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ON THE LIPSCHITZ REGULARITY FOR MINIMA OF FUNCTIONALS DEPENDING ON x, u, AND ∇u UNDER THE BOUNDED SLOPE CONDITION*

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Abstract. We prove the existence of a global Lipschitz minimizer of functionals of the form $\mathcal{I}(u) = \int_\Omega f(\nabla u(x)) + g(x, u(x))\, dx$, $u \in \phi + W_0^{1,1}(\Omega)$, assuming that ϕ satisfies the bounded slope condition (BSC). Our assumptions on the Lagrangian allow the function f to be strongly degenerate.

Key words. bounded slope condition, regularity for minimizers, barriers

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1. Introduction.

In this paper we address the problem of regularity of scalar minimizers of integral functionals of the type

$$\mathcal{L}(u) = \int_\Omega L(x, u, \nabla u)\, dx, \qquad u \in \phi + W_0^{1,1}(\Omega),$$

where Ω is an open, bounded, and convex subset of $\mathbb{R}^n$ and ϕ satisfies the bounded slope condition (BSC). In order to contextualize the problem, we recall some results available in literature. In the pioneering paper [33], it has been proven that, for $L = L(\xi)$ strictly convex and for a given function ϕ satisfying the BSC of rank K (see Definition 4.3 below), there exists a Lipschitz function with the same rank of ϕ that is a minimizer of

$$\mathcal{L}(u) = \int_\Omega L(\nabla u)\, dx \tag{1.1}$$

in the class of Lipschitz functions coinciding with ϕ on $\partial\Omega$. More recently [7], under the same assumptions, Cellina proved that if v is a minimum of the functional (1.1) in $\phi + W_0^{1,1}(\Omega)$, then v is Lipschitz of rank K. A remarkable feature of these results is that neither the uniform convexity nor the growth of L plays any role as it happens in the huge amount of literature on regularity theory in the calculus of variations.

This observation leads naturally to the question of whether the same approach can be used for functionals depending also on (x, u).

In the proofs of both papers cited above, the key tools are a comparison principle between the minimum and the affine functions, the invariance of the minimizers with respect to translations, and a sort of a maximum principle for the gradient of the minimum (see also [22]).

The use of these instruments can be delicate when considering a more general framework. First, we point out that the comparison principle may fail when dropping

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the strict convexity of L, as it happens even in very simple cases where L depends only on the gradient (see Example 1 in [6]). Anyway, the validity of comparison principles for general functionals, depending also on (x, u), has been widely studied, and, in particular, it has been shown that they hold for very general scalar functionals but only for special minimizers (see, for example, [27], [29], [28], [8], [26]). Moreover, another difficulty arises from the fact that the role of the BSC in both [33] and [5] is strictly related to the minimality property of the affine functions, which gets lost in the case of dependence on (x, u, ξ). Last but not least, in this case the minimizers lose both the invariance property with respect to translations and the validity of the maximum principle for their gradients.

Here, we renounce to a general structure of L and study functionals of the form

$$\mathcal{I}(u) = \int_\Omega f(\nabla u) + g(x, u)\, dx, \quad u \in \phi + W_0^{1,1}(\Omega), \tag{1.2}$$

where f is convex and g satisfies suitable assumptions with respect to both variables, and we prove the existence of a Lipschitz minimizer of (1.2); see Theorem 4.8. We underline that our assumptions on f will be very mild: they do not imply particular growth properties of the Lagrangian, and they allow us to consider also functions with anisotropic growth with respect to the gradient variable; see Example 5.3.

The reasons for which we have chosen a functional of the type (1.2) are due to some considerations. On the one hand, many regularity results show that functionals of sum type exhibit better behavior; on the other hand, for such functionals a Haar–Radò type theorem (see [30] and Theorem 2.2 below) was proved that leads to the maximum principle for the gradients. A strong inspiration for the present paper arises from the results in [17] where it has been proven the existence of a Lipschitz minimizer of

$$\mathcal{J}(u) = \int_\Omega f(|\nabla u|) + g(x, u)\, dx, \quad u \in \phi + W_0^{1,1}(\Omega),$$

under the assumptions that Ω is uniformly convex and ϕ satisfies the BSC. In particular, we shall borrow from [17] a comparison principle between a minimizer of $\mathcal{I}(u)$ and suitable minimizers of a conveniently constructed functional (see [7] for the construction of the special minimizers and Theorem 2.4 below for the precise statement of the comparison principle).

We underline that in [17] the radial structure of f is heavily used in the long computations leading to the construction of two barriers that bound the minimizers. Here we remove the assumption of radial structure of f and prove first the existence of Lipschitz barriers and then the existence of a Lipschitz minimizer, assuming only that the boundary datum satisfies the BSC. Finally, we recall that the authors of [17] assume the uniform convexity of f only in some small annuli. Similarly, we assume the uniform convexity of f only in some small regions of the domain.

We point out that many authors faced the question of how to weaken the assumption of everywhere uniform convexity. In particular, there is a vast literature concerning the Lipschitz regularity of the local minimizers of functionals depending on (x, ξ) under the assumption of uniform convexity only at infinity. The study of the regularity of local minimizers of nonuniformly convex functionals started with [9] (see also [21]), where integrands enjoying a p-Laplacian-type structure at infinity were considered. Actually, for nonsmooth functions, the p-uniform convexity expressed as

$$\frac{1}{2}[F(x, \xi_1) + F(x, \xi_2)] \geq F\left(x, \frac{\xi_1 + \xi_2}{2}\right) + \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2,$$

for every $\xi_1, \xi_2 \in \mathbb{R}^{nN} \setminus B(0, \bar{R})$ endpoint of a segment contained in $\mathbb{R}^{nN} \setminus B(0, \bar{R})$, was already revealed to be sufficient for the Lipschitz continuity of the local minimizers (see [18]). Recently, in [1] the authors considered the functional

$$\int_\Omega f(\nabla u(x)) + h(x)u(x)\, dx,$$

with f uniformly convex outside a ball and $h \in L^\infty(\Omega)$, and used the BSC to prove the global Lipschitz regularity of minimizers.

Moreover, we mention the contributions of [25] and [20] and those appearing in [4], [12], [19] for the nonstandard growth condition case. All these results concern Lagrangians regular with respect to the x variable. Later on, in [14] the Lipschitz regularity was obtained also for the solutions of systems with ellipticity conditions at infinity under a Sobolev regularity assumption on x (see also [14], where integral functionals with variable exponent are considered). Finally, we recall the results of higher integrability for the gradients of the minimizers and of higher differentiability for the minimizers themselves in [10] and [11], respectively, under standard and nonstandard growth conditions. In particular, [10] renounced to the Lipschitz regularity assuming a weaker condition on the integrand with respect to x. To conclude, we mention [32] and [13] where a functional not uniformly convex and depending on (x, ξ) with a very special structure. We underline that this class of functionals includes the case of the area functional in the Heisenberg group.

The paper is organized as follows. In section 2 we recall some notation and known results useful for the construction of the barriers. Section 3 is devoted to the regularity properties of the polar function of f which will be used to describe in detail the functions appearing in the comparison principle. Section 4 is the core of the paper since it contains proofs of both the existence of the barriers and the existence of a Lipschitz minimizer. Finally, section 5 is dedicated to some examples and remarks. Once again, we note that our assumptions do not imply any special growth conditions on the integrand, and, in particular, we allow f to behave differently in different directions.

2. Preliminary results and a comparison principle.

We consider an open bounded domain $\Omega \subset \mathbb{R}^n$ and an integral functional on $W^{1,1}(\Omega)$ of the form

$$\mathcal{I}(u) := \int_\Omega \Big[f(\nabla u(x)) + g(x, u(x)) \Big]\, dx \tag{2.1}$$

for some functions $f \colon \mathbb{R}^n \to \mathbb{R}$ and $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$.

DEFINITION 2.1. *A function $u \in W^{1,1}(\Omega)$ is a minimizer of the functional $\mathcal{I}$ if $\mathcal{I}(u) \le \mathcal{I}(v)$ for every $v \in u + W_0^{1,1}(\Omega)$.*

A standard application of the direct method of the calculus of variations ensures the existence of a minimizer of $\mathcal{I}$ in $\phi + W_0^{1,1}(\Omega)$ for any $\phi \in W^{1,1}(\Omega)$, provided f is convex and superlinear, i.e., $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty$, and g satisfies suitable assumptions.

We recall here a special case of a Haar–Radò-type theorem, which has been proven in its general form in [30, Theorem 5.2].

THEOREM 2.2. *Let f be convex and superlinear, and let g be measurable and convex in the second variable. Assume, moreover, that there exists a positive constant K such that*

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R}, \quad v \ge u + K|y - x| \Rightarrow g_v^+(y, v) \ge g_v^+(x, u),$$

where g_v^+ denotes the right derivative of g with respect to the second variable. If there exist two Lipschitz continuous functions $l^-, l^+ \in \phi + W_0^{1,1}(\Omega)$ of rank L on $\bar{\Omega}$ such that

$$l^-(x) \le u(x) \le l^+(x) \quad a.e.\ in\ \Omega,$$

where $u \in \phi + W_0^{1,1}(\Omega)$ is the maximum or the minimum of the minimizers of $\mathcal{I}$, then $|u(x) - u(y)| \le L|x - y|$ for every Lebesgue point x and y.

Remark 2.3. Note that in [29], it has been proven that the pointwise minimum and the pointwise maximum of the minimizers of $\mathcal{I}$ belong to $\phi + W_0^{1,1}(\Omega)$ and are still minimizers of the same functional provided f is superlinear.

We now define the integral functionals $I_{\pm\alpha}$ on $W^{1,1}(\Omega)$ by setting

$$I_{\pm\alpha}(u) := \int_\Omega \Big[f(\nabla u(x)) \pm \alpha u(x) \Big]\, dx,$$

where α is a positive constant. A result by Cellina (see [7]) states that if f is convex and superlinear, then for every $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the functions $\omega_{\pm\alpha}(x) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\omega_{\pm\alpha}(x) := \frac{n}{\pm\alpha} f^*\Big(\pm \alpha \frac{x - x_0}{n} \Big) + c \tag{2.2}$$

are unique minimizers of $I_{\pm\alpha}$ in the sense that $I_{\pm\alpha}(\omega_{\pm\alpha}) < I_{\pm\alpha}(v)$ for every $v \in \omega_{\pm\alpha} + W_0^{1,1}(\Omega)$. We remark that the hypotheses on f guarantee that $\omega_{\pm\alpha} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.

In order to state a comparison result between the minimizers of $\mathcal{I}$ and the minimizers of I_α, $I_{-\alpha}$, we have to be precise about what we mean when we say that two Sobolev functions satisfy an inequality on the boundary of Ω: for given $u, v \in W^{1,1}(\Omega)$, we shall write $u \le v$ on $\partial\Omega$ if $(u - v)^+ \in W_0^{1,1}(\Omega)$.

THEOREM 2.4 ([17, Theorem 2.4]). *Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear, and suppose that $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to x and Lipschitz continuous in the second variable, with Lipschitz constant equal to α, that is,*

$$|g(x, u_1) - g(x, u_2)| \le \alpha |u_1 - u_2|$$

for every $x \in \Omega$ and $u_1, u_2 \in \mathbb{R}$. Let u be a minimizer of $\mathcal{I}$, and let ω_α, $\omega_{-\alpha}$ be as in (2.2) for some x_0 and c. If $u \ge \omega_{+\alpha}$ on $\partial\Omega$, then $u \ge \omega_{+\alpha}$ a.e. in Ω, and if $u \le \omega_{-\alpha}$ on $\partial\Omega$, then $u \le \omega_{-\alpha}$ a.e. in Ω.

We conclude this section with the following lemma that will be instrumental in what follows.

LEMMA 2.5. *Consider a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$, and assume that $\eta \in \partial f(\xi) \cap \partial f(\zeta)$, $\xi \ne \zeta$. Then f is affine on the segment joining ξ and ζ, i.e.,*

$$f(t\xi + (1 - t)\zeta) = t f(\xi) + (1 - t) f(\zeta) \quad \forall t \in [0, 1]. \tag{2.3}$$

Moreover, $\eta \in \partial f(t\xi + (1 - t)\zeta)$ $\forall t \in [0, 1]$.

Proof. Since f is convex, we have

$$\begin{aligned} f(t\xi + (1 - t)\zeta) &\ge f(\xi) + \eta \cdot (t\xi + (1 - t)\zeta - \xi) \\ &= f(\xi) + (1 - t)\eta \cdot (\zeta - \xi) \end{aligned} \tag{2.4}$$

and

$$f(t\xi + (1-t)\zeta) \geq f(\zeta) + \eta \cdot (t\xi + (1-t)\zeta - \zeta)$$
$$= f(\zeta) + t\eta \cdot (\xi - \zeta). \tag{2.5}$$

At this point, we multiply the inequality in (2.4) for t and the inequality in (2.5) for $1-t$ and sum term to term, obtaining

$$f(t\xi + (1-t)\zeta) \geq tf(\xi) + (1-t)f(\zeta).$$

Therefore the equality (2.3) follows thanks to the convexity.

Moreover, by using that $\eta \in \partial f(\xi)$ and that (2.4) is actually an equality, we obtain

$$\begin{aligned} f(\vartheta) &\geq f(\xi) + \eta \cdot (\vartheta - \xi) \\ &= f(t\xi + (1-t)\zeta) + \eta \cdot (\xi - (t\xi + (1-t)\zeta)) + \eta \cdot (\vartheta - \xi) \\ &= f(t\xi + (1-t)\zeta) + \eta \cdot (\vartheta - (t\xi + (1-t)\zeta)) \end{aligned} \tag{2.6}$$

for every $\vartheta \in \mathbb{R}^n$, which concludes the proof. ∎

3. Regularity properties of the polar function. In what follows we give some basic results on the polar f^* of a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ that will be useful in the next section.

We recall that $f^*\colon \mathbb{R}^n \to [-\infty, +\infty]$ is defined by

$$f^*(\xi) := \sup_{x\in\mathbb{R}^n} \{x \cdot \xi - f(x)\} \quad \forall \xi \in \mathbb{R}^n$$

(see [16]) and denote the effective domain of f by

$$\mathrm{dom} f = \{x \in \mathbb{R}^n : f(x) \in \mathbb{R}\}.$$

The following lemma holds.

Lemma 3.1. *Assume that f is convex,* $\mathrm{dom} f = \mathbb{R}^n$*, $f(\xi) \geq 0$, and $f(0) = 0$. Then f is superlinear if and only if* $\mathrm{dom} f^* = \mathbb{R}^n$*.*

Proof. We first prove that the superlinearity of f implies that the effective domain of its polar function f^* is $\mathbb{R}^n$. Indeed, let us assume by contradiction that $\mathrm{dom}\, f^* \neq \mathbb{R}^n$. This implies the existence of $\eta \in \mathbb{R}^n$ with

$$\sup_{\xi\in\mathbb{R}^n} \{\xi \cdot \eta - f(\xi)\} = f^*(\eta) = +\infty.$$

Therefore, we can find a sequence $(\xi_k)_k \subset \mathbb{R}^n$, such that

$$f(\xi_k) + k < \xi_k \cdot \eta \leq |\xi_k||\eta|.$$

Hence, $\lim_k |\xi_k| = +\infty$ and $\lim_k \frac{f(\xi_k)}{|\xi_k|} \leq |\eta|$, which contradicts the superlinearity condition $\lim_{|\eta|\to+\infty} \frac{f(\eta)}{|\eta|} = +\infty$.

In order to prove the reverse implication, we show that if f is not superlinear, then $\mathrm{dom} f^* \subsetneq \mathbb{R}^n$.

Let $k \in \mathbb{R}^n$, $k \neq 0$, such that $\lim_{t\to+\infty} \frac{f(tk)}{t} = l > 0$. Without loss of generality, we assume $k = e_1$ so that $\lim_{t\to+\infty} \frac{f(te_1)}{t} = l$. It follows that

$$f(te_1) \leq lt + c \quad \forall t > 0. \tag{3.1}$$

Consider $\eta = (\eta_1, 0, \dots, 0) \in \mathbb{R}^n, \eta_1 > l$, and prove that $f^*(\eta) = +\infty$:

$$\begin{aligned} f^*(\eta) &= \sup_{\xi\in\mathbb{R}^n} \{\xi\eta - f(\xi)\} = \sup_{\xi\in\mathbb{R}^n} \{\xi_1\eta_1 - f(\xi)\} \\ &\geq \sup_{t\in\mathbb{R}} \{t\eta_1 - f(te_1)\} \geq \sup_{t\in\mathbb{R}} \{t(\eta_1 - l) - c\} = +\infty, \end{aligned} \tag{3.2}$$

where, in the last estimate, we used (3.1). ∎

In the next corollary we list the properties of the polar function f^*.

Corollary 3.2. *Assume that f is convex,* $\mathrm{dom} f = \mathbb{R}^n$, $f(\xi) \geq 0$, *and* $f(0) = 0$. *Moreover, assume that f is superlinear. Then $f^*(\xi)$ verifies the same properties of f.*

Proof. First, observe that it is immediate to verify that f^* is convex, $f^*(\xi) \geq 0$, and $f^*(0) = 0$ since f verifies the same properties. The superlinearity of f implies that $\mathrm{dom}\, f^* = \mathbb{R}^n$ by using Lemma 3.1. Finally, assume by contradiction that f^* is not superlinear. Lemma 3.1 implies that $\mathrm{dom} f^{**} \subsetneq \mathbb{R}^n$, which is absurd since $f^{**} = f$. ∎

We conclude the section with the following.

Lemma 3.3. *Assume that f is convex and that* $\mathrm{dom} f = \mathbb{R}^n$. *Moreover, assume that there exist $\tau_1 < \tau_2$ and $\epsilon > 0$ such that*

$$f(\xi) \geq f(\zeta) + \eta_\zeta(\xi - \zeta) + \frac{\epsilon}{2}|\xi - \zeta|^2 \tag{3.3}$$

for every η_ζ belonging to the subdifferential $\partial f(\zeta)$ and for every $\xi, \zeta \in A_1^c \cap A_2$, where

$$A_i := \{\xi \in \mathbb{R}^n : f(\xi) < \tau_i\}, \quad i = 1, 2.$$

Then f^* is $C^{1,1}((A_1^*)^c \cap A_2^*)$, with

$$A_i^* := \{x \in \mathbb{R}^n : \exists \xi \in A_i : x \in \partial f(\xi)\}, \quad i = 1, 2,$$

and for every $\eta_\xi \in (A_1^*)^c \cap A_2^*$ such that ∇f^* is differentiable, the second derivatives of f^* satisfy

$$|\partial_{ij} f^*(\eta_\xi)| \leq \frac{1}{\epsilon}. \tag{3.4}$$

Proof. First, we observe that (3.3) implies the strict convexity of f on $A_1^c \cap A_2$. As a first step we show that $\partial f^*(\eta)$ is a singleton for every $\eta \in (A_1^*)^c \cap A_2^*$. Let us suppose that it is not. This means that there exist $\xi \neq \zeta$ in $\partial f^*(\eta)$ and therefore $\eta \in \partial f(\xi) \cap \partial f(\zeta)$. By Lemma 2.5 f is affine on the segment joining ξ and ζ, and hence f is not strictly convex on it. We shall obtain a contradiction once we prove that the segment intersects $A_1^c \cap A_2$. By the definitions of A_2^* and $(A_1^*)^c$ we can deduce that at least one of the two points, say ξ, belongs to $A_1^c \cap A_2$ while $\zeta \in A_1^c$. It is enough to conclude that the intersection above is not empty. Since $\partial f^*(\eta)$ is a singleton, from now on we shall denote it by $\nabla f^*(\eta)$.

By assumption (3.3), for any $\xi, \zeta \in A_1^c \cap A_2$, we have

$$f(\xi) - f(\zeta) - \eta_\zeta(\xi - \zeta) \geq \frac{\epsilon}{2}|\xi - \zeta|^2, \tag{3.5}$$

$$f(\zeta) - f(\xi) - \eta_\xi(\zeta - \xi) \geq \frac{\epsilon}{2}|\xi - \zeta|^2 \tag{3.6}$$

for every $\eta_\zeta \in \partial f(\zeta)$ and $\eta_\xi \in \partial f(\xi)$. By adding term by term (3.5) and (3.6), we get

$$(\eta_\xi - \eta_\zeta)(\xi - \zeta) \geq \epsilon|\xi - \zeta|^2.$$

Applying the Cauchy–Schwarz inequality on the left-hand side, dividing both sides by $\epsilon|\xi - \zeta|$, and recalling that $\xi = \nabla f^*(\eta_\xi)$ and $\zeta = \nabla f^*(\eta_\zeta)$, we get

$$|\nabla f^*(\eta_\xi) - \nabla f^*(\eta_\zeta)| \leq \frac{1}{\epsilon}|\eta_\xi - \eta_\zeta|, \tag{3.7}$$

proving that f^* is $C^{1,1}((A_1^*)^c \cap A_2^*)$. The Hessian matrix is then defined almost everywhere in $(A_1^*)^c \cap A_2^*$. In order to estimate its sup norm we fix a point η_ξ where ∇f^* is differentiable. Inequality (3.7) implies that $\forall i, j = 1, \dots, n$,

$$|\partial_i f^*(\eta_\xi) - \partial_i f^*(\eta_\xi + te_j)| \leq \frac{1}{\epsilon}|t|,$$

and hence,

$$|\partial_{ij} f^*(\eta_\xi)| \leq \frac{1}{\epsilon},$$

which completes the proof. ∎

4. Construction of the barriers.

From now on, we assume that the function $f : \mathbb{R}^n \to \mathbb{R}$ in (2.1) satisfies the following hypotheses:

(F1) f is convex, $f(\xi) \geq 0$ and such that $f(0) = 0$;

(F2) $\operatorname{dom} f = \mathbb{R}^n$;

(F3) for every $k \in \mathbb{N}$ there exist $\epsilon_k > 0$ and $\tau_1^{k+1} > \tau_2^k > \tau_1^k > k$ such that

  (i) $f(\xi) \geq f(\zeta) + \eta_\zeta(\xi - \zeta) + \frac{\epsilon_k}{2}|\xi - \zeta|^2$ for every $\xi, \zeta \in (A_1^k)^c \cap \overline{A_2^k}$ and $\eta_\zeta \in \partial f(\zeta)$, where

$$A_i^k := \{\xi \in \mathbb{R}^n : f(\xi) < \tau_i^k\}, \quad i = 1, 2,$$

  are bounded sets.

  (ii) $\exists\, \alpha, R > 0$ such that $\lim_{k \to +\infty} \epsilon_k d_i^k > 2\frac{\alpha}{n} R$, where $d_i^k = \inf_{\partial A^k} \operatorname{dist}(x, \partial A_i^k)$, $i = 1, 2$, and $A^k = \{\xi \in \mathbb{R}^n : f(\xi) < \tau_k\}$ with $\tau_k = \frac{\tau_1^k + \tau_2^k}{2}$.

  (iii) $\frac{1}{\epsilon_k|\xi|} < \frac{1}{R}$ $\forall \xi \in (A_1^k)^c \cap A_2^k$.

Remark 4.1. We note that hypotheses (F1) and (F3)(i) imply that f is not identically equal to zero.

The first result of this section will be the existence of Lipschitz barriers for the minimizers of the functional (2.1) with a fixed boundary datum in $C^{1,1}(\Omega)$ (see Theorem 4.5). Later, in Theorem 4.6 we shall prove that there exists a minimizer of (1.2) inheriting the global Lipschitz regularity of such barriers.

Before stating and proving the mentioned results, we need some definitions.

DEFINITION 4.2. *An open bounded subset Ω of $\mathbb{R}^n$ is R-uniformly convex, $R > 0$, if for every $\gamma \in \partial\Omega$ there exists a vector $b_\gamma \in \mathbb{R}^n$, with $|b_\gamma| = 1$, such that*

$$R\, b_\gamma \cdot (\gamma' - \gamma) \geq \frac{1}{2}|\gamma' - \gamma|^2 \quad \forall \gamma' \in \partial\Omega. \tag{4.1}$$

We recall that the definition of the BSC, introduced by Hartman and Stampacchia in [24], is the following.

Definition 4.3 (BSC). *The function ϕ satisfies the BSC of rank $M \geq 0$ if for every $\gamma \in \partial\Omega$ there exist z_γ^-, $z_\gamma^+ \in \mathbb{R}^n$, and $M \in \mathbb{R}$ such that*

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^- \cdot (\gamma' - \gamma) \leq \phi(\gamma'),$$

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^+ \cdot (\gamma' - \gamma) \geq \phi(\gamma'),$$

and $|z_\gamma^\pm| \leq M$ for every $\gamma \in \partial\Omega$.

Remark 4.4. The BSC implies that ϕ is Lipschitz of rank M. Moreover, it forces Ω to be convex unless ϕ is affine. Necessary and sufficient conditions for the BSC are studied, respectively, in [23] and [31]. In particular, we shall use the characterization of functions satisfying the BSC on uniformly convex sets given in [31] that states that ϕ satisfies the BSC on a uniformly convex set Ω if and only if ϕ is $C^{1,1}$. To be precise, we shall use it on R-uniformly convex sets.

Theorem 4.5. *Let Ω be an open bounded R-uniformly convex set, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfy hypotheses* (F1)–(F3). *Let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in x and Lipschitz continuous in the second variable with Lipschitz constant equal to α.*

For every fixed function $\phi : \Omega \to \mathbb{R} \in C^{1,1}(\Omega)$, there exists a minimizer of the functional (2.1). *Moreover, there exist $\ell^+, \ell^- : \overline{\Omega} \to \mathbb{R}$, both Lipschitz of rank $L = L(R, f, \phi)$, such that*

$$\ell^-(\gamma) = \phi(\gamma) = \ell^+(\gamma) \quad \text{for every } \gamma \in \partial\Omega$$

and

$$\ell^-(x) \leq u(x) \leq \ell^+(x)$$

for almost every $x \in \Omega$ and for every minimum u.

Proof. In order to prove the existence of a minimizer for the functional (2.1), it will be sufficient to prove that f is superlinear or, equivalently, by virtue of Lemma 3.1, that $\mathrm{dom} f^* = \mathbb{R}^n$.

Let us consider the sets A_i^k and A^k involved in assumptions (F3)(i),(ii) and denote

$$(A_i^k)^* := \{x \in \mathbb{R}^n : \exists \xi \in A_i^k : x \in \partial f(\xi)\}, \quad i = 1, 2,$$

and $(A^k)^*$ the set defined analogously. It is easy to check that, thanks to the assumption that $\mathrm{dom} f = \mathbb{R}^n$, all these sets are bounded. Moreover, they enjoy the following useful properties proven below:

1. $\overline{(A_2^k)^*} \subseteq \overline{(A_1^{k+1})^*}$ for every $k \in \mathbb{N}$.
2. $\overline{(A_1^k)^*} \subset (A_2^k)^*$ for every $k \in \mathbb{N}$.
3. There exists a ball $B(0, r) \subseteq (A_1^1)^*$.

The inclusion in property 1 obviously follows from the definitions of the sets themselves. To prove property 2, we first observe that assumptions (F1) and (F2), together with the definitions of A_1^k, A_2^k and the fact that A_i^k are bounded, imply that the set $A_2^k \setminus \overline{A_1^k}$ is open and nonempty. So, consider $\xi \in A_2^k \setminus \overline{A_1^k}$ and $x \in \partial f(\xi)$. Obviously, $x \in (A_2^k)^*$ but $x \notin \overline{(A_1^k)^*}$. Indeed, let us assume by contradiction that $x \in \overline{(A_1^k)^*}$.

Then there exist a sequence $\{x_h\}_{h\in\mathbb{N}} \subset (A_1^k)^*$ such that x_h converges to x and a sequence $\{\xi_h\}_{h\in\mathbb{N}} \subset A_1^k$ such that $x_h \in \partial f(\xi_h)$ $\forall h \in \mathbb{N}$. Up to a subsequence, ξ_h converges to $\zeta \in \overline{A_1^k}$, and the semicontinuity of the subdifferential implies that $x \in \partial f(\zeta)$. Hence $x \in \partial f(\xi) \cap \partial f(\zeta)$ and, arguing as in the proof of Lemma 3.3, we get that f is affine on the segment joining ξ and ζ that has a nontrivial intersection with $A_2^k \setminus \overline{A_1^k}$ contradicting (F3). Therefore the desired inclusion holds true.

Now it remains to prove the third assertion. To this aim, we shall prove that $0 \in \mathrm{int}(\mathrm{A}_1^1)^* \neq \emptyset$. Observe that $0 \in (A_1^1)^*$. Indeed, $0 \in A_1^1$, and assumption (F1) implies that it is a minimum for f. Hence $0 \in \partial f(0)$.

Consider the set A_1^1, and observe that $0 \in \mathrm{int}A_1^1$ and that A_1^1 is a bounded set. Then for every $x \in \mathrm{S}^{n-1}$, the $(n-1)$-dimensional sphere, there exist $\xi_x \in \partial A_1^1$ and $t_x > 0$ such that $t_x x \in \partial f(\xi_x)$. Denote $\bar{t} := \inf\{t_x : x \in \mathrm{S}^{n-1}$ and $t_x x \in \partial f(\xi_x)\}$, and observe that $\bar{t} \neq 0$. Otherwise, there would exist $\{t_{x_h}\}_h \subset \mathbb{R}^+$ and $\{\xi_{x_h}\}_h \subset \partial A_1^1$ such that $t_{x_h} x \in \partial f(\xi_{x_h})$ and, up to subsequences,

$$t_{x_h} \to 0,$$

$$\xi_{x_h} \to \bar{\xi} \in \partial A_1^1.$$

The semicontinuity of the subdifferential would imply that $0 \in \partial f(\bar{\xi})$ and hence $f(\bar{\xi}) = 0$, which is absurd since $f \equiv \tau_1^1 > 1$ on ∂A_1^1.

In order to conclude, observe that for every $0 < s < \bar{t}$ and for every $x \in \mathrm{S}^{n-1}$, there exists $\xi_{x,s} \in \mathrm{int}A_1^1$ such that $sx \in \partial f(\xi_{x,s})$. It follows that $sx \in (A_1^1)^*$, and hence $B(0,r) \subseteq (A_1^1)^*$ for every $0 < r < \bar{t}$.

At this point, to prove that $\mathrm{dom}f^* = \mathbb{R}^n$, it is sufficient to show that $\lambda x \in \mathrm{dom}f^*$ for every $\lambda > r$ and for every $x \in S^{n-1}$.

To this aim, let us fix $x \in S^{n-1}$ and consider $x_i^k \in \partial(A_i^k)^*$, $i = 1, 2$, $k \in \mathbb{N}$, such that there exists $\lambda_i^k > 0$ such that $\lambda_i^k x = x_i^k$. We remark that, thanks to the inclusions 1 and 2, it follows that $\lambda_1^k < \lambda_2^k \leq \lambda_1^{k+1}$ and

$$\lambda_2^h \geq \sum_{k=1,\dots,h} (\lambda_2^k - \lambda_1^k). \tag{4.2}$$

Let $(y_n)_n \subset (A_i^k)^*$ such that $(y_n)_n \to x_i^k$. Then for every y_n there exists $\zeta_n \in A_i^k$ such that $y_n \in \partial f(\zeta_n)$. Up to a subsequence, $\zeta_n \to \zeta \in \overline{A_i^k}$, and hence, thanks to the semicontinuity of ∂f, we obtain $x_i^k \in \partial f(\zeta)$. Arguing similarly we can prove the existence of $\zeta' \in \overline{(A_i^k)^c}$ such that $x_i^k \in \partial f(\zeta')$. It follows that $x_i^k \in \partial f(\eta)$ for every η belonging to the segment joining ζ and ζ' (see Lemma 2.5) that intersects ∂A_i^k. Denoting by ξ_i^k the point on ∂A_i^k such that $x_i^k \in \partial f(\xi_i^k)$, we can apply assumption (F3)(i) to obtain

$$f(\xi_2^k) - f(\xi_1^k) \geq x_1^k(\xi_2^k - \xi_1^k) + \frac{\epsilon_k}{2}|\xi_2^k - \xi_1^k|^2$$

and

$$f(\xi_1^k) - f(\xi_2^k) \geq x_2^k(\xi_1^k - \xi_2^k) + \frac{\epsilon_k}{2}|\xi_2^k - \xi_1^k|^2.$$

Adding these two inequalities term by term, we obtain

$$(x_2^k - x_1^k)(\xi_2^k - \xi_1^k) \geq \epsilon_k|\xi_2^k - \xi_1^k|^2,$$

and, applying the Cauchy–Schwarz inequality on the left-hand side, we get

$$|x_2^k - x_1^k| \geq \epsilon_k |\xi_2^k - \xi_1^k| \geq \epsilon_k (d_1^k + d_2^k).$$

Hence, by the definition of λ_i^k and (4.2), we have

$$\lambda_2^h |x| \geq \sum_{k=1,\ldots,h} (\lambda_2^k - \lambda_1^k)|\,|x| = \sum_{k=1,\ldots,h} |x_2^k - x_1^k| \geq \sum_{k=1,\ldots,h} \epsilon_k (d_1^k + d_2^k).$$

Assumption (F3)(ii) implies that $\lim_h \lambda_2^h |x| = +\infty$, and hence $\operatorname{dom} f^* = \mathbb{R}^n$.

For the second assertion of the theorem, it is sufficient to construct the function ℓ^- (the construction of ℓ^+ follows in exactly the same way).

As we pointed out in Remark 4.4, note that ϕ satisfies the BSC. Fix a point $\gamma \in \partial\Omega$, and consider the vector z_γ^- involved in the definition of the BSC at the point γ. The proof of the theorem will be achieved once we show that there exist x_γ, c_γ such that the set

$$\Omega_{x_\gamma, c_\gamma} := \Big\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \Big(\frac{\alpha}{n} (x - x_\gamma) \Big) + c_\gamma - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0 \Big\} \tag{4.3}$$

contains Ω and $\gamma \in \partial\Omega_{x_\gamma, c_\gamma} \cap \partial\Omega$. In fact, by this last property, we have immediately that the inequality

$$l_{x_\gamma, c_\gamma}(x) := \frac{n}{\alpha} f^* \Big(\frac{\alpha}{n} (x - x_\gamma) \Big) + c_\gamma \leq z_\gamma^- \cdot (x - \gamma) + \phi(\gamma) \leq \phi(x)$$

holds for any $x \in \partial\Omega$. The comparison principle in Theorem 2.4 then implies that

$$l_{x_\gamma, c_\gamma}(x) \leq u(x) \qquad \text{a.e. on } \Omega$$

for every minimizer u with boundary datum ϕ. We get the result by simply setting

$$\ell^-(x) = \sup_{\gamma \in \partial\Omega} l_{x_\gamma, c_\gamma}(x).$$

We divide the core of the proof into different steps.

Step 1. Define the auxiliary domain as follows: fix $a \in \mathbb{R}^n$, $b > 0$, and let

$$\Omega_b := \Big\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \Big(\frac{\alpha}{n} x \Big) - a \cdot x - b < 0 \Big\}.$$

Whenever we assume that (F1) and (F2) hold, Lemma 3.1 implies that Ω_b is bounded for every b and that 0 is contained in its interior for $b > 0$.

Fix $\tau_k \in (\tau_1^k, \tau_2^k)$. For $A_k := \{\xi \in \mathbb{R}^n : f(\xi) < \tau_k\}$, consider $A_k^* := \{x \in \mathbb{R}^n : \exists \xi \in A_k : x \in \partial f(\xi)\}$ for which $(A_1^k)^* \subset A_k^* \subset (A_2^k)^*$ obviously hold, and let k be such that

$$\Big| \nabla f^* \Big(\frac{\alpha}{n} x \Big) \Big| > |a| \quad \text{ for every } \quad \frac{\alpha}{n} x \in \partial A_k^*. \tag{4.4}$$

This is guaranteed, for k sufficiently large, because Corollary 3.2 yields that f^* is superlinear and $\operatorname{dom} f^* = \mathbb{R}^n$ so that

$$\lim_{k \to +\infty} \operatorname{diam}(A_i^k)^* = +\infty, \quad i = 1, 2. \tag{4.5}$$

Now let us select a special domain of type (4.4). Fix k such that (4.4) holds and $\eta \in S^{n-1}$. There exist $\frac{\alpha}{n} x_\eta \in A_k^*$ and $\lambda_\eta > 0$ such that

$$\lambda_\eta \eta = \nabla f^* \Big(\frac{\alpha}{n} x_\eta \Big) - a. \tag{4.6}$$

If we define

$$b_\eta := \frac{n}{\alpha} f^*\left(\frac{\alpha}{n} x_\eta\right) - a \cdot x_\eta,$$

we can deduce from the definitions above that $x_\eta \in \partial\Omega_{b_\eta}$ and therefore, by (4.6), that the outward normal to Ω_{b_η} in x_η is η.

Now we are interested in proving that we can find a ball of radius R contained in Ω_{b_η} that touches $\partial\Omega_{b_\eta}$ in x_η. Hence, in the next step we shall compute the principal curvatures of $\partial\Omega_{b_\eta}$ in a neighborhood of x_η, and in Steps 3 and 4 we will show the existence of such a ball.

Step 2. Since $\partial\Omega_{b_\eta}$ is described by the equation $G(y) = 0$, where

$$G(y) := \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n} y\Big) - a \cdot y - b_\eta, \tag{4.7}$$

the principal curvatures of $\partial\Omega_{b_\eta}$ in a neighborhood of x_η can be found through the calculation of the second derivatives of the function ψ implicitly defined by $G(y) = 0$. Without loss of generality, we can suppose $y_n = \psi(y_1, y_2, \ldots, y_{n-1}) = \psi(\hat{y})$.

Observe that, by Lemma 3.3, the function $f^* \in C^{1,1}((A_2^k)^* \setminus \overline{(A_1^k)^*})$, and hence for a.e. $y \in (A_2^k)^* \setminus \overline{(A_1^k)^*}$ it has second derivatives. Therefore, if we consider the points $y \in \partial\Omega_{b_\eta}$ such that $\frac{\alpha}{n} y \in (A_2^k)^* \setminus \overline{(A_1^k)^*}$, we assume, without restriction, that for $\mathcal{H}^{n-1}$-a.e. y, we can compute the second derivatives of the function $\psi(\hat{y})$.

Otherwise, a simple measure theory argument implies that we can choose τ_k' sufficiently close to τ_k such that the corresponding set Ω_{b_η} satisfies the desired property. Therefore we have to estimate

$$\frac{\partial_{ij}\psi(\hat{y})}{(1 + |\nabla\psi(\hat{y})|^2)^{3/2}} \quad \text{for}\, i, j = 1, \ldots, n-1.$$

By the implicit function theorem, we have $\frac{\alpha}{n} y = \frac{\alpha}{n}(\hat{y}, \psi(\hat{y})) \in {(A_1^k)^*}^c \cap (A_2^k)^*$, so that

$$\begin{aligned}
&\frac{\partial_{ij}\psi}{(1 + |\nabla\psi|^2)^{3/2}} \\
&= \frac{-\partial_{ij}G(\partial_n G)^2 + \partial_{in}G\partial_j G\partial_n G + \partial_{jn}G\partial_i G\partial_n G - \partial_{nn}G\partial_i G\partial_j G}{|\nabla G|^3} \\
&= \left[\sum_{i=1}^n \left[\partial_i f^*\left(\frac{\alpha}{n} y\right) - a_i\right]^2\right]^{-\frac{3}{2}} \cdot \frac{\alpha}{n}\Big\{ - \partial_{i,j} f^*\Big(\frac{\alpha}{n} y\Big)\Big[\partial_n f^*\Big(\frac{\alpha}{n} y\Big) - a_n\Big]^2 \\
&+ \partial_{in} f^*\Big(\frac{\alpha}{n} y\Big)\Big[\partial_j f^*\Big(\frac{\alpha}{n} y\Big) - a_j\Big] \cdot \Big[\partial_n f^*\Big(\frac{\alpha}{n} y\Big) - a_n\Big] \\
&+ \partial_{jn} f^*\Big(\frac{\alpha}{n} y\Big)\Big[\partial_i f^*\Big(\frac{\alpha}{n} y\Big) - a_i\Big] \cdot \Big[\partial_n f^*\Big(\frac{\alpha}{n} y\Big) - a_n\Big] \\
&- \partial_{nn} f^*\Big(\frac{\alpha}{n} y\Big)\Big[\partial_i f^*\Big(\frac{\alpha}{n} y\Big) - a_i\Big] \cdot \Big[\partial_j f^*\Big(\frac{\alpha}{n} y\Big) - a_j\Big]\Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\frac{\partial_{ij}\psi}{(1 + |\nabla\psi|^2)^{3/2}}\right| \le &\left[\sum_{i=1}^n \left[\partial_i f^*\left(\frac{\alpha}{n} y\right) - a_i\right]^2\right]^{-\frac{3}{2}} \\
&\cdot \frac{\alpha}{n}\Big\{ \sum_{i,j=1}^{n-1} \Big|\partial_{ij} f^*\left(\frac{\alpha}{n} y\right)\Big| + \Big|\partial_{nn} f^*\left(\frac{\alpha}{n} y\right)\Big| \\
&\cdot \sum_{i,j=1}^{n} \Big|\Big(\partial_i f^*\left(\frac{\alpha}{n} y\right) - a_i\Big) \cdot \Big(\partial_j f^*\left(\frac{\alpha}{n} y\right) - a_j\Big)\Big|\Big\}.
\end{aligned}$$

Now observe that, thanks to Lemma 3.3, we have $|\partial_{ij} f^*| \leq \frac{1}{\varepsilon\, k}$, and hence by assumption (F3)(iii) we get

$$\frac{|Hf^*(\frac{\alpha}{n}y)|}{|\nabla f^*(\frac{\alpha}{n}y)|} < \frac{1}{R} \quad \forall y \in \partial\Omega_{b_\eta} \text{ and } \frac{\alpha}{n} y \in (A_1^k)^{*^c} \cap (A_2^k)^*.$$

Step 3. Now we fix $\tau_k = \frac{\tau_1^k + \tau_2^k}{2}$ and prove that $\operatorname{dist}(\frac{\alpha}{n}x_\eta, \partial(A_i^k)^*) \geq 2\frac{\alpha}{n}R$, $i = 1, 2$. We follow exactly the same argument that we used to show the superlinearity of f. Let us give the proof for $(A_1^k)^*$; the proof for the other one is similar. To this aim, let us estimate the $\operatorname{dist}(\frac{\alpha}{n}x_\eta, x)$ for $x \in \partial(A_1^k)^*$. Let $\xi \in \partial A_1^k$ and $\xi_\eta \in \partial A^k$, respectively, be such that $x \in \partial f(\xi)$ and $\frac{\alpha}{n}x_\eta \in \partial f(\xi_\eta)$. From assumption (F3)(i) we have that

$$f(\xi_\eta) - f(\xi) \geq x(\xi_\eta - \xi) + \frac{\epsilon_k}{2}|\xi_\eta - \xi|^2,$$

$$f(\xi) - f(\xi_\eta) \geq \frac{\alpha}{n}x_\eta(\xi - \xi_\eta) + \frac{\epsilon_k}{2}|\xi_\eta - \xi|^2.$$

Adding these two inequalities term by term, we obtain

$$\left(x - \frac{\alpha}{n}x_\eta\right)(\xi - \xi_\eta) \geq \epsilon_k|\xi - \xi_\eta|^2,$$

and, applying Cauchy–Schwarz inequality to the left-hand side, we get

$$\left|x - \frac{\alpha}{n}x_\eta\right| \geq \epsilon_k|\xi - \xi_\eta| \geq \epsilon_k d_1^k.$$

Assumption (F3)(ii) implies that $|x - \frac{\alpha}{n}x_\eta| \geq 2\frac{\alpha}{n}R$.

It follows that the ball $B(\frac{\alpha}{n}x_\eta - \frac{\alpha}{n}R\eta, \frac{\alpha}{n}R)$ is certainly contained in $B(\frac{\alpha}{n}x_\eta, 2\frac{\alpha}{n}R)$ and hence in the set $(A_1^k)^{*^c} \cap (A_2^k)^*$.

Step 4. In order to construct the sets $\Omega_{x_\gamma, c_\gamma}$ in (4.3) and the corresponding functions l_{x_γ, c_γ}, choose $a = z_\gamma^-$, $x_\gamma := \gamma - x_\eta$, and $c_\gamma := \phi(\gamma) - \frac{n}{\alpha}f^*(\frac{\alpha}{n}x_\eta)$ so that

$$\begin{aligned}
&\gamma - x_\eta + \Omega_{b_\eta} \\
&= \left\{x : \frac{n}{\alpha}f^*\Big(\frac{\alpha}{n}(x - \gamma + x_\eta)\Big) - z_\gamma^- \cdot (x - \gamma + x_\eta) - b_\eta < 0\right\} \\
&= \left\{x : \frac{n}{\alpha}f^*\Big(\frac{\alpha}{n}(x - x_\gamma)\Big) + \phi(\gamma) - z_\gamma^- \cdot x_\eta - b_\eta - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0\right\} \\
&= \left\{x : \frac{n}{\alpha}f^*\Big(\frac{\alpha}{n}(x - x_\gamma)\Big) + c_\gamma - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0\right\} = \Omega_{x_\gamma, c_\gamma}
\end{aligned}$$

since

$$z_\gamma^- \cdot x_\eta + b_\eta = \frac{n}{\alpha}f^*\Big(\frac{\alpha}{n}x_\eta\Big).$$

Now let us consider $y \in B(x_\eta - R\eta, R)$ so that $\frac{\alpha}{n}y \in B(\frac{\alpha}{n}x_\eta - \frac{\alpha}{n}R\eta, \frac{\alpha}{n}R)$ and hence, from the conclusion of Step 3, $\frac{\alpha}{n}y \in (A_1^k)^{*^c} \cap (A_2^k)^*$. Then for $\mathcal{H}^{n-1}$-a.e. $y \in B(x_\eta - R\eta, R) \cap \partial\Omega_{b_\eta}$ the principal curvatures of $\partial\Omega_{b_\eta}$ can be estimated as in Step 2, with $\frac{1}{R}$. It follows, by the convexity of Ω_{b_η} that $B(x_\eta - R\eta, R) \subseteq \Omega_{b_\eta}$.

In this way, using also the fact that Ω is R-uniformly convex, we have

$$\Omega \subseteq B(\gamma - R\eta, R) = \gamma - x_\eta + B(x_\eta - R\eta, R) \subseteq \gamma - x_\eta + \Omega_{b_\eta} = \Omega_{x_\gamma, c_\gamma}.$$

Moreover, it is immediate to verify that $\gamma \in \partial\Omega_{x_\gamma, c_\gamma} \cap \partial\Omega$, and this concludes the proof. ∎

Now, combining the results in Theorems 2.2 and 4.5, we have the following.

THEOREM 4.6. *Let Ω be an open bounded R-uniformly convex set. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies hypotheses* (F1)–(F3) *and that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in x, convex, and Lipschitz continuous in the second variable, with Lipschitz constant equal to α. Moreover, assume there exists a positive constant K such that*

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R}, \quad v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u),$$

where g_v^+ denotes the right derivative of g with respect to the second variable.

Then, for every $\phi \in C^{1,1}(\Omega)$, there exists a minimizer $u \in \phi + W_0^{1,1}(\Omega)$ of (1.2) *that is actually in $\phi + W_0^{1,\infty}(\Omega)$.*

Proof. By Theorem 4.5 we know there exist two Lipschitz functions l^- and l^+ such that for every minimum u of the functional $\mathcal{I}(u)$ it holds that

$$l^-(x) \leq u(x) \leq l^+(x) \quad \text{a.e. in } \Omega.$$

Then, Theorem 2.2 guarantees that the minimum and the maximum of the minimizers belong to $\phi + W_0^{1,\infty}(\Omega)$. ∎

Let us note that for a fixed $\phi \in C^{1,1}(\Omega)$ with a known Lipschitz constant K, the construction of the barriers as in Theorem 4.5 holds true assuming a weaker condition on the function f. More precisely, we could replace assumption (F3) with the following:

(F3') There exist constants $0 < \tau_1 < \tau_2$ and $\epsilon > 0$ such that

(i) $f(\xi) \geq f(\zeta) + \eta_\zeta(\xi - \zeta) + \frac{\epsilon}{2}|\xi - \zeta|^2$ for every $\xi, \zeta \in (A_1)^c \cap A_2$ and $\eta_\zeta \in \partial f(\zeta)$, where

$$A_i := \{\xi \in \mathbb{R}^n : f(\xi) < \tau_i\}, \quad i = 1, 2.$$

(ii) $\epsilon \cdot d_i > \frac{2\alpha}{n} R$, where $d_i = \inf_{\partial A} \operatorname{dist}(x, \partial A_i)$, $i = 1, 2$, and $A = \{\xi \in \mathbb{R}^n : f(\xi) < \tau\}$, $\tau = \frac{\tau_1 + \tau_2}{2}$.

(iii) $\frac{1}{\epsilon|\xi|} < \frac{1}{R}\ \forall \xi \in (A_1)^c \cap A_2$.

(iv) the set $A_1 \supset B(0, K)$ and is bounded.

We underline that assumption (F3') doesn't require that for any $k \in \mathbb{N}$ we find sets in which (i)–(iii) hold true but only the existence of a unique region in which those conditions are valid. Actually, under assumption (F3'), we are able to construct the barriers as we did in Theorem 4.5 also for the case when f grows only linearly. More precisely we have the following.

THEOREM 4.7. *Let Ω be an open bounded R-uniformly convex set, and let $\phi : \Omega \to \mathbb{R}$ be a $C^{1,1}(\Omega)$ function with Lipschitz constant K. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies hypotheses* (F1), (F2), *and* (F3') *and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ measurable in x, and Lipschitz continuous in the second variable with Lipschitz constant equal to α.*

Then there exist $\ell^+, \ell^- : \overline{\Omega} \to \mathbb{R}$, both Lipschitz of rank $L = L(R, f, \phi)$, such that

$$\ell^-(\gamma) = \phi(\gamma) = \ell^+(\gamma) \quad \text{for every } \gamma \in \partial\Omega$$

and

$$\ell^-(x) \leq u(x) \leq \ell^+(x) \quad \text{for almost every } x \in \Omega$$

for every minimizer u of the functional (1.2).

The proof follows the ideas of the proof of Theorem 4.5 provided in Step 1. We consider only $a \in \mathbb{R}^n$ such that $|a| < K$. The assumption (F3')(iv) implies the boundedness of the sets Ω_b and that for opportune values of b the estimate of the curvature in Step 2 holds.

We conclude this section by underlining that a version of Theorem 4.6 also holds which replaces assumption (F3) with (F3'). Actually, we need to assume that f is superlinear in order to have both the existence of a minimizer and the maximum and the minimum of the minimizers (see Remark 2.3).

THEOREM 4.8. *Let Ω be an open bounded R-uniformly convex set, and let $\phi : \Omega \to \mathbb{R}$ be a $C^{1,1}(\Omega)$ function with Lipschitz constant K. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies hypotheses* (F1), (F2), *and* (F3') *and is superlinear. Moreover, assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the assumption of Theorem* 4.6. *Then there exists a minimizer $\bar{u} \in \phi + W_0^{1,1}(\Omega)$ of $\mathcal{I}(u)$ that is actually in $\phi + W_0^{1,\infty}(\Omega)$.*

5. Remarks and examples.

In this section we comment on the results obtained above.

Since, as we pointed out in the introduction, there is a vast literature dealing with functionals where the Lagrangian is uniform convex outside a ball, we will show that Theorem 4.7 holds under this kind of assumption instead of (F3). Moreover, assuming suitable hypotheses on the function g, we shall obtain that every minimizer of the functional $\mathcal{I}(u)$ is Lipschitz continuous.

THEOREM 5.1. *Let Ω be an open bounded R-uniformly convex set. Assume that f satisfies hypotheses* (F1)–(F2), *and suppose there exist a ball $B = B(0, r)$ and a constant $\epsilon > 0$ such that*

$$f(\xi) \geq f(\zeta) + \eta_\zeta(\xi - \zeta) + \frac{\epsilon}{2}|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^n \setminus B$ and $\eta_\zeta \in \partial f(\zeta)$. Let g be measurable in x and Lipschitz continuous with respect to the second variable. For any $\phi : \Omega \to \mathbb{R} \in C^{1,1}(\Omega)$ there exists a minimizer u of the functional

$$\int_\Omega \left[f(\nabla v) + g(x, v)\right] dx, \quad v \in \phi + W_0^{1,1}(\Omega),$$

and $\ell^+, \ell^- : \overline{\Omega} \to \mathbb{R}$, both Lipschitz of rank $L = L(R, f, \phi)$, such that

$$\ell^-(\gamma) = \phi(\gamma) = \ell^+(\gamma) \quad \text{for every } \gamma \in \partial\Omega$$

and

$$\ell^-(x) \leq u(x) \leq \ell^+(x) \quad \text{for almost every } x \in \Omega.$$

Moreover, if g is convex and there exists a positive constant K such that

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R}, \quad v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u),$$

then every minimizer $u \in \phi + W_0^{1,1}(\Omega)$ is actually in $W_0^{1,\infty}(\Omega)$.

Proof. The assumption of the uniform convexity outside B implies the superlinearity of the function f, and therefore the existence of the minimizer follows.

The existence of the barriers ℓ^- and ℓ^+ follows by applying Theorem 4.7, since assumption (F3') holds provided $A_1 = B$ and $A_2 = \mathbb{R}^N$. Moreover, Theorem 4.6

yields, and hence the existence of a Lipschitz minimizer u follows. To conclude, let us show that every minimizer v in $\phi + W_0^{1,1}(\Omega)$ has the same regularity of u. Let us define

$$\Omega' = \left\{ x \in \Omega : f\left(\frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\right) < \frac{1}{2} f(\nabla u(x)) + \frac{1}{2} f(\nabla v(x)) \right\}$$

and assume that $|\Omega'| > 0$. By using the definition of Ω' and the convexity of both functions f and g, we have

$$\begin{aligned}
&\mathcal{I}\left(\frac{1}{2}u + \frac{1}{2}v\right) = \int_{\Omega'} f\left(\frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\right) + g\left(x, \frac{1}{2}u + \frac{1}{2}v\right) dx \\
&+ \int_{\Omega\setminus\Omega'} f\left(\frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\right) + g\left(x, \frac{1}{2}u + \frac{1}{2}v\right) dx \\
&< \int_{\Omega'} \frac{1}{2}\left(f(\nabla u) + g(x,u)\right) + \frac{1}{2}\left(f(\nabla v) + g(x,v)\right) dx \\
&+ \int_{\Omega\setminus\Omega'} \frac{1}{2}(f(\nabla u) + g(x,u)) + \frac{1}{2}\left(f(\nabla v) + g(x,v)\right) dx \\
&= \frac{1}{2}\mathcal{I}(u) + \frac{1}{2}\mathcal{I}(v) \le \frac{1}{2}\mathcal{I}(u) + \frac{1}{2}\mathcal{I}(u) = \mathcal{I}(u),
\end{aligned} \tag{5.1}$$

which contradicts the minimality of u and, obviously, the minimality of v if we exchange u and v in the last line of the above inequality.

We deduce that $|\Omega'| = 0$, and hence

$$(\nabla u(x), f(\nabla u(x))) \quad \text{and} \quad (\nabla v(x), f(\nabla v(x)))$$

belong to the same face of the epigraph of f for a.e. $x \in \Omega$. It follows that $\nabla u(x) \neq \nabla v(x)$ if and only if $|\nabla u(x)| < r$ and $|\nabla v(x)| < r$, and therefore we have the Lipschitz continuity of v. ∎

Let us now discuss the assumptions on the function f. We remark that (F3) is a key tool in the proofs of the theorems in section 4; in particular, it is fundamental to prove the superlinearity of f (see the first part of the proof of Theorem 4.5).

In the following Example 5.2 we show that for any superlinear ψ, we can construct a function that satisfies assumption (F3) and has the same growth as ψ. Consequently, we deduce that the functions f we have in mind could have any superlinear growth (see also Example 5.3).

Example 5.2. Let $\psi(t) : [0, +\infty) \to \mathbb{R}$ be a superlinear, strictly convex, and C^1 function, and define a strictly increasing sequence $(\tau_k)_{k\in\mathbb{N}}$ such that $\psi'(\tau_k) = k$. Consider the constant $R > 0$ as in (F3)(ii), and fix $\epsilon_k = \frac{8R}{\tau_{k+1} - \tau_k}$. Now define the function $n(t) = [\psi'(t)]$.

For

$$h''(t) = \begin{cases} \epsilon_{n(t)} & \text{if } t \in (\tau_{n(t)}, \frac{\tau_{n(t)} + \tau_{n(t)+1}}{2}), \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

define

$$f(\xi) = h(|\xi|) = \int_0^{|\xi|} \left(\int_0^s h''(t) dt \right) ds.$$

We have

$$h'(t) \leq \sum_{k=0}^{n(t)} \epsilon_k \frac{(\tau_{k+1} - \tau_k)}{2} = 4R(n(t)+1) = 4R([\psi'(t)]+1)$$

and

$$h'(t) \geq \sum_{k=0}^{n(t)-1} \epsilon_k \frac{(\tau_{k+1} - \tau_k)}{2} = 4Rn(t) = 4R[\psi'(t)].$$

It follows that

$$\lim_{|\xi|\to+\infty} \frac{f(\xi)}{\psi(|\xi|)} = \lim_{|\xi|\to+\infty} \frac{h(|\xi|)}{\psi(|\xi|)} = \lim_{t\to+\infty} \frac{h(t)}{\psi(t)} = \lim_{t\to+\infty} \frac{h'(t)}{\psi'(t)}$$

and

$$\lim_{t\to+\infty} \frac{4R[\psi'(t)]}{\psi'(t)} \leq \lim_{t\to+\infty} \frac{h'(t)}{\psi'(t)} \leq \lim_{t\to+\infty} \frac{4R([\psi'(t)]+1)}{\psi'(t)}. \tag{5.3}$$

The fact that the first and last limits in (5.3) are positive and finite implies that f grows as ψ.

We underline that by the definition, the function f satisfies (F1), (F2), and (F3). On the other hand, it does not hold true that there exist a ball B and a constant $\epsilon > 0$ such that

$$f(\xi) \geq f(\zeta) + \eta_\zeta(\xi - \zeta) + \frac{\epsilon}{2}|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^n \setminus B$ and $\eta_\zeta \in \partial f(\zeta)$.

We point out that for simplicity in the previous example, we constructed a radially symmetric function f. It is straightforward to consider functions with different growth in the different directions as in the following example.

Example 5.3. For $\xi = (\xi_1, \xi_2)$, with $\xi_1 \in \mathbb{R}^k, \xi_2 \in \mathbb{R}^{n-k}$, let us consider

$$f(\xi) = h_1(|\xi_1|) + h_2(|\xi_2|),$$

where h_1, h_2 are constructed as in Example 5.2 by means of two functions ψ_1, ψ_2 with independent behaviors at infinity. To clarify this assertion, we consider, for example,

$$\psi_1(|\xi_1|) = |\xi_1|^p \log(e + |\xi_1|), \quad \psi_2(|\xi_|) = |\xi_2|^q,$$

where we underline that the exponents p, q could not be related.

We conclude this section with some examples of functions g satisfying the assumptions of Theorems 4.6 and 4.8.

Example 5.4. It is easy to check that the function

$$g(x, u) = \alpha\sqrt{1 + u^2}, \quad \alpha > 0,$$

fulfills the requirements of the theorems cited above.

Example 5.5. The function

$$g(x,u) = (\lambda u - a(x))u$$

with $a(x) \in W^{1,\infty}(\Omega)$ and $\lambda \geq 0$ is such that the assumptions above are satisfied provided u is assumed to be bounded.

Note that for $f(\nabla u) = \frac{1}{2}|\nabla u|^2$ and such a choice of g, the functional we obtain appears in elasto-plastic torsion problems. See, for example, [2].

Example 5.6. A central problem in image restoration is the reconstruction of an image u from a degraded datum $a(x)$. The most common model linking u to a is the following: $a(x) - Ru = v$, where R is a linear operator typically modeling blur, and v is the noise, and the goal is to minimize a functional whose model case is

$$\mathcal{I}(u) = \int_\Omega f(\nabla u) + v^2 dx.$$

The functional

$$\int_\Omega f(\nabla u) + g(x,u) dx,$$

with

$$g(x,u) = |a(x) - \lambda u|^2, \quad a(x) \in C^1(\bar{\Omega}), \lambda \in \mathbb{R},$$

satisfies the assumptions of Theorems 4.6 and 4.8 provided u is bounded. Moreover, such a functional is of type $\mathcal{I}(u)$.

It is well known that the assumption of the boundedness of the minimizers often appears in the study of higher regularity in the calculus of variations. In [3] the boundedness of the minimizers of functionals of type

$$\int_\Omega f(x,\nabla u) + a(x)u\, dx,$$

with $a(x)$ such that $a(x)u \in L^1$ and f satisfying only a growth assumption from below, is proven provided the boundary datum is bounded. In this spirit, the request of the boundedness of the minimizer u in Examples 5.5 and 5.6 makes sense since, in this paper, we consider a boundary datum ϕ satisfying the bounded slope condition.

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