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# Co-Homology and Intersection Theory for Feynman Integrals

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**Abstract**

Feynman Integrals play a pivotal role in the computation of multi-loop Scattering Amplitudes, and so they are of vital importance for our capabilities of making predictions. Recent advances from multiple experimental sides of fundamental Physics, range from the detection of colliding point-like particles at the Large Hadron Collider to the observation of gravitational waves associated to black holes merging. They challenge us to understand and control better than ever the intimate nature of those integrals.

A key step in the study of Feynman Integrals consists in realizing that their otherwise insurmountable complexity can be significantly mitigated exploiting suitable Integration by Parts identities. Thanks to these it is possible to build a huge set of linear relations among those integrals and exploit them, through extensive linear algebra manipulations, in order to identify a set of independent building blocks, known as Master Integrals. Master Integrals can be thought of as a basis in the full space of integrals which can appear in the calculation.
Equally importantly, Integration by Parts identities allow deriving differential equations for Master Integrals. Differential equations constitute another fundamental tool, since solving them analytically, or numerically, it is possible to determine the expressions and eventually numerical values for the basis of integrals, and thereby for the full family of Feynman Integrals, under consideration.

In this thesis we show how the powerful approach of differential equations, in particular the so called canonical form, can be used in order to evaluate Feynman Integrals arising in some models relevant for Dark Matter detection. The corresponding Master Integrals, involving different masses in internal and external states, are expressed in terms of Generalized Polylogarithms.

Moreover, in this work we discuss the role of a mathematical framework−(probably) not so common to theoretical physicists−known as twisted (Co)Homology or, more colloquially, Intersection Theory. It has recently emerged that this framework offers a new view and a new perspective towards Feynman Integrals, elucidating some of their properties along with offering new computational tools. Notably, within this theory, it is possible to build the so called co-homology intersection number, which acts as a scalar product among Feynman Integrals. Loosely speaking, following a by now well established Amplitude tradition, several important aspects emerge once we embed our construction into the complex plane, and exploit its richness. Intersection numbers turn to be built upon basic ingredients, such as Residues and Stokes' theorem. Thanks to these new techniques the decomposition in terms of Master Integrals can be obtained by means of a−conceptually clear−projection formula.

**Sommario**

Gli integrali di Feynman hanno un ruolo centrale nel calcolo delle Ampiezze di Scattering a molti loop, quindi sono oggetti di vitale importanza per la nostra capacità di fare previsioni. I recenti sviluppi dal lato sperimentale, che coinvolgono avanzamenti che vanno dalla rivelazioni di urti di particelle puntiformi al Large Hadron Collider fino all'osservazione di onde gravitazionali prodotte a seguito della fusione di buchi neri, ci spingono a investigare e capire la natura più profonda di questi integrali meglio di qunato fatto fino ad ora.

Un punto cruciale nello studio degli integrali di Feynamn consiste nel realizzare che la loro complessitaà, altrimenti insormontabile, può essere smussata grazie ad opportune relazioni note come Integration by Parts. Infatti, grazie a queste, è possibile costruire un sistema di equazioni lineari, che può essere manipolato—o, in un certo senso, risolto, grazie a dispendiose manipolazioni algebriche—fino ad identificare degli ingredienti fondamentali noti come Master Integrals. I Master Integrals possono essere pensati come una sorta di base nello spazio di tutti gli integrali che possono essere coinvolti nel calcolo. Altresì, le Integration by Parts consentono di derivare opportune equazioni differenziali per i Master Integrals. Le equazioni differenziali costituiscono un altro strumento di fondamentale importanza; infatti, risolvendole, analiticamente o numericamente, è possibile determinate l'espressione, e da ultimo i valori numerici, per la base di integrali, e quindi per ogni possibile integrale all'interno della famiglia in considerazione.

In questa tesi si mostra come l'approcio basato sulle equazioni differenziali, e in particolare la cosiddetta canonical form, può essere adottato per la valutazione di integrali di Feynman che compaiono in certi modelli rilevanti per l'osservazione della Dark Matter. I corrispondenti Master Integrals, che coinvolgono stati intermedi ed esterni massivi, sono espressi in termini di Generalized Polylogarithms.

Inoltre si discute il ruolo di una nuova teoria matematica—probabilmente non così comune ai fisici teorici—nota come twisted (Co)Homology, o, in gergo, Intersection Theory. Recentemente è emerso che questa teoria offre una nuova visione e prospettiva sugli integrali di Feynman, facendo chiarazza su alcuni loro aspetti e offrendo nuovi strumenti computazionali. In particolare è possibile costruire il cosiddetto co-homology intersection number, che agisce come una sorta di prodotto scalare per gli integrali di Feynman. Approssimativamente, seguendo una tendenza ormai stabilita nello studio delle Ampiezze di Scattering, molti aspetti emergono in modo naturale estendendo lo studio al piano complesso, sfruttandone le sue peculiarità. Gli intersection numbers sono costruiti a partire da semplici ingredienti fondamentali, quali il calcolo dei Residui e il teorema di Stokes. Grazie a queste nuove tecniche la decomposizione in termini di Master Integrals tramite una, concettualmente chiara, formula di proiezione.

This work is based on the research carried on by the author during his Ph.D. studies at Università degli Studi di Padvoa, Dipartimento di Fisica e Astronomia “Galileo Galilei”.

**Articles:**

**1**. Hjalte Frellesvig, Federico Gasparotto, Stefano Laporta, Manoj K. Mandal, Pierpaolo Mastrolia, Luca Mattiazzi, and Sebastian Mizera, *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers*, JHEP 05 (2019), p. 153, doi: 10.1007/JHEP05(2019)153, arXiv: 1901.11510 [hep-ph].[1]

**2**. Hjalte Frellesvig, Federico Gasparotto, Manoj K. Mandal, Pierpaolo Mastrolia, Luca Mattiazzi, and Sebastian Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, Phys. Rev. Lett. 123.20 (2019), p. 201602, doi: 10.1103/PhysRevLett.123.201602, arXiv: 1907.02000 [hep-th].

**3**. Hjalte Frellesvig, Federico Gasparotto, Stefano Laporta, Manoj K. Mandal, Pierpaolo Mastrolia, Luca Mattiazzi, and Sebastian Mizera, *Decomposition of Feynman Integrals by Multivariate Intersection Numbers*, JHEP 03 (2021), p. 027. doi: 10.1007/JHEP03(2021)027, arXiv:2008.04823 [hep-th].

**3**. Raghuveer Garani, Federico Gasparotto, Pierpaolo Mastrolia, Henrik J. Munch, Sergio Palomares-Ruiz, and Amedeo Primo, *Two-photon exchange in leptophilic dark matter scenarios*, JHEP 12 (2021), p. 212, doi: 10.1007/JHEP12(2021)212, arXiv: 2105.12116 [hep-ph].

**4**. Vsevolod Chestnov, Federico Gasparotto, Manoj K. Mandal, Pierpaolo Mastrolia, Saiei J. Matsubara-Heo, Henrik J. Munch, and Nobuki Takayama, *Macaulay Matrix for Feynman Integrals: Linear Relations and Intersection Numbers*, JHEP 09 (2022), p. 187, doi: 10.1007/JHEP09(2022)187, arXiv:2204.12983 [hep-th].

**5**. Vsevolod Chestnov, Hjalte Frellesvig, Federico Gasparotto, Manoj K. Mandal, and Pierpaolo Mastrolia, *Intersection Numbers from Higher-order Partial Differential Equations*, arXiv:2209.01997 [hep-th].

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**6**. Dhimiter D. Canko, Federico Gasparotto, Luca Mattiazzi, Costas G. Papadopoulos, and Nikolaos Syrrakos, *$N^3$LO calculations for $2 \to 2$ processes using Simplified Differential Equations*, SciPost

[1]This work was accepted for publication before the official start of the Ph.D. program. The author was supported by the post-lauream research fellowship: “Supporting Talent and ReSearch at Padova University”−“Diagrammalgebra, The Algebraic Structure of Quantum Interactions”.

Phys. Proc. 7 (2022), p. 028, doi: 10.21468/SciPostPhysProc.7.028,
arXiv: 2110.08110 [hep-ph].

**7**. Federico Gasparotto, and Manoj K. Mandal, *On the Application of Intersection Theory to Feynman Integrals: the multivariate case*, PoS, MA2019 (2022), p. 019, doi: 10.22323/1.383.0019.

# Contents

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# List of Acronyms

**DM** Dark Matter

**DR** Dimensional Regularization

**FIs** Feynman Integrals

**GPLs** Generalized Polylogarithms

**HPLs** Harmonic Polylogarithms

**MIs** Master Integrals

**QCD** Quantum Chromodynamics

**QFT** Quantum Field Theory

**SM** Standard Model of Particle Physics

We use (Co)Homology when we refer *both* to Homology *and* Co-Homology.

# 1

# Introduction

The capability of making quantitative predictions is at the core of Physics [1]. The framework of Quantum Field Theory (QFT), which binds together Quantum Mechanics and Special Relativity, successfully describes at least three of the four forces in Nature at the quantum level—namely the electromagnetic, strong and weak nuclear forces—through the Standard Model (SM) of Particle Physics. Scattering Amplitudes are the key link between theory and experiment, since they allow converting abstract objects—such as quantum fields and Lagrangians—into observables. Following a well established approach, Scattering Amplitudes are traditionally computed via perturbation theory—roughly speaking relying on a series expansion w.r.t. a small parameter—in terms of Feynman diagrams. The Leading Order (LO) in the above-mentioned series offers, very often, an inadequate estimate of the physical quantity under consideration. The situation progressively improves considering the Next-to-Leading Order (NLO) term, Next-to-Next-to-Leading Order (NNLO) term and (if possible) Next-to-Next-to-Next-to-Leading Order (NNNLO) term. A brief look at fig. (1.1)—which, roughly speaking, describes the production rate of the Higgs boson at Hadron Collider—speaks louder than words. Historically, higher order corrections (i.e. beyond LO) related to quantities such as leptonic (in particular electron and muon) magnetic moments kept physicists busy since the early days of QFT [3]. They are still of vital importance nowadays in order to identify possible tiny and elusive signals of New Physics beyond the SM, such as the putative $g-2$ anomaly—see e.g. [4]. Remarkably, given the impressive and recent efforts in the field of gravitational wave physics, higher order precise predictions starts to be, and they will be—with no doubts—in the future, mandatory in General Relativity—the theory which describes, at the classical level, the fourth fundamental force of Nature namely gravity; see e.g. [5] and references therein.

Nothing is for free. These higher order corrections come hand in hand with lots of enigmas and complications, both from the physics and mathematics point of view. The source of complexity can be identified with the presence of ubiquitous, unavoidable and involved dimensionally regulated Feynman Integrals (FIs) [6, 7]. In a certain sense, these integrals describe the presence of

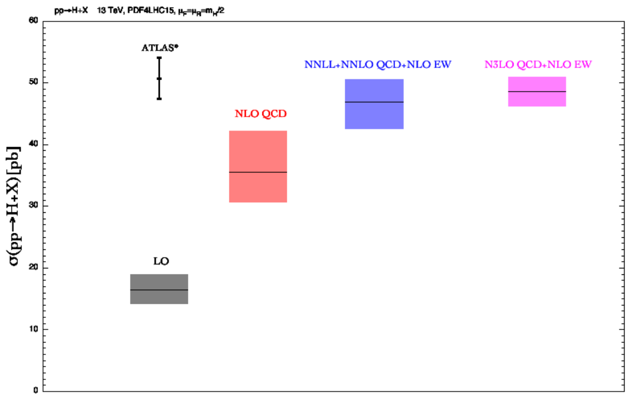



Figure 1.1: **Production rate of Higgs boson at the Large Hadron Collider**. Experimental measurement denoted with ‡, theoretical predictions with color bands. Figure adapted from [2].

virtual—i.e. non-physical—states circulating in closed circuits, dubbed as loops, which emerge in the diagrammatic approach proposed by Feynman.

Uncovering the deepest structure underlying those integrals is of crucial importance. On the one hand, this can boost significantly our ability to make predictions, helping the interplay among theoretical ideas and experimental evidence. On the other hand, it can improve—and, in more radical scenarios, modify—our understanding of QFT offering us new ways of organizing calculations, explaining the surprising simplifications often encountered in concrete computations and even uncovering unexpected relations. Besides this, FIs represent a unique arena in order to test novel mathematical ideas, and they offer a data mine to (dis)prove older conjectures and formulate new ones.

When dealing with FIs, one is confronted with (at least) two unpleasant aspects: their proliferation and the complexity of their evaluation. In this respect it is for sure desirable to discard as much as possible "redundant"—i.e. non independent—objects and focus, or, better, re-express everything, in terms of truly basic building blocks: the so-called Master Integrals (MIs). The quest for this operation is the source of the *algebraic complexity* associated to FIs.
Traditionally this step is accomplished by means of (suitable) Integration By Parts identities (IBPs) fulfilled by dimensionally regulated integrals [8, 9], as well as a certain (extremely efficient variant of) a linear algebra technique [10] which ultimately goes back to Gauss.

The other source of complexity—often referred to as *analytic complexity*—associated to FIs is, in fact, inherent in the evaluation of MIs. In principle, having in mind purely phenomenological applications, (floating point) numbers are all what we need. Nevertheless analytic results can teach us much more. For example, understanding the class of functions describing scattering processes is undoubtedly a deep and fascinating question. In the past years the method of differential equations for MIs [11, 12, 13]—and in particular the so-called canonical form [14]—was proven to be both extremely powerful and insightful—see e.g. [15, 16, 17, 18, 19] for advanced multi-loop, multi-leg and multi-scale applications. Since the early days of these techniques, it was quickly realized that Generalized Polylogarithms (GPLs) [20] play a central role in multi-loop calculus; uncovering their deep mathematical anatomy was vital for the development of the field. Nonetheless, it has long been known [21] that they do not exhaust the full class of functions appearing in amplitudes, and other structures—loosely referred to as elliptic functions—are under scrutiny these days, see e.g. [22] and reference therein.

In recent years, it was observed that many essential aspects of FIs within Dimensional Regularization (DR) are captured by the theory of twisted (Co)Homology. This theory was originally developed by a school of Japanese mathematicians during the last quarter of the previous century (see e.g. the monographs [23, 24])—and it is still prosperous nowadays—mainly to study the properties of hypergeometric integrals. Notably, this theory allows us to build the co-homology intersection number [25, 26, 27, 28, 29, 30, 31], or, simply, intersection number: these latter can be thought of, in practice, as a sort of "scalar product" among FIs. It is therefore possible to express any given FIs in terms of a basis of MIs in a natural way, namely through a simple projection; in this way the brute force system solving procedure is avoided [32, 33, 34, 35, 36].

It is fair to say that, at the time of writing, we are not yet at a stage in which these new techniques are competitive (in terms of performances) with the more mature and well established ones. Nevertheless we believe that twisted (Co)Homology offers a new and comprehensive framework capable to shed new light on FIs. New structures, which could be invisible from more standard approaches, may not be uncovered yet and, at the same time, it is not unreasonable to expect improvements and boost even concerning efficiency in the (not too far) future. Other important aspects of FIs captured by twisted (Co)Homology are considered in the recent literature, see [37, 38] for applications in the context of canonical basis of MIs, [39, 40] for the link among FIs, Twisted (Co)Homology and Generalized Unitarity, as well as the relation with the diagrammatic coaction [41, 42, 43]. We refer the reader to [44, 45, 46, 47, 48] for reviews and comprehensive overview articles on these topics.

Remarkably the same mathematical framework offers penetrating insights also in the context of Amplitudes within String Theory—the best candidate to describe gravity at the quantum level—see [49, 50], as well as Scattering Amplitudes in the Cachazo-Ye-Yuan formulation of QFT [27].

Having at our disposal such a comprehensive and unifying framework, is—to say the least—encouraging and promising; our preliminary successful studies call—rather urge—for further commitment and dedications.

This work is organized as follows. In chapter 2 we review basic notions about FIs in DR, such as IBPs, reduction onto MIs as well as the method of differential equations traditionally employed to evaluate the latter. Beside the standard representation in momentum space, we employ also alternative representations, in particular Baikov representation [51, 52, 53] (and, more briefly, the Lee-Pomeransky representation [54]) since, employing those, the link to twisted (Co)Homology is more transparent.

In chapter 3 we apply part of the machinery introduced, in particular the method of differential equations to evaluate some quantities−namely 2 loop form factors involving masses in intermediate and external states−which turn to be relevant in some models describing Dark Matter (DM) detection.

In chapter 4 we give a gentle overview to the theory of Twisted (Co)Homology−focusing on integrals admitting a univariate representation. We introduce the twisted (co)homology groups and show how to build pairings among those (and their dual)−the so-called intersection numbers−which will play an important role in the following discussion.

In fact, in chapter 5 we show how, thanks to intersection numbers, it is possible to build linear and quadratic relations for certain class of integrals−hypergeometric integrals−often arising in the mathematical literature. Crucially, we also show how, in the case of FIs, we can obtain reduction onto Master Integrals working with maximally cut Baikov representation and intersection numbers.

In chapter 6 we move to the study of twisted Co-Homology associated to integrals admitting a multivariate representation. The pivotal point consists in building a multivariate generalization of the co-homology intersection number. We review several [28, 30, 29] strategies to achieve this goal.

In chapter 7 we derive linear relations among multivariate integrals via intersection numbers. We highlight the peculiarities of FIs, compared to the hypergeometric integrals traditionally considered in the mathematical literature.

In chapter 8 we draw our conclusions, pointing out possible future directions and improvements.

In appendix A we give some details concerning technical aspects concerning the multivariate intersection numbers.

In appendix B we give the expressions for several mathematical functions which appear in the main text.

# 2

# Feynman Integrals

In this chapter we review standard notions about FIs in DR. We start from the very definition of FIs in momentum space (as they arise from Feynman Rules), and we introduce two alternative representations, the Baikov representation and the Lee-Pomeransky representation. We describe how, thanks to IBPs, it is possible to obtain linear relations among FIs and, exploiting them, how it is possible to identify a basis of MIs. We review the powerful method of differential equations, and in particular the so-called "canonical form", which is of crucial importance in order to evaluate MIs themselves.

## 2.1 Integral representation: momentum space

In the traditional approach to perturbative (Q)FT we consider a given multi-loop amplitude $\mathcal{A}$ expressed in terms of Feynman diagrams, and the corresponding analytic expression being dictated by Feynman rules. We denote with $\Gamma^{\mu_1\cdots\mu_n}$ the expression obtained from $\mathcal{A}$, stripping off all the information about external states (e.g. polarization vectors, spinors), i.e. schematically[1]

$$\mathcal{A} \sim \bar{u}\,\Gamma^{\mu_1\cdots\mu_n}\,u\,\epsilon_{\mu_1}\ldots\epsilon_{\mu_n} \rightsquigarrow \Gamma^{\mu_1\cdots\mu_n}. \tag{2.1}$$

$\Gamma^{\mu_1\cdots\mu_n}$ can be further decomposed as

$$\Gamma^{\mu_1\cdots\mu_n} = \sum_i \mathcal{T}_i^{\mu_1\cdots\mu_n}\,\mathcal{F}_i, \tag{2.2}$$

where all the Dirac and Lorentz structures are now incorporated in the $\mathcal{T}_i$. The various $\mathcal{F}_i$, referred to as *form factors*, retain all the loop dependence. The form factors $\mathcal{F}_i$ can be extracted from eq. (2.2), upon the application of suitably chosen *projectors*. Each $\mathcal{F}_i$ is a linear combina-

[1] We consider the case of just two external spinors for ease of notation.

tions of scalar multi-loop integrals: the so-called *Feynman integrals* (FIs); they will be the main focus of this work.

Given $(p_1, \ldots, p_E)$ independent external momenta, a generic $\ell$-loop FI in dimensional regularization reads

$$I_{a_1,a_2,\ldots,a_n} = \int \prod_{j=1}^{\ell} \frac{d^d k_j}{i\,\pi^{d/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}; \tag{2.3}$$

Several comments are in order; the $D_j$s with $1 \le j \le n$, referred to as (inverse) *propagators*, read

$$D_j = q_j^2 - m_j^2, \qquad q_j = \sum_{r=1}^{\ell} \mathbb{A}_{jr} k_r + \sum_{r=1}^{E} \mathbb{B}_{jr} p_r, \qquad 1 \le j \le n; \tag{2.4}$$

The number $n$ amounts to $n = \ell\,(\ell+1)/2 + lE$; the elements of $(D_1, \ldots, D_n)$ are in *one to one* correspondence with the set of scalar products involving at least one loop momentum. For $\ell \ge 2$, the set $(D_1, \ldots, D_n)$ is, in general, larger than the set of denominators appearing in a given Feynman diagrams, and the additional elements are often referred to as *irreducible scalar products* (ISPs).

The indices $(a_1, \ldots, a_n)$ are integers, so $a_i \in \mathbb{Z}$ for $1 \le i \le n$; we will often adopt the convention $|a| = \sum_{i=1}^{n} a_i$.

The collections of all the possible integrals of the form of eq. (2.3), defines an *integral family*. We can organize its elements in *sectors*—often called *topologies*. More precisely, given any FI within an integral family, we can associate to it an integer, say ID, according to the following rule

$$I_{a_1,a_2,\ldots,a_n} \rightsquigarrow \mathtt{ID} = \sum_{j=1}^{n} 2^{j \cdot \Theta(a_j - \frac{1}{2})} - 1; \tag{2.5}$$

integrals with the same ID, fall into the same sector. So a sector is a collection of integrals characterized by the fact that a subset of the propagators appear with powers which are strictly positive integers in eq. (2.3). Such propagators will be often called *denominators*. *Subsectors*-or subtopologies-are characterized by a smaller collection of the above-mentioned subset of propagators[2].

There is a natural way to associate to each integral in a given sector a $\ell$-loop *graph*; the momentum flowing in each internal edge is given by eq. (2.4), while external edges are associated to external momenta; momentum conservation is enforced at each vertex—see [44] for a detailed and proper mathematical discussion.

[2]We will usually adopt the splitting $n = n_{\text{den}} + n_{\text{ISP}}$ and list the denominators first, i.e.

$$(D_1, \ldots, D_n) = (D_1, \ldots, D_{n_{\text{den}}}, D_{n_{\text{den}}+1}, \ldots, D_{n_{\text{den}}+n_{\text{ISP}}}),$$

while dealing with a given sector.

Moreover it is also useful to associate to any given FI the following

$$I_{a_1,a_2,\ldots,a_n} \rightsquigarrow \begin{cases} r &= \sum_{i=1}^{n} a_i\, \Theta\left(a_i - \frac{1}{2}\right), \\ s &= -\sum_{i=1}^{n} a_i\, \Theta\left(-a_i + \frac{1}{2}\right). \end{cases} \tag{2.6}$$

The quantities introduced in eq. (2.6) describes somehow the "*complexity*" of the integral: the bigger they are, the more complicated the corresponding FI is expected to be. When we are just interested in the values of $\mathtt{ID}$, $r$ and $s$, we will employ the notation $I_{\mathtt{ID},r,s}$ or $I_{r,s}$.

Eq. (2.3) is *not* the only possible incarnation of FIs. Via some manipulations we can obtain FIs expressed in different representations compared to eq. (2.3). We will consider in this work mostly the Baikov representation [51, 52, 53] and the Lee-Pomeransky representation [54] (along with the standard Schwinger and Feynman ones). Each of them is useful since some aspects beyond FIs may turn out to be more transparent in one particular representation than in others. We review these representations hereafter, see also the pedagogical treatments in [55, 44].

## 2.2

## Integral representation: Baikov representation

The main observation behind Baikov representation is that eq. (2.3) is a *scalar integral* and the loop momenta appear contracted in the scalar products $k_i \cdot k_k$ and $k_i \cdot p_m$ with $1 \leq i, j \leq \ell$ and $1 \leq m \leq E$. Therefore we may try to rewrite eq. (2.3) in such a way that these scalar products become the integration variables.

### 2.2.1

### One loop case

Let us consider first a $\ell = 1$ FI in Euclidean signature. We introduce two different bases of $\mathbb{R}^d$, say $\mathcal{E} = (e_1, \ldots e_d)$ and $\mathcal{E}' = (p_1, \ldots, p_E, e'_{E+1}, \ldots, e'_d)$. On the one hand $\mathcal{E}$ is the standard orthonormal basis of $\mathbb{R}^d$, i.e.

$$e_i \cdot e_j = \delta_{ij}, \qquad 1 \leq i, j \leq d. \tag{2.7}$$

On the other hand, $\mathcal{E}'$ is built completing $(p_1, \ldots, p_E)$, namely the independent external momenta, to a basis of $\mathbb{R}^d$ in such a way that

$$p_i \cdot e'_j = 0, \qquad 1 \leq i \leq E, \quad E+1 \leq j \leq d; \tag{2.8a}$$

$$e'_i \cdot e'_j = \delta_{ij}, \qquad E+1 \leq i, j \leq d. \tag{2.8b}$$

Let us consider the express the loop momentum $k$ into $\mathcal{E}$ and $\mathcal{E}'$

$$
\begin{aligned}
k &= k^1 e_1 + \cdots + k^E e_E + k^{E+1} e_{E+1} + \cdots + k^d e_d && (2.9a)\\
&= k_{||}^1 p_1 + \cdots + k_{||}^E p_E + k_{\perp}^{E+1} e'_{E+1} + \cdots + k_{\perp}^d e'_d. && (2.9b)
\end{aligned}
$$

The two different set of coordinates are related by the following

$$
\begin{pmatrix} k^1 \\ \vdots \\ k^d \end{pmatrix} = \mathbb{P} \begin{pmatrix} k_{||}^1 \\ \vdots \\ k_{\perp}^d \end{pmatrix}, \tag{2.10}
$$

with the columns of $\mathbb{P}$ being the elements of $\mathcal{E}'$ expressed in the basis $\mathcal{E}'$.

$$
\mathbb{P} = \begin{pmatrix} | & & | & | & & | \\ p_1 & \ldots & p_E & e'_E & \ldots & e'_d \\ | & & | & | & & | \end{pmatrix}. \tag{2.11}
$$

The original integral eq. (2.3) is expressed in terms of $(k^1, \ldots, k^d)$; we can can rearrange it in terms of $(k_{||}^1, \ldots, k_{\perp}^d)$ upon changing the integral measure according to

$$
\frac{d^d k}{\pi^{d/2}} = \det \mathbb{P} \, \frac{d^E k_{||} \, d^{d-E} k_{||}}{\pi^{d/2}}. \tag{2.12}
$$

Thanks to eq. (2.8a) and (2.8b) we have

$$
\det \mathbb{P} = \sqrt{\det \mathbb{P}^{\top} \ \det \mathbb{P}} = \sqrt{\det \mathbb{P}^{\top} \, \mathbb{P}} = \sqrt{\det \mathbb{G}(p_1, \ldots, p_E)} = \sqrt{\mathbf{G}(p_1, \ldots, p_E)} \tag{2.13}
$$

where $\mathbb{G}(p_1, \ldots, p_E)$ is the Gram matrix built upon the external momenta

$$
(\mathbb{G}(p_1, \ldots, p_E))_{ij} = p_i \cdot p_j, \tag{2.14}
$$

and

$$
\mathbf{G}(p_1, \ldots, p_E) = \det \mathbb{G}(p_1, \ldots, p_E). \tag{2.15}
$$

Furthermore, recalling the decomposition eq. (2.9b), we have

$$
\begin{aligned}
k \cdot p_j &= \left( \sum_{i=1}^{E} k_{||}^i p_i + \sum_{i=E+1}^{d} k_{\perp}^i e'_i \right) \cdot p_j \\
&\overset{(2.8a)}{=} \sum_{i=1}^{E} p_j \cdot p_i \, k_{||}^i.
\end{aligned} \tag{2.16}
$$

So we obtain the following

$$\begin{pmatrix} k \cdot p_1 \\ \vdots \\ k \cdot p_E \end{pmatrix} = \mathbb{G}(p_1, \ldots, p_E) \begin{pmatrix} k_{\|}^1 \\ \vdots \\ k_{\|}^E \end{pmatrix}; \tag{2.17}$$

thus

$$\frac{d^d k}{\pi^{d/2}} = (\mathbf{G}(p_1, \ldots, p_E))^{-1/2} \, \frac{\prod_{i=1}^{E} d(k \cdot p_i) \; d^{d-E} k_\perp}{\pi^{d/2}}. \tag{2.18}$$

Furthermore, recalling once again eqs. (2.8a, 2.8b,2.9b) and the shift invariance of the Gram determinant, we have

$$\mathbf{G}(k_1, p_1, \ldots, p_E) = \mathbf{G}\left(\sum_{i=E+1}^{d} k_\perp^i e_i', p_1, \ldots, p_E\right) = \det\begin{pmatrix} k_\perp^2 & \mathbb{0} \\ \mathbb{0} & \mathbb{G}(p_1, \ldots, p_E) \end{pmatrix} = k_\perp^2 \, \mathbf{G}(p_1, \ldots, p_E). \tag{2.19}$$

Using standard $(d-E)$-dimensional spherical coordinates, eq. (2.19) and the fact that $k \cdot k$ and $k_\perp^2$ are related by a linear shift, we have:

$$\begin{aligned} \int_0^\infty d^{d-E} k_\perp &= \frac{\pi^{\frac{d-E}{2}}}{\Gamma\left(\frac{d-E}{2}\right)} \int_0^\infty dk_\perp^2 \, \left(k_\perp^2\right)^{\frac{d-E-2}{2}} \\ &= \frac{\pi^{\frac{d-E}{2}}}{\Gamma\left(\frac{d-E}{2}\right)} \int_\gamma d(k \cdot k) \, \left(\frac{\mathbf{G}(k, p_1, \ldots, p_E)}{\mathbf{G}(p_1, \ldots, p_E)}\right)^{\frac{d-E-2}{2}}, \end{aligned} \tag{2.20}$$

where the integration region $\gamma$ is dictated by the requirement

$$\gamma \;\rightsquigarrow\; \frac{\mathbf{G}(k, p_1, \ldots, p_E)}{\mathbf{G}(p_1, \ldots, p_E)} > 0. \tag{2.21}$$

So, combining everything, we have:

$$\frac{d^d k}{\pi^{\frac{d}{2}}} = \frac{\prod_{i=1}^{E} d(k \cdot p_i) \, d(k \cdot k)}{\pi^{\frac{E}{2}} \, \Gamma\left(\frac{d-E}{2}\right)} \, (\mathbf{G}(p_1, \ldots, p_E))^{\frac{-d+E+1}{2}} \, (\mathbf{G}(k_1, p_1, \ldots, p_E))^{\frac{d-E-2}{2}}. \tag{2.22}$$

Finally, recalling that the set of denominators $\mathbf{z} = \{z_1, \, \ldots, \, z_{E+1}\} = \{D_1, \, \ldots, \, D_{E+1}\}$ are linear in the scalar products $\mathbf{s} = (k \cdot k, k \cdot p_1, \ldots, k \cdot p_E)$, namely

$$\mathbf{z} = \mathbb{A}\mathbf{s} + \mathbf{c}, \tag{2.23}$$

we arrive at the following

$$I^E_{a_1, \ldots a_{E+1}} = \frac{(\mathbf{G}(p_1, \ldots, p_E))^{\frac{-d+E+1}{2}}}{\pi^{\frac{E}{2}} \, \Gamma\left(\frac{d-E}{2}\right) \, \det \mathbb{A}} \int_\gamma (\mathcal{B}(\mathbf{z}))^{\frac{d-E-2}{2}} \prod_{i=1}^{E+1} dz_i \, z_i^{-a_i}, \tag{2.24}$$

where $\mathcal{B}(\mathbf{z})$, the so-called Baikov polynomial, is nothing but

$$\mathcal{B}(\mathbf{z}) = \mathbf{G}(k_1, p_1, \dots, p_E)\Big|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}. \tag{2.25}$$

## 2.2.2

## Multi-loop case

We can extend eq. (2.24) for the case $\ell > 1$; in order to do this we can just apply the argument outlined above focusing on a single loop momentum at a time; at the $i-$th step, the "external" space is spanned by $(k_{i+1},\ \dots,\ k_\ell,\ p_1,\ \dots,\ p_E)$.
We just notice that, once all the loop momenta are exhausted, we obtain a remarkable simplification among Gram determinants arising in the intermediate stages

$$\begin{aligned}\prod_{i=1}^{\ell} \frac{d^d k_i}{\pi^{\frac{d}{2}}} &= \prod_{1\le i,j\le \ell} d(k_i\cdot k_j) \prod_{\substack{1\le i\le \ell\\ 1\le j\le E}} d(k_i\cdot p_j) \prod_{j=1}^{\ell} \frac{\pi^{-\frac{E+\ell-j}{2}}}{\Gamma\left(\frac{d-(E+\ell-j)}{2}\right)} \frac{(\mathbf{G}(k_j,\dots,k_\ell,p_1,\dots,p_E))^{\frac{d-(E+\ell-j)-2}{2}}}{(\mathbf{G}(k_{j+1},\dots,k_\ell,p_1,\dots,p_E))^{\frac{d-(E+\ell-j-1)-2}{2}}}\\ &= \prod_{1\le i,j\le \ell} d(k_i\cdot k_j) \prod_{\substack{1\le i\le \ell\\ 1\le j\le E}} d(k_i\cdot p_j) \frac{\pi^{-\frac{\ell E}{2}-\frac{\ell(\ell-1)}{4}}}{\prod_{j=1}^{\ell}\Gamma\left(\frac{d-(E+\ell-j)}{2}\right)} \frac{(\mathbf{G}(k_1,\dots,k_\ell,p_1,\dots p_E))^{\frac{d-E-\ell-1}{2}}}{(\mathbf{G}(p_1,\dots,p_E))^{\frac{d-E-1}{2}}}.\end{aligned} \tag{2.26}$$

Only the innermost and the outermost survive.

Finally, expressing the scalar products in terms of denominators (cf. eq. (2.23)), we land on

$$I^E_{a_1,\dots,a_n} = \frac{(\mathbf{G}(p_1,\dots,p_E))^{\frac{-d+E+1}{2}}}{\pi^{\frac{\ell E}{2}+\frac{\ell(\ell-1)}{4}} \prod_{j=1}^{\ell}\Gamma\left(\frac{d-E-j+1}{2}\right)\ \det\mathbb{A}} \int_\gamma (\mathcal{B}(\mathbf{z}))^{\frac{d-\ell-E-1}{2}} \prod_{i=1}^{n} dz_i\, z_i^{-a_i}, \tag{2.27}$$

where

$$\mathcal{B}(\mathbf{z}) = \mathbf{G}(k_1,\dots,k_\ell,p_1,\dots,p_E)\big|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}. \tag{2.28}$$

The integration region $\gamma$ in eq. (2.27) deserves a comment.
Applying the same reasoning as in eq. (2.21) to each loop, we obtain the set of constraints

$$\gamma_j \rightsquigarrow \frac{\mathbf{G}(k_j,\dots,k_\ell,p_1,\dots p_E)}{\mathbf{G}(k_{j+1},\dots,k_\ell,p_1,\dots p_E)} > 0, \qquad 1\le j\le \ell; \tag{2.29}$$

so

$$\gamma = \gamma_\ell \cap \dots \cap \gamma_1. \tag{2.30}$$

A point laying on the boundary of $\gamma$ belongs to at least some $\gamma_j$, with $1\le j\le \ell$. By eq. (2.29) this means that

$$\mathbf{G}(k_j,\dots k_\ell,p_1,\dots p_E) = 0, \qquad \text{for some } j. \tag{2.31}$$

Eq. (2.31) tells us that there is degeneracy (i.e. that the vectors $(k_j, \dots k_\ell, p_1, \dots p_E)$ are not independent). In particular it implies a fortiori

$$\mathbf{G}(k_1, \dots k_\ell, p_1, \dots p_E) = 0. \tag{2.32}$$

Therefore the integration region $\gamma$ is such that

$$\mathcal{B}(\partial\gamma) = 0. \tag{2.33}$$

Eq. (2.27) can be continued to Minkowski space, see [56, 57] for detailed discussions. Through this work we will use the MATHEMATICA code associated to [58] for the generation of Baikov polynomials.

As it was shown in [59, 58, 60, 56], one of the main advantages of Baikov representation is that it is well suited for considering *cut integrals*. Roughly speaking a cut integral consists in one or more internal particles particles going "on-shell", and so considering $\frac{1}{D_j} \to \delta(D_j)$ for some $j \in (1, \dots, n_{\text{den}})$. In Baikov representation this prescription simply amounts to consider the residue of the corresponding variable at $0$, i.e. $\delta(D_j) \rightsquigarrow \text{Res}_{z_j=0}(\bullet)$, for some $j \in (1, \dots, n_{\text{den}})$.

## 2.3

## Integral representation: Lee-Pomeransky representation

In order to derive the Lee-Pomeransky representation we have to pass through standard Feynman and Schwinger representations. Let us reconsider eq. (2.3); we assume all the $a_j$ to be positive[3] and recast it as

$$I_{a_1,\dots,a_n} = (-1)^{|a|} \int \prod_{j=1}^{\ell} \frac{d^d k_j}{i\,\pi^{\frac{d}{2}}} \frac{1}{\mathrm{D}_1^{a_1} \dots \mathrm{D}_n^{a_n}}, \tag{2.34}$$

where

$$|a| = \sum_{j=1}^{n} a_j, \qquad \text{and} \qquad \mathrm{D}_j = -D_j = -q_j^2 + m_j^2, \qquad 1 \le j \le n. \tag{2.35}$$

We can then employ the *Schwinger trick* and write

$$\frac{1}{\mathrm{D}^a} = \frac{1}{\Gamma(a)} \int_{\mathbb{R}_+} dz z^{a-1} \exp(-z\,\mathrm{D}), \tag{2.36}$$

and so eq. (2.34) becomes

$$I_{a_1,\dots,a_n} = \frac{(-1)^{|a|}}{\prod_{j=1}^{n}\Gamma(a_j)} \int_{\mathbb{R}_+^n} \left( \prod_{j=1}^{n} dz_j\, z_j^{a_j-1} \right) \int \prod_{j=1}^{\ell} \frac{d^d k_j}{i\,\pi^{d/2}} \exp\left[ -\sum_{i=1}^{n} z_i \left(-q_j^2 + m_j^2\right) \right]. \tag{2.37}$$

[3] In order to fix the ideas we can think $\ell = 1$ FI, or a $\ell > 1$ FI where we *do not* include ISPs. hence $n = n_{\text{den}}$.

For later convenience we rewrite

$$\sum_{i=1}^{n} z_i \left(-q_j^2 + m_j^2\right) = -\sum_{r,s=1}^{\ell} k_r \, \mathbb{M}_{rs} \, k_s + 2 \sum_{r=1}^{\ell} k_r \, \nu_r + J, \tag{2.38}$$

where the indices $r, s$ label different loop momenta, $\mathbb{M}$ is a *symmetric* matrix and Lorentz indices are contracted.
Being $\mathbb{M}$ symmetric there exists and *orthogonal* matrix $\mathbb{O}$ such that

$$\mathbb{M} = \mathbb{O}^{\top} \, \mathbb{D} \, \mathbb{O}, \qquad \mathbb{O}^{\top} \, \mathbb{O} = \mathbb{1}, \tag{2.39}$$

where $\mathbb{D}$ is diagonal.
We can consider the change of variables:

$$k_r \to \sum_{s=1}^{\ell} \mathbb{O}^{\top}_{rs} \, k_s + \mathbb{M}^{-1}_{rs} \, \nu_s, \tag{2.40}$$

in such a way that eq. (2.37) becomes

$$\begin{aligned} I_{a_1,\ldots,a_n} = \frac{(-1)^{|a|}}{\prod_j^n \Gamma(a_j)} \int_{\mathbb{R}^n_+} \left( \prod_{j=1}^{n} dz_j \, z_j^{a_j-1} \right) \exp\left[ -\left( \sum_{r,s=1}^{\ell} \nu_r \, \mathbb{M}_{rs} \, \nu_s + J_s \right) \right] \\ \times \int \prod_{j=1}^{\ell} \frac{d^d k_j}{i \, \pi^{d/2}} \exp\left[ \sum_{r=1}^{\ell} \mathbb{D}_{rr} \, k_r^2 \right]. \end{aligned} \tag{2.41}$$

After performing a *Wick rotation* (see standard textbooks e.g. [61], appendix B), eq. (2.41) is suitable for a gaussian integration; the result reads

$$I_{a_1,\ldots,a_n} = \frac{(-1)^{|a|}}{\prod_{j=1}^n \Gamma(a_j)} \int_{\mathbb{R}^n_+} \left( \prod_{j=1}^{n} dz_j \, z_j^{a_j-1} \right) \exp\left[ -\left( \sum_{r,s=1}^{\ell} \nu_r \, \mathbb{M}_{rs} \, \nu_s + J \right) \right] (\det \mathbb{M})^{-\frac{d}{2}}. \tag{2.42}$$

We introduce the so-called *Symanzik* polynomials

$$\mathcal{U} = \det \mathbb{M}, \qquad \text{and} \qquad \mathcal{F} = \mathcal{U} \left( \sum_{r,s=1}^{\ell} \nu_r \, \mathbb{M}_{rs} \, \nu_s + J \right), \tag{2.43}$$

which are *homogenoues* polynomials in the **z**s variables of degree $\ell$ and $(\ell + 1)$, respectively.
We finally arrive at

$$I_{a_1,\ldots,a_n} = \frac{(-1)^{|a|}}{\prod_{j=1}^n \Gamma(a_j)} \int_{\mathbb{R}^n_+} \left( \prod_{j=1}^{n} dz_j \, z_j^{a_j-1} \right) \exp\left( -\frac{\mathcal{F}}{\mathcal{U}} \right) \mathcal{U}^{-\frac{d}{2}}. \tag{2.44}$$

Eq. (2.44) is the *Schwinger representation*.

Since all the **z**s are non negative, we have the following identity:

$$1 = \int_{-\infty}^{+\infty} dt\, \delta\left(t - \sum_{j=1}^{n} z_j\right) = \int_{0}^{+\infty} dt\, \delta\left(t - \sum_{j=1}^{n} z_j\right). \tag{2.45}$$

Inserting eq. (2.45) in eq. (2.44), rescaling all the integration variables as $z_j \to t\, z_j$ and performing in the $dt$-integration we land on

$$I_{a_1,\ldots,a_n} = (-1)^{|a|} \frac{\Gamma\left(|a| - \frac{\ell\, d}{2}\right)}{\prod_{j=1}^{n}(a_j)} \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^{n} dz_j\, z_j^{a_j - 1}\right) \delta\left(1 - \sum_{j=1}^{n} z_j\right) \frac{\mathcal{U}^{|a| - \frac{(\ell+1)\, d}{2} - 1}}{\mathcal{F}^{|a| - \frac{\ell\, d}{2} - 1}}. \tag{2.46}$$

Eq. (2.46) is known as *Feynman representation*.

Finally we introduce the *Lee-Pomeransky* representation:

$$I_{a_1,\ldots,a_n} = \frac{(-1)^{|a|}}{\prod_{j=1}^{n}\Gamma(a_j)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{(\ell+1)}{2}\, d - |a|\right)} \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^{n} dz_j\, z_j^{a_j - 1}\right) G^{-\frac{d}{2}}, \tag{2.47}$$

where

$$G = \mathcal{U} + \mathcal{F}. \tag{2.48}$$

Once again, inserting the identity: $1 = \int_0^{+\infty} dt\, \delta\left(t - \sum_{j=1}^{n} z_j\right)$ and rescaling $z_j \to t\, z_j$, we can show that eq. (2.47) is equivalent to eq. (2.46).

Eq. (2.46) is clearly ill defined for $a_j < 0$ for some $j \in (1, \ldots, n)$. As discussed in [62, 63], the expressions $\int_0^\infty \frac{dx_j\, x^{a_j}}{\Gamma(a_j)}(\bullet)$ for $a_j \leq 0$ has to be understood as $\int_0^\infty \frac{dx_j\, x^{a_j}}{\Gamma(a_j)}(\bullet) \to (-1)^{a_j} \frac{\partial^{(-a_j)}(\bullet)}{\partial z_j^{(-a_j)}}\bigg|_{z_j=0}$. Moreover, in general the integrand in eq. (2.47) is not, in general, vanishing at the boundary of the integration domain, contrary to Baikov representation.

## 2.4 Integration by Parts Identities

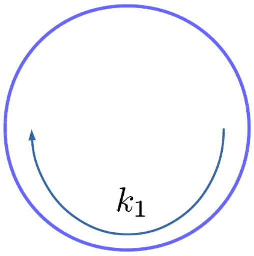
It is for sure desirable to find relations among FIs within a given integral family. On practical grounds, it is certainly convenient to exploit such relations, in order to identify a smaller and minimal set of independent integrals within a given integral family—thus, in a certain sense, avoiding redundancies. Within DR FIs fulfil *Integration by Parts Identities* (IBPs) [8, 9], namely:

$$0 = \int \prod_{j=1}^{\ell} \frac{d^d k_i}{i\pi^{d/2}} \frac{\partial}{\partial k_j^\mu} \left(\frac{\xi^\mu}{D_1^{a_1} \ldots D_n^{a_n}}\right), \tag{2.49}$$

where the vector $\xi^\mu$ belongs to the set $(k_1^\mu, \dots, k_\ell^\mu, p_1^\mu, \dots, p_E^\mu)$ (or linear combinations built upon them).
Performing the algebra in eq. (2.49) (under the integral sign) and reading back the result in terms of FIs, we infer that we obtain *linear identities* among integrals of the following type $I_{\mathtt{ID},r,s}$, $I_{\mathtt{ID},r+1,s}$, $I_{\mathtt{ID},r,s+1}$ and $I_{\mathtt{ID},r+1,s+1}$ (cf. eqs. (2.5,2.6)). The coefficients are *polynomials* in $d$, masses and scalar products among external momenta. Simplifications among numerators and denominators could also produce integrals belonging to subsectors.

Let us consider the following simple example.

**Example.** *One loop massive tadpole. Let us consider the integral family associated with the following graph*[4]




(2.50)

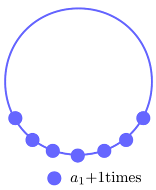
*the (only) denominator present is*

$$D_1 = k_1^2 - m^2. \tag{2.51}$$

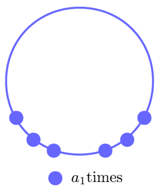
*Eq. (2.49) for $v^\mu = k_1^\mu$ and arbitrary $a_1$ gives:*

$$0 = (d - 2a_1)\, I_{a_1} - 2m^2 a_1\, I_{a_1+1}, \tag{2.52}$$

*which can be rearranged as (a single • denotes a denominator raised to power 2, two • denote a denominator raised to power 3, and so on)*




$$= \frac{d - 2a_1}{2m^2 a_1}$$



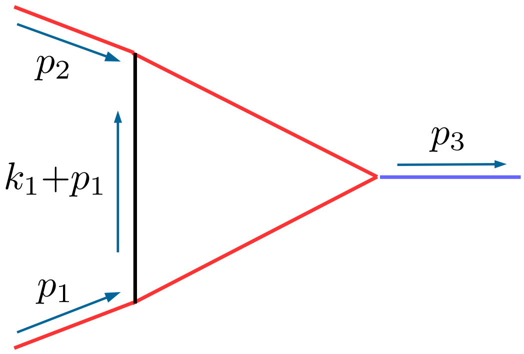

(2.53)

*This implies that applying repeatedly eq. (2.53) any integral within the integral family can be written just in terms of $I_1$.*

Rather than considering IBPs for generic values of $(a_1, a_2, \dots, a_n)$, it is instructive to substitute explicit values and massage the resulting identities a bit.

**Example.** *One loop QED triangle. Let us consider the following graph*[5]

[4] This example is adapted from [64].
[5] This example is adapted from [65].




(2.54)

*The denominators are*

$$D_1 = k_1^2 - m^2, \qquad D_2 = (k_1+p_1)^2, \qquad D_3 = (k_1+p_1+p_2)^2 - m^2. \tag{2.55}$$

*The kinematic is* $p_1^2 = p_2^2 = m^2$ *and* $(p_1+p_2) = p_3^2 = s$.

*Reading eq. (2.49) for the explicit choices* $v^\mu = \{k_1^\mu, p_1^\mu, p_2^\mu\}$ *and* $(a_1, a_2, a_3) = (1,1,1)$ *yields:*

$$0 = (d-4)I_{1,1,1} + \left(s - 2m^2\right) I_{1,1,2} - 2m^2 I_{2,1,1} - I_{0,1,2} - I_{0,2,1}, \tag{2.56a}$$
$$0 = \left(2m^2 - s\right) I_{1,1,2} + 2m^2 I_{2,1,1} + I_{0,1,2} + I_{0,2,1} - I_{1,0,2} - I_{2,0,1}, \tag{2.56b}$$
$$0 = \left(2m^2 - s\right) I_{2,1,1} + 2m^2 I_{1,1,2} - I_{1,0,2} + I_{1,2,0} - I_{2,0,1} + I_{2,1,0}. \tag{2.56c}$$

*Despite its simplicity, this system of equations presents several key aspects of IBPs.*

*After the identifications* $I_{2,1,1} \leftrightarrows I_{1,1,2}$, $I_{2,0,1} \leftrightarrows I_{1,0,2}$ *and* $I_{0,1,2} \leftrightarrows I_{2,1,0}$, *which follow from the invariance of the integrals under shift of the loop momentum, we deduce that*

$$(2.56b) = (2.56c). \tag{2.57}$$

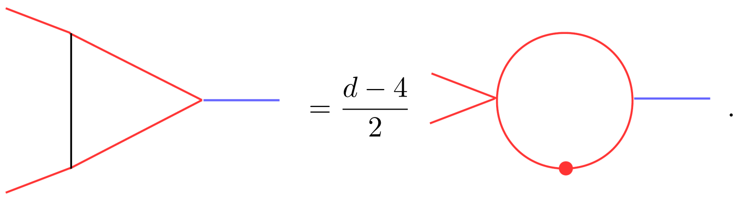
*This means that the last equation does not add any new information to the system, or, in other words, that IBPs contain a huge redundancy.*

*Moreover, summing the first and second equation, we infer*

$$\begin{aligned} 0 &= (2.56a) + (2.56b) \\ &= (d-4)I_{1,1,1} - I_{1,0,2} - I_{2,0,1} \\ &= (d-4)I_{1,1,1} - 2I_{1,0,2}; \end{aligned} \tag{2.58}$$

*where, in the last line, we identified* $I_{1,0,2} \leftrightarrows I_{2,0,1}$ *once again thanks to the invariance upon shift of the integration variable.*

*Diagrammatically this means*



(2.59)

*so, colloquially, we say that "the triangle" is* not independent—*we say it* is reducible *in terms of* (*a suitable*) *"bubble"*.

The previous simple example represent a toy-model for a more systematic treatment of IBPs which goes under the name of *Laporta algorithm* [10].
IBPs can be generated systematically up to certain explicit values of $(r, s)$ for a given sector and all its subsectors (cf. eq. (2.6)). The identities can be cast in a linear system where FIs are considered as unknowns. Remarkably the number of equations grows faster than the number of integrals. We can then sort the integrals according to their *complexity* (as mentioned above, this means that integrals with higher $r$ and $s$ are regarded as more complicated with respect to integrals with lower values of $r$ and $s$). It is possible to process the system via Gauss' elimination, namely scanning it equation by equation and expressing the *most complicated* integral in terms of other integrals, and substituting the resulting relation in all the remaining equations. Eventually, after a back substitution, all the integrals turns to be expressed as *linear combinations* of fewer left-over ones, referred to as *Master Integrals* (MIs). MIs are considered the truly independent objects. Usually the *number* of MIs is several order of magnitudes lower than number of FIs involved in the full system (and in physical applications). In this work we will denote the set of MIs obtained via this system solving procedure as $(\mathcal{J}_1, \ldots, \mathcal{J}_\nu)$. Different variants of the strategy briefly reviewed here are nowadays implemented in several public (and private) computer codes, such as AIR [66], REDUZE [67], FIRE [68], LITERED [69] and KIRA [70]. Let us mention that one of the main obstacles encountered in practical implementations is due to extensive amount of algebraic manipulation required in the system solving procedure. It is often the case that the expressions found in intermediate stages of the calculations are way more involved than the final result. This issue can be overcome relying on the so-called *functional reconstruction techniques* [71, 72]—see also the public implementations FINITEFLOW [73] and FIREFLY [74, 75]. Roughly speaking the idea is to recover the coefficients of the MIs—rational functions in $d$, masses and scalar products among external momenta—from numerical evaluations via a sort of fitting procedure. Assigning explicit values to parameters, the linear system is solved multiple times over (different) finite field(s) $\mathbb{F}_p$ ($p$ prime numbers)[6]: in this way all the operations are *exact*—no numerical instabilities—and *fast*—rational number with large numerators and denominators are absent. Let us mention that, thanks to these techniques, linear-solvers originally designed for the Laporta algorithm become extremely efficient tools also in other contexts.

[6] See also [76] for a previous application of finite fields in the context of IBPs.

## 2.5

## Integration by Parts Identities: Baikov representation

It is instructive to consider IBPs in terms of Baikov representation; this idea was proposed originally in [59] and later improved in several publications [77, 78, 79, 80, 81]. The main advantage of this strategy is that, benefiting from tools of Computational Algebraic Geometry, it is possible to generate a smaller set of IBPs identities—so, in a certain sense, it is possible to "trim" the starting set of identities—doing so the subsequent system solving procedure is facilitated.
Our starting point is the vanishing of a total differential under the integral sign; so the analogue of eq. (2.49) is[7]:

$$0 = \int_\gamma d\left( \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\ldots z_n^{a_n}}\,\xi \right), \tag{2.60}$$

where $\mathcal{B}(\partial\gamma) = 0$, $\xi$ is a $(n-1)$ differential form: $\xi = \sum_{i=1}^{n}(-1)^{i+1}\,\xi_i$, with $\xi_i = \widehat{\xi}\,dz_1\wedge\ldots\wedge\widehat{dz_i}\wedge\ldots\wedge dz_n$ and each $\widehat{\xi}_i$ a polynomial in the variables $\mathbf{z} = (z_1,\ldots,z_n)$ whose coefficients may depend on the kinematic invariants[8].
Rearranging eq. (2.60) we obtain[9]

$$\begin{aligned}
0 &= \int_\gamma d\mathbf{z}\sum_{m=1}^{n}\left[\frac{d-\ell-E-1}{2}\,\partial_m\mathcal{B}\,\frac{\mathcal{B}^{\frac{(d-2)-\ell-E-1}{2}}}{z_1^{a_1}\ldots z_n^{a_n}}\,\widehat{\xi}_m - a_m\frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\ldots z_m^{a_m+1}\ldots z_n^{a_n}}\,\widehat{\xi}_m + \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\ldots z_n^{a_n}}\,\partial_m\,\widehat{\xi}_m\right]\\
&= \int_\gamma d\mathbf{z}\,\frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\ldots z_n^{a_n}}\left[\frac{d-\ell-E-1}{2}\sum_{m=1}^{n}\widehat{\xi}_m\,\frac{\partial_m\mathcal{B}}{\mathcal{B}} - \sum_{i=1}^{n_{\text{den}}}a_i\,\frac{\widehat{\xi}_i}{z_i} - \sum_{j=n_{\text{den}}+1}^{n}a_j\,\frac{\widehat{\xi}_j}{z_j} + \sum_{m=1}^{n}\partial_m\widehat{\xi}_m\right],
\end{aligned} \tag{2.61}$$

where $d\mathbf{z} = dz_1\wedge\ldots\wedge dz_n$.
On the one hand, we notice that the first summand in eq. (2.61) involves $\mathcal{B}^{\frac{(d-2)-\ell-E-1}{2}}$, i.e. we obtain linear relation among integrals living in $(d-2)$ and $d$ dimensions. This can be avoided provided that we can find sets of $n+1$ polynomials $(\widehat{\xi}_1,\ldots,\widehat{\xi}_n,\widehat{\xi}_\mathcal{B})$, fulfilling the following relation

$$\sum_{m=1}^{n}\widehat{\xi}_m\,\partial_m\mathcal{B} = \widehat{\xi}_\mathcal{B}\,\mathcal{B}. \tag{2.62}$$

Eq. (2.62) appears in the context of computational algebraic geometry under the name of syzygy equation. Since $\widehat{\xi}_\mathcal{B}$ can be always retrieved from $(\widehat{\xi}_1,\ldots,\widehat{\xi}_n)$, we will focus on this second, shorter, set.
Sets of polynomials $(\widehat{\xi}_1,\ldots,\widehat{\xi}_n)$ satisfying eq. (2.62) we will denoted it by $M_1$.
On the other hand, the second term in eq. (2.61) is responsible for integrals with denominators

[7]Comparing to the integral representation in eq. (2.27), we remove the common over-all prefactors, without altering integral relations.

[8]In the more refined language of [80], $\widehat{\xi}_i$ belongs to the *ring* $\mathbb{A} = \mathbb{Q}(\mathbf{y})[z_1,\ldots,z_n]$, where $\mathbf{y}$ denotes the kinematic invariants.

[9]See footnote 2 where $n_{\text{den}}$ was introduced.

raised to higher powers. This can be avoided requiring that[10]:

$$\widehat{\xi}_i = z_i\, p_i \qquad 1 \leq i \leq n_{\text{den}}, \tag{2.63}$$

where $(p_1, \ldots, p_{n_{\text{den}}})$ are polynomials in the **z**s variables. The set of polynomials $(\widehat{\xi}_1, \ldots, \widehat{\xi}_n)$ satisfying eq. (2.63) will be denoted by $M_2$[11].
For the case at hand, namely IBPs in Baikov representation, it is easy to solve eq. (2.62) and eq. (2.63) separately, with almost no computational effort, as shown below. The problem is pushed in satisfying both of them simultaneously; this amounts to computing the *module intersection* $M_1 \cap M_2$ [80].

Assuming this is done, namely eq. (2.62) and eq. (2.63) are fulfilled simultaneously, IBPs in Baikov representation read

$$0 = \int_\gamma d\mathbf{z}\, \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1} \ldots z_n^{a_n}} \left[ \frac{d-\ell-E-1}{2}\, \widehat{\xi}_{\mathcal{B}} + \sum_{i=1}^{n_{\text{den}}} \Big( \partial_i \left( z_i\, p_i \right) - a_i\, p_i \Big) + \sum_{j=n_{\text{den}}+1}^{n} \left( \partial_j \widehat{\xi}_j - a_j\, \frac{\widehat{\xi}_j}{z_j} \right) \right]. \tag{2.64}$$

We show here how, for the case at hand, we can build explicitly *generators* for $M_1$ and $M_2$.

Focusing on eq. (2.62) we can exploit the fact that $\mathcal{B}$ is, up to a linear change of variables, the determinant of a symmetric matrix: the *Gram matrix* $\mathbb{G}(k_1, \ldots, k_\ell, p_1, \ldots, p_E)$ (recall that $\det \mathbb{G} = \mathbf{G}$, and cf. eq. (2.28)).
It is straightforward to show that the determinant of a symmetric matrix fulfills the following relation[12]

$$\sum_{k=1}^{\ell+E} (1 + \delta_{jk})\, s_{ik}\, \frac{\partial \det(\mathbb{G})}{\partial s_{jk}} = 2\, \delta_{ij}\, \det(\mathbb{G}). \tag{2.65}$$

Restricting $j$ to $1 \leq j \leq \ell$ and expressing the scalar products in terms of **z**s variables, namely:

$$\mathbf{s} = \mathbb{A}^{-1}(\mathbf{z} - \mathbf{c}), \tag{2.66}$$

then eq. (2.65) reads

$$\sum_{k=1}^{\ell+E} (1 + \delta_{jk})\, s_{ik}\, |_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})} \sum_{m=1}^{n} \frac{\partial \det(\mathbb{G})\, |_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}}{\partial z_m}\, \frac{\partial z_m}{\partial s_{jk}} = 2\, \delta_{ij}\, \det(\mathbb{G})\, |_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}\,, \tag{2.67}$$

[10] See also [82, 83, 84] for syzygy-related ideas in the context of IBPs.
[11] In the language of [80], $M_1$ and $M_2$ are sub-modules of the module $\mathbb{A}^n$.
[12] The authors of [80] acknowledge that idea of exploiting such relations is due to Roman Lee, see also http://mathsketches.blogspot.com/2010/07/blog-post.html.

or equivalently (reintroducing $\mathcal{B} = \det(\mathbb{G})\,|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}$)

$$\sum_{m=1}^{n}\left(\sum_{k=1}^{\ell+E}(1+\delta_{jk})\,s_{ik}\,|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}\,\frac{\partial z_m}{\partial s_{jk}}\right)\frac{\partial \mathcal{B}}{\partial z_m} = 2\,\delta_{ij}\,\mathcal{B}. \tag{2.68}$$

Comparing eq. (2.62) and (2.68) we are left with the following identifications:

$$\widehat{\xi}_m = \left(\sum_{k=1}^{\ell+E}(1+\delta_{jk})\,s_{ik}\,|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}\,\frac{\partial z_m}{\partial s_{jk}}\right) \qquad 1 \le m \le n, \tag{2.69a}$$

$$\widehat{\xi}_B = 2\,\delta_{ij}; \tag{2.69b}$$

with $1 \le j \le \ell$ and $1 \le i \le n$.

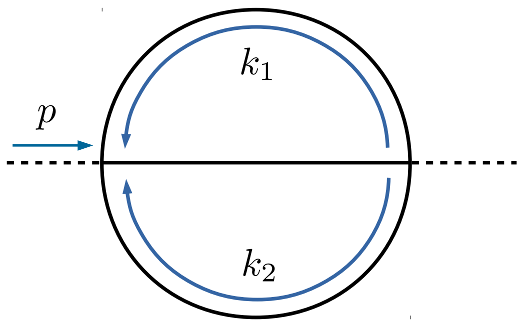
Thus, eq. (2.69a) provides $\ell\cdot n$ solutions of eq. (2.62) and it turns out that they form a generating set of $M_1$.

On the other hand a generating set of $M_2$ is simply given by the rows of the following matrix

$$\left(\begin{array}{cccc|cccc}
z_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
0 & z_1 & \dots & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & 0 & 0 & \ddots & 0\\
0 & 0 & \dots & z_{n_{\text{den}}} & 0 & 0 & \dots & 0\\
\hline
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0\\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0\\
0 & 0 & \ddots & 0 & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{array}\right). \tag{2.70}$$

Finally eqs. (2.62,2.63) can be mutually satisfied considering the *module intersection* of eq. (2.69a) and eq. (2.70).

**Example.** *Two loop massless sunrise. Let us consider the following graph*




(2.71)

*The denominators are chosen as*

$$D_1 = k_1^2, \quad D_2 = k_2^2, \quad D_3 = (k_1 + k_2 - p)^2\,, \tag{2.72}$$

*while the ISPs read*

$$D_4 = (k_1 + p)^2 - 2\,s, \qquad D_5 = (k_2 + p)^2 - 2\,s. \tag{2.73}$$

*The kinematics is* $p^2 = s$.
*The Baikov polynomial is*

$$\begin{aligned} B = &- s^2 z_3 - 2sz_1^2 - 2sz_2^2 - sz_3^2 + 5sz_1z_2 + 3sz_1z_3 + 3sz_2z_3 + sz_1z_4 - 2sz_2z_4 \\ &- sz_3z_4 - 2sz_1z_5 + sz_2z_5 - sz_3z_5 + sz_4z_5 - 3z_1z_2^2 - 2z_2z_4^2 - 2z_1z_5^2 + z_4z_5^2 \\ &- 3z_1^2z_2 + z_1z_2z_3 + 2z_2^2z_4 + 5z_1z_2z_4 - z_2z_3z_4 + 2z_1^2z_5 + z_4^2z_5 + 5z_1z_2z_5 - z_1z_3z_5 \\ &- 3z_1z_4z_5 - 3z_2z_4z_5 + z_3z_4z_5, \end{aligned} \tag{2.74}$$

*and*

$$\frac{d - E - \ell - 1}{2} = \frac{d - 1 - 2 - 1}{2} = \frac{d-4}{2}. \tag{2.75}$$

*The relation between* $\mathbf{z} = \{z_1, \ldots, z_5\}$ *and the scalar products* $\mathbf{s} = \{k_1^2,\, k_2^2,\, k_1{\cdot}k_2,\, k_1{\cdot}p,\, k_2{\cdot}p\}$ *is*

$$\mathbf{z} = \mathbb{A}\mathbf{s} + \mathbf{c} \tag{2.76}$$

*with*

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -2 & -2 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ s \\ -s \\ -s \end{pmatrix}. \tag{2.77}$$

*Let us denote* $\widehat{\xi}^{(i,j)} = (\widehat{\xi}_1, \ldots, \widehat{\xi}_5)$ *the set of polynomials obtained via eq. (2.69a) for a given choice of* $(i, j)$*; we obtain*

$$\begin{aligned} \widehat{\xi}^{(1,1)} &= (2z_1, 0, z_1{-}2z_2{+}z_3{+}z_5, s{+}z_1{+}z_4, 0), \\ \xi^{(1,2)} &= (0, s{-}2z_1{-}2z_2{+}z_3{+}z_4{+}z_5, z_1{-}2z_2{+}z_3{+}z_5, 0, 2s{-}3z_1{-}2z_2{+}z_3{+}2z_4{+}z_5), \\ \widehat{\xi}^{(2,1)} &= (s{-}2z_1{-}2z_2{+}z_3{+}z_4{+}z_5, 0, -2z_1{+}z_2{+}z_3{+}z_4, 2s{-}2z_1{-}3z_2{+}z_3{+}z_4{+}2z_5, 0), \\ \widehat{\xi}^{(2,2)} &= (0, 2z_2, -2z_1{+}z_2{+}z_3{+}z_4, 0, s{+}z_2{+}z_5), \\ \widehat{\xi}^{(3,1)} &= (s{-}z_1{+}z_4, 0, -z_1{-}z_2{+}z_4{+}z_5, 3s{-}z_1{+}z_4, 0), \\ \widehat{\xi}^{(3,2)} &= (0, s{-}z_2{+}z_5, -z_1{-}z_2{+}z_4{+}z_5, 0, 3s{-}z_2{+}z_5). \end{aligned} \tag{2.78}$$

*Therefore* $(\widehat{\xi}^{(1,1)}, \widehat{\xi}^{(1,2)}, \widehat{\xi}^{(2,1)}, \widehat{\xi}^{(2,2)}, \widehat{\xi}^{(3,1)}, \widehat{\xi}^{(3,2)})$ *is a generating set for* $M_1$.

*On the other hand the rows of*

$$\left(\begin{array}{ccc|cc} z_1 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{2.79}$$

*constitute a generating set for $M_2$.*

*Then the module intersection of eq. (2.78) and eq. (2.79)—computed via e.g. SINGULAR*[13] *[86]—yields 9 set of polynomials fulfilling both eq. (2.62) and eq. (2.63).*

*Let us focus on one set of the obtained polynomials*

$$(0, z_2^2+z_1z_2-z_4z_2-z_5z_2, -sz_3+z_2z_3-z_5z_3, 0, -sz_5-sz_1+2sz_2+sz_3-z_5^2+z_1z_5+z_2z_5-z_4z_5); \tag{2.80}$$

*the corresponding IBPs identity, for the explicit choice $(a_1,\, a_2,\, a_3,\, a_4,\, a_5) = (1,\, 1,\, 1,\, 0,\, 0)$ reads*[14]

$$\frac{d-2}{2}\Big( - sI_{1,1,1,0,0} + I_{0,1,1,0,0} + 2I_{1,0,1,0,0} - I_{1,1,1,-1,0} - 2I_{1,1,1,0,-1}\Big) = 0. \tag{2.81}$$

*No square denominator appears. Eq. (2.81) is verified with* LITERED.

## 2.6

## Differential equations for Feynman Integrals: momentum space

Once a minimal set of MIs for a given integral family has been identified, we are left with the problem of its evaluation. Roughly speaking we can group the various possible strategies into two categories, *direct* and *indirect* methods. With direct methods we mean all those approaches which aim at attacking each MI individually, numerically or analytically, often exploiting their deep mathematical structures and benefiting from advanced tools such as *polynomial reduction* and *linear reducibility* [87], along with the implementations in HYPERINT [88], *tropocial integration* [89], *sector decomposition* [90, 91] and its public implementations SECTOR_DECOMPOSITION [91], FIESTA [92] and PYSECDEC [93], just to mention a few.
On contrary, by indirect methods we refer to the idea of solving (analytically or numerically) some (cleverly built) functional relations obeyed by MIs; *dimensional recurrence relations* [94, 52], see also the implementations SUMMERTIME [95] and DREAM [96], *difference equations* [10, 97, 98, 99, 100] and *differential equations* [101, 12, 13]—see also [64, 102, 103, 44, 45] for comprehensive and detailed reviews. In this work we consider the method of differential equations; we summarize the main ideas behind it hereafter.

[13]We used the command SINGULARINTERSECT of its MATHEMATICA interface [85].
[14]We do not apply to this identity any symmetry relations, nor we eliminate dimensionsless integrals.

Let us start with integrals expressed in momentum space eq. (2.3). We assume that a set—or, better in this context, a vector—of MIs has been identified; we denote it with $\mathbf{J} = (\mathcal{J}_1, \ldots, \mathcal{J}_\nu)$. We regard (each entry in) this vector as a function of the internal masses or kinematic invariants, collectively denoted by $\mathbf{y}$, so $\mathbf{J} \rightsquigarrow \mathbf{J}(\mathbf{y})$.

We can then differentiate (under the integral sign) each $\mathcal{J}_i$ w.r.t. a given $y_k \in \mathbf{y}$. The result of this operation will be a linear combination of integrals belonging to the integral family (the action of the derivative is straightforward if we are differentiating w.r.t. any internal mass; in the case of kinematic invariants we have just to rewrite the differential operator in terms of derivative w.r.t. external momenta via the chain rule). Nevertheless we can just IBPs-reduce the above mentioned combination of integrals onto MIs. So far we have

$$\begin{aligned} \partial_{y_k} \mathcal{J}_i(\mathbf{y}) &= \sum_i c_i(\mathbf{y})\, I_i(\mathbf{y}) \\ &\overset{\text{IBPs}}{=} \sum_{j=1}^{\nu} \hat{\mathbf{\Omega}}_{y,ij}(\mathbf{y})\, \mathcal{J}_j(\mathbf{y}). \end{aligned} \tag{2.82}$$

Iterating the procedure for all the MIs, we will obtain a *system of differential equations*

$$\partial_y \mathbf{J}(\mathbf{y}) = \hat{\mathbf{\Omega}}_y(\mathbf{y}) \cdot \mathbf{J}(\mathbf{y}). \tag{2.83}$$

Repeating it for all the $\mathbf{y}$s, we obtain a system of partial differential equations which we can cast in the compact form

$$d_{\mathbf{y}}\mathbf{J}(\mathbf{y}) = \mathbf{\Omega}(\mathbf{y}) \cdot \mathbf{J}(\mathbf{y}), \tag{2.84}$$

where

$$d_{\mathbf{y}}(\bullet) = \sum_{y_k \in \mathbf{y}} \partial_{y_k}(\bullet)\, dy_k \qquad \text{and} \qquad \mathbf{\Omega}(\mathbf{y}) = \sum_{y_k \in \mathbf{y}} \hat{\mathbf{\Omega}}_{y_k}(\mathbf{y}) dy_k. \tag{2.85}$$

Eq. (2.84) deserves some comments. First of all, organizing the vector $\mathbf{J}(\mathbf{y})$ from the simplest integral to the most complicated one—according to the ordering introduced in the Laporta algorithm—it is clear that the resulting system eq. (2.84) is lower block triangular. This is expected since $\partial_y(\bullet)$ cannot produce any new denominator w.r.t. the set already present in the MI it is acting on.

It turns out that not all the differential operators $\partial_y(\bullet)$ are independent. Let us focus on any given MI, say $\mathcal{J}(\mathbf{y}) = \mathcal{J}_{(r,s)}(\mathbf{y})$[15]; rescaling all the masses and all the external momenta according to $(p, m) \to (\lambda\, p, \lambda m)$ (which means $\mathbf{y} \to \lambda^2\, \mathbf{y}$), the integral representation eq. (2.3) reveals that $\mathcal{J}_{(r,s)}(\mathbf{y})$ is an *homogeneous function* of degree $\frac{\ell d}{2} - r + s$, i.e.

$$\mathbf{y} \to \lambda^2 \mathbf{y} \quad \Longrightarrow \quad \mathcal{J}_{(r,s)}(\mathbf{y}) \to \left(\lambda^2\right)^{\frac{\ell d}{2} - r - s}\, \mathcal{J}_{(r,s)}(\mathbf{y}). \tag{2.86}$$

[15]We temporary introduce the dependence on $(r, s)$—see eq. (2.6)—since it will be useful for the considerations in the following.

Then, a theorem by Euler on homogenous functions guarantees that

$$\sum_{y_i \in \mathbf{y}} y_i\, \partial_{y_i} \mathcal{J}_{(r,s)}(\mathbf{y}) = \left( \frac{\ell d}{2} - r + s \right) \mathcal{J}_{(r,s)}(\mathbf{y}). \tag{2.87}$$

Eq. (2.87) tells us precisely that not all the differential operators are independent.

We can consider the matrix-valued version of eq. (2.87), which reads

$$\sum_{y_i \in \mathbf{y}} y_i\, \partial_{y_i} \mathbf{J}(\mathbf{y}) = \sum_{y_i \in \mathbf{y}} y_i\, \hat{\mathbf{\Omega}}_{y_i}(\mathbf{y}) \cdot \mathbf{J}(\mathbf{y}) = \mathrm{Diag}\left( \frac{\ell d}{2} - r_1 + s_1, \dots, \frac{\ell d}{2} - r_\nu + s_\nu \right) \cdot \mathbf{J}(\mathbf{y}), \tag{2.88}$$

Eq. (2.88) represents a non trivial relations for the various $\hat{\mathbf{\Omega}}_y(\mathbf{y})$, and, in practice, it serves as a non trivial check on their correctness. Moreover eq. (2.88) implies that differential equations are trivial in the case of *single scale* problems[16].

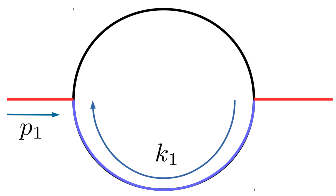
Furthermore the system eq. (2.84) satisfies *integrability condition*, i.e. $d^2_{\mathbf{y}} \mathbf{J}(\mathbf{y}) = 0$, which implies

$$d_{\mathbf{y}} \mathbf{\Omega}(\mathbf{y}) = \mathbf{\Omega}(\mathbf{y}) \wedge \mathbf{\Omega}(\mathbf{y}). \tag{2.89}$$

Once again, eq. (2.89) gives non-trivial checks in practical examples.

In view of the previous discussion, we will work with a set of rescaled adimensional variables $\mathbf{x} = \{x_1, \dots, x_{n-1}\}$ with $x_{i-1} = \frac{y_i}{y_1}$ for $2 \le i \le n$. All the complicated functional dependence of the MIs is encoded in those variables; the dependence on $y_1$ is in fact trivial—it is the only dimensionful variable left—and is dictated by simple dimensional analysis.

**Example.** *The half massive bubble. Let us consider the integral family defined by the following graph*




(2.90)

*the denominators are chose as*

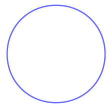
$$D_1 = (k_1 + p_1)^2, \qquad D_2 = k_1^2 - m^2, \tag{2.91}$$

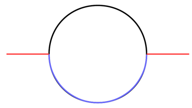
*with* $p_1^2 = s$ *(the internal mass is denoted in blue).*
*Comparing with the main text we have* $\mathbf{y} = (m^2, s)$*; from now on we will drop the the dependence on* $(m^2, s)$ *for ease of notation.*

[16] Nevertheless they can still be applied in such cases, at the price of introducing a new, fictitious, scale-which is eventually set to 0-in order to have a non-trivial system. See e.g. [104, 105, 106].

*Let us consider the set of MIs given by*

$$\mathbf{J} = (\mathcal{J}_1, \mathcal{J}_2) = \Bigg(\quad , \quad \Bigg). \tag{2.92}$$





*It is instructive to consider the action of $\partial_{m^2}(\bullet)$ and $\partial_s(\bullet)$ separately on each element in eq. (2.92).*

*We clearly have*

$$\begin{aligned} \partial_{m^2}\mathcal{J}_1 &= I_{0,2} \overset{\text{IBPs}}{=} \frac{d-2}{m^2}\,\mathcal{J}_1, \\ \partial_{m^2}\mathcal{J}_2 &= I_{1,2} \overset{\text{IBPs}}{=} -\frac{d-2}{m^2(m^2-s)}\,\mathcal{J}_1 + \frac{d-3}{m^2-s}\,\mathcal{J}_2. \end{aligned} \tag{2.93}$$

*In order to consider the action of $\partial_s(\bullet)$ we have just to apply the chain rule $\partial_s(\bullet) = \frac{1}{2s}p^\mu\,\partial_{p^\mu}(\bullet)$; we obtain*[17]

$$\begin{aligned} \partial_s\mathcal{J}_1 &= 0, \\ \partial_s\mathcal{J}_2 &= \frac{1}{2s}\left(I_{2,0} - I_{1,1} + (m^2-s)I_{2,1}\right) \\ &\overset{\text{IBPs}}{=} \frac{d-2}{2(m^2-s)s}\mathcal{J}_1 - \frac{(d-4)s+(d-2)m^2}{m^2-s}\mathcal{J}_2. \end{aligned} \tag{2.94}$$

*Putting in matrix form we have*

$$\hat{\mathbf{\Omega}}_{m^2} = \begin{pmatrix} \frac{d-2}{2m^2} & 0 \\ -\frac{d-2}{2m^2(m^2-s)} & \frac{d-3}{m^2-s} \end{pmatrix}, \qquad \hat{\mathbf{\Omega}}_s = \begin{pmatrix} 0 & 0 \\ \frac{d-2}{2s(m^2-s)} & \frac{(d-2)m^2+(d-4)s}{2s(s-m^2)} \end{pmatrix}. \tag{2.95}$$

*These matrices satisfy the relation*

$$m^2\,\hat{\mathbf{\Omega}}_{m^2} + s\,\hat{\mathbf{\Omega}}_s = \text{Diag}\left(\frac{d}{2}-1, \frac{d}{2}-2\right), \tag{2.96}$$

*as they should.*

*Finally, we can verify the the matrices satisfy the explicit integrability condition*

$$\partial_s\hat{\mathbf{\Omega}}_{m^2} - \partial_{m^2}\hat{\mathbf{\Omega}}_s - [\hat{\mathbf{\Omega}}_s, \hat{\mathbf{\Omega}}_{m^2}] = \mathbb{0}. \tag{2.97}$$

Eq. (2.84) does not exhaust all the inputs needed in order to determine the actual expressions of the MIs. As usual, when dealing with differential equations, we need also an information about the initial conditions or, better, boundary constants. Differential equations for MIs are no different in this respect: we need to determine the boundary values, say $\mathbf{J}_0$, so that the

[17] If needed, rewrite numerators in terms of denominators and simplify common factors

general solution of eq. (2.84) represents the actual expression of the MIs we are aiming to.
On practical calculations, the set of constant $\mathbf{J}_0 = (\mathcal{J}_{0,1}, \ldots, \mathcal{J}_{0,\nu})$ can be determined, for example, requiring that the solution of eq. (2.84) matches the values of the MIs in some particular kinematic limits, where the asymptotic behaviour of the MIs can be obtained by other methods—e.g. *Expansion by Regions* [107]. Nevertheless it is worth stressing that, often, some general considerations such as the expected regularity of MIs in some limits, or the analysis of the differential equation around singular points, provide enough information in order to fix the boundary constants.

All our considerations so far are valid for both numerical and analytic strategies aiming to solve eq. (2.84). Purely numerical approaches [108, 109, 110, 111] and semi-analytic ones [112, 113] together with the implementations DiffExp [114], SeaSide [115] and AMFlow [116] were proven to be extremely effective and powerful in several cutting-edge examples—see e.g. recent works [117, 118, 119, 120, 121].

We discuss here the analytic solution of eq. (2.84), and we tailor the discussion to this approach. First of all we are almost always just interested in the $\epsilon$-expansion of the MIs, say

$$\mathcal{J}_i(\mathbf{x}, \epsilon) = \sum_{k=\text{Min}}^{\text{Max}} \mathcal{J}_{i,k}(\mathbf{x})\, \epsilon^k, \qquad 1 \le i \le \nu, \tag{2.98}$$

where (usually)

$$\epsilon = \frac{4-d}{2}. \tag{2.99}$$

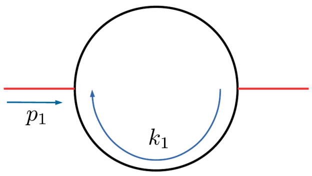
This means that we have just to determine the unknowns in eq. (2.98), without solving the differential equations in full generality.
Moreover, a strategy for solving eq. (2.84) should certainly exploit its lower block triangular structure. Roughly speaking we can start from the simplest block, which, once solved, become the known, non-homogeneous, term of the next-to-simplest block. The next-to-simplest block can in fact be split into an *homogeneous* and a *non-homogeneous* part and can be solved via *Euler variations of constant*. Iterating the procedure bottom-up, we arrive to the final block. In the case of more than one MIs in a given block we are forced to consider a system of coupled differential equations, which can always be cast into a single *higher order* differential equation. On a mathematical ground, no general technique is known in order to obtain a solutions of an (homogeneous) higher order differential equation and we have to rely on guessing or intuition. Nevertheless, for the case of FIs, it was shown that *maximal cut* of a given MI represents the solution of the corresponding homogeneous differential equation [21, 122, 123, 58, 124]. We refer the reader to [64] for the details and many worked out examples of the procedure sketched here.

It turns out that this iterative procedure can be by-passed, and the system can be solved "*all at once*", in a more "*algebraic*" way. In order to explain how this can be done, we need some preliminary considerations.

## 2.6.1

### Canonical Form

It should be clear that the basis of MIs for a given problem is by no means unique. Let us consider the (perhaps most) simple integral family[18][19]:




$$= I_{a_1,a_2} = e^{\epsilon \gamma_E} \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{1}{D_1^{a_1}} \frac{1}{D_2^{a_1}}, \tag{2.100}$$

with

$$D_1 = k_1^2, \qquad D_2 = (k_1 + p_1)^2. \tag{2.101}$$

and $p_1^2 = s$. The integral family we are considering has just $1$ MIs; eq. (2.100) is simple enough that the integration can be performed analytically for arbitrary integers $(a_1, a_2)$, obtaining

$$I_{a_1,a_2} = (-1)^{a_1+a_2} (-s)^{2-a_1-a_2-\epsilon} \frac{\Gamma(a_1 + a_2 - 2 + \epsilon)}{\Gamma(a_1)\,\Gamma(a_2)} \frac{\Gamma(2 - a_1 - \epsilon)\Gamma(2 - a_2 - \epsilon)}{\Gamma(4 - a_1 - a_2 - 2\epsilon)}. \tag{2.102}$$

We can use eq. (2.102) and sample over $(a_1, a_2)$; each choice would correspond to a valid MI. First, let us consider the standard

$$\begin{aligned} I_{1,1} = {} & \frac{1}{\epsilon} + (2 - \log(-s)) + \epsilon \left( \frac{1}{2} \log^2(-s) - 2\log(-s) - \frac{\pi^2}{12} + 4 \right) \\ & + \epsilon^2 \left( -\frac{1}{6} \log^3(-s) + \log^2(-s) + \frac{1}{12} \left(\pi^2 - 48\right) \log(-s) - \frac{7\zeta(3)}{3} - \frac{\pi^2}{6} + 8 \right) + O\left(\epsilon^3\right) \end{aligned} \tag{2.103}$$

and compare with the (at the moment) artificial choice

$$\begin{aligned} (-s)\epsilon I_{2,1} = {} & 1 - \epsilon \log(-s) + \frac{\epsilon^2}{12} \left(6 \log^2(-s) - \pi^2\right) \\ & + \frac{\epsilon^3}{12} \left(-2\log^3(-s) + \pi^2 \log(-s) - 28\zeta(3)\right) + O\left(\epsilon^4\right) \end{aligned} \tag{2.104}$$

The last equation is not only very compact, but it is also organized in a very neat and precise way. Let us attribute a *weight* to the various terms appearing in eqs. (2.103, 2.104). We assign weight $0$ to all rational numbers and rational functions of the kinematics, weight $1$ to $\log(-s)$

[18] The factor $e^{\epsilon\gamma_E}$ is included here in order to have somewhat cleaner formulas in the following discussion. Nevertheless its presence is important if we address more number-theoretical questions related to FIs, see [125, 126].
[19] This example is adapted from [127].

and to $\pi$ and weight $n$ to $\zeta_n$. Moreover we consider that the weight of a product is the sum of the weight of its factors (e.g. the weight of $\pi\ \log(-s)$ is the same as the one of $\log^2(-s)$, and is equal to 2). Assigning then weight $-1$ to $\epsilon$, we notice that each term in eq. (2.104) has weight 0, and so we will refer to it as an expression of *uniform weight* 0. Evidently eq. (2.103) is *not* of uniform weight. Therefore, in general, finding a basis of MIs such that the integrated expression is of uniform weight 0 seems desirable.
This simple analysis suggests that it should be possible to choose a suitable basis of MIs, such that a structure like the one in eq. (2.104) naturally emerges from the differential equation itself. An even more striking fact is that, provided this basis is found, not only the final result is very elegant but also the integration procedure is—to some extent—trivialized.

A remarkable observation was presented in [14]. In this work it was proposed to consider a carefully chosen set of MIs, say $\mathbf{I}(\mathbf{x},\epsilon)$—in general not the set dictated by the Laporta ordering—such that the resulting system of differential equations is cast in the so-called *canonical* form (and the corresponding set of MIs $\mathbf{I}(x,\epsilon)$ is referred to as *canonical basis*)

$$d_{\mathbf{x}}\mathbf{I}(\mathbf{x},\epsilon)=\epsilon\,\mathbf{\Omega}_{\rm c}(\mathbf{x})\,\mathbf{I}(\mathbf{x},\epsilon),\qquad \mathbf{\Omega}_{\rm c}(\mathbf{x})=\sum_{\eta_i\in\mathcal{A}}\mathbb{M}_{\eta_i}\,d\log(\eta_i(\mathbf{x})).\tag{2.105}$$

In eq. (2.105) the regulator $\epsilon$ is factorized from the kinematics. The matrices $\mathbb{M}_\eta$ contained only *rational numbers*, while the set of $d\log$ forms are referred to as *letters*, and their union $\mathcal{A}$ is the *alphabet*. This implies, for example, that each of the term in eq. (2.89) is vanishing independently (since the lhs is proportional to $\epsilon$, while the r.h.s. is proportional to $\epsilon^2$)

$$d_{\mathbf{x}}\mathbf{\Omega}_{\rm c}(\mathbf{x})=0,\qquad \mathbf{\Omega}_{\rm c}(\mathbf{x})\wedge\mathbf{\Omega}_{\rm c}(\mathbf{x})=0.\tag{2.106}$$

Eq. (2.105) can be solved very elegant, convenient and somehow compact way. For the sake of clarity, let us specify the discussion to the case of a single variable $x$.
Let us assume that the boundary vector is represented by $\mathbf{I}(x,\epsilon)|_{x\to0}=\mathbf{I}_0(\epsilon)$. Then the solution of eq. (2.105) (with $\mathbf{x}\to x$) is

$$\mathbf{I}(x,\epsilon)=\mathcal{P}\exp\left(\epsilon\int_0^x\hat{\mathbf{\Omega}}_{\rm c}(t)dt\right)\,\mathbf{I}_0(\epsilon)\tag{2.107}$$

where the *path order exponential* $\mathcal{P}\exp(\bullet)$ appeared, and its first few terms read

$$\mathcal{P}\exp\left(\epsilon\int_0^x\hat{\mathbf{\Omega}}_{\rm c}(t)dt\right)=\mathbb{1}+\epsilon\int_0^x\hat{\mathbf{\Omega}}_{\rm c}(t_1)dt_1+\epsilon^2\int_0^x\hat{\mathbf{\Omega}}_{\rm c}(t_2)dt_2\int_0^{t_2}\hat{\mathbf{\Omega}}_{\rm c}(t_1)dt_2+\mathcal{O}(\epsilon^3).\tag{2.108}$$

We can see that the canonical form offers a clear way to truncate the $\epsilon$ expansion of eq. (2.108), and the integrated expression resemble closely eq. (2.104).
In fact, in the case in which all the arguments of the the $d\log$ form are rational functions of $x$, then eq. (2.108) evaluates to a particular class of functions known as *Generalized Polylogarithms*

(GPLs)[20] [20] ($\log(\bullet), \log^2(\bullet)$ being just special cases). The mathematical aspects of these functions are very well studied and understood and can bee implement in computer codes, such as HPL [131, 132], the MAPLE implementation[133] and POLYLOGTOOLS [134]. Moreover, several efficient numerical routines are available for their evaluation, such us the implementation in HPLOG [135], GINAC [136], CHAPLIN [137], the one of [138], HANDYG [139] , FASTGPL [140]. We briefly review hereafter some of their properties.

Nevertheless, before doing so, we pause for a comment. We may wonder why (and how) we could ever face arguments of $dlog$s which are *not* rational functions. After all differential equations are derived through operations which manifestly involve only rational functions (differentiation, reduction to MIs via Laporta algorithm). The obvious solution to this puzzle is that the canonical basis is related to standard set of MIs (e.g. the one dictated by Laporta criterion) via

$$\mathbf{I}(x,\epsilon) = \mathbf{B}(x,\epsilon)\,\mathbf{J}(x,\epsilon), \qquad \mathbf{B}(x,\epsilon) \rightsquigarrow \text{algebraic function of } x, \tag{2.109}$$

and the entries of $\mathbf{B}(x,\epsilon)$ are algebraic functions (i.e. they are *not* forced to be rational, and the may contain e.g. square roots).
Finding a suitable change of variable(s) which rationalizes simultaneously all the square roots is, in general, a very non-trivial problem−nowadays we have at our disposal the software RATIONALIZEROOTS [141, 142] dedicated to this task. If the arguments are not explicitly rational, then it is not guaranteed that eq. (2.108) (and, in general, iterated integrals) can be expressed in terms of GPLs, see [143] for an explicit counterexample. Let us also stress that eq. (2.105) is not guaranteed to exist for generic FIs. So the situation we will discuss is not the most generic possible set-up; though we are still able to cover a pletora of interesting examples.

GPLs can be introduced in the following recursive way

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;x), \qquad (a_n \neq 0), \tag{2.110}$$

where the various $a_i$, often referred to as *weights*, are elements in $\mathbb{C}$ and

$$G(;x) = 1. \tag{2.111}$$

In the case the rightmost index is $a_n = 0$, we may be worried about a severe divergence at $0$; this is indeed correct, nevertheless we *define*

$$G(\underbrace{0,\dots,0}_{n \text{ times}};x) \equiv \frac{1}{n!}\log^n(x). \tag{2.112}$$

[20] In the study of Differential equations for FIs the class of functions which was systematically studied at the beginning was the one of *Harmonic Polylogarithms* (HPLs) [128]. See also the references in this work for previous incarnations of these functions in related contexts of Physics. More generally the class of functions that are relevant are *iterated integrals* [129, 130].

We will often employ a vector-like notation, where e.g. $\mathbf{a} = (a_1, \ldots, a_n)$[21].
The products of two GPLs (with same argument) is controlled by the so-called *shuffle relations*

$$G(\mathbf{a}; x)\, G(\mathbf{b}; x) = \sum_{\mathbf{c} \in \mathbf{a} \sqcup\!\sqcup \mathbf{b}} G(\mathbf{c}; x), \tag{2.113}$$

where $\mathbf{a} \sqcup\!\sqcup \mathbf{b}$ denotes all the possible lists obtained by joining $\mathbf{a}$ and $\mathbf{b}$, without modifying the relative order among the elements of $\mathbf{a}$ and the ones of $\mathbf{b}$[22]. They are *invariant under rescaling*, i.e. given that $\lambda \in \mathbb{C} \setminus \{0\}$

$$G(\mathbf{a}; x) = G(\lambda \mathbf{a}; \lambda x), \qquad \text{provided } a_n \neq 0. \tag{2.114}$$

Much more than this is known about GPLs; for example they can be equipped with additional structure like the *coproduct* and the *symbol map*, and it is known that they form an *Hopf algebra*; see e.g. [144, 126] and references therein. Such a variety of tools is useful also in practice, in order to study *discontinuities* of GPLs, *derivatives* w.r.t. their weights and in order to capture the−otherwise mysterious−complicated functional identities they satisfy.

## 2.6.2

## Canonical basis via Magnus/Dyson Exponential

In the previous discussion we just assumed that a canonical form exists and it is possible to cast the system in this form. Clearly this is a non trivial task. We discuss now a possible strategy, presented for the first time in [145], in order to cast the system in such a from. We will dub this approach as the *Magnus/Dyson exponential* [146, 147]. Over the years other techniques appeared in order to reach the canonical form; for example, let us mention the analysis basis on the *leading singularity* of the integral, see e.g. [102] and the package DLogBasis [148], various implementation of the strategy of [149], namely Fuchsia [150], epsilon [151] and Libra [152], the programs CANONICA [153, 154] and Initial [155], and the intersection theory-based method [38].

First of all it is instructive to consider a generic change of basis of MIs

$$\mathbf{J}(x, \epsilon) = \mathbf{B}(x, \epsilon)\, \mathbf{F}(x, \epsilon), \qquad \mathbf{B}(x, \epsilon) \rightsquigarrow \text{invertible}, \tag{2.115}$$

[21]The above-mentioned HPLs (cf. footnote 20) are characterized by the fact that $a_i \in (-1, 0, 1)$.
[22]Let us consider $\mathbf{a} = (a_1)$ and $\mathbf{b} = (b_1, b_2)$, then

$$\mathbf{a} \sqcup\!\sqcup \mathbf{b} = \{(a_1, b_1, b_2),\ (b_1, a_1, b_2),\ (b_1, b_2, a_1)\},$$

and so e.g. $(b_2, b_1, a_1) \notin \mathbf{a} \sqcup\!\sqcup \mathbf{b}$ since the order among $b_1$ and $b_2$ is not respected.

and observe that eq. (2.84), for the new basis reads

$$d_x\mathbf{F}(x,\epsilon) = \mathbf{B}^{-1}(x,\epsilon)\cdot(\mathbf{\Omega}(x,\epsilon)\cdot\mathbf{B}(x,\epsilon) - d_x\mathbf{B}(x,\epsilon))\cdot\mathbf{F}(x,\epsilon). \tag{2.116}$$

Our starting point is that—essentially by trial-and-error and experience—it is possible to choose a basis of MIs, say $\mathbf{F}(x,\epsilon)$ such that the system is *linear* in $\epsilon$, i.e.

$$d_x\mathbf{F}(x,\epsilon) = [\mathbf{\Omega}_{x,0}(x) + \epsilon\,\mathbf{\Omega}_{x,1}(x)]\cdot\mathbf{F}(x,\epsilon). \tag{2.117}$$

Therefore (cf. eq. (2.116)) it is sufficient to find a transformation $\mathbf{R}(x)$, with $\mathbf{F}(x,\epsilon) = \mathbf{R}(x)\mathbf{I}(x,\epsilon)$ such that

$$d_x\mathbf{R}(x) = \mathbf{\Omega}_{x,0}(x)\cdot\mathbf{R}(x); \tag{2.118}$$

if this is the case, the system for the new basis $\mathbf{I}(x,\epsilon)$ is $\epsilon$-factorized,

$$d_x\mathbf{I}(x,\epsilon) = \epsilon\,\left[\mathbf{R}^{-1}(x)\cdot\mathbf{\Omega}_{x,1}(x)\cdot\mathbf{R}(x)\right]\cdot\mathbf{I}(x,\epsilon); \tag{2.119}$$

in (very) many practical examples eq. (2.119) is not only $\epsilon$-factorized, but also canonical in the sense of eq. (2.105).
Eq. (2.118) is yet another differential equation, and the reader may wonder whether this is another complicated task: fortunately this is not the case in practice. First of all let us observe that eq. (2.118) does not exhibit any dependence on $\epsilon$. Moreover we further assume (this is often the case in practice) that $\hat{\mathbf{\Omega}}_0(x)$ has the following structure

$$\hat{\mathbf{\Omega}}_{x,0}(x) = \hat{\mathbf{N}}(x) + \hat{\mathbf{D}}(x), \tag{2.120}$$

where $\hat{\mathbf{D}}(x)$ is a *diagonal* matrix and $\hat{\mathbf{N}}(x)$ a *nilpotent* one; to fix the ideas, we assume that it is a *strictly lower triangular* matrix.
We will solve eq. (2.118) in two steps, or, in other words, we consider a solution of eq. (2.118) of the following form (the notation will be clear in a moment)

$$\mathbf{R}(x) = \mathbf{R}_{\mathbf{D}}(x)\cdot\mathbf{R}_{\mathbf{N}'}(x); \tag{2.121}$$

first we look for a transformation $\mathbf{R}_{\mathbf{D}}(x)$ such that

$$d_x\mathbf{R}_{\mathbf{D}}(x) = \mathbf{D}(x)\cdot\mathbf{R}_{\mathbf{D}}(x), \tag{2.122}$$

whose solution reads

$$\mathbf{R}_{\mathbf{D}}(x) = \exp\int_{\bullet}^{x} dx_1\,\hat{\mathbf{D}}(x_1), \tag{2.123}$$

where the integration in eq. (2.123) has to be understood as an indefinite integration
Plugging the ansatz eq. (2.121), with the explicit solution eq. (2.123), into eq. (2.118) we are left

with the following

$$
\begin{aligned}
d_x \mathbf{R}_{\mathbf{N}'}(x) &= \left[\mathbf{R}_{\mathbf{D}}^{-1}(x) \cdot \mathbf{N}(x) \cdot \mathbf{R}_{\mathbf{D}}(x)\right] \cdot \mathbf{R}_{\mathbf{N}}(x) \\
&= \mathbf{N}'(x) \cdot \mathbf{R}_{\mathbf{N}'}(x)
\end{aligned}
\tag{2.124}
$$

we notice that the matrix appearing in eq (2.124), under our assumptions, strictly lower trian-gluar.
The formal solution of eq. (2.124) can be expressed in terms of the Dyson series as (this is morally the same as eq. (2.108))

$$
\mathbf{R}_{\mathbf{N}'}(x) = \mathbb{1} + \sum_{n=1}^{\infty} \mathscr{D}_n[\mathbf{N}'(x)], \tag{2.125}
$$

where

$$
\mathscr{D}_n[\mathbf{N}'(x)] = \int_\bullet^x dt_1 \int_\bullet^{t_1} dt_2 \cdots \int_\bullet^{t_{n-1}} dt_n \hat{\mathbf{N}}'(t_1) \cdot \hat{\mathbf{N}}'(t_2) \ \cdots \ \hat{\mathbf{N}}'(t_n). \tag{2.126}
$$

The remarkable fact is that under our assumptions, namely the fact that $\mathbf{N}(x)$ (and $\mathbf{N}'(x)$ is strictly lower triangular), eq (2.125) involves just a finite number of terms ($\mathscr{D}_n(\bullet)$ vanishes by construction for $n \geq \nu$).

Alternatively, eq. (2.124) can be solved by means of the *Magnus exponential*

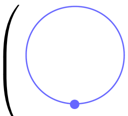
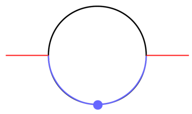
$$
\mathbf{R}_{\mathbf{N}'}(x) = \exp\left(\sum_{n=1}^{\infty} \mathcal{M}_n[\mathbf{N}'(x)]\right), \tag{2.127}
$$

where the various terms in eq. (2.127) are repeated integrals of nested commutators; the first few terms in the series are

$$
\begin{aligned}
\mathcal{M}_1[\mathbf{N}'(x)] &= \int_\bullet^x dt_1 \hat{\mathbf{N}}'(t_1), \\
\mathcal{M}_2[\mathbf{N}'(x)] &= \frac{1}{2} \int_\bullet^x dt_1 \int_\bullet^{x_1} dt_2 \left[\hat{\mathbf{N}}' t_1), \hat{\mathbf{N}}'(t_2)\right], \\
\mathcal{M}_3[\mathbf{N}'(x)] &= \frac{1}{6} \int_\bullet^x dt_1 \int_\bullet^{t_1} dt_2 \int_\bullet^{t_2} dt_3 \left[\hat{\mathbf{N}}'(t_1), \left[\hat{\mathbf{N}}'(t_2), \hat{\mathbf{N}}'(t_3)\right]\right] + \left[\hat{\mathbf{N}}'(t_3), \left[\hat{\mathbf{N}}'(t_2), \hat{\mathbf{N}}'(t_1)\right]\right].
\end{aligned}
\tag{2.128}
$$

The mapping between the terms of eq. (2.127) and eq. (2.125) is shown in [145]. Even if for the sake of our presentation we discussed just the single variable case, the procedure discussed here can be applied to systems depending on more variables. After having identified a basis of MIs which fulfills a system of differential equations linear in $\epsilon$ in all the variables, we can apply the transformation in eq. (2.123) one variable at a time (recalling that once we apply a change of basis to a given set of MIs, this propagates everywhere). Once the systems is in the form of eq. (2.124) for all the variables, we solve eq. (2.124) for each variable (again, once we obtain a given transformation via the Magnus/Dyson exponential, this has to be applied everywhere); see e.g. [156] for details.

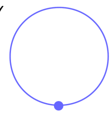
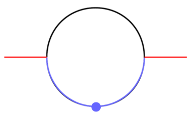
**Example.** *The half massive bubble (continued). Let us start from the following basis of MIs*

$\mathbf{J} = \Bigg($  $,$  $\Bigg).$ (2.129)

*We will consider the dependence on $m^2$ and $x = -s/m^2$.*
*Since $m^2$ is the only dimensionful quantity, its dependence can be simply predicted by dimensional analysis (or obtained from the scaling relation)*

$$\mathcal{J}_1 \sim (m^2)^{-\epsilon}, \qquad \mathcal{J}_2 \sim (m^2)^{-1-\epsilon}. \tag{2.130}$$

*It is useful to work with MIs which has the same $m^2$ scaling, and so we consider*

$\mathbf{J}' = \Bigg($  $, m^2$  $\Bigg).$ (2.131)

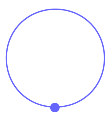
*Furthermore, the analytic evaluation of of $\mathcal{J}_1'$ can be carried out explicitly, obtaining*

$$\begin{aligned} \mathcal{J}_1' &= (m^2)^{-\epsilon}\,\Gamma(\epsilon) \\ &= \frac{1}{\epsilon} - \log(m^2) + \mathcal{O}(\epsilon). \end{aligned} \tag{2.132}$$

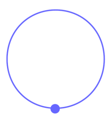
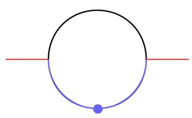
*This result is clearly not of uniform weight 0. We need to consider $\mathcal{J}_1' \to \epsilon\,\mathcal{J}_1' = \mathcal{J}_1''$ in order to remove the $1/\epsilon$ pole. Moreover, we find convenient to slightly modify the integral measure according to*

$$e^{\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \to \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \cdot \frac{(m^2)^{\epsilon}}{\Gamma(1+\epsilon)}. \tag{2.133}$$

*With the explicit choice eq. (2.133), then*

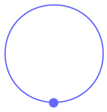
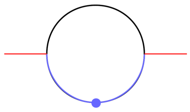
$\epsilon$  $= 1.$ (2.134)

*After all these considerations we can finally move to study the dependence of the MIs w.r.t. the variable $x$, which is the non-trivial one.*
*According to what we said so far, the basis of MIs is*

$\mathbf{J}'' = \Bigg( \epsilon$  $,\ m^2$  $\Bigg),$ (2.135) *and the corresponding differential equation is controlled by*

$$\hat{\mathbf{\Omega}}_x = \begin{pmatrix} 0 & 0 \\ -\frac{1}{x(x+1)} & -\frac{1}{x} - \frac{(x-1)\epsilon}{x(x+1)} \end{pmatrix}. \tag{2.136}$$

*Eq. (2.136) is linear in $\epsilon$; we could naively think that this is a good starting point for the Magnus/Dyson algorithm. However, as a general rule, it is convenient to have as many elements as possible proportional to $\epsilon$, or, equivalently, to have a kernel $\hat{\mathbf{\Omega}}_0$ with as many 0s has possible, in order to have a better convergence of eq. (2.125). In the case at hand, this is achieved rescaling also the second MIs by $\epsilon$, that is to say considering*

$\mathbf{F} = \Bigg( \epsilon$  $, \ \epsilon\, m^2$  $\Bigg)\,;$ (2.137)

*the corresponding system of differential equations is controlled by*

$$\hat{\mathbf{\Omega}}_x = \hat{\mathbf{\Omega}}_{x,0} + \epsilon\, \hat{\mathbf{\Omega}}_{x,1} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{x(x+1)} & -\frac{x-1}{x(x+1)} \end{pmatrix}. \tag{2.138}$$

*So, in this case*

$$\hat{\mathbf{\Omega}}_{x,0} = \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x} \end{pmatrix}. \tag{2.139}$$

*Therefore the desired rotation simply reads*

$$\mathbf{R} = \mathbf{R}_{\mathbf{D}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x} \end{pmatrix}. \tag{2.140}$$

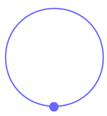
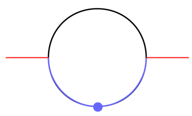
*The resulting transformed matrix is*

$$\mathbf{\Omega}_{\rm c} = \epsilon \left( d\log(x)\, \mathbb{M}_1 + d\log(1+x)\, \mathbb{M}_2 \right), \tag{2.141}$$

*with*

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{M}_2 = \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix}. \tag{2.142}$$

*Unfolding everything back, we find that the following basis of MIs is canonical (going back to the physical variables $(s, m^2)$)*

$\mathbf{I} = \Bigg( \epsilon$  $, \ (-s)\epsilon$  $\Bigg).$ (2.143)

## 2.7 Differential Equations for Feynman Integrals: Baikov representation

As we have done in the case of IBPs, it is interesting to briefly explore differential equations fulfilled by MIs, working with Baikov representation [58, 60].
The general strategy is the one outlined above: namely acting with a differential operator $\partial_y(\bullet)$ on a certain MI, and reduce the resulting combination of FIs back in terms of MIs. Dealing with Baikov representation, the action of $\partial_y(\bullet)$ is—to some extent—non trivial.

In order the fix the ideas, we focus on the dependence on a single variable $y$ .
Therefore we consider the action of $\partial_y(\bullet)$ on a certain MI, say

$$\mathcal{J} = I_{a_1,a_2,\dots,a_n} \sim \left(\mathbf{G}(\mathbf{p}_1,\dots,\mathbf{p}_E)\right)^{\frac{E-d+1}{2}} \int_\gamma \mathcal{B}(\mathbf{z})^{\frac{d-\ell-E-1}{2}} \prod_{i=1}^{n} dz_i\, z^{-a_1} \qquad a_i \rightsquigarrow \text{fixed}. \tag{2.144}$$

It is important to make explicit the $y$ dependence in the various quantities; in order to do this we introduce for a moment the following notations

$$\mathbf{G}(\mathbf{p}_1,\dots,\mathbf{p}_E) \rightsquigarrow \mathbf{G}(y), \qquad \gamma \rightsquigarrow \gamma(y), \qquad \mathcal{B}(\mathbf{z}) \rightsquigarrow \mathcal{B}(\mathbf{z},y), \tag{2.145}$$

since each of them could, in principle, depend on $y$.
The action of $\partial_y(\bullet)$ on $\mathbf{G}(y)$ is trivial; it will just produce a term proportional to the MI itself:

$$\partial_y \mathcal{J} \supset \left(\frac{E-d+1}{2}\frac{\partial_y \mathbf{G}(y)}{\mathbf{G}(y)}\right)\mathcal{J}. \tag{2.146}$$

The action $\partial_y(\bullet)$ on $\gamma(y)$ gives no contribution, since $\mathcal{B}(\partial\gamma) = 0$.

The non trivial point is acting with $\partial_y(\bullet)$ of $\mathcal{B}(\mathbf{z},y)$ under the integral sign[23].
The resulting expression is

$$\begin{aligned}\partial_y \mathcal{J} \supset \int_\gamma d\mathbf{z}\, \frac{\mathcal{B}^{\frac{d-\ell-E-1-2}{2}}}{z_1^{a_1}\dots z_n^{a_n}}\frac{d-\ell-E-1}{2}\,\partial_y\mathcal{B}. &= \int_\gamma d\mathbf{z}\, \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\dots z_n^{a_n}}\frac{d-\ell-E-1}{2}\,\frac{1}{\mathcal{B}}\,\partial_y\mathcal{B} \\ &= \int_\gamma d\mathbf{z}\, \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1}\dots z_n^{a_n}}\frac{d-\ell-E-1}{2}\,\partial_y \log\mathcal{B}.\end{aligned} \tag{2.147}$$

Eq. (2.147) can be interpreted as linear combinations of integrals in $(d-2)$-dimensions, which is *not* what we want.

Once more the issue can be overcome relying on the so-called syzygy equations, and look for

[23] From now one we omit the dependence of $\mathcal{B}(\mathbf{z},y)$, on $(\mathbf{x},y)$. Therefore $\mathcal{B}(\mathbf{z},y) = \mathcal{B}$.

polynomials $(\widehat{\xi}_1, \ldots, \widehat{\xi}_n, \widehat{\xi}_B)$ such that [60]

$$\partial_y \mathcal{B} = \sum_{m=1}^{n} \widehat{\xi}_m \, \partial_m \mathcal{B} + \widehat{\xi}_B \, \mathcal{B}. \tag{2.148}$$

Plugging eq. (2.148) into eq. (2.147) yields

$$\int_\gamma d\mathbf{z} \, \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1} \ldots z_n^{a_n}} \, \frac{d-\ell-E-1}{2} \left( \sum_{m=1}^{n} \widehat{\xi}_m \, \frac{\partial_m \mathcal{B}}{\mathcal{B}} + \widehat{\xi}_B \right) \tag{2.149}$$

The blue term gives combination of $d$-dimensional FIs, and so there is no problem with them. The left-over terms can be reorganized as follows

$$\sum_{m=1}^{n} \int_\gamma d\mathbf{z} \, \frac{\mathcal{B}^{\frac{d-\ell-E-1}{2}}}{z_1^{a_1} \ldots z_n^{a_n}} \, \frac{d-\ell-E-1}{2} \, \widehat{\xi}_m \, \frac{\partial_m \mathcal{B}}{\mathcal{B}} = \sum_{m=1}^{n} \int_\gamma d\mathbf{z} \, \frac{\widehat{\xi}_m}{z_1^{a_1} \ldots z_n^{a_n}} \partial_m \mathcal{B}^{\frac{d-\ell-E-1}{2}}; \tag{2.150}$$

finally integrating by parts and neglecting surface terms, we have

$$\sum_{m=1}^{n} \int_\gamma d\mathbf{z} \, \frac{\widehat{\xi}_m}{z_1^{a_1} \ldots z_n^{a_n}} \partial_m \mathcal{B}^{\frac{d-\ell-E-1}{2}} = - \sum_{m=1}^{n} \int_\gamma d\mathbf{z} \, \mathcal{B}^{\frac{d-\ell-E-1}{2}} \, \partial_m \left( \frac{\widehat{\xi}_m}{z_1^{a_1} \ldots z_n^{a_n}} \right). \tag{2.151}$$

The r.h.s. is finally written as linear combination of $d$-dimensional FIs.
Therefore we summarize our findings saying that the action of $\partial_y(\bullet)$ on any given MIs can be written as linear combination of ($d$-dimensional) FIs, at the price of solving syzygy equations eq. (2.148). The resulting combination can be rewritten in terms of MIs, as it was done in the standard case.

# 3

# Loop Calculus for Dark Matter Models

We discuss in this chapter an application of the method of differential equations for MIs. The context in which these techniques are applied is not the (perhaps) usual perturbative Quantum Chromodynamics (QCD), rather the so called *Dark matter leptophillic scenarios*. Without any pretension of completeness, we give here an extremely concise introduction to this topic. Our goal in doing so is motivating the type of amplitude that we will then process with multi-loop techniques. We will avoid any sort of phenomenological discussion; we refer to the original work [157] and references therein for such considerations. The integral family under consideration is a three-point 2 loop graph, with different masses in internal and external lines. In parallel to this, we will also consider other auxiliary integrals families (three-point 2 loop graph with equal masses and two-point 2 loop graph with different masses) which emerge in suitable kinematic limits of the previous one. We employ the method of canonical basis, obtained by means of the Magnus/Dyson exponential.

## 3.1

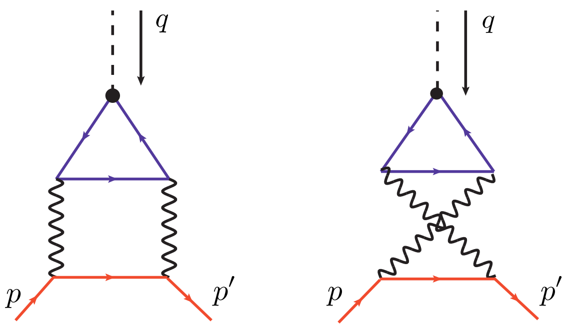
## Physical Model and Form Factors

There is a vast experimental effort devoted to the study of elastic non-relativistic scattering among Dark matter (DM) particles (denoted by $\chi$), and nuclei, (say $A$): so $\chi A \to \chi A$; this type of processes goes under the name of *direct Dark matter searches*; they are complementary to the so called *indirect Dark matter searches* and *Dark matter searches at Collider*—see e.g. the review [158]. In the leptophillic scenarios, DM does not interact directly with quarks, and therefore with nuclei. Rather, the DM-SM interaction is mediated via a field ($\phi$) which couples to *leptons* ($l$)—and hence the name leptophillic.
Limiting ourselves to a scalar mediator ($\phi_S$) or a pseudo-scalar mediator ($\phi_P$), the Lagrangian

describing these models will contain the following terms

$$-\mathcal{L}_S \supset g_S\, \phi_S \bar{l} l + g_\chi\, \phi_S \Gamma_\chi, \qquad -\mathcal{L}_P \supset i g_P\, \phi_P \bar{l} \gamma_5 \ell + g_\chi\, \phi_S \Gamma_\chi. \tag{3.1}$$

where $\Gamma_\chi = \{\chi^\dagger \chi\}$ in the case of scalar DM, and $\Gamma_\chi = \{\bar{\chi}\, \chi, \bar{\chi}\gamma_5\chi\}$ in the case of fermionic DM. Finally, the interaction among leptons and quarks is mediated via photons. Actually, at the energies relevant for experiments, photons cannot access quarks directly, rather we will consider their interactions with nucleons (in other words, the internal structure of nucleons cannot be accessed). The coupling of photons to nucleons is well known and is controlled by the Dirac and Pauli form factors, $F_1$ and $F_2$ respectively. For the phenomenological scenario considered in [157], the contribution proportional to $F_2$ is negligible, and the coupling among photons and nucleons is analogous to the one among photons and quarks, where the only difference is due to the charge−therefore only protons are relevant; see once again [157] for a detailed discussion[1]
Diagrammatically we are lead to the following 2-loop Feynman diagrams[2]:




(3.2)

Leptons (with mass $m_l$) are depicted in blue, nucleons (with mass $m_N$) in red, photons via wiggly lines. The kinematics is $p^2 = p'^2 = m_N^2$ and $q^2 = (p' - p)^2 = t$. The corresponding amplitudes is

$$\mathcal{A}_{S,P}(t; m_N, m_l) = i\, g_{S,P}\, Q_N^2 \sum_{l=e,\mu,\tau} Q_l^2\, \left(\bar{u}_N(p')\, \Gamma_{S,P}(t; m_N, m_l)\, u_N(p)\right). \tag{3.3}$$

Some comments are in order. The subscript $\bullet_{S,P}$ denotes either the scalar contribution (in this case we use $(\bullet_S)$) or the pseudo-scalar one $(\bullet_P)$. $Q_l = 1$ and $Q_N = \{1, 0\}$, for $N = p, n$ (where $p$ denotes the *proton* and $n$ the *neutron*) are the electric charges of lepton and nucleon respectively. Finally, $(e, \mu, \tau)$ stands for electron, muon and tau respectively.
The operator $\Gamma_{S,P}(t, m_N, m_l)$ results from the sum of the two diagrams, which give identical

[1] In the following we will keep the general subscript $(\bullet)_N$ for nucleons, since some of the expressions obtained can be used even in different approximations compared to the one of [157].
[2] The two loop diagrams are the so called *one body interactions*. Other contributions are the *two body interactions*, corresponding to one loop Feynman diagrams. See [157] and references therein for the interplay of the two.

contributions. Its explicit expression is[3]

$$\Gamma_{S,P}(t;m_N,m_l) = -32\,\pi^2\,\alpha_{\rm em}^2 \int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{g_{\mu\rho}\,\gamma^\rho\,(\not{k}_2-m_N)\,g_{\nu\sigma}\,\gamma^\sigma\,\mathrm{Tr}^{\mu\nu}_{S,P}(q,k_1,k_2;m_N,m_l)}{D_1\,D_2\,D_3\,D_4\,D_5\,D_6}\,, \tag{3.4}$$

where $\alpha_{\rm em} = e^2/(4\,\pi)$ and $\mathrm{Tr}^{\mu\nu}_{S,P}$ is the Dirac trace

$$\mathrm{Tr}^{\mu\nu}_{S,P}(q,k_1,k_2;m_N,m_l) = \mathrm{Tr}\Big\{\,(\not{k}_1+\not{q}+m_l)\,\Lambda_{S,P}\,(\not{k}_1+m_l)\,\gamma^\mu\,(\not{k}_1+\not{k}_2+\not{p}'+m_l)\,\gamma^\nu\Big\}\,, \tag{3.5}$$

with $\Lambda_S = \mathbb{1}$ and $\Lambda_P = \gamma_5$. The inverse propagators $D_i$ in eq. (3.4) are

$$D_1 = k_1^2 - m_l^2, \quad D_2 = (k_1+q)^2 - m_l^2, \quad D_3 = k_2^2 - m_N^2, \tag{3.6}$$

$$D_4 = (k_2+p)^2, \quad D_5 = (k_2+p')^2, \quad D_6 = (k_1+k_2+p')^2 - m_l^2. \tag{3.7}$$

From Lorentz invariance, the operators $\Gamma_{S,P}$ must have the form

$$\begin{aligned}\Gamma_{S,P}(t;m_N,m_l) = A_{S,P}(t;m_N,m_l)\,\Lambda_{S,P} + B_{S,P}(t;m_N,m_l)\,\left(\not{p}'+\not{p}\right)\,\Lambda_{S,P}\\ + C_{S,P}(t;m_N,m_l)\,\left(\not{p}'-\not{p}\right)\,\Lambda_{S,P}.\end{aligned} \tag{3.8}$$

Upon using the equations of motions eq. (3.8) can be rewritten as

$$\begin{aligned}\mathcal{F}_S^{1b} &= A_S + 2m_N B_S,\\ \mathcal{F}_P^{1b} &= A_P + 2m_N C_P.\end{aligned} \tag{3.9}$$

So we have one single form factor in the scalar case, and one in the pseudo-scalar one. $\mathcal{F}^{1b}_{S,P}$ can be extracted from $\Gamma_{S,P}$ via the following projection

$$\mathcal{F}^{1b}_{S,P}(t;m_N,m_\ell) = \frac{1}{2\,(p'\pm p)^2}\,\mathrm{Tr}\left\{\Lambda_{S,P}\left(\not{p}\pm m_N\right)\,\Gamma_{S,P}\,\left(\not{p}'\pm m_N\right)\right\}, \tag{3.10}$$

where the sign is $(+1)$ for the scalar case $(S)$, and $(-1)$ for the pseudo-scalar $(P)$. Our goal is to compute the expressions for $\mathcal{F}^{1b}_{S,P}$.

## 3.2 Scalar Integrals Evaluation

Even if the diagrams in (3.2) are finite, from now on we assume the DR scheme with $d = 4-2\epsilon$. The desired results are then recovered in the $\epsilon \to 0$ limit. Working in DR[4] allows us

[3] We write the expressions momentarily in $d = 4$.

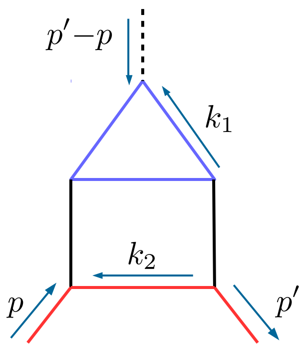
[4] For the $d$-dimensional treatment of $\gamma_5$ we follow the pragmatic approach of [159] and references therein. In particular $\gamma_5 = -i/4!\epsilon_{\mu,\nu,\rho,\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ where the indices are $d-$dimensional. Dirac algebra is performed also with the help of Package-X [160].

to benefit from the techniques described in the previous part of this work, namely IBPs and differential equations for MIs.
In parallel with the integral family associated to (3.2), we consider also an auxiliary integral family with a unique mass dependence, i.e. $m_l^2 = m_N^2 = m^2$—referred to as the *equal mass case*—and one associated with the *soft approximation* $q^\mu \to 0$ (corresponding to $t = 0$). Both these limits are considered *ab-initio* and, in both cases, the resulting integrals depend on one scale less compared to the original problem; simpler expressions are expected for both MIs and form factors. The auxiliary integral family for the equal mass limit serves as useful check for eq. (3.9), since we could verify that the form factors for the full problem (numerically) agree with the simpler ones, once the limit $m_N \to m_l$ is considered. Moreover, the equal mass limit is useful for comparing our results with previous expressions obtained in the literature, offering other important consistency checks. Finally, we mention that the sclar form factor in the soft approximation is of phenomenological interest [157][5].

### 3.2.1

### Different Mass Integral Family and its Canonical Basis

We consider the integral family associated to the following graph:




(3.11)

The denominators read

$$D_1 = k_1^2 - m_l^2, \quad D_2 = (k_1+q)^2 - m_l^2, \quad D_3 = k_2^2 - m_N^2, \tag{3.12}$$

$$D_4 = (k_2+p)^2, \quad D_5 = (k_2+p')^2, \quad D_6 = (k_1+k_2+p')^2 - m_l^2, \tag{3.13}$$

while the ISP is chosen as

$$D_7 = (k_1 - p)^2. \tag{3.14}$$

The kinematics is given by $p^2 = p'^2 = m_N^2$ and $t = (p-p')^2$.

This integral family have been computed originally in [161], in the context of Higgs decay into $\bar{b}\,b$ pair; we recomputed it separately.

[5]The pseudo-scalar form factor in the soft approximation admits an asymptotic expansion which we do not consider here.

In this work we employed Reduze2 [67] and LiteRed [162]. In this case 20 MIs were identified. Employing the strategy described in section 2.6.2, we obtain the following set of canonical MIs

$$
\begin{aligned}
&\mathrm{I}_1 = \epsilon^2\, \mathcal{J}_1, && \mathrm{I}_2 = \epsilon^2\, \mathcal{J}_2,\\
&\mathrm{I}_3 = \epsilon^2\, \lambda_l\, \mathcal{J}_3, && \mathrm{I}_4 = -\epsilon^2\, t\, \mathcal{J}_4,\\
&\mathrm{I}_5 = \epsilon^2 \left(\frac{1}{2}\, (-t+\lambda_l)\, \mathcal{J}_4+\lambda_l\, \mathcal{J}_5\right), && \mathrm{I}_6 = -\epsilon^2\, t\, \mathcal{J}_6,\\
&\mathrm{I}_7 = \frac{\epsilon^2\, m_N^2\, (t+\lambda_l)\, \rho_N}{(\lambda_N+t)\, \rho_l}\, (\mathcal{J}_7+2\,\mathcal{J}_8), && \mathrm{I}_8 = \epsilon^2\, m_N^2\, \mathcal{J}_8,\\
&\mathrm{I}_9 = \epsilon^2\, \lambda_l\, \mathcal{J}_9, && \mathrm{I}_{10} = -\epsilon^2\, t\, \lambda_l\, \mathcal{J}_{10},\\
&\mathrm{I}_{11} = \epsilon^3\, \lambda_N\, \mathcal{J}_{11}, && \mathrm{I}_{12} = \frac{\epsilon^2\, \lambda_l}{4\, t}\Big((t-\lambda_N)\, (\mathcal{J}_4+2\,\mathcal{J}_5)-4\, m_N^2\, \lambda_N\, \mathcal{J}_{12}\Big),\\
&\mathrm{I}_{13} = \epsilon^3\, \lambda_N\, \mathcal{J}_{13}, && \mathrm{I}_{14} = \epsilon^3\, (-1+2\,\epsilon)\, t\, \mathcal{J}_{14},\\
&\mathrm{I}_{15} = \epsilon^3\, \lambda_l\, \lambda_N\, \mathcal{J}_{15}, && \mathrm{I}_{16} = \epsilon^3\, \lambda_N\, \mathcal{J}_{16},\\
&\mathrm{I}_{17} = \epsilon^3\, \lambda_N\, \mathcal{J}_{17}, && \mathrm{I}_{18} = \frac{\epsilon^2}{t}\Big(\lambda_l\, (-t+\lambda_N)\, \mathcal{J}_9+\epsilon\, (t-\lambda_l)\, \lambda_N\, \mathcal{J}_{17}+(-1+2\,\epsilon)\, \lambda_l\, \lambda_N\, \mathcal{J}_{18}\Big),
\end{aligned}
$$
$$
\begin{aligned}
\mathrm{I}_{19} = {}& \frac{\epsilon^2}{2\, t}\Big(t\, (\lambda_l-t)\, \mathcal{J}_3-2\, t\, m_l^2\, (\mathcal{J}_7+2\,\mathcal{J}_8)+\big(4\, t\, m_l^2+\lambda_l\, (\lambda_N-t)\big)\, \mathcal{J}_9+\epsilon\, \frac{4\, t^2\, m_N^2}{\lambda_N+t}\, \mathcal{J}_{16}\\
&+\epsilon\, \big(\lambda_N\, (t-\lambda_l)-4\, t\, m_l^2\big)\, \mathcal{J}_{17} \quad +(2\,\epsilon-1)\, \big(4\, t\, m_l^2+\lambda_l\, \lambda_N-t^2\big)\, \mathcal{J}_{18}+2\, t^2\, \big(m_l^2-m_N^2\big)\, \mathcal{J}_{19}\Big),\\
\mathrm{I}_{20} = {}& -\epsilon^4\, t\, \lambda_N\, \mathcal{J}_{20}.
\end{aligned}
\tag{3.15}
$$

where we introduced the following notations

$$
\lambda_i = \sqrt{-t}\,\sqrt{4\, m_i^2 - t}\,, \qquad \rho_i = \sqrt{\frac{2\, m_i^2 - t - \lambda_i}{m_i^2}}, \qquad \text{with: } i = N, l. \tag{3.16}
$$

and the integrals $(\mathcal{J}_1, \ldots, \mathcal{J}_{20})$ are depicted in fig. (3.1). The square roots are rationalized by the following change of variables

$$
t = -m_l^2\, \frac{(1-x^2)^2}{x^2}, \qquad m_N^2 = m_l^2\, \frac{(1-x^2)^2 y^2}{(1-y^2)^2 x^2}, \tag{3.17}
$$

where $(x, y)$ are chosen to lay in the following region:

$$
0 < x < 1 \quad \bigcap \quad 0 < y < x. \tag{3.18}
$$

This region, in terms of physical variables, corresponds to

$$
t < 0 \quad \bigcap \quad 0 < m_N^2 < m_l^2, \tag{3.19}
$$

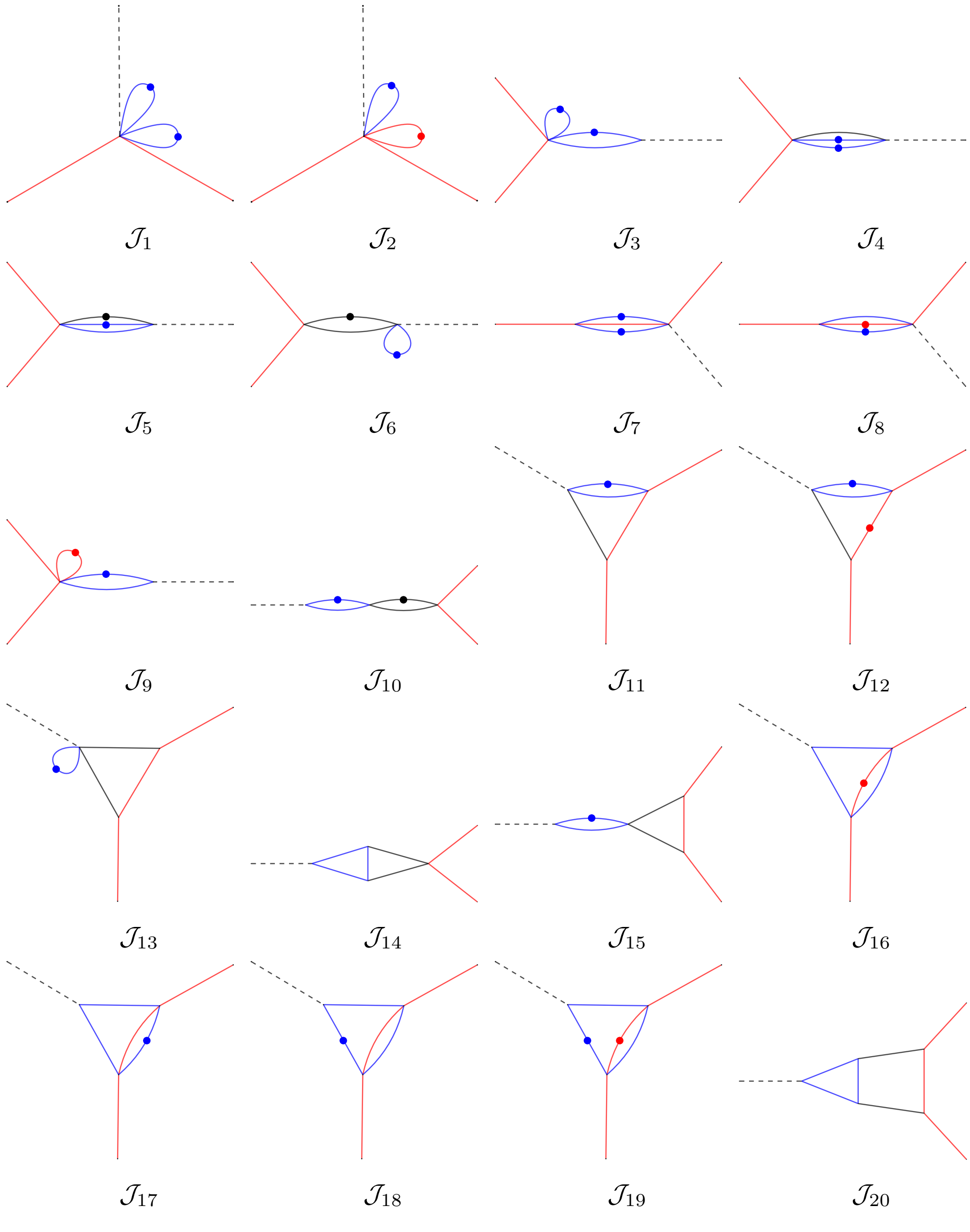



Figure 3.1: **MIs for the integral family in fig. (3.11)**. Dots denote squared propagators.

and the inverse of eq. (3.17) is

$$x = \frac{1}{2}\,(\sqrt{4-\sigma_l} - \sqrt{-\sigma_l}), \quad y = \frac{1}{2}\,(\sqrt{4-\sigma_N} - \sqrt{-\sigma_N}), \quad \sigma_{l,N} = \frac{t}{m_{l,N}^2}. \tag{3.20}$$

The same definition of $(x, y)$ holds also for the inverted mass hierarchy: $0 < m_l^2 < m_N^2$ $(t < 0)$ and so it can be used for the full kinematic region of our interest.
With the explicit choices in eqs. (3.15,3.17), the (system of) differential equation(s) reads

$$d\mathbf{I}(x, y, \epsilon) = \epsilon \boldsymbol{\Omega}_{\rm c}(x, y)\,\mathbf{I}(x, y, \epsilon), \qquad \boldsymbol{\Omega}_{\rm c}(x, y) = \sum_{i=1}^{12} \mathbb{M}_i\, d\log(\eta_i(x, y)) \tag{3.21}$$

with

$$\begin{aligned}
&\eta_1(x,y) = x, && \eta_2(x,y) = 1 + x, && \eta_3(x,y) = 1 - x, \\
&\eta_4(x,y) = 1 + x^2, && \eta_5(x,y) = y, && \eta_6(x,y) = 1 + y, \\
&\eta_7(x,y) = 1 - y, && \eta_8(x,y) = 1 + y^2, && \eta_9(x,y) = x + y, \\
&\eta_{10}(x,y) = x - y, && \eta_{11}(x,y) = 1 + xy, && \eta_{12}(x,y) = 1 - xy.
\end{aligned}$$

The solution of eq. (3.21) (with rational alphabet) leads to GPLs.

The boundary constants are determined thanks to the following considerations[6][7]

- The tadpoles $\mathrm{I}_{1,2}$ and the factorized integral $\mathrm{I}_6$ are obtained by direct integration and provided as an external input to the system of differential equations:
$$\begin{aligned}
&\mathrm{I}_1 = 1\,, \\
&\mathrm{I}_2(x, y, \epsilon) = \left(\frac{(1 - x^2)^2\, y^2}{(1 - y^2)^2\, x^2}\right)^{-\epsilon}, \\
&\mathrm{I}_6(x, \epsilon) = \left(\frac{(1 - x^2)^2}{x^2}\right)^{-\epsilon} \left(1 - \frac{\pi^2}{6}\,\epsilon^2 - 2\,\zeta_3\,\epsilon^3 - \frac{\pi^4}{40}\,\epsilon^4 + \mathcal{O}(\epsilon)^5\right).
\end{aligned}$$

- The boundary constants of the integrals $\mathrm{I}_{3,4,5,9,10,11,12,14,16,17,18,19}$ are determined by imposing their regularity at the pseudo-threshold $t \to 0$. In particular, due to the prefactors that appear in the definition of the canonical MIs in eq. (3.15), $\mathrm{I}_{3,4,5,9,10,14,16,17}$ vanish in this limit.[8] The same conclusion is inferred for $\mathrm{I}_{11,12,18,19}$, analyzing the differential equation in this limit. $\mathrm{I}_{10}$ results to be vanishing in the $t \to 0$ limit, due to the faster convergence of the factorized (canonical) massive bubble, with respect to the massless one.

- The boundary constants of $\mathrm{I}_{7,8}$ are determined thanks to the regularity at $m_N^2 \to 0$; specifically, due to the prefactors in eq. (3.15), these MIs vanish in this limit.

- The boundary constants of $\mathrm{I}_{13,15,20}$ are determined by demanding regularity in the $t \to 4m_N^2$ limit. Thanks to the prefactors in eq. (3.15), these MIs vanish in this limit.

[6] The integral measure is assumed to be $(m_l^2)^\epsilon \int d^d k_i\, (i\pi^{d/2}\Gamma(1+\epsilon))^{-1}$ for each loop, in such a way that $\mathrm{I}_1 = 1$. The original integral measure can be always retrieved at the end.

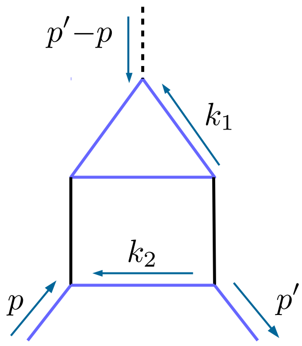
[7] In some cases the analytic expressions were recovered from high-precision numerical evaluations obtained with GiNaC, thanks to the PSLQ algorithm [163, 164] implementation of PolyLogTools.

[8] The fact that $\mathrm{I}_{14}$ vanishes in the $t \to 0$ limit can be inferred from the results of ref. [165].

### 3.2.2

### Equal Mass Integral Family and its Canonical Basis

We consider here the auxiliary integral family




(3.22)

Denominators and the kinematics is the one in subsection 3.2.1, upon considering $m_N^2 = m_l^2 = m^2$.
In this case 15 MIs are identified, and they can be chose as[9]

$$
\begin{aligned}
&\mathrm{I}_1 = \epsilon^2\,\mathcal{J}_1, && \mathrm{I}_2 = \epsilon^2\,\lambda_m\,\mathcal{J}_2,\\
&\mathrm{I}_3 = -\epsilon^2\,t\,\mathcal{J}_3, && \mathrm{I}_4 = \epsilon^2\left(\frac{1}{2}\,\mathcal{J}_3\,\left(\lambda_m+t\right)+\mathcal{J}_4\,\lambda_m\right),\\
&\mathrm{I}_5 = -\epsilon^2\,t\,\mathcal{J}_5, && \mathrm{I}_6 = \epsilon^2\,m^2\,\mathcal{J}_6,\\
&\mathrm{I}_7 = -\epsilon^2\,t\,\lambda_m\,\mathcal{J}_7, && \mathrm{I}_8 = \epsilon^3\,\lambda_m\,\mathcal{J}_8,\\
&\mathrm{I}_9 = \epsilon^2\left(\frac{1}{4}\,\left(4\,m^2-\lambda_m-t\right)\,\left(\mathcal{J}_3+2\,\mathcal{J}_4\right)+m^2\,\left(4\,m^2-t\right)\,\mathcal{J}_9\right), && \mathrm{I}_{10} = \epsilon^3\,\lambda_m\,\mathcal{J}_{10},\\
&\mathrm{I}_{11} = \epsilon^3\,(1-2\,\epsilon)\,t\,\mathcal{J}_{11}, && \mathrm{I}_{12} = \epsilon^3\,t\,\left(t-4\,m^2\right)\,\mathcal{J}_{12},\\
&\mathrm{I}_{13} = \epsilon^3\,\lambda_m\,\mathcal{J}_{13}, &&\\
&\mathrm{I}_{14} = \epsilon^2\left(\left(4\,m^2-\lambda_m-t\right)\,\left(\mathcal{J}_2-\epsilon\,\mathcal{J}_{13}\right)+\left(2\,\epsilon-1\right)\,\left(4\,m^2-t\right)\,\mathcal{J}_{14}\right), &&\\
&\mathrm{I}_{15} = -\epsilon^4\,\lambda_m\,t\,\mathcal{J}_{15}, &&
\end{aligned} \tag{3.23}
$$

where we have introduced the notation

$$\lambda_m = \sqrt{-t}\,\sqrt{4\,m^2-t}. \tag{3.24}$$

and $(\mathcal{J}_1,\ldots,\mathcal{J}_{15})$ are presented in fig. (3.2). Eq. (3.24) is rationalized by the following change of variables

$$t = -m^2\frac{(1-w)^2}{w} \tag{3.25}$$

with $w$ in the following region

$$0 < w < 1, \tag{3.26}$$

[9]The canonical set of MIs for this example has been found by Hernik J. Munch.

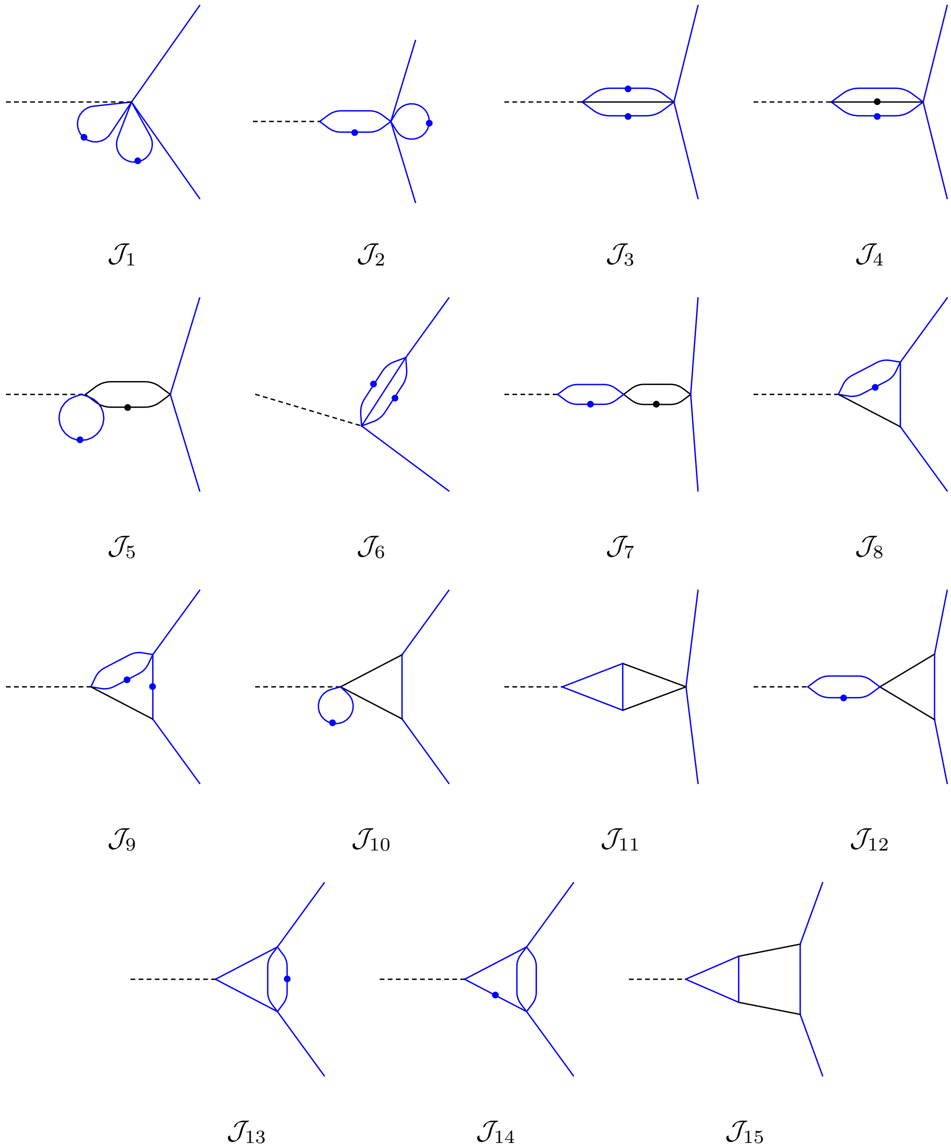



Figure 3.2: **MIs for the integral family in fig. (3.22)**. Dots denote squared propagators.

corresponding to

$$t < 0 \quad \bigcap \quad m^2 > 0. \tag{3.27}$$

The inverse of eq. (3.25) reads

$$w = \frac{\sqrt{4m^2 - t} - \sqrt{-t}}{\sqrt{4m^2 - t} + \sqrt{-t}} \tag{3.28}$$

The canonical differential equation w.r.t. reads

$$d\mathbf{I}(w,\epsilon) = \epsilon \boldsymbol{\Omega}_{\rm c}(w)\mathbf{I}(w,\epsilon), \qquad \boldsymbol{\Omega}_{\rm c}(w) = \sum_{i=1}^{3} \mathbb{M}_i d\log(\eta_i(w)), \tag{3.29}$$

with

$$\eta_1(w) = w, \qquad \eta_2(w) = 1+w, \qquad \eta_3(w) = 1-w. \tag{3.30}$$

Eq. (3.29) can be solved in terms of GPLs (or, better, in terms of HPLs).

The boundary constants are fixed thanks to the following considerations

- The integrals $\mathrm{I}_{1,5,6}$ are provided as an external input. In particular, $\mathrm{I}_{1,5}$ are obtained by direct integration
$$\begin{aligned} \mathrm{I}_1 &= 1\,, \\ \mathrm{I}_5(w,\epsilon) &= \left(\frac{(1-w)^2}{w}\right)^{-\epsilon}\left(1 - \frac{\pi^2}{6}\epsilon^2 - 2\,\zeta_3\,\epsilon^3 - \frac{\pi^4}{40}\epsilon^4 + \mathcal{O}(\epsilon)^5\right)\,, \end{aligned}$$
while $\mathrm{I}_6$ is obtained from the general case as [166]
$$\begin{aligned} \mathrm{I}_6(\epsilon) &= -\epsilon^2\,\frac{\pi^2}{12} + \frac{\epsilon^3}{4}\,\left(2\,\pi^2\,\log(2) - 7\,\zeta_3\right) \\ &+ \frac{\epsilon^4}{360}\left(31\,\pi^4 - 180\,\log^4(2) - 360\,\pi^2\,\log^2(2) - 4320\,\mathrm{Li}_4\left(\frac{1}{2}\right)\right) + \mathcal{O}(\epsilon^5). \end{aligned} \tag{3.31}$$

- The integral $\mathrm{I}_{12}$ is obtained as the product of $\mathrm{I}_2$ and $\mathrm{I}_{10}$.

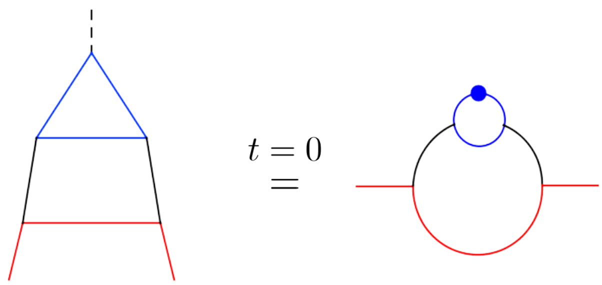
- The boundary constants for $\mathrm{I}_{2,3,4,8,9,11,13,14}$ are determined by the regularity at the pseudo-threshold $t \to 0$. In particular $\mathrm{I}_{2,3,4,8,11,13}$ vanish in this limit due to the prefactors in eq. (3.23). Analyzing the differential equation in the $t \to 0$ limit, we obtain relations among MIs in this limit, namely $\mathrm{I}_9\,|_{t\to 0} = 3\,\mathrm{I}_6$ and $\mathrm{I}_{14}\,|_{t\to 0} = 6\,\mathrm{I}_6$; the latter are sufficient to determine the boundary constants for $\mathrm{I}_{9,14}$. $\mathrm{I}_7$ vanishes due to the factorized canonical bubble.

- The boundary constants for $\mathrm{I}_{10}$ are determined from the regularity at the pseudo-threshold $t \to 4\,m^2$. In particular due to the prefactor in eq. (3.23), $\mathrm{I}_{10}$ vanishes in this limit.

- The boundary constants for $\mathrm{I}_{10}$ are obtained by comparing our results with those in ref. [167].

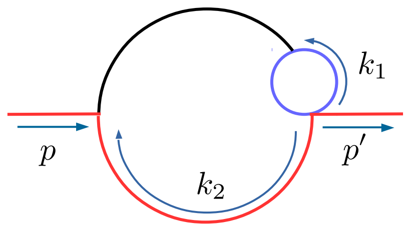
### 3.2.3 Soft Limit and its Canonical Form

We consider here the integral family which is relevant for the case $q^\mu \to 0$ (which implies $t \to 0$). The situation is represented diagrammaticaly by the following



$$(3.32)$$

so, compared to the previous section we will consider a two-point function. The integral family is associated with the following graph:




$$(3.33)$$

The denominators are[10]

$$D_1 = k_1^2 - m_l^2, \quad D_3 = k_2^2 - m_N^2, \quad D_4 = (k_2 + p)^2, \quad D_6 = (k_1 + k_2 + p)^2 - m_l^2, \tag{3.34}$$

and the ISP is

$$D_7 = (k_1 - p)^2. \tag{3.35}$$

[10]The subscripts follow from the original set in subsection 3.2.1.

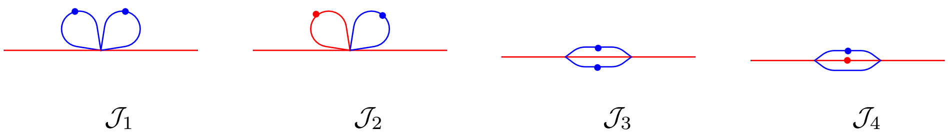



Figure 3.3: **MIs for the integral family in fig. (3.33)**. Dots denote squared propagators.

For this problem 4 MIs are identified. The canonical set of MIs is given by

$$\begin{aligned} \mathrm{I}_1 &= \epsilon^2\, \mathcal{J}_1, & \mathrm{I}_2 &= \epsilon^2\, \mathcal{J}_2\,, \\ \mathrm{I}_3 &= \epsilon^2\, m_l\, m_N\, (\mathcal{J}_3 + 2\,\mathcal{J}_4)\,, & \mathrm{I}_4 &= \epsilon^2\, m_N^2 \mathcal{J}_4, \end{aligned} \tag{3.36}$$

where $(\mathcal{J}_1, \ldots, \mathcal{J}_4)$ are depicted in fig. (3.3). The differential equations is derived w.r.t.

$$z = \frac{m_N}{m_l}, \tag{3.37}$$

and it takes the form

$$d\mathbf{I}(z,\epsilon) = \epsilon \boldsymbol{\Omega}_{\rm c}(z)\, \mathbf{I}(z,\epsilon), \qquad \boldsymbol{\Omega}_{\rm c}(z) = \sum_{i=1}^{3} \mathbb{M}_i\, d\log(\eta_i(z)), \tag{3.38}$$

with

$$\eta_1(z) = z, \qquad \eta_2(z) = 1+z, \qquad \eta_3(z) = 1-z. \tag{3.39}$$

Eq. (3.38) can be solved in terms of GPLs (rather HPLs).

The boundary constants are fixed by means of the following analysis

- $\mathrm{I}_1$ and $\mathrm{I}_2$ are obtained by direct integration

$$\begin{aligned} \mathrm{I}_1 &= 1\,, \\ \mathrm{I}_2(z,\epsilon) &= z^{-2\,\epsilon}\,. \end{aligned}$$

- The boundary conditions for $\mathrm{I}_3$ and $\mathrm{I}_4$ are determined from to their regularity at $m_N \to 0$ [166]. In particular, thanks to the prefactors in eq. (3.36), they vanish in this limit.

## 3.3

## Results

The explicit results for $\mathcal{F}^{1b}_{S,P}$ where we retain all the dependence on $(t, m_N^2, m_l^2)$ are too lengthy to be reported here. Rather we present analytic expressions for the form factors in various limits;

• **Scalar form factor in the equal mass limit:** $\mathcal{F}^{1b}_S(t; m^2, m^2)$.

The variable $w$ is introduced in eq. (3.28) and we use the short-hand notation $G(\mathbf{a}; w) = G_{\mathbf{a}}(w)$. We have $\mathcal{F}^{1b}_S = \frac{\alpha^2_{em}}{\pi^2}\, \hat{\mathcal{F}}^{1b}_S$, with

$$
\begin{aligned}
\hat{\mathcal{F}}^{1b}_S(w) = &-\frac{(20\,w^2+4\,w)}{(w-1)\,(w+1)^2}\,G_0(w) + \frac{w}{(w-1)\,(w+1)}\left[24\,G_{-1,0}(w)+4\,G_{1,0}(w)\right] \\
&+\frac{w}{(w-1)^2\,(w+1)^3}\left[-12\,w^3-38\,w^2+8\,w+2\right]G_{0,0}(w) - \frac{w}{\pi^2\,(w-1)^2}\left[2\,G_{0,0,0}(w)+4\,\zeta_3\right] \\
&+\frac{w\,\zeta_3}{(w-1)^3\,(w+1)}\left[5\,w^2-6\,w+5\right]G_0(w) + \frac{w}{\pi^2\,(w+1)^2}\left[12\,G_{0,0,1}(w)-\frac{4\,\pi^2}{3}\,G_0(w)+16\,G_1(w)\right] \\
&+\frac{w}{(w-1)^2\,(w+1)^2}\left[(-16\,w^2+64\,w-16)\,G_{0,-1,0}(w)+(8\,w^2-48\,w+8)\,G_{1,0,0}(w)+16\,w\,G_{0,1,0}(w)\right] \\
&+\frac{w}{(w-1)\,(w+1)^3}\left[(-14\,w^2+4\,w-14)\,G_{0,1}(w)+\left(\frac{w^2}{3}+2\,w+\frac{1}{3}\right)\,G_{1,0}(w)\,\pi^2\right. \\
&\left.+(2\,w^2+12\,w+2)\,\left(G_{1,0,0,0}(w)-G_{0,0,1,0}(w)\right)+\left(2\,w^2-\frac{20\,w}{3}+2\right)\,\pi^2\right] \\
&+\frac{w}{(w-1)^3\,(w+1)^3}\left[\left(\frac{5\,w^4}{6}-2\,w^3+\frac{23\,w^2}{3}-2\,w+\frac{5}{6}\right)\,G_{0,0}(w)\,\pi^2\right. \\
&+\left(8\,w^4-32\,w^3+112\,w^2-32\,w+8\right)\,G_{0,0,-1,0}(w)+\left(\frac{3\,w^4}{2}-2\,w^3+9\,w^2-2\,w+\frac{3}{2}\right)\,G_{0,0,0,0}(w) \\
&+\left(-9\,w^4+12\,w^3-54\,w^2+12\,w-9\right)\,G_{0,0,0,1}(w)+\left(-2\,w^4+24\,w^3-76\,w^2+24\,w-2\right)\,G_{0,1,0,0}(w) \\
&\left.+\left(\frac{w^4}{18}-\frac{2\,w^3}{45}+\frac{11\,w^2}{45}-\frac{2\,w}{45}+\frac{1}{18}\right)\,\pi^4\right]. \qquad (3.40)
\end{aligned}
$$

Thanks to crossing symmetry, this form factor is related to the one appearing in the Higgs boson decay into a heavy-quarks discussed in [168]. Nevertheless, the form factor considered in this reference receives contribution from several s-channel diagrams, (and so from several integral families). Our expression, instead, is associated to a single integral family.

• **Pseudo-scalar form factor in the equal mass limit:** $\mathcal{F}^{1b}_S(t; m^2, m^2)$.

The variable $w$ is introduced in eq. (3.28) and we use the short-hand notation $G(\mathbf{a}; w) = G_{\mathbf{a}}(w)$.

We have $\mathcal{F}_P^{1b} = \frac{\alpha_{em}^2}{\pi^2}\,\hat{\mathcal{F}}^{1b}_P$, with

$$\begin{aligned}\hat{\mathcal{F}}^{1b}_{\mathcal{P}}(w) = &-\frac{w^2}{(w-1)^3\,(w+1)}\,\left[\frac{2\pi^2}{3}\,G_0(w)+4\,G_{0,0,0}(w)\right]\\ &+\frac{w}{(w-1)^2}\,\left[G_{0,0}(w)-4\,G_{0,0,1}(w)-4\,G_{0,1,0}(w)+8\,G_{1,0,0}(w)+12\,\zeta_3+\frac{\pi^2}{3}\right]\\ &+\frac{w}{(w-1)\,(w+1)}\,\Big[-3\,\zeta_3\,G_0(w)-\frac{\pi^2}{6}\,G_{0,0}(w)-\frac{\pi^2}{3}\,G_{1,0}(w)-\frac{1}{2}\,G_{0,0,0,0}(w)\\ &\qquad+3\,G_{0,0,0,1}(w)+2\,G_{0,0,1,0}(w)-2\,G_{0,1,0,0}(w)-2\,G_{1,0,0,0}(w)-\frac{\pi^4}{45}\Big]\,.\end{aligned} \tag{3.41}$$

Thanks to crossing symmetry, this quantity is related to the pseudo-scalar form factor earlier considered in the context of QCD corrections to heavy quarks form factors in s-channel processes−see $AR$ in [159]. We reproduced the result of the above-mentioned reference with our MIs and suitable projectors. Our result, valid for a t-channel process, is reported in [157] for the first time.

• **Scalar form factor in the soft limit:** $\mathcal{F}_S^{1b}(t\sim 0; m_N^2, m_l^2)$.
The variable $z$ is introduced in eq. (3.37). We have $\mathcal{F}_S^{1b} = \frac{\alpha_{em}^2}{\pi^2}\,\hat{\mathcal{F}}^{1b}_S$, with

$$\hat{\mathcal{F}}^{1b}_S(z) \;=\; -\frac{2}{z}\Big[1-\frac{\ln(z)}{2}+f_S(z)+f_S(-z)\Big]\,, \tag{3.42}$$

$$\text{where}\qquad f_S(\tau) \;=\; \frac{1}{4\,\tau^2}\left(4+3\,\tau+\tau^3\right)\Big[\log|\tau|\log(1+\tau)+\mathrm{Li}_2(-\tau)\Big]. \tag{3.43}$$

• **Scalar form factor:** $\mathcal{F}_S^{1b}(t; m_N^2, m_l^2)$ & **Pseudo-scalar form factor:** $\mathcal{F}_P^{1b}(t; m_N^2, m_l^2)$.
As anticipated above the expression are too lengthy to be reported here. Rather the qualitative behaviour[11] is captured by the plot in fig. (3.4) and fig. (3.5), for the scalar and pseudo-scalar contribution respectively.

[11] The numerical evaluations of the form factor was carried over by P. Mastrolia and A. Primo.

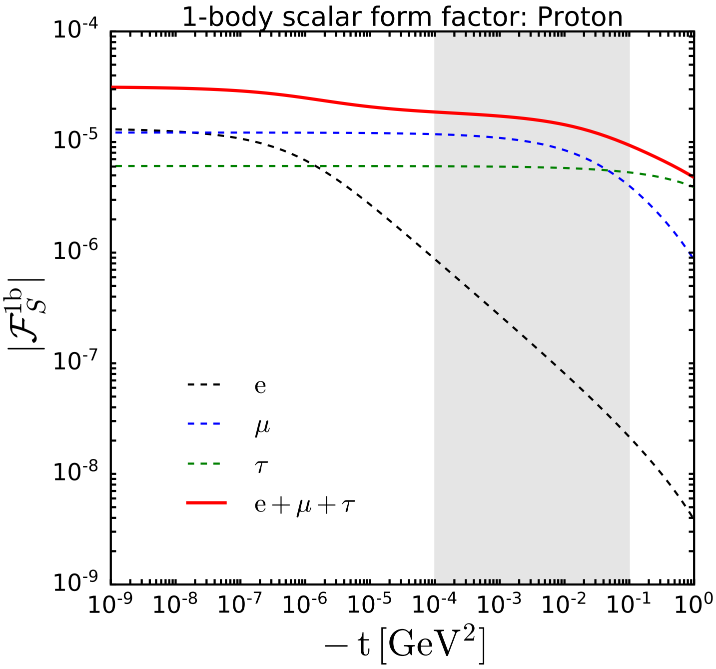



Figure 3.4: **Scalar form factor** $\mathcal{F}_S^{1b}(t; m_N^2, m_l^2)$. The nucleon considered is the proton; $(e, \mu, \tau)$ denote the contributions of electron, muon and tau.

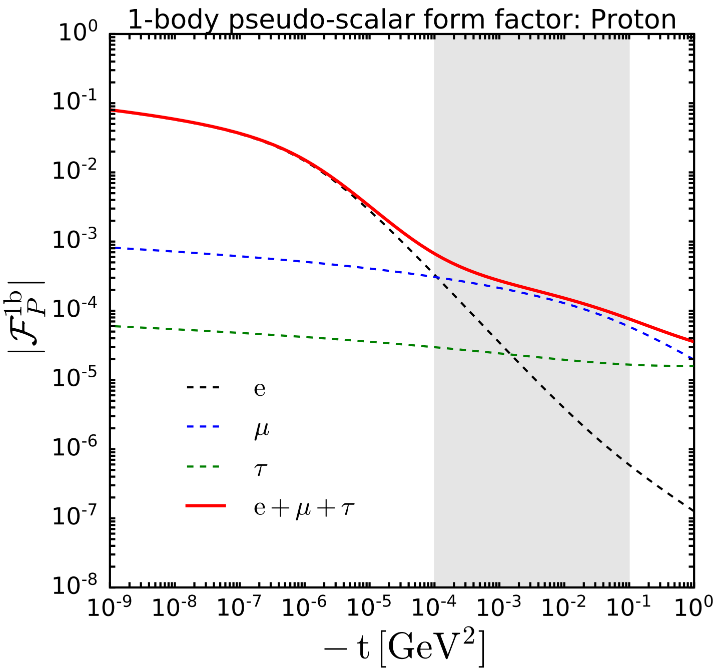



Figure 3.5: **Pseudo-scalar form factor** $\mathcal{F}_P^{1b}(t; m_N^2, m_l^2)$. The nucleon considered is the proton; $(e, \mu, \tau)$ denote the contributions of electron, muon and tau.

# 4

# An Introduction to Twisted (Co)Homology: univariate case

The main goal of this chapter is to give a gentle introduction to twisted Homology and Co-Homology and to study integrals and their properties within this framework. We will focus our discussion mostly on the univariate case (i.e. one-fold integrals)−and later on generalize our results to the multivariate case. After introducing the (co)homology groups, we will review the construction of (co)homology intersection numbers: these last are pairing among the elements of the above-mentioned groups (and their dual) and will play a crucial role in the remaining part of this work. There are several textbooks and reviews on this topic, see e.g. [23, 169, 170]; here we follow the discussion of [24], trying to smooth as much as possible all the mathematical asperities.[1]

Loosely speaking, we will focus on integrals of the form

$$\int_{\Delta} (P_1(z))^{\alpha_1} \dots (P_m(z))^{\alpha_m} \, dz, \tag{4.1}$$

where $P_i(z)$, $1 \leq i \leq m$ are polynomials in z (eventually depending on some external data), $\alpha_i \in \mathbb{C} \setminus \mathbb{Z}$, $1 \leq i \leq m$, and $\Delta$ is some integration contour−not yet specified−in $X = \mathbb{C} \setminus \bigcup_{j=1}^{m} D_j$ with $D_j = \{z \in \mathbb{C} : P_j(z) = 0\}$. One of the main peculiarities of the integrand in eq. (4.1) is that, under our assumptions, $(P_i(z))^{\alpha_i}$ is a multivalued function in $X$.

We introduce here a simple yet interesting example which will guide us through the whole

[1]We benefit a lot from the set of lectures "*Advanced Methods for Scattering Amplitudes*" delivered by S.L. Cacciatori, Y. Goto and P. Mastrolia, Padova−July 2021 as well as from the cycle of seminars held by Y. Goto and K. Matsumoto, Padova−September 2019 and several discussions with S. Mizera.

discussion.

**Example.** *The Euler Beta function. Perhaps the simplest example of integrals of the form of eq. (4.1) is*

$$\begin{aligned} B(p,q) &= \int_0^1 dz\, z^{p-1}\,(1-z)^{q-1}, \qquad \mathrm{Re}(p)>0,\ \text{and}\ \mathrm{Re}(q)>0, \\ &= \int_0^1 z^p\,(1-z)^q\,\frac{dz}{z(1-z)} \end{aligned} \tag{4.2}$$

*We have the clear identifications*[2]*:*

- $P_1(z) = z$*, and* $\alpha_1 = p-1$*;*
- $P_2(z) = 1-z$*, and* $\alpha_2 = q-1$*;*
- $\Delta = (0,1) \subset X = \mathbb{C}\setminus\{0,1\}$.

We will consider here some gymnastic one the Euler Beta function eq. (4.2); technically speaking we will obtain an analytic continuation valid for $(p,q) \notin \mathbb{Z}$, but getting acquainted with the multivalued nature of the integrand will be useful for later constructions.

Let us consider the closed contour $\gamma_{\mathrm{P}}$, dubbed as the Pochhammer contour [171], depicted in fig. (4.1).


Figure 4.1: **The Pochhammer contour denoted,** $\gamma_{\mathrm{P}}$. Branch points are denoted by $\circ$: so $\circ = 0$ or $\circ = 1$; the starting (and final) point, placed at position $\epsilon$ along the $(0,1)$ segment is denoted by: $\bullet$. We assume $\arg(t) = 0$ and $\arg(1-t) = 0$ on each point belonging to $(0,1)$. $C_\circ$ and $C_\circ'$ denote a small circles of radius $\epsilon$ around the point $\circ$. More details are given in the main text.

We aim to rewrite the integral over $\gamma_{\mathrm{P}}$ in terms of the original original integral eq. (4.2). Let us

[2]The importance of the harmless rearrangement in the last line of eq. (4.2) will become more transparent soon.

state the result, and then justify it.

$$\int_{\gamma_{\mathrm{P}}} z^p\,(1-z)^q \frac{dz}{z(1-z)} = \left(1 - e^{2\pi i q} + e^{2\pi i (q+p)} - e^{2\pi i q}\right) \int_0^1 z^p\,(1-z)^q \frac{dz}{z(1-z)}. \tag{4.3}$$

Some comments are in order to explain the r.h.s. of eq. (4.3); firstly we integrate following a straight line from 0 to 1 passing through the point $\bullet$, and this explain the first term $+1$. Next, we turn around the branch point $\circ = 1$ moving counterclockwise along the circle of arbitrary small radius $\epsilon$, say $\mathcal{C}_1$ (cf. caption in fig. (4.1)). Therefore this contribution to the integral is negligible (and the same holds for $C_0$, $C_0'$ and $C_1'$) but $z^p(1-z)^q$ produces an extra $e^{2\pi i q}$. This factor, together with the fact that we travel from 1 to 0 (and not from 0 to 1 as done before) explains the second term $-e^{2\pi i q}$ in eq. (4.3). Following the same logic, keeping track of the orientation of circles and segments, we can derive the other contributions namely $+e^{2\pi i (q+p)}$ and $-e^{2\pi i q}$.
We can recast eq. (4.3) as:

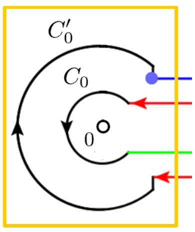
$$\int_{\gamma_{\mathrm{P}}} z^p\,(1-z)^q \frac{dz}{z(1-z)} = \left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right) \int_0^1 z^p\,(1-z)^q \frac{dz}{z(1-z)}. \tag{4.4}$$

Let us introduce

$$\gamma' = \frac{\gamma_{\mathrm{P}}}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)}. \tag{4.5}$$

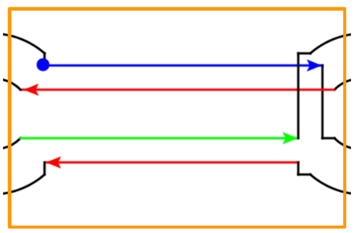
We can split the contour $\gamma'$ as the sum of three contributions, say around $\circ = 0$, the segment $[\epsilon, 1-\epsilon]$ and $\circ = 1$.

Let $C_\epsilon(\circ)$ be an anticlockwise circle of radius $\epsilon$ centered at $\circ$ with starting point laying on the segment $(0,1)$ (with $\arg(\epsilon) = 0$ and $\arg(1-\epsilon) = 0$).
The contribution around $\circ = 0$ is$-$keeping track of the argument of $z$ and $(1-z)$ at the starting point of $C_0$ and $C_0'$ as well as orientation$-$



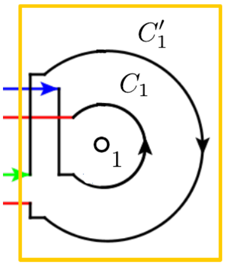

$$= \frac{C_0 + C_0'}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)} = \frac{\left(e^{2\pi i q} - 1\right) C_\epsilon(0)}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)} = -\frac{C_\epsilon(0)}{1 - e^{2\pi i p}}. \tag{4.6}$$

The contribution from the middle segment is



$$= \frac{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)}\,[\epsilon, 1-\epsilon] = [\epsilon, 1-\epsilon]. \tag{4.7}$$

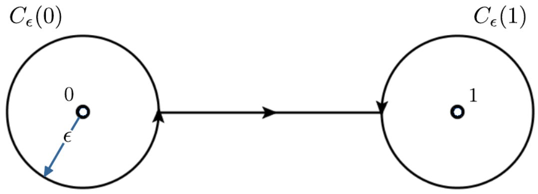
Finally the contribution around $\circ = 1$ is




$$= \frac{C_1 + C_1'}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)} = \frac{\left(1 - e^{2\pi i p}\right) C_\epsilon(1)}{\left(1 - e^{2\pi i q}\right)\left(1 - e^{2\pi i p}\right)} = +\frac{C_\epsilon(1)}{1 - e^{2\pi i q}}. \tag{4.8}$$

Therefore we have

$$\gamma' =$$




$$= -\frac{C_\epsilon(0)}{1 - e^{2\pi i p}} + [\epsilon, 1-\epsilon] + \frac{C_\epsilon(1)}{1 - e^{2\pi i q}}. \tag{4.9}$$

After all these manipulations we are lead to consider the following

$$\int_{\gamma'} u(z)\, \frac{dz}{z(1-z)}, \qquad u(z) = z^p (1-z)^q. \tag{4.10}$$

Eq. (4.10), which is valid for $(p, q) \notin \mathbb{Z}$, reduces to eq. (4.2) when $\mathrm{Re}(p), \mathrm{Re}(q) > 0$ (and $(p, q) \notin \mathbb{Z}$).

Even if eq. (4.10) looks like a simple reshuffling of the original objects at our disposal, it offers an interesting change of perspective.

Eq. (4.10) can be seen as the integral of a multivalued function over a topological path $\gamma'$. On the other hand, eq. (4.10) can be re-considered as the pairing of a single-valued differential form which, in the case at hand, reads

$$\frac{dz}{z(1-z)} \tag{4.11}$$

and

$$-\frac{C_\epsilon(0)}{1 - e^{2\pi i p}} \otimes u(z) + [\epsilon, 1-\epsilon] \otimes u(z) + \frac{C_\epsilon(1)}{1 - e^{2\pi i q}} \otimes u(z). \tag{4.12}$$

Colloquially, eq. (4.12) means that we *assign to each path an information about the branch of the multivalued function* $u$—in the case at hand, the branch is specified by $\arg(z) = \arg(1-z) = 0$ on $[\epsilon, 1-\epsilon]$ as well as at the starting point of $C_\epsilon(0)$ and $C_\epsilon(1)$. Eq. (4.12) is called *twisted*—or *loaded*—path.

Therefore—introducing a notation which will be fully clear only later on—we will focus on the following configuration

$$\begin{aligned} \int_\gamma u(z)\,\varphi(z) &= \langle\,\varphi\,|\,\gamma\otimes u(z)] \\ &\rightsquigarrow \text{Single-valued 1-form} \times \text{Twisted path;} \end{aligned} \tag{4.13}$$

Integrals of the form of eq. (4.13) can be addressed and better understood thanks to the theory of *twisted* Homology and *twisted* Co-Homology, or, more colloquially with the tools offered by *Intersection Theory*—the reason behind this name will be clear later on.

## 4.1

## Basic Aspects of Twisted Homology

We discuss here some basic aspects of twisted Homology. Clearly an important difference compared to "*standard*"—i.e. the non-twisted[3] case—is due to the multivaluedness of the objects we are dealing with. We do not dive into formal proofs, rather we try to give priority to intuition and explicit calculations. In order the fix the ideas, at the beginning we will work explicitly with the Euler Beta integral; we will slightly generalize our results later on.

### 4.1.1

### Boundary Operator

While studying regular homology, we realize that the *boundary operator* $\partial(\bullet)$ plays an important role. So, it seems desirable to understand how we can implement to action of the boundary operator, say $\partial^u(\bullet)$—where the superscript $u$ is used in order to stress that we are dealing with twisted, or loaded, paths—within our context.

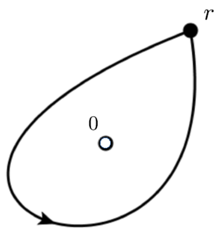
Let us consider first a straight path $(r,s)$ loaded with $u(z)$, we define the action of $\partial^u$ as

$$\begin{aligned} \partial^u\left((r,s)\otimes u(z)\right) &= [s]\otimes u(s) - [r]\otimes u(r) \\ &\rightsquigarrow \; + \textit{Final point} \text{ of the path } \textit{loaded} \text{ with } u(z) \text{ evaluated there} \\ &\qquad - \textit{Initial point} \text{ of the path } \textit{loaded} \text{ with } u(z) \text{ evaluated there.} \end{aligned} \tag{4.14}$$

[3]For a physics-oriented introduction to Homology see e.g. [172].

Let us consider now the closed path $\mathcal{P}_r$ given by:

$$\mathcal{P}_r = -\frac{1}{1-e^{2\pi i p}} \tag{4.15}$$




.

We find:

$$\partial^u \left(\mathcal{P}_r \otimes u(z)\right) = -\frac{1}{1-e^{2\pi i p}}\Big([r]\otimes e^{2\pi i p}u(r) - [r]\otimes u(r)\Big) = [r]\otimes u(r). \tag{4.16}$$

Finally, moving to the loaded path as defined in eq. (4.9)—we have

$$\begin{aligned}
\partial^u\,(4.9) =& \partial^u \left(-\frac{C_\epsilon(0)}{1-e^{2\pi i p}}\otimes u(z) + [\epsilon, 1-\epsilon]\otimes u(z) + \frac{C_\epsilon(1)}{1-e^{2\pi i q}}\otimes u(z)\right)\\
=& -\frac{1}{1-e^{2\pi i p}}\Big([\epsilon]\otimes e^{2\pi i p}u(\epsilon) - [\epsilon]\otimes u(\epsilon)\Big) + \Big([1-\epsilon]\otimes u(1-\epsilon) - [\epsilon]\otimes u(\epsilon)\Big)\\
&+\frac{1}{1-e^{2\pi i q}}\Big([1-\epsilon]\otimes e^{2\pi i q}u(1-\epsilon) - [1-\epsilon]\otimes u(1-\epsilon)\Big)\\
=&[\epsilon]\otimes u(\epsilon) + \Big([1-\epsilon]\otimes u(1-\epsilon) - [\epsilon]\otimes u(\epsilon)\Big) - [1-\epsilon]\otimes u(1-\epsilon)\\
=&0.
\end{aligned} \tag{4.17}$$

We will refer to a *twisted*—or *loaded*—*path* with *zero boundary* as a *twisted*—or *loaded*—*cycle*.

Eq. (4.17) shows that eq. (4.9) is a twisted cycle; conversely $\mathcal{P}_r \otimes u(z)$ is *not* twisted cycle, as eq. (4.16) reveals.
We would like also to go back to $(0,1)\otimes u(z)$ which was, after all, the starting point of our discussion (cf. eq. (4.2)). It has *no boundary inside* $X = \mathbb{C}\setminus\{0,1\}$, and so we declare that it is a twisted cycle as well.
We conclude this paragraph stating that, similarly to the regular (i.e. non-twisted) case, the boundary operator turns to be nilpotent, namely:

$$\partial^u \circ \partial^u = 0. \tag{4.18}$$

We will organize the results from the previous paragraph in the following subsection, introducing other important structures.

### 4.1.2

Twisted Homology Groups

We will focus on the following, slightly more general[4]:

$$u(z) = (z - x_0)^{\alpha_0} \dots (z - x_{m+1})^{\alpha_{m+1}}, \tag{4.19}$$

defined on $X = \mathbb{CP}^1 \setminus \{x_0, \dots, x_{m+1}, x_{m+2} = \infty\}$. We define $\alpha_{m+2} = -\sum_{j=0}^{m+1} \alpha_j$ and, to fix ideas, we assume that the $x_i$ are *real* (i.e. $x_i \in \mathbb{R}$, $0 \leq i \leq m+1$) and ordered in a natural way: $x_i < x_j$ for $i < j$. Moreover we assume that the exponents $\alpha_i$ are *not integers* (namely $\alpha_i \notin \mathbb{Z}$, $0 \leq i \leq m+2$).

Let us introduce a *twisted* $k-$*chains*

$$\sum_{j \in J} a_j \, \Delta_j^k \otimes u, \tag{4.20}$$

where $J$ is some *finite set*, $a_j \in \mathbb{C}$ and we can think at each $\Delta_j^k$ as a point $(k = 0)$, path $(k = 1)$ or curvilinear triangle $(k = 2)$.
Then denoting with $\mathscr{C}_k(X, u)$, or simply—for short—$\mathscr{C}_k$, the set of twisted $k$-chains

$$\mathscr{C}_k(X, u) = \mathscr{C}_k = \left\{ \sum_{j \in J} a_j \, \Delta_j^k \otimes u \right\} \tag{4.21}$$

Then the ($\mathbb{C}$-linear) map $\partial^u$ acts as

$$\begin{aligned} &\partial^u : \mathscr{C}_k \to \mathscr{C}_{k-1}, \\ &\Delta^k \otimes u \to \partial\Delta^k \otimes u\big|_{\partial\Delta^k}. \end{aligned} \tag{4.22}$$

The nilpotency of the boundary operator (cf. eq. (4.18)) implies:

$$\partial^u \circ \partial^u = 0 \quad \Longrightarrow \quad \mathrm{Im}\,(\partial^u : \mathscr{C}_{k+1} \to \mathscr{C}_k) \subset \mathrm{Ker}\,(\partial^u : \mathscr{C}_k \to \mathscr{C}_{k-1})\,; \tag{4.23}$$

elements of the kernal of $\partial^u$ are referred to as *twisted cycles*:

$$\mathrm{Ker}\,(\partial^u : \mathscr{C}_k \to \mathscr{C}_{k-1}) \ni \text{twisted cycle}. \tag{4.24}$$

Eq. (4.23) leads to the definition of the *twisted k-th homology group*:

$$\mathrm{H}_k(X, u) = \mathrm{H}_k = \frac{\mathrm{Ker}\,(\partial^u : \mathscr{C}_k \to \mathscr{C}_{k-1})}{\mathrm{Im}\,(\partial^u : \mathscr{C}_{k+1} \to \mathscr{C}_k)}, \qquad k = 0, 1, 2. \tag{4.25}$$

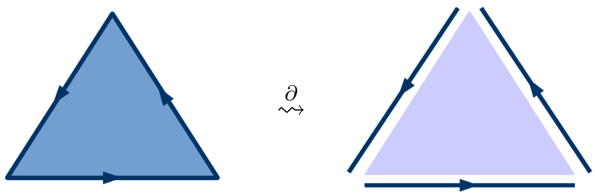
The other $\mathrm{H}^k$ with $k > 2$ are empty, since we cannot consider higher dimensional objects.
The next, natural, problem is to determine the dimension of $\mathrm{H}_0$, $\mathrm{H}_1$ and $\mathrm{H}_2$; we will try to do it

[4] It can be shown that, for instance, the Euler Beta $B(p, q)$ and the hypergeometric ${}_2F_1$ can be cast in this form (cf. appendix B). Beside them, also the Lauricella $F_D$ admits the same representation [173, 174].

in an intuitive way in the following paragraph.

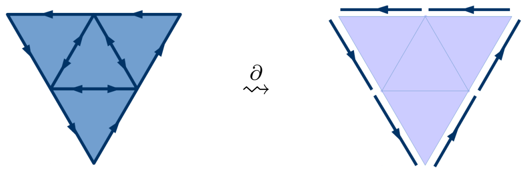
Let us start from $\mathrm{H}_0$; eq. (4.15) and eq. (4.16) shows that every loaded point can be written as the boundary of a suitably chosen loaded path, therefore $\dim \mathrm{H}_0 = 0$.

Moving to $\mathrm{H}_2$, we argue that we *cannot* find an element laying in $\mathrm{Ker}\,(\partial^u : \mathscr{C}_2 \to \mathscr{C}_1)$.
Let us start from a single triangle and compute its boundary:




(4.26)

it is clearly non zero.
We could be tempted to eliminate the boundaries patching up other triangles (with proper orientation). Nevertheless this does not work, since we keep introducing new boundaries at each step, and the problem cannot be solved if we are working with finite sums.




(4.27)

Thus, we conclude that $\dim \mathrm{H}_2 = 0$.

It is a known fact that the *Euler characteristic* of $X$ is equal to the *alternating sum* of the dimensions of the twisted homology groups [23]:

$$\begin{aligned}\chi(X) &= \dim(\mathrm{H}_0) - \dim(\mathrm{H}_1) + \dim(\mathrm{H}_2) \\ &= 0 - \dim(\mathrm{H}_1) + 0 \\ &= -\dim(\mathrm{H}_1).\end{aligned} \tag{4.28}$$

We have

$$\begin{aligned}\chi(X) = \chi(\mathbb{CP}^1 \setminus \{(m+3) \text{ points}\}) &= \chi(\mathbb{CP}^1 \approx \mathrm{S}^2) - \chi(\{(m+3) \text{ points}\}) \\ &= 2 - (m+3) \\ &= -(m+1).\end{aligned} \tag{4.29}$$

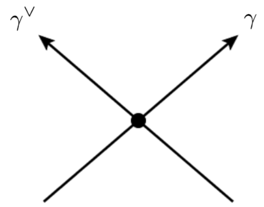
and so:

$$\dim(\mathrm{H}_1) = m + 1. \tag{4.30}$$

### 4.1.3 Homology Intersection Number

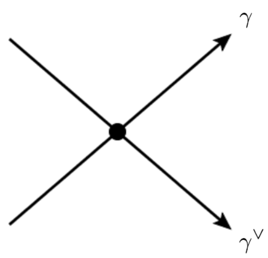
Crucially, it is possible to build a bilinear pairing among elements of $\mathrm{H}_1$ (or better, as we will see, among $\mathrm{H}_1$ and its dual space), referred to as *homology intersection number* [175, 176, 177, 178]. Let us describe how this is built, following a constructive approach.
Starting from the non-twisted case (i.e. $u = 1$), we can define a topological intersection number which describes how many times two paths, say $\gamma$ and $\gamma^\vee$, intersect each other (taking into account the relative orientation).

We have two building blocks:



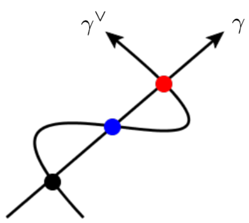

$$= +1, \tag{4.31}$$

or



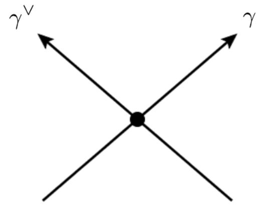

$$= -1. \tag{4.32}$$

We can see that a deformation of one of the two paths does not alter the final result:



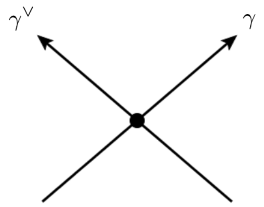

$$= +1 - 1 + 1 = +1. \tag{4.33}$$

Moving to the twisted case, we have to take into account the presence of $u(z)$. Let us consider two twisted—or loaded—cycles, intersecting transversally, say—for concreteness—as in eq. (4.31), at a given point $\bullet$. A naive guess for the intersection pairing could be



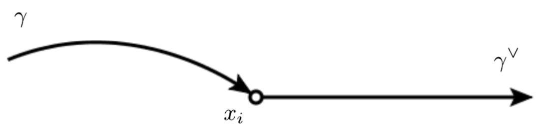

$$= (+1) \cdot u(\bullet) \cdot u(\bullet), \tag{4.34}$$

where we have one factor of $u(z)$ evaluated at $\bullet$ for each cycle.
Differently from the non-twisted case, eq. (4.34) is *not* invariant under small deformation of on of the two cycles ($u(z)$ evaluated at, say, $\bullet$ is different from $u(z)$ evaluated at a different point, say $\blacklozenge \neq \bullet$).
We can cure this problem, correcting eq. (4.34) as:



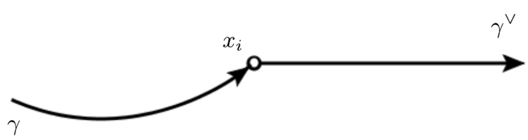

$$= (+1) \cdot u(\bullet) \cdot u^{-1}(\bullet). \tag{4.35}$$

Eq. (4.35) reveals that we *cannot* build a pairing among elements of $\mathrm{H}_1 = \mathrm{H}_1(X, u)$, rather among (elements of) $\mathrm{H}_1(X, u)$ and its *dual space* $\mathrm{H}_1(X, u^{-1})$.
We notice that the factor $u(z) \cdot u^{-1}(z)$ is not trivially $1$, since we could have different phases going around a branch point.
Indeed, the situation around a given branch point, say $x_i$, is delicate. Let us focus on a specific example: the intersection between $\gamma = (x_{i-1}, x_i)$ and $\gamma^\vee = (x_i, x_{i+1})$. Even in the non-twisted case the situation seems ambiguous: on the one hand small deformations may change the final result:




$$\stackrel{?}{=} +1, \tag{4.36}$$

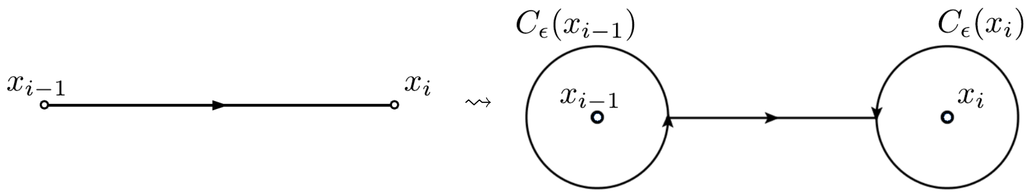
or




$$\stackrel{?}{=} -1. \tag{4.37}$$

On the other hand, declaring that such an intersection number is $0$$-$since, after all, $x_i \notin X$$-$is too drastic: if this is the case, we could deform all the topological cycle and force all the intersection numbers to be $0$. Therefore some sort of regularization$-$say $\mathrm{reg}_h(\bullet)$$-$is needed. It is at this stage that our preliminary discussion becomes useful. It turns out that the following map does the job (cf. eq. (4.9))

$\mathrm{reg}_h$ :




. (4.38)

Therefore the homology intersection number is given by

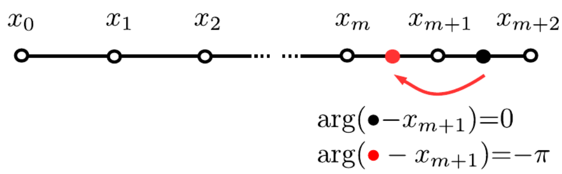
$$\begin{aligned} [\bullet|\bullet] \;:\; & \mathrm{H}_1(X,u)\times \mathrm{H}_1(X,u^{-1}) \to \mathbb{C} \\ & ([\gamma\otimes u(z)|, |\gamma^\vee \otimes u^{-1}(z)]) \to [\gamma\otimes u(z)|\gamma^\vee\otimes u^{-1}(z)] \\ & \qquad = \sum_{\bullet\in\gamma\cap\gamma^\vee} [\mathrm{reg}_h(\gamma)\mid\gamma^\vee]_{\text{top.}}\cdot u(\bullet)\cdot u^{-1}(\bullet). \end{aligned} \tag{4.39}$$

We will give here explicit examples of homology intersection numbers, where the topological cycle are $\gamma = (x_j, x_{j+1})$ and $\gamma^\vee = (x_k, x_{k+1})$ for some $j, k$.

At this stage, it is important to specify which branch of $u(z)$ is loaded on any given cycle, say $(x_p, x_{p+1})$; our prescription is that the branch of $u$ loaded on it is given by the following assignments

$$\arg(z - x_j) = 0 \quad \text{for } j \le p, \qquad \arg(z - x_j) = -\pi, \quad \text{for } j \ge p+1. \tag{4.40}$$

Eq. (4.40) may seem a bit involved at a first glance. Let us try to justify it in the following. Given a point $\bullet \in (x_{m+1}, x_{m+2})$, then all the factors $(z - x_j)|_{z=\bullet} = (\bullet - x_j)$ for $0 \le j \le m+1$ are positive real quantities, and therefore we have the natural choice $\arg(z - x_j) = 0$ for $0 \le j \le m+1$, consistently with eq. (4.40). Next, we consider a $\bullet \in (x_m, x_{m+1})$. It is clear that $(\bullet - x_j)$ for $0 \le j \le m$ are positive and real (and so $\arg(z - x_j) = 0$ for $0 \le j \le m$), while $(\bullet - x_{m+1}) < 0$. Our choice is then $\arg(\bullet - x_{m+1}) = -\pi$. So graphically




. (4.41)

Eq. (4.41) is summarized saying that $u(z)$ is *defined on the lower half plane*. Once again this is exactly what eq. (4.40) is telling us. Iterating this reasoning we get exacly eq. (4.40). We also assume that eq (4.40) holds for $u^{-1}(z)$ as well.
Finally we need to load a branch on $C_\epsilon(x_p)$ (and $C_\epsilon(x_{p+1})$). The branch is assigned by the fact that $\arg(z - x_p)$ $(\arg(z - x_{p+1}))$ at the starting point of $C_\epsilon(x_p)$ (and $C_\epsilon(x_{p+1})$) is given by eq. (4.40)

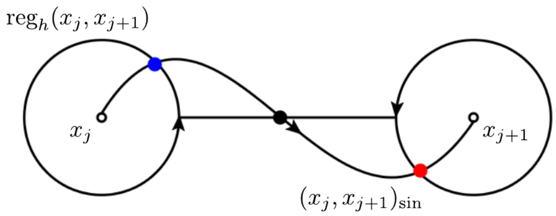
and it increases along $C_\epsilon(x_p)$ (and $C_\epsilon(x_{p+1})$).

We give below some explicit examples.

• **The self-intersection number** $\left[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j+1}) \otimes u^{-1}(z)\right]$.
We start with the (self) intersection $(x_j, x_{j+1}) \otimes u(z)$ and $(x_j, x_{j+1}) \otimes u^{-1}(z)$. On the one hand we have to apply the regularization map $\mathrm{reg}_h(\bullet)$ to the first element, on the other we have to slightly deform the second cycle; this is necessary since we would have to segments laying on top of each other. We claim that the way in which we do it, does not alter the final result (i.e. different deformations lead to the same result).
Let us consider first

$$\left[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j+1})_{\mathrm{sin}} \otimes u^{-1}(z)\right], \tag{4.42}$$

namely:

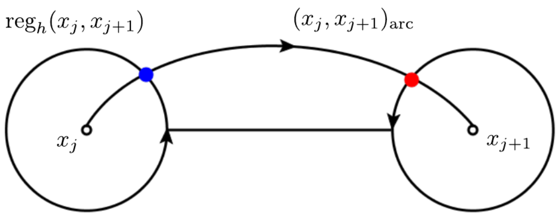



$$= -\frac{(-1)_{\text{top.}}}{1-e^{2\pi i \alpha_j}} + (-1)_{\text{top.}} + \frac{(+1)_{\text{top.}}}{1-e^{2\pi i \alpha_{j+1}}}. \tag{4.43}$$

We could also have

$$\left[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j+1})_{\mathrm{arc}} \otimes u^{-1}(z)\right], \tag{4.44}$$

which corresponds to:




$$= -\frac{(-1)_{\text{top.}}}{1-e^{2\pi i \alpha_j}} + \frac{(+1)_{\text{top.}} \cdot e^{2\pi i \alpha_{j+1}}}{1-e^{2\pi i \alpha_{j+1}}}; \tag{4.45}$$

where the factor $e^{+2\pi i \alpha_{j+1}}$ originates from the difference of the arguments, since $\arg(z - x_{j+1})$ increased flowing along the cricle.
Despite the fact that individual contributions are different, eq. (4.43) and eq. (4.45) sum up to the same result:

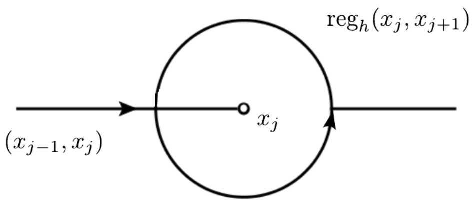
$$\left[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j+1}) \otimes u^{-1}(z)\right] = \frac{1-e^{2\pi i(\alpha_j + \alpha_{j+1})}}{\left(1-e^{2\pi i \alpha_j}\right)\left(1-e^{2\pi i \alpha_{j+1}}\right)}. \tag{4.46}$$

Keeping in mind eq. (4.40) and tracing the variation of $\arg(\bullet)$ along circles we can compute intersection numbers among adjacent cycles.

• **The intersection number** $[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j-1}) \otimes u(z)]$.
Let us consider

$$[(x_j, x_{j+1}) \otimes u(z) \,|\, (x_j, x_{j-1}) \otimes u(z)]\,, \tag{4.47}$$

which is given by

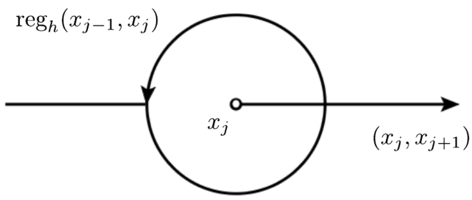



$$= -\frac{(+1)_{\text{top.}} \cdot e^{2\pi i \alpha_j}}{1 - e^{2\pi i \alpha_j}}. \tag{4.48}$$

• **The intersection number** $[(x_{j-1}, x_j) \otimes u(z) \,|\, (x_j, x_{j+1}) \otimes u(z)]$ .
Finally we consider

$$[(x_{j-1}, x_j) \otimes u(z) \,|\, (x_j, x_{j+1}) \otimes u(z)]\,, \tag{4.49}$$

associated to




$$= +\frac{(-1)_{\text{top.}}}{1 - e^{2\pi i \alpha_j}}; \tag{4.50}$$

So we have

$$[(x_{j-1}, x_j) \otimes u(z) \,|\, (x_j, x_{j+1}) \otimes u(z)] = +\frac{(-1)}{1 - e^{2\pi i \alpha_j}}. \tag{4.51}$$

We conclude this section mentioning that intersection numbers involving non-adjacent segments are trivially zero, since the topological intersection is vanishing.

## 4.2

## Basic Aspects of Twisted Co-Homology

### 4.2.1

### Twist and Connection

Let us go back now to our guiding example, namely Euler Beta-like integrals (cf. eq. (4.2)). Given

$$u(z) = z^p(1-z)^q, \qquad \xi \rightsquigarrow \text{single-valued smooth function on } X = \mathbb{C} \setminus \{0,1\}, \tag{4.52}$$

ordinary Stokes' theorem gives

$$\int_0^1 d\left(u(z)\,\xi\right) = 0, \tag{4.53}$$

where we used $u\left(\partial(0,1)\right) = 0$ for $\mathrm{Re}(p,q)$ sufficiently large (see also eq. (2.60), in order to make contact with physics-motivated examples).
Once again, it is useful to slightly rearrange eq. (4.53); performing the algebra under the integral sign we find:

$$\begin{aligned} 0 = \int_0^1 d\left(u(z)\,\xi\right) &= \int_0^1 \left(u(z)d\xi + du(z)\wedge\xi\right) \\ &= \int_0^1 u(z)\,\left(d\xi(z) + \frac{du(z)}{u(z)}\wedge\xi\right); \end{aligned} \tag{4.54}$$

we can further refine the previous expression, introducing the following holomorphic one form—dubbed as *twist*—

$$\omega(z) = d\log u(z) = \frac{du(z)}{u(z)}, \tag{4.55}$$

and the operator—often dubbed as *connection*—

$$\nabla_\omega(\bullet) = d(\bullet) + \omega(z)\wedge\bullet. \tag{4.56}$$

Therefore eq. (4.54) boils down to:

$$0 = \int_0^1 u(z)\,\nabla_\omega\xi. \tag{4.57}$$

Eq. (4.57) implies that shifting the single-valued one form $\varphi$ by $\nabla_\omega\xi$ under the integral sign is harmless, namely

$$\int_0^1 u(z)\,\varphi = \int_0^1 u(z)(\varphi + \nabla_\omega\xi); \tag{4.58}$$

therefor $\varphi$ and $\varphi+\nabla_\omega\xi$ can be considered equivalent

$$\varphi \sim \varphi + \nabla_\omega\xi. \tag{4.59}$$

In fact, studying integrals of the form of eq. (4.13) at the level of equivalence classes is one of the essence of twisted Co-Homology.
We conclude this introductory paragraph noticing showing that $\nabla_\omega$ is *integrable*, i.e.:

$$\nabla_\omega \circ \nabla_\omega = 0. \tag{4.60}$$

An explicit calculation shows:

$$\begin{aligned}
\nabla_\omega \left(\nabla_\omega \xi\right) &= \nabla_\omega \left(d\xi + \omega(z) \wedge \xi\right) \\
&= d\Big(d\xi + \omega(z) \wedge \xi\Big) + \omega(z) \wedge \Big(d\xi + \omega(z) \wedge \xi\Big) \\
&= d\omega(z) \wedge \xi - \cancel{\omega(z) \wedge d\xi} + \cancel{\omega(z) \wedge d\xi} + \omega(z) \wedge \omega(z) \wedge \xi \\
&= \cancelto{0}{d\omega(z)} \wedge \xi + \cancelto{0}{\omega(z) \wedge \omega(z)} \wedge \xi \\
&= 0.
\end{aligned} \tag{4.61}$$

We reorganize these preliminary notions in a more systematic way in the following subsection.

### 4.2.2 Twisted Co-Homology Groups

Let us work for concreteness on the set-up described in eq. (4.19)[5]; we introduce the space of smooth $k-$forms on $X = \mathbb{CP}^1 \setminus \{x_0, \dots x_{m+2}\} = \mathbb{CP}^1 \setminus \mathcal{P}_\omega$[6]

$$\mathscr{E}^k(X) = \mathscr{E}^k = \big\{\text{smooth single-valued } k-\text{forms on } X\big\}, \tag{4.63}$$

then the operator introduced in eq. (4.56) is such that

$$\begin{aligned}
\nabla_\omega : \mathscr{E}^k &\to \mathscr{E}^{k+1}, \\
\xi &\to d\xi + \omega(z) \wedge \xi.
\end{aligned} \tag{4.64}$$

Eq. (4.60) implies

$$\nabla_\omega \circ \nabla_\omega = 0 \quad \Longrightarrow \quad \text{Im}\left(\nabla_\omega : \mathscr{E}^{k-1} \to \mathscr{E}^k\right) \subset \text{Ker}\left(\nabla_\omega : \mathscr{E}^k \to \mathscr{E}^{k+1}\right); \tag{4.65}$$

[5] We stress that this is just a choice in order to simplify the discussion. Even if several mathematical functions fall into this category (cf. footnote (4)), some of the results derived hereafter will be applied in a more generic scenario. For example we will never rely on the fact that the various $x_i$ are real.

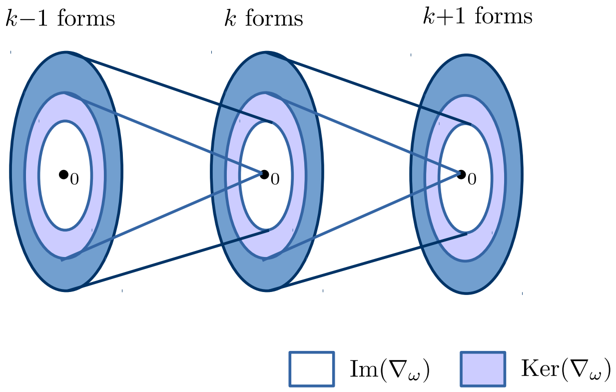
[6] The set of points $\{x_0, \dots x_{m+2}\}$ are now the poles of $\omega(z)$, see also eq. (4.55). Hence the notation

$$\mathcal{P}_\omega = \{x_0, \dots x_{m+2}\}. \tag{4.62}$$

elements laying in the kernel of $\nabla_\omega(\bullet)$ are referred to as *twisted co-cycles*

$$\mathrm{Ker}\left(\nabla_\omega : \mathscr{E}^k \to \mathscr{E}^{k+1}\right) \ni \text{twisted co-cycle}. \tag{4.66}$$

The situation summarized in the following diagram:




(4.67)

We can introduce *twisted $k-th$ co-homology group*

$$\mathrm{H}^k(X, \nabla_\omega) = \mathrm{H}^k = \frac{\mathrm{Ker}\left(\nabla_\omega : \mathscr{E}^k \to \mathscr{E}^{k+1}\right)}{\mathrm{Im}\left(\nabla_\omega : \mathscr{E}^{k-1} \to \mathscr{E}^{k}\right)}, \qquad k = 0, 1, 2. \tag{4.68}$$

Their elements are denoted with $\langle\varphi|$, and two object which differ by shift through $\nabla_\omega(\bullet)$ are declared to be equivalent

$$\mathrm{H}^k \ni \langle\varphi| \, : \, \varphi \sim \varphi + \nabla_\omega \xi. \tag{4.69}$$

We would like now to determine the dimension of $\mathrm{H}^0$, $\mathrm{H}^1$ and $\mathrm{H}^2$; once again we proceed trying to give priority to the intuition.

We claim that $\dim \mathrm{H}^0 = 0$. Let us consider

$$\nabla_\omega f = 0 \tag{4.70}$$

as a differential equation; we infer that a general solution reads

$$f = c/u(z), \tag{4.71}$$

being $c$ a constant, $c \in \mathbb{C}$.
Nevertheless eq. (4.71) multivalued in $X$, the only global solution single-valued solution is $f = 0$, and hence the claim.
Next, we want to show that also $\dim \mathrm{H}^2(X, \nabla_\omega) = 0$; we aim to prove that given any $\eta \in \mathscr{E}^2$ we

can find $\lambda \in \mathscr{E}^1$ such that

$$\nabla_\omega \lambda = \eta. \tag{4.72}$$

By the $\bar{\partial}-$Poincaré lemma[7], there exists a $(1,0)$ form$-$say $\lambda = g\,dz-$such that $\bar{\partial}\lambda = \eta$. Therefore[8]:

$$\begin{aligned} \nabla_\omega \lambda &= (\partial + \bar{\partial})\big(g\,dz\big) + \omega(z) \wedge \big(g dz\big) \\ &= \bar{\partial}\big(g\,dz\big) \\ &= \eta. \end{aligned} \tag{4.73}$$

Once again, we rely on the fact that the Euler characteristic is equal to the alternating sum of the dimensions of the twisted co-homology groups [23],

$$\begin{aligned} \chi(X) &= \dim(\mathrm{H}^0) - \dim(\mathrm{H}^1) + \dim(\mathrm{H}^2) \\ &= 0 - \dim(\mathrm{H}_1) + 0 \\ &= -\dim(\mathrm{H}_1), \end{aligned} \tag{4.74}$$

and therefore[9]:

$$\dim \mathrm{H}^1\,(=\nu)\; = -\chi(X) = m+1. \tag{4.75}$$

So we conclude by saying that $\mathrm{H}^1$ is the only interesting$-$i.e. non empty$-$space. Its elements are $\langle \varphi| \in \mathrm{H}^1$ (see also eq. (4.69)).

Any *holomorphic* one form belongs to $\mathrm{Ker}(\nabla_\omega(\bullet))$ and so to $\mathrm{H}^1$. From now on, unless specify differently, we will focus on those i.e. $\varphi = \varphi(z) = \hat{\varphi}(z)\,dz$, with $\hat{\varphi}(z)$ holomorphic in $X$.

### 4.2.3

### Euler Characteristic via Morse Theory

We discuss here how to compute $\chi(X)$ relying on *Morse* theory$-$or, better, its complex version known as *Picard–Lefschetz* theory[10].

Let us introduce the following *Morse function*

$$\begin{aligned} h_r &: X \to \mathbb{R}, \\ &\quad z \to \mathrm{Re}\left(\log u(z)\right). \end{aligned} \tag{4.76}$$

Colloquially eq. (4.76) gives an "height" to each point of $X$; it is useful since one can deduce topological properties of $X$ studying this function.

[7] See e.g. [179], proposition (1.3.7).

[8] In full generality $d(\bullet) = \partial(\bullet) + \bar{\partial}(\bullet) = \frac{\partial(\bullet)}{\partial z} dz + \frac{\partial(\bullet)}{\partial \bar{z}} d\bar{z}$.

[9] At the moment $\nu$ is a mere symbol. However, as it will become clear in the following, this choice is not an accident (cf. section 2.4). In fact, applying this framework to the case of FIs, $\nu$ will be precisely the number of MIs. The connection among dimensions of (co)homology groups, Euler characteristic and number of MIs has been proposed originally in [54, 180].

[10] For a comprehensive discussion on Morse Theory see e.g. [181].

We also introduce the—at the moment—auxiliary function

$$\begin{aligned} h_c : X &\to \mathbb{C}, \\ z &\to \log u(z); \end{aligned} \tag{4.77}$$

Its role will be clear later on; for the moment we stress that $h_c \neq h_r$.
A point $z_{\text{crt}}$ is a *critical point* of $h_r$ iff

$$dh_r(z_{\text{crt}}) = 0; \tag{4.78}$$

then the following relation holds [181]:

$$\chi(X) = \sum_{j=0}^{2} (-1)^j \ \mathrm{M}_j, \tag{4.79}$$

where $\mathrm{M}_j$ is the *number of critical points* with *Morse index* $j$; intuitively a Morse index indicates the number of downwards directions for the Morse function—i.e. the direction along which $h_r$ decreases—passing through $z_{\text{crt}}$.
The function $\log u(z)$ is *holomorphic* if we are *not* in the proximity of a branch point $x_i$; so in a neighborhood of a critical point $z_{\text{crt}}$, $h_r = \mathrm{Re}\,(\log u(z))$ admits the following expansion (assuming *local coordinate* $w$ around each critical point $z_{\text{crt}}$)

$$h_r(w) = h_r(0) + \mathrm{Re}(w^2) + \dots \ . \tag{4.80}$$

Rewriting eq. (4.80) in terms of real coordinates $w = x + iy$ we find

$$h_r(x, y) = h_r(0, 0) + x^2 - y^2 + \dots \ . \tag{4.81}$$

Eq. (4.81) reveals that each critical point has Morse index equal to *one*.
Therefore eq. (4.79) reduces to

$$\chi(X) = - \ \mathrm{M}_1, \tag{4.82}$$

so $\chi(X)$ is equal to the *number* of critical points, up to a sign.
Finally Cauchy-Riemann equations imply that $z_{\text{crt}}$ is a critical point of $h_r$ iff it is a critical point also for $h_c$[11]. Critical points of $h_c = \log u(z)$, are nothing but *zeros* of

$$\omega(z) = d \log u(z); \tag{4.85}$$

[11]If $z_{\text{crt}} = (x_{\text{crt}}, y_{\text{crt}})$ is a critical point for $h_c$ then it is clear that it is also a critical point for $h_r$, since

$$dh_c|_{(x_{\text{crt}},y_{\text{crt}})} = d\,\mathrm{Re}\ h_c|_{(x_{\text{crt}},y_{\text{crt}})} + i\,d\,\mathrm{Im}\ h_c|_{(x_{\text{crt}},y_{\text{crt}})} = dh_r|_{(x_{\text{crt}},y_{\text{crt}})} + i\,d\,\mathrm{Im}\ h_c|_{(x_{\text{crt}},y_{\text{crt}})} = 0, \tag{4.83}$$

implies that both terms vanish separately, hence $dh_r|_{(x_{\text{crt}},y_{\text{crt}})}$.

Conversely if $(x_{\text{crt}}, y_{\text{crt}})$ is a critical point for $h_r = \mathrm{Re}(\log u)$—i.e. $d_x h_r|_{(x_{\text{crt}},y_{\text{crt}})} = d_y h_r|_{(x_{\text{crt}},y_{\text{crt}})} = 0$—then Cauchy—Riemann equations imply ($h_r = \mathrm{Re}\, h_c$)

$$d_y h_r|_{(x_{\text{crt}},y_{\text{crt}})} = -d_x\ \mathrm{Im}\, h_c|_{(x_{\text{crt}},y_{\text{crt}})} = 0, \qquad d_x h_r|_{(x_{\text{crt}},y_{\text{crt}})} = d_y\ \mathrm{Im}\, h_c|_{(x_{\text{crt}},y_{\text{crt}})} = 0, \tag{4.84}$$

and therefore $dh_c|_{(x_{\text{crt}},y_{\text{crt}})} = d\,\mathrm{Re}\, h_c|_{(x_{\text{crt}},y_{\text{crt}})} + id\,\mathrm{Im}\, h_c|_{(x_{\text{crt}},y_{\text{crt}})} = 0$.

Putting everything together, we have

$$\dim \mathrm{H}^1 \, (= \nu) \; = -\chi(X) = \#\,\text{solutions of: } \omega(z) = 0. \tag{4.86}$$

### 4.2.4

## Co-Homology Intersection Number

Following [25] we can introduce a bilinear, non-degenerate pairing among elements of $\mathrm{H}^1(X, \nabla_\omega)$ and the dual space $\mathrm{H}^1(X, \nabla_{-\omega})$ (notice that $u \to u^{-1}$ produces $\omega \to -\omega$), referred to as *co-homology intersection number*, or, simply (in this work) *intersection number*.

Let us consider $\langle \varphi | \in \mathrm{H}^1(X, \nabla_\omega)$ and $|\varphi^\vee\rangle \in \mathrm{H}^1(X, \nabla_{-\omega})$ with $\varphi$ and $\varphi^\vee$ holomorphic; our first guess would be

$$\langle \varphi \,|\, \varphi^\vee \rangle \stackrel{?}{=} \int_X \varphi(z) \wedge \varphi^\vee(z). \tag{4.87}$$

Unfortunately eq. (4.87) is ill defined; roughly speaking the integrand diverges near the boundary of $X$–i.e. the set of points $\mathcal{P}_\omega = \{x_0, x_1, \ldots, x_{m+2}\}$ removed from $\mathbb{CP}^1$–due to the fact that $\varphi^{(\vee)}$ may have poles at those locations. Furthermore the integrand in eq. (4.87) carries a $dz \wedge dz = 0$ in the numerator, loosely speaking we land in a sort of indeterminate "0/0" problem. Some sort of regularization for $\varphi$, say $\mathrm{reg}_\omega\,(\varphi)$, is required.

Intuitively, we want to replace $\varphi$ with something vanishing in a small neighborhood $V_i$ around each $x_i$, laying in the same equivalence class of the original $\varphi$; so we look for something of the following form

$$\mathrm{reg}_\omega : \varphi \to \mathrm{reg}_\omega(\varphi) = \varphi \pm \nabla_\omega(\bullet) \;\rightsquigarrow\; \text{vanishing on each } V_i. \tag{4.88}$$

Let us consider an holomorphic function $\psi_i$, satisfying:

$$\nabla_\omega \, \psi_i = \varphi, \qquad \text{locally on each } V_i. \tag{4.89}$$

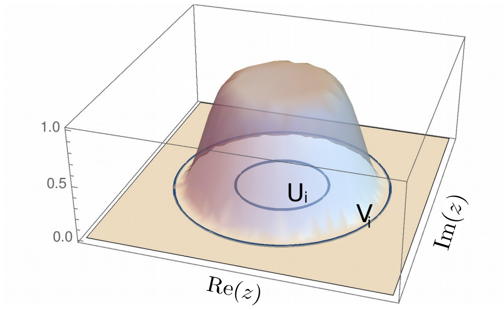
For the moment we just assume that an explicit solution of eq. (4.89) can be found. Therefore we can correct our guess eq. (4.88) as

$$\mathrm{reg}_\omega : \varphi \to \mathrm{reg}_\omega(\varphi) \stackrel{?}{=} \varphi - \sum_i \nabla_\omega(\psi_i) \;\rightsquigarrow\; \text{vanishing on each } V_i \text{ by eq. (4.89)}. \tag{4.90}$$

Eq. (4.93) is almost the final answer; we have just to glue, or interpolate, the various $\psi_i$ together in a proper way, to have something defined on the full $X$. In order to do this, we introduce a *bump function* $h_i(z,\bar{z})$ such that

$$h_i(z,\bar{z}) = \begin{cases} 1 & \text{on } U_i, \\ 0 \leq h_i \leq 1 & \text{on } V_i \setminus U_i, \\ 0 & \text{outside } V_i; \end{cases} \tag{4.91}$$

where $U_i \subset V_i$ is another (small) neighborhood of $x_i$.
The situation is described by the following picture:

$h_1(z,\bar{z}) \rightsquigarrow$  (4.92)


So, the desired regularization map reads

$$\text{reg}_\omega : \varphi \to \text{reg}_\omega(\varphi) = \varphi - \sum_i \nabla_\omega(h_i\,\psi_i). \tag{4.93}$$

Therefore, the intersection pairing is given by

$$\begin{aligned} \langle \varphi\,|\,\varphi^\vee \rangle &= \int_X \text{reg}_\omega(\varphi) \wedge \varphi^\vee \\ &= \int_X \left(\varphi - \sum_i \nabla_\omega(h_i\,\psi_i)\right) \wedge \varphi^\vee. \end{aligned} \tag{4.94}$$

Eq. (4.94) looks quite involved; we will show now that we can simplify it in order to arrive to a (strikingly) simple and compact result.

Let us consider its derivation step-by-step. On the one hand, that $h_i = 0$ on $X \setminus \bigcup_i V_i$ and $\varphi \wedge \varphi^\vee = 0$ there; on the other hand, that $h_i = 1$ on $U_i$ and $(\varphi - \nabla_\omega \psi_i) \wedge \varphi^\vee = (\varphi - \varphi) \wedge \varphi^\vee = 0$ by thanks to (4.89). Therefore the integral eq. (4.94) gets non-vanishing contribution only from the region $\bigcup_i V_i \setminus U_i$, i.e.

$$\langle \varphi\,|\,\varphi^\vee \rangle = \int_{\bigcup_i V_i \setminus U_i} \left(\varphi - \sum_i \nabla_\omega(h_i\,\psi_i)\right) \wedge \varphi^\vee; \tag{4.95}$$

using (once again) the fact that $\varphi \wedge \varphi^\vee = 0$ on $\bigcup_i V_i \setminus U_i$ we obtain

$$\langle \varphi\,|\,\varphi^\vee \rangle = -\sum_i \int_{V_i \setminus U_i} \nabla_\omega\,(h_i \psi_i) \wedge \varphi^\vee; \tag{4.96}$$

since also $\omega \wedge \varphi^\vee = 0$ on $\bigcup_i V_i \setminus U_i$, eq. (4.96) simplifies to

$$\langle \varphi \,|\, \varphi^\vee \rangle = -\sum_i \int_{V_i \setminus U_i} d(h_i \psi_i) \wedge \varphi^\vee = -\sum_i \int_{V_i \setminus U_i} d(h_i \psi_i \, \varphi^\vee); \tag{4.97}$$

finally Stokes' theorem and the very definition eq. (4.91) yield

$$\begin{aligned} \langle \varphi \,|\, \varphi^\vee \rangle = -\sum_i \int_{\partial(V_i \setminus U_i)} h_i \psi_i \, \varphi^\vee &= -\sum_i \int_{\partial V_i} h_i \psi_i \, \varphi^\vee + \sum_i \int_{\partial U_i} h_i \psi_i \, \varphi^\vee \\ &= -\sum_i \int_{\partial V_i} 0 \cdot \psi_i \, \varphi^\vee + \sum_i \int_{\partial U_i} 1 \cdot \psi_i \, \varphi^\vee \\ &= 2\pi i \sum_{x_i \in \mathcal{P}_\omega} \mathrm{Res}_{z=x_i} \left(\psi_i \varphi^\vee\right) . \end{aligned} \tag{4.98}$$

So, the intersection pairing reads[12]:

$$\begin{aligned} \langle \bullet | \bullet \rangle \ : \ \mathrm{H}^1(X, \nabla_\omega) \times \mathrm{H}^1(X, \nabla_{-\omega}) \to \mathbb{C} \\ (\langle \varphi |, | \varphi^\vee \rangle) \to \langle \varphi | \varphi^\vee \rangle = \sum_{x_i \in \mathcal{P}_\omega} \mathrm{Res} \left(\psi_i \varphi^\vee\right) , \end{aligned} \tag{4.99}$$

with

$$\psi_i \ : \ \text{local solution of: } \nabla_\omega \psi_i = \varphi \quad ; \qquad \mathcal{P}_\omega = \text{set of poles of } \omega \, . \tag{4.100}$$

We notice that, compared to the full integral eq. (4.94), in eq. (4.99) only the local behaviour around the set of poles $x_i$ is relevant and the result is built upon small building blocks.

In a more rigorous language, the regularization map $\mathrm{reg}_\omega(\bullet)$ replaces $\varphi$ with its (non holomorphic) compact support version. We could employ a different choice, namely replacing $\varphi^\vee$ with its compact support version–i.e. adopting $\mathrm{reg}_{-\omega}(\bullet)$– this would be an alternative and equivalent legitimate choice.

The intersection pairing is manifestly (bi)linear, i.e. $\langle c_1\varphi_1 + c_2\varphi_2 | \varphi^\vee\rangle = c_1\langle \varphi_1|\varphi^\vee\rangle + c_2\langle\varphi_2|\varphi^\vee\rangle$ (as well as $\langle \varphi | c_1\varphi_1^\vee + c_2\varphi_2^\vee\rangle = c_1\langle\varphi|\varphi_1^\vee\rangle + c_2\langle\varphi|\varphi_2^\vee\rangle$). Moreover we stress that the intersection number is a pairing among elements of $\mathrm{H}^1$ and its dual, and so replacing say $\varphi$ with another representative in the same co-homology class, i.e. $\varphi \to \varphi + \nabla_\omega \xi$ does not alter the result (the same holds for $\varphi^\vee$, upon the replacement $\varphi^\vee \to \varphi^\vee + \nabla_{-\omega}\xi^\vee$).

Above we just assumed that eq. (4.89) could be found, without specifying how this is done in practice. We address this point now.

Assuming local coordinates around each $x_i$, say $z = y + x_i$, the following series expansions hold

$$\omega = \sum_{m=-1}^{\infty} \hat{\omega}_m \, y^m \, dy, \qquad \varphi = \sum_{m=\min_\varphi}^{\infty} \hat{\varphi}_m \, y^m \, dy; \tag{4.101}$$

[12] Comparing to eq. (4.98), we rescaled the r.h.s. by an harmless factor $(2\pi i)^{-1}$. This is usually *not the convention* adopted in the mathematical literature.

We can just look for a solution—around each $x_i$—of the form[13]:

$$\psi_i = \sum_{m=\min_\varphi+1}^{\infty} \psi_m \, y^m. \tag{4.102}$$

We have just to plug the expansion eq. (4.101, 4.102) into $\nabla_\omega \psi = \varphi$; we have:

$$\left( \sum_{m=\min_\varphi+1}^{\infty} m \, \psi_m \, y^{m-1} + \left( \frac{\hat{\omega}_{-1}}{y} + \hat{\omega}_0 + \hat{\omega}_1 \, y + \ldots \right) \cdot \sum_{m=\min_\varphi+1}^{\infty} \psi_m \, y^m \right) = \sum_{m=\min_\varphi} \hat{\varphi}_m \, y^m, \tag{4.103}$$

which, rearranging the products of two sums into a single sum, gives

$$\sum_{m=\min_\varphi}^{\infty} \left( (m+1) \, \psi_{m+1} + \sum_{q=-1}^{m-\min_\varphi-1} \hat{\omega}_q \, \psi_{m-q} \right) y^m = \sum_{m=\min_\varphi}^{\infty} \hat{\varphi}_m \, y^m. \tag{4.104}$$

Requiring that eq. (4.104) holds order by order in $y$, we obtain a *linear system* for the unknown coefficients $\psi_m$ in eq. (4.102).

We notice that eq. (4.104) can be simplified even further, since the unknown coefficients obey a recursion relation—in other words the linear system is *triangular*—and we have

$$\psi_{m+1} = \frac{1}{m+1+\hat{\omega}_{-1}} \left( \hat{\varphi}_m - \sum_{q=0}^{m-\min_\varphi-1} \hat{\omega}_q \, \psi_{m-q} \right), \qquad \psi_{\min_\varphi+1} = \frac{\hat{\varphi}_{\min_\varphi}}{\min_\varphi+1+\hat{\omega}_{-1}}; \tag{4.105}$$

the coefficients appearing on the r.h.s. at the $m-$th step can be considered known.

Even if the coefficients in eq. (4.102) can be computed—in principle—to arbitrary order, it is useful to inspect eq. (4.99). We notice that product $\psi \, \varphi^\vee$ has to be computed up to order: $(-1)$, since we are just interested in its residue.
Therefore, considering the local expansion

$$\varphi^\vee = \sum_{m=\min_{\varphi^\vee}}^{\infty} \hat{\varphi}_m^\vee \, y^m \, dy, \tag{4.106}$$

we infer that it is sufficient to expand $\psi$ up to order: $(-\min_{\varphi^\vee}-1)$; in this way all the meaningful contributions to the residue are captured.

• **The special case of logarithmic differential forms**.
We conclude this section, specifying our discussion to the case of logarithmic forms, i.e. differ-

[13]The minimum of the expansion has to be understood as an "optimal" choice. Clearly a lower minimum would correspond to coefficients which are automatically set to 0.

ential forms such that

$$\min_{\varphi} \geq -1, \qquad \text{and} \qquad \min_{\varphi^\vee} \geq -1; \tag{4.107}$$

around each pole. Such differential forms are often considered in the mathematical literature. If the conditions in eq. (4.107) are satisfied (around each pole), then just the leading terms in eq. (4.101) and eq. (4.106) are relevant, and eq. (4.99) boils down to

$$\langle \varphi | \varphi^\vee \rangle = \sum_{x_i \in \mathcal{P}_\omega} \frac{\mathrm{Res}_{z=x_i}(\varphi)\, \mathrm{Res}_{z=x_i}(\varphi^\vee)}{\mathrm{Res}_{z=x_i}(\omega)}. \tag{4.108}$$

Let us give some simple examples of co-homology intersection numbers constructed starting from eq. (4.19); eq. (4.108) gives

$$\left\langle d\log \frac{z-x_k}{z-x_{k+1}} \middle| d\log \frac{z-x_j}{z-x_{j+1}} \right\rangle = \begin{cases} -1/\alpha_j & \text{if } k = j-1, \\ 1/\alpha_j + 1/\alpha_{j+1} & \text{if } k = j, \\ -1/\alpha_{j+1} & \text{if } k = j+1, \\ 0 & \text{else.} \end{cases} \tag{4.109}$$

Other examples of co-homology intersection numbers will be considered extensively in the following.

## 4.3 Twisted Riemann's Period Relations

In the previous sections we introduced pairings among elements of twisted (co)homology groups and their duals. In the same spirit, an *integral* is the pairing between an element of the twisted co-homology group and twisted homology group

$$\begin{aligned} \langle \bullet | \bullet ] \;:\; & \mathrm{H}^1(X, \nabla_\omega) \times \mathrm{H}_1(X, u) \to \mathbb{C}, \\ & (\langle \varphi |, |\gamma \otimes u(z)]) \to \langle \varphi | \gamma \otimes u(z)] = \int_\gamma u(z)\, \varphi(z). \end{aligned} \tag{4.110}$$

while a *dual integral* is

$$\begin{aligned} [\bullet | \bullet \rangle \;:\; & \mathrm{H}_1(X, u^{-1}) \times \mathrm{H}^1(X, \nabla_{-\omega}) \to \mathbb{C}, \\ & ([\gamma^\vee \otimes u^{-1}(z)|, |\varphi^\vee\rangle) \to [\gamma^\vee \otimes u^{-1}(z) | \varphi^\vee \rangle = \int_{\gamma^\vee} u^{-1}(z)\, \varphi^\vee(z). \end{aligned} \tag{4.111}$$

Let $(\langle e_1|, \dots, \langle e_\nu|)$ a *basis* for $\mathrm{H}^1(X, \nabla_\omega)$ $((|h_1\rangle, \dots, |h_\nu\rangle)$ a *basis* for $\mathrm{H}^1(X, \nabla_{-\omega}))$ and $([\gamma_1 \otimes u(z)|, \dots, [\gamma_\nu \otimes u(z)|)$ a *basis* for $\mathrm{H}_1(X, u)$ $(([\gamma_1^\vee \otimes u^{-1}(z)|, \dots, [\gamma_\nu^\vee \otimes u^{-1}(z)|)$ a *basis* for $\mathrm{H}_1(X, u^{-1}))$ then we define the following matrices

- $\mathbf{C}_{ij} = \langle e_i | h_j \rangle$ ⇝the *co-homology intersection matrix*,
- $\mathbf{H}_{ij} = [\gamma_i \otimes u(z) | \gamma_j^\vee \otimes u^{-1}(z)]$ ⇝the *homology intersection matrix*,
- $\mathbf{P}_{ij} = \langle e_i | \gamma_j \otimes u(z)]$ ⇝the *twisted period matrix*,
- $\mathbf{P}_{ij}^\vee = [\gamma_j^\vee \otimes u(z) | h_i \rangle$ ⇝the *dual twisted perioed matrix*.

All the previous quantities appear tight together in the so-called *Twisted Riemann's Period Relations*[14] [182]

$$\mathbf{C} = \mathbf{P} \cdot \mathbf{H}^{-\top} \cdot \mathbf{P}^{\vee\top}. \tag{4.112}$$

Eq. (4.112) contains a lot of information and reveals deep structures underlying the structure of these integrals. Let us just mention that eq. (4.112) offers yet another computational strategy for the evaluation of co-homology intersection numbers of *logarithmic*[15] (one) forms [27]

$$\langle \varphi | \varphi^\vee \rangle = - \sum_{z_{\rm crt} \in \mathcal{S}_\omega} \mathrm{Res}_{z=y_i} \left( \frac{\hat{\varphi}\, \varphi^\vee}{\partial \hat{\omega}} \right), \tag{4.113}$$

where, now, $\mathcal{S}_\omega$ is the set of *zeros* of $\omega$.

[14] Compared to the original literature [182], the co-homology intersection numbers used in this work have an additional $(2\pi i)^{-1}$ factor. In order to reproduce eq. (4.112) correctly, we have to restore the convention in the literature. This means that we have to multiply the co-homology intersection numbers defined in this work by a factor $2\pi i$.

[15] The formula is not restricted to one forms, and it holds for generic twist, even if it was originally proposed for the case of (degenerate) hyperplanes arrangements.

# 5

# Integral Relations via Intersection Theory: univariate case

The main focus of this chapter will be the study of integral relations obtained thanks to the tools described in chapter 4. Having at our disposal, so far, a theory capable to treat integrals admitting a one-fold representation, we will first consider how to obtain contiguity relations—i.e. linear relations—among special functions often considered in the mathematical literature. These relations are obtained by means of projection via intersection numbers, thanks to the so-called master decomposition formula. We will briefly show how, within this framework, it is possible to derive also quadratic relations among these functions. Next we move to the case of FIs; given our current limitations we consider the case of reduction onto MIs on the maximal cut—i.e. discarding subsectors; once again, thanks to intersection numbers this is done bypassing the system solving strategy described in chapter 2.

## 5.1 Illustrative Example: Euler Beta Function

Let us consider once again the integral

$$I = I_{1,1} = \int_0^1 z^p (1-z)^q \, \frac{dz}{z(1-z)}; \tag{5.1}$$

indeed eq. (5.1) corresponds to the particular choice $(a_1, a_2) = (1,1)$ within the more integral family

$$I_{a_1,a_2} = \int_0^1 z^p (1-z)^q \, \frac{dz}{z^{a_1}(1-z)^{a_2}}, \qquad (a_1, a_2) \in \mathbb{Z}^2. \tag{5.2}$$

On the one hand, linear relations among elements of eq. (5.2) are known in the mathematical literature as *contiguity relations*. On the other hand, the reader acquainted with the physics literature will quickly realize that—at the prize of generating and solving a (in this case) small system of IBPs— any given integral within the ones in the integral family eq. (5.2), can be expressed in terms of a single MI, say[1] $\mathcal{J} = I_{1,1}$. So

$$\begin{aligned} I_{a_1,a_2} &= c_{a_1,a_2}\, I_{1,1} \\ &= c_{a_1,a_2}\, \mathcal{J}. \end{aligned} \tag{5.3}$$

The coefficient $c_{a_1,a_2}$ is determined through the system solving procedure for any given $(a_1, a_2)$.

We will follow here an orthogonal approach, exploiting the twisted co-homology structure underpinning eqs. (5.1,5.2).

The first step consists in focusing on the single valued differential form, or, better, twisted co-cycle associated to each integral, stripping off the twisted cycle which is common to all of them. The linear relations derived at the level of co-cycles will hold for the integrals as well since these are built pairing twisted co-cycles with the common twisted cycle (cf. section 4.3). So in the case at hand

$$I_{a_1,a_2} = \int_0^1 z^p(1-z)^q \frac{dz}{z^{a_1}\,(1-z)^{a_2}} \rightsquigarrow \left\langle \frac{dz}{z^{a_1}(1-z)^{a_2}} \right| = \langle\varphi| \in \mathrm{H}^1(X, \nabla_\omega), \tag{5.4}$$

where

$$X = \mathbb{CP}^1 \setminus \{0, 1, \infty\} \qquad \text{and} \qquad \omega = d\log\left(z^p(1-z)^q\right) = \left(\frac{p}{z} - \frac{q}{1-z}\right) dz. \tag{5.5}$$

Actually, the fact that there is just *one* independent form (modulo IBPs)—i.e. *one* MIs— is, in this approach, a prediction: $\mathrm{H}^1(X, \nabla_\omega)$ is a one dimensional space, as we can infer from a critical points analysis, see eq. (4.75).

Explicitly, we have

$$\nu = 1 = \#\,\text{solutions of: } \omega = 0. \tag{5.6}$$

Consistently with eq. (5.3), we choose the single basis element as

$$\langle e| = \left\langle \frac{dz}{z\,(1-z)} \right| \in \mathrm{H}^1(X, \nabla_\omega). \tag{5.7}$$

Roughly speaking the main idea is that we can think at $c_{a_1,a_2}$ in eq. (5.3) as the projection of any given $\langle\varphi|$ onto $\langle e|$; the coefficient can be extracted thanks to a (IBP-compatible) "scalar product" defined among elements of $\mathrm{H}^1$. Indeed, the intersection number in eq. (4.99) is what we are looking for.

We have

$$c_{a_1,a_2} = \left\langle \frac{dz}{z^{a_1}(1-z)^{a_2}} \middle| \frac{dz}{z(1-z)} \right\rangle \Big/ \left\langle \frac{dz}{z(1-z)} \middle| \frac{dz}{z(1-z)} \right\rangle, \tag{5.8}$$

[1]This choice is just for illustrative purposes.

where, at this stage, the denominator in eq. (5.8) can be seen as sort of normalization.

**Example.** *The explicit case* $(a_1, a_2) = (2, 1)$*. We want to determine the coefficient* $c_{2,1}$ *appearing in*

$$I_{2,1} = c_{2,1}\,\mathcal{J}. \tag{5.9}$$

*Following eq. (5.8), two different intersection numbers are required, namely*

$$\left\langle \frac{dz}{z^2(1-z)} \middle| \frac{dz}{z\,(1-z)} \right\rangle \qquad \textit{and} \qquad \left\langle \frac{dz}{z(1-z)} \middle| \frac{dz}{z\,(1-z)} \right\rangle. \tag{5.10}$$

*We show here the explicit intermediate expressions needed for the evaluation of:*

$$\left\langle \frac{dz}{z^2(1-z)} \middle| \frac{dz}{z\,(1-z)} \right\rangle. \tag{5.11}$$

*We have to solve the differential equation*

$$\nabla_\omega \psi = \frac{dz}{z^2(1-z)}, \qquad \textit{around} \quad \mathcal{P}_\omega = \{0, 1, \infty\}. \tag{5.12}$$

*The explicit solutions around each pole reads*

- $\psi_0 \rightsquigarrow$ *series solution around* $0$ *(local coordinate:* $y = z$*)*
$$\psi_0 = -\frac{1}{(1-p)y} - \frac{q+p-1}{(1-p)p} + \mathcal{O}(y); \tag{5.13}$$
- $\psi_1 \rightsquigarrow$ *series solution around* $1$ *(local coordinate:* $y = z-1$*)*
$$\psi_1 = -\frac{1}{q} + \mathcal{O}(y^0); \tag{5.14}$$
- $\psi_\infty \rightsquigarrow$ *series solution around* $\infty$ *(local coordinate:* $y = 1/z$*)*
$$\psi_\infty = \mathcal{O}(y^2). \tag{5.15}$$

*Let us now focus on the expansion of*

$$\frac{dz}{z\,(1-z)}, \qquad \textit{around} \quad \mathcal{P}_\omega = \{0, 1, \infty\} \tag{5.16}$$

- $\frac{dz}{z(1-z)} \rightsquigarrow$ *series expansion around* $0$ *(local coordinate:* $y = z$*)*
$$\left(\frac{1}{y} + 1 + \mathcal{O}(y)\right)\, dy; \tag{5.17}$$
- $\frac{dz}{z(1-z)} \rightsquigarrow$ *series expansion around* $1$ *(local coordinate:* $y = z-1$*)*
$$\left(-\frac{1}{y} + \mathcal{O}(y^0)\right)\, dy; \tag{5.18}$$
- $\frac{dz}{z(1-z)} \rightsquigarrow$ *series expansion around* $\infty$ *(local coordinate:* $y = 1/z$*)*
$$(1 + \mathcal{O}(y))\, dy. \tag{5.19}$$

*Combining the various expansions as dictated by eq. (4.99), we conclude that just the points* $0$ *and* $1$ *give a non vanishing contribution in the sum of residues*[2]*; we obtain*

$$\left\langle \frac{dz}{z^2(1-z)} \middle| \frac{dz}{z\,(1-z)} \right\rangle = \frac{1-2p-q}{(1-p)p} + \frac{1}{q}. \tag{5.20}$$

*Similarly we can compute*

$$\left\langle \frac{dz}{z\,(1-z)} \middle| \frac{dz}{z\,(1-z)} \right\rangle = \frac{p+q}{p\,q}. \tag{5.21}$$

*Therefore*

$$c_{2,1} = \left\langle \frac{dz}{z^2(1-z)} \middle| \frac{dz}{z(1-z)} \right\rangle \Big/ \left\langle \frac{dz}{z(1-z)} \middle| \frac{dz}{z(1-z)} \right\rangle = \frac{1-p-q}{1-p}. \tag{5.22}$$

## 5.2 Linear Relations and Master Decomposition Formula

Let us consider now the more general case

$$I = \int_\gamma u(z)\,\varphi(z) \tag{5.23}$$

where $\gamma$ is some path in

$$X = \mathbb{CP}^1 \setminus \mathcal{P}_\omega, \qquad \mathcal{P}_\omega \rightsquigarrow \text{set of poles of: } \omega = d\log u. \tag{5.24}$$

such that $u(\partial\gamma) = 0$.
The integral family in eq. (5.23) admits $\nu$ MIs, say $(\mathcal{J}_1, \ldots, \mathcal{J}_\nu)$, where

$$\nu = \#\text{solutions of: } \omega = 0, \qquad \mathcal{J}_i = \int_\gamma u(z)\,e_i(z) \quad 1 \leq i \leq \nu. \tag{5.25}$$

The problem of finding the decomposition of any given integral of the form of eq. (5.23) in terms of the above mentioned basis of MIs, is then translated into the decomposition of $\langle\varphi| \in \mathrm{H}^1(X, \nabla_\omega)$ in terms of $(\langle e_i|, \ldots, \langle e_\nu|) \in \mathrm{H}^1(X, \nabla_\omega)$

$$I = \sum_{i=1}^{\nu} c_i\,\mathcal{J}_i \quad \rightsquigarrow \quad \langle\varphi| = \sum_{i=1}^{\nu} c_i\,\langle e_i|. \tag{5.26}$$

Let us discuss now how to obtain the decomposition in eq. (5.26). We first introduce an *auxiliary dual element* $|\varphi^\vee\rangle \in \mathrm{H}^1(X, \nabla_{-\omega})$ and an *arbitrary dual basis* $(|h_1\rangle, \ldots, |h_\nu\rangle) \in \mathrm{H}^1(X, \nabla_{-\omega})$, and we

[2] In fact a given pole gives a non vanishing contribution iff the following relation holds: $\min_\varphi + \min_{\varphi^\vee} \leq -2$.

build the $(\nu+1)\times(\nu+1)$ matrix, dubbed as $\mathbf{M}$ matrix, as follows

$$\mathbf{M} = \begin{pmatrix} \langle\varphi\,|\,\varphi^\vee\rangle & \langle\varphi\,|\,h_1\rangle & \dots & \langle\varphi\,|\,h_\nu\rangle \\ \langle e_1\,|\,\varphi^\vee\rangle & \langle e_1\,|\,h_1\rangle & \dots & \langle e_1\,|\,h_\nu\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu\,|\,\varphi^\vee\rangle & \langle e_\nu\,|\,h_1\rangle & \dots & \langle e_\nu\,|\,h_\nu\rangle \end{pmatrix} = \begin{pmatrix} \langle\varphi\,|\,\varphi^\vee\rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}. \tag{5.27}$$

We notice that, thanks to the linearity of the intersection pairing, the $\mathbf{M}$ matrix is degenerate, since the first row can be seen as a linear combination of the others. Therefore we have

$$0 = \det\mathbf{M} = \det\begin{pmatrix} \langle\varphi\,|\,\varphi^\vee\rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}. \tag{5.28}$$

Recalling the expression of the determinant of a matrix in terms of its sub-blocks, we have

$$0 = \det\mathbf{M} = \det\mathbf{C}\cdot\Big(\langle\varphi\,|\,\varphi^\vee\rangle - \mathbf{A}^\top\cdot\mathbf{C}^{-1}\cdot\mathbf{B}\Big). \tag{5.29}$$

Since $(\langle e_i|,\dots,\langle e_\nu|)$ and $(|h_1\rangle,\dots,|h_\nu\rangle)$ were chosen to be a basis of $\mathrm{H}^1(X,\nabla_{\pm\omega})$, then $\det\mathbf{C}\neq 0$. Therefore we are left with

$$\langle\varphi\,|\,\varphi^\vee\rangle - \mathbf{A}^\top\cdot\mathbf{C}^{-1}\cdot\mathbf{B} = 0. \tag{5.30}$$

Reinserting the explicit expressions, we have

$$\langle\varphi|\varphi^\vee\rangle = \sum_{i,j=1}^{\nu}\langle\varphi|h_j\rangle\,\left(\mathbf{C}^{-1}\right)_{ji}\,\langle e_i|\varphi^\vee\rangle; \tag{5.31}$$

eq. (5.31) holds not matter what the choice of $|\varphi^\vee\rangle$ is—so we can drop it and pretend the resulting expression to hold; doing this we arrive at the *master decomposition formula*

$$\langle\varphi| = \sum_{i,j=1}^{\nu}\langle\varphi|h_j\rangle\,\left(\mathbf{C}^{-1}\right)_{ji}\,\langle e_i|. \tag{5.32}$$

Eq. (5.32) is precisely of the form of eq. (5.26), namely

$$\langle\varphi| = \sum_{i=1}^{\nu} c_i\,\langle e_i| \qquad \text{with:} \qquad c_i = \sum_{j=1}^{\nu}\langle\varphi|h_j\rangle\,\left(\mathbf{C}^{-1}\right)_{ji} \quad \text{and} \quad \mathbf{C}_{ij} = \langle e_i|h_j\rangle. \tag{5.33}$$

The coefficients of the decomposition are extracted in terms of intersection numbers (see also eq. (5.8); the normalization factor we were referring to is nothing but $\mathbf{C}^{-1}$).
We conclude this paragraph noticing that we can consider again eq. (5.31), with $|\varphi^\vee\rangle$ fixed and letting $\langle\varphi|$ be arbitrary; so we can drop it from the expression and obtain the *dual master decomposition formula*—whose importance will be clear later on—which reads

$$|\varphi^\vee\rangle = \sum_{i,j=1}^{\nu}|h_j\rangle\,\left(\mathbf{C}^{-1}\right)_{ji}\,\langle e_i|\varphi^\vee\rangle, \tag{5.34}$$

and so

$$|\varphi^\vee\rangle = \sum_{j=1}^{\nu} c_j^\vee \, |h_j\rangle \qquad \text{with:} \qquad c_j^\vee = \sum_{i=1}^{\nu} \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\varphi^\vee\rangle \quad \text{and} \quad \mathbf{C}_{ij} = \langle e_i|h_j\rangle. \tag{5.35}$$

## 5.3

## Quadratic Relations and Twisted Riemann's Period Relations

Even if the main focus of this work is devoted to the study of linear relations among integrals, it is worth stressing that twisted (Co)Homology represents a framework where also *quadratic* relations among integrals can be addressed [182, 183, 184, 185, 186], as they are naturally encoded in twisted Riemann's Period Relations eq. (4.112). Since quadratic relations for FIs were recently investigated by the physics community [187, 188, 189, 190, 191, 192, 193, 194, 195], we expect that twisted (Co)Homology could help in understanding more about those identities in the future. Remarkably, twisted Riemann's Period Relations were recently understood to be the mathematical structure underpinning Kaway-Lewellew-Tye relations among closed and open String Amplitudes [49].
In the following, we inspect some explicit example from the mathematical literature, see [169]; we stick to setting introduced in section 4.1 and subsection 4.1.3 (see also appendix B).

**Example.** *The reflection formula for $B(p,q)$. Let us consider the following*

$$u(z) = z^{-p-q}\,(z-1)^q \tag{5.36}$$

*which is in the form of eq. (4.19), provided we consider $m=0$ and*

$$x_0 = 0, \qquad x_1 = 1, \qquad x_2 = \infty; \tag{5.37}$$

$$\alpha_0 = -p-q, \qquad \alpha_1 = q, \qquad \alpha_2 = p. \tag{5.38}$$

*It is clear than $\nu = \dim \mathrm{H}^1 = \mathrm{H}_1 = 1$.*
*Let*

$$|\gamma \otimes u(z)] = |(1,\infty) \otimes u(z)] \qquad \langle e| = \langle f \log(z-1)|; \tag{5.39}$$

*we then consider*

$$\begin{aligned} \mathbf{P} = \mathcal{J} = \langle e|\gamma \otimes u(z)] &= \int_1^\infty z^{-p-q}\,(z-1)^q \,\frac{dz}{z-1} \\ &= B(p,q). \end{aligned} \tag{5.40}$$

*Let us also introduce*

$$[\gamma^\vee \otimes u^{-1}(z)| = [(1,\infty)\otimes u^{-1}(z)|, \qquad |h\rangle = |d\log(z-1)\rangle; \tag{5.41}$$

*next we consider the dual integral*

$$\begin{aligned}\mathbf{P}^{\vee} = \mathcal{J}^{\vee} = [\gamma^{\vee} \otimes u^{-1}(z)|h\rangle &= \int_{1}^{\infty} z^{p+q}(z-1)^{-q}\frac{dz}{z-1} \\ &= B(-p,-q).\end{aligned} \tag{5.42}$$

*Finally we just need the self intersection numbers*

$$\begin{aligned}\mathbf{H} = [\gamma \otimes u(z)|\gamma^{\vee} \otimes u^{-1}(z)] &= [(1,\infty) \otimes u(z)|(1,\infty) \otimes u^{-1}(z)] \\ &= \frac{1-e^{2\pi i(q+p))}}{\left(1-e^{2\pi q}\right)\left(1-e^{2\pi i p}\right)},\end{aligned} \tag{5.43}$$

*and*[3]

$$\begin{aligned}\mathbf{C} = \langle e|h\rangle &= \langle d\log(z-1)|d\log(z-1)\rangle \\ &= 2\pi i\left(\frac{1}{p}+\frac{1}{q}\right).\end{aligned} \tag{5.44}$$

*So eq. (4.112) gives*

$$\begin{gathered}\mathbf{C} = \mathbf{P}\cdot\mathbf{H}^{-\top}\cdot\mathbf{P}^{\vee\top} = \\ 2\pi i\left(\frac{1}{p}+\frac{1}{q}\right) = B(p,q)\,B(-p,-q)\,\frac{\left(1-e^{2\pi q}\right)\left(1-e^{2\pi i p}\right)}{1-e^{2\pi i(q+p))}}.\end{gathered} \tag{5.45}$$

**Example.** *Quadratic relations for hypergeometric ${}_2F_1$. Let us consider the following*

$$u(z) = z^{b-c}(z-x)^{-b}(z-1)^{c-a}, \tag{5.46}$$

*which is in the form of eq. (4.19), with $m=1$ and*

$$x_0 = 0, \qquad x_1 = x, \qquad x_2 = 1, \qquad x_3 = \infty, \tag{5.47}$$

$$\alpha_0 = b-c, \qquad \alpha_1 = -b, \qquad \alpha_2 = c-a, \qquad \alpha_4 = a. \tag{5.48}$$

*Thus we have $\nu = \dim \mathrm{H}^1 = \mathrm{H}_1 = 2$.*
*Let*

$$\langle e_1| = \langle d\log(z-1)|, \qquad |\gamma_1 \otimes u(z)] = |(1,\infty) \otimes u(z)], \tag{5.49}$$

$$\langle e_2| = \langle d\log(z/(z-x))|, \qquad |\gamma_2 \otimes u(z)] = |(0,x) \otimes u(z)]; \tag{5.50}$$

*and*

$$|h_1\rangle = |d\log(z-1)\rangle, \qquad [\gamma_1^{\vee} \otimes u^{-1}(z)] = [(1,\infty) \otimes u^{-1}(z)|, \tag{5.51}$$

$$|h_2\rangle = |d\log(z/(z-x))\rangle, \qquad [\gamma_2^{\vee} \otimes u^{-1}(z)| = [(0,x) \otimes u^{-1}(z)|. \tag{5.52}$$

[3] See the comment in footnote 14.

*Thus we have*

$$\mathbf{P} = \begin{pmatrix} \langle e_1|\gamma_1 \otimes u(z)] & \langle e_1|\gamma_2 \otimes u(z)] \\ \langle e_2|\gamma_1 \otimes u(z)] & \langle e_2|\gamma_2 \otimes u(z)] \end{pmatrix}$$
$$= \begin{pmatrix} B(a, c{-}a)_2F_1(a,b,c;x) & -x^{1-c}e^{-i\pi(c-a-b)}B(1{+}b{-}c, 1{-}b)_2F_1(1{+}b{-}c, 1{+}a{-}c, , 2{-}c; x) \\ -xB(1+a, 1+c-a)_2F_1(1+a, 1+b, 2+c; x) & x^{-c}e^{-i\pi(c-a-b)}B(b-c, -b)_2F_1(b-c, a-c, -c; x) \end{pmatrix} \tag{5.53}$$

*and*

$$
\begin{aligned}
\mathbf{P}^\vee &= \begin{pmatrix} [\gamma_1^\vee \otimes u^{-1}(z)|h_1\rangle & [\gamma_2^\vee \otimes u^{-1}(z)|h_1\rangle \\ [\,\gamma_1^\vee \otimes u^{-1}(z)|h_2\rangle & [\gamma_2^\vee \otimes u^{-1}(z)|h_2\rangle \end{pmatrix} \\
&= \begin{pmatrix} B(-a, a-c)\,{}_2F_1(-a,-b,-c;x) & -x^{1+c}e^{i\pi(c-a-b)}B(1-b+c, 1+b)\,{}_2F_1(1-b+c, 1-a+c, 2+c; x) \\ -xB(1-a, 1+a-c)\,{}_2F_1(1-a, 1-b, 2-c; x) & e^{i\pi(c-a-b)}x^c B(c-b, b)\,{}_2F_1(c-b, c-a, c; x) \end{pmatrix}.
\end{aligned}
\tag{5.54}
$$

*We get*

$$\mathbf{C} = \begin{pmatrix} \langle e_1|h_1\rangle & \langle e_1|h_2\rangle \\ \langle e_2|h_1\rangle & \langle e_2|h_2\rangle \end{pmatrix} = \begin{pmatrix} \langle d\log(z-1)|d\log(z-1)\rangle & \langle d\log(z-1)|d\log(z/(z-x))\rangle \\ \langle d\log(z/(z-x))|d\log(z-1)\rangle & \langle d\log(z/(z-x))|d\log(z/(z-x))\rangle \end{pmatrix}$$
$$= 2\pi i \begin{pmatrix} \frac{1}{c-a}+\frac{1}{a} & 0 \\ 0 & \frac{1}{b-c}-\frac{1}{b} \end{pmatrix}; \tag{5.55}$$

*and*

$$\begin{aligned} \mathbf{H} &= \begin{pmatrix} [\gamma_1 \otimes u(z)|\gamma_1^\vee \otimes u^{-1}(z)] & [\gamma_1 \otimes u(z)|\gamma_2^\vee \otimes u^{-1}(z)] \\ [\ \gamma_2 \otimes u(z)|\gamma_1^\vee \otimes u^{-1}(z)] & [\gamma_2 \otimes u(z)|\gamma_2^\vee \otimes u^{-1}(z)] \end{pmatrix} \\ &= \begin{pmatrix} [(1,\infty)\otimes u(z)|(1,\infty)\otimes u^{-1}(z)] & [(1,\infty)\otimes u(z)|(0,x)\otimes u^{-1}(z)] \\ [\ (0,x)\otimes u(z)|(1,\infty)\otimes u^{-1}(z)] & [(0,x)\otimes u(z)|(0,x)\otimes u^{-1}(z)] \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-e^{2i\pi c}}{(1-e^{2i\pi a})\left(1-e^{2i\pi(c-a)}\right)} & 0 \\ 0 & \frac{1-e^{-2i\pi c}}{\left(1-e^{-2i\pi b}\right)\left(1-e^{2i\pi(b-c)}\right)} \end{pmatrix}. \end{aligned} \tag{5.56}$$

*So the* $(1,1)$ *entry of the identity eq.* (4.112) *gives*[4]

$$\begin{aligned} 1 = {}& \frac{abx^2(a-c)(c-b)}{c^2\,(c^2-1)}\,{}_2F_1(1+a-c,1+b-c;2-c;x)\,{}_2F_1(1-a+c,1-b+c;2+c;x) \\ &+ {}_2F_1(-a,-b;-c;x)\,{}_2F_1(a,b;c;x). \end{aligned} \tag{5.57}$$

*The* $(1,2)$ *entry is*

$$0 = {}_2F_1(1-a,1-b;2-c;x)\,{}_2F_1(a,b;c;x) - {}_2F_1(1+a-c,1+b-c;2-c;x)\,{}_2F_1(c-a,c-b;c;x). \tag{5.58}$$

*The* $(2,1)$ *entry is*

$$\begin{aligned} 0 = {}& {}_2F_1(-a,-b;-c;x)\,{}_2F_1(1+a,1+b;2+c;x) \\ &- {}_2F_1(a-c,b-c;-c;x)\,{}_2F_1(1-a+c,1-b+c;2+c;x) \\ &= (5.58)\big|_{(a,b,c)\to-(a,b,c)}. \end{aligned} \tag{5.59}$$

*Finally the* $(2,2)$ *entry is*

$$\begin{aligned} 1 = {}& \frac{abx^2(a-c)(c-b)}{c^2\,(c^2-1)}\,{}_2F_1(1-a,1-b;2-c;x)\,{}_2F_1(1+a,1+b;2+c;x) \\ &+ {}_2F_1(a-c,b-c;-c;x)\,{}_2F_1(c-a,c-b;c;x). \end{aligned} \tag{5.60}$$

*Eqs.* (5.57,5.58,5.59,5.60) *are numerically verified against* Mathematica.

[4] We present the identities in such a way that the l.h.s. is normalized to 0 or 1.

## 5.4

## Feynman Integrals Reduction via Intersection Numbers: Maximal Cut

We are now in the position of discussing the case of FIs, in particular the reduction onto MIs via co-homology intersection numbers; as we anticipated, this allows us to avoid the explicit system solving procedure required by the standard Laporta algorithm.

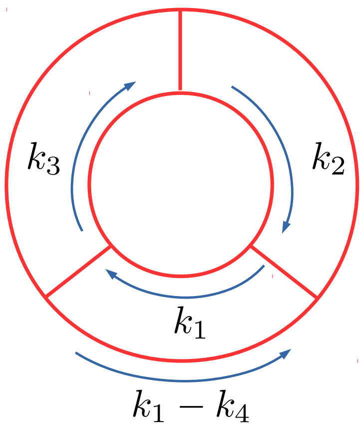
Having at our disposal—at the moment—intersection pairings among 1-forms, we will focus on FIs admitting a one-fold integral representation. Despite this limitation, we can still treat a plethora of interesting examples.
We will investigate maximally cut FIs in Baikov representation; following a pragmatic approach, a maximal cut acts like a filter: given a certain integral family all the integrals belonging to subtopologies are not supported on the cut and they vanish (cf. the discussion at the end of section 2.2). So, we are only sensitive to the *homogeneous* part of the reduction.

The integration variable, denoted by $z$, will be the ISP for the integral family under consideration, and we have the identification

$$u(z) = (\mathcal{B}(z))^{\frac{d-\ell-E-1}{2}}; \tag{5.61}$$

the integration cycle $\gamma$, always assumed to be such that $u(\partial\gamma) = \mathcal{B}(\partial\gamma) = 0$, is never involved explicitly, and so we will be omitted.

**Example.** *A four-loop vacuum graph. We consider the following graph*




(5.62)

*The denominators are chosen as*

$$\begin{aligned}
&D_1 = k_1^2 - 1, && D_2 = k_2^2 - 1, && D_3 = k_3^2 - 1, \\
&D_4 = (k_1 - k_2)^2 - 1, && D_5 = (k_1 - k_3)^2 - 1, && D_6 = (k_2 - k_3)^2 - 1, \\
&D_7 = (k_1 - k_4)^2 - 1, && D_8 = (k_2 - k_4)^2 - 1, && D_9 = (k_3 - k_4)^2 - 1;
\end{aligned} \tag{5.63}$$

*while the ISPs is chosen as*

$$z = D_{10} = k_4^2. \tag{5.64}$$

*The multivalued function and the* twist *associated to it read*

$$u(z) = \left(\frac{z}{2} - \frac{3z^2}{16}\right)^{\frac{d-5}{2}}, \qquad \omega(z) = \frac{(d-5)(3z-4)}{z(3z-8)}\, dz. \tag{5.65}$$

*The equation $\omega = 0$, yields $1$ solution implying $1$ MI (on the maximal cut):*

$$\nu = 1, \qquad \mathcal{P}_\omega = \left\{0, \frac{8}{3}, \infty\right\}. \tag{5.66}$$

*Let us choose the MI and the corresponding differential form as*

$$\mathcal{J} = I_{1,1,1,1,1,1,1,1,1;0} \rightsquigarrow \langle e| = \langle 1\, dz|. \tag{5.67}$$

*We focus on the decomposition of*

$$I_{1,1,1,1,1,1,1,1,1;-1} \rightsquigarrow \langle \varphi| = \langle z\, dz|; \tag{5.68}$$

*our (arbitrary) choice for the dual basis will be:*

$$|h\rangle = |h_1\rangle = |1\, dz\rangle. \tag{5.69}$$

*Eq. (5.32) requires the evaluation of $\langle \varphi | h_1\rangle$ and $\mathbf{C}_1 = \langle e|h_1\rangle$.*

*We have*

$$\langle \varphi | h_1\rangle = \langle z\, dz | 1\, dz\rangle = \frac{64(d-5)}{27(d-6)(d-4)}; \tag{5.70}$$

*and*

$$\mathbf{C}_1 = \langle e|h\rangle = \langle 1\, dz | 1\, dz\rangle = \frac{16(d-5)}{9(d-6)(d-4)}. \tag{5.71}$$

*Then, combining everything together we obtain*

$$\begin{aligned}
\langle \varphi| &= \langle \varphi | h\rangle\, \mathbf{C}_1^{-1}\, \langle e| \\
&= \frac{4}{3}\, \langle e|.
\end{aligned} \tag{5.72}$$

*Eq. (5.72) implies*

$$I_{1,1,1,1,1,1,1,1,1;-1} = \frac{4}{3}\,\mathcal{J}, \tag{5.73}$$

*in agreement with* LiteRed.

*It is also instructive to show explicitly the independence of the final result eq. (5.72) from the choice of dual basis.*
*Let us consider*

$$|h\rangle = |h_2\rangle, \tag{5.74}$$

*then the intersection numbers read*

$$\langle\varphi|h_2\rangle = \langle z\,dz|z\,dz\rangle = \frac{256(d-5)}{81(d-6)(d-4)}, \tag{5.75}$$

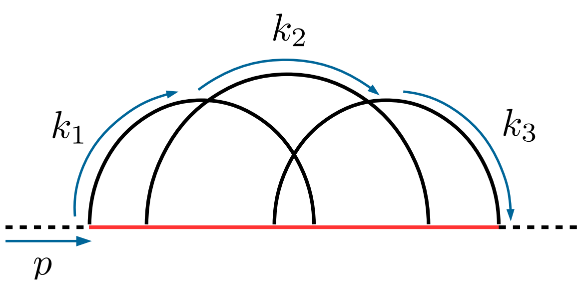
*and*

$$\mathbf{C}_2 = \langle e|h_2\rangle = \langle 1\,dz|z\,dz\rangle = \frac{64(d-5)}{27(d-6)(d-4)}. \tag{5.76}$$

*The individual intersection numbers are changed, but the final decomposition is not:*

$$\begin{aligned}\langle\varphi| &= \langle\varphi|h_2\rangle\,\mathbf{C}_2^{-1}\,\langle e| \\ &= \frac{4}{3}\,\langle e|.\end{aligned} \tag{5.77}$$

**Example.** *Three-loop triple-cross. We consider the following graph:*




(5.78)

*The denominators are chosen as*

$$\begin{aligned}
&D_1 = k_1^2, \qquad D_2 = k_2^2, && D_3 = k_3^2, D_4 = (p-k_1)^2 - 1 \\
&D_5 = (p-k_1-k_2)^2 - 1, && D_6 = (p-k_1-k_2-k_3)^2 - 1, \\
&D_7 = (p-k_2-k_3)^2 - 1, && D_8 = (p-k_3)^2 - 1;
\end{aligned} \tag{5.79}$$

while the ISP is chosen as

$$z = D_9 = k_2 \cdot p. \tag{5.80}$$

The kinematics is $p^2 = s$.

The multivalued function and the *twist* form associated to it read

$$\begin{aligned} u(z) &= \left(\frac{1}{4}z^2(2z-s-3)(2z-s+1)\right)^{\frac{d-5}{2}}, \\ \omega(z) &= (d-5)\left(\frac{1}{2z-s-3}+\frac{1}{2z-s+1}+\frac{1}{z}\right)dz. \end{aligned} \tag{5.81}$$

The number of solution of $\omega=0$ amounts to two, denoting that there are two MIs (on the maximal cut)

$$\nu = 2, \qquad \mathcal{P}_\omega = \left\{0, \frac{s-1}{2}, \frac{s+3}{2}, \infty\right\}. \tag{5.82}$$

Let us consider the following basis of MIs

$$\mathcal{J}_1 = I_{1,1,1,1,1,1,1,1;0} \rightsquigarrow \langle e_1| = \langle 1\,dz|, \qquad \mathcal{J}_2 = I_{1,1,1,1,1,1,1,1;-1} \rightsquigarrow \langle e_2| = \langle z\,dz|. \tag{5.83}$$

We aim to decompose

$$I_{1,1,1,1,1,1,,1,1;-2} \rightsquigarrow \langle\varphi| = \langle z^2\,dz|, \tag{5.84}$$

in terms of the above-mentioned basis.

We employ the following arbitrary choice for the dual basis:

$$|h_1\rangle = \big|dz/z\big\rangle, \qquad |h_2\rangle = \big|2\,dz/(2z-s+1)\big\rangle \tag{5.85}$$

Eq. (5.32) requires the evaluation of $\langle\varphi|h_1\rangle$, $\langle\varphi|h_2\rangle$ and $\mathbf{C}_{ij} = \langle e_i|h_j\rangle$ with $1 \leq i,j \leq 2$.

$$\begin{aligned} \langle\varphi|h_1\rangle &= \big\langle z^2\,dz\big|dz/z\big\rangle \\ &= \frac{(d-4)(d-3)s^3+3(d-4)(d-3)s^2+(d(35d-281)+560)s+3d(11d-89)+536}{32(d-4)(2d-9)(2d-7)}, \end{aligned} \tag{5.86}$$

and

$$\begin{aligned} \langle\varphi|h_2\rangle &= \big\langle z^2\,dz\big|2\,dz/(2z-s+1)\big\rangle \\ &= \frac{-(d-4)(d-3)s^3+(d(25d-203)+408)s^2+(d(21d-167)+328)s+3(d-4)(9d-35)}{32(d-4)(2d-9)(2d-7)}, \end{aligned} \tag{5.87}$$

finally

$$\mathbf{C} = \begin{pmatrix} \frac{s+1}{4(2d-9)} & -\frac{s-3}{4(2d-9)} \\ \frac{(d-4)s^2+2(d-4)s+9d-40}{16(d-4)(2d-9)} & -\frac{(d-4)s^2-2(5d-22)s-3d+12}{16(d-4)(2d-9)} \end{pmatrix}. \tag{5.88}$$

Combining everything together, we get:

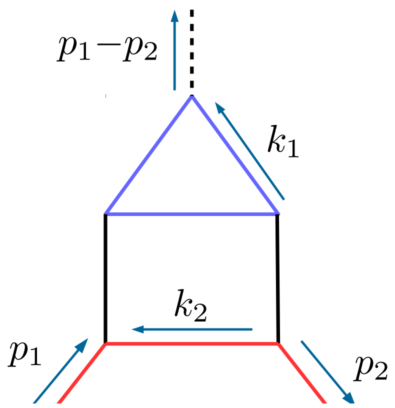
$$
\begin{aligned}
\langle\varphi| &= \sum_{i=1}^{2}\langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i| \\
&= -\frac{(d-4)(s-1)(s+3)}{4(2d-7)} \langle e_1| + \frac{(3d-11)(s+1)}{2(2d-7)}\langle e_2|,
\end{aligned}
\tag{5.89}
$$

which implies

$$
I_{1,1,1,1,1,1,1,1;-2} = -\frac{(d-4)(s-1)(s+3)}{4(2d-7)}\,\mathcal{J}_1 + \frac{(3d-11)(s+1)}{2(2d-7)}\,\mathcal{J}_2, \tag{5.90}
$$

in agreement with LiteRed.

**Example.** *Two loop two masses triangle. We consider the following graph:*




(5.91)

*The denominaotrs are chosen as*

$$
\begin{aligned}
&D_1 = k_1^2 - m_\ell^2, \quad D_2 = (k_1 - p_1 + p_2)^2 - m_\ell^2, \quad D_3 = k_2^2 - m_N^2, \\
&D_4 = (k_2 + p_1)^2, \quad D_5 = k_2 + p_2, \quad D_6 = (k_1 + k_2 + p_2)^2 - m_\ell^2,
\end{aligned}
\tag{5.92}
$$

*while the ISP is chosen as*

$$
z = D_7 = (k_1 - p_1)^2. \tag{5.93}
$$

*The kinematics is such that $p_1^2 = p_2^2 = m_N^2$ and $(p_1 - p_2)^2 = t$.*
*The multivalued function and the* twist *associated to it read*

$$
\begin{aligned}
u(z) &= \left(\frac{t^2}{16}\left(-2m_\ell^2\left(m_N^2 + t + z\right) + m_\ell^4 + \left(-m_N^2 + t + z\right)^2\right)\right)^{\frac{d-5}{2}} \\
\omega(z) &= -\frac{(d-5)\left(m_\ell^2 + m_N^2 - t - z\right)}{-2m_\ell^2\left(m_N^2 + t + z\right) + m_\ell^4 + \left(-m_N^2 + t + z\right)^2}\,dz.
\end{aligned}
\tag{5.94}
$$

*The equation* $\omega = 0$ *admits one solution, implying that there is one MIs (one the maximal cut)*

$$\nu = 1, \qquad \mathcal{P}_\omega = \left\{ (m_\ell - m_N)^2 - t, (m_\ell + m_N)^2 - t, \infty \right\}. \tag{5.95}$$

*Our choice of MIs is*

$$\mathcal{J}_1 = I_{1,1,1,1,1,1;0} \rightsquigarrow \langle e| = \langle 1\, dz|. \tag{5.96}$$

*and we look for the decomposition of*

$$I_{1,1,1,1,1,1;-1} \rightsquigarrow \langle \varphi| = \langle z\, dz|. \tag{5.97}$$

*The choice of the dual basis is*

$$|h\rangle = |1\, dz\rangle. \tag{5.98}$$

*The required intersection number,* $\langle \varphi|h\rangle$ *and* $\langle e|h\rangle$*, read*

$$\langle \varphi|h\rangle = \langle z\, dz|1\, dz\rangle = \frac{4(d-5) m_\ell^2\, m_N^2 \left(m_\ell^2 + m_N^2 - t\right)}{(d-6)(d-4)}, \tag{5.99}$$

*and*

$$\mathbf{C} = \langle e|h\rangle = \langle 1\, dz|1\, dz\rangle = \frac{4(d-5) m_\ell^2\, m_N^2}{(d-6)(d-4)}. \tag{5.100}$$

*Eq. (5.32) reads*

$$\begin{aligned} \langle \varphi| &= \langle \varphi|h\rangle\, \mathbf{C}^{-1}\, \langle e| \\ &= \left(m_\ell^2 + m_N^2 - t\right)\, \langle e|, \end{aligned} \tag{5.101}$$

*and so*

$$I_{1,1,1,1,1,1;-1} = \left(m_\ell^2 + m_N^2 - t\right)\, \mathcal{J}, \tag{5.102}$$

*in agreement with* LiteRed*.*

Even if we do not discuss it here, the same strategy can be applied to obtain differential equations fulfilled by MIs, on the maximal cut [32, 33].

# 6

# An Introduction to Twisted Co-Homology: multivariate case

We are now ready to study integrals admitting a multivariate representation. Since we are ultimately interested in deriving integral relations for FIs, we will focus on the twisted co-homology structure underlying those integrals, extending the discussion of section 5.4. Supported by the treatment in section 4.2, we introduce the twisted co-homology groups and we discuss in detail an algorithm, first proposed in [28], for the evaluation of multivariate intersection numbers. Even if the application of this framework to simple examples is—we believe—an important achievement, we would like to stress that we are far from the final word on the subject, and—very likely—new structures may emerge in the near future. The evaluation of the intersection number is, per se, still an open topic. For example we discuss in some detail the proposal of [30]; this work presents an improvement of the algorithm for the evaluation of multivariate intersection numbers, which may lead to more efficient implementations. We also review the approach of [29]—which is, in some sense, orthogonal to the one mentioned above—intersection numbers are viewed as solution of suitable (system of partial) differential equations, the so-called Secondary Equation.

# 6.1

# Twisted Co-Homology Groups

Let us consider the following (we use the notation $\mathbf{z} = (z_1, \dots, z_n)$):

$$u(\mathbf{z}) = (P_1(\mathbf{z}))^{\alpha_1} \dots (P_m(\mathbf{z}))^{\alpha_m}, \tag{6.1}$$

defined on $X = \mathbb{C}^n \setminus \bigcup_{j=1}^m D_j$ with $D_j = \{\mathbf{z} \in \mathbb{C}^n : P_j(\mathbf{z}) = 0\}$ and $\alpha_m \in \mathbb{C} \setminus \mathbb{Z}$, $1 \le i \le m$. Given eq. (6.1), in order to study the co-homology structure associated to it, we define

$$\omega(\mathbf{z}) = d \log u(\mathbf{z}) = \sum_{i=1}^{n} \hat{\omega}_i(\mathbf{z})\, dz_i, \quad \nabla_\omega(\bullet) = d(\bullet) + \omega \wedge \bullet. \tag{6.2}$$

The various co-homology groups are (cf. the discussion in section 4.2)

$$\mathrm{H}^k = \frac{\mathrm{Ker}(\nabla_\omega : \mathscr{E}^k \to \mathscr{E}^{k+1})}{\mathrm{Im}(\nabla_\omega : \mathscr{E}^{k-1} \to \mathscr{E}^k)}, \qquad k = 0, 1, \dots, 2n. \tag{6.3}$$

Under some assumptions [196]$-$which we always assume to be satisfied$-$once can show that all the groups $\mathrm{H}^{k \neq n}$ are trivial$-$i.e. empty. Being $\mathrm{H}^n$ the only interesting space, we will just consider (equivalence classes) of $n$-forms (we will use the notation $\mathbf{n} = (12 \dots n)$ to denote differential $n-$forms)

$$\langle \varphi^{(\mathbf{n})} | \, : \, \varphi^{(\mathbf{n})} \sim \varphi^{(\mathbf{n})} + \nabla_\omega \xi^{(\mathbf{n-1})} \tag{6.4}$$

with

$$\varphi^{(\mathbf{n})} = \hat{\varphi}(\mathbf{z}) d\mathbf{z}; \tag{6.5}$$

and $\hat{\varphi}(\mathbf{z})$ defined on $X$.

The first natural question concerns the dimension of $\mathrm{H}^n$, i.e.:

$$\nu_{(\mathbf{n})} = \dim \mathrm{H}^n\,. \tag{6.6}$$

Once again we can rely on the counting of critical points, cf. subsection 4.2.3. In fact we can generalize eq. (4.75) to

$$\nu_{(\mathbf{n})} = \#\text{ solutions of: } \omega_1 = 0, \; \dots, \; \omega_n = 0. \tag{6.7}$$

We just mention that eq. (6.7) is equivalent to a system of polynomial equations

$$\begin{cases} \sum_{i=1}^{m} \alpha_i \partial_{z_1} P_i(\mathbf{z}) \left(\prod_{j\neq i} P_j(\mathbf{z})\right) = 0 \\ \qquad \vdots \\ \sum_{i=1}^{m} \alpha_i \partial_{z_n} P_i(\mathbf{z}) \left(\prod_{j\neq i} P_j(\mathbf{z})\right) = 0 \\ 1 - z_0 \prod_{i=1}^{m} P_i(\mathbf{z}) = 0 \end{cases}, \tag{6.8}$$

where the last equation in (6.8) is introduced in order to remove spurious solutions which could otherwise annihilates both the numerator and the denominator of certain $\omega_i$.
Since we are just interested in counting the number of solutions of eq. (6.8) (or, equivalently, eq. (6.7)), and not in the actual solutions, we can rely on tools from Computational Algebraic Geometry such has *Gröbner basis* [54], *Shape Lemma* (see e.g. [197], and [198] for an application to physics in a different context), *Maculay Resultants* (see e.g. [199], and [200] for an application to physics in a different context) or advanced numerical techniques (see e.g. HomotopyContinuation.jl [201] and [202, 203] for an application to physics in a different context) to determine $\nu$. Therefore finding the number of solutions of eq. (6.8) does not represent an obstacle in practice[1]

## 6.2 Co-Homology Intersection Number: multivariate case

### 6.2.1 Co-Homology Intersection Number: Generalized Beta Function

We are now at the main point: the co-homology intersection number for multivariate forms. As we have done in section 4.2, we start from an explicit example—the generalized Beta function—in order to develop some intuition and introduce key concepts[2].
Let us consider

$$B(p,q,r) = \int_{\gamma^{(2)}} z_1^p \, z_2^q \, (1 - z_1 - z_2)^r \, \frac{dz_1 \wedge dz_2}{z_1 \, z_2 \, (1 - z_1 - z_2)}, \tag{6.9}$$

where

$$\gamma^{(\mathbf{2})} = z_1 > 0 \cap z_2 > 0 \cap z_1 + z_2 < 1 \quad \in \quad X \in \mathbb{C}^2 \setminus (z_1 = 0 \cup z_2 = 0 \cup 1 - z_1 - z_2 = 0)\,. \tag{6.10}$$

[1]Nevertheless, the situation could be more delicate when $\nu$ represents the number of MIs emerging from solutions of IBPs system.
[2]See also the very clear discussion of [26], from where this preliminary discussion is adapted. We thank S. Mizera for sharing his notes on this subject.

Eq. (6.9) is simple enough to perform a direct integration w.r.t. $z_1$

$$\begin{aligned} B(p,q,r) &= \int_{\gamma^{(2)}} \left( z_2^q \int_{\gamma^{(1)}} z_1^p \, (1-z_1-z_2)^r \, \frac{dz_1}{z_1\,(1-z_1-z_2)} \right) \frac{dz_2}{z_2} \\ &= \frac{\Gamma(p)\,\Gamma(r)}{\Gamma(p+r)} \int_{\gamma^{(2)}} z_2^{q-1} \, (1-z_2)^{p+r-1} \, \frac{dz_2}{z_2} \\ &= \int_{\gamma^{(2)}} u(z_2) \, \frac{dz_2}{z_2}, \end{aligned} \tag{6.11}$$

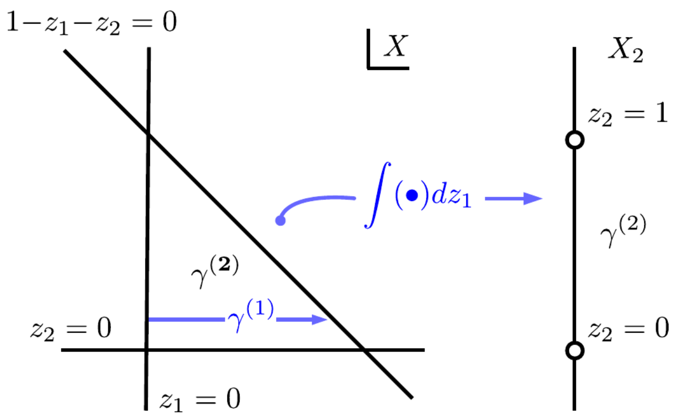
with

$$\begin{aligned} &\gamma^{(\mathbf{1})} = (0, 1-z_2) \in X_1, \qquad && \text{with: } X_1 = \mathbb{C} \setminus \{0, 1-z_2\} = \mathbb{CP}^1 \setminus \{0, 1-z_2, \infty\}, \\ &\gamma^{(2)} = (0,1) \in X_2, && \text{with: } X_2 = \mathbb{C} \setminus \{0,1\} = \mathbb{CP}^1 \setminus \{0,1,\infty\}; \end{aligned} \tag{6.12}$$

and

$$u(z_2) = \frac{\Gamma(p)\,\Gamma(r)}{\Gamma(p+r)} \, z_2^{q-1} \, (1-z_2)^{p+r-1}. \tag{6.13}$$

The situation is summarized in the following [3]




(6.14)

Provided the fact that we are able to integrate out the variable $z_1$, we reduce a 2 variables problem into a 1 variable problem. We can consider the analogue of eq. (4.54)

$$\begin{aligned} 0 = \int_0^1 d\left(u(z_2)\,\xi\right) &= \int_0^1 u(z_2) \left( d\xi + \Omega^{(2)} \wedge \xi \right) \\ &= \int_0^1 u(z_2) \, \nabla_{\Omega^{(2)}}(\xi). \end{aligned} \tag{6.15}$$

with, in this case,

$$\Omega^{(2)} = d \log u(z_2), \qquad \nabla_{\Omega^{(2)}}(\bullet) = d(\bullet) + \Omega^{(2)} \wedge (\bullet). \tag{6.16}$$

[3] Figure adapted from [26]. We depict just the real slice.

Thanks to eqs. (6.15,6.16), we could proceed following the discussion in subsection 4.2.4 and compute the intersection number w.r.t. $z_2$. Nevertheless, it is clear that in a (realistic) more complicated example, we cannot perform an explicit direct integration. Fortunately we can rely on the possibility of obtaining $\Omega^{(2)}$ via univariate intersection numbers w.r.t. $z_1$—thus avoiding a direct integration.

Let us consider the following identification

$$\begin{aligned} u(z_2) &= \int_0^{1-z_2} z_2^q \, z_1^p \, (1-z_1-z_2)^r \, \frac{dz_1}{z_1\,(1-z_1-z_2)} \\ &= \int_{\gamma^{(1)}} u(z_1)\, e^{(\mathbf{1})}, \end{aligned} \tag{6.17}$$

with

$$u(z_1) = z_2^q \, z_1^p \, (1-z_1-z_2)^r, \qquad e^{(\mathbf{1})} = \frac{dz_1}{z_1\,(1-z_1-z_2)}, \qquad \gamma^{(1)} = (0, 1-z_2). \tag{6.18}$$

where, at this stage, $z_2$ is *not* considered as an integration variable, rather it is regarded as an *external parameter*.
Then, $\Omega^{(2)}$, which is what we want to determine, is defined implicitly by the following (cf. eqs. (6.15, 6.16))

$$du(z_2) = \Omega^{(2)} \wedge u(z_2). \tag{6.19}$$

Rewriting $u(z_2)$ as in eq. (6.17), then eq. (6.19) is equivalent to

$$\int_{\gamma^{(1)}} u(z_1)\, \nabla_{\hat{\omega}_2}(e^{(\mathbf{1})}) = \hat{\Omega}^{(2)} \int_{\gamma^{(1)}} u(z_1)\, e^{(\mathbf{1})}, \tag{6.20}$$

with

$$\nabla_{\hat{\omega}_2}(\bullet) = \partial_{z_2}(\bullet) + \hat{\omega}_2 \cdot (\bullet). \tag{6.21}$$

But now we realize that $\hat{\Omega}^{(2)}$ can be computed without relying on any explicit integration, via univariate intersection numbers, w.r.t. the variable $z_1$, employing $\nabla_\omega|_{dz_2=0} = \nabla_{\omega_1}$ as a connection. Therefore we obtain

$$\hat{\Omega}^{(2)} = \frac{\langle \nabla_{\hat{\omega}_2}(e^{(\mathbf{1})}) \,|\, h^{(\mathbf{1})} \rangle}{\langle e^{(\mathbf{1})} \,|\, h^{(\mathbf{1})} \rangle}, \tag{6.22}$$

with arbitrary $|h^{(\mathbf{1})}\rangle \in \mathrm{H}^1_{-\omega_1}$.

The previous construction admits a slight generalization, where $u(z_2)$ is promoted to a vector-valued object

$$u(z_2) = \int_{\gamma^{(1)}} u(z_1)\, e^{(\mathbf{1})} \;\rightsquigarrow\; u(z_2) = \sum_{i=1}^{\nu_{(\mathbf{1})}} u_i(z_2) = \sum_{i=1}^{\nu_{(\mathbf{1})}} \int_{\gamma^{(1)}} u(z_1)\, e_i^{(\mathbf{1})}, \tag{6.23}$$

where

$$\nu_{(\mathbf{1})} = \dim(\mathrm{H}^1(X_1, \nabla_{\pm\omega_1})). \tag{6.24}$$

Eq. (6.23) induces

$$\hat{\Omega}^{(2)}_{ij} = \sum_{k=1}^{\nu_{(\mathbf{1})}} \langle \nabla_{\omega_1} e_i^{(\mathbf{1})} \,|\, h_k^{(\mathbf{1})} \rangle \left(\mathbf{C}^{(\mathbf{1})}\right)^{-1}_{kj}, \qquad \text{with:} \qquad \mathbf{C}^{(\mathbf{1})}_{ij} = \langle e_i^{(\mathbf{1})} \,|\, h_j^{(\mathbf{1})} \rangle, \tag{6.25}$$

and the matrix-valued connection[4]

$$\left(\nabla_{\Omega^{(2)}}\right)_{ij} (\bullet)_j = \delta_{ij}\; d(\bullet)_j + \left(\Omega^{(2)\top}\right)_{ij} \wedge (\bullet)_j. \tag{6.26}$$

We can finally describe a procedure in order to compute the intersection pairing among $\langle \varphi^{(\mathbf{2})}| \in \mathrm{H}^2(X, \nabla_\omega)$ and $\langle \varphi^{(\mathbf{2})\vee}| \in \mathrm{H}^2(X, \nabla_{-\omega})$. We assume

$$\begin{aligned} &(\langle e_1^{(\mathbf{1})}|, \,\ldots,\, \langle e_{\nu_{(1)}}^{(\mathbf{1})}|) \;\rightsquigarrow\; \text{basis for } \mathrm{H}^1(X_1, \nabla_{\omega_1}) \\ &(|h_1^{(\mathbf{1})}\rangle, \,\ldots,\, |h_{\nu_{(1)}}^{(\mathbf{1})}\rangle) \;\rightsquigarrow\; \text{basis for } \mathrm{H}^1(X_1, \nabla_{-\omega_1}), \end{aligned} \tag{6.27}$$

then we can derive the following decompositions

$$\langle \varphi^{(\mathbf{2})}| = \sum_{i=1}^{\nu_{(\mathbf{1})}} \langle e_i^{(\mathbf{1})}| \wedge \langle \varphi_i^{(2)}| \qquad \text{with} \qquad \langle \varphi_i^{(2)}| \in \mathrm{H}^1(X_2, \nabla_{\Omega^{(2)}}), \tag{6.28a}$$

$$|\varphi^{(\mathbf{2})\vee}\rangle = \sum_{j=1}^{\nu_{(\mathbf{1})}} |h_j^{(\mathbf{1})}\rangle \wedge |\varphi_j^{(2)\vee}\rangle \qquad \text{with} \qquad |\varphi_j^{(2)\vee}\rangle \in \mathrm{H}^1(X_2, \nabla_{-\Omega^{(2)}}); \tag{6.28b}$$

where $(\langle \varphi_1^{(2)}|, \ldots, \langle \varphi_{\nu_{(\mathbf{1})}}^{(2)}|)$ (resp. $(|\varphi_1^{(2)\vee}\rangle, \ldots, |\varphi_{\nu_{(\mathbf{1})}}^{(2)\vee}\rangle)$) are derived via the (dual) master decomposition formula eqs. (5.32,5.34).

Next we define

$$\begin{aligned} \langle \varphi^{(\mathbf{2})} \,|\, \varphi^{(\mathbf{2})\vee} \rangle &\stackrel{?}{=} \frac{1}{(2\pi i)^2} \int_{X_2} \left( \int_{X_1} \mathrm{reg}_{\omega_1} \left(e_i^{(\mathbf{1})}\right) \wedge h_j^{(\mathbf{1})} \right) \varphi_i^{(2)} \wedge \varphi_j^{(2)\vee} \\ &\stackrel{?}{=} \frac{1}{2\pi i} \int_{X_2} \mathbf{C}^{(\mathbf{1})}_{ij}\, \varphi_i^{(2)} \wedge \varphi_j^{(2)\vee}. \end{aligned} \tag{6.29}$$

Indeed eq. (6.29) is almost correct; following the discussion in (4.2.4), we have to replace $\varphi_i^{(2)}$ with $\mathrm{reg}_{\Omega^{(2)}}\left(\varphi^{(2)}\right)$, namely its compact support version laying in the same equivalence class. Given the fact that we are dealing with a matrix-valued connection, the explicit realization of

[4] Just consider (sum over repeated indices understood)

$$\begin{aligned} 0 = \int_{\gamma^{(2)}} d\left(u_j(z_2)\,\xi_j\right) = \int_{\gamma^{(2)}} \left(u_j(z_2) d\xi_j + du_j(z_2) \wedge \xi_j\right) &= \int_{\gamma^{(2)}} \left(u_j(z_2) d\xi_j + \Omega^{(2)}_{ji} u_i(z_2) \wedge \xi_j\right) \\ &= \int_{\gamma^{(2)}} u_i(z_2) \left(\delta_{ij}\, d\xi_j + \left(\Omega^{(2)\top}\right)_{ij} \wedge \xi_j\right). \end{aligned}$$

the map is

$$\mathrm{reg}_{\Omega^{(2)}} : \varphi^{(2)} \to \mathrm{reg}_{\Omega^{(2)}} \left(\varphi^{(2)}\right) = \varphi^{(2)} - \sum_{x_i \in \mathcal{P}_{\Omega^{(2)}}} \nabla_{\Omega^{(2)}} \left(h_{x_i}(z_2, \bar{z}_2)\, \psi_{x_i}\right) \tag{6.30}$$

where $\psi_{x_i}$ is a vector-valued solution local solution of[5]

$$\nabla_{\Omega^{(2)}} \left(\psi_{x_i}\right) = \varphi^{(2)} \qquad \text{around } x_i. \tag{6.31}$$

Therefore eq. (6.29) becomes

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \frac{1}{2\pi i} \int_{X_2} \left(\mathrm{reg}_{\Omega^{(2)}}\, \varphi^{(2)}\right)_i \wedge \varphi_j^{(2)\vee} \cdot \mathbf{C}_{ij}^{(\mathbf{1})} \tag{6.32a}$$

$$= \sum_{x_i \in \mathcal{P}_{\Omega^{(2)}}} \mathrm{Res} \left(\psi_{x_i} \cdot \mathbf{C}^{(1)} \cdot \varphi^{(2)\vee}\right); \tag{6.32b}$$

where eq. (6.32b) is derived from eq. (6.32a) following the same steps as in subsection 4.2.4.

**Example.** *A two variables co-homology intersection number. Let us reconsider the situation described in eq. (6.9), namely*

$$u(z_1, z_2) = z_1^p z_2^q (1 - z_1 - z_2)^r \tag{6.33}$$

*then*

$$\omega = \hat{\omega}_1\, dz_1 + \hat{\omega}_2\, dz_2 = \left(\frac{p}{z_1} - \frac{r}{1 - z_1 - z_2}\right) dz_1 + \left(\frac{q}{z_2} - \frac{r}{1 - z_1 - z_2}\right) dz_2, \tag{6.34}$$

*and let*

$$X = \mathbb{C}^2 \setminus \{z_1 = 0 \cup z_2 = 0 \cup 1 - z_1 - z_2 = 0\}; \tag{6.35}$$

*say we want to compute*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{\vee(\mathbf{2})} \rangle = \left\langle \frac{d\mathbf{z}}{z_1 z_2 (1 - z_1 - z_2)} \middle| \frac{d\mathbf{z}}{z_1^2 z_2 (1 - z_1 - z_2)} \right\rangle. \tag{6.36}$$

*The space $X$ can be decomposed in terms of $X_1 = \mathbb{C} \setminus \{0, 1-z_2\} = \mathbb{CP}^1 \setminus \{0, 1-z_2, \infty\}$ (whose coordinate is $z_1$) and $X_2 = \mathbb{C} \setminus \{0, 1\} = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ (whose coordinate is $z_2$).*

*According to the discussion above, we have to choose a basis for $\mathrm{H}^1(X_1, \nabla_{\pm\omega|_{dz_2=0}})$. A critical point analysis reveals*

$$\nu_{(\mathbf{1})} = \dim \left(\mathrm{H}^1(X_1, \nabla_{\pm\omega|_{dz_2=0}})\right) = 1; \tag{6.37}$$

[5] See appendix A for details of this step.

*our choice is*

$$\langle e^{(\mathbf{1})}| = \left\langle \frac{dz_1}{z_1(1-z_1-z_2)} \right| \in \mathrm{H}^1(X_1, \nabla_{\omega_1}), \qquad |h^{(\mathbf{1})}\rangle = \left| \frac{dz_1}{z_1(1-z_1-z_2)} \right\rangle \in \mathrm{H}^1(X_1, \nabla_{-\omega_1}). \tag{6.38}$$

*The univariate intersection number among basis elements is*

$$\mathbf{C}^{(\mathbf{1})} = \langle e^{(\mathbf{1})}|h^{(\mathbf{1})}\rangle = \frac{p+r}{pr\,(z_2-1)^{\,2}}. \tag{6.39}$$

*We have to determine* $\hat{\Omega}^{(2)}$*; aoccrding to eq.* (6.25)

$$\hat{\Omega}^{(2)} = \langle \nabla_{\hat{\omega}_2}(e^{(\mathbf{1})})|h^{(\mathbf{1})}\rangle \cdot \mathbf{C}_{(\mathbf{1})}^{-1} = \frac{1-p-r}{1-z_2} + \frac{q}{z_2}. \tag{6.40}$$

*Given*

$$\langle \varphi^{(\mathbf{2})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2(1-z_1-z_2)} \right|, \qquad |\varphi^{(\mathbf{2})\vee}\rangle = \left| \frac{d\mathbf{z}}{z_1^2 z_2(1-z_1-z_2)} \right\rangle, \tag{6.41}$$

*then the univariate decomposition w.r.t. the variable* $z_1$*−see eqs.* (6.28a, 6.28b)*−gives*

$$\begin{aligned} \langle \varphi^{(2)}| &= \langle \varphi^{(\mathbf{2})}|h^{(\mathbf{1})}\rangle \cdot \mathbf{C}_{(\mathbf{1})}^{-1} = \left\langle \frac{dz_2}{z_2} \right|, \\ |\varphi^{(2)\vee}\rangle &= \mathbf{C}_{(\mathbf{1})}^{-1} \cdot \langle e^{(\mathbf{1})}|\varphi^{(\mathbf{2})\vee}\rangle = \left| \frac{(1+p+r)\,dz_2}{(1+p)(1-z_2)z_2} \right\rangle. \end{aligned} \tag{6.42}$$

*The local solution of eq.* (6.31) *around each pole reads*

- $\psi_0 \rightsquigarrow$ *series solution around* $0$ (*local coordinate:* $y = z$)

$$\psi_0 = \frac{1}{q} + \mathcal{O}(y); \tag{6.43}$$

- $\psi_1 \rightsquigarrow$ *series solution around* $1$ (*local coordinate:* $y = z-1$)

$$\psi_1 = \frac{1}{p+r}\, y - \frac{(p+q+r)}{(p+r)(p+r+1)}\, y^2 + O\left(y^3\right) \tag{6.44}$$

- $\psi_\infty \rightsquigarrow$ *series solution around* $\infty$ (*local coordinate:* $y = \frac{1}{z}$)

$$\psi_\infty = \frac{1}{p+q+r-1} + \mathcal{O}(y). \tag{6.45}$$

*Finally eq.* (6.32b) *yields*

$$\left\langle \frac{d\mathbf{z}}{z_1 z_2(1-z_1-z_2)} \,\middle|\, \frac{d\mathbf{z}}{z_1^2 z_2(1-z_1-z_2)} \right\rangle = \frac{(p+q+r)(p+q+r+1)}{p(p+1)qr}. \tag{6.46}$$

### 6.2.2

## Co-Homology Intersection Number: general case

The above construction suggests that the multivariate intersection number among $n-$forms can be computed via a recursive strategy. We assume that $(n-1)$-intersection numbers are known, or, better, computable; then given

$$\langle\varphi^{(\mathbf{n})}| \in \mathrm{H}^n(X, \nabla_\omega), \qquad |\varphi^{(\mathbf{n})\vee}\rangle \in \mathrm{H}^n(X, \nabla_{-\omega}), \qquad \omega = d\log u, \tag{6.47}$$

and chosen

$$\begin{aligned} &(\langle e_1^{(\mathbf{n-1})}| \ \ldots,\ \langle e_{\nu_{(\mathbf{n-1})}}^{(\mathbf{n-1})}|) \rightsquigarrow \text{basis for } \mathrm{H}^{n-1}(X_{n-1}, \nabla_{\omega|_{dz_n=0}}) \\ &(|h_1^{(\mathbf{n-1})}\rangle \ \ldots,\ |h_{\nu_{(\mathbf{n-1})}}^{(\mathbf{n-1})}\rangle) \rightsquigarrow \text{basis for } \mathrm{H}^{n-1}(X_{n-1}, \nabla_{-\omega|_{dz_n=0}}) \end{aligned} \tag{6.48}$$

the multivariate intersection number is given by

$$\langle\varphi^{(\mathbf{n})}\,|\,\varphi^{(\mathbf{n})\vee}\rangle = \sum_{x_i\in\mathcal{P}_{\Omega^{(n)}}} \mathrm{Res}_{z_n=x_i}\left(\psi_{x_i,i}\cdot \mathbf{C}_{ij}^{(\mathbf{n-1})}\cdot\varphi_j^{(n)\vee}\right), \tag{6.49}$$

where the following are assumed to be known

- $\mathbf{C}_{ij}^{(\mathbf{n-1})} = \mathbf{C}_{(\mathbf{n-1}),ij} = \langle e_i^{(\mathbf{n-1})}\,|\,h_j^{(\mathbf{n-1})}\rangle$;
- $\langle\varphi^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{(\mathbf{n-1})}}\langle e_i^{(\mathbf{n-1})}|\wedge\langle\varphi_i^{(n)}|$ with
$$\langle\varphi_i^{(n)}| = \sum_{j=1}^{\nu_{(\mathbf{n-1})}}\langle\varphi^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle\left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ji}; \tag{6.50}$$
- $|\varphi^{(\mathbf{n})\vee}\rangle = \sum_{i=1}^{\nu_{(\mathbf{n-1})}}|h_i^{(\mathbf{n-1})}\rangle\wedge|\varphi_i^{(n)\vee}\rangle$ with
$$|\varphi_i^{(n)\vee}\rangle = \sum_{j=1}^{\nu_{\mathbf{n-1}}}\left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ij}\langle e_j^{(\mathbf{n-1})}|\varphi^{(\mathbf{n})\vee}\rangle; \tag{6.51}$$
- $\psi_{x_i}$ local solution of (see footnote 5)
$$\nabla_{\Omega^{(n)}}(\psi_{x_i}) = \varphi^{(n)}, \qquad \hat{\Omega}_{ij}^{(n)} = \sum_{k=1}^{\nu_{(\mathbf{n-1})}}\langle\nabla_{\omega_{n-1}}e_i^{(\mathbf{n-1})}|h_k^{(\mathbf{n-1})}\rangle\left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{kj}; \tag{6.52}$$
- $\mathcal{P}_{\Omega^{(n)}}$ the set of poles of $\hat{\Omega}^{(n)}$ (including $\infty$). In order to improve the readability, from now on will adopt the notation
$$\mathcal{P}_n = \mathcal{P}_{\Omega^{(n)}}. \tag{6.53}$$

From the recursive nature of the algorithm, it is clear that intersection numbers among $(n-1)$-forms can be computed building upon intersection numbers among $(n-2)$-forms and so on, till we land on intersection numbers among 1-forms which can be computed as described in subsection 4.2.4. Therefore, the dimension $\nu_{(\mathbf{m})}$ of $\mathrm{H}^m$, as well as (dual) bases $(\langle e_1^{(\mathbf{m})}|,\ \ldots,\ \langle e_{\nu_{(\mathbf{m})}}^{(\mathbf{m})}|)$ and $(|h_1^{(\mathbf{m})}\rangle,\ \ldots,|h_{\nu_{(\mathbf{m})}}^{(\mathbf{m})}\rangle)$, for $\mathbf{m} = \mathbf{1}, \ldots, \mathbf{n-1}$ has to be provided as input.

We give here an elementary example of pseudo-code which illustrates the algorithm[6]

[6] We assume the reader to be familiar with MATHEMATICA syntax. Nevertheless these few lines has to be understood just as a sketch; we omit detailed discussions such as managing of lists.

```
InterX[phi,phiV,omegas_List,e_List,h_List,zvars_List]=
 (* varin is the set of inner variables; varout is the outer variable *)
varin=zvars[[2;;]];
varout=zvars[[1]];

If[Length[varin]>0,

  (* omegain is the set of dlog forms associate to the (n-1) inner variables; omegaout is related to outer variable *)
   omegain=omega[[2;;]];
   omegaout=omega[[1]];

  (* eInner (hInner) are the set of internal (dual) bases w.r.t.  varout *)
   eInner=If[Length[varin]>=2,e[[2;;]],e[[2]]];
   hInner=If[Length[varin]>=2,h[[2;;]],h[[2]]];

  (* eCurrent (hCurrent) are the current (dual) basis *)
   eCurrent=If[Length[varin]==1,eInner,eInner[[1]]];
   hCurrent=If[Length[varin]==1,eInner,hInner[[1]]];

   (* CMatrix is the matrix of intersection numbers among current (dual) basis *)
   CMatrix=InterX[eCurrent,hCurrent,omegain,eInner,hInner,varin];

   (* phiN is the decomposition of phi onto eCurrent *)
   phiN=InterX[phi,hCurrent,omegain,eInner,hInner,varin];
   phiN=phiN.Inverse[C_matrix];

   (* phiVN is the decomposition of phiV onto hCurrent *)
   phiVN=InterX[eCurrent,phiV,omegain,eInner,hInner,varin];
   phiVN=Inverse[CMatrix].phiVN;

    (* Omega is the matrix defining the new connection *)
   Omega=InterX[D[eCurrent,varout]+omegaout*eCurrent,omegain,eInner,hInner,varin].Inverse[
CMatrix];
   ,
    (* inizialization the univariate case *)
   CMatrix={{1}};
   phiN={phi};
   phiVN={phiV};
   Omega={omegas};
   ];

   (* DeqRes solve the differential equation, and compute the resiude around each pole of Omega (summing the results) *)
   out=DeqRes[phiN,phiVN,CMatrix,Omega,varout]
   ];


   (**************************************************)
   (*************** Example of usage ***************)
```

```
(**************************************************)

(* We consider the example in eq (6.46) *)

(* Inizializations *)
phi=1/z1/z2/(1-z1-z2);
phiV=1/z1^2/z2/(1-z1-z2);
omegas={q/z2-r/(1-z1-z2),p/z1-r/(1-z1-z2)};
e={{},{1/z1/(1-z1-z2)}};
h={{},{1/z1/(1-z1-z2)}};
zvars={z2,z1};

(* Call the function *)
InterX[phi,phiV,omegas,e,h,zvars]
(* output *)
(p+q+r)*(p+q+r+1)/p/(p+1)/q/r.
```

Code 6.1: Example of pseudo-code for the evaluation of multivariate intersection numbers. InterX is the functions which computes (multivariate) intersection number. phi (resp. phiV) corresponds to the (dual) form $\varphi^{(\mathbf{n})}$ (resp. $\varphi^{(\mathbf{n})\vee}$). omegas_List is the list of dlogs forms, with $\omega_1$ as rightmost and $\omega_n$ as leftmost. e_List (resp. h_List) corresponds to the list of inner (dual) bases, with $e^{(\mathbf{1})}$ (resp. $h^{(\mathbf{1})}$), as rightmost, $e^{(\mathbf{n-1})}$ (resp. $h^{(\mathbf{n-1})}$) as next-to-leftmost and an empty list—which replaces $e^{(\mathbf{n})}$ (resp. $h^{(\mathbf{n})}$)—as leftmost element. zvars_List is the set of integration variables, with $z_1$ as rightmost and $z_n$ as leftmost.

Let us add a brief comment on eq. (6.49). We notice that the residue contains $\mathbf{C}_{(\mathbf{n-1})}$, which is compensated by $\mathbf{C}^{-1}_{(\mathbf{n-1})}$ contained in $|\varphi^{(n)\vee}\rangle$ (see eq. (6.51)). In practical calculations it is therefore useful to keep this in mind, avoiding spurious objects and minimizing the amount of algebraic manipulations preformed.
Concretely, we can thus consider

$$|\varphi^{(n)\vee}_{\mathbf{C},i}\rangle = \langle e_i^{(\mathbf{n-1})}|\varphi^{(\mathbf{n})\vee}\rangle, \tag{6.54}$$

and

$$\langle\varphi^{(\mathbf{n})}|\varphi^{(\mathbf{n})\vee}\rangle = \sum_{x_i\in\mathcal{P}_n} \mathrm{Res}_{z_n=x_i}\left(\psi_{x_i}\cdot\varphi^{(n)\vee}_{\mathbf{C}}\right). \tag{6.55}$$

Nevertheless, we stress that this does not mean that we can avoid the computation of $\mathbf{C}^{(\mathbf{n-1})}$, since it enters indirectly in the definitions of other quantities, e.g. $\Omega^{(n)}$.

Equivalently, as noticed in [30], eq. (6.54) can be thought of as a decomposition obtained employing a rescaled dual basis, say $(|h^{(\mathbf{n-1})}_{\mathbf{C},1}\rangle,\ldots,|h^{(\mathbf{n-1})}_{\mathbf{C},\nu_{(\mathbf{n-1})}}\rangle)$ related to the old one via

$$|h^{(\mathbf{n-1})}_{\mathbf{C},i}\rangle = \sum_{j=1}^{\nu_{(\mathbf{n-1})}} |h_j^{(\mathbf{n-1})}\rangle(\mathbf{C}^{-1}_{(\mathbf{n-1})})_{ji}, \tag{6.56}$$

such that

$$\mathbf{C}_{ij}^{(\mathbf{n-1})} \equiv \delta_{ij} = \langle e_i^{(\mathbf{n-1})} | h_{\mathbf{C},j}^{(\mathbf{n-1})} \rangle. \tag{6.57}$$

Once again this is, to the best of our knowledge, an a-posteriori choice, and general criteria in order to obtain such a basis are missing.

## 6.3

## Co-Homology Intersection Number: Proposal for Optimization

In the derivation of the multivariate intersection number we consider the decomposition (cf. eqs. (6.50,6.51))

$$\langle \varphi^{(\mathbf{n})} | = \sum_{i=1}^{\nu_{(\mathbf{n-1})}} \langle e_i^{(\mathbf{n-1})} | \wedge \langle \varphi_i^{(n)} |, \tag{6.58a}$$

$$| \varphi^{(\mathbf{n})\vee} \rangle = \sum_{i=1}^{\nu_{(\mathbf{n-1})}} | h_{\mathbf{C},i}^{(\mathbf{n-1})} \rangle \wedge | \varphi_{\mathbf{C},i}^{(n)\vee} \rangle; \tag{6.58b}$$

where $(\langle e_1^{(\mathbf{n-1})} | \ldots, \langle e_{\nu_{(\mathbf{n-1})}}^{(\mathbf{n-1})} |)$ is a basis for $\mathrm{H}^{n-1}(X_{n-1}, \nabla_{\omega|_{dz_n=0}})$, and $(| h_{\mathbf{C},1}^{(\mathbf{n-1})} \rangle, \ldots, | h_{\mathbf{C},\nu_{(\mathbf{n-1})}}^{(\mathbf{n-1})} \rangle)$ is a basis for $\mathrm{H}^{n-1}(X_{n-1}, \nabla_{-\omega|_{dz_n=0}})$ such that eq. (6.57) holds. The notation $\langle \varphi_i^{(n)} |$ and $| \varphi_{\mathbf{C},i}^{(n)\vee} \rangle$, for $1 \leq i \leq \nu_{(\mathbf{n-1})}$ is not accidental; in fact the coefficients of these decompositions can be thought of as equivalence classes themselves [30] [7]

$$\langle \varphi^{(n)} | \;:\; \varphi^{(n)} \sim \varphi^{(n)} + \nabla_{\Omega^{(n)}} \xi, \tag{6.59a}$$

$$| \varphi_{\mathbf{C}}^{(n)\vee} \rangle \;:\; \varphi_{\mathbf{C}}^{(n)\vee} \sim \varphi_{\mathbf{C}}^{(n)\vee} + \nabla_{-\Omega^{(n)}} \xi^{\vee}. \tag{6.59b}$$

This means that we can use the freedom allowed by eqs. (6.59a, 6.59b) in order to simply the subsequent steps in the calculation; we will discuss hereafter a possible strategy, proposed in [30], in order to do this.

We assume in the following that $\Omega^{(n)}$ is *fuchsian*, i.e. all the poles are simple poles[8]. Then we aim to find a different representative of $\varphi^{(n)}$, laying in the same equivalence class but having only *simple poles*; we will denote this representative as $\varphi_{\mathrm{sp}}^{(n)}$ (similarly $\varphi_{\mathbf{C},\mathrm{sp}}^{(n)\vee}$ is the representative with simple poles within the same equivalence class of $\varphi_{\mathbf{C}}^{(n)\vee}$).
One of the advantages of working with $\varphi_{\mathrm{sp}}^{(n)}$ and $\varphi_{\mathrm{sp}}^{(n)\vee}$ is that the pole-by-pole solution of

$$\nabla_{\Omega^{(n)}} (\psi_{x_i}) = \varphi_{\mathrm{sp}}^{(n)}, \qquad \text{around } x_i \in \mathcal{P}_n, \tag{6.60}$$

[7] On the one hand the action of $(\nabla_{\Omega^{(n)}})_{ij} (\bullet)_j$ is as in eq. (6.26). On the other hand $\left(\nabla_{-\Omega^{(n)}}\right)_{ij} (\bullet)_j = \delta_{ij} d(\bullet)_j - \Omega_{ij}^{(n)} \wedge (\bullet)_j$. See [28, 30] for a derivation.

[8] If this is not the case, we can try to employ some gauge-like transformation as done in eq. (2.116) or a different choice for the internal basis.

is bypassed—even the explicit locations of the poles is not needed.
In fact, since $\varphi_{\rm sp}^{(n)}$ and $\varphi_{\mathbf{C},\rm sp}^{(n)\vee}$ have just simple poles, the required solution of eq. (6.60) around each pole consists just in the constant term in the expansion of $\psi_{x_i}$. Nevertheless, we can write a single unique expression which, once expanded around each pole, reproduces the solution of eq. (6.60) at the desired order. Explicitly, we have that

$$\psi = \big(\hat{\Omega}^{(n)}\big)^{-\top} \cdot \hat{\varphi}_{\rm sp}^{(n)} \tag{6.61}$$

once expanded around each $x_i$, reproduces $\psi_{x_i}$ at the desired order.
Once eq. (6.61) is known, the expression for the multivariate intersection number eq. (6.55) becomes

$$\langle \varphi^{(\mathbf{n})} | \varphi^{(\mathbf{n})\vee} \rangle = \sum_{x_i \in \mathcal{P}_n} \mathrm{Res}_{z_n = x_i} \left( \hat{\varphi}_{\rm sp}^{(n)} \cdot \big(\hat{\Omega}^{(n)}\big)^{-1} \cdot \varphi_{\mathbf{C},\rm sp}^{(n)\vee} \right) . \tag{6.62}$$

Eq. (6.62) can massaged further.
Let us introduce the *adjugate* matrix of $\hat{\Omega}^{(n)}$, dubbed as $\mathrm{adj}\,\big(\hat{\Omega}^{(n)}\big)$, which satisfies

$$\hat{\Omega}^{(n)} \cdot \mathrm{adj}\,\big(\hat{\Omega}^{(n)}\big) = \det \hat{\Omega}^{(n)} \cdot \mathbb{1}. \tag{6.63}$$

Furthermore, we consider the following

$$\det \hat{\Omega}^{(n)} = \frac{\mathrm{P}}{\mathrm{Q}}, \tag{6.64}$$

Then the set of *poles* of $\hat{\Omega}^{(n)}$—dubbed as $\mathcal{P}_n$—coincides with the *zeros* of $\mathrm{Q}$ (plus the point at infinity), while the set of *critical points* of $\hat{\Omega}^{(n)}$—dubbed as $\mathcal{S}_n$—is given by the zeros of $\mathrm{P}$. We assume that those two sets are distinct (i.e. they do not have any common element).
With this in mind, we can rewrite eq. (6.62) as

$$\langle \varphi^{(\mathbf{n})} | \varphi^{(\mathbf{n})\vee} \rangle = \sum_{x_i \in \mathcal{P}_n} \mathrm{Res} \left( \frac{\mathrm{Q}}{\mathrm{P}}\, \hat{\varphi}_{\rm sp}^{(n)} \cdot \mathrm{adj}\, \hat{\Omega}^{(n)} \cdot \varphi_{\mathbf{C},\rm sp}^{(n)\vee} \right) . \tag{6.65}$$

We notice that in eq. (6.65) the residue is computed just for the subset $\mathcal{P}_n$ of the full set of poles of the integrand, which is given by $\mathcal{S}_n \bigcup \mathcal{P}_n$. Since the sum of all the residue as to be identically vanishing, we conclude that

$$\langle \varphi^{(\mathbf{n})} | \varphi^{(\mathbf{n})\vee} \rangle = - \sum_{z_{\rm crt} \in \mathcal{S}_n} \mathrm{Res} \left( \frac{\mathrm{Q}}{\mathrm{P}}\, \hat{\varphi}_{\rm sp}^{(n)} \cdot \mathrm{adj}\, \hat{\Omega}^{(n)} \cdot \varphi_{\mathbf{C},\rm sp}^{(n)\vee} \right) . \tag{6.66}$$

So eq. (6.66) is completely localized on the set of critical points of $\hat{\Omega}^{(n)}$, rather than on its poles. We find that eq. (6.66) is an interesting result on its own; nevertheless in Ref. [30] it was noticed that it corresponds to a *global residue*, and can be computed efficiently, see e.g. [204, 30]—once again without knowing the explicit locations of the critical points $\mathcal{S}_n$. So, under our assumptions, we can evaluate the multivariate intersection number via eq. (6.66), without knowing the explicit locations of poles nor the critical points. Since, usually, determining the locations of the poles (or critical points) introduces algebraic extensions—e.g. square roots—this variant of the algorithm allows to avoid these extensions.

• **The reduction to simple poles.**
We postponed the detailed description on how to find the representative $\varphi_{\rm sp}^{(n)}$ with simple poles laying equivalence class $\langle\varphi^{(n)}|$ (as well as $\varphi_{\mathbf{C},\rm sp}^{(n)\vee}$ belonging to $|\varphi_{\mathbf{C},\rm sp}^{(n)\vee}\rangle$). We address here such a problem.

We will heavily rely on the partial fraction decomposition of $\hat{\varphi}^{(n)}$ w.r.t. $z_n$. The partial fraction can be performed even without introducing algebraic extensions; so we do not spoil this feature of the algorithm.
Let us first assume that $\varphi^{(n)}$ has a pole of order $\text{ord}_\infty$ at infinity. This means that the partial fraction decomposition of $\varphi^{(n)}$ contains the monomial $z_n^{\text{ord}_\infty -2}$. We can then consider the following vector-valued function

$$\xi^\infty = \mathbf{c}\, z_n^{\text{ord}_\infty -1}, \tag{6.67}$$

where $\mathbf{c} = (c_1, \ldots, c_{\nu_{(\mathbf{n-1})}})$ is a vector of $\nu_{(\mathbf{n-1})}$ unknown coefficients.
Requiring that in the partial fraction decomposition of

$$\widetilde{\varphi}^{(n)} = \varphi^{(n)} + \nabla_{\Omega^{(n)}}\big(\xi^\infty\big) \tag{6.68}$$

the monomial $z_n^{\text{ord}_\infty -2}$ is absent, we obtain a linear system for the $(c_1, \ldots, c_{\nu_{(\mathbf{n-1})}})$. Provided the fact that this system admits a solution, then the corresponding $\widetilde{\varphi}^{(n)}$ will have a pole of order $\text{ord}_\infty -1$ at infinity. We can iterate the procedure until we land on a simple pole at infinity.

Let us consider now the case of poles at finite positions. Let us assume that an irreducible polynomial $q(z_n)$ of degree $\deg(q)$ appears raised to power $\text{ord}_q$ in the partial fraction decomposition of $\varphi^{(n)}$.
It is sufficient to consider the following vector-valued function

$$\xi^q = \frac{1}{q(z_n)^{\text{ord}_q -1}} \sum_{j=0}^{\deg(q)-1} \mathbf{c}_j z_n^j. \tag{6.69}$$

where each $\mathbf{c}_j = (c_{j,1}, \ldots, c_{j,\nu_{(\mathbf{n-1})}})$ is a vector of $\nu_{(\mathbf{n-1})}$ unknown coefficients.
Requiring that in the partial fraction decomposition of

$$\widetilde{\varphi}^{(n)} = \varphi^{(n)} + \nabla_{\Omega^{(n)}}\big(\xi^q\big) \tag{6.70}$$

the term proportional to $q(z_n)^{-\text{ord}_q}$ is absent, we obtain $\nu_{(n-1)} \cdot \deg_q$ constraints for the unknown $\nu_{(n-1)} \cdot \deg_q$ coefficients. Provided the fact that the resulting linear system admits a solution, the partial fraction decomposition of $\widetilde{\varphi}^{(n)}$ will contain at most $q(z_n)^{-\text{ord}_q +1}$. We can iterate the procedure until we arrive at $\mathrm{ord}_q = 1$.

The same discussion holds for $\varphi^{(n)\vee}$, replacing $\nabla_{\Omega^{(n)}}(\bullet)$ with $\nabla_{-\Omega^{(n)}}(\bullet)$. Notice that, since we assumed that $\Omega^{(n)}$ has just simple pole, this procedure never introduce any *new* higher order pole.

**Example.** *Let us reconsider eq. (6.11)*

$$u(z_1, z_2) = z_1^p z_2^q (1 - z_1 - z_2)^r; \tag{6.71}$$

*A critical point analysis gives*

$$\nu_{(\mathbf{2})} = 1, \qquad \nu_{(\mathbf{1})} = 1; \tag{6.72}$$

*we choose again the internal basis as*

$$\langle e^{(\mathbf{1})}| = \left\langle \frac{dz_1}{z_1(1 - z_1 - z_2)} \right|, \qquad |h^{(\mathbf{1})}\rangle = \left| \frac{dz_1}{z_1(1 - z_1 - z_2)} \right\rangle. \tag{6.73}$$

• *Let us consider the evaluation of*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \left\langle \frac{d\mathbf{z}}{z_1 z_2^2 (1 - z_1 - z_2)} \middle| \frac{d\mathbf{z}}{z_1 z_2 (1 - z_1 - z_2)} \right\rangle \tag{6.74}$$

*With the above-mentioned choice of internal basis we obtain (as reported in eqs. (6.39,6.40))*

$$\mathbf{C}^{(1)} = \frac{p + r}{pr\,(z_2 - 1)^{\,2}}, \tag{6.75}$$

*and*

$$\hat{\Omega}^{(2)} = \frac{1 - p - r}{1 - z_2} + \frac{q}{z_2}. \tag{6.76}$$

*Eq. (6.76) has just simple poles.*
*The decomposition $\langle \varphi^{(\mathbf{2})}| = \langle e^{(\mathbf{1})}| \wedge \langle \varphi^{(2)}|$ and $|\varphi^{(\mathbf{2})\vee}\rangle = |h^{(\mathbf{1})}\rangle \wedge |\varphi^{(2)\vee}\rangle$ yields*

$$\langle \varphi^{(2)}| = \left\langle \frac{dz_2}{z_2^2} \right|, \tag{6.77a}$$

$$|\varphi^{(2)\vee}\rangle = \left| \frac{dz_2}{z_2} \right\rangle. \tag{6.77b}$$

*and so*

$$|\varphi^{(2)\vee}_{\mathbf{C}}\rangle = \left| \frac{(p + r)\, dz_2}{pr\,(z_2 - 1)^{\,2} z_2} \right\rangle. \tag{6.78}$$

*Eqs. (6.77a,6.78) have a double pole at a finite position.*
*We will work first on $\varphi^{(2)}$. Following the description in the main text, we have*

$$q(z_2) = z_2, \qquad \mathrm{ord}_{z_2} = 2. \tag{6.79}$$

*Thus, we consider the following ansatz*

$$\xi^{z_2} = \frac{1}{z_2}\, c, \tag{6.80}$$

*where $c$ is an unknown constant.*
*Let us consider*

$$\widetilde{\varphi}^{(2)} = \varphi^{(2)} + \nabla_{\Omega^{(2)}} \xi^{z_2} \tag{6.81}$$

*and its partial fraction decomposition*

$$\frac{c(p+r-1)}{z_2-1} - \frac{c(p+r-1)}{z_2} + \frac{c(q-1)+1}{z_2^2}; \tag{6.82}$$

*requiring that the blue term is vanishing, we find*

$$c = \frac{1}{1-q}; \tag{6.83}$$

*so*

$$\varphi^{(2)}_{\text{sp}} = \widetilde{\varphi}^{(2)}\big|_{c\to(1-q)^{-1}} = \left( \frac{-p-r+1}{(q-1)\,(z_2-1)} + \frac{p+r-1}{(q-1)z_2} \right) dz_2 \tag{6.84}$$

*is free from higher order poles.*

*We move now to $\varphi^{(2)\vee}_{\mathbf{C}}$; in its partial fraction decomposition we find*

$$q(z_2) = 1 - z_2, \qquad \text{ord}_{1-z_2} = 2. \tag{6.85}$$

*So, we consider*

$$\xi^{1-z_2} = \frac{1}{1-z_2}\, c. \tag{6.86}$$

*Next, we consider*

$$\widetilde{\varphi}^{(2)\vee}_{\mathbf{C}} = \varphi^{(2)\vee}_{\mathbf{C}} + \nabla_{-\Omega^{(2)}} \xi^{1-z_2}, \tag{6.87}$$

*and its partial fraction decomposition which reads*

$$\frac{(p+r)(cpr+1)}{pr\,(z_2-1)^2} + \frac{cpqr-p-r}{pr\,(z_2-1)} - \frac{cpqr-p-r}{pr z_2}; \tag{6.88}$$

*requiring that the blue term is vanishing, we find*

$$c = -\frac{1}{pr}. \tag{6.89}$$

*So*

$$\varphi^{(2)\vee}_{\mathbf{C},\text{sp}} = \widetilde{\varphi}^{(2)\vee}_{\mathbf{C}}\big|_{c\to-(pr)^{-1}} = \left( \frac{p+q+r}{pr\,(1-z_2)} + \frac{p+q+r}{pr z_2} \right) dz_2, \tag{6.90}$$

*has just simple poles.*

*Finally we just need*

$$\det \hat{\Omega}^{(2)} = \hat{\Omega}^{(2)} = \frac{\mathrm{P}}{\mathrm{Q}} = \frac{q-(p+q+r-1)z_2}{(1-z_2)z_2} \tag{6.91}$$

*with* $\mathcal{P}_2 = \{0, 1, \infty\}$ *and* $\mathcal{S}_2 = \{\frac{q}{p+q+r-1}\}$. *Therefore eq. (6.66) gives*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = - \sum_{z_{\mathrm{crt}} \in \mathcal{S}_2} \mathrm{Res}_{\, z_2 = z_{\mathrm{crt}}} \left( \frac{\mathrm{Q}}{\mathrm{P}} \hat{\varphi}^{(2)}_{\mathrm{sp}} \, \varphi^{(2)\vee}_{\mathbf{C},\mathrm{sp}} \right) = \frac{(p+q+r-1)(p+q+r)}{p(q-1)qr}. \tag{6.92}$$

*Let us also briefly mention that eq. (6.92) can be massaged even further—see once again [30]. In fact eq. (6.92) is of the following form:*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = - \sum_{z_{\mathrm{crt}} \in \mathcal{S}_2} \mathrm{Res}_{\, z_2 = z_{\mathrm{crt}}} \left( \frac{h}{\mathrm{P}} \right), \qquad \mathcal{S}_2 \rightsquigarrow \text{zeros of } \mathrm{P}, \tag{6.93}$$

*and* h *can be thought as a ratio of two polynomials in* $z_2$, *say*

$$h = \frac{\mathrm{P}_h}{\mathrm{Q}_h}. \tag{6.94}$$

*Assuming that* P *and* $\mathrm{Q}_h$ *does not have any common zero, then Hilbert's Nullstellensatz guarantees that there exists two polynomials, say* R *and* S *such that*[9]

$$1 = \mathrm{R}\,\mathrm{P} + \mathrm{Q}_h\,\mathrm{S}\,. \tag{6.95}$$

. *Thanks to this step we have:*

$$(6.93) = - \sum_{z_{\mathrm{crt}} \in \mathcal{S}_2} \mathrm{Res}_{\, z_2 = z_{\mathrm{crt}}} \left( \frac{h}{\mathrm{P}} \cdot 1 \right) = - \sum_{z_{\mathrm{crt}} \in \mathcal{S}_2} \mathrm{Res}_{\, z_2 = z_{\mathrm{crt}}} \left( \frac{\mathrm{P}_h\;\mathrm{S}}{\mathrm{P}} \right). \tag{6.96}$$

*Eq. (6.96) is the basis of further manipulations in [30]; we avoid here such a discussion referring the reader to the original work. Let us just mention that* $\mathrm{P}_h$, S *and* P *are polynomials, and so e.g. eq. (6.96) correspond to the residue of the same function, computed at* $z_2 = \infty$[10].

• *Let us also briefly discuss the calculation of*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \left\langle 1 \cdot \frac{d\mathbf{z}}{z_1(1-z_1-z_2)} \middle| \frac{d\mathbf{z}}{z_1 z_2(1-z_1-z_2)} \right\rangle, \tag{6.97}$$

*since it exhibits a new feature.*

*We find that*

$$\langle \varphi^{(2)} | = \langle 1 \cdot dz_2 | \tag{6.98}$$

[9] We accomplished this step via the Mathematica command PolynomialExtendedGCD.

[10] In our basic Mathematica implementation we find this step satisfactory enough in practical examples; nevertheless we expect more refined implementations of [30] to be more efficient.

*which has a pole at infinity with*

$$\text{ord}_\infty = 2. \tag{6.99}$$

*There for we consider the following ansatz*

$$\xi^\infty = cz_2. \tag{6.100}$$

*Next we consider*

$$\widetilde{\varphi}^{(2)} = \varphi^{(2)} + \nabla_{\Omega^{(2)}}\xi^\infty. \tag{6.101}$$

*and its partial fraction decomposition*

$$1 + (p+q+r)c + \frac{c(p+r-1)}{z_2 - 1}; \tag{6.102}$$

*requiring that the blue term is vanishing, we find*

$$c = -\frac{1}{(p+q+r)}. \tag{6.103}$$

*So, finally*

$$\varphi^{(2)}_{\text{sp}} = \widetilde{\varphi}^{(2)}|_{c\to-(p+q+r)^{-1}} = \left(-\frac{p+r-1}{(z_2-1)(p+q+r)}\right) dz_2. \tag{6.104}$$

*Eqs. (6.104,6.90), can be used in eq. (6.66) yielding*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \frac{1}{pr}. \tag{6.105}$$

## 6.4

## Co-Homology Intersection Number: Secondary Equation Approach

We discuss here briefly an alternative approach for the computation of (multivariate) intersection numbers. This approach was proposed in [29], and aim to obtain intersection numbers as solutions of suitable system(s) of (partial) differential equations: the so-called Secondary Equation; this method can be applied irrespectively to compute univariate and multivariate intersection numbers, even though it relies on essential external inputs. We review the methods hereafter.

Let us consider once again eq. (4.112), and rearrange it as

$$\mathbf{H}^\top = \mathbf{P}^{\vee\top} \cdot \mathbf{C}^{-1} \cdot \mathbf{P}. \tag{6.106}$$

Eq. (6.106) offers another possibility of computing co-homology intersection numbers, and, more precisely, the full matrix $\mathbf{C}$. Clearly pretending to have access to $\mathbf{H}$, and$-$even worse$-\mathbf{P}$ and $\mathbf{P}^\vee$ is too demanding. Nevertheless we recall that the $u(\mathbf{z})$ does, in general, depend on other external parameters which are not integration variables, say $\mathbf{x}$−to fix the ideas we can think at the case of FIs where $x$ represents (ratio of) kinematic invariants. On the one hand we expect that $\mathbf{C}$ and $\mathbf{P}^{(\vee)}$ *do depend* on $\mathbf{x}$, on the other hand we assume that $\mathbf{H}$ is *independent* from it−c.f. e.g. the explicit computations in section 4.1.3. In other words we just have

$$d_{\mathbf{x}}\mathbf{H}^{\top} = 0. \tag{6.107}$$

Moreover $\mathbf{P}$ and $\mathbf{P}^{\vee}$ obey the following

$$d_{\mathbf{x}}\mathbf{P} = \mathbf{\Omega} \cdot \mathbf{P}, \qquad d_{\mathbf{x}}\mathbf{P}^{\vee} = \mathbf{\Omega}^{\vee} \cdot \mathbf{P}^{\vee}. \tag{6.108}$$

The matrices $\mathbf{\Omega}^{(\vee)}$ are known as *Pfaffians matrices*−once again, in order to fix the ideas, we can think at the differential equations fulfilled by FIs. These matrices represent the external input for this algorithm[11] Therefore, applying $d_{\mathbf{x}}(\bullet)$ to eq. (6.107), eqs. (6.107, 6.108) imply

$$\mathbf{P}^{\vee\top} \cdot \left(\mathbf{\Omega}^{\vee\top} \cdot \mathbf{C}^{-1} + d_{\mathbf{x}}\mathbf{C}^{-1} + \mathbf{C}^{-1} \cdot \mathbf{\Omega}\right) \cdot \mathbf{P} = \mathbb{0}. \tag{6.109}$$

Then, requiring that $\mathbf{P}$ and $\mathbf{P}^{\vee}$ have full rank, we infer that

$$\mathbf{\Omega}^{\vee\top} \cdot \mathbf{C}^{-1} + d_{\mathbf{x}}\mathbf{C}^{-1} + \mathbf{C}^{-1} \cdot \mathbf{\Omega} = \mathbb{0}, \tag{6.110}$$

or, equivalently[12]

$$d_{\mathbf{x}}\mathbf{C} = \mathbf{\Omega} \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{\Omega}^{\vee\top}. \tag{6.111}$$

Eq. (6.111) is known as *Secondary Equation*, and will plays a key role: solving the latter we can determine $\mathbf{C}$.

Eq. (6.111) represents a system of partial differential equations. Solving such a system is, in general, a formidable task. Fortunately enough it is known that co-homology intersection numbers, and so the full matrix $\mathbf{C}$, are known to be rational function in $\mathbf{x}$ [29][13]. Therefore we just look for *a rational solution*, say $\mathbf{C}_{\mathrm{Rat}}$ of eq. (6.111). This is a less complicated task and it is implemented in the program IntegrableConnection [217].

Finally, even if we determine *a rational solution* $\mathbf{C}_{\mathrm{Rat}}$ of eq. (6.111), there is still an ambiguity due to an over-all multiplicative constant[14] $\kappa$ in order to determine the correct matrix $\mathbf{C}$; so we

[11] It is important to stress that the cases of our interest, namely hypergeometric functions and FIs−once expressed in the Lee-Pomeransky representation (cf. section 2.3), fall into the category of Gelf́and-Kapranov-Zelevinsky (GKZ) systems [205]. Within this framework Pfaffian matrices can be obtained e.g. my means of the Maculay matrix method [36]. For the interplay among GKZ systems and FIs see also [206, 209, 210, 211, 212, 213, 214, 215, 216, 207, 208].

[12] Using $\mathbb{0} = \mathbf{C}^{-1} \cdot d_{\mathbf{x}}\mathbf{C} \cdot \mathbf{C}^{-1} + d_{\mathbf{x}}\mathbf{C}^{-1}$.

[13] We are going to use intersection numbers in order to achieve reduction onto MIs; it is clear that the coefficients of the decomposition obtained through the standard Laporta algorithm are rational functions (cf. section 2.4). So−unless mysterious cancellations occur− we can expect the intersection numbers themselves to be rational functions.

[14] In other words, we have to input some boundary values in eq. (6.111), in order to fully determine $\mathbf{C}$.

have

$$\mathbf{C} = \kappa\, \mathbf{C}_{\mathrm{Rat}}, \tag{6.112}$$

where $\kappa$ is, in general, unknown.

Even if the actual value of $\kappa$ can be, in principle computed−see [218]−we can avoid this final step if we are just interested obtaining linear relations among integrals. The reason for this will become clear in chapter 7.

# 7

# Integral Relations via Intersection Theory: multivariate case

In this chapter we will address the study of *linear* relations among integrals admitting a multivariate representation. At a first sight there is no conceptual difference w.r.t. the univariate case, see the discussion in section 5.2—clearly we have to replace the univariate intersection number employed in section 5.2 with the multivariate intersection number discussed in section 6.2. First we will apply our machinery to the case of hypergeometric integrals, and then we will move to the case of FIs, highlighting some of their peculiarities, particular features as well as possible strategies to overcome them.

## 7.1 Contiguity Relations via Intersection Numbers

We show here how the master decomposition formula and the multivariate intersection numbers can be employed in order to obtain contiguity relations among certain hypergeometric integrals. Such type of relations are often of interest in the mathematical literature, and they constitute the prototype of relations we aim to derive working with FIs.

**Example.** *The hypergeometric ${}_3F_2$. Let us consider the following*

$$u(\mathbf{z}) = z_1^{a_3}(1 - z_1)^{b_2-a_3}\, z_2^{a_2}\,(1 - z_2)^{b_1-a_2}\,(1 - x z_1 z_2)^{-a_1}; \tag{7.1}$$

*eq. (7.1) is associated to the hypergeometric function ${}_3F_2$ (see also appendix B). We hvae*

$$\omega = \hat{\omega}_1 dz_1 + \hat{\omega}_2 dz_2 = \left( \frac{a_3}{z_1} + \frac{a_3 - b_2}{1 - z_1} + \frac{a_1 x z_2}{1 - x z_1 z_2} \right) dz_1 + \left( \frac{a_2}{z_2} + \frac{a_2 - b_1}{1 - z_2} + \frac{a_1 x z_1}{1 - x z_1 z_2} \right) dz_2. \tag{7.2}$$

*A critical point analysis reveals that*

$$\nu_{(\mathbf{2})} = 3, \tag{7.3}$$

*while, for the inner variable $z_1$, we find*

$$\nu_{(\mathbf{1})} = 2; \tag{7.4}$$

*We choose the inner bases as*

$$(\langle e_1^{(\mathbf{1})}|\,, \langle e_2^{(\mathbf{1})}|) = (\langle d\log z_1|,\ \langle d\log(1-z_2)|), \tag{7.5a}$$

$$(|h_1^{(\mathbf{1})}\rangle\,, |h_2^{(\mathbf{1})}\rangle) = (|d\log z_1\rangle,\ |d\log(1-z_2)\rangle). \tag{7.5b}$$

*We aim to achieve the decomposition*

$$\begin{aligned} {}_3F_2(a_1, a_2, a_3 - 1; b_1 + 1, b_2; x) = {} & c_1\, {}_3F_2(a_1, a_2, a_3; b_1 + 1, b_2 + 1; x) \\ & + c_2\, {}_3F_2(a_1, a_2, a_3 + 1; b_1 + 1, b_2 + 1; x) \\ & + c_3\, {}_3F_2(a_1, a_2, a_3; b_1, b_2; x). \end{aligned} \tag{7.6}$$

*with the corresponding twisted co-cycles being*

$${}_3F_2(a_1, a_2, a_3 - 1; b_1 + 1, b_2; x) \rightsquigarrow \frac{1}{B(a_2, 1 + b_1 - a_2)\, B(a_3 - 1, 1 + b_2 - a_3)} \left\langle \frac{d\mathbf{z}}{z_1^2 z_2} \right| = \langle \varphi^{(\mathbf{2})}|, \tag{7.7}$$

*and*

$$\begin{aligned} {}_3F_2(a_1, a_2, a_3; b_1 + 1, b_2 + 1; x) &\rightsquigarrow \frac{1}{B(a_2, 1{+}b_1{-}a_2)\, B(a_3, 1{+}b_2{-}a_3)} \left\langle \frac{d\mathbf{z}}{z_1\, z_2} \right| = \langle e_1^{(\mathbf{2})}|, \\ {}_3F_2(a_1, a_2, a_3 + 1; b_1 + 1, b_2 + 1; x) &\rightsquigarrow \frac{1}{B(a_2, 1{+}b_1{-}a_2)\, B(a_3{+}1, b_2{-}a_3)} \left\langle \frac{d\mathbf{z}}{(1 - z_1)\, z_2} \right| = \langle e_2^{(\mathbf{2})}|, \\ {}_3F_2(a_1, a_2, a_3; b_1, b_2; x) &\rightsquigarrow \frac{1}{B(a_2, b_1{-}a_2)\, B(a_3, b_2{-}a_3)} \left\langle \frac{d\mathbf{z}}{z_1\, z_2\, (1 - z_1)\, (1 - z_2)} \right| = \langle e_3^{(\mathbf{2})}|. \end{aligned} \tag{7.8}$$

*Our choice for the dual basis is*

$$\begin{aligned} B(a_2, 1{+}b_1{-}a_2)\, B(a_3, 1{+}b_2{-}a_3) \left| \frac{d\mathbf{z}}{z_1\, z_2} \right\rangle &= |h_1^{(\mathbf{2})}\rangle, \\ B(a_2, 1{+}b_1{-}a_2)\, B(a_3{+}1, b_2{-}a_3) \left| \frac{d\mathbf{z}}{(1 - z_1)\, z_2} \right\rangle &= |h_2^{(\mathbf{2})}\rangle, \\ B(a_2, b_1{-}a_2)\, B(a_3, b_2{-}a_3) \left| \frac{d\mathbf{z}}{z_1\, z_2\, (1 - z_1)\, (1 - z_2)} \right\rangle &= |h_3^{(\mathbf{2})}\rangle. \end{aligned} \tag{7.9}$$

*The required intersection numbers are*

$$\mathbf{C}^{(\mathbf{2})}_{ij} = \langle e^{(\mathbf{2})}_i | h^{(\mathbf{2})}_j \rangle \qquad 1 \leq i, j \leq 3; \qquad \textit{and} \qquad \langle \varphi^{(\mathbf{2})} | h^{(\mathbf{2})}_j \rangle \qquad 1 \leq j \leq 3. \tag{7.10}$$

*We find*

$$\begin{aligned}
\mathbf{C}^{(\mathbf{2})}_{11} &= \frac{a_1}{a_2}\left(\frac{a_2+a_3-b_1}{a_3\,(a_1-b_1)\,(a_3-b_1)}+\frac{1}{(a_1-a_2)\,(a_2-b_2)}\right)+\frac{(a_2-b_1)}{(a_1-b_1)}\left(\frac{1}{(a_1-a_2)\,(b_2-a_1)}+\frac{1}{a_2a_3}\right),\\
\mathbf{C}^{(\mathbf{2})}_{12} &= \frac{(a_1+a_2)\,a_3b_1+a_3b_2\,(a_1+a_2-b_1)-\left(a_1^2+a_2a_1+a_2^2\right)a_3}{a_2\,(a_1-b_1)\,(a_1-b_2)\,(a_2-b_2)\,(b_2-a_3)},\\
\mathbf{C}^{(\mathbf{2})}_{13} &= \frac{b_1b_2}{a_2a_3\,(a_2-b_1)\,(a_3-b_2)},\\
\mathbf{C}^{(\mathbf{2})}_{21} &= \frac{\left(a_1\,(a_2-b_1-b_2)+(a_2-b_1)\,(a_2-b_2)+a_1^2\right)(a_3-b_2)}{a_2a_3\,(a_1-b_1)\,(a_1-b_2)\,(a_2-b_2)},\\
\mathbf{C}^{(\mathbf{2})}_{22} &= \frac{(a_2-a_3)\,a_1\,(a_2-b_1-b_2)-a_3\,(a_2-b_1)\,(a_2-b_2)+(a_2-a_3)\,a_1^2}{a_2\,(a_1-b_1)\,(a_1-b_2)\,(a_2-b_2)\,(b_2-a_3)},\\
\mathbf{C}^{(\mathbf{2})}_{23} &= \frac{b_1b_2}{a_2a_3\,(a_2-b_1)\,(a_3-b_2)},\\
\mathbf{C}^{(\mathbf{2})}_{31} &= \frac{(a_2-b_1)\,(a_3-b_2)}{a_2a_3b_1b_2},\\
\mathbf{C}^{(\mathbf{2})}_{32} &= \frac{a_3\,(a_2-b_1)}{a_2b_1b_2\,(a_3-b_2)},\\
\mathbf{C}^{(\mathbf{2})}_{33} &= \frac{b_1b_2}{a_2a_3\,(a_2-b_1)\,(a_3-b_2)}.
\end{aligned} \tag{7.11}$$

*and*

$$\begin{aligned}
\langle \varphi^{(\mathbf{2})} | h^{(\mathbf{2})}_1 \rangle &= \frac{a_2a_1^2x(b_1-a_2)+a_2a_1b_1x(a_2-b_1)+a_1\left((a_3-b_1)^2+a_2(2a_3-b_1-1)-a_3+b_1\right)(a_3-b_2)+(a_2-b_1)(b_1-a_3)(-a_3+b_1+1)(a_3-b_2)}{a_2a_3b_2(a_3-b_1-1)(a_3-b_1)(b_1-a_1)},\\
\langle \varphi^{(\mathbf{2})} | h^{(\mathbf{2})}_2 \rangle &= \frac{a_3\,((a_2-b_1)\,(a_3-b_1-1)+a_1\,(a_2+a_3-b_1-1))}{a_2b_2\,(a_1-b_1)\,(-a_3+b_1+1)\,(a_3-b_2)},\\
\langle \varphi^{(\mathbf{2})} | h^{(\mathbf{2})}_3 \rangle &= \frac{b_1\,(b_2\,(a_3-b_1-1)+a_1a_2x)}{a_2a_3\,(a_2-b_1)\,(a_3-b_1-1)\,(a_3-b_2)}.
\end{aligned} \tag{7.12}$$

*Combing everything within the master decomposition formula*

$$\begin{aligned}
\langle \varphi^{(\mathbf{2})} | &= \sum_{i,j=1}^{3} \langle \varphi^{(\mathbf{2})} | h^{(\mathbf{2})}_j \rangle \left(\mathbf{C}^{-1}_{(\mathbf{2})}\right)_{ji} \langle e^{(\mathbf{2})}_i |\\
&= \sum_{i=1}^{3} c_i \, \langle e^{(\mathbf{2})}_i |,
\end{aligned} \tag{7.13}$$

*we obtain*

$$\begin{aligned} c_1 &= \frac{a_3\left(x\left(-a_1-a_2+b_1+b_2\right)+b_2+1\right)+a_1 a_2 x-a_3^2-b_2\left(b_1 x+1\right)}{b_2\left(a_3-b_1-1\right)}, \\ c_2 &= \frac{a_3\left(x\left(a_1+a_2-b_1-b_2\right)+a_3-1\right)}{b_2\left(a_3-b_1-1\right)}, \\ c_3 &= \frac{b_1(x-1)}{a_3-b_1-1}. \end{aligned} \tag{7.14}$$

*The coefficients appearing in eq. (7.14) are (numerically) verified with* MATHEMATICA.

## 7.2

## Feynman Integrals Reduction via Intersection Numbers

We are finally at the point of discussing full reduction (i.e. including subtopologies) of FIs. In view of our subsequent discussion, it is beneficial to recall briefly our starting point—cf. also section 5.4—namely FIs in Baikov representation

$$I_{a_1,\ldots,a_n} = \int_\gamma u(\mathbf{z})\,\varphi(\mathbf{z}), \tag{7.15}$$

with—in general[1]

$$u(\mathbf{z}) = (\mathcal{B}(\mathbf{z}))^{\frac{d-\ell-E-1}{2}}, \qquad \varphi(\mathbf{z}) = \frac{\mathrm{N}(\mathbf{z})}{z_1^{a_1}\ldots z_n^{a_n}} d\mathbf{z}, \qquad (a_1,\ldots,a_n)\in\mathbb{Z}^n, \tag{7.16}$$

and

$$\gamma \in X = \mathbb{C}^n \setminus (\mathcal{B}=0), \quad \text{s.t.} \quad \mathcal{B}(\partial\gamma)=0. \tag{7.17}$$

Some of the $a_i$ can be (strictly) positive—in particular the exponents of the actual denominators $(a_1,\ldots,a_{n_{\text{den}}})$—and this poses a problem, since $\varphi$ is not defined on $X$—colloquially, with an abuse of terminology, we will say that the "poles" of $\varphi(\mathbf{z})$ are *not regulated* by $u(\mathbf{z})$. Thus the full machinery of chapter 6 cannot be directly applied.
One possible solution consists in consider a slightly modified version of the original $u(\mathbf{z})$, namely

$$u(\mathbf{z}) \rightsquigarrow u_{\text{reg}}(\mathbf{z}) = \prod_{i=1}^{n_{\text{den}}} z_i^{\rho_i}\, u(\mathbf{z}), \tag{7.18}$$

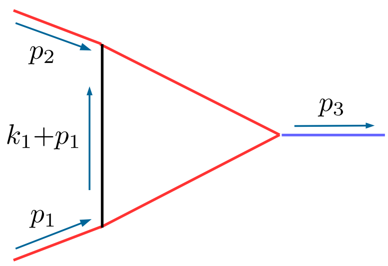
where $\rho_i \notin \mathbb{Z}$, for $1 \le i \le n_{\text{den}}$, are referred to as *regulators*. In this case $X = \mathbb{C}^n \setminus (\mathcal{B} = 0 \bigcup_{i=1}^{n_{\text{den}}} z_i = 0)$, and now $\varphi$ is defined on $X$. This regularization can be consider an analogue of the one of [219].
The reduction onto MIs is then performed working with eq. (7.18); the regulators $\rho_i$ are set to 0 once the coefficients of the reduction are obtained.

[1] In order to fix, we can consider $\mathrm{N}(\mathbf{z})$ as a polynomial in $\mathbf{z}$. Nevertheless we can lift this assumption, and consider the case in which $\mathrm{N}(\mathbf{z})$ has singularities, provided that they are "regulated" by $u(\mathbf{z})$.

Eq. (7.18) has an obvious disadvantage, namely the fact that there are more unphysical parameters involved in the reduction; in order to minimize the amount of algebraic manipulations involved, we often find convenient—and effective in practice—to set all the regulators equal, $\rho_i = \rho$ for $1 \leq i \leq n_{\text{den}}$. As anticipated, an alternative to this regularization procedure is discussed in the refined—and, probably, more appropriate—treatment of [40, 39], where the framework of relative twisted Co-Homology [220] is adopted.

We make here a small remark; we could consider a slightly modified version of eq. (7.18), where we regulate just a *subset* $\Sigma$, with $\Sigma \subset \{1, 2, \ldots, n_{\text{den}}\}$ of the full set of possible poles; $\Sigma$ will be, in practice, associated to a given *sector* (cf. section 2.1). We will denoted the replacement $u(\mathbf{z}) \rightsquigarrow u_\Sigma(\mathbf{z})$ according to eq. (7.18)). If this is the case, we *could not* consider—i.e. we would not sensitive—to differential forms with poles at $z_i$ with $i \notin \Sigma$. The auxiliary object $u_\Sigma(\mathbf{z})$ is useful, in practice, since the corresponding $\omega_\Sigma = d \log u_\Sigma(\mathbf{z})$ dictates the number of MIs in the sector $\Sigma$ (including all its possible subsectors) through a critical point analysis.
Therefore a possible strategy to identify a putative set of MIs consists in—starting from the smallest possible sector(s)— $i)$ consider the corresponding $\log u_\Sigma(\mathbf{z})$, $ii)$ count the number of critical points $\nu_\Sigma$, $iii)$ update the list of MIs accordingly, without over-counting the MIs common to subsectors.
Some comments are in order. First of all, we are completely blind to any sort of symmetry relation, as the ones usually implemented in public codes, therefore the number of independent MIs can be—in principle—reduced further. Second, the same strategy can be adopted in order to determine basis elements for the internal layers, required by the algorithm for the multivariate intersection number described in sec (6.2). Third, and more important, the determination of the number of MIs via (co)homological methods is a topic still under development, and the strategy above has to be considered more a guiding principle. We refer the reader to e.g. the discussion of [54] for the case of non isolated critical points, [221] for subtleties concerning the (loop-by-loop) Baikov representation, [39, 40] for a treatment of FIs directly in momentum space, where the dimension of Co-Homology groups are determined without relying on the critical point analysis and the case of degenerate kinematics is discussed. See also [180, 36, 203] for related, more mathematical, discussions.

**Example.** *One loop QED triangle. Let us consider the integral family associated to the following graph*




(7.19)

*The denominators are*

$$z_1 = D_1 = (k_1 + p_1 + p_2)^2 - m^2, \qquad z_2 = D_2 = k_1^2 - m^2, \qquad z_3 = D_3 = (k_1 + p_1)^2 \tag{7.20}$$

*and the kinematics is* $p_1^2 = p_2^2 = m^2$ *and* $p_3^2 = s$.

*There are in total* 8 *possible sectors, labelled by*

$$\{\mathbf{\Sigma}\} = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}. \tag{7.21}$$

*Then we consider*

$$u_\Sigma(\mathbf{z}) = \prod_{i \in \Sigma} z_i^{\rho_i} u(\mathbf{z}), \qquad \omega_\Sigma = d \log u_\Sigma(\mathbf{z}), \tag{7.22}$$

*associated to each sector.*
*The counting of critical points associated to each regularized twist is*

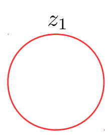
$$\{\nu_\Sigma\} = \{0, 1, 1, 0, 3, 1, 1, 3\}. \tag{7.23}$$

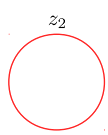
*Eq.* (7.23) *implies that there is* no *MIs without any pole,* one *MIs with pole in* $z_1$ *only,* one *MIs with pole in* $z_2$ *only,* no *MIs with pole in* $z_3$ *only,* three *MIs with poles in either* $z_1$ *or* $z_2$ *or both* (*notice that* two *of them were already counted*), one *MIs with poles in either* $z_1$ *or* $z_3$ *or both* (*notice that* one *was already counted, so no new MIs appears*), one *MIs with poles in either* $z_2$ *or* $z_3$ *or both* (*notice that* one *was already counted, so no new MIs appears*), three *MIs with poles in either* $z_1$ *or* $z_2$ *or* $z_3$ *or in each possible couple or in the full triplet* (*notice that* three *were already counted, so no new MIs appears*).

*So we are left with the following basis*

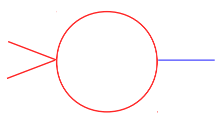
$$\langle e_1^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1} \right|, \quad \langle e_2^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_2} \right|, \quad \langle e_3^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2} \right|, \tag{7.24}$$

*which are nothing but the familiar*

$\langle e_1^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_1 =$ 

, $\quad \langle e_2^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_2 =$ 

, (7.25)

$\langle e_3^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_3 =$  .

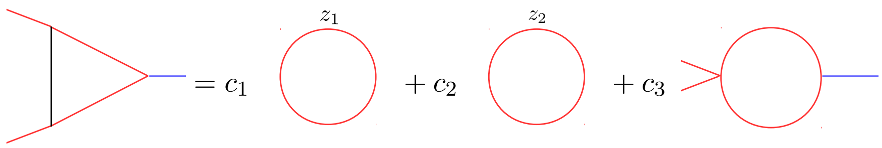
*We will focus on the decomposition of*

$$I_{1,1,1} = c_1\, \mathcal{J}_1 + c_2\, \mathcal{J}_2 + c_3\, \mathcal{J}_3, \tag{7.26}$$

*where the twisted co-cycle associated to $I_{1,1,1}$ is*

$$\langle \varphi^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2 z_3} \right|. \tag{7.27}$$

*The diagrammatic representation of eq. (7.26) is*




(7.28)

*In order to compute multivariate intersection numbers we have to provide explicit choices of bases for one forms w.r.t. $z_1$, and of bases of two forms-w.r.t. $(z_1, z_2)$. Once again a critical point analysis on the regularized twist performed considering $(z_2, z_3)$ and $z_3$ constant respectively, gives*

$$\nu_{(\mathbf{1})} = 2, \qquad \nu_{(\mathbf{2})} = 4; \tag{7.29}$$

*the corresponding bases are*

$$(\langle e_1^{(\mathbf{1})}|, \langle e_2^{(\mathbf{1})}|) = \left( \langle 1 \cdot dz_1|, \left\langle \frac{dz_1}{z_1} \right| \right), \tag{7.30}$$

*and*

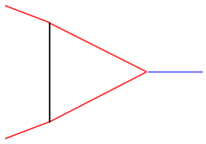
$$(\langle e_1^{(\mathbf{2})}|, \langle e_2^{(\mathbf{2})}|, \langle e_3^{(\mathbf{2})}|, \langle e_4^{(\mathbf{2})}|) = \left( \langle 1 \cdot dz_1 \wedge dz_2|, \left\langle \frac{dz_1 \wedge dz_2}{z_1} \right|, \left\langle \frac{dz_1 \wedge dz_2}{z_2} \right|, \left\langle \frac{dz_1 \wedge dz_2}{z_1 z_2} \right| \right). \tag{7.31}$$

*We consider dual forms* identical *to forms, namely $h_i^{(\bullet)} = e_i^{(\bullet)}$ for all $i$, and $\bullet = \mathbf{1}, \mathbf{2}, \mathbf{3}$.*

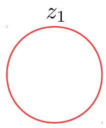
*Computing all the required intersection numbers according to eq. (5.32), we obtain the following $\rho$-dependent coefficients:*

$$\begin{aligned} c_1(\rho) &= \frac{(d+3\rho-3)(d+3\rho-2)(d+6\rho-2)}{2m^2(d+4\rho-4)(d+4\rho-3)(d+4\rho-2)\,(s-4m^2)}, \\ c_2(\rho) &= \frac{(d+3\rho-3)(d+3\rho-2)(d+6\rho-2)}{2m^2(d+4\rho-4)(d+4\rho-3)(d+4\rho-2)\,(s-4m^2)}, \\ c_3(\rho) &= -\frac{(d+3\rho-3)\,\left(2dm^2+8m^2\rho-6m^2+\rho s\right)}{m^2(d+4\rho-4)(d+4\rho-3)\,(s-4m^2)}; \end{aligned} \tag{7.32}$$

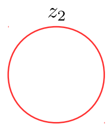
*in the $\rho \to 0$ limit the decomposition boils down to:*



$$= \frac{2-d}{2(d-4)m^2\,(4m^2-s)}$$

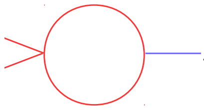



$$+ \frac{2-d}{2(d-4)m^2\,(4m^2-s)}$$




(7.33)

$$+ \frac{2(d-3)}{(d-4)\,(4m^2-s)}$$

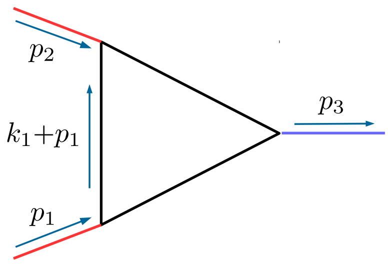


.

*Eq. (7.33) is in agreement with* FiniteFlow *&* LiteRed[2].

Before moving on, we pause for a small comment; despite the fact that the result of the intersection number $\langle \varphi^{(\mathbf{n})} | \varphi^{(\mathbf{n})\vee} \rangle$ is *independent* from the choices of (dual) internal bases (as well as on the ordering of the variables), intermediate expression can severely depend on their choices. At the time being there is no clear criterion which could leads to simple intermediate expressions. We give an explicit example on the impact of different choices of internal bases hereafter (see also the discussion in appendix A).

**Example.** *One loop massless triangle. We consider here the following graph*




(7.34)

*The denominators are chosen as*

$$z_1 = D_1 = (k_1 + p_1 + p_2)^2, \qquad z_2 = D_2 = k_1^2, \qquad z_3 = D_3 = (k_1 + p_1)^2. \tag{7.35}$$

*The kinematics is given by $p_1^2 = p_2^2 = m^2 = 1$ and $p_3^2 = s$.*

*Once again there are 8 possible sectors, labelled by*

$$\{\mathbf{\Sigma}\} = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}. \tag{7.36}$$

[2]We thank G. Fontana and T. Peraro for correspondence on the use of FiniteFlow.

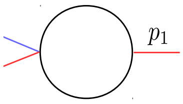
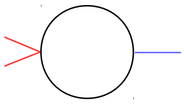
*The counting of critical points is*

$$\{\nu_\Sigma\} = \{0,0,0,0,1,1,1,4\} \tag{7.37}$$

*and the forms are chosen as*

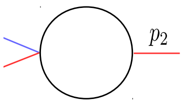
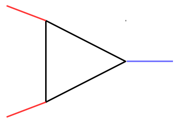
$$\langle e_1^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2} \right|, \quad \langle e_2^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_3} \right|, \quad \langle e_3^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_2 z_3} \right|, \quad \langle e_4^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2 z_3} \right|. \tag{7.38}$$

*Diagrammaticaly they correspond to*

$\langle e_1^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_1 =$  , $\quad \langle e_2^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_2 =$  ,



(7.39)

$\langle e_3^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_3 =$  , $\quad \langle e_4^{(\mathbf{3})}| \rightsquigarrow \mathcal{J}_4 =$  .



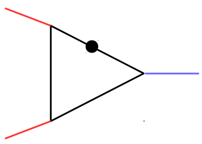
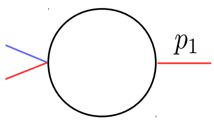
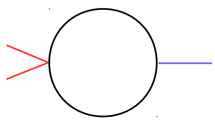
*We focus for concreteness on the decomposition*

$$I_{2,1,1} = c_1\,\mathcal{J}_1 + c_2\,\mathcal{J}_2 + c_3\,\mathcal{J}_3 + c_4\,\mathcal{J}_4, \tag{7.40}$$

*where the co-cycle associate to $I_{2,1,1}$ is*

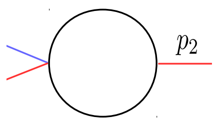
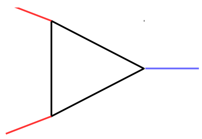
$$\langle \varphi^{(\mathbf{3})}| = \left\langle \frac{d\mathbf{z}}{z_1^2 z_2 z_3} \right|. \tag{7.41}$$

*Eq. (7.40) graphically is nothing but*

 $= c_1$  $+ c_2$ 



(7.42)

$+ c_3$  $+ c_4$  .



*We have to assign specific internal bases w.r.t. $z_1$ and w.r.t. $(z_1, z_2)$; a critical point analysis gives*

$$\nu_{(\mathbf{1})} = 2, \qquad \nu_{(\mathbf{2})} = 4; \tag{7.43}$$

*let us consider the same choice employed in eqs. (7.30,7.31), i.e.:*

$$(\langle e_1^{(\mathbf{1})}|, \langle e_2^{(\mathbf{1})}) = \left( \langle 1 \cdot dz_1|, \left\langle \frac{dz_1}{z_1} \right| \right), \tag{7.44}$$

*and*

$$(\langle e_1^{(\mathbf{2})}|, \langle e_2^{(\mathbf{2})}|, \langle e_3^{(\mathbf{2})}|, \langle e_4^{(\mathbf{2})}|) = \left( \langle 1 \cdot dz_1 \wedge dz_2|, \left\langle \frac{dz_1 \wedge dz_2}{z_1} \right|, \left\langle \frac{dz_1 \wedge dz_2}{z_2} \right|, \left\langle \frac{dz_1 \wedge dz_2}{z_1 z_2} \right| \right). \tag{7.45}$$

*Once again we assume $h_i^{(\bullet)} = e_i^{(\bullet)}$ for all $i$, and $\bullet = \mathbf{1}, \mathbf{2}, \mathbf{3}$.*
*According to the iterative multivariate algorithm, we have to compute*

$$\hat{\Omega}_{ij}^{(3)} = \sum_{k=1}^{\nu_{(\mathbf{2})}} \langle \nabla_{\omega_3} e_i^{(\mathbf{2})} | h_k^{(\mathbf{2})} \rangle \left( \mathbf{C}_{(n-1)}^{-1} \right)_{kj}, \tag{7.46}$$

*and then solve*

$$\nabla_{\Omega^{(3)}} (\psi_{x_i}) = \varphi^{(3)}. \tag{7.47}$$

*Different choices of ( $\langle e_1^{(\mathbf{2})}|$, $\langle e_2^{(\mathbf{2})}|$, $\langle e_3^{(\mathbf{2})}|$, $\langle e_4^{(\mathbf{2})}|$) result in different expressions for $\Omega^{(3)}$; suitable choices can simplify the task of solving eq (7.47).*

*For example eq. (7.45) gives*[3]*:*

$$\hat{\Omega}^{(3)} = \begin{pmatrix} \frac{11}{2z_3} & \frac{3(z_3-1)}{2z_3} & \frac{3(z_3-1)}{2z_3} & 0 \\ -\frac{5}{2(z_3-1)z_3} & \frac{17z_3-3}{2(z_3-1)z_3} & \frac{3(z_3+1)}{2(z_3-1)z_3} & -\frac{15}{z_3-1} \\ -\frac{5}{2(z_3-1)z_3} & \frac{3(z_3+1)}{2(z_3-1)z_3} & \frac{17z_3-3}{2(z_3-1)z_3} & -\frac{15}{z_3-1} \\ -\frac{5(z_3+1)}{2(z_3-1)z_3(z_3^2+3z_3+1)} & \frac{3z_3^2+14z_3+3}{2(z_3-1)z_3(z_3^2+3z_3+1)} & \frac{3z_3^2+14z_3+3}{2(z_3-1)z_3(z_3^2+3z_3+1)} & \frac{3(4z_3^3-5z_3^2-17z_3-2)}{2(z_3-1)z_3(z_3^2+3z_3+1)} \end{pmatrix}, \tag{7.48}$$

*and so $\Omega^{(3)} = \hat{\Omega}^{(3)} dz_3$ has a double pole at infinity.*
*We could consider a rotated basis, say*[4]

$$\begin{pmatrix} \langle e_1^{(\mathbf{2})}| \\ \langle e_2^{(\mathbf{2})}| \\ \langle e_3^{(\mathbf{2})}| \\ \langle e_4^{(\mathbf{2})}| \end{pmatrix}_{\mathbf{T}} = \mathbf{T} \begin{pmatrix} \langle e_1^{(\mathbf{2})}| \\ \langle e_2^{(\mathbf{2})}| \\ \langle e_3^{(\mathbf{2})}| \\ \langle e_4^{(\mathbf{2})}| \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} -\frac{1}{z_3} & \frac{\rho(z_3+1)}{z_3(d+2(\rho-1))} & \frac{\rho(z_3+1)}{z_3(d+2(\rho-1))} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.49}$$

[3] The matrix is reported for the specific value $d = 1, \rho = 3$ and $s = 5$.
[4] The matrix $\mathbf{T}$ is obtained with FUCHSIA.

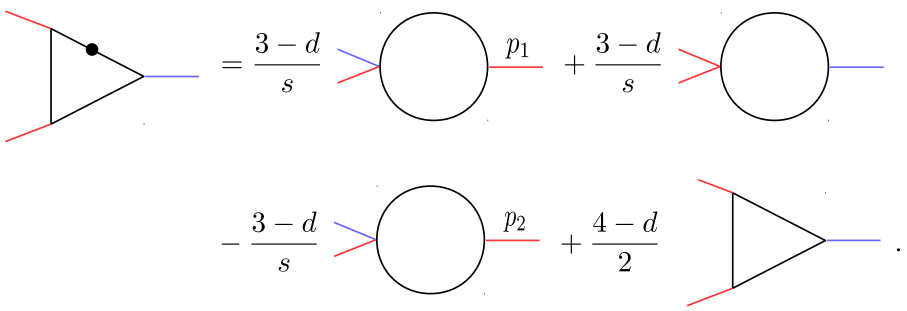
*which leads to*

$$\hat{\Omega}^{(3)}_{\mathbf{T}} = \begin{pmatrix} \frac{3}{2z_3}+\frac{6}{z_3-1} & \frac{24}{5(z_3-1)}-\frac{24}{5z_3} & \frac{24}{5(z_3-1)}-\frac{24}{5z_3} & \frac{18}{z_3}-\frac{36}{z_3-1} \\ \frac{5}{2(z_3-1)} & \frac{3}{z_3}+\frac{4}{z_3-1} & 0 & -\frac{15}{z_3-1} \\ \frac{5}{2(z_3-1)} & 0 & \frac{3}{z_3}+\frac{4}{z_3-1} & -\frac{15}{z_3-1} \\ \frac{-2z_3-3}{2\left(z_3^2+3z_3+1\right)}+\frac{1}{z_3-1} & \frac{4}{5(z_3-1)}-\frac{4(z_3+4)}{5\left(z_3^2+3z_3+1\right)} & \frac{4}{5(z_3-1)}-\frac{4(z_3+4)}{5\left(z_3^2+3z_3+1\right)} & \frac{9(2z_3+3)}{2\left(z_3^2+3z_3+1\right)}-\frac{6}{z_3-1}+\frac{3}{z_3} \end{pmatrix}; \tag{7.50}$$

*in this way* $\Omega^{(3)} = \hat{\Omega}^{(3)} dz_3$ *as at most simple poles (including at infinity).*
*Completing the calculation with the rotated basis, eq.* (5.32) *gives*

$$\begin{aligned} c_1(\rho) &= \frac{(d+3(\rho-1))(d+4(\rho-1))}{(\rho-1)s(d+2(\rho-2))}, \\ c_2(\rho) &= \frac{(d+3(\rho-1))(d+4(\rho-1))}{(\rho-1)s(d+2(\rho-2))}, \\ c_3(\rho) &= -\frac{d+3(\rho-1)}{(\rho-1)s}, \\ c_4(\rho) &= \frac{(d+4(\rho-1))(s(d+2(\rho-2))-2\rho)}{2(\rho-1)s(d+2(\rho-2))}; \end{aligned} \tag{7.51}$$

*in the limit* $\rho \to 0$ *the decomposition boils down to*




(7.52)

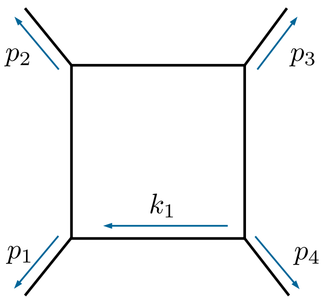
*Eq.* (7.52) *is in agreement with* FiniteFlow *&* LiteRed.

## 7.3 Feynman Integrals Reduction via Intersection Numbers and Unitarity-cuts

The decomposition via intersection numbers can be also combined with unitarity-based method. After having identified all the sectors which contain at least *one* MIs, we can engineer a *spanning set of cuts* defined as the minimal set of cuts such that each MIs appears at least once [59] (MIs which do not contain all the cut-denominators will not contribute to the decomposition on that cut).

The advantages of this combined strategy are clear. Unitarity-cuts are easy to perform working with the Baikov representation and the required intersection number are computationally less expensive since they involved *fewer* integration variables w.r.t. the original problem. Let us also mention that the same MI can appear on different spanning cuts, and so the corresponding coefficient is computed in different ways—i.e. building upon different sets of intersection numbers. This offers a consistency check on our procedure.

**Example.** *One loop massless box via unitarity cuts. Let us consider the integral family associated with the following graph*




(7.53)

*The kinematics is chosen as* $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$, $s = (p_1+p_2)^2$ *and* $t = (p_2+p_3)^2$.

*There are* 16 *possible sectors, labelled by*

$$\begin{aligned}\{\boldsymbol{\Sigma}\} = \{&\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \\ &\{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.\end{aligned} \tag{7.54}$$

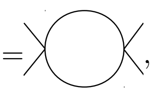
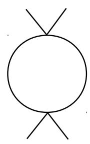
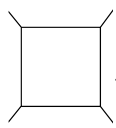
*The counting of critical points associated to each regularized twist is*

$$\{\nu_\Sigma\} = \{0,0,0,0,0,0,1,0,0,1,0,1,1,1,1,3\}, \tag{7.55}$$

*and so the basis elements can be chose as*

$$\langle e_1^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_3}\right|, \qquad \langle e_2^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_2 z_4}\right|, \qquad \langle e_3^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2 z_3 z_4}\right|, \tag{7.56}$$

*diagrammaticaly they correspond to*

$\langle e_1^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_1 =$  , $\quad \langle e_2^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_2 =$  $\quad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_3 =$  . (7.57)

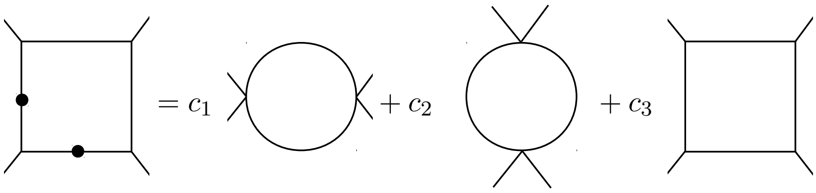
*We focus on the decomposition*

$$I_{2,2,1,1} = c_1 \mathcal{J}_1 + c_2 \mathcal{J}_2 + c_3 \mathcal{J}_3, \tag{7.58}$$

*where the co-cycle associated to* $I_{2,2,1,1}$ *is*[5]

$$\langle \varphi^{(4)}| = \left\langle \frac{d\mathbf{z}}{z_1^2 z_2^2 z_3 z_4} \right| = \left\langle \partial_1 \log u(\mathbf{z}) \, \frac{d\mathbf{z}}{z_1 z_2^2 z_3 z_4} \right| = \left\langle \partial_2 \log u(\mathbf{z}) \, \frac{d\mathbf{z}}{z_1^2 z_2 z_3 z_4} \right|. \tag{7.59}$$

*Eq. (7.58) is nothing but*

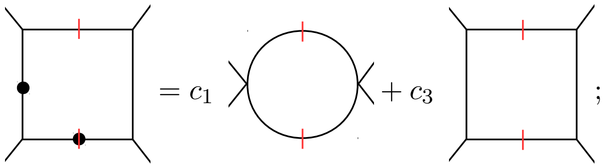


(7.60)

*Eqs. (7.57, 7.58) imply that the full decomposition can be reconstructed from the following spanning set of cuts*

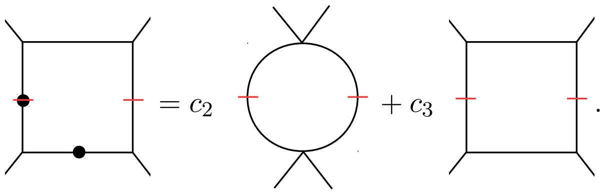
$$\{\mathcal{S}\} = \{\mathrm{Cut}_{1,3}, \mathrm{Cut}_{2,4}\}. \tag{7.61}$$

*Diagrammaticaly this means that on* $\mathrm{Cut}_{1,3}$ *we have*



(7.62)

*while on* $\mathrm{Cut}_{2,4}$



(7.63)

- $\mathrm{Cut}_{1,3}$.

*On this cut we have the following*

$$u(\mathbf{z}) \rightsquigarrow u_{\circlearrowleft}(z_2, z_4) = u(\mathbf{z})|_{z_1=z_3=0}, \tag{7.64}$$

*and its regularized version*[6]

$$u_{\mathrm{reg},\circlearrowleft}(z_2, z_4) = z_2^{\rho_2} z_4^{\rho_4} u_{\circlearrowleft}(z_2, z_4). \tag{7.65}$$

[5]The chain of equalities follows from trivial IBPs in Baikov representation, before applying any sort of regularization.

[6]Notice that the counting of critical points associated to the twist $d \log u_{\mathrm{reg},\circlearrowleft}(z_2, z_4)$ is consistence with the diagrammatic decomposition in eq. (7.62).

*Then the following identifications hold*

$$\langle e_1^{(\mathbf{4})}| \rightsquigarrow \langle e_{1,\circlearrowleft}^{(24)}| = \left\langle 1 \cdot dz_2 \wedge dz_4 \right|, \qquad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \langle e_{3,\circlearrowleft}^{(24)}| = \left\langle \frac{dz_2 \wedge dz_4}{z_2 z_4} \right|. \tag{7.66}$$

*and*

$$\langle \varphi^{(\mathbf{4})}| \rightsquigarrow \langle \varphi_{\circlearrowleft}^{(24)}| = \left\langle \left(\partial_1 \log u(\mathbf{z})\right)|_{z_1=z_3=0} \cdot \frac{dz_2 \wedge dz_4}{z_2^2 z_4} \right|. \tag{7.67}$$

*Choosing $z_2$ as the inner variable, a critical points analysis reveals*

$$\nu_{(2)} = 2, \tag{7.68}$$

*and the inner basis elements are chose as*

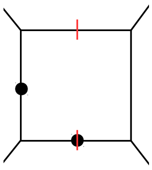
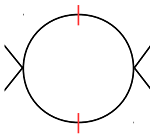
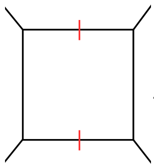
$$(\langle e_1^{(2)}|, \langle e_2^{(2)}) = (\langle 1 \cdot dz_2|, \langle d \log z_2|). \tag{7.69}$$

*We choose the dual bases elements as $h_i^{(\bullet)} = e_i^{(\bullet)}$ for all $i$, $\bullet = (2), (24)$.*
*We can then perform the decomposition according to eq. (5.32), with our new setting described by eqs. (7.65-7.67). We obtain*

$$c_1(\rho) = \frac{4(d+2\rho-5)(d+2(\rho-2))(d+2\rho-3)}{(\rho-1)\, s^3 t\, (d+4(\rho-1))}, \tag{7.70a}$$

$$c_3(\rho) = -\frac{(d+2\rho-5)(s(d+4\rho-6)(d+4(\rho-1))+2\rho t(d+2(\rho-2)))}{(\rho-1)s^2 t(d+4(\rho-1))}. \tag{7.70b}$$

*Therefore, in the $\rho \to 0$ we obtain*


$= -\dfrac{4(d-5)(d-3)}{s^3 t}$

$+ \dfrac{(d-6)(d-5)}{st}$

. (7.71)

• $\mathrm{Cut}_{2,4}$.
*On this cut we have the following*

$$u(\mathbf{z}) \rightsquigarrow u_{\circlearrowleft}(z_1, z_3) = u(\mathbf{z})|_{z_2=z_4=0}, \tag{7.72}$$

*and its regularized version*[7]

$$u_{\mathrm{reg},\circlearrowleft}(z_1, z_3) = z_1^{\rho_1} z_3^{\rho_3} u_{\circlearrowleft}(z_1, z_3). \tag{7.73}$$

*Then we have the identifications*

$$\langle e_2^{(\mathbf{4})}| \rightsquigarrow \langle e_{2,\circlearrowleft}^{(13)}| = \langle 1 \cdot dz_1 \wedge dz_3|, \qquad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \langle e_{3,\circlearrowleft}^{(13)}| = \left\langle \frac{dz_1 \wedge dz_3}{z_1 z_3} \right|, \tag{7.74}$$

[7] Notice that the counting of critical points associated to the twist $d \log u_{\mathrm{reg},\circlearrowleft}(z_1, z_3)$ is consistence with the diagrammatic decomposition in eq. (7.63).

*and*

$$\langle\varphi^{(4)}| \rightsquigarrow \langle\varphi_{\circlearrowleft}^{(13)}| = \left\langle \big(\partial_2 \log u(\mathbf{z})\big)|_{z_2=z_4=0} \cdot \frac{dz_1 \wedge dz_3}{z_1^2 z_3} \right|. \tag{7.75}$$

*Choosing $z_1$ as the inner variable, a critical point analysis reveals*

$$\nu_{(1)} = 2, \tag{7.76}$$

*and the inner basis elements are chosen as*

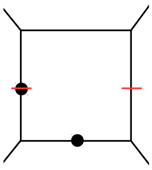
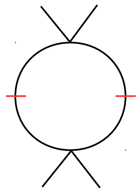
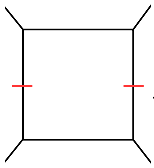
$$(\langle e_1^{(1)}|, \langle e_2^{(1)}|) = (\langle dz_1|, \langle d\log z_1|). \tag{7.77}$$

*We employ $h_i^{(\bullet)} = e_i^{(\bullet)}$ for all $i$, $\bullet = (1), (13)$. Then we can apply eq. (5.32) obtaining*

$$c_2(\rho) = \frac{4(d+2\rho-5)(d+2(\rho-2))(d+2\rho-3)}{(\rho-1)st^3(d+4(\rho-1))}, \tag{7.78a}$$

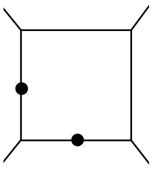
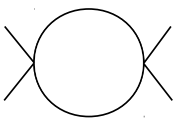
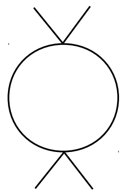
$$c_3(\rho) = -\frac{(d+2\rho-5)(2\rho s(d+2(\rho-2))+t(d+2(2\rho-3))(d+4(\rho-1)))}{(\rho-1)st^2(d+4(\rho-1))}; \tag{7.78b}$$

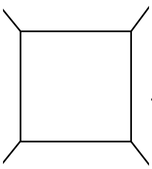
*in the $\rho \to 0$ we obtain*

 $= -\dfrac{4(d-5)(d-3)}{st^3}$  $+ \dfrac{(d-6)(d-5)}{st}$  . (7.79)

*We observe that the coefficients of $\mathcal{J}_3$ in eqs. (7.71,7.79) agree, and this serves as a consistency check.*

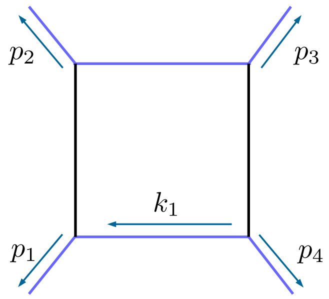
*Combining eqs. (7.71,7.79) we obtain the full decomposition*

 $= -\dfrac{4(d-5)(d-3)}{s^3 t}$  $- \dfrac{4(d-5)(d-3)}{st^3}$ 

$+ \dfrac{(d-6)(d-5)}{st}$  . (7.80)

*Eq. (7.80) is in agreement with* FiniteFlow *&* LiteRed.

**Example.** *One loop Bhabha box. We consider the following graph*




(7.81)

*The denominators are chosen as*

$$z_1 = D_1 = k_1^2 - 1, \qquad z_2 = D_2 = (k_1 - p_1)^2, \tag{7.82}$$
$$z_3 = D_3 = (k_1 - p_1 - p_2)^2 = 1, \qquad z_4 = D_4 = (k_1 - p_1 - p_2 - p_3)^2, \tag{7.83}$$

*while the kinematics is given by* $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 1$, $(p_1 + p_2)^2 = s$ *and* $(p_2 + p_3)^2 = t$.

*There are* 16 *possible sectors, labelled by*

$$\begin{aligned} \{\boldsymbol{\Sigma}\} = \{&\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \\ &\{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}. \end{aligned} \tag{7.84}$$

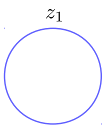
*The counting of critical points associated to each regularized twist, is given by*

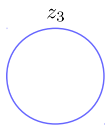
$$\{\nu_\Sigma\} = \{0,1,0,1,0,1,3,1,1,1,1,3,3,3,3,7\}. \tag{7.85}$$

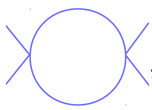
*The basis elements are then*

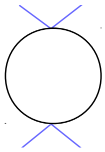
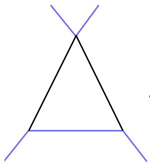
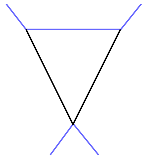
$$\begin{aligned} \langle e_1^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1} \right|, \quad \langle e_2^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_3} \right|, \quad \langle e_3^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_3} \right|, \quad \langle e_4^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_2 z_4} \right|, \\ \langle e_5^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2 z_4} \right|, \quad \langle e_6^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_2 z_3 z_4} \right|, \quad \langle e_7^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2 z_3 z_4} \right|, \end{aligned} \tag{7.86}$$

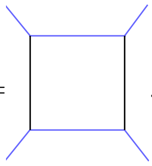
*associated with*

$$\langle e_1^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_1 =$$


$$, \quad \langle e_2^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_2 =$$


$$, \quad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_3 =$$

$$,$$

$$\langle e_4^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_4 =$$

$$, \quad \langle e_5^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_5 =$$

$$, \quad \langle e_6^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_6 =$$

$$, \tag{7.87}$$

$$\langle e_7^{(\mathbf{4})}| \rightsquigarrow \mathcal{J}_7 =$$

$$.$$

*Let us consider the following decomposition*

$$I_{1,2,1,1} = \sum_{i=1}^{7} c_i \, \mathcal{J}_i, \tag{7.88}$$

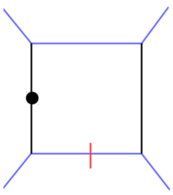
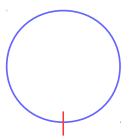
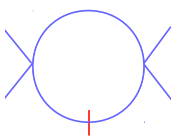
*where the twisted co-cycle associated to* $I_{1,2,1,1}$ *is*[8]

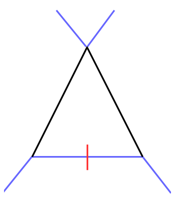
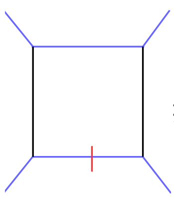
$$\langle \varphi^{(\mathbf{4})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2^2 z_3 z_4} \right| = \left\langle \partial_2 \log u(\mathbf{z}) \cdot \frac{d\mathbf{z}}{z_1 z_2 z_3 z_4} \right|. \tag{7.89}$$

*Eq. (7.88) can be reconstructed from the following spanning set of cuts*

$$\{\mathcal{S}\} = \{\mathrm{Cut}_1, \mathrm{Cut}_3, \mathrm{Cut}_{2,4}\}. \tag{7.90}$$

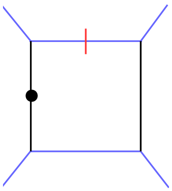
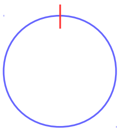
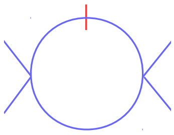
*Diagrammaticaly on* $\mathrm{Cut}_1$ *we have*

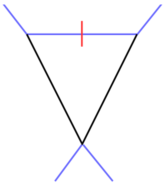
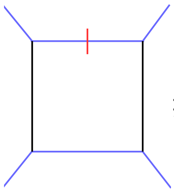

$$= c_1$$

$$+ \, c_3$$


$$c_5$$

$$+ \, c_7$$

$$; \tag{7.91}$$

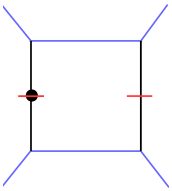
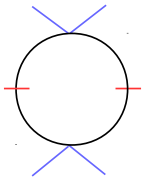
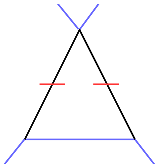
[8]See footnote (5).

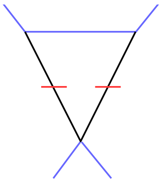
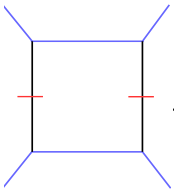
*while on* $\text{Cut}_3$

 $=c_2$  $+\, c_3$ 

$c_6$  $+\, c_7$  ; (7.92)

*finally* $\text{Cut}_{2,4}$

 $=c_4$  $+\, c_5$ 

$c_6$  $+\, c_7$  . (7.93)

• *We consider first* $\text{Cut}_1$.

*On this cut we have the following twist*

$$u(\mathbf{z}) \rightsquigarrow u_{\circlearrowleft}(z_2, z_3, z_4) = u(\mathbf{z})\big|_{z_1=0}, \tag{7.94}$$

*with its regularized version*

$$u_{\text{reg},\circlearrowleft}(z_2, z_3, z_4) = z_2^{\rho_2} z_3^{\rho_3} z_4^{\rho_4} u_{\circlearrowleft}(z_2, z_3, z_4). \tag{7.95}$$

*Furthermore we have the following identifications*

$$\begin{aligned} \langle e_1^{(\mathbf{4})}| \rightsquigarrow \langle e_{1,\circlearrowleft}^{(234)}| &= \langle 1 \cdot dz_2 \wedge dz_3 \wedge dz_4|, \quad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \langle e_{3,\circlearrowleft}^{(234)}| = \left\langle \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_3} \right|, \\ \langle e_5^{(\mathbf{4})}| \rightsquigarrow \langle e_{5,\circlearrowleft}^{(234)}| &= \left\langle \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_2 z_3} \right|, \quad \langle e_7^{(\mathbf{4})}| \rightsquigarrow \langle e_{7,\circlearrowleft}^{(234)}| = \left\langle \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_2 z_3 z_4} \right|, \end{aligned} \tag{7.96}$$

*and*

$$\langle \varphi^{(\mathbf{4})}| \rightsquigarrow \langle \varphi_{\circlearrowleft}^{(234)}| = \left\langle \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_2^2 z_3 z_4} \right|. \tag{7.97}$$

*We have to assign the internal basis w.r.t.* $z_2$ *and w.r.t.* $(z_2, z_3)$*; the counting of critical points is*

$$\nu_{(2)} = 2, \qquad \nu_{(23)} = 4. \tag{7.98}$$

*We employ*

$$(\langle e_1^{(2)}|, \langle e_2^{(2)}|) = (\langle 1 \cdot dz_2|, \langle d\log z_2|), \tag{7.99}$$

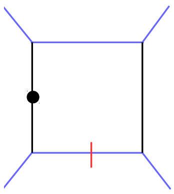
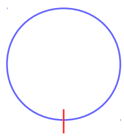
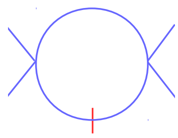
*and*

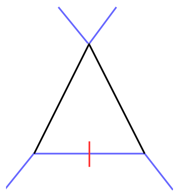
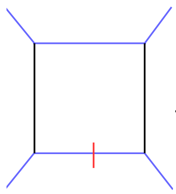
$$(\langle e_1^{(23)}|, \langle e_2^{(23)}|, \langle e_3^{(23)}|, \langle e_4^{(23)}|) = (\langle 1 \cdot dz_2 \wedge dz_3|, \langle d\log z_2 \wedge dz_3|, \langle dz_2 \wedge d\log z_3|, \langle d\log z_2 \wedge d\log z_3|); \tag{7.100}$$

*we will consider* $h_i^{(\bullet)} = e_i^{(\bullet)}$ *for all* $i$*, and* $\bullet = 2, 23$.

*It turns out that with the above-mentioned choices of internal bases* $\hat{\Omega}^{(4)}$ *has just simple poles, and so we can apply at this stage the machinery introduced in section 6.3.*

*In the* $\rho \to 0$ *limit the coefficients read*

 $= \dfrac{(d-2)(d(s-8)-4(s-9))}{4(d-6)(s-4)^2 t}$  $+ \dfrac{4(d-5)(d-3)}{(d-6)(s-4)^2 t}$ 

$+\,0$  $- \dfrac{d-5}{t}$  . (7.101)

• *We move now to* $\mathrm{Cut}_3$.
*We have*

$$u(\mathbf{z}) \rightsquigarrow u_{\circlearrowleft}(z_1, z_2, z_4) = u(\mathbf{z})\big|_{z_3=0}, \tag{7.102}$$

*and its regularized version*

$$u_{\text{reg},\circlearrowleft}(z_1, z_2, z_4) = z_1^{\rho_1} z_2^{\rho_2} z_4^{\rho_4} u_{\circlearrowleft}(z_1, z_2, z_4). \tag{7.103}$$

*Furthermore we have the following identifications*

$$\begin{aligned} &\langle e_2^{(\mathbf{4})}| \rightsquigarrow \langle e_{1,\circlearrowleft}^{(124)}| = \langle 1 \cdot dz_1 \wedge dz_2 \wedge dz_4|, \quad \langle e_3^{(\mathbf{4})}| \rightsquigarrow \langle e_{3,\circlearrowleft}^{(124)}| = \left\langle \frac{dz_1 \wedge dz_2 \wedge dz_4}{z_1}\right|, \\ &\langle e_6^{(\mathbf{4})}| \rightsquigarrow \langle e_{6,\circlearrowleft}^{(124)}| = \left\langle \frac{dz_1 \wedge dz_2 \wedge dz_4}{z_2 z_4}\right|, \quad \langle e_7^{(\mathbf{4})}| \rightsquigarrow \langle e_{7,\circlearrowleft}^{(124)}| = \left\langle \frac{dz_1 \wedge dz_2 \wedge dz_4}{z_1 z_2 z_4}\right|, \end{aligned} \tag{7.104}$$

*and*

$$\langle \varphi^{(\mathbf{4})}| \rightsquigarrow \langle \varphi^{(124)}| = \left\langle \frac{dz_1 \wedge dz_2 \wedge z_4}{z_1 z_2^2 z_4}\right|. \tag{7.105}$$

*For what concerns the internal basis elements, we have*

$$\nu_{(1)} = 2, \quad \nu_{(12)} = 4. \tag{7.106}$$

*and the basis elements read*

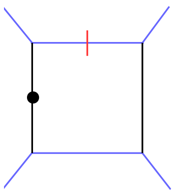
$$(\langle e_1^{(1)}|, \langle e_2^{(1)}|) = (\langle 1 \cdot dz_1|, \langle d\log z_1|), \tag{7.107}$$

*and*

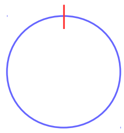
$$(\langle e_1^{(12)}|, \langle e_2^{(12)}|, \langle e_3^{(12)}|, \langle e_4^{(12)}|) = (\langle 1 \cdot dz_1 \wedge dz_2|, \langle d\log z_1 \wedge dz_2|, \langle dz_1 \wedge d\log z_2|, \langle d\log z_1 \wedge d\log z_2|). \tag{7.108}$$

*We will consider* $h_i^{(\bullet)} = e_i^{(\bullet)}$*, for all* $i$ *and* $\bullet = 1, 12$.

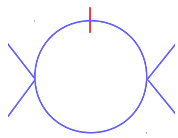
*The coefficients in the* $\rho \to 0$ *limit read*



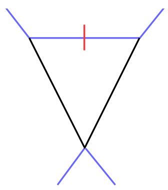
$$= \frac{(d-2)(d(s-8)-4(s-9))}{4(d-6)(s-4)^2 t}$$



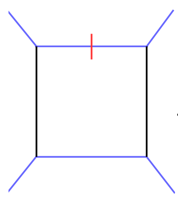
$$+ \frac{4(d-5)(d-3)}{(d-6)(s-4)^2 t}$$



$$+0$$



$$- \frac{d-5}{t}$$



$$. \tag{7.109}$$

• *Finally we consider* $\mathrm{Cut}_{2,4}$.
*We have*

$$u(\mathbf{z}) \rightsquigarrow u_{\circlearrowleft}(z_1, z_3) = u(\mathbf{z})\big|_{z_1 = z_3 = 0}, \tag{7.110}$$

*and its regularized version*

$$u_{\mathrm{reg},\circlearrowleft}(z_1, z_3) = z_1^{\rho_1} z_3^{\rho_3} u_{\circlearrowleft}(z_1, z_3). \tag{7.111}$$

*The following identifications hold*

$$\begin{aligned} \langle e_4^{(\mathbf{4})}| \rightsquigarrow \langle e_{4,\circlearrowleft}^{(13)}| &= \langle 1 \cdot dz_1 \wedge dz_3|, \quad \langle e_5^{(\mathbf{4})}| \rightsquigarrow \langle e_{5,\circlearrowleft}^{(13)}| = \left\langle \frac{dz_1 \wedge dz_3}{z_1} \right| \\ \langle e_6^{(\mathbf{4})}| \rightsquigarrow \langle e_{6,\circlearrowleft}^{(13)}| &= \left\langle \frac{dz_1 \wedge dz_3}{z_3} \right|, \quad e_7^{(\mathbf{4})}| \rightsquigarrow \langle e_{7,\circlearrowleft}^{(13)}| = \left\langle \frac{dz_1 \wedge dz_3}{z_1 z_3} \right|, \end{aligned} \tag{7.112}$$

*and*

$$\langle \varphi^{(\mathbf{4})}| \rightsquigarrow \langle \varphi_{\circlearrowleft}^{(13)}| = \left\langle \left(\partial_2 \log u(\mathbf{z})\right)\big|_{z_2=z_4=0} \cdot \frac{dz_1 \wedge dz_3}{z_1 z_3} \right|. \tag{7.113}$$

*Choosing $z_1$ as inner variable, we have*

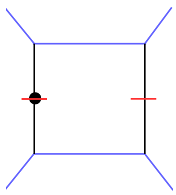
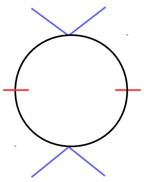
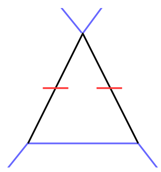
$$\nu_{(1)} = 2, \tag{7.114}$$

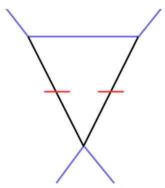
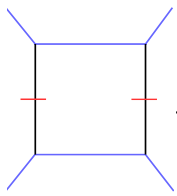
*and*

$$(\langle e_1^{(\mathbf{1})}|, \langle e_2^{(\mathbf{1})}|) = (\langle 1 \cdot dz_1|, \langle d \log z_1|). \tag{7.115}$$

*We will consider $h_i^{(\mathbf{1})} = e_i^{(1)}$, for $i = 1, 2$.*

*The coefficients in the $\rho \to 0$ limit read*

 $= 0$  $+\, 0$ 

$0$  $- \dfrac{d-5}{t}$  . (7.116)

*Combining eqs. (7.101,7.109,7.116) we obtain the full decomposition eq. (7.88), which is in agreement with* FiniteFlow *&* LiteRed.

# 7.4

# Feynman Integrals Reduction via Intersection Numbers and Secondary Equation Approach

In section 6.4 we introduced another method for the evaluation of intersection numbers. We discuss here how it can be combined with the master decomposition formula eq. (5.32), in order to obtain reduction onto MIs.
Let us assume we want to decompose $\langle \varphi^{(\mathbf{n})}|$ in terms of $(\langle e_1^{(\mathbf{n})}|, \ldots, \langle e_\nu^{(\mathbf{n})}|)$. Having introduced the dual basis $(|h_1^{(\mathbf{n})}\rangle, \ldots, |h_\nu^{(\mathbf{n})}\rangle)$, then eq. (5.32) tells us that *two* different sets of intersection

numbers are required

- $\mathrm{set}_1 \rightsquigarrow \langle e_i^{(\mathbf{n})} | h_j^{(\mathbf{n})} \rangle = \mathbf{C}_{ij}$ for $1 \leq i, j \leq \nu$;
- $\mathrm{set}_2 \rightsquigarrow \langle \varphi^{(\mathbf{n})} | h_j^{(\mathbf{n})} \rangle$ for $1 \leq j \leq \nu$.

On the one hand, $\mathrm{set}_1$ can be determined—up to the ovar-all constant $\kappa$—via the procedure outlined above in section 6.4. So, we assume:

$$\mathbf{C} = \kappa\, \mathbf{C}_{\mathrm{rat}}, \tag{7.117}$$

where $\mathbf{C}_{\mathrm{rat}}$ is known at this stage.

On the other hand $\mathrm{set}_2$ can be determined thanks to the same procedure, considering auxiliary bases $(\langle e_1^{(\mathbf{n})\,\mathrm{aux}}|, \ldots, e_\nu^{(\mathbf{n})\,\mathrm{aux}}|) = (\langle e_1^{(\mathbf{n})}|, \ldots, \langle e_{\nu-1}^{(\mathbf{n})}|, \langle \varphi^{(\mathbf{n})}|)$ and $(|h_1^{(\mathbf{n})\,\mathrm{aux}}\rangle, \ldots, |h_\nu^{(\mathbf{n})\,\mathrm{aux}}\rangle) = (|h_1^{(\mathbf{n})}\rangle, \ldots, |h_\nu^{(\mathbf{n})}\rangle)$. So we have

$$\mathbf{C}^{\mathrm{aux}} = \kappa^{\mathrm{aux}}\, \mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}} \tag{7.118}$$

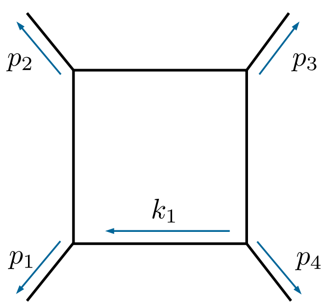
where $\mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}}$ is, at this stage, known. It is clear that $\mathrm{set}_2$ corresponds to the last of row $\mathbf{C}^{\mathrm{aux}}$.

The $(\nu-1)\times(\nu-1)$ left sub-blocks of $\mathbf{C}$ and $\mathbf{C}^{\mathrm{aux}}$ must coincide; given this fact eq. (5.32) implies

$$\mathbf{C}^{\mathrm{aux}} \cdot \mathbf{C}^{-1} = \left(\begin{array}{ccc|c} & & & 0 \\ & \mathbb{1}_{\nu-1} & & \vdots \\ & & & 0 \\ \hline c_1 & c_2 & \ldots & c_\nu \end{array}\right) = \frac{\kappa^{\mathrm{aux}}}{\kappa}\, \mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}} \cdot \mathbf{C}_{\mathrm{Rat}}. \tag{7.119}$$

Therefore the unknowns $\kappa$ and $\kappa^{\mathrm{aux}}$ enter in the coefficients of the decomposition just through the ratio $\kappa^{\mathrm{aux}}/\kappa$. This ratio can be fixed from the $(\nu-1)\times(\nu-1)$ left sub-blocks of eq. (7.119). Explicitly the, say, $(1,1)$ entry of $\mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}} \cdot \mathbf{C}_{\mathrm{Rat}}$ fixes its inverse $\kappa/\kappa^{\mathrm{aux}}$.

**Example.** *One loop massless box in the Lee-Pomeransky representation via the Secondary Equation. Let us consider once again the integral family associated to the following graph*




(7.120)

*The denominator are chosen as*

$$\begin{aligned} z_1 &= D_1 = -k_1^2, & z_2 &= D_2 = -(k_1-p_1)^2, \\ z_3 &= D_3 = -(k_1-p_1-p_2)^2, & z_4 &= D_4 = -(k_1-p_1-p_2-p_3)^2; \end{aligned} \tag{7.121}$$

*the kinematics is* $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$, $s = (p_1+p_2)^2$ *and* $t = (p_2+p_3)^2$.

*In this case we rely on a regularization of Lee-Pomeransky representation. In fact, before applying the machinery introduced in section 6.4, we have to consider some preliminary consideration in order to cast the integrals in a suitable form*[9]*. Let us consider the the integrand associated to* $I_{a_1,a_2,a_3,a_4}$*, namely*

$$I_{a_1,a_2,a_3,a_4} \rightsquigarrow \left(\prod_{i=1}^{4} z_i^{\rho+a_i}\right) (z_1+z_2+z_3+z_4-s\, z_1 z_3 - t\, z_2 z_4)^{-d/2} \frac{d\mathbf{z}}{z_1 z_2 z_3 z_4}. \tag{7.122}$$

*Within this context, we find convenient to consider the rescaling*

$$z_i \to \frac{z_i}{(-s)}, \qquad 1 \le i \le 4, \qquad d = 4-2\epsilon; \tag{7.123}$$

*Under eq. (7.123), eq. (7.122) becomes*

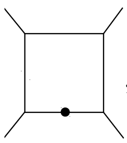
$$\left(\prod_{i=1}^{4} z_i^{\rho}\right) (z_1+z_2+z_3+z_4+z_1 z_3+x z_2 z_4)^{\epsilon} \, (-s)^{2-4\rho-|a|} z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4} \frac{d\mathbf{z}}{G^2 \, z_1 z_2 z_3 z_4}, \tag{7.124}$$

*with* $x = (-t)/(-s)$.

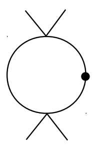
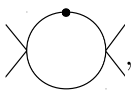
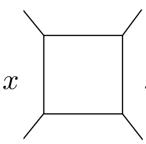
*In this case we have*

$$u(\mathbf{z}) = \left(\prod_{i=1}^{4} z_i^{\rho}\right) (z_1+z_2+z_3+z_4+z_1 z_3+x z_2 z_4.)^{\epsilon} \tag{7.125}$$

*Let us consider the the decomposition of*

$\mathcal{I} = \epsilon(-s)^{3+\epsilon}\, x$  , (7.126)

*in the following basis of MIs*

$\mathcal{J}_1 = (-s)^{1+\epsilon}\, x$  $\mathcal{J}_2 = (-s)^{1+\epsilon}$  , $\mathcal{J}_3 = (-s)^{2+\epsilon}\, \epsilon\, x$  . (7.127)

[9] This regularization is often referred to as "generalized Feynman integrlas", see e.g. section 5 of [36] for a detailed discussion.

*The corresponding twisted co-cycles reads*

$$\langle\varphi^{(\mathbf{4})}| = \frac{\epsilon\,(-s)^{-4\rho}\,z\,\Gamma(2-\epsilon)}{\Gamma(1-2\epsilon-4\rho)\Gamma(1+\rho)^3\Gamma(2+\rho)}\left\langle \frac{z_1 d\mathbf{z}}{G^2}\right|, \tag{7.128}$$

*and*

$$\begin{aligned}
\langle e_1^{(\mathbf{4})}| &= \frac{(-s)^{-4\rho}\,z\,\Gamma(2-\epsilon)}{\Gamma(1-2\epsilon-4\rho)\Gamma(\rho)^2\Gamma(1+\rho)\Gamma(2+\rho)}\left\langle \frac{z_4}{z_1 z_3}\frac{d\mathbf{z}}{G^2}\right|,\\
\langle e_2^{(\mathbf{4})}| &= \frac{(-s)^{-4\rho}\,1\,\Gamma(2-\epsilon)}{\Gamma(1-2\epsilon-4\rho)\Gamma(\rho)^2\Gamma(1+\rho)\Gamma(2+\rho)}\left\langle \frac{z_3}{z_2 z_4}\frac{d\mathbf{z}}{G^2}\right|,\\
\langle e_3^{(\mathbf{4})}| &= \frac{\epsilon\,(-s)^{-4\rho}\,z\,\Gamma(2-\epsilon)}{\Gamma(-2\epsilon-4\rho)\Gamma(1+\rho)^2}\left\langle \frac{d\mathbf{z}}{G^2}\right|.
\end{aligned} \tag{7.129}$$

*We will consider* $h_i^{(\mathbf{4})} = e_i^{(\mathbf{4})}\big|_{(\epsilon,\rho)\to-(\epsilon,\rho)}$, *for* $i = 1, 2, 3$.

*By means of the algorithm presented in* [36], *we obtain*

$$\mathbf{\Omega} = \begin{pmatrix}
-\frac{\rho^2(12x+11)+(x+1)\epsilon^2+7\rho(x+1)\epsilon}{x(x+1)(3\rho+\epsilon)} & -\frac{\rho^2}{(x+1)(3\rho+\epsilon)} & \frac{\rho^2(\rho(x+2)+\epsilon)}{2(\rho+1)x(x+1)\epsilon(3\rho+\epsilon)}\\
\frac{\rho^2}{x(x+1)(3\rho+\epsilon)} & -\frac{\rho^2}{(x+1)(3\rho+\epsilon)} & -\frac{\rho^2(\rho+x(2\rho+\epsilon))}{2(\rho+1)x(x+1)\epsilon(3\rho+\epsilon)}\\
-\frac{2(\rho+1)\epsilon(2\rho+\epsilon)}{x(x+1)(3\rho+\epsilon)} & \frac{2(\rho+1)\epsilon(2\rho+\epsilon)}{(x+1)(3\rho+\epsilon)} & -\frac{\rho^2(5x+7)+\rho(2x+5)\epsilon+\epsilon^2}{x(x+1)(3\rho+\epsilon)}
\end{pmatrix}, \tag{7.130}$$

*and*

$$\mathbf{\Omega}^\vee = \mathbf{\Omega}\big|_{(\epsilon,\rho)\to-(\epsilon,\rho)}. \tag{7.131}$$

*The rational solution to the Secondary Equation* (6.111) *is* $\mathbf{C} = \kappa\,\mathbf{C}_{\mathrm{Rat}}$, *where* $\mathbf{C}_{\mathrm{Rat}}$, *obtained via* [217], *is*

$$\mathbf{C}_{\mathrm{Rat}} = \begin{pmatrix}
-\frac{(2\rho+\epsilon)(4\rho+\epsilon)}{\rho\epsilon} & \frac{\rho}{\epsilon} & -2(\rho-1)\\
\frac{\rho}{\epsilon} & -\frac{(2\rho+\epsilon)(4\rho+\epsilon)}{\rho\epsilon} & -2(\rho-1)\\
2(\rho+1) & 2(\rho+1) & -\frac{4(\rho-1)(\rho+1)\epsilon\left(10\rho^2+\epsilon^2+6\rho\epsilon\right)}{\rho^3}
\end{pmatrix}. \tag{7.132}$$

*Moving then to the auxiliary basis, we input*

$$\mathbf{\Omega}_{\mathrm{aux}} = \begin{pmatrix}
- & \mathbf{\Omega}^\top_{1,\mathrm{aux}} & -\\
- & \mathbf{\Omega}^\top_{2,\mathrm{aux}} & -\\
- & \mathbf{\Omega}^\top_{3,\mathrm{aux}} & -
\end{pmatrix}, \tag{7.133}$$

*where the explicit expressions for* $\mathbf{\Omega}^\top_{1,2,3,\mathrm{aux}}$, *once again obtained through* [36], *are too lengthy to be reported here.*
*A rational solution to the (auxiliary) secondary equation is* $\mathbf{C}^{\mathrm{aux}} = \kappa^{\mathrm{aux}}\,\mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}}$ *with*

$$\mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat}} = \begin{pmatrix}
-\frac{(2\rho+\epsilon)(4\rho+\epsilon)}{\rho\epsilon} & \frac{\rho}{\epsilon} & -2(\rho-1)\\
\frac{\rho}{\epsilon} & -\frac{(2\rho+\epsilon)(4\rho+\epsilon)}{\rho\epsilon} & -2(\rho-1)\\
\frac{2(3\rho+\epsilon+1)(4\rho+2\epsilon+1)\left(6\rho^2+2\rho^2x+\rho x+\rho x\epsilon+\epsilon^2+5\rho\epsilon\right)}{\rho x(2\rho+\epsilon+1)^2} & \frac{2(4\rho+2\epsilon+1)\left(12\rho^2+5\rho+2\epsilon^2+10\rho\epsilon+2\epsilon+1\right)}{(2\rho+\epsilon+1)^2} & \mathbf{C}^{\mathrm{aux}}_{\mathrm{Rat},33}
\end{pmatrix}, \tag{7.134}$$

*and*

$$\begin{aligned}\mathbf{C}^{\text{aux}}_{\text{Rat},33} = &-\frac{4(\rho-1)\epsilon(4\rho+2\epsilon+1)}{\rho^3 x(2\rho+\epsilon+1)^2}\,\big(18\rho^4+6\rho^3+66\rho^4 x+50\rho^3 x+10\rho^2 x \\ &+x\epsilon^4+11\rho x\epsilon^3+2x\epsilon^3+47\rho^2 x\epsilon^2+17\rho x\epsilon^2+x\epsilon^2+91\rho^3 x\epsilon+50\rho^2 x\epsilon \\ &+6\rho x\epsilon+\rho\epsilon^3+8\rho^2\epsilon^2+\rho\epsilon^2+21\rho^3\epsilon+5\rho^2\epsilon\big)\,. \end{aligned} \tag{7.135}$$

*Finally, we evaluate the product* $\mathbf{C}^{\text{aux}}\cdot\mathbf{C}^{-1}$*, which reads:*

$$
\mathbf{C}^{\mathrm{aux}} \cdot \mathbf{C}^{-1} = \frac{\kappa^{\mathrm{aux}}}{\kappa} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2\epsilon(2\rho+\epsilon)(3\rho+\epsilon+1)(4\rho+2\epsilon+1)}{x(2\rho+\epsilon+1)^2(3\rho+\epsilon)} & -\frac{2\rho\epsilon(2\rho+\epsilon)(4\rho+2\epsilon+1)}{(2\rho+\epsilon+1)^2(3\rho+\epsilon)} & \frac{(4\rho+2\epsilon+1)\left(x\left(\rho\left(20\rho^2+15\rho+3\right)+\epsilon^3+(8\rho+2)\epsilon^2+\left(22\rho^2+11\rho+1\right)\epsilon\right)+\rho\left(2\rho(3\rho+1)+\epsilon^2+5\rho\epsilon+\epsilon\right)\right)}{(\rho+1)x(2\rho+\epsilon+1)^2(3\rho+\epsilon)} \end{pmatrix}. \tag{7.136}
$$

*Matching eq (7.136) onto eq. (7.119) we infer* $\kappa/\kappa^{\mathrm{aux}} = 1$. *Finally, considering the last row of eq. (7.136)* *in the $\rho \to 0$ limit, yields the following*

$$\mathcal{I} = -\frac{2\epsilon(1+2\epsilon)}{x(1+\epsilon)}\,\mathcal{J}_1 + 0 \cdot \mathcal{J}_2 + (1+2\epsilon)\,\mathcal{J}_3. \tag{7.137}$$

*Eq. (7.4) cam be verified to be consistent with the reduction provided by* LiteRed.

# 8

# Conclusions and Outlook

Feynman Integrals are pervasive objects in (Quantum) Field Theory; taming their complexity is of primary importance in order to meet the increasing precision required by particle physics experiments and, as it emerged more recently, by the program dedicated to gravitational wave detection.
In order to tackle multi-loop Feynman Integrals, a very convenient−if not mandatory−road map consists in identifying a minimal set of independent building blocks, known as Master Integrals; this task is traditionally accomplished via the solution of a large and computationally challenging linear system of Integration by Parts identities. Once Master Integrals have been identified, we are left with the problem of their evaluation. In this respect, a very powerful approach is the method of canonical differential equations.

In this work we have shown how these techniques can be applied to some (perhaps less conventional to collider physicists) models relevant for detection of Dark Matter particles, involving 2 loop Feynman Integrals and different massive particles both in internal and external states. In particular, the system of differential equations was cast in canonical form thanks to the Dyson/-Magnus exponential. We also computed the Master Integrals ab-initio in several kinematic limits; this was important both for phenomenological aspects, and for comparing our results with the literature. This study confirms once more the flexibility and the wide range of applicability of "Feynman Calculus".

Despite the success over the past years, it is fair to say that the deep mathematical structures controlling multi-loop scattering processes are still being uncovered. Recently, it emerged that the framework of twisted (Co)Homology−originally developed in the mathematical literature in the study of hypergeometric-like integrals−is able to capture several important aspects of Feynman Integrals in Dimensional Regularization.
In this work we have reviewed the basic aspects of this theory. Starting from the univariate case, we introduced the notion of the twisted homology group and the twisted co-homology

group, as well as the corresponding elements, namely twisted cycles and twisted co-cycles. We have discussed how the dimension of the above-mentioned groups can be interpreted in an elegant way in terms of the Euler characteristic. We have discussed how it is possible to build the homology intersection number, i.e. a pairing among elements of the twisted homology group and its dual, as well as co-homology intersection number, i.e. a pairing among elements of the twisted co-homology group and its dual. Finally, we have seen how an integral is interpreted in a natural way as a pairing among a twisted co-cycle and a twisted cycle (and, in the same spirit, a dual integral is a pairing among a dual twisted cycle and a dual twisted co-cycle).
We have studied how linear relations among integrals (admitting an univariate representation) −in particular the reduction of Feynman Integrals in Baikov representation on the Maximal Cut−can be obtained via co-homology intersection numbers and the master decomposition formula. Co-homology intersection numbers act as a "scalar product" in the space of Feynman Integrals. We have briefly touched upon quadratic relations for hypergeometric functions, and in particular how they arise from Twisted Riemann's Period Relations.

We have discussed the twisted co-homology group and the intersection number for twisted co-cycles in the multivariate case. We have considered in detail a recursive algorithm for the evaluation of multivariate co-homology intersection numbers, as well as a proposal for its optimization. Furthermore, we have also seen how co-homology intersection numbers arise as solutions of a suitable system of differential equations: the so-called Secondary Equation.
We have applied these tools in order to derive contiguity relations for hypergeometric integrals; we have also derived full reductions for simple Feynman integrals, highlighting their peculiarities and difficulties compared to the hypoergeometric case.

We hope that our work could help in charting the road towards more refined analysis and more elaborated applications, but we believe that our studies constitute an important starting point in this direction. We are confident that this line of research is of interest for both physicists and mathematicians, and could represent a fruitful and stimulating meeting point for the two communities.
Being far from the last word on the topic, we list here some possible items which are left for future investigations.

**The Role of the representation of Feynman Integrals.** In this work we employed mostly the Baikov representation since the link with the world of twisted (Co)Homology is more transparent. Nevertheless this choice is by no means mandatory. It is known that this representation is not always "faithful"−see e.g. the analysis of [30] section 11.3, where some coalescence among critical points and singular points of the (internal) twist is described. The study can be performed in different representations, e.g. Lee-Pomeransky representation or even in the momentum space representation; it would be interesting to analyze the same physical example, working with different representations and consider the various peculiarities of each of those.
It should also be noticed that the twist associated to Feynman Integrals is not "arbitrary" nor

"completely general", but they have a certain structure which is—so far—ignored. For example, the twist associated to Baikov representation emerges from a Gram determinant; since this fact played an important role in the generation of syzygy equations in the context of Integration by Parts (see section 2.5), it is not hopeless to expect that this simple consideration could offer some insights also on the (Co)Homology side.

**The Counting of Master Integrals via (Co)Homological methods.** There are cases in which e.g. the counting of critical points described in section 6.2 does not match the expectation and the result found by standard Integration by Parts identities and public implementations of the Laporta Algorithm. In this regard, some mathematically solid analysis is needed. The work of [203] represents an encouraging starting point. Generally speaking, experience shows that issues emerge while considering "non-generic" kinematics (by non-generic we mean e.g. equal internal masses; this is in contrast to generic, i.e. different internal masses). It would be interesting to understand why and how subtleties emerge when some generic kinematics reduces to a non-generic one. In this respect, the theory of *restriction* [222] seems to be an appropriate framework to tackle this kind of considerations.

**The Role of Relative Twisted (Co)Homology.** In this work, in order to apply the framework of Twisted (Co)Homology to Feynman Integrals, we employed a suitable regularization. This technical aspect was important in order to perform explicit calculations beyond the maximal cut and obtain successful concrete examples. Nevertheless, as we stressed in the main text, this has obvious computational disadvantages. The freshly proposed framework of relative twisted (Co)Homology [220] seems to be tailored to Feynman Integrals [39, 40]. Loosely speaking, this variant of the theory allows us to avoid the above-mentioned regularization when treating Feynman Integrals (while Dimensional Regularization is still employed). Some concrete calculations within this framework are underway, giving encouraging results.

**The Role of Bases and the Evaluation of Intersection Numbers.** The role of dual bases is so far—to a certain extent—obscure. It would be for sure desirable to identify suitable choice(s) of dual bases "orthogonal" to the ones associated to Master Integrals (i.e. in such a way that $\mathbf{C} = \mathbb{1}$). Such an a priori knowledge would drastically reduce the amount of computations needed to achieve a reduction. Once again, in this respect, relative twisted Co-Homology seems elucidating.
Moreover, considering for concreteness the recursive algorithm of section 6, it would be important to identify a criterion which leads to internal twists which has "good properties"— (cf. also appendix A). More generally, even if intersection numbers themselves are independent of the choice of (dual) internal bases, intermediate results are not and may become even more involved than the final one. Beside this it would be interesting and beneficial to tandem the algorithms of sections 6.3,6.4 with Functional Reconstruction techniques over Finite Fields. Finally, other strategies—not presented here—for the evaluation of intersection numbers are still under development [31] and so we are not at the final point of the search even in this respect.

Broadly speaking Feynman Integrals represent a topic on which several branches of Mathematics converge; it would not be a surprise if some other techniques e.g. the one reviewed in section 2.5, turn out to be beneficial even in the evaluation of intersection numbers. This would represent an opportunity to build bridges among different sub-field of Mathematics which are not explored yet.

**New Insights on Feynman Integrals.** As we have stressed several times, twisted (Co)Homology offers a comprehensive framework for the study of Feynman Integrals in Dimensional Regularization. It would be interesting to explore quadratic relations among (maximally cut) Feynman Integrals, mimicking the ones emerging for hypergeometric functions via Twisted Riemann's Period Relations. More generally, it seems plausible that twisted (Co)Homology could offer new understanding on the relations fulfilled by the type of functions Feynman Integrals evaluate to.
Even if we did not discuss it here explicitly, it should be clear that intersection numbers control also differential equations fulfilled by MIs, which can be obtained by simple projection as well. Therefore, on a more speculative note, it would be interesting to explore whether twisted Co-Homology could offer some insights, or some a-priori knowledge, on the structure−i.e. some particular $\epsilon$ or kinematic dependence−of the coefficients appearing in the differential equation itself. This knowledge could be useful in order to e.g. predict the alphabet of the differential equation or to help function reconstruction techniques, narrowing down the function form of the objects these methods aim to reconstruct.

# A

# Further Details on Multivariate Intersection Number

We give here some details about the algorithm for the multivariate intersection number (cf. section 6.2), and in particular on the solution of the system of differential equations.
For ease of notation, comparing to the main text, we will consider the following replacements

$$\hat{\varphi}^{(n)} \to \hat{\varphi}, \quad \hat{\varphi}_{\mathbf{C}}^{(n)\vee} \to \hat{\varphi}^{\vee}, \quad \psi_{x_i} \to \psi, \quad \hat{\Omega}^{(n)\top} \to \hat{\Omega}^{\top}, \quad \nu_{(\mathbf{n-1})} \to \nu. \tag{A.1}$$

Assuming local coordinates, say $y$, around any given pole $x_i$, the algorithm includes the following steps: $i$) finding the series solution of

$$\partial_y \psi + \hat{\Omega}^{\top} \cdot \psi = \hat{\varphi}; \tag{A.2}$$

and $ii$) computing the residue

$$\text{Res}_{y=0} \left( \psi \cdot \varphi^{\vee} \right). \tag{A.3}$$

We assume the following expansion:

$$\hat{\Omega} = \sum_{i=-1}^{\infty} \hat{\Omega}_i \, y^i, \tag{A.4}$$

and

$$\hat{\varphi} = \sum_{i=\min_{\varphi}}^{\infty} \hat{\varphi}_i \, y^i, \qquad \hat{\varphi}^{\vee} = \sum_{i=\min_{\varphi^{\vee}}}^{\infty} \hat{\varphi}_i^{\vee} \, y^i. \tag{A.5}$$

We look for a solution of eq. (A.2) of the following form

$$\psi = \sum_{j=\text{Min}}^{\text{Max}} \psi_i \, y^i, \tag{A.6}$$

where each $\psi_i$ is a vector of $\nu$ unknowns

$$\psi_i = (\alpha_{i,1}, \ldots, \alpha_{i,\nu})^\top . \tag{A.7}$$

We will set[1] $\mathrm{Min} = \mathtt{Min}\,(\min_\varphi, -1) + 1$; $\mathrm{Max}$ is determined requiring that we have to fix all the coefficients that could give a non vanishing contribution to the residue (see eq. (A.3); therefore we employ $\mathrm{Max} = \max_\psi = -\min_{\varphi^\vee} - 1$. Plugging eq. (A.6) into eq. (A.2) and collecting the terms order by order in $y$ we are lead to the following

$$\begin{pmatrix} \Omega_{-1}^\top + \mathrm{Min}\cdot\mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} \\ \Omega_0^\top & \Omega_{-1}^\top + (\mathrm{Min}+1)\,\mathbb{1} & \cdots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{\mathrm{Max}-\mathrm{Min}-1}^\top & \Omega_{\mathrm{Max}-\mathrm{Min}-2}^\top & \cdots & \Omega_{-1}^\top + \mathrm{Max}\cdot\mathbb{1} \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{Min}} \\ \psi_{\mathrm{Min}+1} \\ \vdots \\ \psi_{\mathrm{Max}} \end{pmatrix} = \begin{pmatrix} \varphi_{\mathrm{Min}-1} \\ \varphi_{\mathrm{Min}} \\ \vdots \\ \varphi_{\mathrm{Max}-1} \end{pmatrix} . \tag{A.8}$$

So, an inspection of eq. (A.8) tells us that if the eigenvalues of $\Omega_{-1}$ are not integer, the solution of eq. (A.2) is guaranteed to exist to all orders in $y$, and the unknowns can be uniquely determined in cascade since the system is block triangular. Nevertheless, we find this condition often to restrictive in practice, i.e. this condition is not always fulfilled by FIs; nevertheless eq. (A.8) indicates precisely that not all the orders contributes.

So, we follow a more pragmatic approach[2]. Without any consideration on the spectrum of $\Omega_{-1}$, we consider a slight modification of eq. (A.8). In fact, rather than considering the two steps $i.$) and $ii.$) separately, we consider the solution of an augmented linear system which combines $i.) \oplus ii.)$ at once. This means that we introduce a new scalar unknown quantity—say $\alpha_{\mathrm{res}}$—and a new single equation

$$\begin{aligned} 0 &= \alpha_{\mathrm{res}} - \mathrm{Res}_{y=0}\left(\psi\cdot\varphi^\vee\right) \\ &= \alpha_{\mathrm{res}} - \sum_{j=\mathrm{Min}}^{\mathrm{Max}} \psi_j\cdot\varphi_{-j-1}^\vee . \end{aligned} \tag{A.9}$$

Next eq. (A.8) becomes

$$\left(\begin{array}{cccc|c} \Omega_{-1}^\top + \mathrm{Min}\cdot\mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} & 0 \\ \Omega_0^\top & \Omega_{-1}^\top + (\mathrm{Min}+1)\cdot\mathbb{1} & \cdots & \mathbb{0} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{\mathrm{Max}-\mathrm{Min}-1}^\top & \Omega_{\mathrm{Max}-\mathrm{Min}-2}^\top & \cdots & \Omega_{-1}^\top + \mathrm{Max}\cdot\mathbb{1} & 0 \\ \hline -\varphi_{-\mathrm{Min}-1}^{\vee\top} & -\varphi_{-\mathrm{Min}-2}^{\vee\top} & \cdots & -\varphi_{-\mathrm{Max}-1}^{\vee\top} & 1 \end{array}\right) \begin{pmatrix} \psi_{\mathrm{Min}} \\ \psi_{\mathrm{Min}+1} \\ \vdots \\ \psi_{\mathrm{Max}} \\ \alpha_{\mathrm{res}} \end{pmatrix} = \begin{pmatrix} \varphi_{\mathrm{Min}-1} \\ \varphi_{\mathrm{Min}} \\ \vdots \\ \varphi_{\mathrm{Max}-1} \\ 0 \end{pmatrix} . \tag{A.10}$$

[1] If $\Omega_{-1}$ has no integer eigenvalue, and so—a fortiori—if 0 is *not* an eigenvalue, then $\Omega_{-1}$ has a trivial kernel and $\mathrm{Min} = \min_\varphi + 1$ would be enough. Since we want to consider also the case in which 0 is eigenvalue, we employ the choice reported in the main text; nevertheless it is a just a prescription which works in practice.

[2] We thank Seva Chestnov for pointing this to us, and for several discussions on this topic—see [223].

Finally, we just solve eq. (A.10) w.r.t. $\alpha_{\rm res}$, eliminating all (and only) the $\alpha_{i,j}$s appearing in it. Comparing eq. (A.8) and eq. (A.10), our approach implies that, even if some the unknowns cannot be uniquely determined (working up to a given order in $y$), the residue—and so the intersection number—is unambiguously determined.

Let us also notice that eq. (A.4) is, per se, a restrictive condition; it should be replaced with the more general

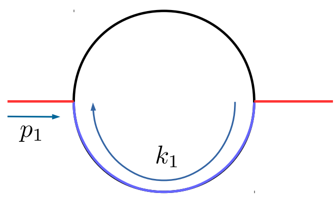
$$\hat{\Omega} = \sum_{i=\min_\Omega}^{\infty} \hat{\Omega}_i \, y^i, \tag{A.11}$$

with, in principle, $\min_\Omega < -1$.

The properties of $\hat{\Omega}$, such as the order of the deepest pole in its expansion, its eigenvalues, etc. are tightly related to the choice of internal basis elements—colloquially $\hat{\Omega}$ controls the (system of) differential equation(s) fulfilled by internal basis elements. Unfortunately conditions on the basis elements such that e.g. eq. (A.4) holds are not known yet. Nevertheless given $\hat{\Omega}$ as in eq. (A.11), we can still apply the method described in eqs. (A.9,A.10), with trivial modifications, i.e. modifying the ansatz with $\mathrm{Min} = \min_\psi = \mathtt{Min}(\min_\varphi, \min_\Omega) + 1$, and solving the augmented linear system[3] w.r.t. $\alpha_{\rm res}$. Alternatively, we can adopt the Moser reduction [224], such as the one implemented in the program FUCHSIA [150] in order to map $\hat{\Omega}$ via a gauge-like transformation $\mathbf{T}$, to a new matrix $\hat{\Omega}_{\mathbf{T}}$ such that eq. (A.4) holds around each pole. We are dealing with series expansions, which could be delicate; our basic implementation relies on the MATHEMATICA command `SeriesData`. In particular `SeriesData[y,0,{},Max+1,Max+1,1]` $(= \mathcal{O}(y^{\mathrm{Max}+1})$), is added to the ansatz $\psi$. In this way the expansion are automatically taken care at least consistently.

We give a simple example of FIs which illustrates some of the features described above.

**Example.** *The one loop half massive bubble. Let us consider the integral family associated to following graph*




(A.12)

*The denominators are*

$$z_1 = D_1 = (k_1 + p)^2, \qquad z_2 = D_2 = k_1^2 - m^2, \tag{A.13}$$

*and the kinematics is $p^2 = s$.*

[3] In this case the structure of the system will not be the one in eq. (A.10), but there is no conceptual difference in the procedure.

*We have the following regularized*

$$u_{\text{reg}}(\mathbf{z}) = z_1^\rho z_2^\rho \left( -\frac{m^4}{4} - \frac{s^2}{4} + \frac{m^2 s}{2} + \frac{m^2}{2}(z_1 - z_2) + \frac{s}{2}(z_1 + z_2) - \frac{1}{4}(z_1^2 + z_2^2) + \frac{z_1 z_2}{2} \right)^{\frac{d-3}{2}}. \tag{A.14}$$

*Counting of critical points gives*

$$\nu_{(\mathbf{1})} = 2, \qquad \nu_{(\mathbf{2})} = 2. \tag{A.15}$$

*We assume the following inner basis*

$$(\langle e_1^{(\mathbf{1})}|, \langle e_1^{(\mathbf{1})}|) = (\langle 1 \cdot dz_1|, \langle d \log z_1|). \tag{A.16}$$

*Let us focus first on*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \left\langle \frac{d\mathbf{z}}{z_1 z_2} \middle| \frac{d\mathbf{z}}{z_1 z_2} \right\rangle. \tag{A.17}$$

*With the above-mentioned choice of internal basis we obtain*

$$\hat{\Omega}^{(2)} = \begin{pmatrix} \frac{d+\rho-2}{2(m^2+z_2)} + \frac{\rho}{z_2} & \frac{\rho}{2} - \frac{\rho s}{2(m^2+z_2)} \\ \frac{-d-\rho+2}{2s(m^2-s+z_2)} + \frac{d+\rho-2}{2s(m^2+z_2)} & \frac{d+2\rho-3}{m^2-s+z_2} - \frac{\rho}{2(m^2+z_2)} + \frac{\rho}{z_2} \end{pmatrix}. \tag{A.18}$$

*Looking at the* $(1,2)$ *entry of eq. (A.18) that* $\Omega^{(2)} = \hat{\Omega}^{(2)} dz_2$ *has a double pole at:* $\infty$ $(\min_\Omega = -2)$*; in the following we will focus on the contribution from this pole.*

*Decomposing the two two-forms in terms of internal one-forms we find*

$$\hat{\varphi}^{(2)} = \begin{pmatrix} 0 & \frac{1}{z_2} \end{pmatrix}, \qquad \hat{\varphi}^{(2)\vee} = \begin{pmatrix} \frac{(d-3)\left(m^2+s+z_2\right)}{z_2(d+\rho-3)(d+\rho-2)} & \frac{(d-3)}{\rho z_2(d+\rho-3)} \end{pmatrix}. \tag{A.19}$$

*We infer that* $\min_\varphi = -1$*, while* $\min_{\varphi^\vee} = -2$*. Therefore we set* $\text{Min} = -1$ *and* $\text{Max} = +1$*. The* *augmented linear system is*

$$\left(\begin{array}{ccccccc}
\frac{1}{2}(-d-3\rho) & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}m^2(d+\rho-2) & \frac{1}{2}(d+\rho-2) & \frac{1}{2}(-d-3\rho+2) & 0 & 0 & 0 & 0 \\
-\frac{1}{2}m^4(d+\rho-2) & -\frac{1}{2}(d+\rho-2)\left(2m^2-s\right) & \frac{1}{2}m^2(d+\rho-2) & \frac{1}{2}(d+\rho-2) & \frac{1}{2}(-d-3\rho+4) & 0 & 0 \\
-\frac{\rho}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\rho s}{2} & -d-\frac{5\rho}{2}+2 & -\frac{\rho}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2}m^2\rho s & (d+2\rho-3)\left(m^2-s\right)-\frac{m^2\rho}{2} & \frac{\rho s}{2} & -d-\frac{5\rho}{2}+3 & -\frac{\rho}{2} & 0 & 0 \\
\hline
0 & 0 & \frac{(d-3)\left(m^2+s\right)}{(d+\rho-3)(d+\rho-2)} & \frac{d-3}{\rho(d+\rho-3)} & \frac{d-3}{(d+\rho-3)(d+\rho-2)} & 0 & 1
\end{array}\right)
\begin{pmatrix} \alpha_{-1,1} \\ \alpha_{-1,2} \\ \alpha_{0,1} \\ \alpha_{0,2} \\ \alpha_{1,1} \\ \alpha_{1,2} \\ \alpha_{\text{res}} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{A.20}$$

*Despite the fact that $\hat{\Omega}^{(2)}_{-2}$ has eigenvalue 0—with multiplicity 2—solving for $\alpha_{\text{res}}$ eliminating all the $\alpha_{i,j}$ appearing in it we obtain the unambiguous*

$$\alpha_{\text{res}} = -\frac{d-3}{\rho(d+\rho-3)(d+2\rho-3)}. \tag{A.21}$$

*Alternatively, we can look for a gauge transformation—or, equivalently, for a new internal basis—which maps $\hat{\Omega}^{(2)}$ to a fuchsian form.*

Fuchsia *produces*

$$\mathbf{T} = \begin{pmatrix} -z_2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.22}$$

*Eq. (A.22) maps the above-mentioned internal basis onto*

$$(\langle e^{(\mathbf{1})}_{\mathbf{T},1}|, \langle e^{(\mathbf{1})}_{\mathbf{T},2}|) = (\langle -dz_1/z_2|, \langle d\log z_1|), \tag{A.23}$$

*which leads to*

$$\hat{\Omega}^{(2)}_{\mathbf{T}} = \begin{pmatrix} \frac{d+\rho-2}{2(m^2+z_2)} + \frac{\rho-1}{z_2} & -\frac{\rho\left(m^2-s\right)}{2m^2 z_2} - \frac{\rho s}{2m^2(m^2+z_2)} \\ \frac{m^2(d+\rho-2)}{2s(m^2+z_2)} + \frac{(d+\rho-2)\left(s-m^2\right)}{2s(m^2-s+z_2)} & \frac{d+2\rho-3}{m^2-s+z_2} - \frac{\rho}{2(m^2+z_2)} + \frac{\rho}{z_2} \end{pmatrix}. \tag{A.24}$$

*Focusing again around the pole $\infty$, then $\hat{\Omega}^{(2)}_{-1}$ has non-integer eigenvalues there, and the analysis can be carried on straightforwardly.*

*Both the procedures (i.e. the choice in eq. (A.16) and the one in eq. (A.23)) yield the same final result*

$$\left\langle \frac{d\mathbf{z}}{z_1 z_2} \middle| \frac{d\mathbf{z}}{z_1 z_2} \right\rangle = \frac{d-3}{\rho^2(d+2\rho-3)}. \tag{A.25}$$

*It is also instructive to consider*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \left\langle \frac{d\mathbf{z}}{z_1^2 z_2} \middle| \frac{d\mathbf{z}}{z_1 z_2} \right\rangle, \tag{A.26}$$

*with the internal basis given by eq. (A.23) which leads to eq. (A.24). We will focus now on the pole: $s-m^2$, where $\hat{\Omega}^{(2)}_{-1}$ has an eigenvalue 0-with multiplicity 1.*

*The coefficients of the two two-forms decomposed in terms of inner one-forms read*

$$\hat{\varphi}^{(2)} = \begin{pmatrix} \frac{d+\rho-2}{(\rho-1)(m^2-s+z_2)^2} & \frac{(d+2\rho-3)\left(m^2+s+z_2\right)}{(\rho-1)z_2(m^2-s+z_2)^2} \end{pmatrix}^\top, \tag{A.27a}$$

$$\hat{\varphi}^{(2)\vee} = \begin{pmatrix} -\frac{(d-3)\left(m^2+s+z_2\right)}{z_2^2(d+\rho-3)(d+\rho-2)} & \frac{d-3}{\rho z_2(d+\rho-3)} \end{pmatrix}^\top; \tag{A.27b}$$

*we have* $\min_\varphi = -2$ *and* $\min_{\varphi^\vee} = 0$*, and so we set* $\mathrm{Min} = \mathrm{Max} = -1$*. The augmented system reads*

$$\left(\begin{array}{c|c} \Omega^{(2)\top}_{-1} - 1 \cdot \mathbb{1} & 0 \\ \hline -\varphi_0^{(2)\vee\top} & 1 \end{array}\right) \begin{pmatrix} \psi_{-1} \\ \alpha_{\mathrm{res}} \end{pmatrix} = \begin{pmatrix} \varphi^{(2)}_{-2} \\ 0 \end{pmatrix}, \tag{A.28}$$

*with explicit expressions*

$$\left(\begin{array}{cc|c} -1 & \frac{(d+\rho-2)\left(s-m^2\right)}{2s} & 0 \\ 0 & d+2\rho-4 & 0 \\ \hline \frac{2(d-3)s}{(d+\rho-3)(d+\rho-2)(m^2-s)^2} & \frac{d-3}{\rho(d+\rho-3)(m^2-s)} & 1 \end{array}\right) \begin{pmatrix} \alpha_{-1,1} \\ \alpha_{-1,2} \\ \alpha_{\mathrm{res}} \end{pmatrix} = \begin{pmatrix} \frac{d+\rho-2}{\rho-1} \\ \frac{2s(d+2\rho-3)}{(\rho-1)(s-m^2)} \\ 0 \end{pmatrix}. \tag{A.29}$$

*In this case all the unknown coefficients appearing in eq. (A.29) can be determined, since* $\Omega^{(2)\top}_{-1} - 1 \cdot \mathbb{1}$ *is invertible (while* $\Omega^{(2)\top}_{-1} - 0 \cdot \mathbb{1}$ *is not, but it does not appear in eq. (A.29)); our procedure gives*

$$\alpha_{\mathrm{res}} = \frac{2(d-3)s}{(\rho-1)\rho(d+2\rho-4)\left(m^2-s\right)^2}. \tag{A.30}$$

*Finally, it is interesting to study a case in which* $\Omega^{(2)\top}_{-1} - 0\mathbb{1}$ *(which is not invertible) appears in the augmented linear system.*

*Let us consider*

$$\langle \varphi^{(\mathbf{2})} | \varphi^{(\mathbf{2})\vee} \rangle = \left\langle \frac{d\mathbf{z}}{z_1^2 z_2} \middle| \frac{d\mathbf{z}}{z_1^2 z_2} \right\rangle, \tag{A.31}$$

*with (once again) the internal basis given by eq. (A.23) which leads to eq. (A.24). We obtain*

$$\hat{\varphi}^{(2)} = \begin{pmatrix} \frac{d+\rho-2}{(\rho-1)(m^2-s+z_2)^2} & \frac{(d+2\rho-3)\left(m^2+s+z_2\right)}{(\rho-1)z_2(m^2-s+z_2)^2} \end{pmatrix}^\top, \tag{A.32a}$$

$$\hat{\varphi}^{(2)\vee} = \begin{pmatrix} -\frac{d-3}{(\rho+1)z_2^2(d+\rho-2)} & \frac{(d-3)\left(m^2+s+z_2\right)}{\rho(\rho+1)z_2(m^2-s+z_2)^2} \end{pmatrix}^\top. \tag{A.32b}$$

*We have* $\min_\varphi = -2$ *and* $\min_{\varphi^\vee} = -2$*, and so we set* $\mathrm{Min} = -1$ *and* $\mathrm{Max} = +1$*; the augmented system is*

$$\begin{pmatrix} \hat{\Omega}^{(2)\top}_{-1} - 1\mathbb{1} & \mathbb{0} & \mathbb{0} & 0 \\ \hat{\Omega}^{(2)\top}_{0} & \hat{\Omega}^{(2)\top}_{-1} - 0\mathbb{1} & \mathbb{0} & 0 \\ \hat{\Omega}^{(2)\top}_{1} & \hat{\Omega}^{(2)\top}_{0} & \hat{\Omega}^{(2)\top}_{-1} + 1\mathbb{1} & 0 \\ -\hat{\varphi}^{(2)\vee\top}_{0} & -\hat{\varphi}^{(2)\vee\top}_{-1} & -\hat{\varphi}^{(2)\vee\top}_{-2} & 1 \end{pmatrix} \begin{pmatrix} \psi_{-1} \\ \psi_0 \\ \psi_1 \\ \alpha_{\mathrm{res}} \end{pmatrix} = \begin{pmatrix} \hat{\varphi}^{(2)}_{-2} \\ \hat{\varphi}^{(2)}_{-1} \\ \hat{\varphi}^{(2)}_{0} \\ 0 \end{pmatrix}. \tag{A.33}$$

*With explicit values we obtain*

$$
\left(\begin{array}{cc|cc|cc|c}
-1 & \frac{(d+\rho-2)\left(s-m^2\right)}{2s} & 0 & 0 & 0 & 0 & 0 \\
0 & d+2\rho-4 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{d+\rho-2}{2s}+\frac{\rho-1}{s-m^2} & \frac{m^2(d+\rho-2)}{2s^2} & 0 & \frac{(d+\rho-2)\left(s-m^2\right)}{2s} & 0 & 0 & 0 \\
0 & \frac{\rho\left(m^2+s\right)}{2s\left(s-m^2\right)} & 0 & d+2\rho-3 & 0 & 0 & 0 \\
\hline
\frac{1-\rho}{\left(m^2-s\right)^2}-\frac{d+\rho-2}{2s^2} & -\frac{m^2(d+\rho-2)}{2s^3} & \frac{d+\rho-2}{2s}+\frac{\rho-1}{s-m^2} & \frac{m^2(d+\rho-2)}{2s^2} & 1 & \frac{(d+\rho-2)\left(s-m^2\right)}{2s} & 0 \\
\frac{\rho}{2m^2s-2s^2} & \frac{1}{2}\rho\left(\frac{1}{s^2}-\frac{2}{\left(m^2-s\right)^2}\right) & 0 & \frac{\rho\left(m^2+s\right)}{2s\left(s-m^2\right)} & 0 & d+2\rho-2 & 0 \\
\hline
\frac{d-3}{(\rho+1)(d+\rho-2)\left(m^2-s\right)^2} & \frac{(d-3)\left(m^2+s\right)}{\rho(\rho+1)\left(m^2-s\right)^3} & 0 & \frac{(d-3)\left(m^2+s\right)}{\rho(\rho+1)\left(m^2-s\right)^2} & 0 & \frac{2(d-3)s}{\rho(\rho+1)\left(m^2-s\right)} & 1
\end{array}\right)
\begin{pmatrix} \alpha_{-1,1} \\ \alpha_{-1,2} \\ \alpha_{0,1} \\ \alpha_{0,2} \\ \alpha_{1,1} \\ \alpha_{1,2} \\ \alpha_{\mathrm{res}} \end{pmatrix}
=
\begin{pmatrix} \frac{d+\rho-2}{\rho-1} \\ \frac{2s(d+2\rho-3)}{(\rho-1)\left(s-m^2\right)} \\ 0 \\ -\frac{(d+2\rho-3)\left(m^2+s\right)}{(\rho-1)\left(s-m^2\right)^2} \\ 0 \\ \frac{(d+2\rho-3)\left(m^2+s\right)}{(\rho-1)\left(s-m^2\right)^3} \\ 0 \end{pmatrix},
\tag{A.34}
$$

*we uniquely determine*

$$\begin{aligned}\alpha_{\text{res}} =&(d-3)\left(d^2m^4+6d^2m^2s+5d^2s^2+4dm^4\rho-6dm^4+32dm^2\rho s-36dm^2s+24d\rho s^2\right.\\&-30ds^2+4m^4\rho^2-12m^4\rho+8m^4+44m^2\rho^2s-96m^2\rho s+52m^2s+28\rho^2s^2\\&\left.-72\rho s^2+44s^2\right)/\left((\rho-1)\rho(\rho+1)(d+2\rho-4)(d+2\rho-2)\left(m^2-s\right)^4\right).\end{aligned} \tag{A.35}$$

*As a consistency check we consider the decomposition*

$$I_{2,1} = c_1\mathcal{J}_1 + c_2\mathcal{J}_2, \tag{A.36}$$

*with*

$$I_{2,1} \rightsquigarrow \langle\varphi^{(\mathbf{2})}| = \left\langle \frac{d\mathbf{z}}{z_1^2 z_2}\right|, \tag{A.37}$$

*and*

$$\mathcal{J}_1 \rightsquigarrow \langle e_1^{(\mathbf{2})}| = \left\langle \frac{d\mathbf{z}}{z_2}\right|, \qquad \mathcal{J}_2 \rightsquigarrow \langle e_2^{(\mathbf{2})}| = \left\langle \frac{d\mathbf{z}}{z_1 z_2}\right|. \tag{A.38}$$

*Eq. (A.36) can be obtained with two different choices of dual basis, namely*

$$(|h^{(\mathbf{2})}_{\bullet,1}\rangle, |h^{(\mathbf{2})}_{\bullet,2}\rangle) = \left(\left|\frac{d\mathbf{z}}{z_2}\right\rangle, \left|\frac{d\mathbf{z}}{z_1 z_2}\right\rangle\right), \tag{A.39}$$

*or*

$$(|h^{(\mathbf{2})}_{\bullet,1}\rangle, |h^{(\mathbf{2})}_{\bullet,2}\rangle) = \left(\left|\frac{d\mathbf{z}}{z_2}\right\rangle, \left|\frac{d\mathbf{z}}{z_1^2 z_2}\right\rangle\right). \tag{A.40}$$

*Both eq. (A.39) (involving eq. (A.26)) and eq. (A.40) (involving eq. (A.31)) give the same decomposition. Moreover the coefficients in the $\rho \to 0$ are in agreement with* LiteRed.

# B

# Special Mathematical Functions

We list here the special mathematical functions which appear in the main text.

- **The Euler Beta Integral.**

$$\begin{aligned} B(p,q) &= \int_0^1 z^p (1-z)^q \, \frac{dz}{z(1-z)} \qquad z = \frac{1}{w} \\ &= \int_1^\infty w^{-p-q} (w-1)^q \, \frac{dw}{w-1}. \end{aligned} \tag{B.1}$$

- **The ${}_2F_1$ Hypergeometric Function.**

$$\begin{aligned} {}_2F_1(a,b,c;x) &= \frac{1}{B(a,c-a)} \int_0^1 z^a \, (1-z)^{c-a} (1-xz)^{-b} \, \frac{dz}{z(1-z)} \qquad z = \frac{1}{w} \\ &= \frac{1}{B(a,c-a)} \int_1^\infty w^{b-c} (w-x)^{-b} (w-1)^{c-a} \, \frac{dw}{w-1}. \end{aligned} \tag{B.2}$$

- **The ${}_3F_2$ Hypergeometric Function.**

$$\begin{aligned} &{}_3F_2(a_1,a_2,a_3 : b_1,b_2,b_3 : x) = \frac{1}{B(a_2,b_1-a_2)B(a_3,b_2-a_3}\times \\ &\int_{(0,1)^2} z_1^{a_3} (1-z_1)^{b_2-a_3} z_2^{a_2} (1-z_2)^{b_1-a_2} (1-xz_1z_2)^{-a_1} \, \frac{dz_1 \wedge dz_2}{z_1(1-z_1)z_2(1-z_2)}. \end{aligned} \tag{B.3}$$

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