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CICLO XXXV

ON FINITE FUEL IRREVERSIBLE INVESTMENT PROBLEMS

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Riassunto

Questa tesi si concentra sui problemi di investimento irreversibile con risorse limitate, motivati dalla loro applicazione su mercati elettrici. Attraverso la tesi studiamo diversi modelli su come aumentare in modo ottimale la produzione di energia con l'obiettivo di ottenere lil massimo guadagno vendendo l'energia prodotta.

Iniziamo il Capitolo 1 descrivendo il modello generale dei problemi di investimento irreversibile con risorse limitate che verrà utilizzato in questa tesi. Esaminiamo i principali risultati sulla soluzione di questo tipo di problemi: l'equazione di Hamilton-Jacobi-Bellman (HJB) e le condizioni stocastiche infinito-dimensionali di Kuhn-Tucker. Nel Capitolo 2 si studia un problema proveniente dal mercato elettrico, dove è considerata una grande azienda che cerca la strategia ottimale per aumentare la potenza rinnovabile installata, in modo da massimizzare l'utilità di vendere l'energia elettrica sul mercato al netto dei costi di installazione, assumendo che gli incrementi nella produzione di energia riducano la media del prezzo dell'elettricità. Come prima cosa, testiamo il reale impatto degli incrementi di potenza installata rinnovabile sul prezzo dell'energia elettrica in Italia e valutiamo quanto la strategia di installazione rinnovabile messa in atto in Italia si sia discostata da quella ottimale ottenuta dal modello nel periodo 2012-2018. Dopodiché, estendiamo il modello con una singola azienda al caso di N aziende e studiamo il gioco cooperativo dal punto di vista di un pianificatore sociale. Inoltre, studiamo la situazione competitiva quando ci sono due società presenti sul mercato. Caratterizziamo esplicitamente l'equilibrio di Nash per la situazione competitiva quando il prezzo dell'energia elettrica non è influenzato dagli incrementi di potenza installata e lo confrontiamo con l'ottimo di Pareto.

Nel capitolo 3 studiamo il problema di investimento irreversibile con risorse limitate in un mercato con N aziende. Dimostriamo l'ottimalità di una politica ammissibile utilizzando la condizione stocastiche generalizzate di Kuhn-Tucker. Dopdiché, riformuliamo il problema in termini di misure per studiare il comportamento asintotico quando il numero di imprese tende all'infinito e dimostriamo l'esistenza di una soluzione in tale approccio.

La tesi si conclude con il Capitolo 4, dove estendiamo la condizione del primo ordine per il problema dell'investimento irreversibile con risorse limitate, considerando il caso in cui c'è impatto sul mercato. Il risultato è dimostrato per il caso particolare in cui esiste un'unica azienda, il processo di shock segue un processo di Ornstein-Uhlenbeck e il funzionale di utilità è lineare sia nella dinamica controllata che nel controllo.

Abstract

This thesis is focused on finite fuel irreversible investment problems, motivated by their application in electricity markets. Through the thesis we study different models of optimally increase the power production in order to obtain maximum utilities by selling the produced energy.

We start Chapter 1 by describing the general model of finite fuel irreversible investment problems that will be used through this thesis. Then, we review the main results on the solution of this type of problems: the Hamilton-Jacobi-Bellman (HJB) equation and infinite dimensional stochastic Kuhn-Tucker conditions. In Chapter 2 we study a particular application to electricity markets of our general model presented in [50], where it is considered a big company that aims to find the optimal strategy of increasing the renewable installed power in order to maximize the utility of selling the electricity in the market net of the installation costs, assuming that the increments in power production reduce the long mean of the electricity price. First, we test the real impact of current renewable installed power in the electricity price in Italy, and assess how much the renewable installation strategy which was put in place in Italy deviated from the optimal one obtained from the model in the period 2012-2018. Then, we extend the single company model to the case of N companies and study the cooperative game of a social planner point of view. Also, we study the competitive situation when two companies are presented in the market. We characterize explicitly the Nash equilibrium for the competitive situation when the electricity price is not affected by installed power increments and we compare it with the Pareto optima.

In Chapter 3 we study the optimal stochastic irreversible investment problem under limited resources in a market with N firms. We prove the optimality of an admissible policy using the generalized stochastic Kuhn-Tucker condition developed in [23]. Then, we reformulate the problem in terms of measures in order to study the asymptotic behavior when the number of firms goes to infinity and we prove the existence of a solution in such a framework.

The thesis ends with Chapter 4, where we extend the first order condition for the finite fuel irreversible investment problem presented in [23], considering market impact. The result is proved for the particular case when there is a single company, the shock process follows an Ornstein-Uhlenbeck process and the utility functional is linear in both the controlled dynamics and the control.

Contents

Introduction

This thesis is inspired by the electricity market problem of optimally increase the power production of a company in order to obtain the maximum gain by selling the produced energy in the market net of some cost, under the constraint of maximum installed power and the assumption that the increments of installed power affect negatively the electricity price. This problem is described by a finite fuel singular control problem, know also as finite fuel irreversible investment problem.

Singular control problems where first introduced by [7, 8], where they found that the optimal solution corresponds to the solution of a *free boundary problem*. Supposing that the value function corresponding to the optimal strategy is smooth up to second order in space, they were able to analyze the Hamilton-Jacobi-Bellman (HJB) equation characterizing the value function. On this line works as [46], [48] [49], [40], [53], [24] solve completely the one dimensional problem of minimizing a convex cost functional when the controlled diffusion is a Brownian motion tracked by a non decreasing process, known as the *monotone follower problem*. This approach is also applied in games involving singular controls as [26, 41, 51] for competitive games and [25] from the point of view of a regulator, where by using similar arguments, it is possible to establish the HJB equation characterizing the value function associated to the Nash equilibrium of the competitive game and the Pareto optima for the cooperative game. Differently from these works, we study the competitive game where there are two companies that aim at maximizing their own utilities, when both players can act simultaneously. We establish a verification theorem for the problem and we explicitly characterize the Nash equilibrium supposing that there is not market impact. We also study the Pareto optima from the regulator point of view and we compare both strategies.

In the theory of intertemporal consumption and portfolio choice, in [42] they replaced the standard consumption space (control space) with the space of right-continuous, increasing functions and proposed to use a new class of utility functionals since the standard time-additive utility functionals are not continuous in the economically appropriate topology. This models are singular control problems of the monotone follower type and they have been approached by deriving first order conditions for optimality on a general semimartingale setting, without relying on any Markovian assumption [5, 6, 3]. However, in this approach the market impact is not considered. We extend the results when there is market impact, considering that the utility functional is linear both in the control and in the controlled dynamics. In this case, we describe the dynamic by an Ornstein-Uhlenbeck process whose drift is affected negatively by the control variable. This approach of first order conditions has also be extend by [23] to the case when there are N-firms and a social planner aims to maximize the total sum of the expected utilities, but without market impact. We also use this approach to prove the optimality of an admissible strategy of the problem of maximize the sum of the utilities of N power producers that increment their installed power and sell the produced energy in the market net of the total installation cost. The study of the asymptotic behavior of games with N-players involving singular control also attracted the attention in the last years [34, 18, 38]. Motivated by the increasing research on this topic, we study the asymptotic model of the N-producer social planner problem and we prove the existence of solutions.

Thesis overview

We start the thesis with Chapter 1 by describing the general model of finite fuel irreversible investment problems that will be used through this dissertation. Then, we review the main results on the solution of this type of problems: the Hamilton-Jacobi-Bellman (HJB) equation and infinite dimensional stochastic Kuhn-Tucker conditions.

In the HJB approach, by relying on a Markovian structure and assuming *the principle of smooth fit*, which supposes that the value function should be smooth up to second order in space, we show the heuristic procedure to obtain the HJB equation characterizing the value function of the finite fuel singular control problem. It will be seen naturally that the optimal solution of this problem corresponds to the solution of a *free boundary problem*, which consists in finding a curve which separates the state space into a *waiting region* and an *action region*. Then, we give the explicit solution of the irreversible investment problem presented in [50], which is the inspiring model of the thesis.

On the other hand, for the first order conditions we start by giving an intuition behind the derivation of the stochastic KKT conditions presented in [5, 6] for the particular case of intertemporal consumption and portfolio choice, stated by [42]. We summarize some useful results developed in this approach as the Bank-El Karoui representation theorem [4], which relates the solution of this problems with the solution of a backward equation. This representation enables to state first order conditions when the finite fuel constraint is not constant and is described by some stochastic process [3, 23]. We finish with the results in [29], where it is proved that the solution of the backward equation of the Bank-El Karoui representation theorem is related to the free boundary problem.

In Chapter 2 we start by validating empirically the assumption that the increments in renewable electricity production reduce the long mean of the electricity price. We suppose that the electricity price evolves according to an Ornstein-Uhlenbeck process which includes an exogenous variable in the drift term, corresponding to the increments in energy production. In order to validate our model, we use a dataset of weekly Italian prices, together with photovoltaic and wind power production, of the six main Italian price zones (North, Central North, Central South, South, Sicily and Sardinia), covering the period 2012–2018. In principle, both photovoltaic and wind power production could have an impact on power prices, so we start by estimating parameters of an autoregresive with exogenous variable (ARX) model where both photovoltaic and wind power production are present as exogenous variables: the parameters of this discrete time model will then be transformed in parameters for the continuous time Ornstein-Uhlenbeck model by standard techniques, see e.g. [17]. Unfortunately, for three price zones we find out that our Ornstein-Uhlenbeck model, even after correcting for price impacts, produce non-independent residuals. This is an obvious indication that the Ornstein-Uhlenbeck model is too simple for these zones, and one should instead use more sophisticated models, like CARMA ones (see e.g. [12]): we leave this part for future research. For the remaining three zones, we find out that, for each zone, at most one of the two renewable sources has an impact: in particular, power price in the North is only impacted by photovoltaic production, and in Sardinia only by wind production, while in Central North is not impacted by any of them. Thus, we are able to model the optimal installation problem for North and Sardinia using the theory existing in [50]. Instead, for the installation problem in Central North, we must solve an instance of the problem with no price impact: this can be derived as a particular case of the results in [50], and results in a much more elementary formulation than the general case treated there. More in detail, we obtain that the function of the capacity which should be hit by the power price in order to make additional installation is in this case equal to a constant, obtained by solving a nonlinear equation. The corresponding optimal strategy should thus be to not install anything until the price threshold is hit, and then to install the maximum possible capacity.

Afterwards we check the effective installation strategy, in the different price zones, against the optimal one obtained theoretically. In doing so, we must take into account the fact that the Italian market is liberalized since about two decades, thus there is not a single producer which can impact prices by him/herself, but rather prices are impacted by the cumulative installation of all the power producers in the market. We thus extend our model by formulating it for N players who can install, in the different price zones, the corresponding impacting renewable power source, monotonically and independently of each other: the resulting power price will be impacted by the sum of all these installations, while each producer will be rewarded by a payoff corresponding to their installation. The resulting N-player nonzero-sum game can be solved with different approaches. A formulation requiring a Nash equilibrium would result in a system of N variational inequalities with $N+1$ variables (see e.g. [26] and references therein), which would be quite difficult to treat analytically. We choose instead to seek for Pareto optima first. One easy way to achieve this is to assume, in analogy with [23], the existence of a "social planner" which maximizes the sum of all the N players' payoffs, under the constraint that the sum of their installed capacity cannot be greater than a given threshold (which obviously represents the physical finite capacity of a territory to support power plants of a given type). We prove that, in our framework, this produces Pareto optima. More in detail, by summing together all the N players' installations in the social planner problem, one obtains the same problem of a single producer, which has a unique solution that represents the optimal cumulative installation of all the combined producers. Though with this approach it is not possible to distinguish the single optimal installations of each producer, we can assess how much the effective cumulative installation strategy which was carried out in Italy during the dataset's period differs from the optimal one which we obtained theoretically. To give an idea of what we instead would get when searching for Nash equilibria, we present the case $N = 2$ and formulate a verification theorem that the value functions of each player should satisfy. Here we want to point out a difference which arises in our problem with respect to the current stream of literature. In fact, in stochastic singular games the usual framework is that a player can act only when the other ones are idle, see e.g. [25, 26, 37, 38]. Here instead we take explicitly into consideration the possibility that both players acts (i.e. install) simultaneously. In the case with no market impact we are able to prove that the strategy where both players install simultaneously is a Nash equilibrium. We found that this equilibrium induces the players to install *before* than when they would have done under a Pareto optimum.

In Chapter 3, we study the optimal social planner point of view of a stochastic irreversible investment problem under limited resources in a market with N firms, in the case when the investment has not impact in the market. This problem is studied and solved in [23], where they state infinite dimensional stochastic Kuhn-Tucker conditions to prove optimality of policies. The optimal solution for the social planner problem proposed by [23] is as follows: every firm increases its investment until it reaches the critical level $\frac{\theta}{N}$, where θ is the constraint of the problem, whenever the initial condition is lower than the critical level. If the initial conditions of some firms are greater than the critical level, then those firms should not invest. Nevertheless, this strategy results to be not admissible when at least one of the firms is over the critical level.

We begin by constructing an admissible strategy and we use the Kuhn-Tucker conditions to prove its optimality. Our admissible optimal strategy $\bar{I}^* = (I_1^*, \dots, I_N^*)$ differs from the one presented in [23] on the definition of the critical level. We solve two explicit cases for a finite number of firms. In the first problem we aim to model the investment on a saturated market. For example, the investment on installing wind turbines, every time will increase to produce the same energy, because good wind places will be occupied and the same turbine will produce less energy in a "worse" place. We consider as revenue in the utility functional the function $R(x,y) = xh(y)$, with $h(y)$ a concave function, and an external shock process driven by a geometric Brownian motion. In the second problem instead we consider as revenue the function $R(x,y) = e^x y^\alpha$ and an external shock process evolving accordingly to an Ornstein-Uhlenbeck process.

Once we prove the optimality of our modified strategy with the new critical level for the finite firm problem, we study the asymptotic version of the problem. First, we redefine the finite firm problem using measures, but differently from works studying limit behaviors of finite player games, we do not consider probability measures, in the sense that instead of use the empirical flow of random measures $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{I_i(t)}$ we use the random measures $\hat{\mu}_t^N := \sum_{i=1}^N \delta_{I_i(t)}$ defined on the Borel sets over $[0,\theta]$ an we consider the cumulative measure $\hat{\nu}_t([a,b]) := \int_{[a,b]} z\hat{\mu}_t^N(dz)$ with $[a,b] \in \mathcal{B}([0,\theta])$. Afterwards, we define the mean field control version of our problem, considering an utility functional which depends on random variables defined on the space

$$\mathcal{V} := \left\{ \nu : \Omega \to D_{[0,\infty)}(\mathcal{M}) \middle| \, \forall s \geq 0, \, \nu_s \text{ is } \mathcal{F}_s\text{-measurable} \right\},$$

where

$$D_{[0,\infty)}(\mathcal{M}) := \Big\{ \mu : [0,\infty) \to \mathcal{M} | \mu_{0-} = \lambda, \; t \to \mu_t([a,b]) \text{ is a cadlag function } \forall \; 0 \leq a < b \leq \theta,$$

$$\int_{(a,b]} \frac{1}{z}\mu_t(dz) \leq \int_{(a,b]} \frac{1}{z}\mu_s(dz) \text{ for every } 0 \leq a < b \leq \theta \text{ and } s \leq t, \; \mu_s([0,\theta]) \leq \mu_t([0,\theta]) \Big\}$$

and

$$\mathcal{M} := \left\{ \mu : \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+ | \text{ for every } 0 \leq a \leq b \leq \theta \text{ , } \mu([a,b]) \leq \theta \; \right\}.$$

endowed with the topology of the weak convergence.

We prove existence of a solution for our mean field control problem in Theorem 3.4.2, by using an extension of Komlos' theorem to the random variables defined in our space $\mathcal{V}$.

The thesis ends with Chapter 4, where we extend the first order condition for the finite fuel irreversible investment problem presented in [23], considering the case when the shock process appearing in the utility functional is affected by the control variable. We consider the particular case when the revenue R of the utility is linear both in the control and in the dynamics, i.e., $R(x,y) = xy$. As control dynamic we consider an Ornstein- Uhlenbeck process where the control affects the drift term. This is exactly the same model discussed in the preview chapter, corresponding to the work in [50], where they solved the problem by using the Hamilton-Jacobi-Bellman (HJB) approach assuming some regularity of the value function and a Markovian structure of the problem. Here instead, we do not consider any Markovian structure of the problem. The principal result that helps us to prove the Kuhn-Tucker conditions is that our utility functional $\mathcal{J}(x,y,I)$ is supported by a subgradient $\nabla\mathcal{J}(x,y,I)$, in the sense that, for any admissible control I and I', the subgradient $\nabla\mathcal{J}(x,y,I)$, satisfies

$$\mathcal{J}(x,y,I)-\mathcal{J}(x,y,I')\leq\langle\nabla\mathcal{J}(x,y,I'),I-I'\rangle.$$

By using the above concavity property of our functional we are able to prove the following sufficient and necessary conditions of optimality for the control variable. Denote by $\mathcal{T}$ the set of stopping times τ with values in $[0,\infty)$ $\mathbb{P}$-a.s. Suppose that there exist a nonnegative Lagrange multiplier measure $d\lambda(\omega,t)$ such that $\mathbb{E}\left[\int_{[0,\infty)}d\lambda(t)\right]<\infty$, and the following conditions are satisfied for some admissible strategy I^*

$$\nabla\mathcal{J}(x,y,I^*)(\tau)\leq\mathbb{E}\left[\int_\tau^\infty d\lambda(s)\middle|\mathcal{F}_\tau\right]\ \mathbb{P}\text{-a.s}\ \forall\tau\in\mathcal{T},\tag{1}$$

$$\int_0^\infty\left(\nabla\mathcal{J}(x,y,\hat{I})(t)-\mathbb{E}\left[\int_t^\infty d\lambda(s)\middle|\mathcal{F}_t\right]\right)dI^*(t)=0\ \mathbb{P}\text{-a.s},\tag{2}$$

$$\mathbb{E}\left[\int_0^\infty(\theta-(y+I^*(t)))\,d\lambda(t)\right]=0,\tag{3}$$

then I^* maximize the utility functional $\mathcal{J}(x,y,I)$.

The Lagrange multiplier measure $d\lambda(\omega,t)$, is such that $\lambda(\omega,t)$ corresponds to the increasing predictable process part of the Doob-Meyer decomposition of the Snell envelope of the subgradient evaluated at the optimum.

Chapter 1

Preliminaries

In this introductory chapter we present the general setting for finite fuel irreversible investment problems and two approaches to characterize the solution of this problems: Hamilton-Jacobi-Bellman (HJB) equation and first order conditions.

Finite fuel irreversible investment problems are singular control problems. Singular control problems where first introduced by [7], where they model a spaceship traveling to a target, whose relative position is changed using some fuel. The observation of the relative position of the target allows to predict the dismiss distance to the target, making it possible to decide whether to use the fuel to approximate the target or not. They aim to find an optimal control procedure which minimizes the sum of all fuel costs together with a cost associated with possibly missing the target. The value of the cost at the optimal control procedure is known as the *value function*. They found that the optimal procedure corresponds to the solution of a *free boundary problem*, in the sense that there exists a curve such that, if for a time s the spaceship is at the "right" side of the curve, then not fuel should be expended. Instead, if the spaceship is at the "left" side of the curve, then some fuel must be used to change the relative position to the target. That curve is such that it separates the state space into a *waiting region* and an *action region*. Those results are extended in [8] to the case when there is a constraint in the available fuel. Singular control problems with a limited resources are called *finite fuel problems*.

In some stochastic control problems, the *dynamic programming principle* suggests "bang-bang" or singular optimal laws, which reduce the associated HJB equation to a free boundary problem for finding the "switching curves" where the bangs occur, or where the support of the singular control lies. Inspired by this observation, in [11] they introduce an heuristic principle, latter called *the principle of smooth fit*, which supposes that the value function should be smooth up to second order in space. They were able to analyze the HJB equation associated to the optimal performance of the control problems: "bounded velocity follower", "monotone follower with finite horizon" and "Finite fuel follower" and solve them explicitly by deriving the *free boundary* and the value functions. Following the same idea of *the principle of smooth fit*, [46], [48] [49], [40], [53], [24] solve completely the one dimensional problem of minimizing a convex cost functional when the controlled diffusion is a Brownian motion tracked by a non decreasing process, known as the *monotone follower problem*. This problems found applications in economy under the name *irreversible investment problems* [45], [48], [49] among others.

On the same line, when the value function is smooth [59] shows that the construction of an optimal control must depend on the solution of a *Skorohod problem*. Also, it has been shown [48], [49] that in the one-dimensional case, a singular stochastic control problem is equivalent to a stopping time problem, in the sense that the value function of the latter is nothing but the derivative in the state variable of the first one.

In the theory of intertemporal consumption and portfolio choice, in [42] they replaced the standard consumption space with the space of right-continuous, increasing functions and proposed to use a new class of utility functionals since the standard time-additive utility functionals are not continuous in the economically appropriate topology. This model are singular control problems of the monotone follower type and in the last decade, they have been approached by deriving first order condition for optimality on a general semimartingale setting, without lying on any Markovian assumption. In [5], they establish an infinite dimensional version of the Kuhn-Tucker conditions. In [6] this method was generalized to the case when the space of control are *optional random measures*. Inspired by that work, in [4] a representation theorem was proved for optional processes.

Following the study of control problems of the monotone follower type, in [3] they consider a dynamic constraint described by some increasing stochastic process and derive a first order characterization for the optimal solution of the problem based on the Snell envelope of the gradient's cost functionals at the optimum. They construct the optimal policy explicitly in terms of the solution of the Bank-El Karoui representation theorem [4]. Extending this result, in [23] they develop infinite dimensional Kuhn-Tucker conditions, when there are N agents and a social planner aims to maximize the total sum of the expected utilities.

This chapter is organized as follows: In Section 1.1. we give the general setting for the singular control problem that we will address thought this thesis. In Section 1.2 we show the HJB approach and how it is applied to solve explicitly solution of the irreversible investment problem presented in [50], which is the inspiring model of this thesis. Finally, Section 1.3, we give the intuition behind the derivation of the first order condition for the particular case of intertemporal consumption and portfolio choice, and we summarize some useful results of this approach, that will be used in the subsequent chapters.

1.1 General setting

On this thesis we consider the following general structure: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space where a one dimensional Brownian motion W is defined and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets. Consider a diffusion process $\left(X^{x,Y^y}(t)\right)_{t\geq 0}$ satisfying the stochastic differential equation

$$\begin{cases} dX^{x,Y^y}(s) = b(X^{x,Y^y}(s), Y^y(s))dt + \sigma dW(s), \\ X^{x,Y^y}(0) = x, \end{cases} \tag{1.1}$$

with $b : \mathbb{R}^2 \to \mathbb{R}$ a function called the drift term and $\sigma > 0$ a constant known as the volatility term. The process $(Y^y(t))_{t\geq 0}$ is such that

$$\begin{cases} Y^y(t) = y + I(t) \\ Y^y(0-) = y \end{cases} \tag{1.2}$$

with $y \geq 0$. The *admissible control* $(I(t))_{t\geq 0}$ is defined on the following space, called the *admissible set*,

$$\mathcal{I}(y) \triangleq \big\{ I : \Omega \times [0,\infty) \to [0,\infty) : I \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{- adapted }, \, t \to I(t) \text{ is increasing, cadlag,} \\ I(0-) = 0 \leq I(t) \leq \theta - y \big\} \tag{1.3}$$

where $\theta > 0$ is a fixed maximum growth level and y the initial investment.

Let us consider the following utility functional

$$\mathcal{J}(x,y,I) \quad = \quad \mathbb{E}\left[\int_0^\infty e^{-\rho s} R(X^{x,Y^y}(s), Y^y(s)) ds - c \int_0^\infty e^{-\rho s} dI_s \right], \tag{1.4}$$

where $\rho > 0$ is a constant called the discount factor and $R : \mathbb{R}^2 \to \mathbb{R}$ is a concave function in the second component, representing the revenue under the diffusion X^{x,Y^y}. The optimal control problem is to find an optimal strategy $I^* \in \mathcal{I}(y)$ such that

$$V(x,y) = \sup_{I \in \mathcal{I}(y)} \mathcal{J}(x,y,I) = \mathcal{J}(x,y,I^*), \tag{1.5}$$

where $V(x,y)$ is known as the value function.

1.2 Hamilton-Jacobi-Bellman equation

In the case of singular control problems the HJB equation associated to the optimal control takes the form of a variational inequality [55, 32] and can be derived assuming the *smooth fit principle* and a Markovian structure of the problem. It is also know that the solution of the variational inequality is related with the solution of a *free boundary problem*. To begin, we will derive the variational inequality for our general setting. Subsequently, we will show the explicit solution for a particular case, developed in [50], where they solve the problem of optimally increase the installed solar power in order to maximize the profit of selling the energy in the market, net of the installation costs, assuming that the electricity price is affected by the installed power. In this model, the drift of the diffusion b is $b(x,y) = \kappa(\zeta - x - \beta y)$, with $\kappa, \beta > 0$, $\zeta \in \mathbb{R}$ and the revenue is $R(x,y) = axy$ with $a > 0$.

In the HJB setting we look for Markovian controls, which in our framework are defined as follows

Definition 1.2.1 *(Markovian control) A control process $I \in \mathcal{I}(y)$ in the form*

$$I(s) = \iota(s, X^{x,Y^y}(s), Y^y(s))$$

for some measurable function ι from $\mathbb{R}^+ \times \mathbb{R}_0^+$ into $\mathbb{R}$, is called Markovian control.

Because of the Markovian structure of the controls, the derivation of the HJB equation is obtained by the following heuristic argument at the initial time $t = 0$: do not apply any action during a time period Δt and then continue optimally or intermediately start to increase the control variable.

To derive the equation related with the first strategy let us apply the dynamic programming principle on an increment Δt to the value function (1.5),

$$V(x,y) \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-\rho t} R(X^{x,Y^y}(t), y)dt + e^{-\rho\Delta t} V(X^{x,Y^y}(\Delta t), y)\right]. \tag{1.6}$$

Employing Ito's formula to the last term of the right-hand side of (1.6), dividing by Δt, and then letting $\Delta t \to 0$, we obtain

$$\mathcal{L}^y V(x,y) - \rho V(x,y) + R(x,y) \leq 0, \tag{1.7}$$

where $\mathcal{L}^y$ is the differential operator associated to the dynamic (1.1) when $y \in \mathbb{R}_0^+$ is considered as fixed, which is written as follow

$$\mathcal{L}^y u(x,y) := b(x,y) D_x u(x,y) + \frac{\sigma^2}{2} D_{xx} u(x,y). \tag{1.8}$$

On the other hand, the strategy obtained by suddenly starting to increase the installed power level is associated with

$$V(x, y+\epsilon) \leq V(x,y) + \epsilon,$$

dividing by ϵ and letting $\epsilon \to 0$ we obtain

$$\frac{\partial V(x,y)}{\partial y} - c \leq 0.$$

The utility obtained in the special case when no action is applied, i.e. $I(s) \equiv 0$ for all $s \geq 0$, is denoted by

$$Q(x,y) := \mathcal{J}(x,y,0) = \mathbb{E}\left[\int_0^{\infty} e^{-\rho s} R(X^{x,Y^y}(s), y) ds\right]. \tag{1.9}$$

It is also a particular solution of (1.7), when there is equality. Finally, we write the variational inequality which the candidate value function should satisfy

$$\max\left\{\mathcal{L}^y w(x,y) - \rho w + R(x,y), \frac{\partial w(x,y)}{\partial y} - c\right\} = 0, \tag{1.10}$$

with boundary condition $w(x,\theta) = Q(x,\theta)$.

Equation (1.10) defines two regions: a waiting region $\mathbb{W}$ where the optimal strategy is to not apply any increment in the control variable

$$\mathbb{W} = \left\{ (x,y) \in \mathbb{R} \times [0,\theta) : \mathcal{L}^y w(x,y) - \rho w + R(x,y) = 0, \frac{\partial w}{\partial y} - c < 0 \right\}, \tag{1.11}$$

and an action region $\mathbb{I}$, given by

$$\mathbb{I} = \left\{ (x,y) \in \mathbb{R} \times [0,\theta) : \mathcal{L}^y w(x,y) - \rho w + R(x,y) \leq 0, \frac{\partial w}{\partial y} - c = 0 \right\}, \tag{1.12}$$

where the optimal strategy is to increase the control variable $I \in \mathcal{I}(y)$.

A verification theorem that relates the solution of the variational inequality (1.10) with the value function (1.5) can be found in [32] when $b(x,y) = \hat{b}(x) + y$ and $R(x,y) = \hat{R}(x) + f(y)$. More results can be found in [50, 30]. The verification theorem allows to exhibit as byproduct an optimal Markovian control, which should be such that it maintains the dynamic $(X^{x,Y}(s), Y^y(s))$ inside the closure $\bar{\mathbb{W}}$ of $\mathbb{W}$ for all $s \geq 0$ with minimal effort.

As it was already mentioned in singular control problems, the HJB equation under some regularity assumptions on the solution, reduces to a *free boundary problem*. The free boundary problem consists in finding a certain curve $F : [0,\theta] \to \mathbb{R}$ such that the waiting region $\mathbb{W}$ and action region $\mathbb{I}$ are separated in the sense that

$$\mathbb{W} = \left\{ (x,y) \in \mathbb{R} \times [0,\theta) : x < F(y) \right\}, \tag{1.13}$$

$$\mathbb{I} = \left\{ (x,y) \in \mathbb{R} \times [0,\theta) : x \geq F(y) \right\}. \tag{1.14}$$

The optimal control can be expressed in terms of the free boundary as follows: when the current value of the diffusion $X^{x,Y^y}(t)$ is sufficiently low, such that $X^{x,Y^y}(t) < F(Y^y(t))$, then the optimal choice is to not increase the control $I(t)$ until the process $X^{x,Y^y}(t)$ crosses $F(Y^y(t))$, passing to the action region, where the optimal choice is to increase the control $I(t)$ in order to maintain the pair $(X^{x,Y^y}(t), Y^y(t))$ not below of the free boundary. Once $X^{x,Y^y}(t) > F(\theta)$ the optimal choice is restricted to increase immediately the process $I(t)$ to the maximum θ.

An explicit solution

In this section we present the explicit characterization of the free boundary of the problem presented in [50]. We explicit the solution of the particular case when the diffusion X^{x,Y^y} is not influenced by the process Y^y. In Chapter 2 we will consider this same dynamics and utility functional to study a competitive situation where two agents aim to maximize their utilities by increasing their controls, where the sum of both controls can not be greater than a given constant.

In [50] they model a big company that produce and sell solar energy in the market. Assuming that the increments on renewable energy reduce the electricity price, they search the optimal installation strategy that maximize the utilities of selling the produced solar energy in the market net of the installation cost of the solar panels. In this case, the electricity price X^{x,Y^y} is a diffusion process, modeled by a Ornstein-Uhlenbeck, where a process Y^y, as in (1.2), representing the current renewable installed power of the company, influences the mean reverting term, with $y \in [0, \theta]$ the initial installed power and θ the maximum allowed power. This is the case of $b(x, y) = \kappa(\zeta - x - \beta y)$, for some constant $\kappa, \beta \geq 0$ and $\zeta \in \mathbb{R}$ in (1.1). Therefore, the electricity price X^{x,Y^y} under renewable energy production impact, evolves accordingly to

$$\begin{cases} dX^{x,Y^y}(s) = \kappa(\zeta - \beta Y^y(s) - X^{x,Y^y}(s))ds + \sigma dW(s) \; s > 0 \\ X^{x,Y^y}(0) = x. \end{cases} \tag{1.15}$$

The stochastic process $Y = (Y^y(s))_{s \geq 0}$ can be increased irreversibly by installing more renewable energy generation devices, starting from an initial installed power $y \geq 0$, until a maximum θ. This strategy is described by the control process $I = (I(s))_{s \geq 0}$ and takes values on the set $\mathcal{I}(y)$ (1.4).

As we already said, the aim of the company is to maximize the expected profits from selling the produced energy in the market, net of the total expected cost of installing a generation device, which for an admissible strategy I, is described by the following utility functional

$$\mathcal{J}(x, y, I) \quad = \quad \mathbb{E}\left[\int_0^\infty e^{-\rho s} X^{x,Y^y}(s) a Y^y(s) ds - \int_0^\infty c e^{-\rho s} dI(s)\right],$$

where $\rho > 0$ is the discount factor, c is the installation cost of 1 MW of technology, $a > 0$ is the conversion factor of the installed device's rated power to the effective produced power per time unit. The objective of the company is to maximize the functional in Equation (1.16) by finding an optimal strategy $I^* \in \mathcal{I}(y)$ such that

$$V(x, y) = \mathcal{J}(x, y, I^*) = \sup_{I \in \mathcal{I}(y)} \mathcal{J}(x, y, I).$$

The variational inequality in this case is

$$\max\left\{\mathcal{L}^y w(x, y) - \rho w(x, y) + axy, \frac{\partial w}{\partial y} - c\right\} = 0, \tag{1.16}$$

with boundary condition $w(x, \theta) = Q(x, \theta)$, as in (1.19), and the differential operator $\mathcal{L}^y$ as defined in (1.8).

It is proved in [50, Theorem 3.2] that under some regularity conditions (linear growth, continue second order partial derivatives in the first component and continuous partial derivative in the second component) the solution of (1.16) identifies with the value function $V(x, y)$. Additionally, it is proved that the waiting region $\mathbb{W}$ and the action region $\mathbb{I}$ are separated by the strictly increasing function $F : [0, \theta] \to \mathbb{R}$ [50, Corollary 4.5], called the free boundary. By setting $\hat{F}(y) = F(y) + \beta y$, the free boundary is characterized by the ordinary differential equation [50, Proposition 4.4 and Corollary 4.5]

$$\begin{cases} \hat{F}'(y) = \beta \times \dfrac{N(y, \hat{F}(y))}{D(y, \hat{F}(y))}, \; y \in [0, \theta) \\ \hat{F}(\theta) = \hat{x}. \end{cases} \tag{1.17}$$

where

$$N(y,z) \quad = \quad \left(\psi(z)\psi^{''}(z)-\psi^{'}(z)^2\right)\left(\frac{\rho+2\kappa}{\rho}\psi^{'}(z)+\left((\rho+\kappa)\left(c-\hat{Q}(z,y)\right)\psi^{''}(z)+\psi^{'}(z)\right)\right),$$

$$D(y,x)=\psi(x)\left((\rho+\kappa)(c-\hat{Q}(x,y))\left(\psi^{'}(x)\psi^{'''}(x)-\psi^{''}(x)^2\right)+\psi(x)\psi^{'''}(x)-\psi^{'}(x)\psi^{''}(x)\right)$$

and the function ψ is the strictly increasing and positive fundamental solution of the homogeneous equation $\mathcal{L}w(x,y)-\rho w(x,y)=0$ (see in [30, Lemma 4.3] or in [50, Lemma A.1]), given by

$$\psi(x) \quad = \quad \frac{1}{\Gamma(\frac{\rho}{\kappa})}\int_0^\infty t^{\frac{\rho}{\kappa}-1}e^{-\frac{t^2}{2}-(\frac{x-\zeta}{\sigma}\sqrt{2\kappa})t}dt \tag{1.18}$$

and

$$\hat{Q}(x,y)=\frac{a\zeta\kappa+a\rho x-a\beta(\rho+2\kappa)y}{\rho(\rho+\kappa)}. \tag{1.19}$$

On the other hand, the boundary condition $\hat{x}$ in (1.17) is the unique solution of

$$\psi'(x)(c-\hat{Q}(x,\theta))+(\rho+\kappa)^{-1}\psi(x) \quad = \quad 0. \tag{1.20}$$

Remark 1.2.2 *The solution $\hat{x}$ is such that $\hat{x}\in\left(\bar{c},\bar{c}+\frac{\psi(\bar{c})}{\psi'(\bar{c})}\right)$, with $\bar{c}=c(\rho+\kappa)-\frac{\zeta\kappa-\beta(\rho+2\kappa)\theta}{\rho}$ [50, Lemma 4.2].*

When there is not impact, i.e., $\beta=0$, we have $\hat{F}(y)\equiv F(y)$, then from (1.17) every $y\in[0,\theta)$, $F^{'}(y)\equiv 0$, hence the free boundary is a constant with value $F(y)=\hat{x}$, with $\hat{x}$ the same solution of (1.20), considering $\beta=0$ in the function $\hat{Q}(x,y)$ defined in (1.19). Notice that in this case $\hat{Q}$ does not depend on y.

In this case, the candidate value function is given by

$$w(x,y)=\begin{cases}A(y)\psi(x)+Q(x,y)\text{ , if }(x,y)\in\mathbb{W}\cup(\{\theta\}\times(-\infty,\hat{x}))\\ Q(x,\theta)-c(\theta-y)\text{ , if }(x,y)\in\mathbb{I}\cup(\{\theta\}\times(\hat{x},\infty))\end{cases}, \tag{1.21}$$

with $Q(x,y)$ defined in Equation (1.9), $\psi(x)$ given by Equation (1.18) and $A(y)$ given by

$$A(y)=\frac{\theta-y}{(\rho+\kappa)\psi^{'}(\hat{x})}. \tag{1.22}$$

The optimal control is written as (see [50, Theorem 4.8])

$$I^*(t) = \begin{cases} 0 \ , \ t \in [0, \tau) \\ \theta - y \ , \ t \geq \tau \end{cases} \quad , \tag{1.23}$$

with $\tau = \inf\{t \geq 0, X(t) \geq \hat{x}\}$.

1.3 First order conditions

The case when the diffusion X^{x,Y^y} is not influenced by the process Y^y, i.e., when $b(x, y) = \hat{b}(x)$ and the revenue function $R(x, \cdot)$ is concave and satisfies the Inada conditons, is of special interest in economy. This is the case of the problem of intertemporal consumption and portfolio choice and it was formulated as a singular control problem by [42]. In [5], they establish necessary and sufficient condition of optimality for this types of problems in the form of infinite dimensional Kuhn-Tucker conditions. In [6] the Kuhn-Tucker conditions were extended to the case when the control variable is the distribution function of a non negative optional random measure. The definition of optional random measures is as follows:

Definition 1.3.1 *(Optional random measure [44]) Let $\mathcal{V}$ be the space of positive finite measures on $[0, T]$ with the topology of weak convergence. An optional random measure is a $\mathcal{V}$-valued random variable μ such that the process $\mu_t(\omega) := \mu(\omega, [0, t])$ is adapted.*

To prove existence of optimal controls on this framework, [6] uses a version of the Komlos' Theorem for random measures, which states that for every sequence $(\nu^n)_{n \in \mathbb{N}} \in \mathcal{V}$ there exists a subsequence $(\nu^{n'})_{n' \in \mathbb{N}}$ converging weakly in Cesaro sense to a random variable $\nu \in \mathcal{V}$, i.e,

$$\frac{1}{n'} \sum_{k=1}^{n'} \nu^k \to \nu \text{ as } n' \to \infty.$$

The formal lemma states as follows:

Lemma 1.3.2 *[44, Lemma 3.5] Let $\mathcal{V}_t$ be the space of optional random measures in $[0, T]$ and μ^n be optional random measures such that $\sup_n \mathbb{E}\mu^n_T < \infty$. Then, there exist an optional random measure μ with $\mu_T \in L^1$ and a subsequence $\mu^{n'}$ such that all its further subsequence are Cesaro convergent in $\mathcal{V}_t$ to μ a.s.*

Inspired on the model proposed by [42], in [5] it is studied the following problem: one considers an economic agent living from time 0 up to some time $T \geq 0$ which decides how much of a perishable consumption good to consume at each time $t \in [0, T]$. Following HHK the set of cumulative consumption plans is

$$\mathcal{C} := \Big\{ C : [0,T] \to \mathbb{R} | C \text{ is a non negative, nondecreasing right- continuous} \Big\}.$$

They assume as given a complete set of forward markets, where the consumption good is traded at some deterministic price $q(t)$ $i \in [0,T]$. The agent buys his preferred consumption plan at time 0. They assume that q is a continuous, strictly positive function. Then, the corresponding price functional $\mathcal{Q}(C) := \int_0^T q(u)dC(u)$ is a linear functional on $\mathcal{C}$, which is continuous with respect to the weak topology. The agent is endowed with some capital $w \geq 0$ and his budget set is given by:

$$\mathcal{A}(w) := \Big\{ C \in \mathcal{C} | \mathcal{Q}(C) \leq w \Big\}.$$

In contrast to the standard models, the agent does not obtain utility from his current consumption $dC(t)$, but from an index of past consumption $Y^C(t)$ which may be interpreted as his current standard of living. Following HHK, they assume this index to be given by:

$$Y^C(t) := y(t) + \int_0^t k(t,s)dC(s),$$

for some nonnegative, continuous functions k and y. The utility associated with a consumption plan $C \in \mathcal{C}$ is given by the functional:

$$U(C) := \int_0^T u(s, Y^C(s))ds$$

where $u : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. The optimization problem consists in

$$\text{Minimize} U(C) \text{ over } C \text{ subjected to } C \in \mathcal{A}(w). \tag{1.24}$$

To prove existence and uniqueness of the problem, they prove that the functional $U(\cdot)$ is continuous on $\mathcal{C}$ equipped with the topology of weak convergence of measures on $([0,T], \mathcal{B}([0,T]))$ and that $\mathcal{A}(w)$ is compact with respect to the weak topology. Assuming that $u(t,\cdot)$ is strictly monotone and strictly concave for every $t \in [0,T]$ and if $C \to Y^C$ is injective they obtain uniqueness. To characterized the optimal consumption plan for problem they establish Kuhn-Tucker like necessary and sufficient first order condition, which states as follows

Theorem 1.3.3 *[5, Theorem 4.2] Necesary and sufficient condition for a consumption plan C^* to solve* (1.24) *are:*

(i) $\mathcal{Q}(C^*) = w$

(ii) $\int_t^T \partial_y u(s, Y^{C^*}(s))k(s,t)ds \leq Mq(t)$ $\forall t \in [0,T]$

(iii) $\int_t^T \partial_y u(s, Y^{C^*}(s))k(s,t)ds = Mq(t)$ $\forall t \in$ *supp* dC^*

for some constant $M = M(w, T) \geq 0$.

The result above is generalized in [6] when the consumption plans are optinal random measures. Inspired by the explicit characterization of optimal consumption plans analyzed in [6], in [4] they formulate a representation theorem with application to optimization and obstacle problem. Under the following

Assumption 1.3.4 *The mapping* $f : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ *satisfies:*

i. For each $\omega \in \Omega$ *and any* $t \in [0, T]$*, the function* $f(\omega, t, \cdot) : \mathbb{R} \to \mathbb{R}$ *is continuous and strictly decreasing from* $+\infty$ *to* $-\infty$.

ii. For any $l \in \mathbb{R}$*, the stochastic process* $f(\cdot, \cdot, l) : \Omega \times [0, T] \to \mathbb{R}$ *is measurable with*

$$\mathbb{E}\left[\int_0^T |f(t, l)| dt\right] < +\infty$$

They found that:

Theorem 1.3.5 *[4, Theorem 3] Under Assumption 1.3.4, every optional process* X *of class (D) which is lower semicontinuous in expectation with* $X(T) = 0$ *admits a representation of the form*

$$X(t) = \mathbb{E}\left[\int_t^T f\left(s, \sup_{u\in[t,s]} l(u)\right) ds \Big| \mathcal{F}_t\right], t \in [0, T] \tag{1.25}$$

for some suitable optional process l *taking values in* $\mathbb{R} \cup \{\infty\}$ *which satisfies the integrability condition* $f(t, \sup_{u\in[s,t]} l(u))\mathbb{1}_{[S,T)}(t) \in L^1(\mathbb{P} \otimes dt)$ *for any stopping time* $S \leq T$.

With this theorem it is possible to relate the optimal solution of our singular control problem to the solution of a backward stochastic differential equation. Using this arguments, in [3], the irreversible investment problem under dynamic fuel constraint was solved. They construct a characterization of the optimal control based on the Snell envelope of the utility functional's gradients at the optimum. Assuming that the revenue function $R(x, \cdot)$ is strictly concave and differentiable in y, they denote by $\nabla \mathcal{J}(x, y, I)_s$ the subgradient of $\mathcal{J}(x, y, I)$, defined by [3, Lemma 2.1]

$$\nabla \mathcal{J}(x, y, I)(t) := \mathbb{E}\left[\int_t^\infty e^{-\rho s} R_y(X^x(s), Y^y(s)) ds \Big| \mathcal{F}_t\right] - ce^{-\rho t}\ , \ t \geq 0.$$

The Snell envelope $\mathbb{S}(I)$ of the subgradient $\nabla \mathcal{J}(x, y, I)$ is given by

$$\mathbb{S}(I)_\tau := \operatorname*{ess\,sup}_{t\in[\tau,\infty)} \mathbb{E}\left[\nabla\mathcal{J}(x,y,I)_t \middle| \mathcal{F}_\tau\right].$$

By the Doob-Meyer decomposition, $\mathbb{S}(I)_\tau$ can be written as $\mathbb{S}(I)_\tau = M(I) + A(I)$, where $M(I)$ is an uniformly integrable martingale and $A(I)$ is a increasing predictable process. They found that the optimal solution is characterized by the conditions [3, Theorem 2.2]

$$\begin{cases} I^* \text{ is flat off } \{\nabla\mathcal{J}(x,y,I^*) = \mathbb{S}(I^*)\} \\ A(I^*)\text{is flat off}\{I^* = \theta\}. \end{cases} \tag{1.26}$$

Remark 1.3.6 *Recall that an increasing process λ is flat off a set $B \in \mathcal{F}_\infty \otimes \mathcal{B}([0,\infty))$ if the induced measure $d\lambda$ almost surely does not charge the set B, i.e., $\mathbb{E}\left[\int_0^\infty \mathbb{1}_B d\lambda\right] = 0$.*

Moreover, the optimal solution is given in terms of a progressively measurable random process l specifying an upper bound which the optimal control should respect granted enough fuel is left to do so. The upper bound l is characterized by the backward equation

$$\mathbb{E}\left[\int_t^\infty R_y(X^x(s), \sup_{u\in[t,s)} l(u))ds \middle| \mathcal{F}_t\right] = ce^{-\rho t}, \tag{1.27}$$

which corresponds to the optional solution of the representation problem of [4] (see (1.25)). The optimal solution is then given by [3, Theorem 3.1]

$$I^*(t) = \sup_{u\in[0,t)} (l(u) \wedge \theta) \vee y.$$

The results in [3] were extended in [23] to the social planner problem, which consists of maximizing the sum of the utilities of N firms under dynamic fuel constraint. The problem is to find an optimal vector $\bar{I}^*$ in

$$\mathcal{I}^N(y) \triangleq \Bigg\{ \bar{I} : [0,\infty)\times\Omega \to \mathbb{R}_+^N : I \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{- adapted }, t \to I_t \text{ is nondecreasing, left continuous,}$$

$$\text{such that } I_i(0) = y^i \text{ and } \sum_{i=1}^N I_i(t) \leq \theta, \text{ for all } t \in [0,\infty) \Bigg\}.$$

where $y = (y^1, \ldots, y^N)$ states as the vector of initial conditions, such that

$$\mathcal{J}_{SP}(x,y,\bar{I}^*) = \sup_{\bar{I}\in\mathcal{I}^N(y)} \sum_{i=1}^N \mathcal{J}_i(x,y_i,I_i) \tag{1.28}$$

where for each $i = 1, \dots, N$, $\mathcal{J}_i(x, y_i, I_i)$ is as in (1.4). They prove existence and uniqueness of optimal irreversible investment policies and by using the concavity of the profit functionals they characterize the solution for the social planner problem as the unique solution of some Kuhn-Tucker conditions. In this framework the Lagrange multiplier takes the form of a nonnegative optional random measure on $[0, \infty)$, which is flat off the set of times for which the constraint is binding. They also show that the induced measure dA from (1.26), coincide with the Lagrange multiplier of the Kuhn-Tucker condition for the social planner problem. The Kuhn-Tucker conditions and the sufficient optimality theorem states as follows

Theorem 1.3.7 *([23, Theorem 3.3]) if there exists a non-negative Lagrange multiplier random measure* $\lambda(\omega, dt)$ *on* $\mathcal{B}([0,\infty))$ *such that* $\mathbb{E}\int_{[0,\infty)} d\lambda(t) < \infty$ *and an admissible* $\bar{I}^* \in \mathcal{I}^N(y)$ *such that the following conditions*

$$\nabla \mathcal{J}_i(x, y, I_i^*)(\tau) \leq \mathbb{E}\left[\int_\tau^\infty d\lambda(s) \middle| \mathcal{F}_\tau \right] \ \mathbb{P}\text{-}a.s\ \forall \tau \in \mathcal{T},$$
$$\int_0^\infty \left(\nabla \mathcal{J}_i(x, y, I_i^*)(t) - \mathbb{E}\left[\int_t^\infty d\lambda(s) \middle| \mathcal{F}_t \right] \right) dI_i^*(t) = 0\ \mathbb{P}\text{-}a.s,$$
$$\mathbb{E}\left[\int_0^\infty \left(\theta - \sum_{i=1}^N I_i^*(t) \right) d\lambda(t) \right] = 0,$$

are satisfied, then I^* *is a solution for the social planner problem* (1.28).

According to [23] the optimal solution for this problem is

$$I_i^*(s) = \sup_{0 \leq u < s} \left(l(u) \wedge \hat{\beta} \right) \vee y^i\ ,\ i = 1, \dots, N \tag{1.29}$$

where $l(s)$ is the unique solution to the backward stochastic differential equation (1.27).

Nevertheless, the solution (1.29) does not always produce admissible controls. In Chapter 3, we prove optimality for an admissible control and in Chapter 4 we also extend those condition to the case when X^x is influenced by the increment in the control I, proving sufficient and necessary conditions of optimality.

To conclude this review, in [29] it is shown that the free boundary of the singular control problem is related with the solution of the backward equation as $F(X^x(s)) = l(s)$, where $X^x(s)$ is an uncontrolled diffusion [29, Theorem 3.9]. Using the representation Theorem 1.25, and assuming the strong Markov property, they found that the free boundary is the unique solution of the equation [29, Theorem 3.11]

$$\psi(x) \int_x^{\bar{x}} \left(\int_{\underline{x}}^z R_w(y, F(z)) \psi(y) m^{'}(y) dy \right) \frac{S^{'}(z) dz}{\psi^2(z)} \quad = \quad c, \tag{1.30}$$

where $R_w(x, w)$ stands for the partial derivative in the second component, $\underline{x}$ and $\bar{x}$ are the end points of the domain of X^x, the function $\psi(y)$ is the increasing solution of the equation $\mathcal{L}u(x) = \rho u(x)$, where $\mathcal{L}$ is the infinitesimal generator of the diffusion X^x (see (1.8)) and $m(dx)$ and $s(dx)$ are the speed measure and the scale function measure of X^x, respectively.

Chapter 2

Optimal installation of renewable electricity sources

Based on [2]: Awerkin, A., Vargiolu, T., *Optimal installation of renewable electricity sources: the case of Italy*, Decision in Economics and Finance 44, 1179-1209 (2021).

In this chapter we present an application of irreversible investment problems. This work is based on the paper developed in [50], which we already present in Chapter 1, subsection 1.2 as an explicit solution of a singular control problem using the Hamilton-Jacobi-Bellman (HJB) equation. The paper [50] describes the irreversible installation problem of photovoltaic panels for an infinitely-lived profit maximizing power-producing company, willing to maximize the profits from selling electricity in the market. The power price model used in that paper assumes that the company is a large market player, so its installation has a negative impact on power price. More in detail, the power price is assumed to follow an additive mean-reverting process (so that power price could possibly be negative, as it happens in reality), where the long-term mean decreases as the cumulative installation increases. The resulting optimal strategy is to install the minimal capacity so that the power price is always lower than a given nonlinear function of the capacity, which is characterized by solving an ordinary differential equation deriving from a free-boundary problem.

In Section 2.1 we validate empirically the assumption that the increments in renewable installed power affect negatively the electricity price, i.e., the increments in the control variable I reduce the long mean term of the process describing the evolution of the electricity price. To do so, we consider the Ornstein-Uhlenbeck (O-U) process, including an exogenous increasing process influencing the mean reverting term, which is interpreted as the current renewable installed power. In Section 2.2 we estimate the parameters of this model by using real data of electricity prices and energy production from photovoltaic and wind power plants from the six main Italian price zones. In Section 2.3 we extend the results of [50] to the case when N players can produce electricity by installing renewable power plants. To this extent, we analyze both the concepts of Pareto optima and of Nash equilibria. For this latter, we present a verification theorem in the 2-player case, and an explicit characterization of a Nash equilibrium in the case when there is no price impact. Finally, in Section 2.4 we present some numerical results where we describe the analytical optimal strategy and compare it with the real installation strategy that was put in place in Italy.

2.1 The electricity price model

It is common in literature to model electricity prices via a mean-reverting behavior, and to include (jump) terms representing the seasonal fluctuations and daily spikes, cf. [15, 19, 35, 61] among others. Here, in analogy with [50], we do not represent the spikes and seasonal fluctuations with the following argument: the installation time of solar panels or wind turbines usually takes several days or weeks, which makes the power producers indifferent of daily or weekly spikes. Also, the high lifespan of renewable power plants and the underlying infinite time horizon setting allow us to neglect the seasonal patterns. We therefore assume that the electricity's fundamental price has solely a mean-reverting behavior, and evolves according to an Ornstein-Uhlenbeck (O-U) process[1]. We are also neglecting the stochastic and seasonal effects of renewable power production. In fact, photovoltaic production has obvious seasonal patterns (solar panels do not produce power during the night and produce less in winter than in summer), and both solar and wind power plants are subject to the randomness affecting weather conditions. However, since here we are interested to a long-term optimal behaviour, we interpret the average electricity produced in a generic unit of time as proportional to the installed power. All of this can be mathematically justified if we interpret our fundamental price to be, for example, a weekly average price as e.g. in [16, 33, 36], who used this representation exactly to get rid of daily and weekly seasonalities.

In order to represent price impact of renewables in power prices, which is more and more observed in several national power markets, we follow the common stream in literature (also in analogy with [50]) and represent renewable capacity installation as a non-decreasing process, thus resulting in a singular control problem. This is also analogous to other papers modeling price impact: for example, in problems of optimal execution, [9] and [10] take into account a multiplicative and transient price impact, whereas [39] considers an exponential parametrization in a geometric Brownian motion setting allowing for a permanent price impact. Also, a price impact model has been studied by [1], motivated by an irreversible capital accumulation problem with permanent price impact, and by [30], in which the authors consider an extraction problem with Ornstein-Uhlenbeck dynamics and transient price impact. In all of the aforementioned papers on price impact models dealing with singular stochastic controls [1, 9, 10, 30, 39], the agents' actions can lead to an immediate jump in the underlying price process, whereas in our setting, it cannot. Our model is instead analogous to [20, 21], which show how to incorporate a market impact due to cross-border trading in electricity markets, and to [57], which models the price impact of wind electricity production on power prices. In these latter models, price impact is localized on the drift of the power price.

We assume that the fundamental electricity price $S^x(s)$, in absence of increments on the level of renewable installed power, evolves accordingly to an Ornstein-Uhlenbeck (O-U) process

$$\begin{cases} dS^x(s) = \kappa\left(\zeta - S^x(s)\right) ds + \sigma dW(s) \; s > 0 \\ S^x(0) = x \end{cases}, \tag{2.1}$$

for some constants $\kappa, \sigma, x > 0$ and $\zeta \in \mathbb{R}$, where $(W(s))_{s\geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, more rigorous definition and detailed assumptions will be given in the next section.

We represent the increment on the current installed power level with the sum of increasing processes $Y_i^{y_i}$, where y_i is the initial installed power and the index i stands for the renewable power

[1] We allow for negative prices by modeling the electricity price via an Ornstein-Uhlenbeck process. Indeed, negative electricity prices can be observed in some markets, for example in Germany, cf. [60].

source type, which in our case are sun and wind. We relate $Y_1^{y_1}$ with solar energy and $Y_2^{y_2}$ with wind energy. We assume that the increment in the current renewable installed power affects the electricity price by reducing the mean level instantaneously at time s by $\sum_{i=1}^{2} \beta^i Y_i^{y_i}(s)$ for some $\beta^i > 0$ [50], with $i \in \{1, 2\}$. Therefore the spot price $S^{x,I}(s)$ evolves according to

$$\begin{cases} dS^{x,I}(s) = \kappa(\zeta - \sum_{i=1,2} \beta^i Y_i^{y_i}(s) - S^{x,I}(s))ds + \sigma dW(s) \ s > 0 \\ S^{x,I}(0) = x. \end{cases} \tag{2.2}$$

The explicit solution of (2.2) between two times τ and t, with $0 \leq \tau < t$ is given by

$$\begin{aligned} S^{x,I}(t) &= e^{\kappa(\tau-t)} S^{x,I}(\tau) + \kappa \int_\tau^t e^{\kappa(s-t)} \left(\zeta - \sum_{i=1,2} \beta^i Y_i^{y_i}(s) \right) ds + \int_\tau^t e^{\kappa(s-t)} \sigma dW(s) \\ &= e^{\kappa(\tau-t)} S^{x,I}(\tau) + \zeta(1 - e^{\kappa(\tau-t)}) - \kappa \int_\tau^t e^{\kappa(s-t)} \sum_{i=1,2} \beta^i Y_i^{y_i}(s) ds + \int_\tau^t e^{\kappa(s-t)} \sigma dW(s). \end{aligned} \tag{2.3}$$

The discrete time version of (2.3), on a time grid $0 = t_0 < t_1 < \ldots$, with constant time step $\Delta t = t_{n+1} - t_n$ results in the ARX(1) model

$$X(t_{n+1}) = a + bX(t_n) + \sum_{i=1,2} u^i Z^i(t_n) + \delta\epsilon(t_n). \tag{2.4}$$

where $X(t_0), X(t_1), X(t_2), \ldots$ and $Z^i(t_0), Z^i(t_1), Z^i(t_2), \ldots$ are the observation on the time grid, of process $S^{x,I}$ and $Y_i^{y_i}$ respectively. The random variables $(\epsilon(t_n))_{n=\{0,\ldots,N\}} \sim \mathcal{N}(0, 1)$ are iid and the coefficients a, b, u^1, u^2 and δ are related with κ, ζ, β^1, β^2 and σ by

$$\begin{cases} a = \zeta(1 - e^{-\kappa\Delta t}) \\ b = e^{-\kappa\Delta t} \\ u^1 = -\beta^1(1 - e^{-\kappa\Delta t}) \\ u^2 = -\beta^2(1 - e^{-\kappa\Delta t}) \\ \delta = \frac{\sigma\sqrt{1-e^{-2\kappa\Delta t}}}{\sqrt{2\kappa}} \end{cases} . \tag{2.5}$$

The estimation of the discrete time parameters a, b, δ and u_i, $i = 1, 2$ can be obtained from ordinary least squares, which gives maximum likelihood estimators. Then, the continuous time parameters κ, ζ, σ and β^i with $i = 1, 2$ can be estimated by solving Equations (2.5) [17].

2.2 Parameter estimation for Italian zonal prices

In this section we estimate the parameters of the model in Equation (2.4) using real Italian data of energy price and current installed power.

2.2.1 The dataset

We have data from six main price zones of Italy, which are North, Central North, Central South, South, Sicily and Sardinia. For every zone we have weekly measurements of average energy price in €/MWh, together with photovoltaic and wind energy production in MWh. The time series goes from 07/05/2012 to 25/06/2018, week 19/2012 to 26/2018, corresponding to $N = 321$ observations. The time series of current photovoltaic and wind installed power is instead available with a much lower frequency (i.e. year by year). In order to obtain a time series consistent with the weekly granularity of price and production, we estimate the installed power to be proportional to the running maximum of the photovoltaic and wind energy production of whole Italy, respectively. Summarizing, we use for estimation of the model in Equation (2.4), for every particular zone, the data summarized in Table 2.1.

Variable Type	Nomenclature	Description
Time step observation	$t_1, \dots, t_N$	Weeks when the quantities are observed, $N = 321$.
Response variable	$X(t_0), \dots, X(t_N)$	Electricity price in €/MWh relative to an Italian price zone.
Explanatory variable	$Z^1(t_0), \dots, Z^1(t_N)$	Current installed photovoltaic power in MW, estimated as $Z^1(t_i) = \max(E^1(t_0), \dots, E^1(t_i))$, $i \in \{1, \dots, N\}$, where $E^1(t_i)$ is the sum of the produced energy on the six zones at the observation time t_i.
	$Z^2(t_0), \dots, Z^2(t_N)$	Current installed wind power in MW, estimated as $Z^2(t_i) = \max\{E^2(t_0), \dots, E^2(t_i)\}$, $i \in \{1, \dots, N\}$, where $E^1(t_i)$ is the sum of the produced energy on the six zones at the observation time t_i.

Table 2.1: The data used for parameter estimation of Equation (2.4).

2.2.2 Results

Using ordinary least squares considering the data described above and then setting $\Delta t = t_{i+1} - t_i = \frac{1}{52}$ for all $i = 0, \dots, 320$, we obtain, by Equations (2.5), the continuous time parameters for the O-U model with an exogenous impact in the mean reverting term, for every zone. Table 2.2 shows the estimation results by zone.

Zone		parameters					Box Pierce test
		κ	ζ	β^1	β^2	σ	p-value
North	Value	*** 10.6056	*** 133.0670	* 0.0148	0.0012	*** 47.7527	0.6101
	s.e.	2.1437	32.2392	0.0082	0.0031	2.3741	
Central North	Value	*** 10.9960	*** 120.4933	0.0112	0.0027	*** 45.5106	0.2702
	s.e.	2.1599	30.1593	0.0076	0.0029	2.1413	
Central South	Value	*** 13.2276	*** 100.3647	0.0052	** 0.0056	*** 45.4237	0.0093
	s.e.	2.3958	27.3713	0.0069	0.0026	2.05040	
South	Value	*** 11.4996	*** 98.5810	0.0059	* 0.0047	*** 41.5805	0.0086
	s.e.	2.2004	26.9193	0.0068	0.0026	1.7715	
Sicily	Value	*** 14.1614	** 173.0264	0.0124	*** 0.0107	*** 81.4377	0.0132
	s.e.	2.5146	46.9427	0.0120	0.0044	6.4833	
Sardinia	Value	*** 18.4580	*** 94.7809	0.0020	** 0.0129	*** 68.2290	0.1216
	s.e.	2.9547	33.1946	0.0085	0.0031	4.2260	

Table 2.2: Estimated parameters for the Ornstein Uhlenbeck. Significance code: *** = $p < 0.01$, ** = $p < 0.05$, * = $p < 0.1$.

In Table 2.2, under each parameter we observe the value of every estimator and its respective standard error. Moreover, for each price zone, we include the results of the Box-Pierce test to check the independence of the residuals. This test rejects the independence hypothesis for p-values less than 0.05. According to the results in Table 2.2, the Central South, South and Sicily zones present correlation in the residuals, therefore the proposed O-U model for electricity price is not the right choice, this is an obvious indication that the O-U model is too simple for these zones, and one should instead use more sophisticated models, like CARMA ones (see e.g. [12]): we leave this part for future research. On the other hand the North, Central North and Sardinia zones have independent residuals implying that the model is able to explain the behavior of the electricity price. Regarding the parameters significance for this latter three zones, only the North and Sardinia zones present price impact: in the North there is only photovoltaic impact while in Sardinia only wind impact. We re-estimate the parameters considering only the zones which pass the Box-Pierce test and with only the significant price impact parameters. Table 2.3 summarizes the obtained results.

Zone		parameters					Box Pierce tests
		κ	ζ	β^1	β^2	σ	p-value
North	Value	*** 10.3702	*** 140.5894	** 0.0172	0	*** 47.6586	0.6206
	s.e.	2.0514	26.4732	0.0054		2.3747	
Central North	Value	*** 9.2648	*** 55.6085	0	0	*** 65.9346	0.2771
	s.e.	1.9273	2.8265			4.6367	
Sardinia	Value	*** 18.5248	*** 102.4620	0	*** 0.0123	*** 68.2889	0.1296
	s.e.	2.9510	6.6813		0.0017	4.2260	

Table 2.3: Significant estimated parameters for Ornstein Uhlenbeck . Significance code: $*** = p < 0.01$, $** = p < 0.05$, $* = p < 0.1$

2.3 A market with N producers

The second aim of our work is to check the effective installation strategy, in the different price zones, against the optimal one obtained theoretically. In doing so, we must take into account the fact that the Italian market is liberalized since about two decades, thus there is not a single producer which can impact prices by him/herself, but rather prices are impacted by the cumulative installation of all the power producers in the market. We thus extend our model by formulating it for N players who can install, in the different price zones, the corresponding impacting renewable power source, monotonically and independently of each other: the resulting power price will be impacted by the sum of all these installations, while each producer will be rewarded by a payoff corresponding to their installation. The resulting N-player nonzero-sum game can be solved with different approaches. A formulation requiring a Nash equilibrium would result in a system of N variational inequalities with $N+1$ variables (see e.g. [26] and references therein), which would be quite difficult to treat analytically. We choose instead to seek for Pareto optima first. One easy way to achieve this is to assume, in analogy with [23], the existence of a "social planner" which maximizes the sum of all the N players' payoffs, under the constraint that the sum of their installed capacity cannot be greater than a given threshold (which obviously represents the physical finite capacity of a territory to support power plants of a given type). We prove that, in our framework, this produces Pareto optima. More in detail, by summing together all the N players' installations in the social planner problem, one obtains the same problem of a single producer, which has a unique solution that represents the optimal cumulative installation of all the combined producers. Though with this approach it is not possible to distinguish the single optimal installations of each producer, we can assess how much the effective cumulative installation strategy which was carried out in Italy during the dataset's period differs from the optimal one which we obtained theoretically. To give an idea of what we instead would get when searching for Nash equilibria, we present the case $N=2$ and formulate a verification theorem that the value functions of each player should satisfy. Here we want to point out a difference which arises in our problem with respect to the current stream of literature. In fact, in stochastic singular games the usual framework is that a player can act only when the other ones are idle, see e.g. [25, 26, 37, 38]. Here instead we take explicitly into consideration the possibility that both players acts (i.e. install) simultaneously. This possibility will be confirmed in Section 2.3.2, where (in the case with no market impact) we present a Nash equilibrium where both players install simultaneously. Another peculiarity is that this equilibrium induces the players to install *before* than when they would have done under a Pareto optimum. This is the converse phenomenon of what observed e.g. in [25], where instead players following a Nash equilibrium act later than players following a Pareto optimum.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space where a one-dimensional Brownian motion W is defined and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets. Consider a market with N producers, indexed by $i = 1, \ldots, N$. The cumulative irreversible installation strategy of the producer i up to time s, denoted by $I_i(s)$, is an adapted, nondecreasing, cadlag process, such that $I_i(0) = 0$. We assume that the aggregated installation of the N firms is allowed to increase until a total maximum constant power θ, that is,

$$\sum_{i=1}^{N} (y_i + I_i(s)) \leq \theta \ \mathbb{P}-\text{ a.s., } s \in [0, \infty), \tag{2.6}$$

where y_i is the initial installed power for the firm i and indicate by $\bar{y} = (y_1, \ldots, y_N)$ the vector of the initial conditions. We denote by $\mathcal{I}_N$ the set of admissible strategies of all the players

$$\mathcal{I}_N \triangleq \{\bar{I} : [0,\infty) \times \Omega \to [0,\infty)^N \text{ , non decreasing, left continuous adapted process} \\ \text{with } I_i(0-) = 0, \ \mathbb{P}\text{-a.s., } \textstyle\sum_{i=1}^{N} (y_i + I_i(s)) \leq \theta\}.$$

and notice that each player is constrained, in its strategy, by the installation strategies of the other players.

2.3.1 Pareto optima

We now consider the cooperative situation of a social planner, where the problem consists into finding a efficient installation strategy $\bar{I}^* \in \mathcal{I}_N$ which maximizes the aggregate expected profit, net of investment cost [23]. While in many liberalized markets there is not a single being which can *impose* a given strategy to all the players, this is equivalent to solving a cooperative game with the maximum possible coalition containing all the players.

The social planner problem, therefore is expressed as

$$V_{SP} = \sup_{\bar{I} \in \mathcal{I}_N} \mathcal{J}_{SP}(\bar{I}), \tag{2.7}$$

where

$$\mathcal{J}_{SP}(\bar{I}) = \sum_{i=1}^{N} \mathcal{J}_i(I_i) \tag{2.8}$$

and for $i = 1, 2, \ldots, N$,

$$\mathcal{J}_i(x, \bar{y}, \bar{I}) \quad = \quad \mathbb{E}\left[\int_0^{\infty} e^{-\rho s} S^{x,\bar{y},\bar{I}}(s) a(y_i + I_i(s)) ds - c \int_0^{\infty} e^{-\rho s} dI_i(s)\right], \tag{2.9}$$

where ρ, a and c are the same defined in (1.16).

For the N firms, Pareto optimality is defined as follows:

Definition 2.3.1 *(Pareto optima) An admissible strategy $\bar{I}^* \in \mathcal{I}^N$ is called Pareto optima if the set of inequalities*

$$\mathcal{J}_i(x, \bar{y}, \bar{I}) \geq \mathcal{J}_i(x, \bar{y}, \bar{I}^*)$$

for $i = 1, \ldots, N$, where at least one of the inequalities is strict, does not allow for any admissible solution $\bar{I} \in \mathcal{I}$.

The process $S^{x,\bar{y},\bar{I}}(s)$ is the electricity price affected by the sum of the installations of all the agents which, in analogy with the one-player case, we assume to follow an O-U process with an exogenous mean reverting term, whose dynamics is given by

$$\begin{cases} dS^{x,\bar{y},\bar{I}}(s) = \kappa(\zeta - \beta \sum_{i=1}^{N} (y_i + I_i(s)) - S^{x,\bar{y},\bar{I}}(s)) ds + \sigma dW(s) \ s > 0 \ , \\ S^{x,\bar{y},\bar{I}}(0) = x. \end{cases} \tag{2.10}$$

Call now $\nu(t) = \sum_{i=1}^{N} I_i(t)$ and $\gamma = \sum_{i=1}^{N} y_i$: then, by substituting on the social planner functional (2.8), we get

$$\begin{aligned}
\mathcal{J}_{SP}(\bar{I}) &= \sum_{i=1}^{N} \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^{x,\bar{y},\bar{I}}(s) a(y_i + I_i(s)) ds - c\int_0^\infty e^{-\rho s} dI_i(s)\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^{x,\bar{y},\bar{I}}(s) a\left(\sum_{i=1}^{N} y_i + \sum_{i=1}^{N} I_i(s)\right) ds - c\int_0^\infty e^{-\rho s} d\left(\sum_{i=1}^{N} I_i(s)\right)\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^{x,\bar{y},\bar{I}}(s) a(\gamma + \nu(s)) ds - c\int_0^\infty e^{-\rho s} d\nu(s)\right].
\end{aligned}$$

Observe that we have the same optimal control problem as in the single company case (Chapter 1, Subsection 1.2), therefore we can guess that the optimal solution for the social planner will be equal to that for the single company. In fact, the aggregate optimal strategy for the N producer of a given region results to be Pareto optimal (see Lemma 2.3.2 below).

Lemma 2.3.2 *If $\bar{I}^* \in \arg\max \mathcal{J}_{SP}(\bar{I})$, then $\bar{I}^*$ is Pareto optimal.*

Proof. Suppose $\bar{I}^* \in \arg\max \mathcal{J}_{SP}(\bar{I})$ and assume $\bar{I}^*$ is not Pareto optimal, then there exist $\hat{I}$ such that,

$$\mathcal{J}_i(\hat{I}_i) \geq \mathcal{J}_i(I_i^*) \ , \forall \ i \in \{1, \dots, N\} \tag{2.11}$$

where at least one inequality is strict. Then,

$$\sum_{i=1}^{N} \mathcal{J}_i(\hat{I}_i) > \sum_{i=1}^{N} \mathcal{J}_i(I_i^*), \tag{2.12}$$

contradicting the fact that $\bar{I}^*$ is maximizing. ■

As already said in the Introduction, with this approach it is not possible to distinguish the single optimal installations of each producer, as we can only characterize the cumulative installation $\nu(t) = \sum_{i=1}^{N} I_i(t)$, while the single components $I_i(t)$ remain to be determined. Therefore, it could be possible that an optimal solution is such that a single firm install how much it can, while the others remain idle. However, our aim is about to study the aggregate cumulative installation strategy of the N firms and then compare the analytical optimal solution with the real effective aggregate installation strategy which was carried out in Italy during the time period covered by the dataset.

In the next subsections, instead, we compare these Pareto optima, obtained by assuming that players would cooperate to achieve the maximum cumulative payoff, with Nash equilibria, which instead assume that players compete actively to individually maximize their own payoff.

2.3.2 Nash equilibria in the case $N = 2$

The Pareto optima found previously for the social planner problem assume a collaboration between players: nevertheless, it could be also possible to have competition in the market between the players, therefore it makes sense to study the non cooperative case and search for Nash equilibria. In particular we solve the case with two players and we compare both results.

The formulation for the competitive game with two players states as follows: the electricity price evolves according to (2.10) and every player aims to maximize its own utility (2.9). In this case, in analogy with [37], we will look for a subset of the admissible strategies $\mathcal{I}_2$, which we describe next.

Definition 2.3.3 *(Markovian strategy and admissible control set) A strategy $I(t) \in \mathcal{I}$ is called Markovian if $I(t) = I(S(t), Y^1(t-), Y^2(t-))$ for all $t \geq 0$, where I is a deterministic function of the states immediately before time t. We define the admissible set of Markovian strategies as follows*

$$\mathcal{I}_2^M := \{I_1, I_2 \in \mathcal{I}_2 \mid (I_1, I_2) \text{ are Markovian strategies}\} \subset \mathcal{I}_2.$$

Definition 2.3.4 *(Markovian Nash equilibrium) We say that $\bar{I}^* = (I_1^*, I_2^*) \in \mathcal{I}_2^M$ is a Markovian Nash equilibrium if and only if for every $x \in \mathbb{R}$ and $\bar{y} = (y_1, y_2) \in [0, \theta] \times [0, \theta]$, we have*

$$|\mathcal{J}_i(x, \bar{y}, \bar{I}^*)| < \infty \ , \ i = 1, 2$$

and

$$\begin{cases} \mathcal{J}_1(x, \bar{y}, I_1^*, I_2^*) \geq \mathcal{J}_1(x, \bar{y}, I_1, I_2^*) \text{ for any } I_1, \text{ such that } (I_1, I_2^*) \in \mathcal{I}_2^M, \\ \mathcal{J}_2(x, \bar{y}, I_1^*, I_2^*) \geq \mathcal{J}_2(x, \bar{y}, I_1^*, I_2) \text{ for any } I_2, \text{ such that } (I_1^*, I_2) \in \mathcal{I}_2^M. \end{cases} \tag{2.13}$$

The value function corresponding to the Nash equilibrium for each player i is defined as

$$V_i(x, \bar{y}) := \mathcal{J}(x, \bar{y}, \bar{I}^*).$$

We derive the HJB equation following this heuristic argument: by the Markovian structure it is enough to observe the case at time $t = 0$. For agent i, it can decide to do not increase the current level of installed power and also player j, i.e., the strategy is $\bar{I} = \bar{I}^0 \equiv (0, 0)$ and both continue optimally. In this case, the control problem reduces to the single player case and we have

$$V_i(x, \bar{y}) \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-\rho s} a S^{x, \bar{y}, \bar{I}^0}(s) y_i ds + e^{-\rho \Delta t} V_i(S^{x, \bar{y}, \bar{I}^0}(\Delta t), \bar{y})\right] \ ,$$

leading to

$$\mathcal{L}^{\bar{y}} V_i(x, \bar{y}) - \rho V_i(x, \bar{y}) + a x y_i \leq 0 \ ,$$

with $\mathcal{L}^{\bar{y}}$ the differential operator defined by

$$\mathcal{L}^{\bar{y}}u(x,\bar{y}) = \sigma\frac{\partial^2 u(x,\bar{y})}{\partial x^2} + \kappa\left(\zeta - x - \beta\sum_{i=1}^{2} y_i\right)\frac{\partial u(x,\bar{y})}{\partial x}.$$

Conversely, player i can decide to increase its level by ϵ while player j does not increase its level, then both continue optimally, which is associated with

$$V_i(x,\bar{y}) \geq V_i(x,\bar{y}+e_i\epsilon) - c\epsilon,$$

where e_i is the canonical vector in the direction i. Dividing by ϵ and $\epsilon \downarrow 0$, we get

$$0 \geq \frac{\partial V_i(x,\bar{y})}{\partial y_i} - c.$$

Let us assume instead that player i decides to not increase its level while player j increases its level. By definition of Nash equilibrium, player i is not expected to suffer a loss, therefore

$$V_i(x,\bar{y}) \geq V_i(x,\bar{y}+e_j\epsilon),$$

where e_j is the canonical vector in the direction j. Dividing the above expression by ϵ and letting $\epsilon \downarrow 0$, we obtain

$$\frac{\partial V_i(x,\bar{y})}{\partial y_j} \leq 0.$$

Finally, if instead both players decide to increase their level by ϵ and continue optimally, this is associated with

$$V_i(x,\bar{y}) \geq V_i(x,\bar{y}+(1,1)\epsilon) - c\epsilon, \tag{2.14}$$

dividing by ϵ and $\epsilon \downarrow 0$, we get

$$0 \geq \frac{\partial V_i(x,\bar{y})}{\partial y_i} + \frac{\partial V_i(x,\bar{y})}{\partial y_j} - c. \tag{2.15}$$

The above arguments suggest that the value function of player $i = 1,2$, $V_i(x,\bar{y})$ should be identified with a solution of the following variational inequality

$$\begin{cases} \max\left\{\mathcal{L}^{\bar{y}}w_i(x,\bar{y}) - \rho w_i(x,\bar{y}) + axy_i, \frac{\partial w_i(x,\bar{y})}{\partial y_i} - c\right\} = 0, & (x,\bar{y}) \in \mathbb{W}_j \\ \max\left\{\frac{\partial w_i}{\partial y_j}, \sum_{k=1}^{2}\frac{\partial w_i(x,\bar{y})}{\partial y_k} - c\right\} = 0, & (x,\bar{y}) \in \mathbb{I}_j \end{cases} \tag{2.16}$$

with $i \neq j$ and with the boundary condition $w_i(x,\bar{y}) = Q_i(x,\bar{y})$ whenever $\sum_{i=1}^{2} y_i = \theta$, where

$$\begin{aligned} Q_i(x,\bar{y}) &:= \mathcal{J}_i(x,\bar{y},\bar{I}^0) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} a S^{x,\bar{y},\bar{I}^0}(s) y_i ds\right] \\ &= \frac{axy_i}{\rho+\kappa} + \frac{a\zeta\kappa y_i}{\rho(\rho+\kappa)} - \frac{a\kappa\beta y_i \sum_{i=1}^2 y_i}{\rho(\rho+\kappa)}. \end{aligned}$$

With reference to (2.16), we introduce the waiting and installation regions, for Markovian Nash equilibria, defined as follows [37].

Definition 2.3.5 *(Installation and waiting regions) The installation region of player i is defined as the set of points $\mathbb{I}_i \subseteq \mathbb{R}\times[0,\theta]^2$ such that $dI_i^*(t) \neq 0$ if and only if $(X(t), Y_1(t-), Y_2(t-)) \in \mathbb{I}_i$, and its waiting region as $\mathbb{W}_i = \mathbb{I}_i^c$.*

Remark 2.3.6 *We point out that, in stochastic singular games, the usual framework is that a player can act only when the other ones are idle, i.e. $\mathbb{I}_i \cap \mathbb{I}_j = \emptyset$ for all $i \neq j$, see e.g. [25, 26, 37, 38]. Here instead the variational inequality* (2.16)*, and the argument before it, takes explicitly into consideration the possibility that both players acts (i.e. install) simultaneously. This possibility will be confirmed in Section 5.3, where the presented Nash equilibrium will even have both players acting and waiting simultaneously, i.e. $\mathbb{I}_i = \mathbb{I}_j$.*

Now we establish a verification theorem for the value function.

Theorem 2.3.7 *(Verification theorem) For any $i = 1,2$, suppose $\bar{I}^* \in \mathcal{I}_2^M$, the corresponding $w^i(\cdot) = \mathcal{J}(\cdot;\bar{I}^*)$ satisfies the following:*

(i) $w_i \in C^0(\mathbb{R}\times[0,\theta]^2) \cap C^{2,1,1}(\mathbb{W}_j)$, with $j \neq i$;

(ii) w_i satisfies the growth condition

$$|w_i(x,y_1,y_2)| \leq K(1+|x|); \tag{2.17}$$

(iii) w_i satisfies Equation (2.16)*, with $i \neq j$, with the boundary condition $w_i(x,\bar{y}) = Q_i(x,\bar{y})$, whenever $\sum_{i=1}^2 y_i = \theta$;*

then $\bar{I}^*$ is a Nash equilibrium with value function w_i for each player $i = 1,2$.

Remark 2.3.8 *Differently from the one-player case, where the value function is required to be of class C^2 (or at least smooth enough for the Ito formula to be applied) in the whole domain, here each candidate value function w_i is required to be smooth only in the continuation region $\mathbb{W}_j$ of the other player as, under a Nash equilibrium, the state will not exit from there. In fact, player j will not deviate from I_j^*, thus making $\mathbb{I}_j$ inaccessible: for this reason, player i will be allowed to change its controls only in $\mathbb{W}_j$. This is analogous with other results on singular control games based on variational inequalities, see e.g. [26, 37, 38]*

Proof. Let $(x,\bar{y}) \in \mathbb{R}\times[0,\theta)^2$ be given and fixed, and $(I_i, I_j^*) = \bar{I} \in \mathcal{I}_2^M$. Denote by $\Delta I^i(s) = I_i(s) - I_i(s-)$ and I_i^c the continuous part of the strategy I. Define $\tau_{R,N} := \tau_R \wedge N$, where $\tau_R = \inf\{s > 0 : S^{x,\bar{y}} \notin (-R,R)\}$. Applying the Ito formula to $e^{-\rho\tau_{R,N}} w_i(S^{x,\bar{y}}(\tau_{R,N}), Y_i(\tau_{R,N}), Y_j^*(\tau_{R,N}))$ we have

$$e^{-\rho\tau_{R,N}} w_i(S^{x,\bar{y},\bar{I}}(\tau_{R,N}), Y_i(\tau_{R,N}), Y_j^*(\tau_{R,N})) - w_i(x, y_i, y_j) = \tag{2.18}$$

$$\int_0^{\tau_{R,N}} \Big(-\rho e^{-\rho s} w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s)) + e^{-\rho s}\mathcal{L}^{\bar{y}} w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))\Big)\, ds \tag{2.19}$$

$$+ \int_0^{\tau_{R,N}} \sigma \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial x} dW(s)$$

$$+ \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_i} dI_i^c(s) + \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_j} dI_j^{*c}(s)$$

$$+ \sum_{0\le s\le \tau_{R,N}} e^{-\rho s} \Big[w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s)) - w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s-), Y_j^*(s-))\Big]. \tag{2.20}$$

Set $\Delta Y_k(s) = Y_k(s) - Y_k(s-)$, $k = 1,2$ and notice that

$$\begin{aligned} &w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s)) - w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s-), Y_j^*(s-)) \\ &= \int_0^1 \left[\frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_i}\Delta Y_i(u) + \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_j}\Delta Y_j(u)\right] du. \end{aligned}$$

Considering the above expression, taking expectation in (2.20), observing that the process

$$\left(\int_0^{\tau} \sigma \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial x} dW(s)\right)_{\tau\ge 0}$$

is a martingale and using assumptions (ii), we have

$$\begin{aligned} &w_i(x, y_i, y_j) + K\mathbb{E}\left[e^{-\rho\tau_{R,N}}\Big(1 + |S^{x,\bar{y},\bar{I}}(\tau)|\Big)\right] \ge \\ &= \mathbb{E}\Bigg[\int_0^{\tau_{R,N}} \Big(\rho e^{-\rho s} w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s)) - e^{-\rho s}\mathcal{L}^{\bar{y}} w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))\Big)\, ds \\ &- \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_i} dI_i^c(s) - \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y}}(s), Y_i(s), Y_j^*(s))}{\partial y_j} dI_j^{*c}(s) \\ &- \sum_{0\le s\le \tau_{R,N}} e^{-\rho s} \int_0^1 \left[\frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_i}\Delta Y_i(u) + \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_j}\Delta Y_j(u)\right] du\Bigg]. \end{aligned}$$

Using the variational equation of assumption (iii), we get

$$
\begin{aligned}
& w_i(x, y_i, y_j) + K\mathbb{E}\left[e^{-\rho\tau_{R,N}}\left(1 + |S^{x,\bar{y},\bar{I}}(\tau)|\right)\right] \\
& \geq \mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s} a S^{x,\bar{y},\bar{I}}(s) Y_i(s) ds \right. \\
& - \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_i} dI_i^c(s) - \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_j} dI_j^{*c}(s) \\
& \left. - \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^1 \left[\frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_i} \Delta Y_i(u) + \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_j} \Delta Y_j(u)\right] du \right] \\
& \geq \mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s} a S^{x,\bar{y},\bar{I}}(s) Y_i(s) ds - \int_0^{\tau_{R,N}} e^{-\rho s} \frac{\partial w_i(S^{x,\bar{y},\bar{I}}(s), Y_i(s), Y_j^*(s))}{\partial y_i} dI_i^c(s) \right. \\
& \left. - \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^1 \left[\frac{\partial w_i(S^{x,\bar{y},\bar{I}}(u), Y_i(u), Y_j^*(u))}{\partial y_i} \Delta Y_i(u)\right] du \right] \\
& \geq \mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s} a S^{x,\bar{y},\bar{I}}(s) Y_i(s) ds - c \int_0^{\tau_{R,N}} e^{-\rho s} dI_i(s)\right].
\end{aligned}
$$

We can apply the dominated convergence theorem in the last expression since (see proof [50, Theorem 3.2] for the computations of the following estimates)

$$
\mathbb{E}\left[\int_0^{\tau} e^{-\rho s} a S^{x,\bar{y},\bar{I}}(s) Y_i(s) ds - c \int_0^{\tau} e^{-\rho s} dI_i(s)\right] \leq \theta \int_0^{\infty} e^{-\rho s} \left(|S^{x,\bar{y},,\bar{I}^0}(s)| + \kappa\beta\theta s\right) ds + c\theta
$$

and

$$
\mathbb{E}\left[e^{-\rho\tau_{R,N}}\left(1 + |S^{x,\bar{y},\bar{I}}(\tau_{R,N})|\right)\right] \leq C_1 \mathbb{E}\left[e^{-\rho\tau_{R,N}}(1 + \tau_{R,N})\right] + C_3 \mathbb{E}\left[e^{-\rho\tau_{R,N}}\right]^{1/2} (1 + x^2). \quad (2.21)
$$

Letting $N \uparrow \infty$ and $R \uparrow \infty$, we get

$$
\mathcal{J}(x, \bar{y}, I_i, I_j^*) \leq w_i(x, \bar{y}),
$$

for all I_i such that $(I_i, I_j^*) \in \mathcal{I}_2^M$, therefore $\bar{I}^*$ is a Markovian Nash equilibrium.

■

Remark 2.3.9 *It is also possible to suppose that players i and j increase their installation not equally but at different rates, λ and $(1-\lambda)$ respectively, with $\lambda \in [0,1]$. In this case the variational inequality for player i change when $(x, \bar{y}) \in \mathbb{I}_j$, becoming*

$$
\begin{cases}
\max\left\{\mathcal{L}^{\bar{y}} w_i(x, \bar{y}) - \rho w_i(x, \bar{y}) + axy_i, \frac{\partial w_i(x,\bar{y})}{\partial y_i} - c\right\} = 0, & (x, \bar{y}) \in \mathbb{W}_j \\
\max\left\{\frac{\partial w_i}{\partial y_j}, \lambda \frac{\partial w_i(x,\bar{y})}{\partial y_i} + (1-\lambda)\frac{\partial w_i(x,\bar{y})}{\partial y_j} - \lambda c\right\} = 0, & (x, \bar{y}) \in \mathbb{I}_j \ .
\end{cases} \quad (2.22)
$$

The verification theorem in this case is analogous to the case that we already saw, corresponding to equal rate of installing $\lambda = \frac{1}{2}$*, therefore we omit any proof.*

2.3.3 The case $\beta = 0$: comparison between Pareto optimum and Nash equilibrium

While a complete characterization of Nash equilibria in the general case appears to be technically very challenging and is beyond the scope of this article, here we analyze the case without price impact, i.e. with $\beta = 0$. Inspired by the one-player optimal control and by the N-players Pareto optima, we search for a Nash equilibrium $\bar{I}^*$ where the players, which have initial installation equal to $(Y_1(0), Y_2(0)) = (y_1, y_2)$, wait until the price surpasses a boundary x^* to be determined, and then they make together a cumulative installation which completely saturates the total capacity θ. Following the arguments of the previous subsection, we assume that they share equally this additional installation.

More in detail, we define

$$\tau^* := \inf\{t \geq 0 \mid S(t) \geq x^*\} \tag{2.23}$$

and describe the Nash equilibrium $\bar{I}^*$ as

$$\bar{I}^*(t) := \frac{1}{2}(\theta - y_1 - y_2)(1,1)\mathbf{1}_{t \geq \tau^*}. \tag{2.24}$$

Obviously, in this case $\mathbb{I}_1 = \mathbb{I}_2 = (x^*, +\infty) \times [0, \theta]^2$. For each player $i = 1, 2$, the value function which corresponds to this strategy can be computed as follows:

$$\begin{aligned} w_i(x, \bar{y}) &= \mathbb{E}\left[\int_0^{\tau^*} ae^{-\rho s} S^{x,\bar{y}}(s) y_i ds + \int_{\tau^*}^{\infty} ae^{-\rho s} S^{x,\bar{y}}(s)\left(y_i + \frac{\theta - y_i - y_j}{2}\right) ds - \frac{ce^{-\rho\tau^*}(\theta - y_i - y_j)}{2}\right] \\ &= Q_i(x, \bar{y}) + \frac{1}{2}\mathbb{E}\left[e^{-\rho\tau^*}\int_0^{\infty} ae^{-\rho s} S^{x,\bar{y}}(\tau^* + s)(\theta - y_i - y_j) ds - ce^{-\rho\tau^*}(\theta - y_i - y_j)\right] \\ &= Q_i(x, \bar{y}) + \frac{1}{2}\mathbb{E}\left[e^{-\rho\tau^*} Q_i(S^{x,\bar{y}}(\tau^*), \theta - y_i - y_j, y_j) - ce^{-\rho\tau^*}(\theta - y_i - y_j)\right] \end{aligned}$$

where in the last equality we use the strong Markov property for the process S. Now, if $x < x^*$, then $\tau^* > 0$ and $\mathbb{E}[e^{-\rho\tau^*}] = \frac{\psi(x)}{\psi(x^*)}$, with ψ as in Equation (1.18) [14, Chapter 7.2], and

$$\begin{aligned} w_i(x, \bar{y}) &= Q_i(x, \bar{y}) + \frac{1}{2}\mathbb{E}\left[e^{-\rho\tau^*}\right](Q_i(x^*, \theta - y_i - y_j, y_j) - c(\theta - y_i - y_j)) \\ &= Q_i(x, \bar{y}) + \frac{\psi(x)}{2\psi(x^*)}(Q_i(x^*, \theta - y_i - y_j, y_j) - c(\theta - y_i - y_j)). \end{aligned}$$

Instead, when $x \geq x^*$, then $\tau^* \equiv 0$ and

$$w_i(x, \bar{y}) = Q_i(x, \bar{y}) + \frac{1}{2}(Q_i(x, \theta - y_i - y_j, y_j) - c(\theta - y_i - y_j)).$$

Therefore, for a given level x^*, the value function for the strategy (2.24) is given by

$$w_i(x,\bar{y}) = \begin{cases} Q_i(x,\bar{y}) + \frac{\psi(x)}{2\psi(x^*)}\left(Q(x^*,\theta-y_i-y_j) - c(\theta-y_i-y_j)\right), & x < x^* \\ Q_i(x,\bar{y}) + \frac{1}{2}\left(Q(x,\theta-y_i-y_j) - c(\theta-y_i-y_j)\right), & x^* \geq x \end{cases} \tag{2.25}$$

If we let $x^* := \hat{x}$ as the solution of Equation (1.20), then the corresponding strategy is one of the Pareto optima found in Lemma 2.3.1. However, if we plug the candidate value functions of Equation (2.25) into the variational inequality (2.16), it turns out that this choice does *not* give a Nash equilibrium. Instead, a Nash equilibrium is achieved when we let

$$x^* := \bar{c} = \frac{c(\rho+\kappa)}{a} - \frac{\xi\kappa}{\rho} \tag{2.26}$$

Proposition 2.3.10 *If $x^* = \bar{c}$ defined in Equation* (2.26)*, then the strategy* (2.24) *is a Nash equilibrium and the value function for player $i = 1,2$ is given by* (2.25).

Proof. The function $w_i \in C^0(\mathbb{R}\times[0,\theta]^2)\cap C^{2,1,1}(\mathbb{W}_j)$ by direct computations and it has linear growth by [50, Theorem 3.2, Lemma 4.6]. Let us check that it satisfies the variational inequality (2.16). First of all, the boundary condition $w_i(x,y_i,y_j) = R(x,y_i+y_j)$ whenever $y_i+y_j=\theta$ is fulfilled by direct computations.

Then, for player $i=1,2$, in order to verify the variational inequality (2.16), we distinguish two cases.

Case 1: For player i, $(x,\bar{y})\in\mathbb{W}_j$. In this case we also have $(x,\bar{y})\in\mathbb{W}_i$ and $x < x^* = \bar{c}$. We expect w_i satisfies $\mathcal{L}^{\bar{y}}w_i - \rho w_i + axy_i = 0$: in fact,

$$\begin{aligned}
\mathcal{L}^{\bar{y}}w_i(x,\bar{y}) - \rho w_i(x,\bar{y}) + axy_i &= \mathcal{L}^{\bar{y}}(Q_i(x,\bar{y}) + \frac{\psi(x)}{2\psi(x^*)}(Q(x^*,y_i+y_j) - c(\theta-y_i-y_j))) \\
&\quad -\rho(Q_i(x,\bar{y}) + \frac{\psi(x)}{2\psi(x^*)}(Q(x^*,y_i+y_j) - c(\theta-y_i-y_j))) + axy_i \\
&= \left(\mathcal{L}^{\bar{y}} - \rho\right)Q_i(x,\bar{y}) + axy_i \\
&= \frac{a\kappa(\zeta-x)y_i}{\rho+\kappa} - \frac{\rho axy_i}{\rho+\kappa} - \frac{a\zeta\kappa y_i}{\rho+\kappa} + axy_i = 0.
\end{aligned}$$

Also, when $x < \bar{c}$ we should have $\frac{\partial w_i}{\partial y_i} - c \leq 0$, and in fact

$$\begin{aligned}
\frac{\partial w_i(x,y_i,y_j)}{\partial y_i} - c &= \frac{a}{\rho+\kappa}\left(x + \frac{\xi\kappa}{\rho}\right) + \frac{\psi(x)}{2\psi(x^*)}\left(-\frac{a}{\rho+\kappa}\left(x^* + \frac{\xi\kappa}{\rho}\right) + c\right) - c \\
&\leq \left(\frac{a}{\rho+\kappa}\left(x + \frac{\xi\kappa}{\rho}\right) - c\right)\left(1 - \frac{\psi(x)}{2\psi(x^*)}\right) = \\
&= \frac{a}{\rho+\kappa}(x-\bar{c})\left(1 - \frac{\psi(x)}{2\psi(x^*)}\right) < 0
\end{aligned}$$

as ψ is strictly increasing.

Case 2: For player i, when $(x, \bar{y}) \in \mathbb{I}_j$ then also $(x, \bar{y}) \in \mathbb{I}_i$. We expect $\frac{\partial w_i(x,y_i,y_j)}{\partial y_i} + \frac{\partial w_i(x,y_i,y_j)}{\partial y_j} - c = 0$: in fact,

$$\begin{aligned}\sum_{k=i,j} \frac{\partial w_i(x, y_i, y_j)}{\partial y_k} - c &= \frac{\partial (Q_i(x, y_i, y_j) - Q(x, \theta - y_i - y_j)/2)}{\partial y_i} + \frac{c}{2} + \\ &+ \frac{\partial (Q_i(x, y_i, y_j) - Q(x, \theta - y_i - y_j)/2)}{\partial y_j} + \frac{c}{2} - c = \\ &= \frac{ax}{(\rho+\kappa)} + \frac{a\zeta\kappa}{\rho(\rho+\kappa)} - \frac{ax}{(\rho+\kappa)} - \frac{a\zeta\kappa}{\rho(\rho+\kappa)} = 0.\end{aligned}$$

On the other hand, when $x \geq \bar{c}$ we also expect that $\frac{\partial w_i(x,y_i,y_j)}{\partial y_j} \leq 0$: in fact,

$$\frac{\partial w_i(x, y_i, y_j)}{\partial y_j} = \frac{1}{2}\left(-\frac{ax}{\rho+\kappa} - \frac{a\zeta\kappa}{\rho(\rho+\kappa)} + c\right) = \frac{a}{2(\rho+\kappa)}(\bar{c} - x) \leq 0$$

■

Remark 2.3.11 *Since, after Remark 1.2.1 in Chapter 1, we have $\bar{c} < \hat{x}$, this means that the search for a Nash equilibrium induces the agents to perform an earlier installation with respect to the cooperative behavior of the Pareto optimum seen in the previous section. This phenomenon is the converse of the one observed in [25], where instead the Nash equilibrium's action regions are contained in the Pareto optima's ones, i.e. agents wait more under the Nash equilibrium than under the Pareto optimum. By continuity, we expect a similar behavior also for the case $\beta > 0$, at least for low values of β: in other words, also in the case when price impact is present, competitive Nash equilibria will induce players to install earlier than when they would install under a cooperative Pareto optimum. We reserve to investigate this topic furtherly in future research.*

Remark 2.3.12 *If we consider that both player install at different rates as in remark 2.3.9, we should describe the Nash equilibrium as*

$$\bar{I}^*(t) = (\theta - y_1 - y_2)(\lambda, (1-\lambda))\mathbb{1}_{t \geq \tau^*} \tag{2.27}$$

where τ^* is defined as in (2.23)*. The value function for this strategy and proof that* (2.27) *is a Nash equilibria are analogous to the case presented above and therefore we omit any proof. Moreover, the critical price x^* is the same as in the case of equal installation rates.*

2.4 Numerical verification

In this section we solve numerically Equation (1.17), using the parameters' values estimated in Section 3 for the North, Central North and Sardinia zones.

Following the spirit of Section 2.3.1, we treat the pool of producers in each zone as a coalition maximizing the cumulative payoff and thus realizing a Pareto optimum. We choose not to report results about Nash equilibria, as the analysis in Sections 2.3.2 and 2.3.3 contains only partial results; however, after Remark 2.3.11, we expect that a free boundary relative to a Nash equilibrium would always be located on the left of the Pareto optimum, which instead we explicitly describe below.

Recall from Table 2.3 that the price impact in the North zone is due to photovoltaic power production, while in Sardinia is due to wind power production. Both are cases when the parameter impact is $\beta > 0$, which we describe in Section 1.2, Chapter 1. On the other hand, Central North has not price impact from power production (at least from these two renewable sources), so here we are in the case $\beta = 0$ described also in Section 1.2, Chapter 1.

The parameters c and a presented in (1.16) should be selected according to the type of renewable energy which has an impact on the corresponding price zone. In the case of photovoltaic power we consider a yearly average of the installation cost of 1 MW of the prices available in the market, see e.g. [58]. On the other hand, for the wind power installation cost we consider the invested money on an offshore wind park that is being installed in Sardinia [52]. In both cases we adjust for the presence of government incentives for renewable energy installation (usually under the form of tax benefits), therefore we consider around a 40% and a 60% of the real investment cost c of photovoltaic and wind power, respectively, for our numerical simulation. The parameter a is the effective power produced during a representative year: as we consider a yearly scale for simulation, the parameter a will convert our weekly data of produced power into yearly effective produced power. Additionally, the a value depends on the type of produced power. This information is available e.g. in [28, Chapter 4]. The parameter ρ is the discount factor for the electricity market and is the same in the three cases: no impact, photovoltaic and wind power impact. The parameter θ is the effective power that can be produced considering the real installed power of the respective type of energy. In the case of the estimated parameters κ, ζ, β and σ, we choose a value from the 95% confidence interval, based on better heuristic numerical performance simulation criteria.

We summarize in Table 2.4 the parameters considered for the numerical simulations.

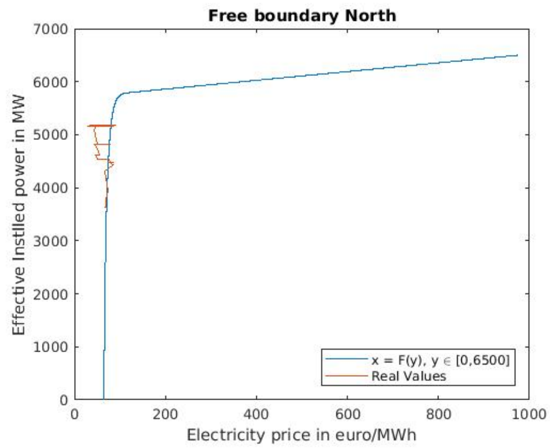
Zone	Parameters' values							
	κ	ζ	β	σ	c	a	θ	ρ
North	6.7	124.7	0.0091	47.7	290000	1400	6500	0.1
Central North	5.6029	50.2381	0	58.9796	290000	1400	6500	0.1
Sardinia	13.213	115.1565	0.0091	68.2889	1944400	7508	5700	0.1

Table 2.4: Parameter values used for the North, Central North and Sardinia zones.

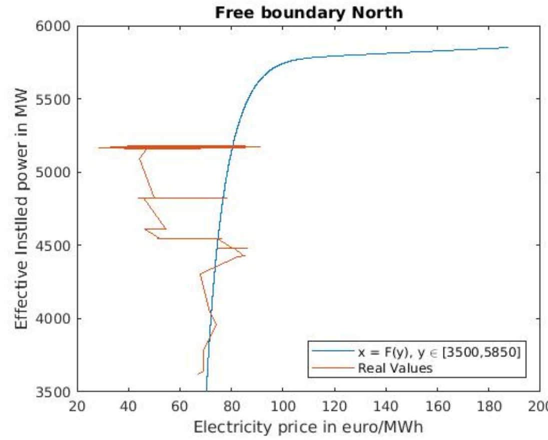
For the Central North case, we consider the cost of photovoltaic installation, because it is the main renewable energy produced in this zone.

2.4.1 North

We solved the ordinary differential equation (1.17) using the data in Table 2.4 for the North, using the backward Euler scheme with step $h = 0.5$ and initial condition $\hat{F}(\theta) = 976.4$ €/MWh, which was obtained by solving Equation (1.20) with the bisection method considering as initial points the extremes of the interval on Remark 1.2.2. The graph of the solution for the free boundary $F(y) = x$ is presented in Figure 2.1a, with a detail on realized power prices in Figure 2.1b.



(a) Simulated free boundary and real data for the North



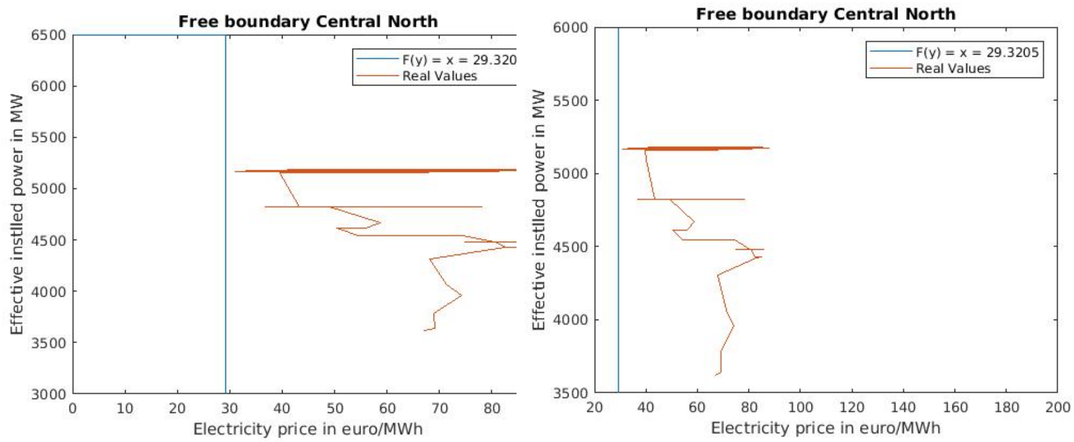
(b) Detail of free boundary and real data for the North

In Figure 2.1a, the point at zero installation level corresponds to $F(0) = 64.9$ €/MWh. The red irregular line corresponds to the realized trajectory $t \to (X(t), Y(t))$, i.e. to the values of electricity price vs effective photovoltaic installed power in the North: from it we can see that, at the beginning of the observation period (2012), the installed power[2] was already around 3600 MW. Instead, the blue smooth line corresponds to the computed free boundary $F(y) = x$, which expresses the optimal installation strategy in the following sense: when the electricity price $S^x(t)$ is lower than $F(Y(t))$, i.e. when we are in the waiting region (see (1.13)), no installation should be done and it is necessary to wait until the price $S^x(t)$ crosses $F(Y(t))$ to optimally increase the installed power level. When the electricity price $S^x(t)$ is between $F(0)$ and $F(\theta)$, enough power should be installed to move the pair price-installation in the up-direction until reaching the free boundary F. In the extreme case when $S^x(t) \geq F(\theta)$ the energy producer should install instantaneously the maximum allowed power θ. In the detailed Figure 2.1b we can observe the strategy followed in the North zone: the installation level from 3500 MW until 4500 MW was approximately optimal, in the sense that the pair price-installed power was around the free boundary F, with possibly some missed gain opportunities when, between 4300 and 4500 MW, the price was deep into the installation region; nevertheless, the rise in renewable installation from 4500 MW to 4800 MW was at the end done with a power price which resulted lower than what should be the optimal one. At around 4800 MW, there was an optimal no installation procedure until the price entered again the installation region: again, the consequent installation strategy was executed with some delay, resulting in a non-optimal strategy. At the end of the installation (around 5200 MW), we can see that the pair price-installed power moved again deep into the installation region: we should then expect an increment in installation.

[2] recall that Y is really just an estimation of the installed power, which is officially given with yearly granularity; moreover, Y is expressed in units of rated power, i.e. in production equivalent to a power plant always producing the power Y

2.4.2 Central North

In this case we do not have price impact, hence the constant free boundary $F(y) \equiv \bar{x}$ was obtained solving Equation (1.20). As before, we used the bisection method considering as initial points the extremes of the interval described in Remark 1.2.2. The obtained value is $F(y) = \bar{x} = 29.3205$ €/MWh.



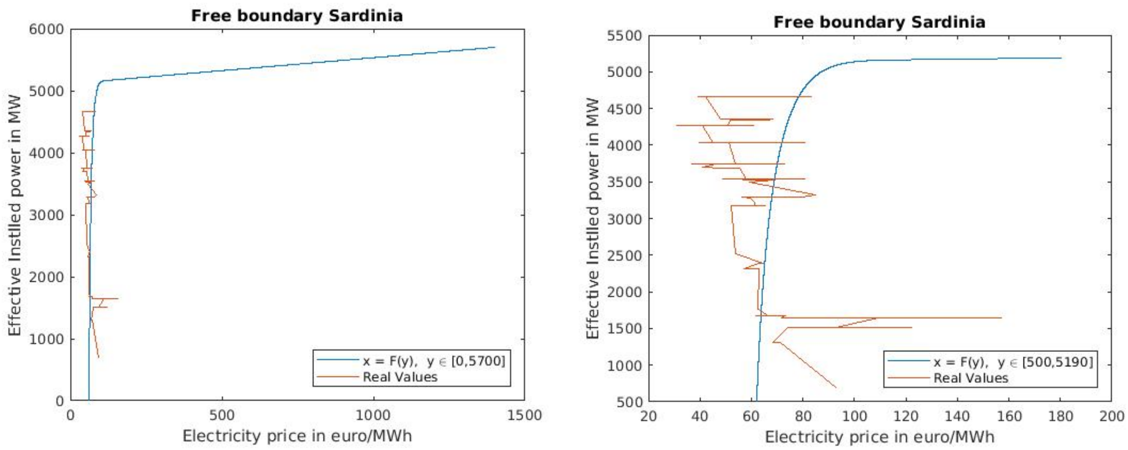
(a) Simulated free boundary and real data for Central North

(b) Detail of free boundary and real data for Central North

In Figure 2.2a the vertical blue line corresponds to the constant free boundary $\bar{x} = 29.3205$ €/MWh, while the red irregular line with the realized values of price-installation action that was put in place in the Central North zone. In this case, the optimal strategy is described as follows: for electricity prices less than $\bar{x}$, no increments on the installation level should be done. Conversely, when the electricity price is grater or equal to $\bar{x}$ the producer should increment the installation level up to the maximum level allowed for photovoltaic power (here we posed $\theta = 6500$ MW). As we can clearly see on Figure 2.2a, the electricity price has always been greater than $\bar{x}$ in the observation period; however, the increments on the installation level was not high enough to arrive to the maximum level $\theta = 6500$ MW, therefore the performed installation was not optimal.

2.4.3 Sardinia

As in the North case, we solved the differential equation (1.17) using the data in Table 2.4 for Sardinia, using the backward Euler scheme with step $h = 0.2$ and initial condition $\hat{F}(\theta) = 1453.3$ €/MWh, which was obtained by solving Equation (1.20) using the bisection method and considering as initial points the extremes of the interval in Remark 1.2.2. The graph of the solution for the free boundary $F(y) = x$ is presented in Figure 2.3a.



(a) Simulated free boundary and real data for Sardinia (b) Detail free boundary and real data for Sardinia

In Figure 2.3a the point at zero installation level corresponds to $F(0) = 61.5199$ €/MWh . The red irregular line corresponds with the realized values of electricity price vs effective wind installed power in Sardinia, from which we can see that the installed wind power at the beginning of the observation period was already around 600 MW. The blue smooth line corresponds to the simulated free boundary $F(y) = x$, which expresses the optimal installation strategy as was already explained for the North case. In the detailed Figure 2.3b we can observe the strategy followed in the Sardinia zone: until the level 1600 MW the power price was very deeply into the installation region, but the installation increments were not high enough to be optimal. Optimality came between the levels 1600 MW and 2400 MW, where the performed strategy was to effectively maintain the pair price-installed power around the free boundary F. However, the subsequent increments were not optimal, in the sense that the installed power was often increased in periods where the electricity price was too low, and in other situations the power price entered deeply in the installation region without the installed capacity following that trend, or rather doing it with some delay.

2.4.4 Discussion

We must start by saying that we did not expect optimality in the installation strategy. In fact, firstly this strategy has been carried out by very diverse market operators, including hundreds of thousands of private citizens mounting photovoltaic panels on the roof of their houses, thus not necessarily by rational agents which solved the procedure shown in Sections 4 and 5. Moreover, we must also say that renewable power plants like photovoltaic panels or wind turbines often meet irrational resistances by municipalities, especially when performed at an industrial level: more in detail, photovoltaic farms are perceived to "steal land" from agriculture (see e.g. [27]), while high wind turbines are generically perceived as "ugly" (together with many other perceived drawbacks, see the exaustive monography [22] on this).

Despite all these possible adverse effects we saw that, in the North and Sardinia price zones, part of the realized trajectory of power price and installed capacity was very near to the optimal free boundary, while in other periods the installation was put in place in moments when power price was not the optimal one — possibly, the installation was planned when the power price was high and deep into the installation region (time periods like this have been described both in the North as in Sardinia, see Sections 6.1 and 6.3) but the installation was delayed by adverse effects like e.g. the ones described above. Summarizing, in these two regions the final installation level resulting at the end of the observation period (2018) seems consistent with the price levels reached during the period.

It is instead difficult to reach such a conclusion in the Central North region: in fact, in that case the realized trajectory of power price and installed capacity was always deeply into the installation region, as the power price was always above the constant free boundary $F(y) = \bar{x}$ which resulted in this case: the optimal strategy should then have been to install immediately the maximum possible capacity. We did observe a rise in installed renewable power during the period, which was obviously not optimal in the execution time (which spanned several years), given the peculiar nature of the free boundary. However, in analogy to what already said for the North and Sardinia price zones, it is possible that the performed installation, which at the end took place during the observation period, has been planned in advance but delayed by the same adverse effects cited above.

Chapter 3

The case with no market impact

In this chapter we study the optimal social planner point of view of a stochastic irreversible investment problem under limited resources in a market with N firms, considering that the investment has not impact in the market. This problem is studied and solved in [23], where they state infinite dimensional stochastic Kuhn-Tucker conditions to prove optimality of policies.

The optimal solution for the social planner problem proposed by [23], is defined as: every firm increases its investment until it reaches the critical level $\frac{\theta}{N}$, where θ is the constraint of the problem, whenever of course, the initial condition is lower that the critical level. If the initial condition of some firms is greater than the critical level, then those firms should not invest. Nevertheless, this strategy results to be not admissible when at least one of the firms is over the critical level. We begin by constructing an admissible strategy and we use the Kuhn-Tucker conditions to prove its optimality. Our admissible optimal strategy $\bar{I}^* = (I_1^*, \ldots, I_N^*)$ differs from the one presented in [23] on the definition of the critical level. We define it in Lemma 3.2.1 on this chapter and then in Theorem 3.2.3 we show its optimality. To characterize the Lagrange multiplier of the problem it is necessary to compute the Snell envelope of the utility functional gradients and find its Doob-Meyer decomposition. It turns out that the increasing predictable process of this decomposition is the Lagrange multiplier that we need to complete the proof of optimality by using Kuhn-Tucker conditions.

Once we know the solution for the N firm problem, we move into study the asymptotic behavior of the problem. First we redifine the N firm problem using measures, but differently from works studying limit behaviors of N player problems, we do not consider probability measures, in the sense that instead of using the empirical flow of random measures $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{I_i(t)}$ we use the random measures $\hat{\mu}_t^N := \sum_{i=1}^N \delta_{I_i(t)}$ defined on the Borel sets over $[0,\theta]$. Then, we consider the cumulative measure $\hat{\nu}_t([a,b]) := \int_{[a,b]} z\hat{\mu}_t^N(dz)$ with $[a,b] \in \mathcal{B}([0,\theta])$. In Lemma 3.3.5 we study the convergence of the measures associated to the optimal strategy for the N firm problem. Afterwards, we define the mean field control version of our problem, considering an utility functional which depends on random variables defined on the space

$$\mathcal{V} := \left\{ \nu : \Omega \to D_{[0,\infty)}(\mathcal{M}) \middle| \, \forall s \geq 0, \, \nu_s \text{ is } \mathcal{F}_s\text{-measurable} \right\},$$

where

$$D_{[0,\infty)}(\mathcal{M}) := \Big\{ \mu : [0,\infty) \to \mathcal{M} | \mu_{0-} = \lambda, \; t \to \mu_t([a,b]) \text{ is a cadlag function } \forall \; 0 \leq a < b \leq \theta,$$

$$\int_{(a,b]} \frac{1}{z}\mu_t(dz) \leq \int_{(a,b]} \frac{1}{z}\mu_s(dz) \text{ for every } 0 \leq a < b \leq \theta \text{ and } s \leq t,\ \mu_s([0,\theta]) \leq \mu_t([0,\theta])\Big\}$$

and

$$\mathcal{M} := \left\{\mu : \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+ | \text{ for every } 0 \leq a \leq b \leq \theta \ ,\ \mu([a,b]) \leq \theta \ \right\}.$$

endowed with the topology of the weak convergence.

We prove existence of a solution for our mean field control problem in Theorem 3.4.2, by using an extension of Komlos' theorem to the random variables defined in our space $\mathcal{V}$.

This chapter is organized as follows: in Section 3.1 we give the general setup of the problem. In Section 3.2 we prove the optimality of a policy using the generalized stochastic Kuhn-Tucker condition developed in [23]. In Section 3.2.1 we solve two explicit cases: when the shock process is a geometric Brownian motion and the revenue of the utility functional is given by $R(x,y) = xh(y)$, with $h(y)$ a concave function; and when the shock process is an Ornstein-Uhlenbeck process and the revenue is given by $R(x,y) = e^x y^\alpha$. In Section 3.3 we formulate the problem in terms of measures in order to study the asymptotic behavior when the number of firms goes to infinity. Finally, in Section 3.4 we prove existence of the mean field optimal control and we establish if the N firms solution converge to the solution of the limit problem.

3.1 N- firms: general set up of the problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space where a one dimensional Brownian motion W is defined and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets. Consider a market with N firms. The cumulative irreversible investment of firm i, $i = 1, 2, \ldots, N$, denoted by $I_i(t)$ is an adapted, non-decreasing, cadlag process such that $I_i(0) = y_i > 0$ and $\sum_{i=1}^N I_i(t) \leq \theta$. Set $\bar{y} = (y_1, \ldots, y_N)$ as the vector of initial conditions and let us define the control set

$$\mathcal{I}^N(\bar{y}) \triangleq \Big\{\bar{I} : [0,\infty) \times \Omega \to \mathbb{R}_+^N : I \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{- adapted , } t \to I_t \text{ is nondecreasing, left continuous,}$$

$$\text{such that } I_i(0) = y_i \text{ and } \sum_{i=1}^N I_i(t) \leq \theta, \text{ for all } t \in [0,\infty)\Big\}.$$

Let $S^x(t)$ be an external shock process, progressively measurable with respect to $\mathcal{F}_t$, with initial condition $S^x(0) = x$. For an admissible strategy $\bar{I} \in \mathcal{I}^N(\bar{y})$, for each $i = 1, \ldots, N$, define the utility functional

$$\mathcal{J}_i(x, I_i) \quad = \quad \mathbb{E}\left[\int_0^\infty e^{\rho s} R(S^x(s), I_i(s))ds - c\int_0^\infty e^{-\rho s} dI_i(s)\right] \tag{3.1}$$

with $\rho > 0$ a discount factor. The function $R : \mathbb{R}^2 \to \mathbb{R}$ represents the revenue under the shock S. We aim to find an optimal strategy $\bar{I}^* \in \mathcal{I}^N(\bar{y})$ such that

$$V_{SP}(x) \quad := \quad \sup_{\bar{I}\in\mathcal{I}^N(\bar{y})} \mathcal{J}_{SP}(x,\bar{I}) = \mathcal{J}_{SP}(x,\bar{I}^*), \tag{3.2}$$

where

$$\mathcal{J}_{SP}(x,\bar{I}) \quad := \quad \sum_{i=1}^{N} \mathcal{J}_i(x,I_i) \tag{3.3}$$

This is know as the social planner problem.

The running utility R satisfies the following concavity and regularity assumptions.

Assumption 3.1.1

i. For all fixed $x\in\mathbb{R}$, the function $R(x,\cdot)$ is concave, positive and non-decreasing and with $R(x,0)=0$. Moreover, it has the continuous partial derivative $R_y(x,y)$ satisfying the Inada conditions

$$\lim_{y\to 0} R_y(x,y) = \infty, \qquad \lim_{y\to\infty} R_y(x,y) = 0$$

ii. The process $(\omega,t)\to e^{-\rho t}R(S^x(t),\theta)$ is $d\mathbb{P}\otimes dt$ integrable.

Under Assumption 3.1.1, the utilities (3.1) $\mathcal{J}_i(x,I_i)$ are well defined and finite for all admissible plans [23].

3.2 The optimal strategy

According to [23] the optimal solution for the social planner problem is

$$I_i^*(s) = \sup_{0\le u<s}\left(l(u)\wedge\hat{\beta}\right)\vee y_i \ , \ i=1,\dots,N \tag{3.4}$$

where $l(s)$ is the unique solution to the backward stochastic differential equation

$$\mathbb{E}\left[\int_\tau^\infty e^{\rho s}R_y\left(S^x(s),\sup_{u\in[\tau,s)} l(u)\right)ds\middle|\mathcal{F}_\tau\right] = ce^{-\rho\tau} \qquad \forall\tau\in\mathcal{T} \tag{3.5}$$

and $\hat{\beta}=\frac{\theta}{N}$. Nevertheless this result is true only when $y_i\le\hat{\beta}$, $\forall i\in\{1,\dots,N\}$.

Instead, if for some i, $\frac{\theta}{N}\le y_i\le\theta$, we need another definition for $\hat{\beta}$ to obtain an admissible strategy. For example, consider the model in [23, Section 5], with the simplification that all firms have the same utility functionals and consider $N=3$, $\theta=1$, $y_1=0.1$, $y_2=0.2$ and $y_3=0.5$. As in this case $l(u)$ is a geometric Brownian motion, we can select its initial condition, drift and volatility parameters such that the event $\sup_{u\in[0,s)} l(u) > 1/3$, for some $s \geq 0$, will occur with positive probability. Let us look at the behavior of the strategy (3.4) for $t \geq s$

$$I_1^*(t) = \frac{1}{3} \vee \frac{1}{10} = \frac{1}{3}\ ,\ I_2^*(t) = \frac{1}{3} \vee \frac{1}{5} = \frac{1}{3}\ ,\ I_3^*(t) = \frac{1}{3} \vee \frac{1}{2} = \frac{1}{2},$$

but $I_1^*(t) + I_2^*(t) + I_3^*(t) = \frac{7}{6} > 1 = \theta$ and therefore $\bar{I}^* = (I_1^* + I_2^* + I_3^*) \notin \mathcal{I}^3(\bar{y})$.

In our problem we are able to increment the sum of the N-agent strategy no more than $\theta - \sum_{i=1}^N y_i$. The concavity of R (and Eq. (3.4)) implies that the social planner decides to increase first those players who has lower initial conditions: in this case we can define $\hat{\beta}$ and the optimal strategy $\bar{I}^* \in \mathcal{I}^N(\bar{y})$ in the following way.

Lemma 3.2.1 *For $t \geq 0$, consider the process $\bar{I}^*$ with components defined in Equation (3.4) with $\hat{\beta} \leq \frac{\theta}{N}$. There exists a subset $M \subseteq \{1, \ldots, N\}$ and an optimal constant*

$$\hat{\beta} = \frac{\theta - \sum_{i\in M} y_i}{N - |M|},$$

such that $y_j < \hat{\beta} \leq y_i$, for all $i \in M$, $j \notin M$ and by substituting this $\hat{\beta}$ in Equation (3.4) we have

$$I_i^*(t) = \begin{cases} y_i & ,\forall i \in M \\ \left(\sup_{0\leq u<t} l(u) \wedge \hat{\beta}\right) \vee y_i & ,\forall i \notin M \end{cases}, \tag{3.6}$$

moreover $\bar{I}^* \in \mathcal{I}^N(\bar{y})$ and for all $\beta < \hat{\beta}$, $I_i(t) := \left(\sup_{0\leq u<t} l(u) \wedge \beta\right) \vee y_i \leq I_i^*(t)$.

Proof. We construct the set M iteratively. For the initial step $k = 0$, define $M_0 = \{i = 1, \ldots, N | y_i > \beta_0\}$ with $\beta_0 = \frac{\theta}{N}$ and set $\hat{\beta}_0 = \frac{\theta - \sum_{i\in M_0} y_i}{|M_0^c|}$. Observe that $\hat{\beta}_0 \leq \beta_0$:

$$\hat{\beta}_0 = \frac{\theta - \sum_{i\in M_0} y_i}{N - |M_0|} \leq \frac{\theta - |M_0|\theta/N}{N - |M_0|} = \frac{\theta}{N} = \beta_0. \tag{3.7}$$

For the subsequent step k, we distinguish two cases:

Case 1: $\forall i \in M_k^c$, $y_i \leq \hat{\beta}_k$. If $i \in M_k$, then $\sup_{0\leq u<t} l(u) \wedge \hat{\beta} \leq \hat{\beta} \leq y_i$ for all $t \geq 0$ and

$$I_i^*(t) = y_i. \tag{3.8}$$

On the other hand, if $i \in M_k^c$, then for all $t \geq 0$

$$I_i^*(t) = \left(\sup_{0\leq u<t} l(u) \wedge \hat{\beta}_k\right) \vee y_i. \tag{3.9}$$

Case 2: $\exists i \in M_k^c,\ y_i > \hat{\beta}_k$. Set $M_{k+1} = \left\{ i = 1, \dots, N | y_i > \hat{\beta}_k \right\}$, define $\hat{\beta}_{k+1} = \frac{\theta - \sum_{i \in M_{k+1}} y_i}{N - |M_{k+1}|}$ and iterate until satisfying the condition of Case 1. As $M_k \subset M_{k+1} \subseteq \{1, \dots, N\}\ \forall k$, then we will converge to case 1 in a finite number of steps $\hat{k}$.

Let then $M := M_{\hat{k}}$ and $\hat{\beta} := \hat{\beta}_{\hat{k}}$. In addition to Equations (3.8) and (3.9), we have for all $t \geq 0$

$$\begin{aligned} \sum_{i=1}^N I_i^*(t) &= \sum_{i \in M} I_i^*(t) + \sum_{i \in M^c} I_i^*(t) \\ &= \sum_{i \in M} y_i + \sum_{i \in M^c} \left(\sup_{0 \leq u < t} l(u) \wedge \hat{\beta} \right) \vee y_i \\ &\leq \sum_{i \in M} y_i + \sum_{i \in M^c} \hat{\beta} \\ &= \sum_{i \in M} y_i + \sum_{i \in M^c} \left(\frac{\theta - \sum_{i \in M} y_i}{N - |M|} \right) \\ &= \sum_{i \in M} y_i + \frac{|M^c|}{|M^c|}\theta - \frac{|M^c|}{|M^c|} \sum_{i \in M} y_i = \theta \end{aligned}$$

and therefore $\bar{I}^* \in \mathcal{I}^N(\bar{y})$. ■

To prove the optimality of the control $\bar{I}^*$ defined in (3.6) we use [23, Theorem 3.3] (see Chapter 1), which states that if there exists a non-negative random measure $\lambda(\omega, dt)$ on $\mathcal{B}([0, \infty))$ such that $\mathbb{E} \int_{[0,\infty)} d\lambda(t) < \infty$ and an admissible $\bar{I}^* \in \mathcal{I}^N(\bar{y})$ such that the following conditions

$$\nabla \mathcal{J}_i(x, I_i^*)(\tau) \leq \mathbb{E} \left[\int_\tau^\infty d\lambda(s) \bigg| \mathcal{F}_\tau \right] \ \mathbb{P}\text{-a.s } \forall \tau \in \mathcal{T}, \tag{3.10}$$

$$\int_0^\infty \left(\nabla \mathcal{J}_i(x, I_i^*)(t) - \mathbb{E} \left[\int_t^\infty d\lambda(s) \bigg| \mathcal{F}_t \right] \right) I_i^*(t) = 0 \ \mathbb{P}\text{-a.s}, \tag{3.11}$$

$$\mathbb{E} \left[\int_0^\infty \left(\theta - \sum_{i=1}^N I_i^*(t) \right) d\lambda(t) \right] = 0, \tag{3.12}$$

are satisfied, then $\bar{I}^*$ is a solution for the social planner problem (3.2).

By [3], the utility functionals (3.1) are supported by the subgradient

$$\nabla \mathcal{J}_i(x, I_i)(t) = \mathbb{E} \left[\int_t^\infty e^{-\rho s} R_y(S^x(s), I_i(s)) ds \bigg| \mathcal{F}_t \right] - c e^{-\rho t}. \tag{3.13}$$

for $t \in [0, \infty)$ in the sense that we have

$$\mathcal{J}(x, I_i) - \mathcal{J}(x, K_i) \leq \langle \nabla \mathcal{J}(x, K_i), I_i - K_i \rangle$$

for all admissible strategies I_i, K_i.

To obtain the Lagrange multiplier measure we need to compute the Doob-Meyer decomposition of the Snell envelope of the subgradient of the utility functional at the optimum. To do so, we start proving the following lemma.

Lemma 3.2.2 *For any stopping time $\tau \in \mathcal{T}$ and $\tau \leq t$, the Snell envelope of*

$$\Psi(t) := \mathbb{E}\left[\int_t^\infty e^{-\rho s}\frac{1}{|M^c|}\sum_{i\in M^c} R_y(S^x(s), I_i^*(s))ds - ce^{-\rho t}\Big|\mathcal{F}_t\right] \tag{3.14}$$

is

$$\mathbb{S}(\bar{I}^*)_\tau := \operatorname*{ess\,sup}_{t\in[\tau,\infty)} \mathbb{E}\left[\Psi(t)\Big|\mathcal{F}_\tau\right] = \mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}\left(R_y\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right)ds\Big|\mathcal{F}_\tau\right] \tag{3.15}$$

and is a supermartingale of class (D), with $\delta(\tau) := \inf\left\{s \geq \tau : l(s) > \hat{\beta}\right\}$ and the usual convention that the infimum of an empty set is infinity.

Proof. Proceeding as in Theorem 3.1 in [3] we have for all $t \geq \tau$,

$$\begin{aligned}
\mathbb{E}\left[\Psi(t)\Big|\mathcal{F}_\tau\right] &= \mathbb{E}\left[\int_t^\infty e^{-\rho s}\frac{1}{|M^c|}\sum_{i\in M^c} R_y(S^x(s), I_i^*(s))ds - ce^{-\rho t}\Big|\mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\int_t^\infty e^{-\rho s}\frac{1}{|M^c|}\sum_{i\in M^c} R_y\left(S^x(s), \left(\sup_{0\leq u<s} l(u)\wedge\hat{\beta}\right)\vee y_i\right)ds\right. \\
&\quad \left. -\int_t^\infty e^{-\rho s}R_y\left(S^x(s), \sup_{t\leq u<s} l(u)\right)ds\Big|\mathcal{F}_\tau\right] \\
&\leq \mathbb{E}\left[\int_t^\infty e^{-\rho s}R_y\left(S^x(s), \sup_{0\leq u<s} l(u)\wedge\hat{\beta}\right)ds - \int_t^\infty e^{-\rho s}R_y\left(S^x(s), \sup_{t\leq u\leq s} l(u)\right)ds\Big|\mathcal{F}_\tau\right] \\
&\leq \mathbb{E}\left[\int_t^\infty e^{-\rho s}R_y\left(S^x(s), \sup_{0\leq u<s} l(u)\wedge\hat{\beta}\right)ds - \int_t^\infty e^{-\rho s}R_y\left(S^x(s), \sup_{\tau\leq u\leq s} l(u)\right)ds\Big|\mathcal{F}_\tau\right] \\
&\leq \mathbb{E}\left[\int_\tau^\infty e^{-\rho s}\left(R_y\left(S^x(s), \sup_{0\leq u<s} l(u)\wedge\hat{\beta}\right) - R_y\left(S^x(s), \sup_{\tau\leq u\leq s} l(u)\right)\right)\vee 0ds\Big|\mathcal{F}_\tau\right].
\end{aligned}$$

The first inequality comes from $I_i^*(s) \geq \sup_{0\leq u<s} l(u)\wedge\hat{\beta}$ for all $i = 1,\ldots,N$, while the second inequality comes from $\sup_{t\leq u<s} l(u) \leq \sup_{\tau\leq u<s} l(u)$. The third one is a direct computation. As the last expression does not depend on t anymore, we have an upper bound for the Snell envelope $\mathbb{S}(\bar{I}^*)_\tau$. In fact it coincides with this envelope since we have equality in the above estimates for

$$t = \delta(\tau) := \inf\left\{s \geq \tau : l(s) > \hat{\beta}\right\}.$$

Indeed the first inequality is an equality by definition of $\delta(\tau)$ and M^c. The second inequality becomes an equality noticing that $\delta(\tau)$ is a time of increase for l, then $\sup_{\tau\le u\le s} l(u) = \sup_{\delta(\tau)\le u\le s} l(u)$, for $s\in(\delta(\tau),\infty)$. The third equality holds since

$$R_y\left(x, \sup_{0\le u<s} l(u)\wedge\hat{\beta}\right) - R_y\left(x, \sup_{\tau\le u\le s} l(u)\right) \text{ is } \begin{cases} \le 0 \text{ for } s\in(\tau,\delta(\tau)], \\ \ge 0 \text{ for } s\in(\delta(\tau),\infty) \end{cases} \tag{3.16}$$

by definition of $\delta(\tau)$. In fact, if $s\in(\tau,\delta(\tau)]$, then

$$R_y\left(x, \sup_{0\le u<s} l(u)\wedge\hat{\beta}\right) - R_y\left(x, \sup_{\tau\le u\le s} l(u)\right) \le R_y\left(x, \sup_{0\le u<s} l(u)\right) - R_y\left(x, \sup_{\tau\le u\le s} l(u)\right) \le 0.$$

On the other hand, if $s\in(\delta(\tau),\infty)$, then

$$R_y\left(x, \sup_{0\le u<s} l(u)\wedge\hat{\beta}\right) - R_y\left(x, \sup_{\tau\le u\le s} l(u)\right) = R_y(x,\hat{\beta}) - R_y\left(x, \sup_{\tau\le u\le s} l(u)\right) \ge 0.$$

Therefore we conclude expression (3.15). Moreover, since $\mathbb{S}(\bar{I}^*)_\tau$ is a supermartingale by definition, let us check that it is of class (D): recall that a cadlag supermartingale Z is of class (D) if $Z_0 = 0$ and the collection $\{Z_T | T \text{ finite-valued stopping time }\}$ is uniformly integrable. We have for any stopping time $\tau\in\mathcal{T}$

$$\begin{aligned} |\mathbb{S}(\bar{I}^*)_\tau| &\le \mathbb{E}\left[\int_0^\infty e^{-\rho s}\frac{1}{|M^c|}\sum_{i\in M^c} R_y\left(S^x(s), \sup_{u\in[0,s)}\left(l(u)\wedge\hat{\beta}\right)\vee y^i\right) ds\Big|\mathcal{F}_\tau\right] + c \\ &\le \sum_{i\in M^c}\frac{1}{y^i\vee l(0)}\mathbb{E}\left[\int_0^\infty e^{-\rho s} R\left(S^x(s),\hat{\beta}\right) ds\Big|\mathcal{F}_\tau\right] + c, \end{aligned}$$

by concavity and definition of M^c. As the random variable $\int_0^\infty e^{-\rho s} R\left(S^x(s),\hat{\beta}\right) ds \in L^1$, then by Theorem 3.1, Chapter 2 in [56], the family $\left\{\mathbb{E}\left[\int_0^\infty e^{-\rho s} R\left(S^x(s),\hat{\beta}\right) ds\big|\mathcal{F}_t\right]\Big| t\ge 0\right\}$ is uniformly integrable and hence by Theorem 3.2, Chapter 2 in [56], the family $\left\{\mathbb{E}\left[\int_0^\infty e^{-\rho s} R\left(S^x(s),\hat{\beta}\right) ds\big|\mathcal{F}_\tau\right]\Big| \tau\in\mathcal{T}\right\}$ is uniformly integrable.

■

Theorem 3.2.3 *The process* $\bar{I}^*$ (3.6) *is optimal for the social planner problem* (3.2).

Proof. To prove optimality, it is enough to check the stochastic Kuhn-Tucker conditions for N players (3.10), (3.11) and (3.12). Take $\tau\in\mathcal{T}$ fixed and define the stopping time

$$\delta(\tau) := \inf\left\{s\ge\tau : l(s) > \hat{\beta}\right\}, \tag{3.17}$$

notice that $\delta(\tau)$ is a time of increase for $\sup_{\tau\le u<s} l(u)$, $s>\delta(\tau)$ and for every $i\in M^c\cup M$

$$I_i^*(s)\ge \sup_{\tau\le u<s} l(u) \qquad \text{for } s\in(\tau,\delta(\tau)]$$

with equality if and only if τ is a time of investment for agent $i\in M^c$.

Fix $i\in M\cup M^c$ and consider

$$\begin{aligned}
\nabla\mathcal{J}_i(x,I_i^*)(\tau) &= \mathbb{E}\left[\int_\tau^\infty e^{-\rho s}R_y(S^x(s),I_i^*(s))ds\middle|\mathcal{F}_\tau\right]-ce^{-\rho\tau}\\
&= \mathbb{E}\left[\int_\tau^{\delta(\tau)} e^{-\rho s}R_y(S^x(s),I_i^*(s))ds\middle|\mathcal{F}_\tau\right]\\
&\quad+\mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}R_y(S^x(s),I_i^*(s))ds\middle|\mathcal{F}_\tau\right]-ce^{-\rho\tau}\\
&\le \mathbb{E}\left[\int_\tau^{\delta(\tau)} e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[0,s)} l(u)\wedge\hat\beta\right)ds\middle|\mathcal{F}_\tau\right]\\
&\quad+\mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[0,s)} l(u)\wedge\hat\beta\right)ds\middle|\mathcal{F}_\tau\right]-ce^{-\rho\tau}\\
&\le \mathbb{E}\left[\int_\tau^{\delta(\tau)} e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[\tau,s)} l(u)\right)ds\middle|\mathcal{F}_\tau\right]\\
&\quad+\mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[0,s)} l(u)\wedge\hat\beta\right)ds\middle|\mathcal{F}_\tau\right]-ce^{-\rho\tau}\\
&= \mathbb{E}\left[\int_\tau^\infty e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[\tau,s)} l(u)\right)ds\middle|\mathcal{F}_\tau\right]\\
&\quad-\mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[\tau,s)} l(u)\right)ds\middle|\mathcal{F}_\tau\right]\\
&\quad+\mathbb{E}\left[\int_{\delta(\tau)}^\infty e^{-\rho s}R_y\left(S^x(s),\sup_{u\in[0,s)} l(u)\wedge\hat\beta\right)ds\middle|\mathcal{F}_\tau\right]-ce^{-\rho\tau},
\end{aligned}$$

where the second inequality follows from $I_i^*(s)\ge\sup_{u\in[0,s)} l(u)\wedge\hat\beta\ge\sup_{u\in[\tau,\delta(\tau))} l(u)\wedge\hat\beta=\sup_{u\in[\tau,\delta(\tau))} l(u)$ for $s\in(\tau,\delta(\tau)]$. The equality holds for the above estimates if and only if τ is a time of increase for I_i^*, that is, $dI_i^*(\tau)>0$: in fact if I_i^* increases then it does not remain on the initial condition, thus $I_i^*(t)=\sup_{0\le u<t} l(u)\wedge\hat\beta$ and $\sup_{u\in[0,s)} l(u)=\sup_{u\in[\tau,\delta(\tau))} l(u)$ for $s\in(\tau,\delta(\tau)]$. Notice that this is true only when $i\in M^c$. Observe also that $\sup_{\tau\le u<s} li(u)=\sup_{\delta(\tau)\le u<s} li(u)$ for $s>\delta(\tau)$ and $\mathcal{F}_\tau\subset\mathcal{F}_{\delta(\tau)}$, then the second integral in the last equality becomes

$$\begin{aligned}\mathbb{E}\left[\int_{\delta(\tau)}^{\infty} e^{-\rho s} R_y\left(S^x(s), \sup_{\tau\le u<s} l(u)\right) ds \middle| \mathcal{F}_\tau\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_{\delta(\tau)}^{\infty} e^{-\rho s} R_y\left(S^x(s), \sup_{\delta(\tau)\le u<s} l(u)\right) ds \middle| \mathcal{F}_{\delta(\tau)}\right]\middle| \mathcal{F}_\tau\right] \\ &= \mathbb{E}\left[ce^{-\rho\delta(\tau)}\middle|\mathcal{F}_\tau\right] = \mathbb{E}\left[c\int_{\delta(\tau)}^{\infty} \rho e^{-\rho s} ds \middle| \mathcal{F}_\tau\right],\end{aligned}$$

therefore,

$$\begin{aligned}\nabla\mathcal{J}_i(x, I_i^*)(\tau) &\le ce^{-\rho\tau} + \mathbb{E}\left[\int_{\delta(\tau)}^{\infty}\left(e^{-\rho s} R_y\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho e^{-\rho s}\right) ds \middle| \mathcal{F}_\tau\right] - ce^{-\rho\tau} \\ &= \mathbb{E}\left[\int_{\delta(\tau)}^{\infty} e^{-\rho s}\left(R_y\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds \middle| \mathcal{F}_\tau\right].\end{aligned}$$

By Lemma 3.2.2 we know that the last expression coincides with the Snell envelope $\mathbb{S}(\bar{I}^*)_\tau$ of the process

$$\mathbb{E}\left[\int_t^{\infty} e^{-\rho s}\frac{1}{|M^c|}\sum_{i\in M^c} R_y\left(S^x(s), I_i^*(s)\right) ds - ce^{-\rho t}\middle| \mathcal{F}_\tau\right]$$

and recalling that it is a supermartingale of class (D) it has a unique Doob-Meyer decomposition $\mathbb{S}(\bar{I}^*)_t = M(t) - A(t)$, where $M(t)$ is a martingale and $A(t)$ is an increasing predictable process. By convenience let us write

$$\mathbb{S}(\bar{I}^*)_t = \mathbb{E}\left[\int_t^{\infty} dA(s)\middle|\mathcal{F}_t\right],$$

since $\mathbb{S}(\bar{I}^*)_\infty = 0$. If for $(s,t) \subseteq \mathbb{R}_0^+$ we set as Lagrange measure $\lambda(\cdot,(s,t]) := A(t) - A(s)$, we conclude that $\bar{I}^*$ satisfies condition (3.10) for all $i \in M \cup M^c$. Also for all $i \in M^c$ condition (3.11) is satisfied. For all $i \in M$, I_i^* never grows and in fact $dI_i^* \equiv 0$ and therefore condition (3.11) is also satisfied.

Now we write explicitly the Lagrange measure λ, computing the increasing process A of the Doob-Meyer decomposition and we prove condition (3.12). For a fixed t and $\tau \ge t$ consider the process

$$\begin{aligned}\varphi(\tau) &:= \mathbb{E}\left[\int_\tau^{\infty} e^{-\rho s}\left(\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds \middle| \mathcal{F}_\tau\right] \\ &\quad + \int_t^{\tau} e^{-\rho s}\left(\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds,\end{aligned} \tag{3.18}$$

also let us define

$$\sigma(t) = \inf\left\{s > t : l(s) \leq \hat{\beta}\right\}.$$

For the stopped time $\tau \wedge \sigma(t)$ we can write

$$\varphi(\tau \wedge \sigma(t)) \quad = \quad \mathbb{S}(\bar{I}^*)_{\tau\wedge\sigma(t)} + \int_t^{\tau\wedge\sigma(t)} e^{-\rho s}\left(R_y\left(S^x(s), \sup_{u\in[0,s)} l(u) \wedge \hat{\beta}\right) - c\rho\right) ds \quad (3.19)$$

since $\delta(\tau\wedge\sigma(t)) = \tau\wedge\sigma(t)$ for all $\tau \geq t$. We have that the process $\varphi(\tau\wedge\sigma(t))$ is an $\mathcal{F}_{\tau\wedge\sigma(t)}$-martingale: in fact, using definition (3.19) we have for $t \leq v < \tau \wedge \sigma(t)$

$$\begin{aligned}
\mathbb{E}\left[\varphi(\tau\wedge\sigma(t))\Big|\mathcal{F}_v\right] &= \mathbb{E}\left[\int_{\tau\wedge\sigma(t)}^{\infty} e^{-\rho s}\left(R_y\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds\right. \\
&\quad \left. + \int_t^{\tau\wedge\sigma(t)} e^{-\rho s}\left(R_y\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds\Big|\mathcal{F}_v\right] \\
&= \mathbb{E}\left[\int_v^{\infty} e^{-\rho s}\left(\left(S^x(s), \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds\right. \\
&\quad \left. + \int_t^{v} e^{-\rho s}\left(R_y\left(S^x(s) \sup_{u\in[0,s)} l(u)\wedge\hat{\beta}\right) - c\rho\right) ds\Big|\mathcal{F}_v\right] = \varphi(v),
\end{aligned}$$

as the second integral is $\mathcal{F}_v$ - measurable. Hence $\{\varphi(\tau\wedge\sigma(t))\}_{\tau\geq t}$ is an $\mathcal{F}_{\tau\wedge\sigma(t)}$- martingale, then by [56, Chapter II, Corollary 3.6, remark 1] it is an $\mathcal{F}_\tau$-martingale.

On the other hand, the integrand in the second integral in (3.19) is equal to

$$e^{-\rho s} R_y\left(S^x(s), \hat{\beta} - c\rho\right) \quad (3.20)$$

since for all $s \in [t, \sigma(t)]$, $\hat{\beta} \geq \sup_{u\in[0,s)} l(u) \wedge \hat{\beta} \geq \sup_{u\in[t,s)} l(u) \wedge \hat{\beta} = \hat{\beta}$ and it is non negative for almost every $\omega \in \Omega$ (see Lemma 4.1 in [23] with $\theta(\omega, t) \equiv \hat{\beta}$). Hence by uniqueness of the Doob-Meyer decomposition, we may conclude

$$dA(s) = e^{-\rho s}\left[R_y\left(S^x(s), \hat{\beta}\right) - c\rho\right] ds,\ s \in [t, \sigma(t))\ ,\ t \geq 0 \quad (3.21)$$

it follows that A increases only on the set $S := \left\{s \geq 0 : l(s) > \hat{\beta}\right\}$, which is a subset of $\{s \geq 0 : \sum_{i=1}^N I_i^*(t) = \theta\}$, in fact for $\tau \in S$, $\sum_{i=1}^N I_i^*(\tau) = \sum_{i\in M^c} \hat{\beta} + \sum_{i\in M} y_i = \theta$. Therefore, setting as Lagrange measure

$$d\lambda(\cdot, s) \quad = \quad e^{-\rho s}\left[R_y\left(S^x(s), \hat{\beta}\right) - c\rho\right] \mathbb{1}_{\{l(s)>\hat{\beta}\}} ds \quad (3.22)$$

we conclude the proof. ■

3.2.1 Some specific models

In this section we compute explicitly the optimal solution for two given shock processes: geometric Brownian motion and exponential Ornstein-Uhlenbeck, and two given revenue functions: $R(x,y) = xy^\alpha$ and $R(x,y) = xh(y)$, respectively. First, we derive the solution when there is a single player and then, assuming that each component of a N-players game has the same utility functional, we write the solution for the cooperative game of the social planer problem.

Optimal strategy for geometric Brownian motion shock process

In this case we aim to model the investment on a saturated market. For example, if y is the investment on installing wind turbines, every time we should pay more and more to produced the same energy, because good wind places will be occupied and the same turbine will produce less energy in a "worse" place.

Let us consider as shock process the geometric Brownian motion $S^x(s) = xe^{\mu s+\sigma W(s)}$, which satisfies the stochastic differential equation

$$\begin{cases} dS^x(s) = \left(\mu + \frac{\sigma^2}{2}\right) S^x(s)ds + \sigma S^x(s)dW(s) \quad s > 0, \\ S^x(0) = x \end{cases} \tag{3.23}$$

where $x, \sigma > 0$ and $\mu \in \mathbb{R}$. Consider as revenue

$$R(x,y) = xh(y)$$

where $h(y)$ is a concave, non decreasing, with $h(0) = 0$ and $h'(y) > 0$ decreasing. The utility functional for every firm i takes the form

$$\mathcal{J}_i(I_i) \quad = \quad \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s)h(I_i(s))ds - c\int_0^\infty e^{-\rho s} dI_i(s)\right]. \tag{3.24}$$

In this case, $I(s)$ represents the increments on investment in wind turbines, while $h(y)$ is the energy produced with that investment. As $h'(y)$ is decreasing the more we invest, the less we will be the increment in the production.

According to [3, Theorem 3.1], when there is just one firm, i.e., $N = 1$, the unique optimal strategy which maximize (3.24) is given by

$$I^*(t) = \sup_{s\in[0,t)} \{l(s) \wedge \theta\} \vee y, \tag{3.25}$$

where the process $l(s)$ is the unique solution to the backward stochastic differential equation

$$\mathbb{E}\left[\int_\tau^\infty e^{-\rho s} S^x(s)h'\left(\sup_{u\in[\tau,s)} l(u)\right) ds \Big| \mathcal{F}_\tau\right] = e^{-\rho\tau}, \forall\tau \in \mathcal{T}. \tag{3.26}$$

Thanks to [29] we know that the process $l(s)$ is such that $l(s) = b(S^x(s))$, where b is the unique solution to the integral equation

$$\psi(x) \int_x^{\bar{x}} \left(\int_{\underline{x}}^z R_w(y, F^{-1}(z)) \psi(y) m^{'}(y) dy \right) \frac{S^{'}(z) dz}{\psi^2(z)} \quad = \quad 1, \tag{3.27}$$

where, in our case, $R_w(x, w) = xh'(w)/c$. The integration limits $\bar{x}$ and $\underline{x}$ are the end points of the domain for the geometric Brownian motion, i.e., $\bar{x} = \infty$ and $\underline{x} = 0$, $m^{'}(y)dy$ is the speed measure and $S^{'}(z)$ the scale function, both associated with the geometric Brownian motion, which are respectively [31]

$$s(dx) = S^{'}(x) dx = exp \left(- \int_1^x \frac{2}{\sigma^2 z^2} \left(\mu + \frac{\sigma^2}{2} \right) z dz \right) dx = x^{-1-2\mu/\sigma^2} dx$$

and

$$m(dx) = \frac{2dx}{\sigma^2 x^2 S'(x)} = \frac{2}{\sigma^2} x^{-1+2\mu/\sigma^2} dx.$$

The function $\psi(x)$ is the increasing solution of

$$\mathcal{L}w = \rho w,$$

where $\mathcal{L}$ is the infinitesimal generator for the geometric Brownian motion, i.e.,

$$\mathcal{L}u(x) \quad = \quad \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + x \left(\mu + \frac{\sigma^2}{2} \right) \frac{\partial u}{\partial x}.$$

In this case we have

$$\psi(x) = x^{m_1},$$

where m_1 is the positive root of

$$m^2 + \frac{2\mu}{\sigma^2} m - \frac{2\rho}{\sigma^2} = 0.$$

Then, Equation (3.27) becomes

$$x^{m_1} \int_x^{\infty} \left(\int_0^z y h'(b(z)) y^{m_1} \frac{2}{\sigma^2} y^{-1+2\mu/\sigma^2} dy \right) \frac{z^{-1-2\mu/\sigma^2} dz}{z^{2m_1}} = c$$

Observe that since h' is monotone decreasing then there exist its inverse $(h')^{-1}$. To find the function b suppose $z^{-m_1}h'(b(z)) \to 0$ as $z \to \infty$, then

$$\int_x^\infty \frac{z^{-m_1}}{m_1 + 2\mu/\sigma^2 + 1} h'(b(z))dz = c\frac{\sigma^2}{2}x^{-m_1}$$
$$-\frac{x^{-m_1}}{m_1 + 2\mu/\sigma^2 + 1} h'(b(x)) = -m_1 c\frac{\sigma^2}{2}x^{-m_1-1}$$
$$h'(b(x)) = m_1(m_1 + 2\mu/\sigma^2 + 1)c\frac{\sigma^2}{2}x^{-1}$$
$$b(x) = (h')^{-1}\left((m_1 + 2\mu/\sigma^2 + 1)c\frac{\sigma^2}{2}x^{-1}\right).$$

Therefore,

$$l(s) = (h')^{-1}\left(m_1(m_1 + 2\mu/\sigma^2 + 1)c\frac{\sigma^2}{2}e^{-\mu s - \sigma W(s)}\right). \tag{3.28}$$

By Theorem 3.2.3, the solution to (3.24) is

$$I_i^*(t) = \begin{cases} y^i & , \forall i \in M \\ \left(\sup_{0\le u<t} l(u) \wedge \hat{\beta}\right) \vee y^i & , \forall i \notin M \end{cases}, \tag{3.29}$$

with l as in (3.28); $\hat{\beta}$ and M defined in Lemma (3.2.1) as

$$\hat{\beta} = \frac{\theta - \sum_{i\in M} y^i}{N - |M|},$$

such that $y^j < \hat{\beta} \le y^i$, for all $i \in M$, $j \notin M$.

The solution of the single player case can also be obtained by the Hamilton-Jacobi-Bellman equation, where it is obtained the expression for the free boundary F. The computation by this approach are presented in the Appendix 5, Section 5.1

Optimal strategy for the Exponential Ornstein-Uhlenbeck shock process

Let us consider that the shock process influencing the utility functional is an exponential O-U $S^x(t) = e^{X^x(t)}$, where $X^x(t)$ is an O-U process. which satisfy the stochastic differential equation

$$dX(t) \quad = \quad \kappa(\zeta - X(t))dt + \sigma dW(t),$$

with $\kappa, \sigma > 0$ and $\zeta \in \mathbb{R}$.

Let us consider the revenue $R(x, y) = e^x y^\alpha$, with $\alpha \in (0, 1)$. Therefore we have the following utility function for every player $i = 1, \dots, N$

$$\mathcal{J}_i(x,y,I) = \mathbb{E}\left[\int_0^\infty e^{-\rho s}e^{X^x(s)}(Y_i^y)^\alpha(s)ds - c\int_0^\infty e^{-\rho s}dI_i(s)\right], \tag{3.30}$$

As we have seen in the previous model and in Chapter 1, Section 1.3, the optimal strategy when there is just one agent $N = 1$ is given by

$$I^*(t) = \sup_{u\in[0,t)} (l(s)\wedge\theta)\vee y,$$

where the process $l(s)$ is related with the free boundary solution by $l(s) = F^{-1}(X^x(s))$ [29] . We use the integral equation for the free boundary described in [29] to solve our control problem. In Appendix 5 we obtain the same result using the Hamilton-Jacobi-Bellman approach.

It is established in [29, Theorem 3.11] that the free boundary (F^{-1}), which characterizes the optimal solution for our problem, solves the integral equation (3.27), where, in our case, $R_w(x,w) = \alpha e^x w^{\alpha-1}/c$, with $\alpha \in (0,1)$, the function ψ is the strictly increasing fundamental solution (1.18), $\bar{x}$ and $\underline{x}$ are the end points of the domain for the O-U process, i.e., $\bar{x} = \infty$ and $\underline{x} = -\infty$, $m^{'}(y)dy$ is the speed measure and $S^{'}(z)$ the scale function, both associated with the O-U process, which are respectively [31]

$$\begin{cases} m^{'}(x)dx = \frac{2}{\sigma^2}e^{\frac{2\kappa x}{\sigma^2}(\zeta-\frac{x}{2})}dx \\ S^{'}(x) = e^{-\frac{2\kappa x}{\sigma^2}(\zeta-\frac{x}{2})} \end{cases}. \tag{3.31}$$

Then, Equation (3.27) becomes

$$\int_x^\infty \left(\int_{-\infty}^z \frac{\alpha}{c}e^y\left(F^{-1}(z)\right)^{\alpha-1}\psi(y)m^{'}(y)dy\right)\frac{S^{'}(z)dz}{\psi^2(z)} = \frac{1}{\psi(x)}, \tag{3.32}$$

as it is valid for every $x \in \mathbb{R}$, we take the derivative of (3.32) on both sides, then

$$-\left(\int_{-\infty}^x \frac{\alpha}{c}e^y\left(F^{-1}(x)\right)^{\alpha-1}\psi(y)m^{'}(y)dy\right)\frac{S^{'}(x)}{\psi^2(x)} = -\frac{\psi^{'}(x)}{\psi^2(x)}.$$

From the last equality we are able to solve for $F^{-1}(x)$,

$$F^{-1}(x) = \left(\left(\int_{-\infty}^x \frac{\alpha}{c}e^y\psi(y)m^{'}(y)dy\right)\frac{S^{'}(x)}{\psi^{'}(x)}\right)^{1/(1-\alpha)}. \tag{3.33}$$

The last expression can be written in a different way, by comparing it with the result (5.48) obtained in the appendix 5, Section 5.2. We should have

$$\left(\left(\int_{-\infty}^x \frac{\alpha}{c}e^y\psi(y)m^{'}(y)dy\right)\frac{S^{'}(x)}{\psi^{'}(x)}\right)^{1/(1-\alpha)} = \left(\frac{\alpha}{c}\left(R(x) - R^{'}(x)\frac{\psi(x)}{\psi^{'}(x)}\right)\right)^{1/(1-\alpha)},$$

hence

$$\left(\int_{-\infty}^{x} e^{y}\psi(y)m^{'}(y)dy\right) S'(x) \quad = \quad R(x)\psi^{'}(x) - R^{'}(x)\psi(x). \tag{3.34}$$

The function $R(x)$, is called the resolvent and satisfies the following relation [31]

$$R(x) \quad = \quad \mathbb{E}\left[\int_{0}^{\infty} e^{-\rho s} e^{X^{x}(s)} ds\right] = \int_{-\infty}^{\infty} e^{y} G(x,y) m^{'}(y)dy, \tag{3.35}$$

where

$$G(x,y) = W^{-1} \cdot \begin{cases} \psi(x)\phi(y) & x \leq y \\ \phi(x)\psi(y) & x \geq y \end{cases}$$

with

$$W \quad := \quad \frac{\psi^{'}(x)\phi(x) - \phi^{'}(x)\psi(x)}{S'(x)}.$$

By differentiating Equation (3.35), we have

$$R^{'} \quad = \quad \int_{-\infty}^{\infty} e^{y} G^{'}(x,y) m^{'}(y)dy,$$

where

$$G^{'}(x,y) = W^{-1} \cdot \begin{cases} \psi^{'}(x)\phi(y) & x \leq y \\ \phi^{'}(x)\psi(y) & x \geq y \end{cases} \quad .$$

Finally we can conclude that the equality (3.34) is true: in fact

$$\begin{aligned} R(x)\psi^{'}(x) - R^{'}(x)\psi(x) \quad &= \quad \left(\int_{-\infty}^{x} e^{y} W^{-1}\phi(x)\psi(y)m^{'}(y)dy + \int_{-x}^{\infty} e^{y} W^{-1}\psi(x)\phi(y)m^{'}(y)dy\right)\psi^{'}(x) \\ &\quad - \left(\int_{-\infty}^{x} e^{y} W^{-1}\phi^{'}(x)\psi(y)m^{'}(y)dy + \int_{-x}^{\infty} e^{y} W^{-1}\psi^{'}(x)\phi(y)m^{'}(y)dy\right)\psi(x) \\ &= \int_{-\infty}^{x} e^{y} W^{-1}\left(\phi(x)\psi(y)\psi^{'}(x) - \phi^{'}(x)\psi(y)\psi(x)\right) m^{'}(y)dy \\ &= \int_{-\infty}^{x} e^{y} \frac{S^{'}(x)}{\psi'(x)\phi(x) - \phi'(x)\psi(x)}\psi(y)\left(\phi(x)\psi^{'}(x) - \phi^{'}(x)\psi(x)\right) m^{'}(y)dy \\ &= \int_{-\infty}^{x} e^{y} S^{'}(x)\psi(y)m^{'}(y)dy. \end{aligned}$$

Hence, for the N- player cooperative game, the optimal solution $\bar{I}^*$ is given by Theorem 3.2.3

$$I_i^*(t) = \begin{cases} y^i & ,\forall i \in M \\ \left(\sup_{0\le u<t} l(u) \wedge \hat{\beta}\right) \vee y^i & ,\forall i \notin M \end{cases}, \tag{3.36}$$

where $l(s) = F^{-1}(X^x(s))$, with F^-1 as in (3.33); $\hat{\beta}$ and M defined in Lemma (3.2.1) as

$$\hat{\beta} = \frac{\theta - \sum_{i\in M} y^i}{N - |M|},$$

such that $y^j < \hat{\beta} \le y^i$, for all $i \in M$, $j \notin M$.

3.3 Pre-limit formulation

In this section we formulate the social planner problem studied above, but interpreting the strategy of the social planner as a random measure. For reasons that will be clear later we should change the Assumption 3.1.1 of $\lim_{y\to 0} R_y(x,y) = \infty$ and ask for differentiability at zero instead. Therefore, in this section we assume

Assumption 3.3.1

i. For all fixed $x \in \mathbb{R}$, the function $R(x,\cdot)$ is concave, positive and non-decreasing and with $R(x,0) = 0$. Moreover, it has the continuous partial derivative $R_y(x,y)$ satisfying

$$\lim_{y\to 0} R_y(x,y) < \infty, \qquad \lim_{y\to\infty} R_y(x,y) = 0$$

ii. The process $(\omega,t) \to e^{-\rho t} R(S^x(t),\theta)$ is $d\mathbb{P}\otimes dt$ integrable.

Consider the running utility $R(x,y) = xh(y)$ and the following utility functional

$$\mathcal{J}_{SP}(x,\bar{I}) = \sum_{i=1}^{N} \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) h(I_i(s))ds - c\int_0^\infty e^{-\rho s} dI_i(s)\right] \tag{3.37}$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) \sum_{i=1}^{N} h(I_i(s))ds - c\int_0^\infty e^{-\rho s} \sum_{i=1}^{N} dI_i(s)\right], \tag{3.38}$$

where S^x is a diffusion process and assume it is non negative. Recall that the social planer problem is to maximize the utility functional (3.38) over the admissible set

$$\mathcal{I}^N(\bar{y}) \triangleq \Big\{ \bar{I} : [0,\infty) \times \Omega \to \mathbb{R}_+^N, \text{ nondecreasing, cadlag , adapted process,}$$

$$\text{such that } I_i(0) = y_i \text{ and } \sum_{i=1}^{N} I_i(t) \le \theta, \mathbb{P}\text{-a.s., for all } t \in [0,\infty)\Big\},$$

Let us define for every $t \ge 0$ the flow of random measures $m_t^N : \Omega \times \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+$ as

$$m_t^N := \sum_{i=1}^{N} \delta_{I_i(t)}, \tag{3.39}$$

then we can write

$$\sum_{i=1}^{N} I_i(t) = \int_0^\infty z m_t^N(dz). \tag{3.40}$$

For $t \ge 0$ let us define the cumulative measure $\nu_t^N : \Omega \times \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+$ for $[0,b] \subseteq \mathbb{R}^+$ as

$$\nu_t^N([0,b]) := \int_{[0,b]} z m_t^N(dz) = \int_0^\infty z \mathbb{1}_{[0,b]}(z) m_t^N(dz) = \sum_{i=1}^{N} I_i(t) \mathbb{1}_{[0,b]}(I_i(t)). \tag{3.41}$$

Notice that for every $[a,b] \subseteq \mathbb{R}^+$

$$\nu_t^N((a,b]) \quad = \quad \nu_t^N([0,b]) - \nu_t^N([0,a]) = \int_{(a,b]} z m_t^N(dz).$$

Observe that we can rewrite m_t^N in terms of ν_t^N on $(0,\theta] = [0,\theta] \setminus \{0\}$ as

$$m_t^N((a,b]) = \int_{(a,b]} \frac{1}{u} \nu_t^N(du). \tag{3.42}$$

for all $0 \le a < b \le \theta$.

In addition, the admissibility conditions for the strategies $\bar{I} \in \mathcal{I}(\bar{y})$ can be rewritten for the measures m_t^N and ν_t^N as follows: the constraint condition becomes

$$\nu_t^N([a,b]) \le \theta \text{ , for every } [a,b] \subseteq \mathbb{R}^+ \text{ and } t \ge 0. \tag{3.43}$$

The non decreasing property of $\bar{I}$ can be expressed as monotonicity conditions, indicated in the following lemma.

Lemma 3.3.2 *The measures m_t^N and ν_t^N satisfy the following monotonicity condition: $\forall s < t$ and $b \le \theta$,*

$$m_t^N([0,b]) \le m_s^N([0,b]) \tag{3.44}$$

and

$$\nu_s^N([0,\theta]) \leq \nu_t^N([0,\theta]). \tag{3.45}$$

Proof. The equality in condition (3.44) is immediate if for every $i \in \{1,\dots,N\}$ we have $I_i(t) \geq I_i(s) > b$ or $b \geq I_i(t) \geq I_i(s)$. If for some $J \subset \{1,\dots,N\}$ we have $I_j(t) > b \geq I_j(s)$, with $j \in J$, then

$$\begin{aligned} m_t^N([0,b]) &= \sum_{i \notin J} \mathbb{1}_{[0,b]}(I_i(t)) + \sum_{j \in J} \mathbb{1}_{[0,b]}(I_j(t)) \\ &\leq \sum_{i \notin J} \mathbb{1}_{[0,b]}(I_i(s)) + \sum_{j \in J} \mathbb{1}_{[0,b]}(I_j(s)) \\ &= m_s^N([0,b]). \end{aligned}$$

The monotonicity condition for $\nu_s^N([0,\theta])$ is a direct consequence of the non decreasing property of the admissible processes $\bar{I}$, indeed

$$\nu_s^N([0,\theta]) = \sum_{i=1}^N I_i(s)\mathbb{1}_{[0,\theta]}(I_i(s)) \leq \sum_{i=1}^N I_i(t)\mathbb{1}_{[0,\theta]}(I_i(t)) = \nu_t^N([0,\theta]).$$

■

Remark 3.3.3 *The monotonicity property for ν_t^N is not true for an arbitrary interval $[0,b]$ with $b<\theta$. In fact, for some $J \subset \{1,\dots,N\}$, when $I_j(t) > b \geq I_j(s)$, with $j \in J$, then*

$$\nu_t^N([0,b]) = \sum_{i=1}^N I_i(t)\mathbb{1}_{[0,b]}(I_i(t)) = \sum_{i \notin J} I_i(t)\mathbb{1}_{[0,b]}(I_i(t)) + \sum_{i \in J} I_i(t)\mathbb{1}_{[0,b]}(I_i(t))$$

and

$$\sum_{i \notin J} I_i(t)\mathbb{1}_{[0,b]}(I_i(t)) \geq \sum_{i \notin J} I_i(s)\mathbb{1}_{[0,b]}(I_i(s))$$

and

$$\sum_{i \in J} I_i(t)\mathbb{1}_{[0,b]}(I_i(t)) \leq \sum_{i \in J} I_i(s)\mathbb{1}_{[0,b]}(I_i(s)).$$

Finally, we can rewrite (3.38) using (3.41) as

$$\mathcal{J}_{SP}(x,\nu^N) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) \left(\int_{[0,\theta]} \frac{h(y)}{y}\nu_s^N(dy)\right) ds - c\int_0^\infty e^{-\rho s} d\nu_s^N([0,\theta])\right], \tag{3.46}$$

where ν^N is an element of the set

$$\mathcal{V}^N(\bar{y}) := \Big\{\nu : \Omega \times [0,\infty) \times \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+ \mid \nu \to \nu_t \text{ is an adapted measure valued process },$$

$$\forall\ 0 \le a < b \le \theta \text{ and } t \ge 0,\ t \to \nu_t((a,b]) \text{ is cadlag, } \int_{(a,b]} \frac{1}{u}\nu_t(du) \le \int_{(a,b]} \frac{1}{u}\nu_s(du),\ \forall\ 0 \le s \le t,$$

$$\nu_s([0,\theta]) \le \nu_t([0,\theta])\ ,\ \nu_{0-}([a,b]) = \sum_{i=1}^{N} y_i \mathbb{1}_{[a,b]}(y_i) \text{ and } \nu_t([a,b]) \le \theta\ \forall t \ge 0 \text{ a.s } \Big\}.$$

Under Assumption 3.3.1, the functional (3.46) is well defined. In fact,

$$\int_{[0,\theta]} \frac{h(y)}{y}\nu_s^N(dy)ds < \infty$$

because as $y \to 0$, $\frac{h(y)}{y} \to h'(0)$. Then, from the bound θ of the admissible measures ν^N, it is immediate that

$$\mathcal{J}_{SP}(x, \nu^N) < \infty.$$

We look for

$$V_{SP}(x) := \sup_{\nu \in \mathcal{V}^N(\bar{y})} \mathcal{J}_{SP}(x, \nu). \tag{3.47}$$

Let us call $\hat{\nu}_t^N$ the cumulative measure associated to the optimal control $\bar{I}^*$, that is

$$\begin{aligned}\hat{\nu}_t^N([0,a]) &= \sum_{i=1}^{N} \left(\left\{ \sup_{u \in [0,t)} l(u) \wedge \hat{\beta} \right\} \vee y_i \right) \mathbb{1}_{[0,a]} \left(\left\{ \sup_{u \in [0,t)} l(u) \wedge \hat{\beta} \right\} \vee y_i \right) \\ &= \sum_{i \in M^c} \left(\left\{ \sup_{u \in [0,t)} l(u) \wedge \hat{\beta} \right\} \vee y_i \right) \mathbb{1}_{[0,a]} \left(\left\{ \sup_{u \in [0,t)} l(u) \wedge \hat{\beta} \right\} \vee y_i \right) + \sum_{i \in M} y_i \mathbb{1}_{[0,a]}(y_i) \end{aligned} \tag{3.48}$$

where M is the set of the index of initial conditions that are over a certain level $\hat{\beta}$.

So far, we know the solution for (3.47) for a finite number of agents N (Theorem 3.2.3), but what would happen in the limit case when $N \to \infty$?. Inspired by mean field control theory, we are interested in studying the limit case of this problem. We start by doing the following assumption on the initial conditions

Assumption 3.3.4 *(Initial conditions)*

i. The initial conditions y_i, $i = 1, \dots, N$ remains the same as $N \to \infty$.

ii. As $N \to \infty$, there exist an m such that all the further initial conditions y_i for $i \ge m$ are zero.

The measure $\hat{\nu}_t^N$ defined in (3.48), when N is finite depends on the constant $\hat{\beta}$ and the set M, defined in Lemma 3.2.1, which depend on N. For the constant $\hat{\beta}$ we have $\hat{\beta} \leq \frac{\theta}{N}$. Then, $\lim_{N\to\infty} \hat{\beta}^N = 0$ and therefore the set M becomes

$$M = \{i \in \mathbb{N} | y_i > 0\}. \tag{3.49}$$

In the following lemma we determine what is $\lim_{N\to} \hat{\nu}^N$.

Lemma 3.3.5 *Let ν^N, $\hat{\nu}^N \in \mathcal{V}^N(\bar{y})$, where $\hat{\nu}^N$ is the measure associated with the optimal control* (3.36) *and suppose that for some $\epsilon > 0$, $\mathbb{E}[S^x(\epsilon)] \neq c\rho/h'(0)$. Under Assumption 3.3.4, we have for all $0 < a \leq \theta$ and $t \geq 0$*

$$\limsup_{N\to\infty} \nu_t^N([0,a]) \leq \theta,$$

and or all $0 < a \leq \theta$ and $t \geq 0$

$$\hat{\nu}_t([0,a]) := \lim_{N\to\infty} \hat{\nu}_t^N([0,a]) = (\theta - C_1)\, \mathbb{1}_{[0,a]}(0) + \sum_{i\in M} y_i \mathbb{1}_{[0,a]}(y_i) \tag{3.50}$$

where M is the indexes of all the initial condition greater than zero as in (3.49) *and $C_1 = \sum_{i\in M} y_i \mathbb{1}_{[0,a]}(0)$.*

Proof. By direct computation we find that ν_t^N is bounded when $N \to \infty$, in fact for every $a \leq \theta$ and every $\bar{I} \in \mathcal{I}^N(\bar{y})$, we have

$$\nu_t^N([0,a]) = \sum_{i=1}^N I_i(t)\mathbb{1}_{[0,a]}(I_i(t)) \leq \sum_{i=1}^N I_i(t)\mathbb{1}_{[0,\theta]}(I_i(t)) = \nu_t^N([0,\theta]) \leq \theta,$$

therefore

$$\limsup_{N\to\infty} \nu_t^N([0,a]) \leq \theta.$$

Let us call $\hat{\beta}^N$ the optimal constant defined in Lemma 3.2.1 and M^N the set of initial conditions greater than $\hat{\beta}^N$. For the optimal control $\bar{I}^*$, the measure $\hat{\nu}_t^N$ is written as

$$(\hat{\nu})_t^N([0,a]) \quad = \quad \sum_{i\in(M^N)^c} \left(\left\{\sup_{u\in[0,t)} l(u) \wedge \hat{\beta}\right\} \vee y_i\right) \mathbb{1}_{[0,a]} \left(\left\{\sup_{u\in[0,t)} l(u) \wedge \hat{\beta}^N\right\} \vee y_i\right) + \sum_{i\in M^N} y_i \mathbb{1}_{[0,a]}(y_i)$$

Define $\tau^N = \inf\left\{s \geq 0 |\ \hat{\beta}^N < l(s)\right\}$. For every $t > \tau^N$, we have

$$\begin{aligned}
\nu_t^N([0,a]) &= \sum_{i\in(M^N)^c} \hat{\beta}\mathbb{1}_{[0,a]}(\hat{\beta}^N) + \sum_{i\in M^N} y_i \mathbb{1}_{[0,a]}(y_i) \\
&= \sum_{i\in(M^N)^c} \left(\frac{\theta - \sum_{j\in M^N} y_j}{|(M^N)^c|}\right)\mathbb{1}_{[0,a]}(\hat{\beta}^N) + \sum_{i\in M^N} y_i \mathbb{1}_{[0,a]}(y_i) \\
&= \sum_{i\in(M^N)^c} \left(\frac{\theta}{|M^c|}\right)\mathbb{1}_{[0,a]}(\hat{\beta}) - \sum_{i\in M^c}\sum_{j\in M} \left(\frac{y_j}{|M^c|}\right)\mathbb{1}_{[0,a]}(\hat{\beta}) + \sum_{i\in M} y_i \mathbb{1}_{[0,a]}(y_i) \\
&= \theta\mathbb{1}_{[0,a]}(\hat{\beta}) - \sum_{j\in M} y_j \mathbb{1}_{[0,a]}(\hat{\beta}) + \sum_{i\in M} y_i \mathbb{1}_{[0,a]}(y_i)
\end{aligned}$$

Observe $\hat{\beta}^N < \frac{\theta}{N}$, then $\hat{\beta}^N \to 0$ as $N \to \infty$. We suppose S^x is non negative, then $l(s) \geq 0$ for $s \geq 0$. We exclude the case $l(s) = 0$ since if there exist an interval $[0, \epsilon)$ such that $l(s) = 0$, $s \in [0, \epsilon)$ we will have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) h'(\sup_{u\in[0,s)} l(u)) ds\right] &= \mathbb{E}\left[\int_0^\epsilon e^{-\rho s} S^x(s) h'(0) ds\right] + \mathbb{E}\left[\int_\epsilon^\infty e^{-\rho s} S^x(s) h'(\sup_{u\in[\epsilon,s)} l(u)) ds\right] \\
&= h'(0)\mathbb{E}\left[\int_0^\epsilon e^{-\rho s} S^x(s) ds\right] + \mathbb{E}\left[\mathbb{E}\left[\int_\epsilon^\infty e^{-\rho s} S^x(s) h'(\sup_{u\in[\epsilon,s)} l(u)) ds \Big| \mathcal{F}_\epsilon\right]\right],
\end{aligned}$$

then $\forall \epsilon' < \epsilon$ we have

$$h'(0)\int_0^{\epsilon'} \mathbb{E}\left[e^{-\rho s} S^x(s)\right] ds + c e^{-\rho\epsilon'} = c$$

implying that $\mathbb{E}\left[S^x(\epsilon')\right] = \rho c / h'(0)$, which contradicts our hypothesis. Then $l(s) > 0$ and therefore $\tau^N \to 0$. Moreover, as $\hat{\beta}^N \to 0$ as $N \to \infty$, the set M^N becomes the set of all initial condition greater than zero. Let us consider the case $M = \emptyset$. In the limit case this means that all the initial condition are zero. It is immediate that

$$\lim_{N\to\infty} \nu_t^N([0,a]) = \theta\mathbb{1}_{[0,a]}(0). \tag{3.51}$$

Consider now the case $M \neq \emptyset$. If $|M| < \infty$, it is also immediate that $\sum_{i\in M} y_i \mathbb{1}_{[0,a]}(\hat{\beta})$ and $\sum_{i\in M} y_i \mathbb{1}_{[0,a]}(y_i)$ are some finite numbers and (3.50) holds. Due to Assumption 3.3.4 the case $|M| = \infty$ can not occur. Setting $C_1 = \sum_{j\in M} y_j$, and letting $N \to \infty$, we conclude

$$\lim_{N\to\infty} \nu_t^N([0,a]) = (\theta - C_1)\,\mathbb{1}_{[0,a]}(0) + \sum_{i\in M} y_i \mathbb{1}_{[0,a]}(y_i) \quad \text{for } t \geq 0. \tag{3.52}$$

■

Remark 3.3.6 *We change Assumption 3.1.1 for Assumption 3.3.1, because if we assume that* $\lim_{y\to 0} R_y(x,y) = \infty$*, then the integral on the functional* (3.46)

$$\int_{[0,\theta]} \frac{h(y)}{y} \nu_s(dy)$$

is not well defined. We could integrate instead on $(0,\theta]$ *and assume that the initial distribution* ν_{0-} *satisfies for every* $0 \le a < b \le \theta$*,*

$$\int_{(a,b]} \frac{1}{y} \nu_{0-}(dy) < \infty.$$

In this case, by changing the set of integration of the above integral we can consider the functional as pre-limit formulation

$$\hat{\mathcal{J}}_{SP}(x,\nu^N) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) \left(\int_{(0,\theta]} \frac{h(y)}{y} \nu_s^N(dy)\right) ds - c\int_0^\infty e^{-\rho s} d\nu_s^N([0,\theta])\right].$$

which is well defined, since the monotonicity property of ν^N *and the bound* θ *of the admissible measures* ν^N *implies*

$$\mathcal{J}_{SP}(x,\nu^N) \le \mathbb{E}\left[(\max_{y\in(0,\theta]} h(y)) \int_0^\infty e^{-\rho s} S^x(s) \left(\int_{(0,\theta]} \frac{1}{y} \nu_{0-}^N(dy)\right) ds - c\int_0^\infty e^{-\rho s} d\nu_s^N([0,\theta])\right] < \infty.$$

Nevertheless, we lose information at zero. For example, this functional will never "see" the mass at zero of the limit measure computed in Lemma 3.3.5 given by $\hat{\nu} = (\theta - C_1)\delta_{\{0\}} + \sum_{y_i\in M} y_i \delta_{\{y_i\}}$*.*

3.4 Mean field optimal control approximation

In this section we discuss the limit case of the N agents problem described above. We start defining the set of admissible strategies $\mathcal{I}^N(\bar{y})$ in terms of a random flow of measures ν_t, which we expect to relate with ν_t^N. Let us define the set $\mathcal{M}$ of finite measures, such that

$$\mathcal{M} := \left\{\mu : \mathcal{B}([0,\theta]) \to \mathbb{R}_0^+ | \text{ for every } 0 \le a \le b \le \theta \text{ , } \mu([a,b]) \le \theta \right\}.$$

and we endow $\mathcal{M}$ with the topology of the weak convergence.

Also define the set $D_{[0,\infty)}(\mathcal{M})$ as

$$D_{[0,\infty)}(\mathcal{M}) := \Big\{\mu : [0,\infty) \to \mathcal{M} | \mu_{0-} = \lambda,\ t \to \mu_t([a,b]) \text{ is a cadlag function } \forall\ 0 \le a < b \le \theta,$$

$$\int_{(a,b]} \frac{1}{z}\mu_t(dz) \le \int_{(a,b]} \frac{1}{z}\mu_s(dz) \text{ for every } 0 \le a < b \le \theta \text{ and } s \le t,\ \mu_s([0,\theta]) \le \mu_t([0,\theta])\Big\}.$$

where λ is the initial distribution and we endow $D_{[0,\infty)}(\mathcal{M})$ with the topology of the weak convergence.

Let us call $\mathcal{V}$ the set of admissible random measures, defined as

$$\mathcal{V} := \left\{ \nu : \Omega \to D_{[0,\infty)}(\mathcal{M}) \middle| \ \forall s \geq 0,\ \nu_s \text{ is } \mathcal{F}_s\text{-measurable} \right\}.$$

Define the utility functional as

$$\mathcal{J}(x,\nu) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) \left(\int_{(0,\theta]} \frac{h(y)}{y} \nu_s(dy) \right) ds - c \int_0^\infty e^{-\rho s} d\nu_s([0,\theta]) \right], \tag{3.53}$$

with $\nu \in \mathcal{V}$ and $h(y)$ is such that $R(x,y) = g(x)h(y)$ satisfies Assumption 3.3.1, with $g(x)$ a continuous increasing function.

Our objective is to find a random measure $\nu^* \in \mathcal{V}$ such that $\nu^* \in \arg\max_{\nu \in \mathcal{V}} \mathcal{J}(x,\nu)$. We call

$$\mathcal{J}(x,\nu^*) := V(x) \tag{3.54}$$

the value function.

Let us move to the existence of such a measure. Inspired by [44, Lemma 3.5] we prove a version of Komlos' theorem for $D_{[0,\infty)}(\mathcal{M})$-valued random variables. Recall that a subsequence a^n is Cesaro convergent if $\bar{a}^n = \frac{1}{n}\sum_{i=1}^n a^i$ is convergent.

Lemma 3.4.1 *Let ν^n be a sequence in $\mathcal{V}$, such that $\liminf_{n\to\infty} \mathbb{E}\left[\nu_t^n([0,r])\right] < \infty$, for all $t \geq 0$ and $r \in [0,\theta]$. Then there exist a random variable ν of $\mathcal{V}$ with $\mathbb{E}\left[\nu_t([0,\theta])\right] < \infty$ for all $t \geq 0$ and a subsequence $\nu^{n'}$ such that*

$$\bar{\nu}_t^{n'} \to \nu_t \quad \text{in } \mathcal{M} \quad \mathbb{P}\text{-a.s. for each } t \geq 0$$

and

$$\bar{\nu}_\cdot^{n'}([0,\theta]) \to \nu_\cdot([0,\theta]) \qquad \text{in } D_{[0,\infty)}(\mathcal{M}) \ \mathbb{P}\text{-a.s.}$$

All the previous convergences are in Cesaro sense.

Proof. Let $Q_1 = \{t_1, t_2, \ldots\}$ be a countable set dense in $[0,\infty)$ and $Q_2 = \{r_1, r_2, \ldots\}$ be countable set dense in $[0,\theta]$ with $\theta \in Q_2$. Consider a sequence $(\nu^n)_{n\in\mathbb{N}} \in \mathcal{V}$ and for each fixed $t_i \in Q_1$ and $r_j \in Q_2$, consider the sequence $(\nu_{t_i}^n([0,r_j]))_{n\in\mathbb{N}}$. By Komlos' theorem, there exists a subsequence $(\nu_{t_i}^{n'}([0,r_j])_{n'\in\mathbb{N}}$, converging in Cesaro sense to a real integrable random variable $\xi_{i,j}$. Now we build the desired subsequence with a diagonal procedure in analogy with [44, Lemma 3.5]. Consider a subsequence ν^{n_k} of $\nu_{t_i}^n([0,r_j])$, and take $\nu_{t_i}^{n_k}([0,r_{j+1}])$. Applying Komlos' theorem again, we find a subsequence $\nu^{n_k'}$ converging in Cesaro sense to an integrable random variable $\xi_{i,j+1}$, such that we also have

$$\lim_{n_k'\to\infty} \bar{\nu}_{t_i}^{n_k'}([0,r_j]) = \xi_{i,j} \qquad \text{a.s.}$$

Consider now a subsequence ν^{n_m} of $\nu^n_{t_i}([0,r_j])$, and take $\nu^{n_m}_{t_{i+1}}([0,r_j])$. Applying Komlos' theorem, there exist a subsequence $\nu^{n'_m}$ converging in Cesaro sense to an integrable random variable $\xi_{i+1,j}$ and such that

$$\lim_{n'_m\to\infty} \bar{\nu}^{n'_m}_{t_i}([0,r_j]) = \xi_{i,j} \qquad \text{a.s.}$$

Repeating the process in both dimensions, we find a subsequence $(\nu^{n'}_{t_i}([0,r_j]))_{n'\in\mathbb{N}}$ converging in Cesaro sense to a random variable ξ_{t_i,r_j}, for all $t_i \in Q_1$, $r_j \in Q_2$ a.s.

Consider the process

$$\nu_t([0,r]) = \lim_{s\searrow t} \xi_{s,r} := \lim_{s\searrow t}\left(\limsup_{t_i\searrow s, r_j\searrow r} \xi_{t_i,r_j}\right) \quad \text{, where } t_i \in Q_1,\ r_j \in Q_2 \text{ and } t < t_i,\ r < r_j. \tag{3.55}$$

Since $\nu_t([0,r])$ is obtained as the limit of $(\nu^n)_n \subseteq \mathcal{V}$, then $\nu_t([0,r])$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. Moreover, for every $r \in [0,\theta]$, $\nu_t([0,r]) \leq \theta$ and it inherits the monotonicity properties of each member of this sequence. More in detail, it is immediate to verify that

$$\nu_t([0,r]) \gtreqless \nu_t([0,r']) \Leftrightarrow r \gtreqless r' \text{ for all } t \in Q_1,\ r,\ r' \in Q_2 \tag{3.56}$$

and

$$\int_{(0,r]} \frac{1}{z}\nu_t(dz) \lesseqgtr \int_{(0,r]} \frac{1}{z}\nu_s(dz) \Leftrightarrow t \gtreqless s. \tag{3.57}$$

For $t, s \notin Q_1$, $r, r' \notin Q_2$, (3.56) and (3.57) follow from the definition (3.55). Furthermore, for each $\omega \in \Omega$ and $r \in [0,\theta]$, the function $\nu_\cdot[0,r]$ is a cadlag function, since it is defined as the right limit, therefore it is right continuous and by the monotonicity property (3.57) the left limit exists. Therefore $\nu \in \mathcal{V}$.

We prove that for each $t \geq 0$, the measure ν_t is such that,

$$\bar{\nu}^{n'}_t([0,\cdot]) \to \nu_t([0,\cdot]) \text{ as } n' \to \infty,$$

for all points of continuity of the map $r \to \nu_t([0,r])$. Notice $r \to \nu_t([0,r])$ is non decreasing, $\nu_t(\{0\}) = 0$ and $\nu_t([0,\theta]) \leq \theta$, then ν_t is discontinuous in at most a countable set. Let r be a point of continuity of ν_{t_i} and $\nu^{n'}_{t_i}$, with $t_i \in Q_1$, then it follows

$$\begin{aligned}|\nu_{t_i}([0,r]) - \bar{\nu}^{n'}_{t_i}([0,r])| \;&\leq\; |\nu_{t_i}([0,r]) - \nu_{t_i}([0,r_j])| + |\nu_{t_i}([0,r_j]) - \bar{\nu}^{n'}_{t_i}([0,r_j])| \\ &\quad + |\bar{\nu}^{n'}_{t_i}([0,r_j]) - \bar{\nu}^{n'}_{t_i}([0,r])| \leq 3\epsilon\end{aligned}$$

and therefore $\nu^{n'}_{t_i} \to \nu_{t_i}$ a.s in $\mathcal{M}$. As for each $\omega \in \Omega$, $t \to \nu_t([0,r))$ is a cadlag functions, the result is true also for those point $t \notin Q_1$. In fact, for every t and $\forall\epsilon > 0$, there exist $t_i > t$ such that $t_i - t < \delta$ implies

$$\begin{aligned}|\nu_t([0,r]) - \bar{\nu}_t^{n'}([0,r])| &\leq |\nu_t([0,r]) - \nu_{t_i}([0,r])| + |\nu_{t_i}([0,r]) - \bar{\nu}_{t_i}^{n'}([0,r])| \\ &\quad + |\bar{\nu}_{t_i}^{n'}([0,r]) - \bar{\nu}_t^{n'}([0,r])| \leq 3\epsilon.\end{aligned}$$

Analogously, we prove now that

$$\bar{\nu}_\cdot^{n'}([0,\theta]) \to \nu_\cdot([0,\theta]) \quad \text{ in } D_{[0,\infty)}(\mathcal{M}) \text{ a.s.}$$

for all points of continuity of the map $t \to \nu_t([0,\theta])$. Notice that for each $\omega \in \Omega$, $t \to \nu_t([0,\theta])$ is a non decreasing cadlag function and $\forall t \geq 0$ $\nu_t([0,\theta]) < \infty$, then $\nu_\cdot([0,\theta])$ is discontinuous in at most a countable set. We know that $\xi_{i,\theta} = \lim_{n' \to \infty} \bar{\nu}_{t_i}^{n'}([0,\theta])$, then $\forall \epsilon > 0$, $\exists M > 0$ such that $n' > M$, $|\bar{\nu}_{t_i}^{n'}([0,\theta]) - \xi_{i,\theta}| < \epsilon$. Let t be a point of continuity of $\nu_\cdot([0,\theta])$ and $\nu_\cdot^{n'}([0,\theta])$, then it follows

$$\begin{aligned}|\nu_t([0,\theta]) - \bar{\nu}_t^{n'}([0,\theta])| &\leq |\nu_t([0,\theta]) - \nu_{t_i}([0,\theta])| + |\nu_{t_i}([0,\theta]) - \bar{\nu}_{t_i}^{n'}([0,\theta])| \\ &\quad + |\bar{\nu}_{t_i}^{n'}([0,\theta]) - \bar{\nu}_t^{n'}([0,\theta])| \leq 3\epsilon\end{aligned}$$

and therefore $\nu_\cdot^{n'}[0,\theta] \to \nu_\cdot[0,\theta]$ in $D_{[0,\infty)}(\mathcal{M})$ a.s.

■

Now we are able to prove existence of an admissible optimal measure for (3.53).

Theorem 3.4.2 *There exist a measure $\nu^* \in \mathcal{V}$ maximizing* (3.53).

Proof. Consider a maximizing sequence $(\nu^n)_{n\in\mathbb{N}} \in \mathcal{V}$ such that

$$\lim_{n\to\infty} \mathcal{J}(x, \nu^n) = V(x)$$

and $\liminf_{n\to\infty} \mathbb{E}\left[\nu_t^n([0,r])\right] < \infty$ since $\mathbb{E}\left[\nu_t^n([0,r])\right] \leq \theta$ for all $t \geq 0$ and $r \in [0,\theta]$. By Lemma 3.4.1, there exists a subsequence $(\nu^{n'})_{n'\in\mathbb{N}}$, such that, for a fixed t, $\nu_t^{n'}$ converges $\mathbb{P}$-a.e in Cesaro sense to $\nu_t^* \in \mathcal{M}$ and $\nu_t^n[0,\theta]$ converges a.e. in Cesaro sense to $\nu_t^*[0,\theta]$ for all $t \geq 0$. Observe that for every $\omega \in \Omega$, $t \geq 0$ and $r \in [0,\theta]$, $\bar{\nu}^{n'} \in D_{[0,\infty)}(\mathcal{M})$ by convexity of $D_{[0,\infty)}(\mathcal{M})$. Moreover, we have

$$\begin{aligned}|\mathcal{J}(x,\nu^{n'})| &\leq \mathbb{E}\left[\left|\int_0^\infty e^{-\rho s} S^x(s)\left(\int_{(0,\theta]} g(y)\nu_s^{n'}(dy)\right)ds\right| + \left|c\int_0^\infty e^{-\rho s} d\nu_s^{n'}((0,\theta])\right|\right] && (3.58)\\ &\leq C_g \mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s)\nu_s^{n'}((0,\theta])ds + c\int_0^\infty e^{-\rho s} d\nu_s^{n'}((0,\theta])\right] && (3.59)\\ &\leq C_g\theta\mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s)ds\right] + c\theta && (3.60)\\ &= \theta(C_g C_x + c) && (3.61)\end{aligned}$$

where $C_g = \max_{x\in[0,\theta]} g(x)$ and by the integrability assumption 3.1.1 we can write $\mathbb{E}\left[\int_0^\infty e^{-\rho s} S^x(s) ds\right] = C_x$. By a direct application of Lemma 3.4.1 and the dominated convergence we have,

$$\mathcal{J}(x, \nu^*) = \lim_{n'\to\infty} \frac{1}{n'} \sum_{k=1}^{n'} J(x, \nu^{n'}) = V(x). \tag{3.62}$$

■

Chapter 4

The case with market impact

In this chapter we extend the stochastic Kuhn-Tucker conditions presented in [23], to the case when the dynamic is controlled, i.e., considering market impact. We consider the particular case when the revenue R of the utility is linear both in the control and in the dynamics, i.e., $R(x, y) = xy$. As controlled dynamics we consider an Ornstein-Uhlenbeck process where the control affects the drift term. This is exactly the same model discussed in the preview chapter, corresponding to the work in [50], where they solved the problem by using the Hamilton-Jacobi-Bellman (HJB) approach assuming some regularity of the value function and a Markovian structure of the problem.

Here instead, we do not consider any Markovian structure of the problem. The principal result that helps us to prove the Kuhn-Tucker conditions is that our utility functional $\mathcal{J}(x, y, I)$ is supported by a subgradient $\nabla\mathcal{J}(x, y, I)$, in the sense that, for any admissible control I and I', the subgradient $\nabla\mathcal{J}(x, y, I)$, satisfies

$$\mathcal{J}(x, y, I) - \mathcal{J}(x, y, I') \leq \langle \nabla\mathcal{J}(x, y, I'), I - I' \rangle.$$

By using the above concave property of our functional we are able to prove the following sufficient conditions of optimality for the control variable. Denote by $\mathcal{T}$ the set of stopping times τ with values in $[0, \infty)$ $\mathbb{P}$-a.s. Suppose that there exist a nonnegative Lagrange multiplier measure $d\lambda(\omega, t)$ such that $\mathbb{E}\left[\int_{[0,\infty)} d\lambda(t)\right] < \infty$, and the following conditions are satisfied for some admissible strategy I^*

$$\nabla\mathcal{J}(x, y, I^*)(\tau) \leq \mathbb{E}\left[\int_\tau^\infty d\lambda(s) \middle| \mathcal{F}_\tau\right] \ \mathbb{P}\text{-a.s } \forall\tau \in \mathcal{T}, \tag{4.1}$$

$$\int_0^\infty \left(\nabla\mathcal{J}(x, y, \hat{I})(t) - \mathbb{E}\left[\int_t^\infty d\lambda(s) \middle| \mathcal{F}_t\right]\right) dI^*(t) = 0 \ \mathbb{P}\text{-a.s}, \tag{4.2}$$

$$\mathbb{E}\left[\int_0^\infty (\theta - (y + I^*(t)))\, d\lambda(t)\right] = 0, \tag{4.3}$$

then I^* maximizes the utility functional $\mathcal{J}(x, y, I)$.

Then, by using arguments as those on [3, 23], we prove that the above conditions are also necessary. The proof is divided in two steps: first we prove that the optimal solution I^* solves the linearized problem

$$\sup_{I\in\mathcal{I}} \mathbb{E}\left[\int_0^\infty \nabla\mathcal{J}(x,y,I^*)(s)dI\right],$$

where $\mathcal{I}$ is the set of admissible strategies. Afterwards, that solution is characterized by the flat off conditions (4.2) and (4.3). To define a Lagrange multiplier measure, we consider the Snell envelope of the subgradient at the optimum. In this case, the Snell envelope is of class D, and therefore we can decompose it in a martingale M and an increasing predictable process λ. The measure $d\lambda$ plays the role of the Lagrange multiplier measure.

This chapter is organized as follows: in Section 1 we give the setting of our problem. In Section 2 we establish a concavity property of our utility functional by proving that it is supported by some subgradient. Finally, we prove sufficient and necessary conditions of optimality in the form of Kuhn-Tucker conditions.

4.1 Setup of the problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space where a one-dimensional Brownian motion W is defined and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets. Let us define a stochastic process $Y = (Y^y(s))_{s\geq t}$, with initial condition $y \in [0,\theta]$. This process can be increased irreversibly starting from y up to a maximum θ, This strategy is described by the control process $I = (I(s))_{s\geq t}$ and it takes values on the set $\mathcal{I}(y)$ of admissible strategies, defined by

$$\mathcal{I}(y) \triangleq \{I : [0,\infty)\times\Omega \to [0,\infty) : I \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{- adapted }, t \to I_t \text{ is increasing, cadlag,}$$
$$\text{with } I_{0-} = 0 \leq I_t \leq \theta - y\}.$$

Hence the process Y is written as follow

$$Y^y(s) = y + I(s).$$

We consider that a shock process X^{x,Y^y}, in absence of any control action, evolves accordingly to an Ornstein-Uhlenbeck process. We suppose that the increments on the control variable affect negatively the mean-reverting term of the shock process, hence, we have the following dynamics for the process X^{x,Y^y}:

$$\begin{cases} dX^{x,Y^y}(t) = \kappa(\zeta - \beta Y^y(t) - X^{x,Y^y}(t))dt + \sigma dW(t) \\ X^{x,Y^y}(0) = x, \end{cases} \tag{4.4}$$

where $\zeta \in \mathbb{R}$, $\kappa > 0$, and $\sigma > 0$.

Consider the utility functional

$$\mathcal{J}(x,y,I) \quad = \quad \mathbb{E}\left[\int_0^\infty e^{-\rho s}X^{x,Y^y}(s)Y^y(s)ds - \int_0^\infty e^{-\rho s}dI(s)\right], \tag{4.5}$$

with $\rho > 0$. We aim to find an admissible control $I \in \mathcal{I}(y)$ maximizing the previous functional, i.e.

$$V(x,y) = \mathcal{J}(x,y,I^*) = \sup_{I \in \mathcal{I}(y)} \mathcal{J}(x,y,I). \tag{4.6}$$

4.2 Stochastic Kuhn-Tucker conditions with controlled shock process

On [23] they present first order conditions for optimality for a social planner problem with N players under constraints. This result is inspired by the first order conditions for optimality under fuel constraint presented in [3]. Roughly speaking the condition of optimality comes from the fact that at the optimal policy the subgradient of the cost functional coincides with its Snell envelope. Recall that the Snell envelope of a process X is the smallest supermartingale U which dominates X, i.e., $X_s \leq U_s$ for all $s \geq 0$. Both [23] and [3] work with a utility functional affected by an uncontrolled shock process. The novelty of our work is to consider a controlled shock process and state the same first order condition as in [23].

We start defining the subgradient for our utility functional and then we prove that if the subgradients of the utility functional at some admissible control $I^* \in \mathcal{I}(y)$ satisfies some conditions, then the control I^* is optimal for our control problem in the sense that it solves (4.6).

The solution of (4.4) can be expressed as

$$\begin{aligned} X^{x,Y^y}(t) &= e^{-\kappa t}x + \kappa \int_0^t e^{\kappa(s-t)}(\zeta - \beta(y + I(s)))ds + \int_0^t e^{\kappa(s-t)}\sigma dW(s) \\ &= X^x(t) - \kappa\beta \int_0^t e^{\kappa(s-t)}(y + I(s))ds, \end{aligned} \tag{4.7}$$

where X^x is the solution of the Ornstein-Uhlenbeck process without price impact, i.e., when $\beta = 0$. Rewriting the cost functional (4.5), using the explicit solution (4.7), we obtain

$$\mathcal{J}(x,y,I) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} X^{x,Y^y}(t)(y + I(t))dt - \int_0^\infty c e^{-\rho t} dI(t)\right] \tag{4.8}$$

$$\begin{aligned} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(X^x(t) - \kappa\beta\int_0^t e^{\kappa(s-t)}(y + I(s))ds\right)(y + I(t))dt - \int_0^\infty c e^{-\rho t} dI(t)\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} X^x(t)(y + I(t))dt - \kappa\beta \int_0^\infty \int_0^t e^{-\rho t} e^{\kappa(s-t)}(y + I(s))(y + I(t))dsdt \right. \\ &\quad \left. - \int_0^\infty c e^{-\rho t} dI(t)\right]. \end{aligned} \tag{4.9}$$

The following lemma is a useful computation that we will apply in the proof of the theorem that states the sufficient conditions for optimality.

Lemma 4.2.1 *For all I, $I' \in \mathcal{I}$, we have*

$$\mathbb{E}\left[\int_0^\infty \int_0^t e^{-\rho t} e^{\kappa(s-t)} \beta\kappa((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right] \geq 0. \tag{4.10}$$

Proof. Observe that

$$\mathbb{E}\left[\int_0^\infty \int_0^t e^{-\rho t} e^{\kappa(s-t)} \beta\kappa((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right] \tag{4.11}$$

$$= \mathbb{E}\left[\int_0^\infty \int_0^\infty K(s,t) \beta\kappa((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right], \tag{4.12}$$

where the function $K : [0, \infty) \times [0, \infty) \to \mathbb{R}_0^+$ is defined by

$$K(s,t) = \begin{cases} \frac{1}{2} e^{-\rho t} e^{\kappa(s-t)} & \text{if } 0 \leq s \leq t < \infty \\ \frac{1}{2} e^{-\rho s} e^{\kappa(t-s)} & \text{if } 0 \leq t \leq s < \infty \end{cases} = \frac{1}{2} e^{-\frac{\rho}{2}(t+s)} e^{-(\frac{\rho}{2}+\kappa)|t-s|} \tag{4.13}$$

is a positive definite kernel. In fact $K(s,t)$ a is symmetric continuous function. To check the positive definiteness, notice that $e^{-(\frac{\rho}{2}+\kappa)|t-s|}$ corresponds to the Laplacian kernel which is positive definite [13, Corallary 3.3]. On the other hand, $K(s,t) = e^{-\frac{\rho}{2}(t+s)}$ is also positive definite,

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j e^{-\frac{\rho}{2}(x_i + x_j)} = \left(\sum_{i=1}^n a_i e^{-\frac{\rho}{s} x_i}\right)^2 \geq 0. \tag{4.14}$$

Therefore, as the product of positive definite kernels is again a positive definite kernel we conclude that $K(s,t)$ is a positive definite kernel. Then, from Mercer's Theorem [54] we can write (4.12) as follows,

$$\mathbb{E}\left[\int_0^\infty \int_0^\infty K(s,t) \beta\kappa((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right] \tag{4.15}$$

$$= \mathbb{E}\left[\int_0^\infty \int_0^\infty \sum_{j=1}^\infty \lambda_j e_i(s) e_j(t) \beta\kappa((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right] \tag{4.16}$$

$$= \beta\kappa \sum_{j=1}^\infty \lambda_j \mathbb{E}\left[\left(\int_0^\infty e_i(s)(I(s) - I^{'}(s)))ds\right)^2\right] \geq 0. \tag{4.17}$$

since β, κ, λ_j are positive constants and $(e_i(\cdot))_i$ is a orthonormal basis of $L^2([0,\infty])$. ■

Theorem 4.2.2 *For all $x \in \mathbb{R}$, $y \in \mathbb{R}_0^+$, and any $I \in \mathcal{I}$ the utility function* (4.5) *is supported by the subgradient*

$$\nabla \mathcal{J}(x,y,I)(u) = \mathbb{E}\left[\int_u^\infty e^{-\rho t}X^x(t)dt - \int_u^\infty e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dt \right. \tag{4.18}$$
$$\left. - \int_u^\infty e^{-\rho t}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt \middle| \mathcal{F}_u\right] - \int_0^u e^{\kappa t}e^{-u(\rho+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt - ce^{-\rho u}$$

in the sense that for any $I^{'} \in \mathcal{I}$, $\nabla\mathcal{J}(x,y,I)$ satisfies the following subgradient property

$$\mathcal{J}(x,y,I) - \mathcal{J}(x,y,I^{'}) \leq \langle \nabla\mathcal{J}(x,y,I'), I - I^{'}\rangle. \tag{4.19}$$

The bracket operator in (4.19) is defined as

$$\langle \alpha, \beta\rangle = \mathbb{E}\left[\int_{[0,T)} \alpha(t)d\beta(t)\right].$$

Proof. Heuristically, we expect that $\langle \nabla\mathcal{J}(x,y,I), I^{'}\rangle = (d\mathcal{J})(x,y,I,I^{'})$, where $(d\mathcal{J})(x,y,I,I^{'})$ is the Gateaux derivative in the direction $I^{'} \in \mathcal{I}$. By definition

$$\begin{aligned}
(d\mathcal{J})(x,y,I,I^{'}) &= \lim_{\epsilon\to 0}\frac{\mathcal{J}(x,y,I+\epsilon I^{'}) - \mathcal{J}(x,y,I)}{\epsilon} && (4.20)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}\left[\int_0^\infty e^{-\rho t}(X^x(t)(y+I(t)+\epsilon I^{'}(t))dt - \int_0^\infty ce^{-\rho t}d(I(t)+\epsilon I^{'}(t))\right.\\
&\quad -\kappa\beta\int_0^\infty\int_0^t e^{-\rho t}e^{\kappa(s-t)}(y+I(s)+\epsilon I^{'}(s))(y+I(t)+\epsilon I^{'}(t))dsdt\\
&\quad -\int_0^\infty e^{-\rho t}(X^x(t)(y+I(t))dt + \int_0^\infty ce^{-\rho t}dI(t)\\
&\quad \left. +\kappa\beta\int_0^\infty\int_0^t e^{-\rho t}e^{\kappa(s-t)}(y+I(s))(y+I(t))dsdt\right]\\
&= \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^\infty e^{-\rho t}X^x(t)I^{'}(t)dt - \int_0^\infty ce^{-\rho t}dI^{'}(t)\right.\\
&\quad -\kappa\beta\int_0^\infty\int_0^t e^{-\rho t}e^{\kappa(s-t)}(y+I(s))I^{'}(t)dsdt\\
&\quad -\kappa\beta\int_0^\infty\int_0^t e^{-\rho t}e^{\kappa(s-t)}(y+I(t))I^{'}(s)dsdt\\
&\quad \left. -\kappa\beta\int_0^\infty\int_0^t e^{-\rho t}e^{\kappa(s-t)}\epsilon I^{'}(s)I^{'}(t)dsdt\right]\\
&= \mathbb{E}\left[\int_0^\infty\int_0^t -\beta\kappa e^{-\rho t}e^{\kappa(s-t)}(y+I(t))I^{'}(s)dsdt + \int_0^\infty e^{-\rho t}X^x(t)I^{'}(t)dt\right.\\
&\quad \left. + \int_0^\infty\int_0^t -\beta\kappa e^{-\rho t}e^{\kappa(s-t)}(y+I(s))I^{'}(t)dsdt - c\int_0^\infty e^{-\rho t}dI^{'}(t)\right], && (4.21)
\end{aligned}$$

applying Fubini's Theorem to the first, second and third terms in (4.21), we get for the first term

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty \int_0^t -e^{-\rho t}\beta\kappa(y+I(t))e^{\kappa(s-t)}I^{'}(s)dsdt\right] &= \mathbb{E}\left[\int_0^\infty \int_0^t \int_0^s -e^{-\rho t}\beta\kappa(y+I(t))e^{\kappa(s-t)}dI^{'}(u)dsdt\right] \\ &= \mathbb{E}\left[\int_0^\infty \int_u^\infty \int_u^t -e^{-\rho t}\beta\kappa(y+I(t))e^{\kappa(s-t)}dsdtdI^{'}(u)\right] \\ &= \mathbb{E}\left[\int_0^\infty \int_u^\infty -e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dtdI^{'}(u)\right]\end{aligned}$$

for the second

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}X^x(t)I^{'}(t)dt\right] = \mathbb{E}\left[\int_0^\infty \int_0^t e^{-\rho t}X^x(t)dI^{'}(u)dt = \int_0^\infty \int_u^\infty e^{-\rho t}X^x(t)dtdI^{'}(u)\right]$$

and for the third

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty \int_0^t -e^{-\rho t}\beta\kappa(y+I(s))e^{\kappa(s-t)}I^{'}(t)dsdt\right] &= \mathbb{E}\left[\int_0^\infty \int_0^t \int_0^t -e^{-\rho t}\beta\kappa(y+I(s))e^{\kappa(s-t)}dI^{'}(u)dsdt\right] \\ &= \mathbb{E}\left[\int_0^\infty \int_0^u \int_u^\infty -e^{-\rho t}\beta\kappa(y+I(s))e^{\kappa(s-t)}dtdsdI^{'}(u)\right] \\ &\quad +\mathbb{E}\left[\int_0^\infty \int_u^\infty \int_s^\infty -e^{-\rho t}\beta\kappa(y+I(s))e^{\kappa(s-t)}dtdsdI^{'}(u)\right] \\ &= \mathbb{E}\left[\int_0^\infty \int_0^u -e^{\kappa s}e^{-u(\delta+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))dsdI^{'}(u)\right] \\ &\quad +\mathbb{E}\left[\int_0^\infty \int_u^\infty -e^{-\rho s}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))dsdI^{'}(u)\right],\end{aligned}$$

therefore, (4.21) can be written as

$$
\begin{aligned}
(d\mathcal{J})(x,y,I,I^{'}) &= \mathbb{E}\left[\int_0^\infty\int_u^\infty -e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dtdI^{'}(u)+\int_0^\infty\int_u^\infty e^{-\rho t}X^x(t)dtdI^{'}(u)\right.\\
&\quad -\int_0^\infty\int_0^u e^{\kappa s}e^{-u(\rho+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))dsdI^{'}(u)-\int_0^\infty\int_u^\infty e^{-\rho s}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))dsdI^{'}(u)\\
&\quad \left.-c\int_0^\infty e^{-\rho t}dI^{'}(t)\right]\\
&= \mathbb{E}\left[\int_0^\infty\left(\int_u^\infty -e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dt+\int_u^\infty e^{-\rho t}X^x(t)dt\right.\right.\\
&\quad -\int_0^u e^{\kappa s}e^{-u(\rho+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))ds-\int_u^\infty e^{-\rho s}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(s))ds\\
&\quad \left.\left.-ce^{-\rho u}\right)dI^{'}(u)\right]\\
&= \mathbb{E}\left[\int_0^\infty\left(\mathbb{E}\left[\int_u^\infty e^{-\rho t}X^x(t)dt-\int_u^\infty e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dt\right.\right.\right.\\
&\quad \left.-\int_u^\infty e^{-\rho t}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt\right|\mathcal{F}_u\Bigg]-\int_0^u e^{\kappa t}e^{-u(\rho+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt\\
&\quad \left.\left.-ce^{-\rho u}\right)dI^{'}(u)\right].
\end{aligned}
$$

Let us call

$$
\begin{aligned}
\nabla\mathcal{J}(x,y,I)(u) &= \mathbb{E}\left[\int_u^\infty e^{-\rho t}X^x(t)dt-\int_u^\infty e^{-\rho t}\beta(y+I(t))(1-e^{\kappa(u-t)})dt\right.\\
&\quad \left.-\int_u^\infty e^{-\rho t}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt\right|\mathcal{F}_u\Bigg]-\int_0^u e^{\kappa t}e^{-u(\rho+\kappa)}\frac{\beta\kappa}{(\rho+\kappa)}(y+I(t))dt-ce^{-\rho u}.
\end{aligned}
\tag{4.22}
$$

On the other hand, using the expression (4.21) we have

$$
\begin{aligned}
\langle \nabla \mathcal{J}(x,y,I^{'}), I - I^{'} \rangle &= \mathbb{E}\left[\int_0^\infty \int_0^t -\beta\kappa e^{-\rho t} e^{\kappa(s-t)}(y + I^{'}(t))(I(s) - I^{'}(s))dsdt\right. \\
&+ \int_0^\infty \int_0^t -\beta\kappa e^{-\rho t} e^{\kappa(s-t)}(y + I^{'}(s))(I(t) - I^{'}(t))dsdt \\
&+ \left.\int_0^\infty e^{-\rho t} X^x(t)(I(t) - I^{'}(t))dt - c\int_0^\infty e^{-\rho t} d(I(t) - I^{'}(t))\right] \\
&= \mathbb{E}\left[\int_0^\infty \int_0^t -\beta\kappa e^{-\rho t} e^{\kappa(s-t)}(yI(s) - yI^{'}(s) + yI(t) - yI^{'}(t) + I(s)I^{'}(t) + I^{'}(s)I(t)\right. \\
&- \left. 2I^{'}(t)I^{'}(s))dsdt + \int_0^\infty e^{-\rho t} X^x(t)(I(t) - I^{'}(t))dt - c\int_0^\infty e^{-\rho t} d(I - I^{'})(t)\right] \\
&= \mathbb{E}\left[\int_0^\infty \int_0^t -\beta\kappa e^{-\rho t} e^{\kappa(s-t)}(I(s)I^{'}(t) + I(t)I^{'}(s) - I(t)I(s) - I^{'}(s)I^{'}(t))dsdt\right] \\
&+ \mathcal{J}(x,y,I) - \mathcal{J}(x,y,I^{'}) \\
&= \mathbb{E}\left[\int_0^\infty \int_0^t \beta\kappa e^{-\rho t} e^{\kappa(s-t)}((I(t) - I^{'}(t))(I(s) - I^{'}(s)))dsdt\right] + \mathcal{J}(x,y,I) - \mathcal{J}(x,y,I^{'}) \\
&\geq \mathcal{J}(x,y,I) - \mathcal{J}(x,y,I^{'}),
\end{aligned}
$$

where the last inequality comes from Lemma 4.2.1.

■

4.2.1 Sufficient and necessary conditions

With the concavity property just proved (4.19), we are able to establish sufficient and necessary conditions of optimality for the problem (4.6). Let us start with the following result for sufficient conditions.

Theorem 4.2.3 *Denote by $\mathcal{T}$ the set of stopping times τ with values in $[0,\infty)$ $\mathbb{P}$-a.s. Suppose that there exists a nonnegative Lagrange multiplier measure $d\lambda(\omega,t)$ such that $\mathbb{E}\left[\int_{[0,\infty)} d\lambda(t)\right] < \infty$, and the following conditions are satisfied for some $I^* \in \mathcal{I}(y)$*

$$\nabla \mathcal{J}(x,y,I^*)(\tau) \leq \mathbb{E}\left[\int_\tau^\infty d\lambda(s) \middle| \mathcal{F}_\tau\right] \ \mathbb{P}\text{-}a.s \ \forall \tau \in \mathcal{T}, \tag{4.23}$$

$$\int_0^\infty \left(\nabla \mathcal{J}(x,y,\hat{I})(t) - \mathbb{E}\left[\int_t^\infty d\lambda(s) \middle| \mathcal{F}_t\right]\right) dI^*(t) = 0 \ \mathbb{P}\text{-}a.s, \tag{4.24}$$

$$\mathbb{E}\left[\int_0^\infty (\theta - (y + I^*(t)))\, d\lambda(t)\right] = 0, \tag{4.25}$$

then I^* is optimal for (4.6).

Proof. Let I^* satisfy (4.23), (4.24) and (4.25) and $I \in \mathcal{I}(y)$ be an arbitrary admissible control. By (4.19) we have

$$\begin{aligned}
\mathcal{J}(x,y,I) - \mathcal{J}(x,y,I^*) &\leq \mathbb{E}\left[\int_{[0.\infty)} \nabla \mathcal{J}(x,y,I^*)(t) d(I(t) - I^*(t))\right] \\
&\leq \mathbb{E}\left[\int_{[0.\infty)} \mathbb{E}\left[\int_{[t,\infty)} d\lambda(s) \middle| \mathcal{F}_t\right] d(I(t) - I^*(t))\right] \qquad (4.26)\\
&= \mathbb{E}\left[\int_{[0,\infty)} \left[\int_{[t,\infty)} d\lambda(s)\right] d(I(t) - I^*(t))\right] \\
&= \mathbb{E}\left[\int_{[0,\infty)} (I(s) - I^*(s)) d\lambda(s)\right] \qquad (4.27)\\
&= \mathbb{E}\left[\int_{[0,\infty)} (I(s) - (\theta - y)) d\lambda(s)\right] \qquad (4.28)\\
&\leq 0
\end{aligned}$$

where (4.26) is an application of (4.24), equality (4.27) is an application of Fubini's Theorem and inequality (4.28) an application of condition (4.25). ■

Now we move on to prove that conditions (4.23), (4.24) and (4.25) are also necessary for optimality. To do so we follow the argument in [23]. We can think $\nabla \mathcal{J}(x,y,I)$ as the optional projection of the product measurable process

$$\begin{aligned}
\Phi(\omega,t) &= \int_t^\infty e^{-\rho s} X^x(\omega,s) ds - \int_t^\infty e^{-\rho s} \beta(y + I(\omega,s)) \left((1 - e^{\kappa(t-s)}) + \frac{\kappa}{(\rho+\kappa)}\right) ds \\
&\quad - \int_0^t e^{\kappa s} e^{-t(\rho+\kappa)} \frac{\beta\kappa}{(\rho+\kappa)} (y + I(\omega,s)) ds - c e^{-\rho t}
\end{aligned}$$

for $\omega \in \Omega$ and $t \in [0,\infty)$. Hence $\nabla \mathcal{J}(x,y,I)$ is uniquely determined up to $\mathbb{P}$-indistinguishability and it holds

$$\mathbb{E}\left[\int_{[0,\infty)} \nabla \mathcal{J}(x,y,I)(t) dI(t)\right] = \mathbb{E}\left[\int_{[0,\infty)} \Phi(t) dI(t)\right] \qquad (4.29)$$

for $I \in \mathcal{I}(y)$ [43] (see also [23, Remark 3.2]).

Observe also that

$$\begin{aligned}
\mathbb{E}\left[|\Phi(t)|\right] &\leq \mathbb{E}\left[\int_t^\infty |e^{-\rho s} X^x(s)| ds\right] + \int_t^\infty e^{-\rho s} \beta\theta \left((1 - e^{\kappa(t-s)}) + \frac{\kappa}{(\rho+\kappa)}\right) ds \\
&\quad + \int_0^t e^{\kappa s} e^{-t(\rho+\kappa)} \frac{\beta\kappa}{(\rho+\kappa)} \theta ds + c \\
&\leq C_1|1 + x| + C_2, \qquad (4.30)
\end{aligned}$$

with C_1, $C_2 > 0$.

We have the following result for necessary conditions of optimality.

Theorem 4.2.4 *If I^* is optimal for the problem (4.6), then it satisfies the Kuhn-Tucker conditions (4.23), (4.24) and (4.25) for some non negative Lagrange multiplier $d\lambda(\omega,t)$ such that $\mathbb{E}\left[\int_{[0,\infty)} d\lambda(t)\right] < \infty$.*

Proof. We follow the same procedure as in [23, Theorem 3.4]. Let $I^* \in \mathcal{I}(y)$ be optimal.

Step 1. We show that the optimal policy I^* solves the linearized problem with finite value

$$\sup_{I\in\mathcal{I}(y)} \mathbb{E}\left[\int_{[0,T)} \Phi^*(s)dI(s)\right] = \sup_{I\in\mathcal{I}(y)} \mathbb{E}\left[\int_{[0,T)} \nabla\mathcal{J}(x,y,I^*)(s)dI(s)\right] \tag{4.31}$$

where Φ^* is the product-measurable process associated to $\nabla\mathcal{J}(x,y,I^*)$. For $I \in \mathcal{I}(y)$, define $I^\epsilon := \epsilon I + (1-\epsilon)I^*$ and Φ^ϵ be the product measurable process associated to $\nabla\mathcal{J}(x,y,I^\epsilon)$. Then $\lim_{\epsilon\to 0} I^\epsilon(t) = I^*(t)$ $\mathbb{P}$-a.s., as well as $\lim_{\epsilon\to 0}\Phi^\epsilon(t) = \Phi^*(t)$ $\mathbb{P}$-a.s. By optimality of I^* and the concavity property (4.19), we have

$$\begin{aligned} 0 &\geq \frac{1}{\epsilon}\mathcal{J}(x,y,I^\epsilon) - \mathcal{J}(x,y,I^*) \\ &\geq \frac{1}{\epsilon}\mathbb{E}\left[\int_{[0,\infty)} \nabla\mathcal{J}(x,y,I^\epsilon)(s)d(I^\epsilon - I^*)(s)\right] \\ &= \mathbb{E}\left[\int_{[0,\infty)} \nabla\mathcal{J}(x,y,I^\epsilon)(s)d(I - I^*)(s)\right] \\ &= \mathbb{E}\left[\int_{[0,\infty)} \Phi^\epsilon(s)d(I - I^*)(s)\right]. \end{aligned} \tag{4.32}$$

Moreover

$$\int_{[0,\infty)} \Phi^\epsilon(s)d(I-I^*)(s) \geq \int_{[0,\infty)} \Phi(s)d(I-I^*)(s),$$

since

$$\begin{cases} I^\epsilon(t) \leq I(t) \text{ on } \{t : I(t) - I^*(t) \geq 0\}, \\ I^\epsilon(t) > I(t) \text{ on } \{t : I(t) - I^*(t) < 0\}. \end{cases} \tag{4.33}$$

Furthermore by (4.30) we can ensure that $\int_{[0,\infty)} \Phi(s)d(I-I^*)(s)$ is $\mathbb{P}$-integrable. By applying Fatou's lemma and (4.32), we have

$$\mathbb{E}\left[\int_{[0,\infty)}\Phi^*(s)d(I-I^*)(s)\right]\leq\liminf_{\epsilon\to 0}\mathbb{E}\left[\int_{[0,\infty)}\Phi^\epsilon(s)d(I-I^*)(s)\right]\leq 0. \tag{4.34}$$

Therefore, I^* solves (4.31) with finite value.

Step 2. Now we characterize this solution with conditions as (4.24) and (4.25). Define

$$\Psi(t) \quad := \quad \operatorname*{ess\,sup}_{\tau\in[t,\infty)}\mathbb{E}\left[\nabla\mathcal{J}(x,y,I^*)(\tau)\Big|\mathcal{F}_t\right] \tag{4.35}$$

It is immediate that Ψ is a supermartingale, by (4.30) $\Psi(t)$ is integrable and by [56, Theorem 3.1, Theorem 3.2, Chapter 2] it is uniformly integrable, then it is of class (D) and. Hence Ψ has a unique Doob-Meyer decomposition $\Psi(t)=M(t)-\lambda(t)$, where M is a martingale and λ is a strictly increasing predictable process.

Observe that for any $I\in\mathcal{I}(y)$, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty\nabla\mathcal{J}(x,y,I^*)(s)dI(s)\right] &\leq \mathbb{E}\left[\int_0^\infty\Psi(s)dI(s)\right]=\mathbb{E}\left[\int_0^\infty\left(\int_s^\infty d\lambda(u)\right)dI(s)\right]\\ &= \mathbb{E}\left[\int_0^\infty I(s)d\lambda(s)\right]\leq(\theta-y)\mathbb{E}\left[\int_0^\infty d\lambda(s)\right],\end{aligned} \tag{4.36}$$

where in the first equality we used $\Psi(t)=\mathbb{E}\left[\int_t^\infty d\lambda(s)|\mathcal{F}_t\right]$ since $\Psi(\infty)=0$, while the second one is an application of Fubini's theorem.

Define now the stopping time

$$\tau:=\inf\{t\in[0,\infty);\nabla\mathcal{J}(x,y,I^*)(t)=\Psi(t)\}$$

and set the admissible strategy

$$\hat{I}(t)=(\theta-y)\mathbb{1}_{(\tau,\infty)}(t).$$

By proof in [3, Lemma 2.5] $d\lambda$ is supported by the set $\{t\geq 0|\nabla\mathcal{J}(x,y,I^*)(t)=\Psi(t)\}$ almost surely. For this strategy we have equality in the inequalities of (4.36), then

$$\mathbb{E}\left[\int_0^\infty\mathcal{J}(x,y,I^*)(t)d\hat{I}(t)\right] \quad = \quad \mathbb{E}\left[\int_0^\infty\hat{I}(t)d\lambda(t)\right]=(\theta-y)\mathbb{E}\left[\int_0^\infty d\lambda(t)\right].$$

Therefore, the inequalities in (4.36) become equalities and we have that the optimal strategy I^* satisfies the following conditions.

$$\begin{cases}\mathbb{E}\left[\left(\int_0^\infty\nabla\mathcal{J}(x,y,I^*)(s)dI(s)-\int_0^\infty I(s)d\lambda(s)\right)dI^*(s)\right]=0\\ ,\mathbb{E}\left[\int_0^\infty(\theta-(y+I^*(s)))\,d\lambda(s)\right]=0.\end{cases} \tag{4.37}$$

Hence, (4.37) implies conditions (4.24) and (4.25), while (4.23) is satisfied by considering the Doob-Meyer decomposition of the Snell envelope of the subgradient at the optimum I^*. This conclude the proof. ■

Chapter 5

Appendix

5.1 The optimal solution for the model in Subsection 3.2.1 by the Hamilton-Jacobi-Bellman equation for one firm

In this section we solve the optimal control problem presented in Subsection 3.2.1 using the HJB approach without rigorous proofs of the results, aiming to compare this approach with the one used in Subsection 3.2.1.

Suppose $\mu + \frac{\sigma^2}{2} < \rho$ and let us compute the utility for a non installation strategy ($I^0(t) \equiv y$, $\forall t \in [0, \infty)$)

$$\begin{aligned} Q(x,y) := \mathcal{J}(x,y,I^0) &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} S^x(t) h(y) dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} x e^{\mu t + \sigma W(t)} h(y) dt\right] \\ &= x h(y) \int_0^\infty e^{t(\mu-\rho)} \mathbb{E}\left[e^{\sigma W(t)}\right] dt \\ &= x h(y) \int_0^\infty e^{t(\mu-\rho)} e^{\sigma^2 t/2} dt \\ &= \frac{x h(y)}{\rho - \mu - \sigma^2/2}. \end{aligned}$$

5.1.1 Variational inequality

The first inequality on the variational inequality comes from

$$V(x,y) \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-\rho s} S^x(s) h(y) ds + e^{-\rho \Delta t} V(S^x(\Delta t), y)\right]. \tag{5.1}$$

Applying Ito's formula to $e^{-\rho \Delta t} V(S^x(\Delta t), y)$

$$e^{-\rho\Delta t}V(S^x(\Delta t),y) = V(x,y) + \int_0^{\Delta t}(-\rho e^{-\rho u}V(x,y) + e^{-\rho u}\mathcal{L}V(S^x(u),y))du \quad (5.2)$$

$$+ \int_0^{\Delta t} e^{-\rho u}D_xV(S^x(u),y)\sigma S^x(u)dW(u), \quad (5.3)$$

with

$$\mathcal{L}u(x,y) = \frac{\sigma^2x^2}{2}\frac{\partial^2 u}{\partial x^2} + x\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}. \quad (5.4)$$

Replacing (5.3) in (5.1), we have

$$0 \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-\rho s}S^x(s)y^\alpha ds + \int_0^{\Delta t}(-\rho e^{-\rho u}V(x,y) + e^{-\rho u}\mathcal{L}V(S^x(u),y))du \right. \quad (5.5)$$

$$\left. + \int_0^{\Delta t} e^{-\rho u}D_xV(S^x(u),y)\sigma S^x(u)dW(u)\right], \quad (5.6)$$

where the stochastic integral is a martingale. Then, dividing by Δt and $\Delta t \to 0$, we obtain

$$0 \geq xh(y) - \rho V(x,y) + \mathcal{L}V(x,y). \quad (5.7)$$

On the other hand, we have

$$V(x,y) \geq V(x,y+\epsilon) - c\epsilon. \quad (5.8)$$

Dividing by ϵ and $\epsilon \to 0$, we get

$$V_y(x,y) - c \leq 0. \quad (5.9)$$

This suggests that the value function V should identify with an appropriate solution of the following variational inequality

$$\max\{xh(y) - \rho w(x,y) + \mathcal{L}w(x,y), w_y(x,y) - c\} = 0 \quad (5.10)$$

with boundary condition $w(x,\theta) = Q(x,\theta)$. Equation (5.10) defines a waiting region $\mathbb{W}$ and an installation region $\mathbb{I}$, given by

$$\mathbb{W} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : xh(y) - \rho w(x,y) + \mathcal{L}w(x,y) = 0, w_y(x,y) - c < 0\} \quad (5.11)$$

$$\mathbb{I} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : xh(y) - \rho w(x,y) + \mathcal{L}w(x,y) \leq 0, w_y(x,y) - c = 0\} \quad (5.12)$$

5.1.2 Constructing an optimal solution

Let us suppose that there exists a function $F : [0, \theta] \to \mathbb{R}$ which separates the installation and waiting region, called the free boundary, such that,

$$\mathbb{W} = \{(x, y) \in \mathbb{R} \times [0, \theta) : 0 < x < F(y)\}, \tag{5.13}$$
$$\mathbb{I} = \{(x, y) \in \mathbb{R} \times [0, \theta) : x \geq F(y)\}. \tag{5.14}$$

For all $(x, y) \in \mathbb{W}$, the candidate value function w should satisfy,

$$xh(y) - \rho w(x, y) + \mathcal{L}w(x, y) = 0. \tag{5.15}$$

The function $Q(x, y)$ is a particular solution of (5.15). Let us study the homogeneous equation

$$-\rho w(x, y) + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} + x \left(\mu + \frac{\sigma^2}{2}\right) \frac{\partial w}{\partial x} = 0 \tag{5.16}$$
$$\Rightarrow x^2 \frac{\partial^2 w}{\partial x^2} + x \left(\frac{2\mu}{\sigma^2} + 1\right) \frac{\partial w}{\partial x} - \frac{2\rho}{\sigma^2} \rho w(x, y) = 0. \tag{5.17}$$

It is well know that the two fundamental solution of the equation (5.17) can be found with the trial solution $w(x, y) = x^m$, which leads the following equation for the exponent m:

$$m^2 + \frac{2\mu}{\sigma^2} m - \frac{2\rho}{\sigma^2} = 0, \tag{5.18}$$

then,

$$\begin{cases} m_1 = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} + 2\rho} > -\frac{\mu}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2}} = 0 \\ m_2 = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} + 2\rho} < 0 \end{cases} . \tag{5.19}$$

Therefore a candidate solution for $(x, y) \in \mathbb{W}$ is

$$w(x, y) = A(y)x^{m_1} + B(y)x^{m_2} + Q(x, y), \tag{5.20}$$

with A , B : $[0, \theta] \to \mathbb{R}$. Nevertheless, when $x \to 0$, $w(x, y) \to 0$ and $x^{m_2} \to \infty$, we should have $B(y) = 0$. Thus,

$$w(x, y) = A(y)x^{m_1} + Q(x, y). \tag{5.21}$$

On the other hand, on $\mathbb{I}$, $w(x, y)$, should satisfies

$$w_y(x, y) - c = 0, \tag{5.22}$$

implying

$$w_{xy}(x, y) = 0. \tag{5.23}$$

For $(x, y) \in \bar{\mathbb{W}} \cap \mathbb{I}$, i.e. for $x = F(y)$, in light of (5.22) and (5.23), our candidate value function $w(x, y)$ should satisfy

$$A'(y)F(y)^{m_1} + \frac{F(y)h'(y)}{\rho - \mu - \sigma^2/2} - c = 0 \tag{5.24}$$

and

$$m_1 A'(y)F(y)^{m_1 - 1} + \frac{h'(y)}{\rho - \mu - \sigma^2/2} = 0. \tag{5.25}$$

Substituting the expression for $A'(y)$ obtained from (5.24) into (5.25), we obtain

$$F(y) = \frac{cm_1(\mu + \sigma^2/2 - \rho)}{h'(y)(1 - m_1)} \tag{5.26}$$

Define $\bar{x} = F(\theta)$. The inverse function F^{-1} of (5.26) defined on $[0, \bar{x}]$ is

$$F^{-1}(x) \quad = \quad (h'^{-1}) \left(\frac{cm_1(\mu + \sigma^2/2 - \rho)}{x(1 - m_1)} \right).$$

By the property of the roots of (5.18), we can rewrite F^{-1} as

$$F^{-1}(x) \quad = \quad (h'^{-1}) \left(\frac{cm_1(m1 + 2\mu/\sigma^2 + 1)}{x} \right). \tag{5.27}$$

Observe that $F^{-1}(S^x(u))$ coincides with the process $l(u)$ on $[0, \bar{x}]$, computed in Subsection 3.2.1, Equation (3.28).

5.1.3 The optimal strategy and the value function

The candidate installation region $\mathbb{I}$ is divided in

$$\mathbb{I}_1 = \{(x, y) \in \mathbb{R}_0^+ \times [0, \theta) : x \in [F(y), F(\theta))\}$$

and

$$\mathbb{I}_2 = \{(x, y) \in \mathbb{R}_0^+ \times [0, \theta) : x \geq F(\theta)\}.$$

We can express the candidate value function $w(x, y) : \mathbb{R}_0^+ \times [0, \theta] \to \mathbb{R}$ as

$$w(x,y) = \begin{cases} A(y)x^{m_1} + R(x,y) & , x \in \mathbb{W} \cup ([0,\bar{x}) \times \theta); \\ A(F^{-1}(x))x^{m_1} + R(x,y) - c(F^{-1}(x) - y) & , x \in \mathbb{I}_1; \\ R(x,\theta) - c(\theta - y) & , x \in \mathbb{I}_2 \cup ([\bar{x},\infty) \times \theta). \end{cases} \tag{5.28}$$

The optimal strategy is

$$I^*(t) = \left(\sup_{u\in[0,t)} F^{-1}(S^x(u)) \wedge \theta \right) \vee y \tag{5.29}$$

$$= \left(F^{-1}\left(\sup_{u\in[0,t)} S^x(u) \right) \wedge \theta \right) \vee y, \tag{5.30}$$

Since (5.27) is an increasing function, then $\sup_{u\in[0,t)} F^{-1}(S^x(u)) = F^{-1}(\sup_{u\in[0,t)} S^x(u))$.

5.2 The optimal solution for the model in Subsection 3.2.1 by the Hamilton-Jacobi-Bellman equation for one firm

5.2.1 Verification theorem

In this section we aim to characterize the value function of the problem presented in Subsection 3.2.1 Chapter 3

$$V(x,y) = \mathcal{J}(x,y,I^*) = \sup_{I\in\mathcal{I}[0,\infty)} \mathcal{J}(x,y,I). \tag{5.31}$$

providing a verification theorem. Observe that for a non installation strategy I^0, we have

$$\mathcal{J}(x,y,0) = \mathbb{E}\left[\int_0^\infty e^{-\rho s} e^{X^x(s)} y^\alpha ds\right] = y^\alpha \int_0^\infty e^{-\rho s} \mathbb{E}\left[e^{X^x(s)}\right] ds.$$

The last expectation can be written as

$$\begin{aligned} \mathbb{E}\left[e^{X^x(s)}\right] &= \mathbb{E}\left[e^{xe^{-\kappa s} + \zeta(1-e^{-\kappa s}) + \sigma\int_0^s e^{-\kappa(s-\tau)}dW(\tau)}\right] \\ &= e^{xe^{-\kappa s} + \zeta(1-e^{-\kappa s})} \mathbb{E}\left[e^{\int_0^s \sigma e^{-\kappa(s-\tau)}dW(\tau)}\right]. \end{aligned}$$

Recall now $\int_0^s \sigma e^{-\kappa(s-\tau)} dW(\tau) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa s})\right)$. Therefore the last expectation can be expressed in terms of the moment generating function of a normal distributed variable. Hence we have

$$\mathbb{E}\left[e^{\int_0^s \sigma e^{-\kappa(s-\tau)}dW(\tau)}\right] \quad = \quad e^{\frac{\sigma^2}{4\kappa}(1-e^{-2\kappa s})}$$

and therefore

$$\mathcal{J}(x,y,0) \quad = \quad y^{\alpha}\int_0^{\infty} e^{-\rho s}e^{xe^{-\kappa s}+\zeta(1-e^{-\kappa s})+\frac{\sigma^2}{4\kappa}(1-e^{-2\kappa s})}ds.$$

We set

$$Q(x) \quad := \quad \int_0^{\infty} e^{-\rho s}e^{xe^{-\kappa s}+\zeta(1-e^{-\kappa s})+\frac{\sigma^2}{4\kappa}(1-e^{-2\kappa s})}ds \tag{5.32}$$

and finally,

$$\mathcal{J}(x,y,0) \quad = \quad y^{\alpha}Q(x). \tag{5.33}$$

The derivation of the variational inequality is analogous to that in Section 1.2, Chapter 1. Therefore, the equation for the strategy in which it is optimal to install is derived from the dynamic programming principle, while the equation of immediately increase the power level is obtained by increasing by ϵ the optimal control. We arrive to the following variational inequality which should satisfied the candidate value function $w(x,y)$

$$\max\{e^x y^{\alpha} - \rho w(x,y) + \mathcal{L}w(x,y), w_y(x,y) - c\} = 0, \tag{5.34}$$

where $\mathcal{L}$ is the infinitesimal generator of the the Ornstein-Uhlenbeck process and the boundary condition $w(x,\theta) = \theta^{\alpha}Q(x)$.

Proposition 5.2.1 *There exist a constant $K > 0$ such that for all $(x,y) \in \mathbb{R}\times[0,\theta]$ one has*

$$|V(x,y)| \leq e^{K(|x|+1)}. \tag{5.35}$$

Moreover, $V(x,\theta) = y^{\alpha}Q(\theta)$ and V is increasing in x.

Proof. Let $(x,y) \in \mathbb{R}\times[0,\theta]$ be given and fixed. To prove the lower bound of the value function V we take the non installation strategy I^0 and since $y \in [0,\theta]$, we obtain

$$V(x,y) \geq Q(x)y^{\alpha} \geq -e^{K_1(|x|+1)}$$

for some $K_1 > 0$. To determine the upper bound, we apply Ito's formula to find that

$$
\begin{aligned}
|e^{-\rho t}e^{X^x(t)}| &\leq |x| + \rho\int_0^t e^{-\rho s}e^{X(s)}ds + \int_0^t e^{-\rho s}\kappa(|\zeta| - |X(s)|)e^{X^x(s)}ds \\
&+\frac{\sigma^2}{2}\int_0^t e^{-\rho s}e^{X^x(s)}ds + \sigma\int_0^t e^{-\rho s}e^{X^x(s)}dW(s).
\end{aligned}
$$

Implying that,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\geq 0}|e^{-\rho t}e^{X^x(t)}|\right] &\leq |x| + C_1\left(\int_0^\infty e^{-\rho s}\mathbb{E}\left[e^{X^x(s)}\right]ds + \int_0^\infty e^{-\rho s}\mathbb{E}\left[X^x(s)e^{X^x(s)}\right]ds\right) \\
&+\sigma\mathbb{E}\left[\sup_{t\geq 0}\int_0^t e^{-\rho s}e^{X^x(s)}dW(s)\right]
\end{aligned}
$$

for some C_1, then by an application of the Burkholder-Davis-Gundy inequality [47, Theorem 3.8, Chapter 3] yields

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\geq 0}|e^{-\rho t}e^{X^x(t)}|\right] &\leq |x| + C_1\left(\int_0^\infty e^{-\rho s}\mathbb{E}\left[e^{X^x(s)}\right]ds + \int_0^\infty e^{-\rho s}\mathbb{E}\left[X^x(s)e^{X^x(s)}\right]ds\right) \\
&+C_2\mathbb{E}\left[\left(\int_0^\infty e^{-2\rho s}e^{2X(s)}ds\right)^{1/2}\right]
\end{aligned}
$$

for some constant $C_2 > 0$ and therefore

$$
\mathbb{E}\left[\sup_{t\geq 0}|e^{-\rho t}e^{X^x(t)}|\right] \leq e^{C(1+|x|)}, \tag{5.36}
$$

for some constant $C > 0$, since it follows from standard calculations that $\mathbb{E}\left[\left(\int_0^\infty e^{-2\rho s}e^{2X(s)}ds\right)^{1/2}\right] \leq \left(\mathbb{E}\left[\int_0^\infty e^{-2\rho s}e^{2X(s)}ds\right]\right)^{1/2}$ $\mathbb{E}\left[X^x(s)e^{X^x(s)}\right] \leq \mathbb{E}\left[\sup_{s\geq 0}X^x(s)e^{X^x(s)}\right] \leq C_3\mathbb{E}\left[\left(\int_0^\infty \sigma^2 e^{X^x(s)}ds\right)^{1/2}\right]$ and $\mathbb{E}\left[e^{X(s)}\right] \leq e^{C_4(1+|x|)}$, for some constant $C_3, C_4 > 0$. Hence we find that for any $I \in \mathcal{I}(y)$

$$
\mathcal{J}(x,y,I) \leq \mathbb{E}\left[\int_0^\infty e^{-\rho s}e^{X^x(s)}(Y^y(s))^\alpha ds\right] \leq \theta^\alpha\mathbb{E}\left[\int_0^\infty |e^{-\rho s}e^{X^x(s)}|ds\right] \leq e^{K_2(1+|x|)}
$$

We prove the exponential growth by setting $K = \max\{K_1, K_2\}$.

If $y = \theta$, then the only admissible strategy is I^0, thus $V(x,y) = \theta^\alpha Q(x)$. To prove that $x \to V(x,y)$ is increasing, let $x_1 > x_2$, then $\mathcal{J}(x_1,y,I) \geq \mathcal{J}(x_2,y,I)$, implying $V(x_1,y) \geq V(x_2,y)$. ■

Theorem 5.2.2 *(Verification theorem). Suppose there exists a function* $w : \mathbb{R}\times[0,\theta] \to \mathbb{R} \in C^{2,1}(\mathbb{R}\times[0,\theta])$ *such that* w *solves the variational inequality* (5.10) *with boundary condition* $w(x,\theta) =$

$\theta^\alpha Q(x)$ and satisfies the growth condition (5.35). Then $w \geq v$ on $\mathbb{R}\times[0,\theta]$. Moreover, suppose that for all initial values $(x,y)\in\mathbb{R}\times[0,\theta)$, there exists a process $I^*\in\mathcal{I}(y)$ such that

$$(X^x(t),(Y^y)^*(t))\in\bar{\mathbb{W}}, \forall t\geq 0, \qquad \mathbb{P}\text{-}a.s., \tag{5.37}$$

$$I^*(t)=\int_{0^-}^t \mathbb{1}_{\{(X^x(t),(Y^y)^*(t))\in\mathbb{I}\}}dI^*(s) \qquad \forall t\geq 0,\ \mathbb{P}\text{-}a.s., \tag{5.38}$$

where $(Y^y)^*(t)$ is the process associated to $I^*(t)$. Then we have

$$V(x,y)=w(x,y),(x,y)\in\mathbb{R}\times[0,\theta],$$

and I^* is optimal, that is, $V(x,y)=\mathcal{J}(x,y,I^*)$.

Proof. (Sketch) The proof of the theorem is analogous to the proof of [50, Theorem 3.2]. In the first step it is shown that $w\geq V$ on $\mathbb{R}\times[0,\theta)$. In the second step it is shown that $w\leq V$ on $\mathbb{R}\times[0,\theta)$ and the optimality of I^* satisfying (5.37) and (5.38).

Step 1. Let $(x,y)\in\mathbb{R}\times[0,\theta)$ be given and fixed and $I\in\mathcal{I}(y)$. For $N>0$ set $\tau_{R,N}:=\tau_R\wedge N$, where $\tau_R:=\inf\{s>0: X^x(s)\notin(-R,R)\}$. Applying Ito's formula to $e^{-\rho\tau_{R,N}}w(X^x(\tau_{R,N}),Y^y(\tau_{R,N}))$ and using the hypothesis of exponential growth of the function w and that it solves the variational inequality (5.10), we arrive to

$$\mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s}e^{X(s)}(Y^y(s))^\alpha ds - c\int_0^{\tau_{R,N}} e^{-\rho s}dI(s)\right]\leq w(x,y)+\mathbb{E}\left[e^{-\rho\tau_{R,N}}e^{K(1+|X^x(\tau_{R,N})|)}\right] \tag{5.39}$$

For the expectation in the right hand side we have the

$$\begin{aligned}\mathbb{E}\left[e^{-\rho\tau_{R,N}}e^{K(1+|X^x(\tau_{R,N})|)}\right] &\leq \mathbb{E}\left[e^K e^{-\rho\tau_{R,N}/2}\sup_{t\geq 0}e^{-\rho t/2}e^{K|X^x(t)|}\right]\\ &\leq \mathbb{E}\left[e^K e^{-\rho\tau_{R,N}}\right]^{1/2}\mathbb{E}\left[\sup_{t\geq 0}e^{-\rho t}e^{2K|X^x(t)|}\right]^{1/2}\\ &\leq \mathbb{E}\left[e^K e^{-\rho\tau_{R,N}}\right]^{1/2}e^{C(1+|x|)/2},\end{aligned}$$

where we have used Holder inequality and the estimation in (5.36). Therefore

$$\lim_{N\uparrow\infty}\lim_{R\uparrow\infty}\mathbb{E}\left[e^{-\rho\tau_{R,N}}e^{K(1+|X^x(\tau_{R,N})|)}\right]=0.$$

Observe also that

$$\Big|\int_0^{\tau_{R,N}} e^{-\rho s}e^{X(s)}(Y^y(s))^\alpha ds - c\int_0^{\tau_{R,N}} e^{-\rho s}dI(s)\Big|\leq \theta^\alpha\int_0^\infty e^{-\rho s}e^{X^x(s)}+c\theta.$$

Then, by the dominated convergence theorem we get

$$\mathcal{J}(x, y, I) \leq w(x, y).$$

Since $I \in \mathcal{I}(y)$ is arbitrary we have

$$V(x, y) \leq w(x, y).$$

Step 2. Consider an admissible process $I^* \in \mathcal{I}(y)$ such that it satisfies (5.37) and (5.38). Following the procedure of step 1, we obtain that the inequality (5.39) become equality, taking limits we find that $\mathcal{J}(x, y, I^*) \geq w(x, y)$. Since clearly $V(x, y) \geq \mathcal{J}(x, y, I^*)$, then $V(x, y) \geq w(x, y)$. By step 1, we conclude $V(x, y) = w(x, y)$ and I^* is optimal. ■

5.2.2 Constructing the optimal solution

To begin we make the educated guess that there exists an injective function $F : [0, \theta] \to \mathbb{R}$, called the free boundary, which separate the waiting and installation regions, such that

$$\mathbb{W} = \{(x, y) \in \mathbb{R} \times [0, \theta) : x < F(y)\}, \tag{5.40}$$
$$\mathbb{I} = \{(x, y) \in \mathbb{R} \times [0, \theta) : x \geq F(y)\}. \tag{5.41}$$

On the waiting region $\mathbb{W}$, the value function must satisfy

$$\frac{\sigma^2}{2}\frac{\partial^2 w(x, y)}{\partial x^2} + \kappa(\zeta - x)\frac{\partial w(x, y)}{\partial x} - \rho w(x, y) + e^x y^\alpha = 0. \tag{5.42}$$

A particular solution of (5.42) is $y^\alpha Q(x)$. On the other hand, the two fundamental solutions of the homogeneous equation

$$\frac{\sigma^2}{2}\frac{\partial^2 w(x, y)}{\partial x^2} + \kappa(\zeta - x)\frac{\partial w(x, y)}{\partial x} - \rho w(x, y) = 0 \tag{5.43}$$

are given by the strictly positive increasing function

$$\psi(x) = e^{\frac{\kappa(x-\zeta)^2}{2\sigma^2}} D_{-\frac{\rho}{\kappa}}\left(-\frac{x-\zeta}{\sigma}\sqrt{2\kappa}\right)$$

and the strictly decreasing positive function

$$\phi(x) = e^{\frac{\kappa(x-\zeta)^2}{2\sigma^2}} D_{-\frac{\rho}{\kappa}}\left(\frac{x-\zeta}{\sigma}\sqrt{2\kappa}\right)$$

with

$$D_\gamma(x) = \frac{e^{-x^2/4}}{\Gamma(-\gamma)} \int_0^\infty t^{-\gamma-1} e^{-\frac{t^2}{2} - xt} dt,$$

$\gamma < 0$. Therefore a general solution of Equation (5.42) is given by

$$w(x, y) = A(y)\psi(x) + B(y)\phi(x) + y^\alpha Q(x).$$

Notice that for a fixed $y \geq 0$, the solution ϕ grows exponentially fast as $x \to -\infty$, then by the structure of the waiting region we should have $B(y) \equiv 0$, hence our candidate solution is

$$w(x, y) = A(y)\psi(x) + y^\alpha Q(x).$$

Furthermore, on $\mathbb{I}$, $w(x, y)$, should satisfy

$$w_y(x, y) - c = 0, \tag{5.44}$$

implying

$$w_{xy}(x, y) = 0. \tag{5.45}$$

For $(x, y) \in \bar{\mathbb{W}} \cap \mathbb{I}$, i.e. for $x = F(y)$, in light of (5.44) and (5.45), our candidate value function $w(x, y)$ should satisfy

$$A'(y)\psi(F(y)) + \alpha y^{\alpha-1} Q(F(y)) - c = 0 \tag{5.46}$$

and

$$A'(y)\psi^{'}(F(y)) + \alpha y^{\alpha-1} Q^{'}(F(y)) = 0. \tag{5.47}$$

Substituting the expression for $A'(y)$ obtained from (5.46) into (5.47), we obtain

$$\frac{c - \alpha y^{\alpha-1} Q(F(y))}{\psi(F(y))} = -\frac{\alpha y^{\alpha-1} Q^{'}(F(y))}{\psi^{'}(F(y))},$$

Define $\bar{x} = F(\theta)$. The inverse function F^{-1} of F defined on $[0, \bar{x}]$ is

$$F^{-1}(x) = \left(\frac{\alpha}{c} \left(Q(x) - Q^{'}(x) \frac{\psi(x)}{\psi^{'}(x)} \right) \right)^{1/(1-\alpha)}. \tag{5.48}$$

Remark 5.2.3 *Even though we are not considering impact by the control in the shock process, in this case the free boundary is not a constant as in the O-U case seen in Chapter 2, this is because the revenue is not linear on the control.*

5.2.3 The optimal strategy an the value function: verification

In the following the initial price level at which the agent start to increase the level of investment is denoted by $x_0 = F(0)$ and we define $\bar{x} = F(\theta)$. Since F is strictly increasing, its inverse exist and it is denoted by F^{-1}. We divide the installation region (5.41) $\mathbb{I}$ into

$$\mathbb{I}_1 := \{(x,y) \in \mathbb{R} \times [0,\theta) : x \in [F(y), \bar{x})\},$$

and

$$\mathbb{I}_2 := \{(x,y) \in \mathbb{R} \times [0,\theta) : x \geq \bar{x}\},$$

.

Following the previous discussion, we establish the candidate value function $w : \mathbb{R} \times [0,\theta] \to \mathbb{R}$ as

$$w(x,y) = \begin{cases} A(y)\psi(x) + Q(x)y^{\alpha} & \text{if } x \in \mathbb{W} \cup ((\infty, \bar{x}) \times \bar{x}) \\ A(F^{-1}(x))\psi(F^{-1}(x)) + Q(F^{-1}(x))y^{\alpha} - c(F^{-1}(x) - \theta), & \text{if } x \in \mathbb{I}_1, \\ y^{\alpha}Q(x) - c(\theta - y) & \text{if } c \in \mathbb{I}_2 \cup ([\bar{x}, \infty) \times \theta) \end{cases} \tag{5.49}$$

Proposition 5.2.4 *The function w from* (5.49) *is a $C^{2,1}(\mathbb{R} \times [0,\theta])$ solution to*

$$\max\{\mathcal{L}w(x,y) - \rho w(x,y) + e^{x}y^{\alpha}, w_y(x,y) - c = 0\}, \tag{5.50}$$

with boundary condition $w(x,\theta) = \theta^{\alpha}Q(x)$,

Proof. The proof is analogous to [50, Proposition 4.7]. ■
The optimal strategy is

$$I^*(t) = \left(\sup_{u \in [0,t)} F^{-1}(S^x(u)) \wedge \theta \right) \vee y \tag{5.51}$$

$$= \left(F^{-1}\left(\sup_{u \in [0,t)} S^x(u) \right) \wedge \theta \right) \vee y, \tag{5.52}$$

with F^{-1} defined in (5.48).

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