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On the structure of the solution of continuous-time algebraic Riccati equations with closed-loop eigenvalues on the imaginary axis. ⋆

L. Ntogramatzidis * V. Arumugam * A. Ferrante **

* *School of Electrical Engineering, Computing and Mathematical Sciences Curtin University, Perth WA 6845, Australia. (E-mails: L.Ntogramatzidis@curtin.edu.au, Vishnuram.Arumugam@student.curtin.edu.au).*

** *Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo, 6/B – I-35131 Padova, Italy. (E-mail: augusto@dei.unipd.it)*

Abstract: This paper proposes a decomposition of the continuous-time algebraic Riccati equation aimed at eliminating the problem of the presence of closed-loop eigenvalues on the imaginary axis. In particular, we show that it is possible to parameterize the the entire set of solutions of the given Riccati equation in terms of the solutions of a reduced-order Riccati equation, which is associated to a Hamiltonian matrix with no eigenvalues on the imaginary axis, and some free parameters arising from the presence of imaginary eigenvalues of the Hamiltonian matrix.

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1. INTRODUCTION

Riccati equations play a fundamental role in countless branches of engineering and applied mathematics, including network analysis, optimal control and filtering, spectral factorization, stochastic realisation to name only a few. Monographs devoted to the study of Riccati equations include Reid (1972); Willems et al. (1991); Lancaster and Rodman (1995); Ionescu et al. (1999); Abou-Kandil et al. (2003).

In particular, many reduction techniques have been proposed in the literature for both the continuous and the discrete-time algebraic/differential/difference Riccati equations, see e.g. Mita (1985); Fujinaka et al. (1987); Hansson et al. (1999); Ferrante (2004); Ferrante and Wimmer (2007); Ntogramatzidis et al. (2015) and the references cited therein. Some of these are tailored to the calculation of a specific solution (for example the stabilizing solution of a continuous/discrete algebraic Riccati equations), while some of them seek to determine the entire set of solutions by solving reduced-order Riccati equations by eliminating parts of this equation which are traditionally considered to lead to theoretical or numerical problems. An important example is the discrete-time algebraic Riccati equation with associated extended symplectic pencil with generalized eigenvalues at the origin or on the unit circle, Ferrante and Wimmer (2007); Ntogramatzidis et al. (2015).

The main purpose of this paper is to determine the entire set of Hermitian solutions of the continuous-time algebraic Riccati equation (ARE) in the case where the associated Hamiltonian matrix has eigenvalues on the imaginary axis. To this end, we propose a reduction methodology whose aim is to decompose any ARE in such a way that it can be solved in terms of a reduced-order ARE associated with a Hamiltonian matrix without purely imaginary eigenvalues and a linear equation. As a consequence all the (Hermitian) solutions of the original equation may be, in turn, decomposed in a part with arbitrary entries, a part obtained by solving a linear equation and a part that can be obtained by solving the reduced-order ARE. This task is accomplished by decomposing the eigenspace of the Hamiltonian matrix associated with each purely imaginary eigenvalue as the direct sum of two subspaces. These two subspaces give rise to two reduction procedures which lead to a complete decomposition of the family of Hermitian solutions. In terms of spectral factorization, Hamiltonian matrices with imaginary eigenvalues are associated with spectra having zeros on the imaginary axis. Therefore, the corresponding spectral factorization problem is particularly delicate, see Ferrante (2005), see also Baggio and Ferrante (2019, 2016a,b) for the discrete-time counterpart where the spectra have zeros on the unit circle. The associated LQ optimal control problem is peculiar because the optimal solution, if it exists, is not stabilizing.

This paper considers a continuous-time ARE with complex coefficients. The reason for this is that the two reduction procedures are applied for each imaginary eigenvalue of the Hamiltonian matrix, and at each reduction the size of the corresponding reduced-order ARE decreases. The changes of coordinates that yield both these decompositions are, in general, complex valued, so when applying, say, the second procedure on a reduced-order ARE obtained at the end of the first one, the coefficients of such equation are, in general, complex.

Notation. We denote by $\mathbb{I}$ the set of imaginary numbers. Given a complex vector $z \in \mathbb{C}^n$, we denote by $\bar{z}$ the complex conjugate of z, and by z^* the conjugate transpose of z. Given a square and invertible complex-valued matrix M, since $(M^*)^{-1} = (M^{-1})^*$, we denote by M^{-*} the inverse of M^*.

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2. MAIN RESULTS

This paper is concerned with the study of the set of Hermitian solutions of the continuous-time algebraic Riccati equation

$$XA + A^* X - X B R^{-1} B^* X + Q = 0, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $Q \in \mathbb{C}^{n\times n}$ and $R \in \mathbb{C}^{m\times m}$, under the assumptions

$$Q = Q^* \geq 0 \quad \text{and} \quad R = R^* > 0.$$

It is well-known that the structure of the Hermitian solutions of (1) is strictly related with the so-called Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix},$$

see e.g. Lancaster and Rodman (1995), Zhou et al. (1996) and Ionescu et al. (1999). For example, if (A, B) is a reachable pair and H has no eigenvalues on the imaginary axis $\mathbb{I}$, the Riccati equation (1) has a maximal solution $X_+ = X_+^* \geq 0$ such that the eigenvalues of the closed-loop matrix $A_+ = A - BR^{-1}B^* X_+$ are all in the left-half complex plane. We recall that all the eigenvalues of H are mirrored with respect to the imaginary axis, so that if λ is an eigenvalue of H, then also $-\lambda^*$ is an eigenvalue of H.

The objective of this paper is to obtain a decomposition of (1) whose purpose is to obtain the complete set of solutions from the set of solutions of a reduced-order Riccati equation whose Hamiltonian matrix has no eigenvalues on the imaginary axis.

Remark 2.1. Other more general forms of the continuous-time algebraic Riccati equation have been considered in the literature. For example, the one associated with a LQ optimal control problem involving a cross-penalty term S between the state and the control evolution in the running cost of the performance index reads as

$$XA + A^* X - (S + XB) R^{-1} (S^* + B^* X) + Q = 0.$$

In this case, it is required that the matrix $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$ be Hermitian and positive semidefinite and that the matrix R be positive definite. This equation can be re-written in the form of (1) by considering, in place of A and Q, the matrices $A - BR^{-1}S^*$ and $Q - SR^{-1}S^*$, respectively.

For all $\lambda \in \mathbb{C}$, we define the subspace of $\mathbb{C}^{2n}$ as

$$\mathcal{E}_\lambda = \ker(H - \lambda I),$$

and we recall that $\mathcal{E}_\lambda \neq \{0\}$ if and only if $\lambda \in \mathbb{C}$ is an eigenvalue of H. In this case, $\mathcal{E}_\lambda$ is the eigenspace of H associated with the eigenvalue λ. The decomposition described in this paper hinges on a decomposition of the eigenspaces of the Hamiltonian matrix H given in the following lemma.

Lemma 2.1. Let $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ with $v_1, v_2 \in \mathbb{C}^n$. Let $\lambda \in \mathbb{I}$. Then, $v \in \mathcal{E}_\lambda$ if and only if $\left[\begin{smallmatrix} v_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{E}_\lambda$ and $\left[\begin{smallmatrix} 0 \\ v_2 \end{smallmatrix}\right] \in \mathcal{E}_\lambda$. Moreover, a basis matrix of $\mathcal{E}_\lambda$ is given by $\left[\begin{smallmatrix} K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right]$, where K_1 is a basis matrix of the kernel of $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ and K_2 is a basis of the kernel of $\left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right]$.

Proof: We have $v \in \mathcal{E}_\lambda$ if and only if

$$\begin{bmatrix} -BR^{-1}B^* & A - \lambda I \\ -A^* - \lambda I & -Q \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = 0. \tag{2}$$

Let $T = [\, T_1 \; T_2 \,]$ be a unitary matrix such that the columns of T_2 form an orthonormal basis of $\ker Q$. Thus,

$$\tilde{Q} = T^* Q T = \begin{bmatrix} \tilde{Q}_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{A} = T^* A T = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{Q}_0$ is non-singular. Let

$$M_\lambda = \begin{bmatrix} \tilde{A}_{11} - \lambda I \\ \tilde{A}_{21} \end{bmatrix} \qquad N_\lambda = \begin{bmatrix} \tilde{A}_{12} \\ \tilde{A}_{22} - \lambda I \end{bmatrix}.$$

Since $\lambda^* = -\lambda$ (which follows from λ being purely imaginary), eq. (2) can be written as

$$\begin{bmatrix} -BR^{-1}B^* & M_\lambda & N_\lambda \\ -M_\lambda^* & -\tilde{Q}_0 & 0 \\ -N_\lambda^* & 0 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ v_{11} \\ v_{12} \end{bmatrix} = 0,$$

where the vector $v_1 = \left[\begin{smallmatrix} v_{11} \\ v_{12} \end{smallmatrix}\right]$ is partitioned conformably with Q. From the third it follows that $N_\lambda^* v_2 = 0$. From the second we find $v_{11} = -\tilde{Q}_0^{-1} M_\lambda^* v_2$. This expression can be substituted into the first, and premultiplying both sides of the equation thus obtained by v_2^* and taking into account that $N_\lambda^* v_2 = 0$ yields

$$-v_2^* (BR^{-1}B^* + M_\lambda \tilde{Q}_0^{-1} M_\lambda^*) v_2 = 0. \tag{3}$$

Since both R^{-1} and $\tilde{Q}_0^{-1}$ are positive definite, the quadratic form $v_2^* (BR^{-1}B^* + M_\lambda \tilde{Q}_0^{-1} M_\lambda^*) v_2$ is positive definite, so that (3) yields $B^* v_2 = 0$ and $M_\lambda^* v_2 = 0$. Since we have also $N_\lambda^* v_2 = 0$, we can conclude that

$$v_2 \in \ker \begin{bmatrix} -B^* \\ -A^* - \lambda I \end{bmatrix} = \ker \begin{bmatrix} -BR^{-1}B^* \\ -A^* - \lambda I \end{bmatrix},$$

which also implies that $\left[\begin{smallmatrix} 0 \\ v_2 \end{smallmatrix}\right] \in \mathcal{E}_\lambda$. From $v_{11} = 0$ and $N_\lambda v_{12} = 0$, we also have $(A - \lambda I) v_1 = 0$ and $Q v_1 = 0$, so that $\left[\begin{smallmatrix} v_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{E}_\lambda$. ■

We now introduce two reduction procedures aimed at eliminating the eigenvalues of the Hamiltonian matrix on the imaginary axis. The first reduction procedure is aimed at eliminating the subspace $\ker \left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right]$ spanned by the columns of K_2, if present. The decomposition that emerges from this reduction procedure allows to express the solution of the Riccati equation in terms of an arbitrary part, a part that solves a reduced-order Riccati equation, a part that is obtained by solving a linear equation, and, in those situations where the solution of such linear equation is not unique, another part that solves a reduced-order Riccati equation.

It is possible that the Hamiltonian matrix of the reduced-order Riccati equation still contains eigenvalues on the imaginary axis. This occurs, in particular, when the subspace $\ker \left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ spanned by the columns of K_1 is not zero. In this case, the second reduction procedure needs to be applied to this reduced-order Riccati equation. If initially $\ker \left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right] = \{0\}$ and $\ker \left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right] \neq \{0\}$, only the second procedure has to be carried out, and we immediately obtain a reduced order Riccati equation with a Hamiltonian matrix devoid of eigenvalues on the imaginary axis.

2.1 Reduction associated with K_2

In both reduction procedures, we address the case where $\lambda = 0$ and the case where $\lambda \in \mathbb{I} \setminus \{0\}$ separately. Let us therefore begin by considering $\lambda = 0$. We introduce a change of basis given by $T = [\, T_1 \; T_2 \,]$, where the columns of T_1 are an orthonormal basis for $\operatorname{im} K_2 = \ker \left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right]$ and T is unitary. Thus, the subspace $\operatorname{im} K_2$ in the new basis is written as $\operatorname{im}\begin{bmatrix} I \\ 0 \end{bmatrix}$. In other words, $T^* K_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

In this case, $-A^* K_2 = \lambda K_2 = 0$ implies that

$$T^* A\, T = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}. \tag{4}$$

Moreover, since $B^* K_2 = 0$, we have $T^* B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$. Consider the decomposition of $\tilde{X} = T^* X T$ and $\tilde{Q} = T^* Q T$ into block matrices whose sizes are compatible with the decomposition in (4), i.e.,

$$\tilde{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \qquad \tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}.$$

It follows that (1) can be written with respect to this basis as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0 & A_{21}^* \\ 0 & A_{22}^* \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} - \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} R^{-1} [\,0 \;\; B_2^*\,] \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} = 0.$$

This equation is equivalent to the three equations

$$X_{12} A_{21} + A_{21}^* X_{12}^* - X_{12} B_2 R^{-1} B_2^* X_{12}^* + Q_{11} = 0 \tag{5}$$

$$X_{12} A_{22} + A_{21}^* X_{22}^* - X_{12} B_2 R^{-1} B_2^* X_{22} + Q_{12} = 0 \tag{6}$$

$$X_{22} A_{22} + A_{22}^* X_{22}^* - X_{22} B_2 R^{-1} B_2^* X_{22} + Q_{22} = 0. \tag{7}$$

We notice the following facts:

- None of these equations depend on X_{11}. Thus, X_{11} is arbitrary.
- Equation (7) is a reduced-order continuous-time ARE. Its solution does not depend on equations (5-6). If this equation does not admit solutions, the original Riccati equation has no solutions. Notice that this equation may still be associated with a Hamiltonian matrix with eigenvalues on the imaginary axis due to the presence of K_1. In this case, the second reduction procedure needs to be applied to this equation.
- Once X_{22} is computed from (7), we can substitute it into (6), which then becomes a linear equation in X_{12}:

  $$X_{12} A_{X_{22}} = -A_{21}^* X_{22} - Q_{12},$$

  where the matrix $A_{X_{22}} \stackrel{\text{def}}{=} A_{22} - B_2 R^{-1} B_2^* X_{22}$ is the closed-loop matrix relative to the subsystem 22. Let

  $$\Gamma \stackrel{\text{def}}{=} -A_{21}^* X_{22} - Q_{12},$$

  so that the latter can be written as

  $$X_{12} A_{X_{22}} = \Gamma.$$

  This equation admits solutions if and only if

  $$\ker A_{X_{22}} \subseteq \ker \Gamma. \tag{8}$$

  If this condition is not satisfied, (6) does not admit solutions, and the original Riccati equation does not admit solutions. If (8) is satisfied and $A_{X_{22}}$ is not singular, (6) has only one solution $\hat{X}_{12} = \Gamma A_{X_{22}}^{\dagger}$. It is sufficient to check whether this solution also satisfies (5). If it does not, the original Riccati equation does not admit solutions, while if the only solution $\hat{X}_{12}$ of (6) also solves (5), we have parameterized the solutions of the algebraic Riccati equation into

  $$\begin{bmatrix} X_{11} & \hat{X}_{12} \\ \hat{X}_{12}^* & X_{22} \end{bmatrix}$$

  where X_{11} is arbitrary, X_{22} is the solution of a reduced-order Riccati equation and $\hat{X}_{12}$ is the only solution that satisfies simultaneously (6) and (5).

  We may also have the case in which $X_{12} A_{X_{22}} = \Gamma$ admits infinite solutions. The set of its solutions is parameterized as

  $$X_{12} = \hat{X}_{12} + K\,\Delta,$$

  where $\hat{X}_{12} = \Gamma A_{X_{22}}^{\dagger}$ and $\operatorname{im} \Delta^* = \ker A_{X_{22}}^*$. By substitution of $X_{12} = \hat{X}_{12} + K\,\Delta$ into (5) we obtain a new reduced-order Riccati equation in K, which reads as

  $$K \hat{A}_{21} + \hat{A}_{21}^* K^* - K\,\Delta\, B_2 R^{-1} B_2^* \Delta^* K^* + \Omega = 0, \tag{9}$$

  where $\hat{A}_{21} \stackrel{\text{def}}{=} \Delta\,(A_{21} - B_2 R^{-1} B_2^*)\,\hat{X}_{12}^*$ and

  $$\Omega \stackrel{\text{def}}{=} \hat{X}_{12} A_{21} + A_{21}^* \hat{X}_{12}^* - \hat{X}_{12} B_2 R^{-1} B_2^* \hat{X}_{12}^* + Q_{11}.$$

Example 2.1. Consider (1) with the following matrices:

$$A = \begin{bmatrix} 0 & -6 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}, \quad Q = \operatorname{diag}\{0, 16, 0\}, \quad R = 1.$$

It is easy to see that the Hamiltonian matrix has eigenvalues on the imaginary axis, and in particular at zero. We have $K_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We change coordinates using the orthogonal matrix

$$T = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right],$$

and we obtain

$$T^* A\, T = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -6 \\ 0 & -2 & -1 \end{array}\right], \quad T^* B = \left[\begin{array}{c} 0 \\ \hline 5 \\ 6 \end{array}\right], \quad T^* Q\, T = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 16 \end{array}\right].$$

We therefore define

$$A_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; A_{22} = \begin{bmatrix} 0 & -6 \\ -2 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \; Q_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}.$$

We compute the solution of the reduced-order Riccati equation that corresponds to the matrices A_{22}, B_2, Q_{22}, R. Its associated Hamiltonian matrix now does not have eigenvalues on the imaginary axis, and we obtain

$$X_{22} = \begin{bmatrix} 4.5033 & -4.4565 \\ -4.4565 & 4.9991 \end{bmatrix}.$$

Since the Hamiltonian matrix of (A_{22}, B_2, Q_{22}, R) does not have eigenvalues at zero, the closed-loop matrix of this subsystem, $A_{X_{22}}$, is non-singular. We can therefore compute X_{12} as

$$X_{12} = \underbrace{(-A_{21}^* X_{22} - Q_{12})}_{=0} A_{X_{22}}^{-1} = 0.$$

Notice that this solution satisfies (5). It follows that

$$X = T \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 4.5033 & -4.4565 \\ 0 & -4.4565 & 4.9991 \end{array}\right] T^* = \left[\begin{array}{cc|c} 4.5033 & -4.4565 & 0 \\ -4.4565 & 4.9991 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

is a symmetric positive semidefinite solution of (1).

Example 2.2. Consider (1) with the following matrices:

$$A = \begin{bmatrix} -2 & -1 & 0 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad R = 1.$$

It is easy to see that the Hamiltonian matrix has a double eigenvalue at zero, but its geometric multiplicity is 1, since $\ker H = \text{span}\left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top\right\}$. We have $K_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, while K_1 is empty. We change coordinates using the orthogonal matrix

$$T = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right],$$

and we obtain

$$T^* A\, T = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & -2 & -1 \\ -1 & -2 & 0 \end{array}\right], \quad T^* B = \left[\begin{array}{c} 0 \\ \hline 1 \\ 2 \end{array}\right], \quad T^* Q\, T = \left[\begin{array}{c|cc} 1 & 0 & 2 \\ \hline 0 & 0 & 0 \\ 2 & 0 & 4 \end{array}\right].$$

We therefore define

$$A_{21} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; A_{22} = \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$Q_{11} = 1, \quad Q_{12} = \begin{bmatrix} 0 & 2 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

We compute the solution of the reduced-order Riccati equation for the matrices A_{22}, B_2, Q_{22}, R, whose Hamiltonian matrix now does not have eigenvalues on the imaginary axis, and we obtain

$$X_{22} = \begin{bmatrix} 0.4960 & -0.7034 \\ -0.7034 & 1.5143 \end{bmatrix}.$$

Since the Hamiltonian matrix of (A_{22}, B_2, Q_{22}, R) does not have eigenvalues at zero, the closed-loop matrix of this subsystem, $A_{X_{22}}$, is non-singular. We can therefore compute X_{12} as

$$X_{12} = (-A_{21}^*\, X_{22} - Q_{12})\, A_{X_{22}}^{-1} = \begin{bmatrix} 0.7121 & -0.4047 \end{bmatrix}.$$

However, a direct substitution shows that this solution does not satisfy (5). It follows that the original Riccati equation does not admit Hermitian solutions.

Let us now consider the case of $\lambda \in \mathbb{I} \setminus \{0\}$. We introduce a change of basis $T = [\, T_1 \; T_2 \; T_3 \,]$, where $T_1 = K_2$ is a basis matrix for $\ker\left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right]$, $T_2 = \overline{T}_1$ and T_3 is such that T is invertible. With this choice, matrix T is not, in general, unitary. It is easy to see that

$$T^* A\, T^{-*} = \begin{bmatrix} \lambda I & 0 & 0 \\ 0 & \bar{\lambda} I & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad T^* B = \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix}.$$

We can partition in this new basis Q and the solution X of the Riccati equation conformably as

$$T^* Q\, T^{-*} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}, \quad T^* X\, T^{-*} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix}.$$

Since λ is purely imaginary, we have $\lambda + \bar{\lambda} = 0$, so that, replacing these partitioned matrices into (1) yields the 6 equations

$$X_{13} A_{31} + A_{31}^* X_{13}^* - X_{13} B_3 R^{-1} B_3^* X_{13}^* + Q_{11} = 0 \quad (10)$$

$$2\bar{\lambda} X_{12} + X_{13} A_{32} + A_{31}^* X_{23}^* - X_{13} B_3 R^{-1} B_3^* X_{23}^* + Q_{12} = 0 \quad (11)$$

$$X_{13} A_{33} + \bar{\lambda} X_{13} + A_{31}^* X_{33} - X_{13} B_3 R^{-1} B_3^* X_{33}^* + Q_{13} = 0 \quad (12)$$

$$X_{23} A_{32} + A_{32}^* X_{23}^* - X_{23} B_3 R^{-1} B_3^* X_{23}^* + Q_{22} = 0 \quad (13)$$

$$X_{23} A_{33} + \lambda X_{23} + A_{32}^* X_{33} - X_{23} B_3 R^{-1} B_3^* X_{33} + Q_{23} = 0 \quad (14)$$

$$X_{33} A_{33} + A_{33}^* X_{33} - X_{33} B_3 R^{-1} B_3^* X_{33} + Q_{33} = 0. \quad (15)$$

Notice that X_{11} and X_{22} do not appear in these equations. Thus, their values are completely arbitrary. Notice also that the last equation depends only on X_{33}, and has the structure of a reduced-order Riccati equation, associated with the closed-loop matrix

$$A_{X_{33}} = A_{33} - B_3\, R^{-1}\, B_3^*\, X_{33}.$$

It follows that (12) can be written as

$$X_{13}\, (A_{X_{33}} + \bar{\lambda}\, I) = -A_{31}^*\, X_{33} - Q_{13}.$$

This equation admits solutions if and only if $\ker(A_{X_{33}} + \bar{\lambda}\, I) \subseteq \ker(-A_{31}^*\, X_{33} - Q_{13})$. If it does not admit solutions, the original Riccati equation does not have solutions. The set of its solutions is parameterized in terms of the matrix K_{13} as $X_{13} = \hat{X}_{13} + K_{13}\, \Delta_{13}$, where $\Delta_{13}\, (A_{X_{33}} + \bar{\lambda}\, I) = 0$. Replacing this set of solutions into (10), we obtain a reduced-order Riccati equation in K_{13} which reads as

$$K_{13}\, \hat{A}_{13} + \hat{A}_{13}^*\, K_{13}^* - K_{13}\, \Delta_{13}\, B_3\, R^{-1}\, B_3^*\, \Delta_{13}^*\, K_{13}^* + \Omega_1 = 0,$$

where $\hat{A}_{13} \stackrel{\text{def}}{=} \Delta_{13}\, (A_{31} - B_3\, R^{-1}\, B_3^*\, \hat{X}_{13}^*)$ and

$$\Omega_1 = \hat{X}_{13}\, A_{31} + A_{31}^*\, \hat{X}_{13}^* - \hat{X}_{13} B_3\, R^{-1}\, B_3^*\, \hat{X}_{13}^* + Q_{11}.$$

Likewise, (14) can be re-written as

$$X_{23}\, (A_{X_{33}} + \lambda\, I) = -A_{32}^*\, X_{33} - Q_{23}.$$

This equation admits solutions if and only if $\ker(A_{X_{33}} + \lambda\, I) \subseteq \ker(-A_{32}^*\, X_{33} - Q_{23})$. If it does not admit solutions, the original Riccati equation does not have solutions. The set of its solutions is parameterized in terms of the matrix K_{23} as $X_{23} = \hat{X}_{13} + K_{23}\, \Delta_{23}$, where $\Delta_{23}\, (A_{X_{33}} + \lambda\, I) = 0$. Replacing this set of solutions into (10), we obtain a reduced-order Riccati equation in K_{13} which reads as

$$K_{23}\, \hat{A}_{23} + \hat{A}_{23}^*\, K_{23}^* - K_{23}\, \Delta_{23}\, B_3\, R^{-1}\, B_3^*\, \Delta_{23}^*\, K_{23}^* + \Omega_2 = 0,$$

where $\hat{A}_{23} \stackrel{\text{def}}{=} \Delta_{23}\, (A_{32} - B_3\, R^{-1}\, B_3^*\, \hat{X}_{23}^*)$ and

$$\Omega_2 = \hat{X}_{23}\, A_{32} + A_{32}^*\, \hat{X}_{23}^* - \hat{X}_{23} B_3\, R^{-1}\, B_3^*\, \hat{X}_{23}^* + Q_{22}.$$

2.2 Reduction associated with K_1

Consider the case $\lambda = 0$. We introduce a change of basis in $\mathbb{C}^n$ given by $T = [\, T_1 \; T_2 \,]$, where T_1 is an orthonormal basis for K_1 and T is unitary. Thus, the subspace $\operatorname{im} K_1$ in the new basis is written as $\operatorname{im}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. Since $(A - \lambda\, I)\, K_1 = 0$, we have also $A\, K_1 = \lambda\, K_1 = 0$, which can be written in the new basis as

$$(T^* A T)\, (T^* K_1) = \lambda\, T^* K_1 = 0. \quad (16)$$

Partitioning $T^* A T$ as $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$, (16) becomes

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} I \\ 0 \end{bmatrix} = 0,$$

which leads to $A_{11} = \lambda\, I = 0$ and $A_{22} = 0$. Thus, in the new basis

$$\tilde{A} = T^* A T = \begin{bmatrix} I\, \lambda & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

From $Q\, K_1 = 0$ and the fact that Q is Hermitian, we find that $\tilde{Q} = T^* Q\, T = \operatorname{diag}\{0, Q_{22}\}$.

Let us also introduce the partitioning

$$\tilde{X} = T^* X\, T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \qquad \tilde{B} = T^* B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The Riccati equation in this new basis reads as

$$\tilde{X}\,\tilde{A} + \tilde{A}^*\,\tilde{X} - \tilde{X}\,\tilde{B}\,R^{-1}\,\tilde{B}^*\,\tilde{X} + \tilde{Q} = 0.$$

The partitioned structure of the Riccati equation leads to the following three equations:

$$\begin{aligned}
&-(X_{11}B_1 + X_{12}B_2)\,R^{-1}\,(B_1^* X_{11} + B_2^* X_{12}^*) = 0 \\
&(X_{11}A_{12} + X_{12}A_{22}) - (X_{11}B_1 + X_{12}B_2)\,R^{-1}\,(B_1^* X_{12} + B_2^* X_{22}^*) = 0 \\
&(X_{12}^* A_{12} + X_{22}A_{22}) + (A_{12}^* X_{12} + A_{22}^* X_{22}) \\
&\qquad -(X_{12}^*\,B_1 + X_{22}B_2)\,R^{-1}\,(B_1^* X_{12} + B_2^* X_{22}^*) + Q_{22} = 0.
\end{aligned}$$

The first yields $X_{11}B_1 + X_{12}B_2 = 0$, which once substituted into the second yields $X_{11}A_{12} + X_{12}A_{22} = 0$. These two equations can be written together as

$$\begin{bmatrix} A_{12}^* & A_{22}^* \\ B_1^* & B_2^* \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12}^* \end{bmatrix} = 0.$$

On the other hand, since the first elimination procedure has already been carried out, the nullspace of the matrix

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_{12}^* & A_{22}^* \\ B_1^* & B_2^* \end{bmatrix}$$

is the origin. This implies that the submatrices X_{11} and X_{12} are zero. Therefore, the third equation can be written as

$$X_{22}A_{22} + A_{22}^* X_{22} - X_{22}B_2\,R^{-1}\,B_2^* X_{22}^* + Q_{22} = 0,$$

which is a reduced-order Riccati equation.

We now consider the case where $\lambda \in \mathbb{I}\backslash\{0\}$. Let $T = [\, T_1 \;\; T_2 \;\; T_3 \,]$ be a change of coordinates in $\mathbb{C}^n$, where $T_1 = K_1$ is a basis matrix of $\ker \left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ and $T_2 = \overline{K}_1$. We find

$$T^{-1} A\, T = \begin{bmatrix} \lambda I & 0 & A_{13} \\ 0 & \bar{\lambda} I & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad T^{-1} Q\, T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}.$$

Partitioning B and X in the new basis as

$$T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad \text{and} \quad T^{-1} X\, T = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix},$$

we can substitute these partitioned matrices in the Riccati equation written in the new basis. Recalling that $\lambda + \bar{\lambda} = 0$, from the submatrices in position $(1,1)$ we obtain the equation

$$-(X_{11}B_1 + X_{12}B_2 + X_{13}B_3)R^{-1}(B_1^* X_{11} + B_2^* X_{12}^* + B_3^* X_{13}^*) = 0,$$

which gives

$$B_1^* X_{11} + B_2^* X_{12}^* + B_3^* X_{13}^* = 0. \tag{17}$$

It follows that the equation in position $(1,2)$ becomes $2\,\bar{\lambda}\,X_{12} = 0$, so that $X_{12} = 0$. The equation in position $(2,2)$ becomes

$$-(X_{12}B_1 + X_{22}B_2 + X_{23}B_3)R^{-1}(B_1^* X_{12} + B_2^* X_{22}^* + B_3^* X_{23}^*) = 0,$$

which gives

$$B_1^* X_{12} + B_2^* X_{22}^* + B_3^* X_{23}^* = 0. \tag{18}$$

Taking into account that $X_{12} = 0$, the term in position $(2,3)$ yields

$$X_{22}\,A_{23} + X_{23}\,A_{33} + \lambda\,X_{23} = 0. \tag{19}$$

Likewise, the term in position $(1,3)$ yields

$$X_{11}\,A_{13} + X_{13}\,A_{33} + \bar{\lambda}\,X_{13} = 0. \tag{20}$$

Writing (17) and (20) together gives

$$\begin{bmatrix} A_{13}^* & A_{33}^* + \lambda I \\ B_1^* & B_3^* \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{13}^* \end{bmatrix} = 0.$$

Since $X_{12}^* = 0$, we can rewrite the same equation as

$$\begin{bmatrix} A_{13}^* & A_{23}^* & A_{33}^* + \lambda I \\ B_1^* & B_2^* & B_3^* \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12}^* \\ X_{13}^* \end{bmatrix} = 0.$$

On the other hand, carrying out the procedure for the elimination of K_1 after the elimination of K_2 has been carried out means that $\ker \left[\begin{smallmatrix} -B^* \\ -A^*-\lambda I \end{smallmatrix}\right] = \{0\}$, so that in this basis

$$\ker \begin{bmatrix} -B_1^* & -B_2^* & -B_3^* \\ 0 & 0 & 0 \\ 0 & -2\,\lambda I & 0 \\ -A_{13}^* & -A_{23}^* & -A_{33}^* - \lambda I \end{bmatrix} = \{0\}.$$

Since $X_{12} = 0$, it follows that X_{11} and X_{13} are both zero.

Writing (18) and (19) gives

$$\begin{bmatrix} A_{23}^* & A_{33}^* + \bar{\lambda} I \\ B_1^* & B_3^* \end{bmatrix} \begin{bmatrix} X_{22} \\ X_{23}^* \end{bmatrix} = 0.$$

Since $X_{12}^* = 0$, we can rewrite the same equation as

$$\begin{bmatrix} A_{13}^* & A_{23}^* & A_{33}^* + \lambda I \\ B_1^* & B_2^* & B_3^* \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{22} \\ X_{23}^* \end{bmatrix} = 0.$$

Since λ is an eigenvalue of H, such is also $\bar{\lambda}$. It follows that $\ker \left[\begin{smallmatrix} -B^* \\ -A^*-\bar{\lambda} I \end{smallmatrix}\right] = \{0\}$, which can be re-written as

$$\ker \begin{bmatrix} -B_1^* & -B_2^* & -B_3^* \\ -2\,\bar{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ -A_{13}^* & -A_{23}^* & -A_{33}^* - \bar{\lambda} I \end{bmatrix} = \{0\}.$$

Since $X_{12} = 0$, we obtain $X_{22} = 0$ and $X_{23} = 0$. With these results, the equation in position $(3,3)$ becomes

$$X_{33}A_{33} + A_{33}^* X_{33} - X_{33}B_3 R^{-1} B_3^* X_{33} + Q_{33} = 0. \tag{21}$$

It follows that the solution of the original Riccati equation is

$$T \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & X_{33} \end{array}\right] T^{-1},$$

where X_{33} is a solution of the reduced-order Riccati equation (21).

Example 2.3. Consider (1) with the following matrices:

$$A = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & -10 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \quad Q = \mathrm{diag}\{1, 0, 0, 0\}, \quad R = 1.$$

It is easy to see that the Hamiltonian matrix has double eigenvalues on the imaginary axis, and in particular at $\pm 10\,i$. Let

$\lambda = 10\,i$. We have $\ker\begin{bmatrix} -B^* \\ -A^*-\lambda I \end{bmatrix} = \{0\}$ and $\ker\begin{bmatrix} A-\lambda I \\ Q \end{bmatrix} \neq \{0\}$. Thus, we need to perform the second reduction, which is relative to K_1. To this end, we find that a basis for $\ker\begin{bmatrix} A-\lambda I \\ Q \end{bmatrix}$ is given by $[\,0\ 0\ 1\ i\,]^\top$, so that with the change of coordinate matrix

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \end{bmatrix}$$

we obtain

$$\tilde{A} = T^{-1} A\, T = \left[\begin{array}{c|c|cc} 10\,i & 0 & 0 & -\sqrt{2}/2 \\ \hline 0 & -10\,i & 0 & -\sqrt{2}/2 \\ \hline 0 & 0 & 0 & 6 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

and $\tilde{Q} = T^{-1}\,Q\,T = \mathrm{diag}\{0, 0, 1, 0\}$, so that

$$A_{13} = A_{23} = \begin{bmatrix} 0 & -\sqrt{2}/2 \end{bmatrix},\ A_{33} = \begin{bmatrix} 0 & 6 \\ 3 & 0 \end{bmatrix}$$

and $Q_{33} = \mathrm{diag}\{1, 0\}$. Finally, we have

$$\tilde{B} = T^{-1}\,B = \left[\begin{array}{c} -\frac{1}{\sqrt{2}} \\ \hline -\frac{1}{\sqrt{2}} \\ \hline 0 \\ 5 \end{array}\right],$$

so that $B_3 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. The reduced-order Riccati equation (21) admits 4 solutions. Indeed, defining

$$a = \frac{3+\sqrt{34}}{25},\ b = \frac{2\,\sqrt{3\,(3+\sqrt{34})}}{25},\ c = \frac{1}{25}\,\sqrt{34 + \frac{34\,\sqrt{34}}{3}},$$

we obtain two real solutions

$$X_{33}^{1,2} = \begin{bmatrix} a & \pm b \\ \pm b & \pm c \end{bmatrix}.$$

Defining

$$d = \frac{3-\sqrt{34}}{25},\ e = \frac{2\,i}{25}\,\sqrt{3\,(\sqrt{34}-3)},\ f = \frac{i}{25}\,\sqrt{\frac{34\,(\sqrt{34}-3)}{3}},$$

we obtain the complex solutions

$$X_{33}^{3,4} = \begin{bmatrix} d & \pm e \\ \pm e & \mp f \end{bmatrix}.$$

For each of these solutions $X_{33}^{i,j}$, we can build $\tilde{X}_{i,j} = \mathrm{diag}\{0, 0, X_{33}^{i,j}\}$, and we have that in the original basis

$$X_{i,j} = T\,\tilde{X}_{i,j}\,T^{-1} = \begin{bmatrix} X_{33}^{i,j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are all the solutions of (1).

3. CONCLUSION

In this paper we have developed a methodology aimed at decomposing the continuous-time ARE associated with a Hamiltonian matrix with imaginary eigenvalues. This decomposition allows to compute the corresponding solutions and to understand their structure. In particular, we have shown that all the solutions may be constructed by suitably combining some free parameters, the solution of a linear equation and the solutions of a reduced-order ARE whose Hamiltonian matrix has no purely imaginary eigenvalues. Future work includes the design of a robust algorithmic framework to deliver the entire set of Hermitian/symmetric solutions of the continuous-time Riccati equation.

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