.$$

If $p = 2$, then $\mathbf{SO}_{2n}(\mathbb{k})$ consists of those matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{SL}_{2n}(\mathbb{k})$ satisfying (3.4) and such that the diagonal terms in ${}^t C J_n A$ and ${}^t B J_n D$ are 0. In both cases $F = \text{Fr}_q$ and

$$\mathbf{G} = [\mathbf{SO}_{2n}(q), \mathbf{SO}_{2n}(q)]/Z([\mathbf{SO}_{2n}(q), \mathbf{SO}_{2n}(q)]) = \mathbf{P}\Omega_{2n}(q).$$

3.2. On normalizers. In Subsection 5.3 we shall need the finite unitary groups $\mathbf{SU}_n(q)$ and $\mathbf{GU}_n(q)$, and the following folklore fact. We consider:

- $\mathbb{G} = \mathbf{SL}_n(\mathbb{k})$, $q_0 = p^{m_0}$ with $m_0 | m$ so that q is a power of q_0 ;
- $F_0: \mathbf{GL}_n(\mathbb{k}) \rightarrow \mathbf{GL}_n(\mathbb{k})$ is defined either by $F_0(A) := \text{Fr}_{q_0}(A)$ or by $F_0(A) := \text{Fr}_{q_0^{1/2}}(\phi(A))$, for $A \in \mathbf{GL}_n(\mathbb{k})$, ϕ as in (3.1), the latter occurring only for m_0 even, in which case, we denote as usual $\mathbf{SU}_n(q_0^{1/2}) = \mathbb{G}^{F_0}$ and $\mathbf{GU}_n(q_0^{1/2}) = \mathbf{GL}_n(\mathbb{k})^{F_0}$.

Proposition 3.1. $N_{\mathbf{GL}_n(q)}(\mathbb{G}^{F_0}) = Z(\mathbf{GL}_n(q))\mathbf{GL}_n(\mathbb{k})^{F_0}$ for both choices of F_0 .

Proof. We prove that $N_{\mathbf{GL}_n(q)}(\mathbb{G}^{F_0}) \leq Z(\mathbf{GL}_n(q))\mathbf{GL}_n(\mathbb{k})^{F_0}$, the other inclusion being immediate. Let $g \in N_{\mathbf{GL}_n(q)}(\mathbb{G}^{F_0})$. For any $y \in \mathbb{G}^{F_0}$ there holds $F_0(gyg^{-1}) = ygy^{-1}$, that is $z := g^{-1}F_0(g) \in C_{\mathbf{GL}_n(q)}(\mathbb{G}^{F_0})$. Now, \mathbb{G}^{F_0} contains regular unipotent elements in \mathbb{U} and \mathbb{U}^- , so it follows from [26, Lemma 5.3] that $C_{\mathbf{GL}_n(q)}(\mathbb{G}^{F_0}) = Z(\mathbf{GL}_n(q))$. In addition, F_0 restricts to a Steinberg endomorphism on the connected group $Z(\mathbf{GL}_n(\mathbb{k})) \simeq \mathbb{k}^\times$, hence Lang-Steinberg theorem [22, Theorem 21.7] is in force and there exists $\zeta \text{id} \in$

$Z(\mathbf{GL}_n(\mathbb{k}))$ such that $\zeta^{-1}F_0(\zeta)\text{id} = z$. Hence, $\zeta^{-1}g \in \mathbf{GL}_n(\mathbb{k})^{F_0} \leq \mathbf{GL}_n(q)$ and so $\zeta\text{id} \in Z(\mathbf{GL}_n(\mathbb{k})) \cap \mathbf{GL}_n(q) = Z(\mathbf{GL}_n(q))$. The claim follows. \square

3.3. The subgroup K . We introduce a subgroup K of \mathbb{G} that will be useful in Sections 4 and 6. In this Subsection \mathbb{G} is one of the groups

$$\mathbf{Sp}_{2n}(\mathbb{k}), n \geq 2; \quad \mathbf{SO}_{2n}(\mathbb{k}), n \geq 3, \quad \text{or} \quad \mathbf{SO}_{2n+1}(\mathbb{k}), n \geq 3,$$

where $p \neq 2$ when $\mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k})$. We set $n' = 2n$ if $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$ or $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{k})$ and $n' = 2n + 1$ if $\mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k})$, so that $\mathbb{G} \leq \mathbf{GL}_{n'}(\mathbb{k})$.

Recall ϕ from (3.1). Then K is the image of the injective group morphism

$$j: \mathbf{GL}_n(q) \rightarrow \mathbb{G}^F, \quad A \mapsto \begin{cases} \begin{pmatrix} A & 0 \\ 0 & \phi(A) \end{pmatrix}, & \text{if } \mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k}), \text{ or } \mathbf{Sp}_{2n}(\mathbb{k}), \\ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A) \end{pmatrix}, & \text{if } \mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k}). \end{cases}$$

3.4. Cuspidal classes in the Weyl group. Let $S = \{s_\alpha : \alpha \in \Delta\}$, so that (W, S) is a Coxeter group. Given $J \subset \Delta$, we set

- $W_J = \langle s_\alpha : \alpha \in J \rangle$;
- \mathbb{P}_J = the standard parabolic subgroup of \mathbb{G} determined by J ;
- \mathbb{L}_J = the standard (reductive) Levi subgroup of \mathbb{P}_J .

Definition 3.2. [16, 3.1.1] A conjugacy class \mathcal{C} in W is called *cuspidal* if $\mathcal{C} \cap W_J = \emptyset$ for all proper subsets J of S ; an element is cuspidal if its conjugacy class is so.

A decomposition of $w \in W$ is a family $\Gamma = (\gamma_j)_{j \in \mathbb{I}_l}$ in Φ , such that

$$(3.5) \quad w = s_{\gamma_1} \cdots s_{\gamma_l},$$

where s_{γ_j} is the corresponding reflection and l is minimal (with this property). Then l is denoted by $\ell^a(w)$ and is called the absolute length of w . By a result of Kostant, see [24], Γ is then a linearly independent family and

$$(3.6) \quad \ell^a(w) = \text{rk}(\text{id} - w)$$

in the natural representation of W . By [16, Exercise 3.16], we have

$$(3.7) \quad w \text{ is cuspidal} \iff \ell^a(w) = \text{rk } \mathbb{G}.$$

Notice that $\ell^a(w) = \text{rk } \mathbb{G}$ means that w has no fixed points.

Given a decomposition Γ of w , we set

$$(3.8) \quad \Psi_\Gamma = \Phi \cap (\mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_l),$$

$$(3.9) \quad \mathbb{G}_\Gamma = \langle \mathbb{T}, \mathbb{U}_\beta : \beta \in \Psi_\Gamma \rangle.$$

Clearly, Ψ_Γ is a root subsystem of Φ and \mathbb{G}_Γ is a connected reductive subgroup of \mathbb{G} . If Γ and Γ' are different decompositions of the same w , then the subsystems Ψ_Γ and $\Psi_{\Gamma'}$, and the subgroups \mathbb{G}_Γ and $\mathbb{G}_{\Gamma'}$, might be different.

Remark 3.3. If $w \in W$ is cuspidal, then \mathbb{G}_Γ is semisimple for any decomposition Γ of w , by (3.7).

Remark 3.4. If $w \in W_J$ for some $J \subset S$, then there is a decomposition F such that $\mathbb{G}_F \leq \mathbb{L}_J$.

Indeed, any decomposition of w in W_J is necessarily a decomposition in W , by (3.6). For, w acts trivially in $(\mathbb{R}J)^\perp$, hence $\text{rk}(\text{id} - w) = \text{rk}(\text{id} - w)|_{\mathbb{R}J}$.

3.5. F -stable tori. Here we assume that \mathbb{G} is connected reductive and F is a Frobenius map. By [22, Proposition 25.1], there is a bijection from the set of \mathbb{G}^F -conjugacy classes of F -stable maximal tori to the set of conjugacy classes in W , described as follows. Let \mathbb{T}' be an F -stable torus in \mathbb{G} . Then $\mathbb{T}' = g\mathbb{T}g^{-1}$ for some $g \in \mathbb{G}$ such that $g^{-1}F(g) \in N(\mathbb{T})$. Let

$$(3.10) \quad w = \text{class of } g^{-1}F(g) \in N(\mathbb{T})/\mathbb{T} \simeq W.$$

The assignment $\mathbb{T}' \mapsto w$ gives rise to the mentioned bijection. We set

$$(3.11) \quad \mathbb{T}_w := g\mathbb{T}g^{-1}.$$

Remark 3.5. Let \mathbb{T}' be an F -stable maximal torus in \mathbb{G} such that $\mathbb{T}' \mapsto w$ in the correspondence above and let \mathbb{G}' be the derived group of \mathbb{G} . Then $\mathbb{T}' \cap \mathbb{G}'$ is an F -stable maximal torus in \mathbb{G}' and $\mathbb{T}' \cap \mathbb{G}' \mapsto w$.

Indeed, $\mathbb{T} \cap \mathbb{G}'$ is a split torus of \mathbb{G}' . The element $g \in \mathbb{G}$ such that $\mathbb{T}' = g\mathbb{T}g^{-1}$ and $g^{-1}F(g) \in N_{\mathbb{G}}(\mathbb{T})$ is a representative of w can be written as $g = g'z$ where $g' \in \mathbb{G}'$ and z is central. Then $\mathbb{T}' \cap \mathbb{G}' = g'(\mathbb{T} \cap \mathbb{G}')(g')^{-1}$ and $(g')^{-1}F(g') \in N_{\mathbb{G}'}(\mathbb{T} \cap \mathbb{G}')$ is a representative of w .

Definition 3.6. An F -stable maximal torus is *cuspidal* if the corresponding conjugacy class in W as above is cuspidal.

Example 3.7. Let $w \in W$ be a Coxeter element, i.e. a product

$$(3.12) \quad w = s_1 \dots s_\theta,$$

where $(s_i)_{i \in \mathbb{I}_\theta}$ is a numeration of S ; this provides a decomposition F of w . Then the class of w is cuspidal and $\mathbb{G}_F = \mathbb{G}$. If $W = \mathbb{S}_n$, the conjugacy class of w is the only cuspidal class in W , [16, §3.1.2].

Definition 3.8. A *Coxeter torus* is an F -stable maximal torus that corresponds to the conjugacy class containing a Coxeter element.

By abuse of terminology the intersection of a Coxeter torus of \mathbb{G} with \mathbb{G}^F will be called a Coxeter torus of \mathbb{G}^F .

3.6. Semisimple classes. Here \mathbb{G} is connected and reductive, unless otherwise stated, F is a Frobenius map and \mathbb{T} is an F -stable torus such that $F(t) = t^q$ for $t \in \mathbb{T}$.

Let $x \in \mathbf{G} = \mathbb{G}^F/Z(\mathbb{G}^F)$ be semisimple non-trivial and pick $\mathbf{x} \in \mathbb{G}^F$ a representative of x , thus \mathbf{x} is semisimple but not central. Let $\mathcal{O} = \mathcal{O}_{\mathbf{x}}^{\mathbf{G}}$ and $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$; there is an epimorphism of racks $\mathfrak{D} \rightarrow \mathcal{O}$.

Let $\mathbf{y} \in \mathbb{G}^F$ be semisimple. By [22, Proposition 26.6], there exists an F -stable maximal torus \mathbb{T}' containing \mathbf{y} ; however, not all F -stable maximal tori intersecting $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ are necessarily \mathbb{G}^F -conjugated to \mathbb{T}' . Consequently we assign to $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ the set $\mathcal{S}_{\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}}$ of all conjugacy classes \mathcal{C} in W corresponding to F -stable maximal tori \mathbb{T}' that intersect $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$.

Remark 3.9. Assume that \mathbb{G} is simple and that we are not in the cases excluded in [22, Theorem 24.17]. If $\mathcal{C} \in \mathcal{S}_{\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}}$, then $\mathcal{O}_{\mathbf{y}}^{[\mathbb{G}^F, \mathbb{G}^F]}$ intersects an F -stable maximal torus \mathbb{T}'' in \mathbb{G}^F corresponding to an element in \mathcal{C} . Indeed, let $g \in \mathbb{G}^F$ be such that $g^{-1}F(g) \in N_{\mathbb{G}}(\mathbb{T})$ represents an element in \mathcal{C} , and let $\mathbb{T}' = g\mathbb{T}g^{-1}$. Then there exists $l \in \mathbb{G}^F$ such that $l \triangleright \mathbf{y} \in \mathbb{T}'$. Now, $\mathbb{G}^F = \mathbb{T}'F[\mathbb{G}^F, \mathbb{G}^F]$, [22, Corollary 24.2, Proposition 24.15, Proposition 24.21], so l decomposes as $l = t_1 l_1$ with $t_1 \in \mathbb{T}'$ and $l_1 \in [\mathbb{G}^F, \mathbb{G}^F]$, and $l_1 \triangleright \mathbf{y} \in t_1^{-1}g\mathbb{T}g^{-1}t_1 = \mathbb{T}''$, where $g^{-1}t_1F(t_1^{-1})F(g) = g^{-1}F(g)$ represents an element in \mathcal{C} .

For our aim, it is convenient to introduce the following notion.

Definition 3.10. A semisimple conjugacy class $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ in \mathbb{G}^F is called *cuspidal* if the set $\mathcal{S}_{\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}}$ consists of cuspidal conjugacy classes in W . In other words, all F -stable maximal tori intersecting $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ are cuspidal.

Also, $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ is called a *Coxeter class* if it only intersects Coxeter tori. Necessarily, $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ is then cuspidal.

Remark 3.11. Since $Z(\mathbb{G}^F) = Z(\mathbb{G})^F$ is contained in every torus of \mathbb{G}^F , the class $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ is cuspidal, respectively Coxeter, if and only if its projection \mathcal{O}' in $\mathbb{G}^F/Z(\mathbb{G}^F)$ intersects only cuspidal tori, respectively Coxeter tori in $\mathbb{G}^F/Z(\mathbb{G}^F)$. We will thus call also \mathcal{O}' cuspidal, respectively Coxeter. In particular, if \mathbb{G} is simply-connected and \mathfrak{D} is cuspidal, \mathcal{O} will be called cuspidal.

If \mathbb{G} is not simply-connected, Remark 3.9 guarantees that $\mathcal{O}_{\mathbf{y}}^{\mathbb{G}^F}$ is cuspidal, respectively Coxeter, if and only if $\mathcal{O}_{\mathbf{y}}^{[\mathbb{G}^F, \mathbb{G}^F]}$ intersects only cuspidal tori, respectively Coxeter tori in \mathbb{G}^F , and we will call also $\mathcal{O}_{\mathbf{y}}^{[\mathbb{G}^F, \mathbb{G}^F]}$ cuspidal, respectively Coxeter.

Proposition 3.12. *Assume \mathbb{G} is simply-connected. If \mathbb{T}' is a maximal F -stable torus that intersects \mathfrak{D} , \mathcal{C} is the conjugacy class in W corresponding to*

\mathbb{T}' and Γ is a decomposition of $w \in \mathcal{C}$, then \mathfrak{D} intersects \mathbb{G}_Γ^F . In particular, the following are equivalent:

- (a) \mathfrak{D} is not cuspidal.
- (b) \mathfrak{D} intersects a proper standard Levi subgroup \mathbb{L} .

Proof. We assume $\mathbf{x} \in \mathfrak{D} \cap \mathbb{T}'$. Since F is a Frobenius automorphism, \mathbb{G}_Γ is F -stable. Pick a representative \dot{w} of w ; by definition, it belongs to \mathbb{G}_Γ . By the Lang-Steinberg Theorem there is $h \in \mathbb{G}_\Gamma$ such that $h^{-1}F(h) = \dot{w}$. The tori \mathbb{T}' and $h\mathbb{T}h^{-1}$ are \mathbb{G}^F -conjugate since they both map to w , cf. (3.10), that is, there exists $y \in \mathbb{G}^F$ such that $y \triangleright \mathbb{T}' = h\mathbb{T}h^{-1} \leq \mathbb{G}_\Gamma$ and so $y \triangleright \mathbf{x} \in \mathbb{G}_\Gamma \cap \mathfrak{D}$.

If \mathfrak{D} is not cuspidal, then pick \mathcal{C} non-cuspidal and apply Remark 3.4. Conversely, if $\mathbf{x}' \in \mathfrak{D} \cap \mathbb{L}$, then there is an F -stable maximal torus \mathbb{T}' of \mathbb{L} that contains \mathbf{x}' . Hence $\mathbb{T}' = u\mathbb{T}u^{-1}$ for some $u \in \mathbb{L}$ such that $\sigma := u^{-1}F(u) \in N_{\mathbb{L}}(\mathbb{T}) \leq N_{\mathbb{G}}(\mathbb{T})$; so that the class of σ belongs to the Weyl group of \mathbb{L} . \square

Recall that a semisimple element y is *regular* if its centraliser $C_{\mathbb{G}}(y)$ consists of semisimple elements, or equivalently, if the irreducible component $C_{\mathbb{G}}(y)^\circ$ of $C_{\mathbb{G}}(y)$ containing the identity is a torus, [27, II.11]. This occurs if and only if y lives in a unique maximal torus. If our $\mathbf{x} \in \mathbb{G}^F$ is regular, then $C_{\mathbb{G}}(\mathbf{x})^\circ$ is the unique F -stable maximal torus containing \mathbf{x} .

Proposition 3.13. *If \mathbf{x} is a cuspidal element, then it is regular.*

We thank Gunter Malle for suggesting us the following proof.

Proof. Let \mathbb{T}_0 be an F -stable maximal torus of \mathbb{G} containing \mathbf{x} , let $g \in \mathbb{G}$ be such that $\mathbb{T}_0 = g\mathbb{T}g^{-1}$ and let w be the corresponding Weyl group element, i.e., $g^{-1}F(g) \in w\mathbb{T}$ as in (3.10). The F -stable maximal tori in \mathbb{G} containing \mathbf{x} are also the F -stable maximal tori in the connected reductive group $C = C_{\mathbb{G}}(\mathbf{x})^\circ$. Every F -stable maximal torus in C is of the form $c\mathbb{T}_0c^{-1}$ for some $c \in C$ such that $c^{-1}F(c) \in N_C(\mathbb{T}_0) \leq gN_{\mathbb{G}}(\mathbb{T})g^{-1}$. Let $W_C = N_C(\mathbb{T}_0)/\mathbb{T}_0$ be the Weyl group of C . We claim that W_C is trivial. Assume for a contradiction that W_C is non-trivial. Let s be a reflection in W_C and let $c \in C$ be such that $c^{-1}F(c) = \dot{s}$, a representative of s in $N_C(\mathbb{T}_0)$. Then, $\dot{s}' := g^{-1}\dot{s}g$ would represent a reflection s' in W and $c\mathbb{T}_0c^{-1} = cg\mathbb{T}g^{-1}c^{-1}$ is an F -stable maximal torus of \mathbb{G} , containing \mathbf{x} and corresponding to $g^{-1}c^{-1}F(c)F(g) = (g^{-1}c^{-1}F(c)g)(g^{-1}F(g)) \in s'w\mathbb{T}$. Therefore, $s'w$ is cuspidal by hypothesis on \mathbf{x} . However, the characteristic polynomial of a cuspidal element is a product of cyclotomic polynomials

different from $(X - 1)$, therefore its value at 0 is 1, see (3.6) and (3.7). On the other hand, $\det(s'w) = -\det(w)$. Hence, $s'w$ and w can not be both cuspidal elements in W , contradicting our assumption on \mathbf{x} . Therefore W_C has no reflections and $C = \mathbb{T}_0$ is the unique maximal torus containing \mathbf{x} . \square

The following well-known result is instrumental to apply Lemma 2.9.

Lemma 3.14. *Assume \mathbb{G} is simply-connected.*

(a) $\mathbf{x}^q \in \mathfrak{D}$. (b) If $\mathbf{x}^q = \mathbf{x}$, then $\mathfrak{D} \cap \mathbb{T}^F \neq \emptyset$.

Proof. First, $\mathcal{O}_{\mathbf{x}}^{\mathbb{G}}$ is F -stable: if $\mathbf{x} = h\mathbf{y}h^{-1}$ for some $h \in \mathbb{G}$, then

$$F(\mathbf{y}) = F(h)^{-1}\mathbf{x}F(h) \in \mathcal{O}_{\mathbf{x}}^{\mathbb{G}}.$$

Since \mathbf{x} is semisimple, there are $t \in \mathbb{T}$ and $g \in \mathbb{G}$ such that $\mathbf{x} = gtg^{-1}$. Thus $t^q = F(t) \in \mathcal{O}_{\mathbf{x}}^{\mathbb{G}}$ and consequently $\mathbf{x}^q \in \mathcal{O}_{\mathbf{x}}^{\mathbb{G}} \cap \mathbb{G}^F = \mathfrak{D}$; here the last equality holds because $C_{\mathbb{G}}(\mathbf{x})$ is connected, \mathbb{G} being simply connected, cf. [20, §2.11, §8.5]. Finally, if $\mathbf{x}^q = \mathbf{x}$, then $t^q = t \in \mathbb{T}^F \cap \mathcal{O}_{\mathbf{x}}^{\mathbb{G}} \subset \mathfrak{D}$ by the same reason. \square

4. SPLIT CONJUGACY CLASSES

We keep the notation from §3.6, namely \mathbb{G} is simple and simply connected, but not of type A. Also F is a Frobenius map; \mathbb{T} is an F -stable torus such that $F(t) = t^q$ for $t \in \mathbb{T}$; $e \neq x \in \mathbf{G} = \mathbb{G}^F/Z(\mathbb{G}^F)$ is semisimple; $\mathbf{x} \in \mathbb{G}^F$ a representative of x ; $\mathcal{O} = \mathcal{O}_x^{\mathbf{G}}$ and $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$. Thus there is an epimorphism of racks $\mathfrak{D} \twoheadrightarrow \mathcal{O}$.

We assume additionally that $\mathfrak{D} \cap \mathbb{T}^F \neq \emptyset$. Without loss of generality, we suppose that $\mathbf{x} \in \mathbb{T}^F$, i.e., \mathbf{x} is split. Adapting the proof of [3, Lemma 3.9] for type A, but with more work, we deal with such classes.

We will need to consider separately the following particular situation:

$$(4.1) \quad \mathbb{G} \text{ is of type } B_{\theta}, \quad q \text{ is odd, } \mathbf{x} \text{ satisfies } s_{\alpha_j}(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } j < \theta, \\ \alpha_{\theta}^{\vee}(-1)\mathbf{x} & \text{if } j = \theta. \end{cases}$$

Here $\theta \geq 2$. When this is the case, then \mathbf{x} has the form

$$(4.2) \quad \mathbf{x} = \left(\prod_{i \in \mathbb{I}_{\theta-1}} \alpha_i^{\vee}((-1)^i) \right) \alpha_{\theta}^{\vee}(\eta), \quad \text{where } \eta^2 = (-1)^{\theta}.$$

Notice that if θ is odd, then such a \mathbf{x} belongs to \mathbb{G}^F iff $q \equiv 1 \pmod{4}$.

Here is the main result of this Section:

Theorem 4.1. *Assume that $q > 2$; \mathbb{G} is not of type A_{θ} ; $q \notin \{3, 5, 7\}$ if we are in (4.1); and $\mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$ intersects the split torus \mathbb{T}^F . Then \mathcal{O} collapses.*

When $q = 2$, \mathbb{T}^F is trivial and the class of \mathbf{x} could not intersect it.

4.1. **Proof of Theorem 4.1.** This follows from Lemmata 4.2 and 4.3.

Lemma 4.2. *Assume that $q > 2$ and that we are **not** in the situation (4.1). Then \mathcal{O} is of type C.*

TABLE 3. Center of some \mathbb{G} , q odd; $\zeta \in \mathbb{F}_q^\times$ has order 4

| type | q | $Z(\mathbb{G})$ |
|---|-----------------------|---|
| B_θ | | $\langle \alpha_\theta^\vee(-1) \rangle$ |
| $C_\theta, \theta > 2$ | | $\left\langle \prod_{i \text{ odd}} \alpha_i^\vee(-1) \right\rangle$ |
| $D_\theta,$ $\theta \in 2\mathbb{Z} + 1$ | $q \equiv 1 \pmod{4}$ | $\left\langle \prod_{i \text{ odd}, i \leq \theta-2} \alpha_i^\vee(-1) \alpha_{\theta-1}^\vee(\zeta) \alpha_\theta^\vee(\zeta^3) \right\rangle$ |
| | $q \equiv 3 \pmod{4}$ | $\langle \alpha_{\theta-1}^\vee(-1) \alpha_\theta^\vee(-1) \rangle$ |
| $D_\theta, \theta \in 2\mathbb{Z}$ | | $\left\langle \prod_{i \text{ odd}} \alpha_i^\vee(-1), \alpha_{\theta-1}^\vee(-1) \alpha_\theta^\vee(-1) \right\rangle$ |
| E_7 | | $\langle \alpha_2^\vee(-1) \alpha_5^\vee(-1) \alpha_7^\vee(-1) \rangle$ |

Proof. Recall that $\mathbf{x} \in \mathbb{T}^F$. We will rely on the proof of [5, Lemmata 4.1, 4.2]. It is shown there that, for any simple root α such that $s_\alpha(\mathbf{x}) \neq \mathbf{x}$, the subrack $\mathbf{Y} = \mathbf{x} \mathbb{U}_\alpha^F \amalg s_\alpha(\mathbf{x}) \mathbb{U}_\alpha^F$ of $\mathcal{O}_\mathbf{x}^{\mathbb{G}^F}$ is of type C. We claim that we can choose α such that the restriction of the projection $\pi : \mathbf{Y} \rightarrow \mathcal{O}$ is injective. If \mathbb{G} is of type E_8, F_4, G_2 , then $Z(\mathbb{G})$ is trivial and $\mathbf{G} = \mathbb{G}^F$. Let $u \neq v \in \mathbf{Y}$ such that $\pi(u) = \pi(v)$, i.e. there is $z \in Z(\mathbb{G}^F)$, $z \neq 1$, such $u = zv$. Hence either $u = \mathbf{x} x_\alpha(a)$ and $v = s_\alpha(\mathbf{x}) x_\alpha(a)$, or vice versa. In any case, $\mathbf{x} \stackrel{\star}{=} z s_\alpha(\mathbf{x})$. Applying s_α , we get $z^2 = 1$. Thus \mathbb{G} is not of type E_6 (here $Z(\mathbb{G}) \simeq \mathbb{Z}/3$); and q should be odd. By \star , we have

$$(4.3) \quad \omega_i(\mathbf{x}) = \omega_i(z s_\alpha(\mathbf{x})) = \omega_i(z) s_\alpha(\omega_i)(\mathbf{x}), \quad i \in \mathbb{I}_\theta.$$

Say $\alpha = \alpha_j$, $j \in \mathbb{I}_\theta$. Then $s_\alpha(\omega_i) = \omega_i$ when $i \neq j$, hence $\omega_i(z) = 1$. Now such z exists only in the situation (4.1), see the shape of $Z(\mathbb{G})$ in Table 3. \square

Lemma 4.3. *If we **are** in situation (4.1) with $q = 9$, then \mathcal{O} is of type C. If we **are** in situation (4.1) with $q > 9$, then \mathcal{O} is of type D.*

Proof. We deal first with $\theta = 2$; now $B_2 = C_2$ and $\mathbb{G} = \mathbf{Sp}_4(\mathbb{k})$. Let $K \simeq \mathbf{GL}_2(q)$ be the subgroup of \mathbb{G}^F which is the image of j as in §3.3. Thus we have a monomorphism of groups $\mathbf{GL}_2(q)/\{\pm 1\} \rightarrow \mathbf{G} = \mathbf{PSp}_4(q)$.

Let $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{GL}_2(q)$; by our assumption on \mathbf{x} , we know that either $\mathbf{x} \stackrel{*}{=} j(y)$ or $\mathbf{x} = -j(y)$. Let $\varpi : \mathbf{GL}_2(q) \rightarrow \mathbf{PGL}_2(q)$ be the canonical projection and $y = \varpi(y)$. Then we have a surjective map of racks

$$\mathcal{O} \cap K/\{\pm 1\} \rightarrow \mathcal{O}_y^{\mathbf{PGL}_2(q)}.$$

Therefore it is enough to prove that $\mathcal{O}_y^{\mathbf{PGL}_2(q)}$ is of type C if $q = 9$ and of type D for $q > 9$.

Let $q = 9$. Then $\mathbf{PGL}_2(9) \simeq \mathbb{A}_6$, and through this isomorphism the class $\mathcal{O}_y^{\mathbf{PGL}_2(q)}$ corresponds to the class labeled by $(1^2, 2^2)$, which is of type C by Example 2.6.

Let now $q > 9$. If $q \equiv 3 \pmod{4}$, then [6, Corollary 5.4 (b)] applies¹. Assume then that $q \equiv 1 \pmod{4}$. Let $\zeta \in \mathbb{F}_q^\times$ be a primitive 4-th root of 1 and let $u \in \mathbf{PGL}_2(q)$ be the class of $\begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}$. Then

$$\mathcal{O}_y^{\mathbf{PGL}_2(q)} = \mathcal{O}_u^{\mathbf{PGL}_2(q)} = \mathcal{O}_u^{\mathbf{PSL}_2(q)}$$

which is of type D by [6, Corollary 5.4 (a)].

Assume next that $\theta > 2$. Here

$$\mathbf{G} = \mathbf{P}\Omega_{2\theta+1}(q) = \mathbb{G}^F/Z(\mathbb{G}^F) \simeq [\mathbf{SO}_{2\theta+1}(q), \mathbf{SO}_{2\theta+1}(q)].$$

We identify $\mathbf{P}\Omega_5(q)$ with a subgroup of $\mathbf{P}\Omega_{2\theta+1}(q)$ via the inclusion

$$\mathbf{SO}_5(q) \hookrightarrow \mathbf{SO}_{2\theta+1}(q), \quad \begin{pmatrix} A & e & B \\ f & k & g \\ C & h & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & e & 0 & B \\ 0 & \text{id}_{\theta-2} & 0 & 0 & 0 \\ f & 0 & k & 0 & g \\ 0 & 0 & 0 & \text{id}_{\theta-2} & 0 \\ C & 0 & h & 0 & D \end{pmatrix},$$

$k \in \mathbb{F}_q^\times$, $A, B, C, D \in \mathbb{F}_q^{2 \times 2}$, etc. Fix $t_\theta \in \mathbb{T}$ of the shape (4.2) and analogously t_2 of the shape (4.2) but for type B_2 . If $\pi : \mathbb{G}^F \rightarrow \mathbf{G}$ is the projection, then

$$\pi(t_\theta) = \text{diag}(-\text{id}_\theta, 1, -\text{id}_\theta) = \pi(t_2)\gamma,$$

$$\text{where } \gamma = \text{diag}(\text{id}_2, -\text{id}_{\theta-2}, 1, -\text{id}_{\theta-2}, \text{id}_2).$$

Here diag refers to a diagonal of blocks. Then

$$\mathcal{O} = \mathcal{O}_{\pi(t_\theta)}^{\mathbf{G}} \geq \mathcal{O}_{\pi(t_\theta)}^{\mathbf{P}\Omega_5(q) \times \langle \gamma \rangle} \simeq \mathcal{O}_{\pi(t_2)}^{\mathbf{P}\Omega_5(q)}$$

which is of type D by the preceding argument. Hence \mathcal{O} is of type D. \square

¹Notice that Corollary 5.4 (b) in *loc. cit.* refers implicitly to the class of involutions in $\mathbf{PGL}_2(q)$ not in $\mathbf{PSL}_2(q)$, as is transparent from the proof.

Lemma 4.4. *If we **are** in situation (4.1) with $n = 2$ and $q = 3$, then \mathcal{O} is austere, hence *kthulhu*.*

Proof. Indeed, $\mathbf{PSp}_4(3) \simeq \mathbf{PSU}_4(2)$ and the semisimple class we are dealing with in the former group corresponds to the unipotent class of type $(2, 1^2)$ in the latter one, which is austere by [4, Lemma 5.2]. \square

Remark 4.5. Assume that $n = 2$. If we **are** in the situation (4.1) with $q = 5$, then calculations with GAP show that \mathcal{O} is austere, hence *kthulhu*. The evidence obtained by performing different computations seems to indicate that in the case $q = 7$, the class is also *kthulhu*.

4.2. Split classes in orthogonal groups. Theorem 4.1 was proved by assuming that \mathbb{G} is simply connected. For recursive arguments on the orthogonal groups we need an analogous statement for the orbits of split elements in $\mathbb{G}^F = \mathbf{SO}_{n'}(q)$ for the action of $[\mathbb{G}^F, \mathbb{G}^F]$ for $n' = 2n$ or $2n + 1$. Let $\mathbb{T} \leq \mathbb{G}$ be the subgroup of diagonal matrices

$$\begin{aligned} \text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}), & \quad \text{if } n' = 2n, \\ \text{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}), & \quad \text{if } n' = 2n + 1. \end{aligned}$$

Recall Remark 2.1.

Lemma 4.6. *If $y \in \mathbb{T}^F - Z(\mathbf{SO}_{n'}(q))$, then $\mathcal{O}_y^{[\mathbf{SO}_{n'}(q), \mathbf{SO}_{n'}(q)]}$ collapses, except in the situation (4.1), i.e., when $n' = 2n + 1$ and $t_i = -1$ for $i \in \mathbb{I}_n$.*

Proof. There is always a simple root α so that the proof of [5, Lemma 4.2] carries over. If $t_i \neq t_{i+1}$ for some $i < n$, then take $\alpha = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$; if, instead, $t_i = t_{i+1}$ for all $i < n$, then our assumptions imply $t_n \neq t_n^{-1}$ and we take $\alpha = \alpha_n$, i.e., ε_n when $n' = 2n + 1$ and $\varepsilon_{n-1} + \varepsilon_n$ when $n' = 2n$. The argument works also in types B_2 and D_3 . \square

5. THE SPECIAL LINEAR GROUPS

In this Section $\mathbb{G} = \mathbf{SL}_n(\mathbb{k})$, that is, we deal with semisimple classes in $\mathbf{G} = \mathbf{PSL}_n(q)$. As in §3.6, $e \neq x \in \mathbf{G}$ is semisimple, $\mathbf{x} \in \mathbb{G}^F - Z(\mathbb{G}^F)$ is a representative of x , $\mathcal{O} = \mathcal{O}_x^{\mathbf{G}}$ and $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$. There is an epimorphism of racks $\mathfrak{D} \twoheadrightarrow \mathcal{O}$.

For inductive arguments, we will also consider classes of elements in $\mathbf{GL}_n(q)$. As observed in [1, Remark 4.1], for any semisimple element $y \in \mathbf{GL}_n(q)$, we have $\mathcal{O}_y^{\mathbf{GL}_n(q)} = \mathcal{O}_y^{\mathbf{SL}_n(q)}$.

Definition 5.1. We say that $A \in \mathbf{GL}_n(q)$ is *irreducible* if its characteristic polynomial p_A is irreducible; necessarily A is regular semisimple.

From our previous work, we know:

Remark 5.2. (i) [3, Theorem 1.1] If $n = 2$, and $q \notin \{2, 3, 4, 5, 9\}$, then any \mathcal{O} not listed in Table 1 collapses.

(ii) [3, Props. 5.4, 5.5] If $n = 3$ and \mathbf{x} is irreducible, then \mathcal{O} is kthulhu.

(iii) [3, Theorem 1.1] If $n \geq 3$ and \mathbf{x} is not irreducible, then \mathcal{O} collapses.

Remark 5.3. Let $n = 2$. We record information on the semisimple classes with $q \in \{2, 3, 4, 5, 9\}$ for recursive arguments. Recall that $\mathbf{PSL}_2(q)$ has two conjugacy classes of maximal tori: the split one, of order $(q-1)/(2, q-1)$ and the Coxeter torus, of order $(q+1)/(2, q+1)$, that contains the irreducible elements.

- If $q = 2$, then $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$ and the semisimple elements are the 3-cycles, that form an abelian rack; if $q = 3$, then $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$ and the semisimple elements have order 2 and form an abelian rack.
- If $q = 4$, then $\mathbf{PSL}_2(4) \simeq \mathbb{A}_5$. The irreducible elements have order 5 and form two conjugacy classes that are sober by [14, Remark 3.2 (b) and (c)]. The split semisimple elements form the conjugacy class of 3-cycles which is of type C by Example 2.7.
- If $q = 5$, then $\mathbf{PSL}_2(5) \simeq \mathbb{A}_5$. The irreducible elements form the conjugacy class of 3-cycles which is of type C by Example 2.7. The split semisimple elements are the involutions in the class $(1, 2^2)$ which is sober because its intersection with any subgroup of \mathbb{A}_5 is either trivial, abelian or indecomposable.
- If $q = 9$, then $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$. The irreducible elements have order 5 and form two conjugacy classes that are sober by [14, Remark 3.2 (b) and (c)]. The split semisimple elements are the involutions in the class $(1^2, 2^2)$ which is of type C by Example 2.6.

Our main result in this Section is:

Theorem 5.4. *Let $\mathcal{O} \neq \{e\}$ be a semisimple conjugacy class in $\mathbf{PSL}_n(q)$. Then any \mathcal{O} not listed in Table 1 collapses.*

By Remark 5.2, we will consider conjugacy classes of irreducible elements assuming $n > 3$. We will see that if n is prime, then such classes are kthulhu by Proposition 5.15, otherwise, they are of type C by Proposition 5.9.

We start by a classical result whose proof we include for completeness.

Lemma 5.5. *Let $n \geq 2$ and $\epsilon = \pm 1$. If $P(X) = X^n + \epsilon \in \mathbb{F}_p[X]$ is irreducible over \mathbb{F}_q , then $n = 2$, $\epsilon = 1$ and $q \equiv 3 \pmod{4}$.*

Proof. First, $\epsilon = 1$ and q is odd, otherwise $P(1) = 0$; and $n = 2b$ is even, otherwise $P(-1) = 0$. Also, $q \equiv 3 \pmod{4}$, otherwise $-1 = \xi^2$ for some $\xi \in \mathbb{F}_q$ and $P(X) = (X^b + \xi)(X^b - \xi)$ would be reducible. Let now $n = 2^h a$ where a is odd, and let $\Phi_d(X)$ be the d -th cyclotomic polynomial. We have the factorization over \mathbb{Z} , hence over \mathbb{F}_p ,

$$(X^n - 1)P(X) = X^{2n} - 1 = \prod_{d|2n} \Phi_d(X) \implies P(X) = \prod_{d|2n, d \nmid n} \Phi_d(X).$$

Thus $\Phi_{2^{h+1}}(X) | P(X)$, hence they are equal and $n = 2^h$. Finally, if $X^{2^h} + 1$ is irreducible over \mathbb{F}_q for $q \equiv 3 \pmod{4}$, then $h = 1$ by [23, Theorem 1]. \square

5.1. Coxeter tori. We assume in the rest of this Section that $n > 3$ and that \mathbf{x} is *irreducible*. In this case $W = \mathbb{S}_n$ and by Proposition 3.12 and Example 3.7 every irreducible class intersects every Coxeter torus. We fix the n -cycle

$$w = (1, 2, \dots, n).$$

By technical reasons, we fix a Coxeter torus \mathbb{T}_w in $\mathbf{GL}_n(\mathbb{k})$; then $\mathbb{T}_w \cap \mathbf{SL}_n(\mathbb{k})$ is a Coxeter torus in $\mathbf{SL}_n(\mathbb{k})$ by Remark 3.5. By [22, Example 25.4], we have

$$(5.1) \quad |\mathbb{T}_w^F| = q^n - 1 = (n)_q(q - 1), \quad |\mathbb{T}_w^F \cap \mathbf{SL}_n(q)| = (n)_q.$$

The group \mathbb{T}_w^F is isomorphic to $\mathbb{F}_{q^n}^\times$, hence it is cyclic; further, any cyclic subgroup of $\mathbf{GL}_n(q)$ of order $q^n - 1$ is conjugated to \mathbb{T}_w^F [19, Example 1.13].

Remark 5.6. ([21, Satz II.7.3] and [25, Theorem 2.3.5 and below]). We have

$$\begin{aligned} N_{\mathbf{GL}_n(q)}(\mathbb{T}_w) &= N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F) \simeq \mathbb{T}_w^F \rtimes C_W(w), \\ N_{\mathbf{SL}_n(q)}(\mathbb{T}_w^F \cap \mathbf{SL}_n(q)) &= N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F) \cap \mathbf{SL}_n(q), \end{aligned}$$

with $C_W(w) \simeq \mathbb{Z}/n$. Let σ be a generator of $C_W(w)$ identified as a subgroup of $N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F)$; σ can be chosen so that $\sigma \triangleright \mathbf{y} = \mathbf{y}^q$ for any $\mathbf{y} \in \mathbb{T}_w^F$.

Lemma 5.7. *Let $\mathbf{y} \in \mathbb{T}_w^F$ be irreducible in $\mathbf{GL}_n(q)$. Then*

$$(5.2) \quad \mathcal{O}_{\mathbf{y}}^{\mathbf{GL}_n(q)} \cap \mathbb{T}_w^F = \mathcal{O}_{\mathbf{y}}^{N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F)} = \langle \sigma \rangle \triangleright \mathbf{y} = \{\mathbf{y}, \mathbf{y}^q, \dots, \mathbf{y}^{q^j}, \dots, \mathbf{y}^{q^{n-1}}\},$$

$$(5.3) \quad \mathcal{O}_{\mathbf{y}}^{N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F)} = \mathcal{O}_{\mathbf{y}}^{\mathbf{SL}_n(q)} \cap \mathbb{T}_w^F = \mathcal{O}_{\mathbf{y}}^{N_{\mathbf{SL}_n(q)}(\mathbb{T}_w^F)}$$

Proof. If $\mathbf{z} \in \mathbb{T}_w^F \cap \mathcal{O}_{\mathbf{y}}^{\mathbf{GL}_n(q)}$, then there is $g \in \mathbf{GL}_n(q)$ such that $gyg^{-1} = \mathbf{z}$, so $gC_{\mathbf{GL}_n(q)}(\mathbf{y})g^{-1} = C_{\mathbf{GL}_n(q)}(\mathbf{z})$, that is, $g \in N_{\mathbf{GL}_n(q)}(\mathbb{T}_w)$ since clearly \mathbf{z} is also irreducible. This and Remark 5.6 imply (5.2). Since $\mathcal{O}_{\mathbf{y}}^{\mathbf{SL}_n(q)} = \mathcal{O}_{\mathbf{y}}^{\mathbf{GL}_n(q)}$, the centraliser argument as above gives (5.3). \square

We investigate when two elements in an irreducible class have the same image through the natural projection $\pi: \mathbf{SL}_n(q) \rightarrow \mathbf{PSL}_n(q)$. Recall that $\mathbb{G}_n(\mathbb{F}_q)$ is the group of n -th roots of unity in \mathbb{F}_q .

Lemma 5.8. *(Just for this Lemma, $n \geq 3$). Let $\mathbf{y}, \mathbf{z} \in \mathfrak{D}$ such that $\pi(\mathbf{y}) = \pi(\mathbf{z})$, i.e., $\mathbf{y} = \lambda \mathbf{z}$ for some $1 \neq \lambda \in \mathbb{G}_n(\mathbb{F}_q)$. Then*

- (i) *There exists $j \in \mathbb{I}_{n-1}$ such that $\lambda \mathbf{z} = \mathbf{z}^{q^j}$.*
- (ii) *Let $j \in \mathbb{I}_{n-1}$ be minimal satisfying $\mathbf{z}^{q^j} = \lambda \mathbf{z}$ for some $1 \neq \lambda \in \mathbb{G}_n(\mathbb{F}_q)$. Then $j|n$ and λ is a primitive $\frac{n}{j}$ -th root of 1.*
- (iii) *Let $j \in \mathbb{I}_{n-1}$ be minimal satisfying $\mathbf{z}^{q^j} = \lambda \mathbf{z}$ for some $1 \neq \lambda \in \mathbb{G}_n(\mathbb{F}_q)$ and let $a := \frac{n}{j}$. Then the characteristic polynomial $p_{\mathbf{z}} \in \mathbb{F}_q[X^a]$. This observation rectifies [3, Remark 3.1 (d)].*
- (iv) *Let $j \in \mathbb{I}_n$ be minimal satisfying $\pi(\mathbf{z}^{q^j}) = \pi(\mathbf{z})$. Then $j \neq 1$.*

Proof. (i): The elements \mathbf{y} and \mathbf{z} lie in the same (unique) maximal torus, so $\mathbf{y} = \mathbf{z}^{q^j}$ for some $j \in \mathbb{I}_{1,n}$ by Lemma 5.7. Therefore, $\lambda \mathbf{z} = \mathbf{z}^{q^j}$ and $\lambda \neq 1$ implies $j < n$.

(ii): If $n = aj + b$, with $a \geq 1$ and $0 \leq b < j$, then

$$\mathbf{z} = \mathbf{z}^{q^n} = \mathbf{z}^{q^{aj+b}} = (\mathbf{z}^{q^{aj}})^{q^b} = (\lambda^a \mathbf{z})^{q^b} = \lambda^a \mathbf{z}^{q^b}$$

that is, $\mathbf{z}^{q^b} = \lambda^{-a} \mathbf{z}$. Hence $b = 0$ by minimality and $\lambda^a = 1$. Now, if $\lambda^c = 1$ with $c \in \mathbb{N}$, then $\mathbf{z}^{q^{cj}} = \lambda^c \mathbf{z} = \mathbf{z}$, hence $c \geq a = \frac{n}{j}$.

(iii): By assumption $p_{\mathbf{z}} = p_{\mathbf{z}^{q^j}} = p_{\lambda \mathbf{z}}$. If $p_{\mathbf{z}}(X) = X^n + c_{n-1}X^{n-1} + \dots + c_0$, then $p_{\lambda \mathbf{z}}(X) = X^n + \lambda c_{n-1}X^{n-1} + \dots + \lambda^n c_0$. Thus $p_{\mathbf{z}}(X) = p_{\lambda \mathbf{z}}(X)$ if and only if $c_h = 0$ for all $h \notin \frac{n}{j}\mathbb{Z}$.

(iv): If $j = 1$ then $p_{\mathbf{z}}$ would be $X^n + (-1)^n$ by (iii). By Lemma 5.5, $n = 2$, a contradiction. \square

5.2. Irreducible elements of $\mathbf{SL}_n(q)$, n not a prime. In this Subsection we assume that $n = cd$, for some $c, d \in \mathbb{N}_{\geq 2}$. Given $\mathbf{S} \in \mathbf{SL}_d(q)$ irreducible, we consider $\mathbf{y} = \text{diag}(\mathbf{S}, \dots, \mathbf{S}) \in \mathbf{SL}_n(q)$. Then $C_{\mathbf{GL}_n(q)}(\mathbf{y}) \simeq \mathbf{GL}_c(q^d)$.

We claim that a Coxeter torus \tilde{T} of $C_{\mathbf{GL}_n(q)}(\mathbf{y})$ remains a Coxeter torus in $\mathbf{GL}_n(q)$ hence $T := \tilde{T} \cap \mathbf{SL}_n(q)$ is a Coxeter torus in $\mathbf{SL}_n(q)$.

Indeed, by (5.1), we have $|\tilde{T}| = ((q^c)^d - 1) = (q^n - 1)$. Since \tilde{T} is cyclic, it is conjugated to \mathbb{T}_w^F as claimed after (5.1). Thus, $|T| = (n)_q$.

In this subsection we will assume that \mathbf{x} lies in a Coxeter torus T of \mathbb{G}^F arising from some \mathbf{y} as above.

Proposition 5.9. *If \mathbf{x} is irreducible, then $\mathcal{O} = \mathcal{O}_x^{\mathbf{PSL}_n(q)}$ is of type C.*

Proof. Let $n = cd$ with c prime and $d \geq 2$. We set:

$$\begin{aligned}\widetilde{M} &:= C_{\mathbf{GL}_n(q)}(\mathbf{y}) \simeq \mathbf{GL}_c(q^d); \\ M &:= C_{\mathbf{SL}_n(q)}(\mathbf{y}) = \widetilde{M} \cap \mathbf{SL}_n(q); \\ M_1 &:= [\widetilde{M}, \widetilde{M}] \simeq \mathbf{SL}_c(q^d).\end{aligned}$$

Thus $M_1 \leq M \leq \widetilde{M}$. Lemma 5.7 gives

$$\begin{aligned}\mathcal{O}_{\mathbf{x}}^{\mathbf{GL}_n(q)} \cap T &= \mathcal{O}_{\mathbf{x}}^{\mathbf{SL}_n(q)} \cap T = \{\mathbf{x}, \mathbf{x}^q, \dots, \mathbf{x}^{q^j}, \dots, \mathbf{x}^{q^{n-1}}\} \\ \mathcal{O}_{\mathbf{x}}^M \cap T &= \mathcal{O}_{\mathbf{x}}^{M_1} \cap T = \mathcal{O}_{\mathbf{x}}^{\widetilde{M}} \cap T = \{\mathbf{x}, \mathbf{x}^{q^d}, \dots, \mathbf{x}^{q^{d(c-1)}}\}.\end{aligned}$$

Hence $\mathbf{x}^q \in \mathcal{D} \cap T$ but $\mathbf{x}^q \notin \mathcal{O}_{\mathbf{x}}^M$. We claim that $\mathcal{O}_{\mathbf{x}^q}^M \not\subseteq N_{\widetilde{M}}(T)$. Suppose the contrary. Then, $\langle \mathcal{O}_{\mathbf{x}^q}^M \rangle$ would be a non-central, normal subgroup of $\widetilde{M} \simeq \mathbf{GL}_c(q^d)$. Then $\mathbf{SL}_c(q^d) \simeq M_1 \leq \langle \mathcal{O}_{\mathbf{x}^q}^M \rangle \leq N_{\widetilde{M}}(T)$, and so $T \cap M_1$ would be normal in M_1 , a contradiction.

We pick $\mathbf{s} \in \mathcal{O}_{\mathbf{x}^q}^M \setminus N_{\widetilde{M}}(T)$ and set

$$s := \pi(\mathbf{s}) \in \mathcal{O}; \quad r := \pi(\mathbf{x}) \in \mathcal{O}; \quad H := \langle r, s, \pi(M_1) \rangle.$$

We claim that r, s and H satisfy the assumptions of Lemma 2.3. First, $\mathbf{s} \notin N_{\widetilde{M}}(T)$ implies $s \triangleright r \neq r$, i.e., (2.2) holds. Indeed, $s \triangleright r = r$ would give $\mathbf{s} \triangleright \mathbf{x} \in \mathbb{Z}(\mathbb{G}^F)\mathbf{x}$ that combined with $T = C_{\mathbf{GL}_n(q)}(\mathbf{x})$ would force $\mathbf{s} \triangleright T = T$.

In addition, $\langle \mathcal{O}_{\mathbf{x}}^{M_1}, \mathcal{O}_{\mathbf{s}}^{M_1} \rangle = \langle \mathcal{O}_{\mathbf{x}}^{\widetilde{M}}, \mathcal{O}_{\mathbf{s}}^{\widetilde{M}} \rangle$ is a non-central, normal subgroup of $\widetilde{M} \simeq \mathbf{GL}_c(q^d)$, hence $\langle M_1, \mathbf{x}, \mathbf{s} \rangle \leq \langle \mathcal{O}_{\mathbf{x}}^{M_1}, \mathcal{O}_{\mathbf{s}}^{M_1} \rangle$ and therefore

$$H = \langle \pi(M_1), r, s \rangle \leq \langle \mathcal{O}_r^{\pi(M_1)}, \mathcal{O}_s^{\pi(M_1)} \rangle \leq \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle \leq H.$$

That is, (2.3) holds and $H \leq \langle \mathcal{O} \rangle$.

Observe that $\pi(M) \simeq M/Z(\mathbf{SL}_n(q)) \cap M$ projects onto $\mathbf{PSL}_c(q^d)$, so the orbits $\mathcal{O}_x^{\pi(M)}$ and $\mathcal{O}_{x^q}^{\pi(M)}$ project onto non-trivial orbits in $\mathbf{PSL}_c(q^d)$, and therefore $|\mathcal{O}_r^H| \geq |\mathcal{O}_r^{\pi(M)}| > 2$ and $|\mathcal{O}_s^H| \geq |\mathcal{O}_s^{\pi(M)}| > 2$, i.e., (2.4) holds.

We finally analyse $\mathcal{O}_r^H \cap \mathcal{O}_s^H$. First of all, $\pi(M_1) \leq H \leq \pi(M)$ and $\mathcal{O}_{\mathbf{x}}^{M_1} = \mathcal{O}_{\mathbf{x}}^M$ imply that $\mathcal{O}_r^H = \mathcal{O}_r^{\pi(M)}$. Similarly, $\mathcal{O}_s^H = \mathcal{O}_s^{\pi(M)}$.

If $\mathcal{O}_r^{\pi(M)} \cap \mathcal{O}_s^{\pi(M)} = \emptyset$, then we are done. Otherwise,

$$x \in \mathcal{O}_s^{\pi(M)} \cap \pi(T) = \mathcal{O}_{x^q}^{\pi(M)} \cap \pi(T) = \{x^q, (x^q)^{q^d}, \dots, (x^q)^{q^{d(c-1)}}\}.$$

Therefore there exists $l \in \mathbb{I}_{0, c-1}$ such that $\mathbf{x}^{q^{dl+1}} \in \mathbb{G}_n(\mathbb{F}_q)\mathbf{x}$. Lemma 5.8 (iv) gives $l \neq 0$. Let $j \in \mathbb{I}_{n-1}$ be minimal satisfying $\mathbf{x}^{q^j} \in \mathbb{G}_n(\mathbb{F}_q)\mathbf{x}$. Then $j|n$ by Lemma 5.8 (ii) whose argument shows that j divides also $dl+1$. Hence, $(j, d) = 1$. Since $j > 1$ by Lemma 5.8 (iv) again, this can occur only if $j = c$ and $(c, d) = 1$. In this case, d has a prime factor c' different from c

and we may repeat the whole construction replacing c by c' and d by $\frac{n}{c'}$. As $j = c \neq c'$, we get that $\mathcal{O}_r^{\pi(M)} \cap \mathcal{O}_s^{\pi(M)} = \emptyset$. The hypotheses of Lemma 2.3 were verified, hence \mathcal{O} is of type C. \square

5.3. Irreducible elements of $\mathbf{SL}_n(q)$, $n > 3$ prime. Here $n > 3$ is prime. Recall that $e \neq x \in \mathbf{G} = \mathbf{PSL}_n(q)$, $\mathbf{x} \in \mathbb{G}^F - Z(\mathbb{G}^F)$ is a representative of x which is irreducible and belongs to the Coxeter torus $T := \mathbb{T}_w^F \cap \mathbf{SL}_n(q)$; we set $\mathcal{O} = \mathcal{O}_x^{\mathbf{G}}$ and $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$. There is an epimorphism of racks $\mathfrak{D} \rightarrow \mathcal{O}$.

We will analyse all possible subgroups of $\mathbf{GL}_n(q)$ intersecting \mathfrak{D} . We start by a few well-known arithmetic results instrumental for our analysis.

Lemma 5.10. *Let n be an odd prime number.*

- (i) *If $(n, q - 1) = 1$, then $(n, q^n - 1) = (n, (n)_q) = 1$.*
- (ii) *If $(n, q - 1) = n$, then $(n^2, (n)_q) = (n, (n)_q) = n$.*
- (iii) *$(q - 1, (n)_q) = (n, q - 1)$.*
- (iv) *$(n(q - 1), (n)_q) = (n, q - 1)$.*

Proof. (i) and (ii) are [25, Lemma 4.1.1], whilst (iii) follows from the Euclidean algorithm. We prove (iv). Combining (i), (ii) and (iii) we obtain

$$(n, (n)_q) = (n, q - 1) = (q - 1, (n)_q),$$

hence $(n(q - 1), (n)_q) = 1$ if $(n, q - 1) = 1$, and $(n(q - 1), (n)_q) \in \{n, n^2\}$ if $(n, q - 1) = n$. In this case, we discard $(n(q - 1), (n)_q) = n^2$ using (ii). \square

Recall $\sigma \in N_{\mathbf{GL}_n(q)}(\mathbb{T}_w)$ from Remark 5.6.

Lemma 5.11. *Let n be a prime.*

- (i) *Let $g \in N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F) \setminus \mathbb{T}_w^F$. Then $|g|$ divides $n(q - 1)$.*
- (ii) *$\mathfrak{D} \cap N_{\mathbb{G}^F}(\mathbb{T}_w) \subset \mathbb{T}_w^F$.*

Proof. (i) By Remark 5.6, there are $k \in \mathbb{I}_{1, n-1}$ and $t \in \mathbb{T}_w^F$ such that $g = \sigma^k t$. Then

$$(\sigma^k t)^n = \left(\prod_{\tau \in \langle \sigma^k \rangle} \tau \triangleright t \right) \sigma^{nk} = \prod_{\tau \in \langle \sigma \rangle} \tau \triangleright t = \left(\prod_{i=1}^n t^{q^i} \right) = t^{(n)_q}$$

by a direct computation. Hence $|g|$ divides $n(q - 1)$.

(ii) Let $g \in \mathfrak{D} \cap N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F) = \mathfrak{D} \cap N_{\mathbb{G}^F}(\mathbb{T}_w^F)$. Recall that $|\mathbf{x}|$ divides $(n)_q$. If $g \notin \mathbb{T}_w^F$, then $|g|$ divides $(n(q - 1), (n)_q)$ by (i). By Lemma 5.10 (iv) $|g|$ divides $(n, q - 1)$, so g is central, contradicting its irreducibility. \square

We recall that a *primitive* prime divisor of $q^n - 1$ is a prime number ℓ such that $\ell | (q^n - 1)$ and $\ell \nmid (q^e - 1)$ for every $e \in \mathbb{I}_{n-1}$

Lemma 5.12. *(Here n is any odd prime). Let y be an irreducible semisimple element in $\mathbf{GL}_n(q)$. Then,*

- (i) *Either there exists a primitive prime divisor ℓ of $q^n - 1$ dividing $|y|$ or else $|y|$ divides $n(q - 1)$, it does not divide $(q - 1)$ and $n|q - 1$.*
- (ii) *If $y \in \mathbf{SL}_n(q)$, then there always exists a primitive prime divisor ℓ of $q^n - 1$ dividing $|y|$.*

Proof. (i) If for every prime divisor ℓ of $|y|$ there is an $e \in \mathbb{I}_{n-1}$ such that ℓ divides $q^e - 1 = (q - 1)(e)_q$ then, any such ℓ divides $(q - 1)((n)_q, (e)_q)$ for some $e < n$. The latter equals $(q - 1)(e, n)_q = q - 1$ by the Euclidean algorithm, so $\ell|(q - 1)$ for any prime divisor ℓ of $|y|$.

Since y is irreducible, $|y|$ cannot divide $q - 1$, so there is a prime divisor ℓ_0 of $|y|$ dividing $q - 1$ and $(n)_q$. By Lemma 5.10 (iii) this is possible only if $n|(q - 1)$ and in this case $\ell_0 = n$. Then, Lemma 5.10 (ii) implies that $|y|$ divides $n(q - 1)$.

(ii) If $y \in \mathbf{SL}_n(q)$ then $|y|$ divides $(n)_q$. Assume, for a contradiction, that no primitive prime divisor of $q^n - 1$ divides $|y|$. Then, $|y|$ would divide $(n(q - 1), (n)_q) = (n, q - 1)$ by (i) and Lemma 5.10 (iii). Thus, y cannot be irreducible. \square

In the terminology of [17, Definition 1.2], Lemma 5.12 says that if n is an odd prime, then all irreducible elements in $\mathbf{SL}_n(q)$ are $\text{ppd}(n, q; n)$ -elements. The following result is a consequence of [17].

Lemma 5.13. *Let ℓ be a primitive prime divisor of $q^n - 1$ dividing $|x|$ and let $H \leq \mathbf{GL}_n(q)$ be such that $x \in H$. Then H occurs in the following list.*

- (a) $\mathbf{SL}_n(q_0) \leq H \leq N_{\mathbf{GL}_n(q)}(\mathbf{SL}_n(q_0))$ where $q_0 = p^{m_0}$ with $m = m_0 d$, $d \in \mathbb{N}$ and $(d, n) = 1$.
- (b) $\mathbf{SU}_n(q_0^{1/2}) \leq H \leq N_{\mathbf{GL}_n(q)}(\mathbf{SU}_n(q_0))$ where $q_0 = p^{m_0}$ a square with $m = m_0 d$, $d \in \mathbb{N}$ and $(d, n) = 1$.
- (c) $H \leq N_{\mathbf{GL}_n(q)}(\mathbb{T}_w^F) = N_{\mathbf{GL}_n(q)}(T)$, and ℓ divides $|H|$.
- (d) $H/(H \cap Z(\mathbf{GL}_n(q))) \simeq M_{11}$, $n = 5$, $\ell = 11$, and $q^5 \equiv 1 \pmod{11}$.
- (e) $H/(H \cap Z(\mathbf{GL}_n(q))) \simeq M_{23}$, or M_{24} , $n = 11$, $\ell = 23$, $q^{11} \equiv 1 \pmod{23}$.
- (f) $\mathbf{PSL}_2(\ell) \leq H/(H \cap Z(\mathbf{GL}_n(q))) \leq \mathbf{PGL}_2(\ell)$, for $\ell \geq 7$, $n = \frac{1}{2}(\ell - 1)$ and $q^n \equiv 1 \pmod{\ell}$.

Proof. The main result in [17] states that the subgroups of $\mathbf{GL}_d(q)$ containing a $\text{ppd}(d, q; e)$ -element, for some $\frac{1}{2}d < e \leq d$ are precisely those occurring in the Examples 2.1, \dots , 2.9 listed therein. We extract the cases satisfying $d = e = n$ an odd prime > 3 .

- Example 2.1 (b) and (d) and Example 2.5 are discarded because they occur for either d or e even.
- Examples 2.1 (a) and (c) are (a) and (b) in our list.
- Example 2.2 does not occur because it requires an H -stable subspace of the natural representation of $\mathbf{GL}_n(q)$ and $\mathbf{x} \in H$ is irreducible.
- Examples 2.3 and Examples 2.4 (a) are discarded as they require $e \neq d$.
- Example 2.4 (b) is the case (c) in our list.
- Example 2.6 (a) is discarded because it requires the prime $\ell = n+1$, which is impossible because $n > 2$.
- Examples 2.6 (b) and (c) are collected in [17, Tables 2,3,4]. In Table 2, d is even. In Tables 3 and 4 the number e is never odd > 3 .
- Examples 2.7 are listed in [17, Table 5]. The column with $\ell = e + 1$, is immediately discarded, just as all rows for which d is not a prime number. We are left with the three possible choices for $H' \simeq H/H \cap Z(\mathbf{GL}_n(q))$ listed in (d) and (e).
- Examples 2.8 are listed in [17, Table 6] and are discarded because $e \neq d$.
- Examples 2.9 are listed in [17, Tables 7,8] and if d is a prime, then e is even. In Table 8, we discard all cases for which $\ell = e + 1$ and we are left with the case (f) in our list. \square

Lemma 5.14. *Let $\mathbb{G} = \mathbf{SL}_n(\mathbb{k})$, $m = m_0 d$ with $(d, n) = 1$ and $q_0 = p^{m_0}$.*

- (i) *If $(n, q_0 - 1) = n$, then $Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \cap \mathbf{SL}_n(q) = \mathbf{SL}_n(q_0)$.*
- (ii) *If m_0 is even and $(n, q_0^{1/2} + 1) = n$, then*

$$Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \cap \mathbf{SL}_n(q) = \mathbf{SU}_n(q_0^{1/2}).$$

Proof. (i) We prove \subset . Let $\zeta \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $g \in \mathbf{GL}_n(q_0)$ be such that $\zeta g \in \mathbf{SL}_n(q)$. Now, $|\zeta|$ divides $n(q_0 - 1)$ because $\zeta^{-n} = \det(g) \in \mathbb{F}_{q_0}^\times$. It also divides $q - 1$ because $\zeta \in \mathbb{F}_q^\times$. Hence it divides

$$(n(q_0 - 1), q - 1) = (q_0 - 1)(n, (d)_{q_0}).$$

By the assumptions on n and q_0 , we have $(d)_{q_0} \equiv d \pmod{n}$. Since $(d, n) = 1$, then $|\zeta|$ divides $q_0 - 1$, i.e., $\zeta \in \mathbb{F}_{q_0}^\times$ and $\zeta g \in \mathbf{SL}_n(q_0)$.

(ii) We prove \subset . Let $\zeta \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $g \in \mathbf{GU}_n(q_0^{1/2})$ be such that $(\zeta \text{id}_n)g \in \mathbf{SL}_n(q)$. Now, as $g = \text{Fr}_{q_0^{1/2}} \phi(g)$, we have $(\det g)^{q_0^{1/2}+1} = 1$. Hence $|\zeta|$ divides $n(q_0^{1/2} + 1)$ and also $q - 1$ because $\zeta \in \mathbb{F}_q^\times$, and so it divides

$$(n(q_0^{1/2} + 1), q - 1) = (q_0^{1/2} + 1) \left(n, (q_0^{1/2} - 1)(d)_{q_0^{1/2}} \right).$$

However, n is an odd prime dividing $q_0^{1/2} + 1$ so it does not divide $q_0^{1/2} - 1$; also, $(n, (d)_{q_0^{1/2}}) = 1$ by the argument in (i) applied to $q_0^{1/2}$. Thus, $|\zeta|$ divides

$q_0^{1/2} + 1$, that is, $\zeta \text{id}_n \in Z(\mathbf{GU}_n(q_0^{1/2}))$, so $\zeta g \in \mathbf{SL}_n(q) \cap \mathbf{GU}_n(q_0^{1/2}) = \mathbf{SU}_n(q_0^{1/2})$. \square

In this Subsection $\tilde{\pi}: \mathbf{GL}_n(q) \rightarrow \mathbf{PGL}_n(q)$ is the natural projection, whose restriction to $\mathbf{SL}_n(q)$ is π .

Proposition 5.15. *Let \mathbf{x} be an irreducible element in the Coxeter torus $T = \mathbb{T}_w^F$. Then \mathcal{O} is kthulhu.*

Proof. We consider all possible intersections $\mathcal{O} \cap \mathbf{M}$ for every $\mathbf{M} \leq \mathbf{G}$ containing x . All such groups have the form $\mathbf{M} = \pi(H \cap \mathbf{SL}_n(q))$ for some $H \leq \mathbf{GL}_n(q)$ containing \mathbf{x} . We will show that either $\mathcal{O} \cap \mathbf{M} = \mathcal{O}_x^{\mathbf{M}}$ or else $\mathcal{O} \cap \mathbf{M}$ is an abelian subrack. This implies that \mathcal{O} is kthulhu.

For our analysis, we will make use of the following auxiliary facts:

Claim 1. $C_{\mathbf{PSL}_n(q)}(x) \cap \mathcal{O} = \{x, x^q, \dots, x^{q^{n-1}}\}$.

Indeed, Lemma 5.7 gives

$$C_{\mathbf{SL}_n(q)}(\mathbf{x}) \cap \mathfrak{D} = \{\mathbf{x}, \mathbf{x}^q, \dots, \mathbf{x}^{q^{n-1}}\}.$$

We describe $C_{\mathbf{PSL}_n(q)}(x)$. If $\mathbf{z} \in \mathbf{SL}_n(q)$ satisfies $\mathbf{z}\mathbf{x}\mathbf{z}^{-1} \in \mathbb{G}_m(\mathbb{F}_q)\mathbf{x} \cap \mathfrak{D}$, then by primality of n and Lemma 5.8 (ii) and (iv) we conclude that $\mathbf{z}\mathbf{x}\mathbf{z}^{-1} = \mathbf{x}$, and so $C_{\mathbf{PSL}_n(q)}(x) = \pi(C_{\mathbf{SL}_n(q)}(\mathbf{x})) = \pi(T)$, whence the claim.

Claim 2. *If $\tilde{\pi}(H)$ is simple, then $\tilde{\pi}(H) \leq \mathbf{PSL}_n(q)$. In particular, we may assume $H \leq \mathbf{SL}_n(q)$.*

Indeed, if $\tilde{\pi}(H)$ is simple, then

$$\tilde{\pi}(H) = [\tilde{\pi}(H), \tilde{\pi}(H)] = \tilde{\pi}([H, H]) \leq \tilde{\pi}([\mathbf{GL}_n(q), \mathbf{GL}_n(q)]) = \pi(\mathbf{SL}_n(q)).$$

We set from now on $H_1 := H \cap \mathbf{SL}_n(q)$ and inspect all possible $\mathbf{M} = \pi(H_1)$ where H runs through the list of subgroups from Lemma 5.13 containing \mathbf{x} , with ℓ a primitive prime divisor of $q^n - 1$ dividing $|\mathbf{x}|$. The numbering of items is as in Lemma 5.13.

Case (a). Here $q = p^m$, $q_0 = p^{m_0}$ where $m = m_0 d$, $d \in \mathbb{N}$ and $(n, d) = 1$. Proposition 3.1 gives $N_{\mathbf{GL}_n(q)}(\mathbf{SL}_n(q_0)) = Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0)$ so

$$(5.4) \quad \mathbf{SL}_n(q_0) \leq H_1 \leq Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \cap \mathbf{SL}_n(q).$$

We will first show that

$$(5.5) \quad \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \cap \mathbf{SL}_n(q) = \mathcal{O}_x^{\mathbf{SL}_n(q_0)}.$$

If $(n, q_0 - 1) = n$, then the inclusions in (5.4) are all equalities by Lemma 5.14 (i). In this case

$$\begin{aligned} \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \cap \mathbf{SL}_n(q) &= \mathfrak{D} \cap \mathbf{SL}_n(q_0) \\ &= \mathcal{O}_x^{\mathbf{SL}_n(\mathbb{k})} \cap \mathbf{SL}_n(q_0) = \mathcal{O}_x^{\mathbf{SL}_n(q_0)} \end{aligned}$$

where the last two equalities follow from [22, Theorem 21.11] and [20, §2.11]. Assume now that $(n, q_0 - 1) = 1$. Since $\mathbf{x} \in H_1 \leq Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0)$, there are $z = \zeta \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $\mathbf{y} \in \mathbf{GL}_n(q_0)$ such that

$$\mathbf{x} = z\mathbf{y}.$$

Consider $\mathbf{x}_1 \in \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0)$. Let $z_1 = \zeta_1 \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $\mathbf{y}_1 \in \mathbf{GL}_n(q_0)$ be such that $\mathbf{x}_1 = z_1\mathbf{y}_1$. By construction, $|\zeta|$ and $|\zeta_1|$ divide $n(q_0 - 1)$ because $\mathbf{x}, \mathbf{x}_1 \in \mathbf{SL}_n(q)$. Since \mathbf{x} is irreducible, \mathbf{y} and \mathbf{y}_1 are again irreducible in $\mathbf{GL}_n(q)$, whence in $\mathbf{GL}_n(q_0)$, because they are regular and lie in a Coxeter torus of $\mathbf{GL}_n(q)$. Let $\{\eta^{q^j} : j \in \mathbb{I}_{0, n-1}\} \subset \mathbb{F}_{q_0^n}$ and $\{\eta_1^{q^j} : j \in \mathbb{I}_{0, n-1}\} \subset \mathbb{F}_{q_0^n}$ be the sets of eigenvalues of \mathbf{y} and \mathbf{y}_1 , respectively, so $\{\zeta\eta^{q^j} : j \in \mathbb{I}_{0, n-1}\}$ and $\{\zeta_1\eta_1^{q^j} : j \in \mathbb{I}_{0, n-1}\}$ are the sets of eigenvalues of \mathbf{x} and \mathbf{x}_1 , respectively. Then

$$\{\zeta\eta^{q^j} : j \in \mathbb{I}_{0, n-1}\} = \{\zeta_1\eta_1^{q^j} : j \in \mathbb{I}_{0, n-1}\}$$

and so $\zeta\eta = \zeta_1\eta_1^{q^{j_0}}$ for some j_0 . Therefore $|\zeta_1\zeta^{-1}| = |\eta\eta_1^{-q^{j_0}}|$ divides $(n(q_0 - 1), q_0^n - 1) = (q_0 - 1)(n, (n)_{q_0}) = q_0 - 1$, where the last equality follows from Lemma 5.10 (i). In other words, $\zeta_1 \in \zeta\mathbb{F}_{q_0}^\times$, and $z^{-1}\mathbf{x}_1$ is a regular semisimple matrix in $\mathbf{GL}_n(q_0)$ with the same eigenvalues as \mathbf{y} , and it is therefore $\mathbf{SL}_n(q_0)$ -conjugate to \mathbf{y} . Hence,

$$\mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \subset z\mathcal{O}_y^{\mathbf{SL}_n(q_0)} = \mathcal{O}_x^{\mathbf{SL}_n(q_0)}.$$

Let now $x' = \pi(\mathbf{x}') \in \mathcal{O} \cap \mathbf{M}$. Then, $z'\mathbf{x}' \in \mathfrak{D}$ for some $z' \in Z(\mathbf{SL}_n(q))$ and $\mathbf{x}' \in Z(\mathbf{SL}_n(q))H_1$, that is,

$$z'\mathbf{x}' \in \mathfrak{D} \cap Z(\mathbf{SL}_n(q))H_1 \subset \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GL}_n(q_0) \cap \mathbf{SL}_n(q) = \mathcal{O}_x^{H_1}.$$

where the equality follows from (5.4) and (5.5). Thus, $\mathbf{x}' \in Z(\mathbf{SL}_n(q))\mathcal{O}_x^{H_1}$ and $x' \in \mathcal{O}_x^{\pi(H_1)} = \mathcal{O}_x^{\mathbf{M}}$, showing that $\mathcal{O} \cap \mathbf{M} = \mathcal{O}_x^{\mathbf{M}}$.

Case (b). Here $q = p^m$, $q_0 = p^{m_0}$ where $m_0|m$, m_0 is even and $(n, d) = 1$. We use the same strategy as in case (a).

Proposition 3.1 gives $N_{\mathbf{GL}_n(q)}(\mathbf{SU}_n(q_0^{1/2})) = Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2})$ so

$$(5.6) \quad \mathbf{SU}_n(q_0^{1/2}) \leq H_1 \leq Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \cap \mathbf{SL}_n(q).$$

We will first show that

$$(5.7) \quad \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \cap \mathbf{SL}_n(q) = \mathcal{O}_x^{\mathbf{SU}_n(q_0^{1/2})}.$$

If $(n, q_0^{1/2} + 1) = n$, then the inclusions in (5.6) are all equalities by Lemma 5.14 (ii). In this case

$$\begin{aligned} \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \cap \mathbf{SL}_n(q) &= \mathfrak{D} \cap \mathbf{SU}_n(q_0^{1/2}) \\ &= \mathcal{O}_x^{\mathbf{SL}_n(\mathbb{k})} \cap \mathbf{SU}_n(q_0^{1/2}) = \mathcal{O}_x^{\mathbf{SU}_n(q_0^{1/2})} \end{aligned}$$

where the last two equalities follow from [22, Theorem 21.11] and [20, §2.11].

Assume now that $(n, q_0^{1/2} + 1) = 1$. Since $\mathbf{x} \in H_1 \leq Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2})$, there are $z = \zeta \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $\mathbf{y} \in \mathbf{GU}_n(q_0) \leq \mathbf{GL}_n(q_0)$ such that

$$\mathbf{x} = z\mathbf{y}.$$

Consider $\mathbf{x}_1 \in \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2})$. Let $z_1 = \zeta_1 \text{id}_n \in Z(\mathbf{GL}_n(q))$ and $\mathbf{y}_1 \in \mathbf{GU}_n(q_0) \leq \mathbf{GL}_n(q_0)$ be such that $\mathbf{x}_1 = z_1\mathbf{y}_1$. By construction, $|\zeta|$ and $|\zeta_1|$ divide $n(q_0^{1/2} + 1)$ because $\mathbf{x}, \mathbf{x}_1 \in \mathbf{SL}_n(q)$ and $\det(g)^{q_0^{1/2} + 1} = 1$ for any $g \in \mathbf{GU}_n(q_0^{1/2})$.

Since \mathbf{x} is irreducible, \mathbf{y} and \mathbf{y}_1 are again irreducible in $\mathbf{GL}_n(q)$, whence in $\mathbf{GL}_n(q_0)$. We show that $|\mathbf{y}|$ cannot divide $n(q_0 - 1)$. Indeed, if this were the case, then we would have $\mathbf{y}^{q_0 - 1} = \xi \text{id}_n$ for some $\xi \in \mathbb{G}_n(\mathbb{F}_q^\times)$, with $\xi \neq 1$. Since $\mathbf{y}^{q_0} \in \mathcal{O}_y^{\mathbf{GL}_n(q_0)}$, the characteristic polynomial $p_{\mathbf{y}}$ would be $X^n - \det(\mathbf{y}) = X^n - \xi^{-n}$, which is not irreducible. Hence, by Lemma 5.12 (i) there is a primitive prime divisor ℓ of $|\mathbf{y}|$ dividing $q_0^n - 1$.

Let $F_0: \mathbf{GL}_n(\mathbb{k}) \rightarrow \mathbf{GL}_n(\mathbb{k})$ be given by $F_0(A) := \text{Fr}_{q_0^{1/2}} \phi(A)$, for $A \in \mathbf{GL}_n(\mathbb{k})$, cf. Subsection 3.2. By [22, Proposition 26.6] there exists an F_0 -stable torus \mathbb{T}' in $\mathbf{GL}_n(\mathbb{k})$ containing \mathbf{y} . Let $\mathcal{T} = \mathbb{T}' \cap \mathbf{GU}_n(q_0^{1/2})$. By [22, Proposition 25.3 (c)] and an analysis of ϕ -classes in the symmetric group, there is a partition λ of n such that

$$|\mathcal{T}| = \prod_{\lambda_i \text{ even}} (q_0^{\lambda_i/2} - 1) \prod_{\lambda_i \text{ odd}} (q_0^{\lambda_i/2} + 1).$$

The latter divides $\prod_{\lambda_i \text{ even}} (q_0^{\lambda_i/2} - 1) \prod_{\lambda_i \text{ odd}} (q_0^{\lambda_i} - 1)$ and is divisible by the primitive prime divisor ℓ of $q_0^n - 1$. Hence, $\lambda = (n)$ and $|\mathcal{T}| = (q_0^{n/2} + 1)$.

Now we proceed as in case (a): considering the set of eigenvalues for \mathbf{x} and \mathbf{x}_1 and of \mathbf{y} and \mathbf{y}_1 , we deduce that $|\zeta_1 \zeta^{-1}|$ divides

$$\left(n(q_0^{1/2} + 1), q_0^{n/2} + 1 \right) = (q_0^{1/2} + 1)(n, (n)_{-q_0^{1/2}}) = (q_0^{1/2} + 1)$$

where $(n, (n)_{-q_0^{1/2}}) = 1$ because $(n)_{-q_0^{1/2}}$ divides $q_0^{n/2} + 1$ and $q_0^{1/2}$ is not a root of $X^n + 1 = (X + 1)^n$ in \mathbb{F}_n by our assumption on q_0 and n .

Hence, $\mathbf{z}_1 \in zZ(\mathbf{GU}_n(q_0^{1/2}))$, and $z^{-1}\mathbf{x}_1$ is a regular semisimple matrix in $\mathbf{GU}_n(q_0^{1/2})$ with the same eigenvalues as \mathbf{y} , and it is therefore $\mathbf{SU}_n(q_0)$ -conjugate to \mathbf{y} by [20, §2.11, §8.5]. Hence,

$$\mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \subset z\mathcal{O}_y^{\mathbf{SU}_n(q_0^{1/2})} = \mathcal{O}_x^{\mathbf{SU}_n(q_0^{1/2})}.$$

Let now $x' = \pi(\mathbf{x}') \in \mathcal{O} \cap \mathbf{M}$. Then, $z'\mathbf{x}' \in \mathfrak{D}$ for some $z' \in Z(\mathbf{SL}_n(q))$ and $\mathbf{x}' \in Z(\mathbf{SL}_n(q))H_1$, that is,

$$z'\mathbf{x}' \in \mathfrak{D} \cap Z(\mathbf{SL}_n(q))H_1 \subset \mathfrak{D} \cap Z(\mathbf{GL}_n(q))\mathbf{GU}_n(q_0^{1/2}) \cap \mathbf{SL}_n(q) = \mathcal{O}_x^{H_1}.$$

where the equality follows from (5.6) and (5.7). Thus, $\mathbf{x}' \in Z(\mathbf{SL}_n(q))\mathcal{O}_x^{H_1}$ and $x' \in \mathcal{O}_x^{\pi(H_1)} = \mathcal{O}_x^{\mathbf{M}}$, showing that $\mathcal{O} \cap \mathbf{M} = \mathcal{O}_x^{\mathbf{M}}$.

Case (c) In this case, $\mathbf{M} \leq \pi(N_{\mathbf{SL}_n(q)}(T)) = N_{\mathbf{G}}(\pi(T))$, where the second equality follows because $Z(\mathbf{SL}_n(q)) \leq T$. If $y = \pi(\mathbf{y}) \in \mathcal{O} \cap \mathbf{M}$ then there is $\mathbf{z} \in Z(\mathbf{SL}_n(q))$ such that $\mathbf{y} \in \mathcal{O}_{\mathbf{z}\mathbf{x}}^{\mathbf{SL}_n(q)} \cap N_{\mathbf{SL}_n(q)}(T)$, and $\mathbf{z}\mathbf{x}$ is again irreducible. By Lemma 5.11 we see that $\mathbf{y} \in T$, so $\mathcal{O} \cap \mathbf{M} \subset \pi(T)$ is abelian.

Case (d) In this case $n = 5$ and $\ell = 11$ and $\tilde{\pi}(H) = M_{11} = \pi(H_1)$. We show that $\mathcal{O} \cap M_{11} = \mathcal{O}_x^{M_{11}}$. The only elements whose order is divisible by ℓ in M_{11} are of order 11, so $|x| = 11$. There are two classes of such elements in M_{11} , say $\mathcal{O}_x^{M_{11}}$ and $\mathcal{O}_y^{M_{11}}$. If $\mathcal{O} \cap M_{11} = \mathcal{O}_x^{M_{11}} \cup \mathcal{O}_y^{M_{11}}$, then $\langle x \rangle - 1 \subset \mathcal{O} \cap M_{11} \cap C_{\mathbf{PSL}_n(q)}(x) = \{x, x^q, x^{q^2}, x^{q^3}, x^{q^4}\}$, a contradiction.

Case (e) In this case $n = 11$ and $\ell = 23$. The only elements of order divisible by ℓ in $\mathbf{M} = M_{23}$ or M_{24} have order 23 and there are 2 conjugacy classes of such elements. We proceed as in case (c).

Case (f) In this case $\ell|(q^n - 1)$ and $\mathbf{M} = \pi(H_1) \simeq H_1/H_1 \cap Z(\mathbf{SL}_n(q))$, and

$$\mathbf{PSL}_2(\ell) \leq H_1/H_1 \cap Z(\mathbf{SL}_n(q)) \leq \mathbf{PGL}_2(\ell).$$

As $[\mathbf{PGL}_2(\ell) : \mathbf{PSL}_2(\ell)] \leq 2$, we have $\mathbf{M} \simeq \mathbf{PGL}_2(\ell)$ or $\mathbf{M} \simeq \mathbf{PSL}_2(\ell)$. In both cases, $|x| = \ell$ and we claim that $\mathcal{O} \cap \mathbf{M} = \mathcal{O}_x^{\mathbf{M}}$. In $\mathbf{PGL}_2(\ell)$ all non-trivial unipotent elements are conjugate and the claim follows. Let $y \in \mathcal{O} \cap \mathbf{PSL}_2(\ell)$. By replacing y with a representative lying in the same Borel subgroup of $\mathbf{PSL}_2(\ell)$ as x , we can ensure that

$$y \in C_{\mathbf{PSL}_2(\ell)}(x) \cap \mathcal{O} \subset C_{\mathbf{PSL}_n(q)}(x) \cap \mathcal{O} = \{x, x^q, \dots, x^{q^{n-1}}\}.$$

Without loss of generality we may assume that x is the class of $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$, for some $\xi \in \mathbb{F}_\ell^\times$ so y is the class of $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}^{q^j} = \begin{pmatrix} 1 & q^j \xi \\ 0 & 1 \end{pmatrix}$ for some $j \in \mathbb{I}_{n-1}$. By assumption $q \equiv q^{n+1} \pmod{\ell}$ hence q is a square modulo ℓ . Therefore x and y are conjugate in $\mathbf{PSL}_2(\ell)$, whence the claim. \square

Remark 5.16. Consider either $g \in M_{11}$, $|g| = 11$, or $g \in M_{23}$ or M_{24} , $|g| = 23$. Then the classes $\mathcal{O}_g^{M_{11}}$, $\mathcal{O}_g^{M_{23}}$ or $\mathcal{O}_g^{M_{24}}$ are contained either in $\mathcal{O}_g^{\mathbf{PSL}_5(q)}$ for some q , or in $\mathcal{O}_g^{\mathbf{PSL}_{11}(q')}$ for some q' , respectively, according to [17] and Claim 2, see Cases (d) and (e). By Proposition 5.15, since g is irreducible in all cases, $\mathcal{O}_g^{\mathbf{G}}$ is kthulhu, where \mathbf{G} is either $\mathbf{PSL}_5(q)$ or $\mathbf{PSL}_{11}(q')$. Hence so are $\mathcal{O}_g^{M_{11}}$, $\mathcal{O}_g^{M_{23}}$ and $\mathcal{O}_g^{M_{24}}$, as was previously proved in [10, Teorema 3.26].

6. SEMISIMPLE CONJUGACY CLASSES REPRESENTED IN K

In this section we deal with semisimple conjugacy classes intersecting the subgroup K which is the image of the map $j: \mathbf{GL}_n(q) \rightarrow \mathbb{G}^F$ introduced in §3.3. We give parallel proofs for two classes of simple groups:

- $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{k})$ with $n \geq 2$; here $\mathbf{G} := \mathbb{G}^F/Z(\mathbb{G}^F)$ and $\pi: \mathbb{G}^F \rightarrow \mathbf{G}$ denotes the standard projection.
- $\mathbb{G} = \mathbf{SO}_{n'}(\mathbb{k})$ with $n' = 2n$ and $n \geq 4$, or $n' = 2n + 1$ and $n \geq 3$; here $\mathbf{G} := [\mathbb{G}^F, \mathbb{G}^F]/Z(\mathbb{G}^F)$ and $\pi: [\mathbb{G}^F, \mathbb{G}^F] \rightarrow \mathbf{G}$ is the standard projection.

In the symplectic case, $\mathbb{G}^F = [\mathbb{G}^F, \mathbb{G}^F]$ so for brevity of the exposition we write $[\mathbb{G}^F, \mathbb{G}^F]$ in both cases. We also consider such groups with smaller n sometimes for the sake of recursive arguments.

We shall consider a semisimple class \mathcal{O} in \mathbf{G} , a class \mathfrak{D} in $[\mathbb{G}^F, \mathbb{G}^F]$ such that $\pi(\mathfrak{D}) = \mathcal{O}$ and assume that it exists $A \in \mathbf{GL}_n(q)$ such that $j(A) \in \mathfrak{D}$.

Here are the main results of this Section:

Theorem 6.1. *Let $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{k})$, $n \geq 2$, and let $A \in \mathbf{GL}_n(q) - Z(\mathbf{GL}_n(q))$ be a semisimple element, which is not an involution if $n = 2$ and $q \leq 7$. Then $\mathcal{O} = \mathcal{O}_{\pi(j(A))}^{\mathbf{G}}$ collapses.*

Theorem 6.2. *Let $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$ or $\mathbf{SO}_{2n+1}(\mathbb{k})$ with $n \geq 3$ in both cases and let $A \in \mathbf{GL}_n(q) - Z(\mathbf{GL}_n(q))$ be a semisimple element.*

Assume in addition that $j(A)$ does not correspond to situation (4.1) if $q \in \{3, 5, 7\}$. Then $\mathcal{O} = \mathcal{O}_{\pi(j(A))}^{\mathbf{G}}$ collapses.

These theorems are proved in Subsection 6.2 after we deal in Subsection 6.1 with the case when A is irreducible.

In the orthogonal case, we consider the orbit $\mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ for later applications even if $j(A)$ does not necessarily belong to $[\mathbb{G}^F, \mathbb{G}^F]$, as in Remark 2.1. See Lemmata 6.4 and 6.5.

We start by some general considerations.

Lemma 6.3. *Let $A \in \mathbf{GL}_n(q)$ be a semisimple element.*

- (i) $\mathcal{O}_{j(A)}^{j(\mathbf{SL}_n(q))} = \mathcal{O}_{j(A)}^{[K, K]} = \mathcal{O}_{j(A)}^K = \mathcal{O}_{j(A)}^{j(\mathbf{GL}_n(q))}$.
- (ii) *If A is irreducible, then either $\mathcal{O}_A^{\mathbf{GL}_n(q)} = \mathcal{O}_{A^{-1}}^{\mathbf{GL}_n(q)}$ or else $j(A)$ is regular in $\mathbf{GL}_{n'}(q)$.*
- (iii) *If A is irreducible, then either $\mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]} = \mathcal{O}_{j(A^{-1})}^{[\mathbb{G}^F, \mathbb{G}^F]}$, or else $j(A)$ is regular.*

Proof. (i) is a consequence of the inclusions

$$j(\mathbf{SL}_n(q)) \simeq [K, K] \leq K \simeq \mathbf{GL}_n(q).$$

(ii): If ζ^{q^h} , $h \in \mathbb{I}_{0,n-1}$, are the (distinct) eigenvalues of A in \mathbb{k} , then $\zeta^{\pm q^h}$ for $h \in \mathbb{I}_{0,n-1}$ (together with 1 when $n' = 2n + 1$) are the eigenvalues of $j(A)$. Assume that $j(A)$ is not regular in $\mathbf{GL}_{n'}(q)$; hence A and A^{-1} have a common eigenvalue. Then the sets of eigenvalues of A and A^{-1} coincide by irreducibility, that is A and A^{-1} are conjugate in $\mathbf{GL}_n(\mathbb{k})$. Since centralisers in $\mathbf{GL}_n(\mathbb{k})$ are connected, [20, p. 19], $\mathcal{O}_A^{\mathbf{GL}_n(q)} = \mathcal{O}_{A^{-1}}^{\mathbf{GL}_n(q)}$. (iii) follows from (i) and (ii) and the inclusion $[K, K] \leq K \cap [\mathbb{G}^F, \mathbb{G}^F]$. \square

6.1. $A \in \mathbf{GL}_n(q)$ is irreducible. We first analyze this case.

Lemma 6.4. (Here $n \geq 3$ when $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$ or $\mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k})$). Let $A \in \mathbf{GL}_n(q)$ be an irreducible element such that

- $j(A)$ is not regular in $\mathbf{GL}_{n'}(q)$,
- $p_A(X) \neq X^2 + 1$ when $\mathbb{G} = \mathbf{Sp}_4(\mathbb{k})$ and $q \equiv 3 \pmod{4}$.

Then

- (i) $\mathcal{O} := \mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ collapses.
- (ii) If $j(A) \in \mathfrak{D} \subseteq [\mathbb{G}^F, \mathbb{G}^F]$, then \mathcal{O} collapses.

Proof. Observe that (ii) follows directly from (i), that we prove. The initial discussion is valid for both orthogonal and symplectic groups. Recall ϕ from (3.1). The irreducibility assumption in A ensures that the eigenvalues of A are all distinct, i.e., A is regular semisimple, so we may assume that A is the companion matrix of its characteristic (and minimal) polynomial $p_A = X^n + a_{n-1}X^{n-1} + \dots + a_0$. That is, A , $\phi(A)$, tA , ${}^tA^{-1}$, A^{-1} , and $\phi(A^{-1})$ have the following shape:

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}, \quad \phi(A) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \\ -1/a_0 & -a_{n-1}/a_0 & \dots & -a_2/a_0 & -a_1/a_0 \end{pmatrix},$$

$${}^tA = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \dots & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}, \quad {}^tA^{-1} = \begin{pmatrix} -a_1/a_0 & -a_2/a_0 & \dots & -a_{n-1}/a_0 & -1/a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} -a_1/a_0 & 1 & 0 & \dots & 0 \\ -a_2/a_0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n-1}/a_0 & 0 & 0 & \dots & 1 \\ -1/a_0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \phi(A^{-1}) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \dots & \dots & -a_0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Also $A \neq A^{-1}$, otherwise A would have eigenvalues ± 1 , contradicting irreducibility. We consider the disjoint subracks:

$$R_1 := \left\{ \begin{pmatrix} A & Y \\ 0 & \phi(A) \end{pmatrix} \in \mathcal{O} \right\}, \quad R_2 := \left\{ \begin{pmatrix} A^{-1} & Y \\ 0 & \phi(A^{-1}) \end{pmatrix} \in \mathcal{O} \right\}, \quad \text{if } n' = 2n;$$

$$R_1 := \left\{ \begin{pmatrix} A & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A) \end{pmatrix} \in \mathcal{O} \right\}, \quad R_2 := \left\{ \begin{pmatrix} A^{-1} & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A^{-1}) \end{pmatrix} \in \mathcal{O} \right\}, \quad \text{if } n' = 2n + 1.$$

Now $R_1 \neq \emptyset$ by construction, and $R_2 \neq \emptyset$ by Lemma 6.3 (iii). It is easy to see that $R_i \triangleright R_j = R_j$, $1 \leq i, j \leq 2$. We continue with each group separately.

Case 1. $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{k})$. Let

$$r_1 := \begin{pmatrix} \text{id}_n & J_n \\ 0 & \text{id}_n \end{pmatrix} \triangleright j(A) = \begin{pmatrix} A & -AJ_n + {}^t A^{-1} J_n \\ 0 & J_n {}^t A^{-1} J_n \end{pmatrix} \in R_1, \quad r_2 := j(A^{-1}) \in R_2.$$

A direct calculation shows that

$$r_1 r_2 := \begin{pmatrix} \text{id}_n & -A {}^t A J_n + J_n \\ 0 & \text{id}_n \end{pmatrix} \quad r_2 r_1 := \begin{pmatrix} \text{id}_n & -J_n + A^{-1} {}^t A^{-1} J_n \\ 0 & \text{id}_n \end{pmatrix}$$

so $r_1 r_2 = r_2 r_1$ if and only if $2 \text{id}_n = A^{-1} {}^t A^{-1} + A {}^t A$. Let us verify that such an equality never holds. Comparing the diagonal entries we obtain $a_i^2 = (-1)^{i+1} a_0^{2i} (1 - a_0^2)$ for $i > 0$, whereas comparing the entries in the first row we obtain $a_1(a_2 + a_0^3) = 0$ and $a_1 a_l = -a_0^3 a_{l-1}$ for $l > 2$. The conditions $a_2 + a_0^3 = 0$ and $a_2^2 = -a_0^4 (1 - a_0^2)$ lead to a contradiction, hence necessarily $a_1 = 0$ and so $a_l = 0$ for any $l > 0$ and $a_0^2 = 1$. In other words, $p_A(X) = X^n \pm 1$. By Lemma 5.5, this is possible only if $n = 2$, $q \equiv 3 \pmod{4}$ and $p_A(X) = X^2 + 1$, which is excluded by hypothesis.

Then, $r_1 \triangleright r_2 \neq r_2$ and, for $H := \langle r_1, r_2 \rangle$ we have $\mathcal{O}_{r_1}^H \cap \mathcal{O}_{r_2}^H \subset R_1 \cap R_2 = \emptyset$ because $A^2 \neq \text{id}$. If $p = 2$, then $|r_1| = |r_2|$ is odd and $\mathcal{O}_{j(A)}^{\mathbf{Sp}_{2n}(q)}$ is of type C by Remark 2.4 (b). If, instead, p is odd, then $r_1 r_2 \neq r_2 r_1$ implies $(r_1 r_2)^2 \neq (r_2 r_1)^2$ as they are p -elements, so $\mathcal{O}_{j(A)}^{\mathbf{Sp}_{2n}(q)}$ is of type D.

We claim that the restriction of the projection $\pi: \mathbf{Sp}_{2n}(q) \rightarrow \mathbf{G}$ to $R_1 \amalg R_2$ is injective. Indeed, this could fail only if $A^2 = \pm 1$, but since A is irreducible, we would have $A^2 = -1$ which would give $p_A(X) = X^2 + 1$, with $q \equiv 3 \pmod{4}$, i.e., the discarded case. Hence $\mathcal{O}_{\pi(j(A))}^{\mathbf{G}}$ collapses.

Case 2. $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$ or $\mathbf{SO}_{2n+1}(\mathbb{k})$. For $n \geq 3$ we consider the matrices:

$$E := \begin{cases} \text{diag}(\text{id}_{\frac{n}{2}}, -\text{id}_{\frac{n}{2}}) & \text{if } n \text{ is even,} \\ \text{diag}(\text{id}_{[\frac{n}{2}]}, 0, -\text{id}_{[\frac{n}{2}]}) & \text{if } n \text{ is odd,} \end{cases}$$

$$U := \begin{cases} \begin{pmatrix} \text{id}_n & E \\ 0 & \text{id}_n \end{pmatrix} & \text{if } \mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k}), \\ \begin{pmatrix} \text{id}_n & 0 & E \\ 0 & 1 & 0 \\ 0 & 0 & \text{id}_n \end{pmatrix} & \text{if } \mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k}). \end{cases}$$

Then $U \in [\mathbb{G}^F, \mathbb{G}^F]$ by [22, Theorem 24.15, Proposition 24.21] and we consider the elements $r_i \in R_i$, $i = 1, 2$:

$$r_1 := U \triangleright j(A) \in R_1 = \begin{cases} \begin{pmatrix} A & -AE + E\phi(A) \\ 0 & \phi(A) \end{pmatrix} & \text{if } \mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k}), \\ \begin{pmatrix} A & 0 & -AE + E\phi(A) \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A) \end{pmatrix} & \text{if } \mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k}), \end{cases}$$

$$r_2 := j(A^{-1}) \in R_2.$$

A direct calculation shows that $r_1 r_2 = r_2 r_1$ if and only if

$$(6.1) \quad 2E = AE\phi(A^{-1}) + A^{-1}E\phi(A).$$

We verify that this never happens. Assume first that p is odd. By looking at the $(1, 1)$ -entries we see that (6.1) never holds if $n \geq 3$. Since $r_1 r_2$ and $r_2 r_1$ are p -elements, it follows that $\pi(r_1 r_2)^2 \neq \pi(r_2 r_1)^2$. The restriction of π to $R_1 \amalg R_2$ is injective because $A^2 = -\text{id}$ with A irreducible would imply $n = 2$, a discarded case. Hence $\mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$ is of type D.

Assume that $p = 2$, so $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$. Then (6.1) amounts to $A^2 E \stackrel{\star}{=} E\phi(A)^2$. If $n \geq 4$, by looking at the first row we see that \star never holds. If $n = 3$, then (6.1) holds only when $a_2 = a_0^{-1}$, $a_1 = a_0^2$, but in this case

$$p_A(X) = X^3 + a_0^{-1}X^2 + a_0^2X + a_0 = (X + a_0)^2(X + a_0^{-1})$$

is not irreducible. Since $|r_1|$ is odd and π is injective, Remark 2.4 (b) applies and so $\mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$ is of type C. \square

Lemma 6.5. (Here $n \geq 3$ for $\mathbb{G} = \mathbf{SO}_{2n}(\mathbb{k})$ and $n \geq 2$ for $\mathbb{G} = \mathbf{SO}_{2n+1}(\mathbb{k})$ or $\mathbf{Sp}_{2n}(\mathbb{k})$). Let $A \in \mathbf{GL}_n(q)$ be an irreducible element such that $j(A)$ is regular in $\mathbf{GL}_{n'}(q)$. Then

- (i) $\mathcal{O} := \mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ collapses.
- (ii) If $j(A) \in \mathfrak{D} \subseteq [\mathbb{G}^F, \mathbb{G}^F]$, then \mathcal{O} collapses.

Proof. Since A is irreducible, $\mathcal{O}_A^{\mathbf{GL}_n(q)} = \mathcal{O}_{A^q}^{\mathbf{GL}_n(q)}$. We have $\mathcal{O} = \mathcal{O}_{j(A^q)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ by Lemma 6.3 (i) so we can consider the disjoint subtracks:

$$R_1 := \left\{ \begin{pmatrix} A & Y \\ 0 & \phi(A) \end{pmatrix} \in \mathcal{O} \right\}, \quad R_2 := \left\{ \begin{pmatrix} A^q & Y \\ 0 & \phi(A^q) \end{pmatrix} \in \mathcal{O} \right\}, \quad \text{if } n' = 2n;$$

$$R_1 := \left\{ \begin{pmatrix} A & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A) \end{pmatrix} \in \mathcal{O} \right\}, \quad R_2 := \left\{ \begin{pmatrix} A^q & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & \phi(A^q) \end{pmatrix} \in \mathcal{O} \right\}, \quad \text{if } n' = 2n + 1.$$

Then $R_i \triangleright R_j \subseteq R_j$ for $1 \leq i, j \leq 2$.

Let $r_1 = j(A) \in R_1$. Since $j(A^q)$ is regular, $C_{\mathbf{GL}_{n'}(q)}(r_1)$ consists of semisimple elements. Thus, there exists $u \in [\mathbb{G}^F, \mathbb{G}^F]$ unipotent block upper triangular, with identity diagonal blocks of size n, n if $n' = 2n$ and $n, 1, n$ if $n' = 2n + 1$, such that $r_2 := u \triangleright j(A^q) \in R_2 \setminus \{j(A^q)\}$. Observe that

$r_2 = j(A^q)v$ for some non-trivial block upper triangular unipotent element v . Now, $r_1j(A^q) = j(A^q)r_1$ and $v \notin C_{\mathbf{GL}_{n'}(q)}(r_1)$ because the latter consists of semisimple elements. Hence, $r_1r_2 \neq r_2r_1$.

If $p = 2$, then $|A|$ is odd and $\mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ is of type C by Remark 2.4 (b).

Let p be odd. Then $H := \langle r_1, r_2 \rangle = \langle r_1, v \rangle = \langle r_2, v \rangle$, with v a p -element. Thus $|\mathcal{O}_{r_i}^H| \geq |\mathcal{O}_{r_i}^{\langle v \rangle}| \geq 3$ for $i = 1, 2$, so $\mathcal{O}_{j(A)}^{[\mathbb{G}^F, \mathbb{G}^F]}$ is of type C by Lemma 2.3.

Since A is irreducible, $A \neq A^q$, hence the restriction of π to $R_1 \amalg R_2$ is injective, giving (ii). \square

Lemma 6.6. *Let $\mathbb{G} = \mathbf{Sp}_4(\mathbb{k})$, let $q \equiv 3 \pmod{4}$ and let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\mathcal{O} = \mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$ is of type D, hence it collapses.*

Proof. By assumption, $\pi(j(A))$ is an involution. Let $s_{c,d} := \begin{pmatrix} c & d \\ d & -c \end{pmatrix} \in \mathbb{F}_q^{2 \times 2}$, where $(c, d) \in \mathbb{F}_q^2$. A direct calculation shows that

$$As_{c,d}A^{-1} = s_{-c,-d} = -s_{c,d}.$$

If, in addition $s_{c,d} \in \mathbf{GL}_2(q)$, then

$$\pi(j(A)) \triangleright \pi(j(s_{c,d})) = \pi(j(s_{c,d})); \quad \text{also} \quad \phi(s_{c,d}) = \frac{-1}{c^2 + d^2} s_{c,-d}.$$

We pick $(a, b) \in \mathbb{F}_q^2$ such that $a^2 + b^2 = -1$. Since $q \equiv 3 \pmod{4}$, we have $ab \neq 0$. As $s_{a,b}$ is semisimple, with same trace and determinant as A , it lies in $\mathcal{O}_A^{\mathbf{SL}_2(q)}$, so $\pi(j(s_{a,b})) \in \mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$.

Consider the disjoint, non-empty subtracks

$$R_1 := \left\{ \pi \left(\begin{pmatrix} A & X \\ 0 & -A \end{pmatrix} \in \mathcal{O}_{\pi(j(A))}^{\mathbb{G}} \right) \right\}, \quad R_2 := \left\{ \pi \left(\begin{pmatrix} s_{a,b} & X \\ 0 & s_{a,-b} \end{pmatrix} \in \mathcal{O}_{\pi(j(A))}^{\mathbb{G}} \right) \right\}$$

of $\mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$. Then $R_i \triangleright R_j = R_j$ for $i, j \in \{1, 2\}$.

We set

$$r := \pi \left(\begin{pmatrix} \text{id}_2 & \text{id}_2 \\ 0 & \text{id}_2 \end{pmatrix} \triangleright \pi(j(A)) \right) = \pi \left(\begin{pmatrix} A & -2A \\ 0 & -A \end{pmatrix} \right) \in R_1, \quad s := \pi(j(s_{a,b})) \in R_2.$$

Now $ab \neq 0$ implies that $s_{a,b}s_{a,-b}$ is not diagonal, hence

$$(rs)^2 = \pi \left(\begin{pmatrix} \text{id}_2 & 2(\text{id}_2 + s_{a,b}s_{a,-b}) \\ 0 & \text{id}_2 \end{pmatrix} \right) \neq \pi \left(\begin{pmatrix} \text{id}_2 & -2(\text{id}_2 + s_{a,b}s_{a,-b}) \\ 0 & \text{id}_2 \end{pmatrix} \right) = (sr)^2,$$

so $\mathcal{O}_{\pi(j(A))}^{\mathbb{G}}$ is of type D. \square

6.2. Proofs of Theorems 6.1 and 6.2. We now drop the irreducibility assumption and proceed to prove the main results of this Section.

Proof of Theorem 6.1. For A irreducible, this is Lemmata 6.4, 6.5 and 6.6.

If A is not irreducible, then we may assume that A is a block diagonal matrix $\text{diag}(A_1, \dots, A_f)$ where the A_i 's are irreducible. If they are all of size 1, then $j(A)$ lies in a \mathbb{F}_q -split torus and Proposition 4.1 applies. If, instead, one of the matrices A_i has size $n_i \geq 2$, then $n > 2$ and A_i is non-central in $\mathbf{GL}_{n_i}(q)$ because it is irreducible. Lemmata 6.4, 6.5 and 6.6 imply that $\mathcal{O}_{j(A_i)}^{\mathbf{Sp}_{2n_i}(q)}$ collapses. The statement follows from injectivity of the composition of rack maps:

$$\left(\prod_{l=1}^{i-1} \{j(A_l)\} \right) \times \mathcal{O}_{j(A_i)}^{\mathbf{Sp}_{2n_i}(q)} \times \left(\prod_{m=i+1}^f \{j(A_m)\} \right) \rightarrow \mathcal{O}_{j(A)}^{\mathbf{Sp}_{2n}(q)} \rightarrow \mathcal{O}_{\pi(j(A))}^{\mathbf{G}}. \quad \square$$

Proof of Theorem 6.2. For A irreducible, this is Lemmata 6.4 and 6.5.

If A is not irreducible, then we may assume that A is a block diagonal matrix $\text{diag}(A_1, \dots, A_f)$ where $f > 1$ and each A_i is an irreducible $n_i \times n_i$ -matrix.

If $q = 2$, then A lies in $\mathbf{SL}_n(q)$ and is not irreducible, so the rack inclusion $\mathcal{O}_A^{\mathbf{SL}_n(q)} \hookrightarrow \mathcal{O}_{j(A)}^{[\mathbf{SO}_{n'}(q), \mathbf{SO}_{n'}(q)]}$ combined with [3, Theorem 1.1] gives the claim because $\mathbf{P}\Omega_{n'}(q) = [\mathbf{SO}_{n'}(q), \mathbf{SO}_{n'}(q)]$.

If $n_i = 1$ for all i , then $j(A)$ lies in a \mathbb{F}_q -split torus and Lemma 4.6 applies.

Therefore, we assume from now on that $q > 2$ and $n_i \geq 2$ for some i .

If $n_i \geq 3$ for some i , then $\mathcal{O}_{j(A_i)}^{[\mathbf{SO}_{2n_i}(q), \mathbf{SO}_{2n_i}(q)]}$ collapses by Lemmata 6.4 and 6.5. Then the claim follows because of the injectivity of the composition of the rack morphisms

$$\begin{aligned} \left(\prod_{l=1}^{i-1} \{j(A_l)\} \right) \times \mathcal{O}_{j(A_i)}^{[\mathbf{SO}_{2n_i}(q), \mathbf{SO}_{2n_i}(q)]} \times \left(\prod_{l=i+1}^f \{j(A_l)\} \right) &\rightarrow \mathcal{O}_{j(A)}^{[\mathbf{SO}_{n'}(q), \mathbf{SO}_{n'}(q)]} \\ &\rightarrow \mathcal{O}_{\pi(j(A))}^{\mathbf{G}}. \end{aligned}$$

From now on we assume that $n_1 = 2$, and $n_i \leq 2$ for all i .

If $n_i = 1$ for some i , say $i = 2$, then $A_2 = (c)$ for some $c \in \mathbb{F}_q^\times$. Since A_1 is irreducible, it is regular and has no eigenvalues in \mathbb{F}_q . Thus the block diagonal matrix $\tilde{A}_1 = \text{diag}(A_1, c)$ has 3 distinct eigenvalues in $\overline{\mathbb{F}_q}$. The matrices $\begin{pmatrix} A_1 & v \\ 0 & c \end{pmatrix}$, $v \in \mathbb{F}_q^2$, have the same eigenvalues, hence they lie in $\mathcal{O}_{\tilde{A}_1}^{\mathbf{GL}_3(q)} = \mathcal{O}_{\tilde{A}_1}^{\mathbf{SL}_3(q)}$, cf. Remark 2.1. Consider the map $j : \mathbf{GL}_3(q) \rightarrow \mathbf{SO}_6(q)$. We claim that $j \begin{pmatrix} A_1^{-1} & 0 \\ 0 & c \end{pmatrix} \in \mathcal{O} := \mathcal{O}_{j(\tilde{A}_1)}^{[\mathbf{SO}_6(q), \mathbf{SO}_6(q)]}$.

Indeed, there is a representative g of a suitable $w \in W$ in the normaliser of the torus of diagonal matrices in $[\mathbf{SO}_6(q), \mathbf{SO}_6(q)]$ that satisfies $g \triangleright j(\tilde{A}_1) =$

$j \begin{pmatrix} {}^t A_1^{-1} & 0 \\ 0 & c \end{pmatrix}$. Also, ${}^t A_1^{-1} \in \mathcal{O}_{A_1^{-1}}^{\mathbf{SL}_3(q)}$, hence $j \begin{pmatrix} {}^t A_1^{-1} & 0 \\ 0 & c \end{pmatrix} \in \mathcal{O}$. Thus

$$R_1 = \left\{ j \begin{pmatrix} A_1 & v \\ 0 & c \end{pmatrix} : v \in \mathbb{F}_q^2 \right\}, \quad R_2 = \left\{ j \begin{pmatrix} A_1^{-1} & v \\ 0 & c \end{pmatrix} : v \in \mathbb{F}_q^2 \right\}$$

are subracks of \mathcal{O} , which are disjoint because A_1 is irreducible. Clearly, $R_i \triangleright R_j \subset R_j$, $1 \leq i, j \leq 2$. Pick $0 \neq v \in \mathbb{F}_q^2$ and set:

$$r := j \begin{pmatrix} A_1 & 0 \\ 0 & c \end{pmatrix} \in R_1; \quad s := j \begin{pmatrix} A_1^{-1} & v \\ 0 & c \end{pmatrix} \in R_2.$$

By a direct calculation, $rs = sr$ implies that c is an eigenvalue of A_1 , a contradiction. Similarly, $(rs)^2 = (sr)^2$ iff $c^2 = -1$, which can occur only if q is even or $q \equiv 1 \pmod{4}$. If q is even, then \mathcal{O} is of type C by Remark 2.4 (b). If $q \equiv 3 \pmod{4}$ or $q \equiv 1 \pmod{4}$ and $c^2 \neq -1$, then \mathcal{O} is of type D.

Assume that $q \equiv 1 \pmod{4}$ and $c^2 = -1$. We claim that

$$(6.2) \quad 3 = |\{s, rsr^{-1}, r^2sr^{-2}\}| \leq \left| \mathcal{O}_s^{(r,s)} \right|,$$

$$(6.3) \quad 3 = |\{r, srs^{-1}, r s r s^{-1} r^{-1}\}| \leq \left| \mathcal{O}_r^{(r,s)} \right|$$

By a direct calculation, $r^2sr^{-2} = s$ iff $A_1^2v = -v = c^2v$, hence c or $-c$ is an eigenvalue of A_1 , a contradiction because they both lie in \mathbb{F}_q ; (6.2) follows. Similarly, $r s r s^{-1} r^{-1} = s r s^{-1}$ iff $(A_1 - c)^2v = 0$, thus c is an eigenvalue of A_1 , a contradiction. Now $s r s^{-1} \neq r$ implies $r s r s^{-1} r^{-1} \neq r$ and (6.3) follows. Hence $\mathcal{O}_{j(\tilde{A}_1)}^{[\mathbf{SO}_3(q), \mathbf{SO}_3(q)]}$ is of type C by Lemma 2.3. Since the composition

$$\begin{aligned} (R_1 \amalg R_2) \times \left(\prod_{l=3}^f \{j(A_l)\} \right) &\rightarrow \mathcal{O}_{j(\tilde{A}_1)}^{[\mathbf{SO}_3(q), \mathbf{SO}_3(q)]} \times \left(\prod_{l=3}^f \{j(A_l)\} \right) \\ &\rightarrow \mathcal{O}_{j(A)}^{[\mathbf{SO}_{n'}(q), \mathbf{SO}_{n'}(q)]} \rightarrow \mathcal{O} = \mathcal{O}_{\pi(j(A))}^{\mathbb{G}} \end{aligned}$$

is an injective morphism of racks, the statement is proved in this case.

There remains the case $n_i = 2$ for all i . It suffices to assume that $f = 2$, so $\mathbb{G} = \mathbf{SO}_8(q)$, and that A_1 and A_2 are the companion matrices of their characteristic polynomials $p_{A_1} = X^2 + aX + b$ and $p_{A_2} = X^2 + cX + d$, so

$$(6.4) \quad A_1 := \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & -d \\ 1 & -c \end{pmatrix}, \quad A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

As in the previous step, there is an element in $[\mathbf{SO}_8(q), \mathbf{SO}_8(q)]$ mapping $j(A)$ to $j \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_2 \end{pmatrix}$. We consider the subracks of $\mathcal{O}_{j(A)}^{[\mathbf{SO}_8(q), \mathbf{SO}_8(q)]}$ given by

$$\begin{aligned} R_1 &:= \left\{ j \begin{pmatrix} A_1 & M \\ 0 & A_2 \end{pmatrix} \in \mathcal{O}_{j(A)}^{[\mathbf{SO}_8(q), \mathbf{SO}_8(q)]} \right\}, \\ R_2 &:= \left\{ j \begin{pmatrix} A_1^{-1} & M \\ 0 & A_2 \end{pmatrix} \in \mathcal{O}_{j(A)}^{[\mathbf{SO}_8(q), \mathbf{SO}_8(q)]} \right\} \end{aligned}$$

which are disjoint since A_1 is irreducible. Clearly, $R_i \triangleright R_j \subset R_j$, $1 \leq i, j \leq 2$. Let $u := \begin{pmatrix} \text{id}_2 & \text{id}_2 \\ 0 & \text{id}_2 \end{pmatrix} \in \mathbf{SL}_4(q)$ and consider

$$r := j(u) \triangleright j(A) = j \begin{pmatrix} A_1 & A_2 - A_1 \\ 0 & A_2 \end{pmatrix} \in R_1, \quad s := j \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_2 \end{pmatrix} \in R_2.$$

A direct calculation in $\mathbf{GL}_4(q)$ shows that $(rs)^2 = (sr)^2$ if and only if

$$(6.5) \quad (A_2 - A_1)A_2(\text{id}_2 + A_2^2) = A_1^{-1}(A_2 - A_1)(\text{id}_2 + A_2^2).$$

Now, $\det(\text{id}_2 + A_2^2) = 0$ implies that there exists $0 \neq v \in \mathbb{F}_q^2$ such that $A_2^2 v = -v$. By the irreducibility of A_2 , we get $p_{A_2} = X^2 + 1$, i.e., $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Assume that $\det(\text{id}_2 + A_2^2) \neq 0$. Then (6.5) is equivalent to $A_2 - A_1 = A_1^{-1} - A_2^{-1}$, which is equivalent to $A_1 = A_2$ by virtue of (6.4). Therefore, if $A_1 \neq A_2$, possibly interchanging the roles of A_1 and A_2 , we can make sure that (6.5) is not satisfied, hence $(rs)^2 \neq (sr)^2$, so $\mathcal{O}_A^{\mathbf{SL}_4(q)}$ is of type D . If $A_1 = A_2 \neq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then we interchange A_1 and A_1^{-1} and argue as above. In all cases, the restriction of π to $R_1 \amalg R_2$ is injective, so $\mathcal{O}_{\pi(j(A))}^{\mathbf{G}}$ is of type D .

Finally, if $A_1 = A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $A \in \mathbf{SL}_4(q)$ and $\mathcal{O}_{\pi_{\mathbf{SL}_4(q)}(A)}^{\mathbf{PSL}_4(q)}$ is of type D by [3, Lemmata 3.15, 3.16, 3.17], and $\mathcal{O}_{\pi(j(A))}^{\pi(K)}$ projects onto it. Whence $\mathcal{O}_{\pi(j(A))}^{\pi(K)}$ is also of type D . \square

7. THE SYMPLECTIC GROUPS

In this Section, $\mathbb{G} = \mathbf{Sp}_{2n}(\mathbb{k})$, $n \geq 2$. Recall that $e \neq x \in \mathbf{G}$ is semisimple, $\mathbb{G}^F \ni \mathbf{x} \mapsto x$, $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbb{G}^F}$ and $\mathcal{O} = \mathcal{O}_{\mathbf{x}}^{\mathbf{G}}$. Here is the main result of this Section:

Theorem 7.1. *Let $\mathbf{x} \notin Z(\mathbb{G})$ be semisimple. Then \mathcal{O} collapses unless $n = 2$, $q \in \{3, 5, 7\}$ and \mathbf{x} is an involution.*

Classes represented in K have been discussed in Section 6. We deal in Subsection 7.1 with cuspidal classes that are not Coxeter, and then with Coxeter classes in Subsection 7.2. Theorem 7.1 is proved in Subsection 7.3.

7.1. Cuspidal classes. Here we discuss the semisimple classes that are cuspidal but not Coxeter. Below we use without further notice that a cuspidal class could not meet a standard Levi subgroup by Proposition 3.12. We start by the following observation: two semisimple symplectic matrices conjugated in $\mathbf{GL}_{2n}(q)$ are then conjugated in $\mathbf{Sp}_{2n}(q)$.

Lemma 7.2. *In either of the following cases, \mathfrak{D} is not cuspidal: (a) Some eigenvalue of \mathbf{x} lies in \mathbb{F}_q . (b) $|x| \in \{2, 3, 4\}$.*

Proof. (a). Indeed, if $\lambda \in \mathbb{F}_q$ is an eigenvalue of \mathbf{x} , then so is λ^{-1} , hence \mathfrak{D} contains an element of the form $\begin{pmatrix} \lambda & & & \\ & A' & B' & \\ & C' & D' & \\ & & & \lambda^{-1} \end{pmatrix}$ which belongs to a Levi subgroup isomorphic to $\mathbf{Sp}_{2(n-1)}(\mathbb{k}) \times \mathbb{k}^\times$. Thus \mathfrak{D} is not cuspidal.

(b). By (a), we may assume that \mathbf{x} has no eigenvalues in \mathbb{F}_q , so ± 1 are excluded. If $|\mathbf{x}| \in \{2, 3, 4\}$, then \mathbf{x} has at most 2 distinct eigenvalues, namely the two primitive roots of 1, so it is not cuspidal by Proposition 3.13. \square

7.1.1. *Cuspidal classes in the Weyl group.* As is well-known, the Weyl group is $W = (\mathbb{Z}/2)^n \rtimes \mathbb{S}_n$; let $(e_j)_{j \in \mathbb{I}_n}$ be the canonical basis of $(\mathbb{Z}/2)^n$. We identify W with a subgroup of \mathbb{S}_{2n} as in [11]:

$$\begin{aligned} W &\simeq \{\varsigma \in \mathbb{S}_{2n} : \varsigma(2n+1-j) = 2n+1-\varsigma(j), j \in \mathbb{I}_n\}, \\ \mathbb{S}_n \ni \sigma &\mapsto \sigma', \quad \sigma'(j) = \begin{cases} \sigma(j), & \text{if } j \in \mathbb{I}_n, \\ 2n+1-\varsigma(2n+1-j), & \text{if } j \in \mathbb{I}_{n+1,2n}; \end{cases} \\ e_j &\mapsto \tau_j = (j \ 2n+1-j), \quad j \in \mathbb{I}_n. \end{aligned}$$

Given $h \leq k$ in \mathbb{I}_n , we consider the $2(k-h+1)$ -cycle in \mathbb{S}_{2n} defined by

$$(7.1) \quad \mathbf{c}_{h,k} = (h \ h+1 \ \dots \ k \ 2n+1-h \ 2n-h \ \dots \ 2n+1-k).$$

Evidently, $\mathbf{c}_{h,k} \in W$. Let now $\boldsymbol{\lambda} = (\mathbf{d}_1, \dots, \mathbf{d}_t)$ be a partition of n , denoted $\boldsymbol{\lambda} \vdash n$, with $\mathbf{d}_1 \geq \dots \geq \mathbf{d}_t$. Set

$$(7.2) \quad \mathbf{c}_\boldsymbol{\lambda} = \mathbf{c}_{1,\mathbf{d}_1} \mathbf{c}_{\mathbf{d}_1+1,\mathbf{d}_1+\mathbf{d}_2} \dots \mathbf{c}_{\mathbf{d}_1+\dots+\mathbf{d}_{t-1}+1,n} \in W.$$

By the identification above, we can rephrase [16, Proposition 3.4.6]:

Proposition 7.3. *The conjugacy class of such $\mathbf{c}_\boldsymbol{\lambda}$ is cuspidal. The family $\mathbf{c}_\boldsymbol{\lambda}$, $\boldsymbol{\lambda} \vdash n$, is a complete set of representatives of the cuspidal conjugacy classes of W .* \square

For instance, if $\boldsymbol{\lambda} = (n)$, then $\mathbf{c}_\boldsymbol{\lambda}$ is a Coxeter element.

7.1.2. *Cuspidal, but not Coxeter, classes.* Let $\boldsymbol{\lambda} = (\mathbf{d}_1, \dots, \mathbf{d}_t) \vdash n$, with $\mathbf{d}_1 \geq \dots \geq \mathbf{d}_t$. Let $\mathbb{G}_\boldsymbol{\lambda}$ be the image of the injective morphism of groups

$$\begin{aligned} &\mathbf{Sp}_{2\mathbf{d}_1}(\mathbb{k}) \times \mathbf{Sp}_{2\mathbf{d}_2}(\mathbb{k}) \times \dots \times \mathbf{Sp}_{2\mathbf{d}_t}(\mathbb{k}) \longrightarrow \mathbf{Sp}_{2n}(\mathbb{k}), \\ &\left(\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \dots, \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \right) \right) \longmapsto \begin{pmatrix} A_1 & & & & & & & & B_1 \\ & \ddots & & & & & & & \ddots \\ & & & A_t & B_t & & & & \ddots \\ & & & C_t & D_t & & & & \ddots \\ & & & & & & & & \ddots \\ & & & & & & & & \ddots \\ C_1 & & & & & & & & D_1 \end{pmatrix}. \end{aligned}$$

Claim. $\mathbf{c}_\boldsymbol{\lambda}$ has a decomposition Γ such that $\mathbb{G}_\Gamma = \mathbb{G}_\boldsymbol{\lambda}$, cf. (3.9).

Proof. First, $w_j = c_{d_1+d_2+\dots+d_{j-1}+1, d_1+d_2+\dots+d_j}$ is a Coxeter element of the factor $\mathbf{Sp}_{2d_j}(\mathbb{k})$ of \mathbb{G}_λ . Up to appropriate identifications, the union of decompositions $\Gamma_1, \dots, \Gamma_t$ of w_1, \dots, w_t is a decomposition of c_λ . This implies the claim. \square

Lemma 7.4. *If the conjugacy class \mathfrak{D} in \mathbb{G}^F is cuspidal but not Coxeter, then it is of type C, hence it collapses.*

Proof. By Proposition 3.12, there is partition $\lambda \neq (n)$ such that \mathfrak{D} intersects $\mathbb{G}_\lambda^F = \mathbf{Sp}_{2d_1}(q) \times \mathbf{Sp}_{2d_2}(q) \times \dots \times \mathbf{Sp}_{2d_t}(q)$. Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_t) \in \mathbb{G}_\lambda^F \cap \mathfrak{D}$, with $\mathbf{x}_j \in \mathbf{Sp}_{2d_j}(q)$ for all j , so that

$$\mathcal{O}_{\mathbf{x}}^{\mathbb{G}_\lambda^F} = \mathcal{O}_{\mathbf{x}_1}^{\mathbf{Sp}_{2d_1}(q)} \times \mathcal{O}_{\mathbf{x}_2}^{\mathbf{Sp}_{2d_2}(q)} \times \dots \times \mathcal{O}_{\mathbf{x}_t}^{\mathbf{Sp}_{2d_t}(q)} \leq \mathfrak{D}.$$

We claim that $\mathbf{x}_j \notin Z(\mathbf{Sp}_{2d_j}(q))$ for all $j \in \mathbb{I}_t$. Indeed, if $\mathbf{x}_j \in Z(\mathbf{Sp}_{2d_j}(q))$ for some j , then \mathbf{x} belongs to the torus $\mathbb{T}_{c_\lambda^j}$, cf. (3.11), where $c_\lambda^j \in W$ is defined as c_λ in (7.2) but omitting $c_{d_1+\dots+d_{j-1}+1, d_1+\dots+d_j}$. This is a contradiction because c_λ^j is not cuspidal proving the claim. Then the lemma follows from Lemma 2.9, by Remark 2.10 and Lemma 3.14. Indeed, $\mathbf{x}_j \neq \mathbf{x}_j^q$, otherwise \mathbf{x} would lie in a non-cuspidal torus by Lemma 7.2 \square

Lemma 7.5. *If the conjugacy class \mathfrak{D} in \mathbb{G}^F is cuspidal but not Coxeter, then the conjugacy class \mathcal{O} in \mathbf{G} is of type C, hence it collapses.*

Proof. Let $\pi : \mathbf{Sp}_{2n}(q) \rightarrow \mathbf{PSp}_{2n}(q)$ be the canonical projection. We may assume that q is odd, thus $\ker \pi = \{\pm \text{id}\}$. Keep the notation of Lemma 7.4.

Claim 1. *The Lemma holds for $t > 2$.*

Indeed, the Lemma 2.9 provides a subrack of \mathfrak{D} of type C with the form

$$Y = \left(\{\mathbf{x}_1\} \times \mathcal{O}_{\mathbf{x}_2}^{\mathbf{Sp}_{2d_2}(q)} \times \{\mathbf{x}_3\} \right) \amalg \left(\{\mathbf{x}_1^q\} \times \mathcal{O}_{\mathbf{x}_2}^{\mathbf{Sp}_{2d_2}(q)} \times \{\mathbf{x}_3\} \right)$$

and clearly the restriction of π to Y is injective.

Claim 2. *The Lemma holds for $t = 2$.*

If $\mathbf{x}_1 \neq -\mathbf{x}_1^q$, then the restriction of π to the subrack of type C

$$Y = \left(\{\mathbf{x}_1\} \times \mathcal{O}_{\mathbf{x}_2}^{\mathbf{Sp}_{2d_2}(q)} \right) \amalg \left(\{\mathbf{x}_1^q\} \times \mathcal{O}_{\mathbf{x}_2}^{\mathbf{Sp}_{2d_2}(q)} \right)$$

is injective; similarly if $\mathbf{x}_2 \neq -\mathbf{x}_2^q$. Thus we may assume that $\mathbf{x}_1 = -\mathbf{x}_1^q$ and $\mathbf{x}_2 = -\mathbf{x}_2^q$. Now \mathbf{x}_1 lives in a Coxeter torus \mathbb{T}_1^F in $\mathbf{Sp}_{2d_1}(q)$. By [22, Proposition 25.3] and [16, §3.4.3], we have

$$|\mathbb{T}_1^F| = q^{d_1} + 1.$$

Hence $|x_1|$ divides $(2(q-1), q^{d_1} + 1) = \begin{cases} 2 & \text{if } q^{d_1} \equiv 1 \pmod{4}, \\ 4 & \text{if } q^{d_1} \equiv 3 \pmod{4}. \end{cases}$

By symmetry, we may assume that the same holds for x_2 . Hence $|x|$ divides 4; this contradicts Lemma 7.2. \square

7.2. Coxeter classes in $\mathbf{Sp}_{2n}(q)$. Let $x \in T' = \mathbb{T}_w^F$ be a Coxeter element and let $\mathfrak{D} = \mathcal{O}_x^{\mathbb{G}^F}$. Hence x is regular and its order divides $q^n + 1$, so $x^{q^n} = x^{-1}$. Arguing as in [5, §2.5] we see that $\mathfrak{D} \cap T' = \{x^{\pm q^j}, j \in \mathbb{I}_{0, n-1}\}$, and that the action of w raises x to x^q . If $\xi \in \overline{\mathbb{F}}_q$ is an eigenvalue of x , then all other eigenvalues of x are of the form $\{\xi^{q^j}, j \in \mathbb{I}_{0, 2n-1}\} = \{\xi^{\pm q^j}, j \in \mathbb{I}_{0, n-1}\}$, with $\xi^{q^n} = \xi^{-1}$ and they are all distinct by Proposition 3.13.

Lemma 7.6. *Assume q is odd. If x is a Coxeter element, then $-x \notin \mathfrak{D}$.*

Proof. If $-x \in \mathfrak{D}$, then with notation as above, $-\xi$ is an eigenvalue of x , so $-\xi = \xi^{q^j}$ or $-\xi = \xi^{-q^j}$ for some $j < n$. In the first case $\xi^{q^{2j}} = (-\xi)^{q^j} = \xi$, whilst in the second $\xi^{q^{2j}} = (-\xi^{-1})^{q^j} = -(\xi^{q^j})^{-1} = \xi$, with $2j < 2n$ in both cases, contradicting regularity of x . \square

Lemma 7.7. *Let x be a Coxeter element in \mathbb{G}^F . Then \mathcal{O} is of type C.*

Proof. Let H be a subgroup of \mathbb{G}^F isomorphic to $\mathbf{SL}_2(q^n)$, which exists by [21, II Satz 9.24]. Any non-split torus of $T' \leq H$ has order $q^n + 1$ by (5.1), hence it is a Coxeter torus in \mathbb{G}^F , [16, 3.4.3]. Therefore

$$|\mathfrak{D} \cap T'| = |\{x^{\pm q^j}, j \in \mathbb{I}_{0, n-1}\}| = 2n, \quad |\mathcal{O}_x^H \cap T'| \leq 2,$$

so the intersection $\mathfrak{D} \cap H$ is not a single H -conjugacy class. Assume $x \in H$. Since $H \simeq \mathbf{SL}_2(q^n)$ with $n \geq 2$, the group $H/Z(H)$ is simple, so the non-central normal subgroup $\langle \mathcal{O}_x^H \rangle$ coincides with H . In addition, $|\mathcal{O}_x^H| = q^n(q^n - 1) > 4$, so \mathfrak{D} is of type C by Lemma 2.8. The restriction of the projection $\pi: \mathbb{G}^F \rightarrow \mathbf{G}$ to \mathfrak{D} is injective by Lemma 7.6, so $\pi(\mathfrak{D}) = \mathcal{O}$ is of type C as well. \square

7.3. The general case. Let \mathbb{L} be a split F -stable Levi subgroup of \mathbb{G} . Then, there exist $f > 0$, $m \geq 0$ and n_i for $i \in \mathbb{I}_f$ satisfying $n = e + \sum_{i=1}^f n_i$ such that \mathbb{L} is isomorphic the image of the injective morphism of groups

$$(7.3) \quad \tilde{j}: \mathbf{GL}_{n_1}(\mathbb{k}) \times \cdots \times \mathbf{GL}_{n_f}(\mathbb{k}) \times \mathbf{Sp}_{2e}(\mathbb{k}) \rightarrow \mathbf{Sp}_{2n}(\mathbb{k})$$

$$(7.4) \quad (A_1, \dots, A_f, A) \mapsto \text{diag}(A_1, \dots, A_r, A, \phi(A_f), \dots, \phi(A_1))$$

Proof of Theorem 7.1. If $n = 2$, $q = 3$ and x is a non-central involution, then \mathcal{O} is kthulhu by Lemma 4.4. Assume that $q \notin \{5, 7\}$ if $n = 2$ and x is a non-central involution. If x is cuspidal but not Coxeter, we invoke Lemma 7.5, whilst if x is Coxeter, then the claim follows from Lemma 7.7.

If \mathbf{x} is not cuspidal, then by Proposition 3.12 we may assume that $\mathbf{x} \in \mathbb{L}^F$ for a proper standard Levi subgroup \mathbb{L} of \mathbb{G} . Let \tilde{j} be as in (7.3) and let $\mathbf{x} = \tilde{j}(\mathbf{x}_1, \dots, \mathbf{x}_f, \mathbf{y})$. Taking n_i , for $i \in \mathbb{I}_f$ and e to be minimal, and possibly increasing f , we assume that each \mathbf{x}_i is irreducible in $\mathbf{GL}_{n_i}(q)$ and \mathbf{y} is cuspidal in $\mathbf{Sp}_{2e}(q)$. Under these assumptions $\mathbf{x}_i \in Z(\mathbf{GL}_{n_i}(q))$ if and only if $n_i = 1$. If $e = 0$ the statement follows from Theorem 6.1. If $e \geq 2$, then we consider the rack embedding

$$(7.5) \quad \{\mathbf{x}_1\} \times \cdots \times \{\mathbf{x}_f\} \times \mathcal{O}_{\mathbf{y}}^{\mathbf{Sp}_{2e}(q)} \rightarrow \mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_{2n}(q)} \rightarrow \mathcal{O}_x^{\mathbf{PSP}_{2n}(q)}$$

and invoke either Lemma 7.5 or Lemma 7.7.

Assume from now on that $e = 1$, i.e., \mathbf{y} is irreducible in $\mathbf{Sp}_2(q) \simeq \mathbf{SL}_2(q)$. If there exists and n_i such that $n_i > 1$, then we consider the rack embedding

$$(7.6) \quad \{\mathbf{x}_1\} \times \cdots \times \mathcal{O}_{\mathbf{x}_i}^{\mathbf{Sp}_{2n_i}(q)} \times \cdots \times \{\mathbf{x}_f\} \times \{\mathbf{y}\} \rightarrow \mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_{2n}(q)} \rightarrow \mathcal{O}_x^{\mathbf{PSP}_{2n}(q)}$$

and invoke Theorem 6.1.

There remains the case in which $n_i = 1$ for every i . We assume that $f = 1$, for if $f > 1$ we can use the rack injection

$$\{\mathbf{x}_1\} \times \cdots \times \{\mathbf{x}_{f-1}\} \times \mathcal{O}_{\tilde{j}(\mathbf{x}_f, \mathbf{y})}^{\mathbf{Sp}_{2(n_f+1)}(q)} \rightarrow \mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_{2n}(q)} \rightarrow \mathcal{O}_x^{\mathbf{PSP}_{2n}(q)}.$$

Since \mathbf{y} is irreducible, it lies in a non-split maximal torus, so its order divides $q + 1$ and so $\mathbf{y}^q = \mathbf{y}^{-1} \in \mathcal{O}_{\mathbf{y}}^{\mathbf{SL}_2(q)}$. Also, if $p_{\mathbf{y}} = X^2 - zX + 1$, then $z \neq \pm 2$. We may assume that $\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$ so $\mathbf{y}^{-1} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{x} = \text{diag}(\lambda, \mathbf{y}, \lambda^{-1})$. We consider the following subracks of $\mathfrak{D} = \mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_4(q)}$:

$$R := \left\{ \mathbf{x}' = \begin{pmatrix} \lambda & * & & \\ 0 & \mathbf{y} & * & \\ 0 & 0 & \lambda^{-1} & \end{pmatrix} : \mathbf{x}' \in \mathfrak{D} \right\}, \quad S := \left\{ \mathbf{x}' = \begin{pmatrix} \lambda^{-1} & * & * & \\ 0 & \mathbf{y}^{-1} & * & \\ 0 & 0 & 0 & \lambda \end{pmatrix} : \mathbf{x}' \in \mathfrak{D} \right\}.$$

By construction, $R \triangleright S \subset S$ and $S \triangleright R \subset R$. Observe that $R \cap S = \emptyset$; otherwise $\mathbf{y} = \mathbf{y}^{-1}$ and $p = 2$, but in this case \mathbf{y} would not be semisimple. Let $M \in \mathbf{SL}_2(q)$ be such that $M \triangleright \mathbf{y} = \mathbf{y}^{-1}$ and let

$$r := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \triangleright \mathbf{x} = \begin{pmatrix} \lambda & -\lambda & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & z & z-\lambda^{-1} \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \in R,$$

$$s := \begin{pmatrix} 0 & 0 & 1 \\ 0 & M & 0 \\ -1 & 0 & 0 \end{pmatrix} \triangleright \mathbf{x} = \text{diag}(\lambda^{-1}, \mathbf{y}^{-1}, \lambda) \in S.$$

A direct calculation shows that $rs = sr$ only if $\lambda^2 = 1$ and $z = \pm 2$, a discarded case. Taking $H := \langle r, s \rangle$, we see that $\mathcal{O}_r^H \cap \mathcal{O}_s^H \subset R \cap S = \emptyset$. Thus, if $p = 2$, then $|\mathbf{x}|$ is odd so $\mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_4(q)}$ is of type C by Remark 2.4. If p is odd, then $(rs)^2 \neq (sr)^2$ because rs and sr are upper triangular unipotent matrices by construction, and so $\mathcal{O}_{\mathbf{x}}^{\mathbf{Sp}_4(q)}$ is of type D. We claim that the restriction of π to $R \amalg S$ is injective: indeed injectivity could fail only for $\lambda^2 + 1 = 0$ and $z = 0$ but in this case, $\lambda \in \mathbb{F}_q$ would be a root of $p_{\mathbf{y}}$ which is irreducible, a contradiction. \square

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