Working Paper Series, N. 1, March 2015

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UNIVERSITÀ DEGLI STUDI DI PADOVA
DIPARTIMENTO DI SCIENZE STATISTICHE

Final version (2015-03-18) Submitted to Physica A 1-11; third revision 6-14.

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1 Introduction

Shannon's entropy is a basic conceptual and physical tool to understand statistical properties of systems. In statistics, it is used for prediction purposes in different areas in order to represent the natural variability of a population (system) with reference to qualitative and quantitative factors or to describe its decomposition, in multivariate contexts, highlighting some relationships (dependence among factors) and their relative strength. The basic reference is a distribution of probabilities $p_i,\ i = 1, 2, \cdots, K$, among K possible alternative states of a system, and there is no need to introduce space or time environments. This special feature is extremely relevant in the sequel.

Shannon's entropy H is a mean value of *local entropies*, $\log(1/p_i)$, based on a particular information unit of measure, a log concave function of the ratio $1/p_i$, namely, $H = \sum_{i=1}^{K} p_i \log(1/p_i)$ for $p_i > 0$ and $\sum_{i=1}^{K} p_i = 1$. The local entropy $\log(1/p_i)$ depicts a level of rarity of event i within the possible system states and, therefore, the level of information to detect or represent it. In an unconstrained system, we notice two polar frameworks: a zero entropy situation, $H = 0$, corresponding to a degenerate distribution, where only one event or state is possible, for instance event i with probability $p_i = 1$ and $p_j = 0$ for $j \neq i$, or, conversely, a maximal entropy situation, $H = \log K$, under the most uninformative distribution, $p_i = 1/K;\ i = 1, 2, \cdots, K$.

Entropy index H, or its normalized version $S = H/\log K$, denotes a measure of variability of a distribution of events stemming from a system. Nevertheless, it does not explain why those events are related to a specific system.

We may consider a further interpretation of *local entropy* in a system. It measures a hypothetical *attraction force* that the system exerts in maintaining the event i as a member of it. Index H denotes the mean-field attraction expressed within the current system. Under a degenerate distribution, we have a coherence among different interpretations: The variability is absent (zero), the rarity of the only possible event is minimal, $\log(1/p_i) = 0$, and the attraction force is absent (zero). Conversely, for small positive values of p_i, the contribution to variability is high, the rarity of the specific event is relevant and the attraction force to be a member of the system is significant. An equilibrium with maximal mean entropic attraction H is obtained under the uniform distribution $p_i = 1/K$ in an unconstrained environment. If the cardinality K of a homogeneous system (uniform distribution) augments, then an increasing entropy due to expansion of $\log K$ confirms that the maximal entropy is a monotone function of complexity.

In constrained environments over space or time, the distributions of events that define an observable equilibrium may be different from the uniform hypothesis. Correspondingly, the related function of local entropies is not invariant, $\log(1/p_i) \neq \log K$. Notice that with the term 'equilibrium' we denote special distributions, or their monotonic transformations, obtained under complex frameworks that may be represented, not exclusively, through equations and related initial conditions.

In this paper, we examine two different experiments that exhibit two equilibria, a first one over space and a second one over time. The proposed static equilibrium over space refers to the shape of a homogeneous chain supported at its ends, under the force of gravity. It is well known that the mathematical description of the previous form is a catenary function.

An apparently different situation is the aggregate dynamic expansion of a viral agent in a human population over time. In this case, under regularity conditions in a homogeneous population, the pertinent mathematical description is a probability density, a logistic distribution of events (Verhulst,1838) that depicts the time when we observe the change of state of individuals.

Both examples may have similar or analogous alternatives. In the first case, for instance, the catenary function properly defines, by reflection, the catenary arch under local uniform compression. In the second one, we may consider many other situations characterized by diffusion of innovations in a socio-economic context or growth models in naturalistic frameworks that may be conceived as variants of logistic distribution; for instance, the Bass models (see in particular Bass, 1969 and Bass, 1994).

The main aim of this paper is to prove that the dynamic equilibrium of a logistic distribution over time may be linked to a corresponding *local entropy function*, which is a catenary. In this way, the local attraction of each event in the system over time or the reciprocal contagion rate over time has a common representation with the static equilibria of suspended chains or catenary arches.

Conversely, starting from a static catenary, we prove that the local cardinality of the chain links, for unitary variations over space, defines, under the gravity force, a logistic distribution through a *local entropy function*.

We highlight that both examples refer to specific constraints. The chain has rigid and homogeneous links; it is not free but suspended at two separate ends. In the contagion or diffusion experiment, the susceptible population is homogeneous and limited. In both cases, the role of entropy and the ‘attraction force’ that it implies are the common bricks. The proposed connection, between static and dynamic equilibria in these two reduced but general cases, is in agreement with Verlinde’s conjecture on the origin of gravity as an effect of entropic force (see Verlinde, 2011).

A further issue of interest for both static and dynamic contexts, previously reduced to a common interpretative key, is the treatment of systematic deviations from uniformity or homogeneity assumptions. The generalized Bass model, GBM (see Bass, 1994), and the related perturbed logistic model may be easily converted to perturbed catenaries in order to take into account different intervention functions in controlling dynamics of viral or diffusion of innovation expansions or, in the static domain, the presence of local non-uniform compression due to design constraints. In this perspective, the proposed solution, through the local entropy function of a perturbed logistic or a GBM, simplifies the operative construction of weighted catenaries as characterized in Osserman (2010).

The paper is organized as follows. Section 2 presents the catenary function, some historical aspects, applications in architecture and its recent weighted extensions by Osserman (2010). Section 3 introduces basic growth models and a diffusion of innovations perspective focusing on logistic and Bass models and their isomorphism. Section 4 establishes the fundamental result of this paper: the connection between a static equilibrium framework, described through a catenary, and the corresponding dynamic equilibria typical of logistic and related models in growth or diffusion of innovations contexts. In particular, Sub-section 4.4 proposes a physical interpretation of the correspondence between the above-mentioned equilibria drawing on Verlinde’s conjecture on gravity conceived as an entropic force. Section 5 introduces a more tractable definition of weighted catenaries via the generalized logistic model. In Section 6, we propose our conclusions.

2 The catenary: Definition and main characteristics

It is sometimes surprising to discover how certain natural and social phenomena may refer to common interpretations in a way that is universal. The multiplicity of manifestations of natural phenomena, from the deepest levels of physical forms of matter to the elementary and complex biology of life, and to more sophisticated linguistic devices typical of social systems, apparently results in specific and extreme fragmentation of knowledge not attributable to common denominators. It should be noted, however, that from the time of ancient Greek philosophers, the attempt to propose ways to partially unify languages and knowledge was always at the centre of Western cultural traditions.

Here, we propose a careful consideration of a typical function of mathematical analysis, the catenary. This function, referable to the hyperbolic cosine,[1] directly describes different phenomena such as homogeneous ropes or chains supported at both ends and subjected to the action of gravity, a geodetic section of a soap film of minimum area subtended by circles of support and the shape of the cables of suspended power lines. Its simple transformation, reflection, describes, under uniform loads, arches in many buildings and the supporting arches of several bridges.

2.1 Some historical aspects

The term used for the catenary function comes from the Latin word *catena* and was introduced by Huygens in a letter to Leibniz in 1690. Similar terms include ‘funicular’, ‘alysoid’ and ‘chainette’. Galileo also examined this curve. In the *Dialogue of the Second Day*, he established that a chain suspended at both ends can be seen as a parabola. However, in the *Dialogue of the Fourth Day*, he corrects his previous sentence by noting that the approximation to the parabola is obtained in the presence of a small curvature. Evidence that the catenary is never a parabola was given in a paper published by Jungius (post mortem) in 1669. In 1675, Hooke published a solution through an encrypted anagram in Latin and first pointed to the dualism between the catenary and its reflected version, thought to form an optimal arch with uniform loads. An outstanding application of his result, in collaboration with Wren, is the dome of St Paul’s Cathedral in London.

2.2 Architecture and Gaudí

The use of the catenary arch in architecture, from its characterization as an ideal self-sustaining arch in relation to uniform loads, draws upon the premises of Hooke, but its actual use is much more recent.

A structured and systematic contribution was made at the turn of the nineteenth and twentieth centuries. Across Europe, various architectural movements arose simultaneously with consequent cultural and political effects. The *Art Nouveau* movement in France, the *Liberty* in Italy and the *Jugendstil* in Germany and Austria were flanked by the so-called *Catalan Modernism*, which was proposed as an artistic movement in various fields of expression, architecture, painting, sculpture and decorative arts (glass processing, textiles, iron and wood). This movement occurred with the use of irregular forms and sinuous floral ornament contrasted with the prevailing classicism of linear forms. This emphasis took its inspiration from organic shapes and forms of expression of life. Catalan Modernism also had political implications connected with the autonomous role of the Barcelona region under the European ‘Industrial Revolution’ active at that time. Many works of Catalan Modernism are found in the Eixample district in Barcelona.

The outstanding creator of this current of thought was Antoni Gaudí (1852 - 1926). He built several buildings including a dozen that are defined by UNESCO as

[1]The hyperbolic cosine was introduced in 1760 by Vincenzo Riccati (mathematician and physicist, 1707–1775, a son of the mathematician Jacopo Riccati) alongside the other hyperbolic functions. It is defined as the arithmetic average of two exponential functions; namely, $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

World Heritage Sites. The work to which he devoted a large part of his energy for decades, La Sagrada Família, is a basilica still under construction.

The architecture of Gaudí is based on the discovery of unusual shapes and unpredictable nature that have great connotations: It is a biomorphism making use of various materials with forged craftsmanship (brick, stone, concrete, ceramics, glass and iron). Among these natural forms systematically reconstructed, in which the colours, light and ventilation play a balancing role, there is a wide use of the catenary arch, because it ensures a static equilibrium, which is a typical choice of Nature. See, for example, the Milà and Batlló houses and La Sagrada Família itself, with their underlying static models based on complex reflected catenaries.

2.3 Classic and weighted catenary

The function that defines the *catenary* takes the form

$$y = c(x) = a\cosh\left(\frac{x-x_0}{a}\right) = \frac{a}{2}\left(e^{\frac{x-x_0}{a}} + e^{-\frac{x-x_0}{a}}\right), \qquad a > 0, \tag{1}$$

where $\cosh(\cdot)$ is called the hyperbolic cosine and x_0 is its minimum point.

A more general expression that allows simple transformations within the coordinate system, is

$$y = a\cosh\left(\frac{x-\alpha}{b}\right) + \beta \qquad a, b > 0; \quad \alpha, \beta \in R. \tag{2}$$

Usually, the parameters α and β in Equation (2) are specified as a function of suspension points of the chain and its length. The derivation of a standard catenary, for $b = a$, is based on the definition of a homogeneous chain with an internal force, a tension, that acts in every point of it, and an external force, the gravity, which determines the internal tensions and their variations along the curve. A reflected version of the catenary solves the dual equilibrium problem (i.e., the construction of a freestanding arch with uniform loads). In this case, we have $b = |a|$ and $a < 0$.

A standard derivation of the catenary function is well known under the assumption of constant mass per unit length. More generally, we argue that a direct solution for a weighted chain is not straightforward. In Osserman (2010), a weighted chain C is introduced as a pair of functions $(f(x), \rho(s))$, where $f(x)$ is a suitable smooth function on an interval $x_1 \leq x \leq x_2$, s is arc length along the curve $y = f(x)$ and $\rho(s)$ is a positive continuous function of s, called the 'density' function of C. A weighted catenary is a weighted chain C in which (i) $-\infty < f'(x_1) < 0 < f'(x_2) < \infty$, and (ii) $f''(x)$ is continuous and positive. As stated in Osserman (2010), if function $f(x)$ satisfies conditions (i) and (ii), there exists a function $\rho(s)$ such that $(f(x), \rho(s))$ is a weighted catenary where $\rho(s(x)) = \frac{Hf''(x)}{\sqrt{1+f'(x)^2}}$, $0 < H = \frac{W}{|f'(x_2)|-|f'(x_1)|}$ and $W = \int_{x_1}^{x_2} \rho(s(x)) \frac{ds}{dx} dx$ is the total weight of the chain.

The proposed theorem is very useful in order to determine the particular density function $\rho(s(x))$ for a known function $f(x)$ satisfying conditions (i) and (ii). Unfortunately, the converse theorem, which states $f(x)$ as a function of $\rho(s(x))$, is not derived, as the $f(x)$ definition is quite general. Notice that this is a central point in determining an accurate curve-fitting approach. Following Osserman's example motivated by the Gateway Arch, the characterization of its structure has to be based on a flexible $f(x)$ controlled by a suitable $\rho(s(x))$ and not vice versa.

In the final part of the present paper, Section 5, we derive a solution based on a simple transformation of a dual problem related to a GBM. The weighted catenary based on the GBM and the related intervention function, $x(t)$, define a function $f(x)$ in the sense of Osserman (2010) so that the corresponding 'density' $\rho(s(x))$ is easy to determine.

3 Some growth models

The phenomena of growth are found in many natural and social contexts. In support of these processes, some key aspects that define a common conceptual basis are identified. These systems consist of a large number of units or agents that are connected in terms of relational forms through appropriate languages that determine, *lato sensu*, the internal architecture of the *attraction forces*; see, for instance, Couzin (2007) and Pentland (2010) in different fields. A key example is a swarm of bees. The organization of these systems does not require a central control. For instance, the initiatives linked to the search for food are based on the flexible recruitment of worker bees. Based on preliminary information provided by some explorer bees through conventional dances, many other workers take steps to reach the site, dynamically involving other bees. Initialization of information process and its expansion, through local interaction, are basic features. In migrations of birds or movements of schools of fish, the evolutions of these systems of animals seem to respond to rather simple and universal forms. For example, information acquired by local members in relation to the presence of a predator is processed collectively, by closing ranks to facilitate the transmission of information and replication of imitative behaviour quickly based on defence veering that do not need a 'conductor'. The system is delimited by the common language that defines the contour of the internal *attraction forces*. The growth of a cell culture in a laboratory (Verhulst, 1838), under suitable testing conditions, is based on autocatalytic splitting mechanisms. These systems tend to saturation with respect to the capacity of the environment. The dynamics of the spread of a viral agent in a human population are a function of types of organization and networks of social relationships (Barabasi, 2002). The susceptible come into contact with a carrier and in turn spread the viral agent, temporarily saturating the subpopulation that may be affected. The launch of a new product, a technological innovation or a fad, responds to similar mechanisms (Rogers, 2003; Granovetter, 1978; Bass, 1969; Bass, 1994; Guseo, 2008; Guseo, 2009; Guseo, 2010; Guseo, 2011). Some opinion leaders quickly adopt innovation based on the systematic actions over time of corporate communications; then the imitative mechanism is activated in parallel by word-of-mouth (WOM), a very powerful tool that is critical to the success or the failure of almost all business initiatives. In all previous systems, 'internal attraction' among agents allows the definition of physical, chemical, social, biological or economic relationships with correlated networks.

The mathematical and statistical models that are used in these contexts are generally included in special subfamilies of Riccati[2] equations. The key paradigms are represented by the logistic model (Verhulst, 1838) and the Bass model Bass (1969). We see the basic features of both.

3.1 Logistic distribution

The logistic equation expresses the aggregate growth over time of a biological system in a constrained environment. This equation defines the instantaneous growth of a phenomenon as an interaction between the cumulative extension of the current process and the residue that may still be activated. Consider as an example the formation of a new biochemical compound in terms of certain components and specific catalysts. Let us define the total mass m obtained at the end of the reaction, $z(t)$ the compound obtained until time t, and $z'(t)$ the instantaneous quantity of the product made at time t. The equation that governs the dynamics may take the following form:

$$z'(t) = \frac{r}{m} z(t)(m - z(t)), \quad t \in R, \tag{3}$$

where the parameter $r > 0$ controls the speed of the reaction. Note that the fraction $z(t)/m$ stimulates, through an interaction (multiplicative interaction), the residual process represented by the difference $(m - z(t))$. The above equation is completed with an initial *positive* condition $z(0) = z_0 > 0$. Under the position $t_p = \frac{1}{r} \log \frac{m-z_0}{z_0}$, the solution is

$$z(t) = m\, L(t) = m \frac{1}{1 + e^{-r(t-t_p)}}, \tag{4}$$

where $L(t)$ represents the distribution function of the corresponding logistic distribution over time. The instantaneous growth rate is

$$z'(t) = m\, l(t) = m \frac{r e^{-r(t-t_p)}}{\left(1 + e^{-r(t-t_p)}\right)^2}, \tag{5}$$

where $l(t)$ is the associated logistic density function. It can be shown that t_p is the time to peak, and it is obviously connected with the initial condition: $z_0 = m/(1 + e^{rt_p})$.

The prevailing applications of the logistic model are usually defined within naturalistic fields, where growth phenomena over time are nonlinearly determined by constrained environments.

3.2 Bass's distribution

The Bass equation (Bass, 1969) has established itself in a very different context with respect to the former, in the area of diffusion of innovation processes in socio-economic systems (Rogers, 2003). This is a typical field of applied mathematics in

[2]Count Jacopo Riccati, renowned Italian mathematician, was born in Venice in 1676 and lived mainly in Castelfranco Veneto, where he was superintendent (mayor) for a decade. He was a man of great erudition and he had contacts with many European mathematicians of the time (among them, different components of the Bernoulli family as well as Hermann, Agnesi and Vallisneri). He is known, among other things, for the equation that now bears his name: $x' = ax^2 + bx + c$. He died in 1754.

the quantitative marketing tradition (the 'Bass model' has about 160 millions of results in Google). Bass suggests that success in marketing a product or service depends very much on the contribution of two main types of consumers: innovators who are directly sensitive to the actions of institutional communication (advertising campaigns and incentives) and imitators who basically ignore this channel of information and prefer to support their decisions to adopt based on interpersonal relationships. In the latter case, the main communication channel is the interaction between adopters and susceptibles through *word-of-mouth* (WOM) in a broad sense. WOM denotes not only simple verbal communication but also all of the signs and gestures that humans use broadly (Pentland, 2010). The equation proposed for government of the dynamics takes the following form:

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t)), \quad t \in [0, +\infty), \tag{6}$$

where, $z'(t)$ denotes the instantaneous adoptions at time t, $z(t)$ the cumulative adoptions until time t, m the asymptotic market potential ($\lim_{t\to+\infty} z(t) = m$), the parameter $p > 0$ represents the contribution of innovators directly proportional to the residual market $(m - z(t))$ and $q > 0$ rules, so modulated by the relative knowledge about the product $z(t)/m$, the access to the residual market $(m - z(t))$ due to the imitators and related word-of-mouth. The above equation is completed with an initial condition that is natural for processes of this type, $z(0) = 0$.

Following the total probability law, a parallel hazard description with three sub-populations and related conditional probabilities of adoption – innovators (1), imitators $z(t)/m$, and neutrals (0) –, is useful for interpretation, namely, $h(t) = z'(t)/(m - z(t)) = (p \cdot 1 + q \cdot (z(t)/m) + (1 - p - q) \cdot 0)$. The obtained hazard is a probability mixture of three latent components. The solution of the cumulative Bass model, expressed by Equation (6), is

$$z(t) = m\frac{(1 - e^{-(p+q)t})}{1 + \frac{q}{p}e^{-(p+q)t}}, \qquad t \in [0, +\infty), \quad 0 < p < q, \tag{7}$$

and the corresponding rate version is

$$z'(t) = m\frac{(p+q)^2 e^{-(p+q)t}}{p\left(1 + \frac{q}{p}e^{-(p+q)t}\right)^2}, \qquad t \in [0, +\infty), \quad 0 < p < q. \tag{8}$$

We can compare the logistic model with that of Bass using a reparameterization of Equations (7) and (8) (i.e., $r = p + q$, $t_p = (\ln q/p)/(p + q)$) or, equivalently, $q = re^{rt_p}/(1 + e^{rt_p})$, $p = r/(1 + e^{rt_p})$.

We obtain, therefore,

$$z(t) = m\, B(t) = m\frac{(1 - e^{-rt})}{1 + e^{-r(t-t_p)}}, \qquad t \in [0, +\infty), \quad 0 < r, \tag{9}$$

where $B(t)$ is the Bass distribution function, and

$$z'(t) = m\, b(t) = m\frac{r\left(e^{-rt} + e^{-r(t-t_p)}\right)}{\left(1 + e^{-r(t-t_p)}\right)^2}, \qquad t \in [0, +\infty), \quad 0 < r, \tag{10}$$

the corresponding rate function with $b(t)$ the Bass (probability) density.

The cumulative Bass model (9) can be determined by a monotonic transformation of the logistic model (isomorphism):

$$B(t) = L(t)\,(1 - e^{-rt}), \qquad t \in [0, +\infty). \tag{11}$$

Notice that in probability theory equation $(1-e^{-rt})$ denotes the distribution function of a monomolecular process and that a product of two or more distribution functions is always a distribution function. For $t \to +\infty$, the logistic and Bass models are asymptotically equivalent, because e^{-rt}, for $r > 0$, tends rapidly to zero. Of course, they differ in the right neighbourhood of zero where the initial positive condition of the logistic exerts its main effect.

As expressed in Equation (11), the logistic model and the Bass model are isomorphic but not equivalent. The main difference can be concentrated in the role of the mechanism of initialization. In the logistic model, this effect is entirely concentrated at the time $t = 0$ by the positive initial condition, $z(0) = z_0 > 0$. In the Bass model, the mechanism of external initialization is based on a 'seeding effect' distributed over time, for $t \geq 0$, and governed by the parameter r, namely, $(1 - e^{-rt})$.

It may be interesting to examine the nature of the basic logistic model within a Complex Systems perspective. In Guseo (2008-2011) some special Cellular Automata models, under a mean-field approximation, may explain a large class of aggregate growth models. The emphasis, at the micro level, is related to the description of heterogeneous agents in a system. The emergent macro levels, related to the aggregate temporal description of a Complex System, define special Riccati equations and the corresponding solutions as distributions over time. The logistic and the Bass models are basically key examples included in previous more general distributions related to growth processes.

4 Catenary and diffusion models

A simple transformation of the catenary function, here proposed for the first time and based essentially on the squared reciprocal transformation, reveals its entropic nature, related to the local entropy concept, and represents the logistic distribution. As explained in Section 3, the latter is the head of a family of models that describes and predicts the growth of biological systems in a constrained environment. It also depicts the diffusion of ideas and knowledge in a social body as well as the market penetration of specific technological innovations or fashions in special areas or countries.

4.1 Static and dynamic equilibria

The proposed above-mentioned connection between the static equilibrium of the catenary and the dynamic one typical of a logistic diffusion process requires a deeper understanding of such an observation. Is it a pure accident, or could it be included in a more general context? In Sub-section 4.4, we discuss some aspects related to the Verlinde's conjecture of the origin of gravity (see Verlinde, 2011) as a consequence of the entropic force due to a change of entropy referred to the distribution of matter. Local entropy or information is based on a monotonic concave transformation of a ratio $1/p_i$, usually $\log(1/p_i)$, where p_i is the probability of an event. The ratio $1/p_i$ describes a measure of separation of event i from the central body of a distribution. A suggestive interpretation of the ratio $1/p_i$ points to the implicit forces that physically determine a distribution of events of a system. Those events exhibit a sort of reciprocal 'attraction' in order to be members of the observed system. Low levels of p_i denote events that are strongly attracted by the system, and, vice versa more probable events belong to the system with a low-level 'attraction'. There is no effort to include them in the generating system: they are not rare, they are typical outcomes of the system. Here, the key argument is based on the meaning of the square root of the reciprocal of a logistic probability density of events in a system. This is eventually a measure of local entropy or information in a broad sense. It allows a direct interpretation of the evolutionary character of a logistic diffusion as a dynamic equilibrium equivalent to a static one over time characterized by a catenary which is usually defined over space where the gravity force, as an emergent phenomenon, exerts its macroscopic effect.

In the following sub-sections, we examine the proposed connections in the two possible directions.

4.2 From the logistic to the catenary over time

It is important to see now the analytic relationship between the catenary model over time and the basic diffusion models previously introduced. For convenience, we examine the connection with the logistic model, knowing that this moves, in accordance with the isomorphism, as the Bass model.

The density of the logistic distribution (see Equation (5)) can be exactly expressed through the hyperbolic cosine:

$$\begin{aligned} l(t) &= \frac{re^{-r(t-t_p)}}{(1+e^{-r(t-t_p)})^2} \\ &= \frac{re^{r(t+t_p)}}{(e^{rt}+e^{rt_p})^2} = \frac{re^{r(t+t_p)}}{e^{2rt}+2e^{r(t+t_p)}+e^{2rt_p}} \\ &= \frac{\frac{r}{2}}{1+\frac{1}{2}e^{r(t-t_p)}+\frac{1}{2}e^{-r(t-t_p)}} = \frac{\frac{r}{2}}{1+\cosh(r(t-t_p))} \\ &= \frac{\frac{r}{4}}{(\cosh(r(t-t_p))+1)/2} = \frac{r}{4}\cosh^{-2}\left(\frac{r(t-t_p)}{2}\right). \end{aligned} \tag{12}$$

For $a = \frac{2}{r}$, the transformation that connects the density of the logistic to the catenary over time is immediately obtained; namely,

$$\frac{\sqrt{\frac{a}{2}}}{\sqrt{l(t)}} = a\cosh\left(\frac{t-t_p}{a}\right) = c(t). \tag{13}$$

Similarly, based on Equations (5) and (10), it follows the relationship $l(t) = b(t) - h(t)$, where $h(t) = re^{-rt}/(1+e^{-r(t-t_p)})^2$, so we can express the immediate connection between the Bass density and the catenary, specifically,

$$\frac{\sqrt{\frac{a}{2}}}{\sqrt{b(t)-h(t)}} = a\cosh\left(\frac{t-t_p}{a}\right) = c(t). \tag{14}$$

By neglecting the monotonic transformation induced by the square root, the reciprocal of a probability density denotes a measure of the rarity of an event at time t.

The main question here is the meaning of the ratio $\sqrt{\frac{a}{2}}/\sqrt{l(t)}$ whose behaviour in terms of t is a catenary in the relevant space (time in this case). As explained in the introductory Sub-section 4.1, it is possible to interpret such a ratio as a local measure of information or *local entropy* related to the events' positioning on a time scale with the focal point in t_p. Shannon's entropy is an average of such a local measure of information, i.e., $H = \sum_i p_i \log(1/p_i)$, where the log function is a device to introduce a unit of measure for information (in particular, $\log_2(\cdot)$ for the binary digit). A similar device may be a concave function like the square root by obtaining an analogous measure, $G = \sum_i p_i \sqrt{(1/p_i)}$. Within a first-order approximation, $G \simeq 1 + H/2$, the measure G is an affine transformation of H. The choice between one of them is therefore a conventional statement. The ratio $1/\sqrt{p_i}$ or $1/\sqrt{l(t)}$ denotes the degree of local entropy that separates events' time positioning with reference to the focal point t_p. The local symmetry of $l(t)$ and $b(t)$ around t_p is well known, and related local entropies are therefore symmetric.

Let us turn again to the basic interpretation of the logistic and Bass models as emergent aggregate behaviour of special complex systems. The reciprocal of a probability describes an information notion of events that are considered as *part of a common complex system*. This ratio exhibits the intensity of a kind of *attraction* that links together those events as members of a statistical population or a statistical complex system. Under the special logistic behaviour, the corresponding squared root reciprocal of the probability density of events is a catenary over time.

4.3 From the catenary to the logistic over space

Vice versa. Is the observed equilibrium of a suspended uniform chain over space an entropic equilibrium? In order to answer this central question, we can study the local length $n(x)$ of a suspended chain for a unitary increment of space argument x if its shape behaves as a catenary function, $c(x) = a\cosh\left(\frac{x-x_0}{a}\right)$. The first derivative of $c(x)$ is

$$c'(x) = \sinh\left(\frac{x-x_0}{a}\right) \tag{15}$$

and, therefore, the first-order approximation of the local length $n(x)$ for a unitary increment of x is

$$n(x) = K^{1/2}(x) = \sqrt{1+\sinh^2\left(\frac{x-x_0}{a}\right)} = \cosh\left(\frac{x-x_0}{a}\right). \tag{16}$$

The function $n(x)$, proportional to the catenary $c(x)$, describes the local length through the number of links in the interval $(x, x+1)$ if we assume their unitary length, and, therefore, we have

$$n(x) = \frac{1}{\frac{1}{n(x)}} = \frac{1}{\frac{1}{K^{1/2}(x)}} = \frac{1}{\sqrt{p(x)}}, \tag{17}$$

where the ratio $p(x) = 1/n^2(x)$ denotes a probability. Moreover, Equation (17) specifies that $n(x) = 1/\sqrt{p(x)}$ denotes also a local entropy.

A similar monotonic transformation $\tilde{n}(x)$ of previous cardinality $n(x)$ may be the following:

$$\tilde{n}(x) = \log K(x) = \log \frac{1}{\frac{1}{K(x)}} = \log \frac{1}{p(x)}. \tag{18}$$

Both functions, $n(x) = 1/\sqrt{p(x)}$ and $\tilde{n}(x) = \log 1/p(x)$, denote local entropies. In particular, $\log K(x)$ is the maximal entropy value for a generic discrete distribution over $K(x)$ points, and this extreme value is obtained for the uniform distribution $1/K(x)$. In other words, the equilibrium of a suspended chain denotes an entropic equilibrium.

As a clear consequence, we have that a catenary always defines a local entropy function based on a logistic probability density. In fact, we can express $p(x)$ as a function of $n(x)$ following Equations (16) and (17), namely,

$$p(x) = \frac{1}{n^2(x)} = \cosh^{-2}\left(\frac{x - x_0}{a}\right). \tag{19}$$

In other words, see Equation (12) in space domain, the probability $p(x)$ is proportional to $l(x)$. Function $p(x)$ is not a generic density; it is a logistic density distribution over space. This important aspect explains that the local entropies $n(x)$ of a spatial catenary, implicitly defining the corresponding probability density $p(x)$, express the normalized internal level of tension (a kind of attraction) among local links at the space coordinate x.

4.4 A physical interpretation of static and dynamic equilibria

Equation (1) depicts, in Cartesian coordinates over space x, the shape of a suspended chain under the force of gravity. This equilibrium is a traditional solution due to the effect of the gravitational force on a system with a special structure based on homogeneous links in terms of shape, density and mass.

In previous Sub-section 4.3, we have stated that the classical catenary solution has an entropic nature due to the well-known maximizing principle related to its equilibrium. The proposed result may be embedded into recent advances in theoretical physics. Following the thought-provoking paper by Verlinde (2011) – which summarizes and overcomes analogous results by Bekenstein (1973, 1981, 2003), Hawking (1975), and Padmanabhan (2010) – we can assume that gravity is an emergent phenomenon like space-time geometry. Gravity arises as an entropic force, once space and time have emerged. Verlinde (2011) identifies a cause for gravity: 'It is driven by differences in entropy, in whatever way defined, and a consequence of the statistical averaged random dynamics at the microscopic level.' It is this differences in entropy that cause motion in order to reach equilibrium.

Following Verlinde's approach, we may interpret the local length $n(x)$ of a catenary, for a unitary variation of space at x, as a measure of *local entropy* related to the internal tension between adjacent links. Links near suspension points have a high tension (a high entropy) that keeps them connected in the system (chain). The links near the lower part of the chain have a small tension (small entropy) to include them in the system. Conversely, we can interpret the reciprocal of the local lenght of a catenary as a probability that governs the disposition of a link in a suspended chain under the gravity force so that high probabilities denote the links in the central positions and low probabilities depict links at the periphery of the system.

At the same time and with the same basic mechanism, the disposition over time of events in a simple growth model, within a constrained environment, is driven by the same catenary function that describes the reciprocal attraction or local entropy of each event within the common system of events.

This is the same, for example, for the events that define the aggregate behaviour of a complex system of increasing yeast cells in a limited environment, the adoption of a technological innovation in a regional market or the diffusion of a new fad among young people. In diffusion of innovation contexts, for instance, adoptions are extremely difficult at the beginning or at the end of a life-cycle. In this case, the internal psychological reasons of the involved agents may be different at the beginning of the process, because few customers are aware of the existence of the product or its quality. At the end, the adoption rate is again low but for different internal reasons; most potential adopters have already adopted the product, and the residual market is limited. Nevertheless, from an aggregate point of view, agents' decisions – not their psychological attitudes – are what matter. Under the logistic (or Bass) framework, the contribution of imitators is summarized by sub-equation $qy(t)(1 - y(t))$ (i.e., a symmetric function of y). It is a mean description that does not take into account local latent motivations of agents but their behaviour due to the interactions depicted by $qy(t)(1-y(t))$ as a relevant set of specific relationships that sustain positive decisions under a common attraction force that include both the share of adopters $y(t)$ and the share of future adopters $(1 - y(t))$.

As a final remark, the invariance between the static equilibrium of a suspended chain or a catenary arch and the logistic process over time defining a saturating equilibrium is coherent with Verlinde's conjecture and is explained by a common idea, the *local entropy*, and a common principle: the entropic force and its variations in the reference domain, time or space.

5 Perturbed catenary and diffusion models

Both classic catenary and logistic function are based on homogeneity of links or agents in a regular environment. In many static situations, it is necessary to implement non-uniform loads or links in the space domain. This issue may be easily discussed through the properties of a GBM or the corresponding perturbed logistic by introducing a control function over space that is non-uniform.

A logistic equation perturbed by an exogenous intervention function $x(t)$ acting on the basic Equation (3) is

$$z'(t) = r\frac{z(t)}{m}(m - z(t))x(t), \quad t \in R, \tag{20}$$

where $x(t)$ is an integrable function in a limited range. For a more general case, see Bass et al. [?].

Its solution, under initial condition $z(0) = z_0 > 0$ or, $t_p = \frac{1}{r}\log\frac{m-z_0}{z_0}$, is

$$z'(t) = mg(t) = m\frac{rx(t)e^{-r\left(\int_0^t x(\tau)d\tau - t_p\right)}}{\left(1 + e^{-r\left(\int_0^t x(\tau)d\tau - t_p\right)}\right)^2}. \tag{21}$$

The corresponding perturbed catenary, for $a = 2/r$, is, therefore,

$$f(t) = \frac{\sqrt{\frac{a}{2}}}{\sqrt{g(t)}} = \frac{1}{\sqrt{x(t)}}\, a \cosh\left(\frac{\int_0^t x(\tau)d\tau - t_p}{a}\right). \tag{22}$$

For $x(t) = 1$, we obtain the pure logistic density and the related catenary representation. For $x(t) \neq 1$, the standard logistic equilibrium is modified, yielding a weighted catenary. Through Equation (22), we directly give an explicit solution for a weighted catenary, function of an external input $x(t)$ that is not uniform in general. Following the theory by Osserman (2010), we can obtain, through $f(t)$, the corresponding 'density' $\rho(s(t))$.

6 Final remarks and discussion

This article emphasizes the unexpected but consistent connection between the catenary function (which describes the static equilibrium of a chain, rope, or cable suspended between two ends and subjected to gravity), or its dual form obtained by reflection, the catenary arch (which optimizes the static equilibrium of uniform loads), and the logistic function (which expresses the aggregate evolution of the growth of biological or social systems in general under the effect of an internal attraction force that determines the existence of a well-identified entity, the system itself). The latter is a dynamic equilibrium chosen by natural evolutionary growth processes.

Moreover, a perturbed logistic model may easily determine a corresponding perturbed catenary with non-uniform loads, and this may be a practical way to identify local loads under fixed design points.

Gaudí's intuition about the static equilibrium of tensions or compression forces is extended here to cover a strong correlation with the dynamic ones related to aggregate diffusions in socio-economic or biological systems. Both respond to a common mathematical concept, the catenary, as defined by Equation (1). This special invariance may be explained at a deeper level through *local entropy* and, in particular, through Verlinde's conjecture that obtains gravity as an effect of entropic force.

The analysis of the catenary in Sub-section 4.3, from the point of view of information theory, parallels the well-known example proposed in Verlinde (2011) for describing an entropic force through a polymer molecule immersed in a heat bath. Monomers are all of the same length, and the point of attachment with each other makes them free to move in any direction. The same is exhibited by the homogeneous links in a suspended chain. The chain equilibrium under gravity is a maximal entropy configuration. Analogously, the polymer in a heat bath assumes the random configuration with higher entropy in comparison with a stretched one that moves it out of the equilibrium.

In particular, when we apply an external force to a chain that takes it out of equilibrium configuration (catenary), the statistical tendency that the chain return to a configuration with higher entropy will define a macroscopic force that points to an opposite direction (in this case, gravity). The direction and intensity of entropic force is justified by a fewer number of states at a lower entropy when the chain is 'contracted in space', rather than when it is in a longer configuration, the equilibrium catenary configuration.

In other words, Nature does not separate static equilibria from the dynamic equilibria of biological or social growth processes governed by logistic family distributions and their extensions over time. Both are a result of an entropic force defined to reach the maximal entropy configuration in a constrained environment and may be interpreted as a gravity force in the space domain and as an attraction force among agents or events within a biological, physical or social system over time.

Riferimenti bibliografici

A-L. Barabasi, A-L (2002). *Linked: the new science of networks*, MA: Perseus Cambridge.

F.M. Bass, F.M. (1969). A new product growth model for consumer durables, *Management Science* 15(1) 215–227.

F.M. Bass, F.M., Krishnan, T., Jain, D. (1994). Why the Bass model fits without decision variables, *Marketing Science* 13 203–223.

Bekenstein, J.D. (1973). Black Holes and Entropy, *Physical Review D* 7 2333–2346.

Bekenstein, J.D. (1981). Universal Upper Bound on the Entropy-to-Energy Ratio for Bounded Systems, *Physical Review D* 23 287–298.

Bekenstein, J.D. (2003). Information in the Holographic Universe, *Scientific American* 289 2 August (2003) 58–65.

Couzin, I. (2007). Collective minds, *Nature* 445 15/2/2007 715.

Granovetter, M. (1978). Threshold Models of Collective Behavior, *American Journal of Sociology* 83 1420–1443.

Guseo, R., Guidolin, M. (2008). Cellular Automata and Riccati Equation Models for Diffusion of Innovations, *Statistical Methods and Applications* 17 (3) 291–308.

Guseo, R., Guidolin, M. (2009). Modelling a dynamic market potential: A class of automata networks for diffusion of innovations, *Technological Forecasting and Social Change* 76 (6) 806–820.

Guseo, R., Guidolin, M. (2010). Cellular Automata with network incubation in information technology diffusion, *Physica A* 389 2422–2433.

Guseo, R., Guidolin, M. (2011). Market potential dynamics in innovation diffusion: Modelling the synergy between two driving forces, *Technological Forecasting and Social Change* 78 (1) 13–24.

Hawking, S.W. (1975). Particle Creation by Black Holes, *Commun. Math. Phys.* 43 199–220.

Osserman, R. (2010). Mathematics of the Gateway Arch, *Notices of the American Mathematical Society* 57(2) (2010) 220–229.

Padmanabhan, T. (2010). Thermodynamical aspects of gravity: new insights, *Reports on Progress in Physics* 73 046901.

Pentland, A. (2010). To Signal is Human, *American Scientist* 98(3) May-June 2010 203-211.

Rogers, E.M. (2003). *Diffusion of Innovations*, 5th ed., Free Press, New York.

Verhulst, F. (1838). Notice sur la loi que la population suit dans son accroissement. *Correspondance Mathématique et Physique* 10 113-121.

Verlinde, E. (2011). On the Origin of Gravity and the Laws of Newton. *Journal of High Energy Physics* vol 2011 4(29) . doi: 10.1007/JHEP04(2011)029.

Acknowledgements

I would like to aknowledge Giacomo Guseo for introducing me to the Gaudí's work and many other colleagues for useful comments and suggestions. Among them, Alessandra Dalla Valle, Md. Abud Darda, Claudia Furlan, Mariangela Guidolin, Umberto Magagnoli, Carmelo Majorana, Cinzia Mortarino, Fortunato Pesarin and Luca Sartore.

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