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Modeling competition among pharmaceutical drugs

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Keywords: competition, innovation diffusion, dynamic market potential, communication network, nonlinear regression

Contents

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1 Introduction

The diffusion of an innovation often has to cope with the rise of many competitors that generate huge contest effects, increase or contraction in the market potential size, changes in the evolutionary dynamics of some brands, reduction or expansion in the life cycle length, and anticipation in the time entrance of additional products in the market. These effects can be modeled only if they are included in a single complex system that should be able to correctly identify competition and contextual forces.

We cannot observe this complex system where single agents (consumers) may interact and share pieces of information regarding alternative technologies, comparable solutions, similar devices, etc. Conversely, we observe the resulting aggregate dynamics (level reached by diffusion, e.g., number of packages sold), and our analysis must be based upon this level of observability.

Usually, the diffusion processes of these products in a marketplace have a limited time-horizon defining particular life cycles with different internal dynamics. We observe poor performances at the beginning of the process after launch due to limited acceptance of a new proposal that interacts with previous knowledge and related agents' lifestyles. Similarly, but for different reasons, we notice a pronounced decrease in sales at the end of the commercial life cycle when the product is perceived as an old non-efficient solution. Following this qualitative reasoning, for modeling

and predictive purposes, we may exclude the direct use of ARMA-like (or VARMA) processes, which are strongly based on weak stationary conditions after some differencing.

The pharmaceutical market is an important example of competition among alternative drugs. The products can differ to a larger extent, whenever they are based upon different active compounds, or they can differ only at the commercial level, when the same active compound is sold by competing firms. Moreover, this market differs from other markets, since in many countries the cost of essential/vital drugs is paid by a welfare system. Therefore, price levels do not directly influence physicians' prescriptions. In addition, in Italy, the Ministry of Health negotiates with pharmaceutical firms the price to be paid by the national health service.

The aim of this paper is to build and apply a competition model for pharmaceutical drugs (source: IMS Health Italy). In particular, we focus on different drugs with the same active compound, based upon glimepiride. This is exactly the problem of substitute products (brands) competing for the same patients. The results will be compared with the outcomes obtained applying alternative models. We emphasize that, in this case, a 'good' model, in addition to describing the data and providing reliable forecasts, should highlight the key features of the competition among the analyzed drugs.

A specific method for studying the dynamics of these special markets is based upon the diffusion of innovation methodology, which is strongly related to system analysis and epidemiological modeling tools.

The models due to Bass (Bass, 1969; Bass et al., 1994) represent an essential step for the development of aggregate univariate diffusion patterns and a huge number of extensions originated from them (see, among others, Meade and Islam, 2006; Peres et al., 2010).

The main contributions that conversely pertain to competition modeling are really sparse. Krishnan et al. (2000), Savin and Terwiesch (2005), and, recently, Guseo and Mortarino (2012) and Guseo and Mortarino (2014b) are relevant proposals that describe competition with a differential representation admitting a closed-form solution. The differential representation is typical of the models proposed in the marketing literature, where an aggregate parsimonious description of real adoption processes, based on interpretable parameters, is essential to capture the features and predict the future evolution of the market under study. The simplicity of the model's structure is obtained by introducing plausible assumptions about the behavior of the agents playing a role within the market. In addition, a tractable solution for estimation and prediction makes a simple validation of the model easy to implement through sales data. The relevant issue in this research topic is to build an adequately large set of models to describe the different characteristics of the diffusion process. Confirmation or rejection of the assumptions underneath each model is then attained by fitting available observed data and by a relative comparison of the models' performances.

Models available in the literature to describe diffusion of competing products in a common market assume that the market potential, i.e., the total number of adoptions that a product will ultimately reach at the end of its life cycle, is invariant throughout the overall life cycle since the products' launch.

This assumption, however, is almost always unrealistic. In general, knowledge and awareness of a product are not immediately disseminated throughout eligible adopters since the market entrance of a pioneering brand. Moreover, new brands are often followed by other competitors, and their launch may affect awareness of the products. The topic arises from the consideration that awareness of a product and adoption are diffusion processes. Awareness is a prerequisite for adoption, and the degree of penetration of a product into the market is limited by the degree of diffusion of knowledge about its existence/properties. For this reason, the market potential should be better described as a dynamic process instead of a fixed constant, as discussed in Guseo and Guidolin (2009) for the univariate case (without competition effects).

In Section 2, the standard Bass model (Bass, 1969) is briefly illustrated with its extension (Guseo and Guidolin, 2009) introducing a dynamic market potential. The underlying reasons motivating this extension are also presented. In Section 3, we discuss how the competition model proposed in Guseo and Mortarino (2014b) can be extended to incorporate the dynamic market potential. Section 4 illustrates the application of the new model to the description of competition between two antidiabetic drugs. A discussion about the improvement obtained for these data with the proposed model is also given. Section 5 contains concluding remarks. In Appendix 1, the details about the closed-form solution of the proposed model are given. Finally, a SARMAX refinement for the model fitting is presented in Appendix 2.

2 A possible form for the dynamic market potential

The simpler form of a univariate diffusion of innovation model is given by a Bass model (Bass, 1969). The differential representation is defined through the equation

$$z'(t) = m\left[p + q\frac{z(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right], \tag{1}$$

where $z(t)$ and $z'(t) = \partial z(t)/\partial t$ represent the cumulative sales and the instantaneous sales at time t, respectively. Parameter m is the fixed market potential (the asymptotic level of cumulative sales or the total number of adoptions at the end of the life cycle).
Equation (1) makes explicit that, at each time point, the increase in sales is proportional to the residual market, $m - z(t)$. The proportionality factor is affected by a fixed effect, p, and by a time-varying effect, $qz(t)/m$. The former does not depend upon the degree of diffusion reached by the process, and p is called the innovative coefficient. The higher this value, the more rapid the takeoff of the life cycle, to describe a process where exogenous factors, like advertising or institutional communication efforts, push the diffusion. The latter effect, $qz(t)/m$, depends upon the degree of saturation of the market and describes how previous sales exert word-of-mouth to promote further diffusion. The coefficient q is called the imitative coefficient. The higher q, the more important the word-of-mouth to increase diffusion.

Under the initial condition $z(0) = 0$, and defining $z(t) = 0$ for $t < 0$, the explicit solution of Equation (1) is

$$z(t) = m\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}\, e^{-(p+q)t}}, \quad t > 0. \tag{2}$$

Although model (1) and its successive extensions proved to be extremely valuable in describing innovation diffusion processes, an important limitation is given by the definition of the market potential, m, as a fixed constant. This assumption conflicts with the common perception that knowledge may be time dependent. Some attempts were proposed in the literature to overcome this limitation.

In some papers, the dynamic market potential is modeled as a function of exogenous observed variables (see, e.g., Kim et al., 1999 and the references cited). In other cases, it is assumed to be a function of time only (e.g., Sharif and Ramanathan, 1981; Centrone et al., 2007; Meyer and Ausubel, 1999).

Here, we will follow the approach by Guseo and Guidolin (2009). In principle, the market potential can be any function $m(t)$. However, a parsimonious and intuitive method for specifying the form of $m(t)$ arises when we look at the communication network spreading information about the products. The number of potential adopters of a product can be thought of, at each time point, as the size of the aware agents' group. We describe awareness of the product as transmitted knowledge through a network that describes the specific contacts among agents eventually 'speaking' about the products. This approach is linked to the literature about social networks, often represented with random graph models where nodes denote individual social actors (agents) and edges denote specific relationships between the actors (Handcock and Gile, 2010). Many contributions assume observability of the edges, either complete or partial, through sample data.

Conversely, in our approach this communication network evolving in time is latent and does not have to be observed or described in detail, due also to the high costs of relational data. The focus is on the number of informed agents (active nodes). This is a key point, since we want to deal with all the situations where the communication network is *product-specific* (people usually choose to talk with someone - and not with someone else - according to the topic of the conversation). In that case, the content-driven network is totally unobservable or it is very difficult to obtain reliable *pertinent* data.

In Guseo and Guidolin (2009), the formalized structure of such a network is described, and it is explained in detail how this interpretation may lead to the following dynamic market potential function:

$$m(t) = K\sqrt{\frac{1 - e^{-(p_c+q_c)t}}{1 + \frac{q_c}{p_c}\, e^{-(p_c+q_c)t}}}, \quad t > 0, \tag{3}$$

where K is the upper asymptotic potential (directly related to the network's size), and p_c and q_c are evolutionary parameters describing how fast communication spreads through the network.

Observe that the expression under the square root in Equation (3) represents itself the core of a Bass (Bass, 1969) model (Equation (2)) describing the latent dif-

fusion process of communication. This is an S-shaped curve, a distribution function, whose peakedness varies according to the product's communication features.

3 The proposed model

The proposed model describes the diffusion of two competing brands. They are supposed to be similar enough to share common market potential, whose size grows in time as described in Section 2. A common market potential assumption is suitable in situations when the products are substitutes competing for the same adopters. Whenever competition, conversely, concerns products different enough to preserve product-specific market potentials, the family of Lotka-Volterra models should be preferred, although these structures do not allow a closed-form solution (Abramson and Zanette, 1998).

Denoted by $z_i(t)$, $i = 1, 2$, the cumulative sales at time t of brand i, and by $z_i'(t) = \partial z_i(t)/\partial t$, $i = 1, 2$, the instantaneous sales, respectively, the model is given by:

$$\begin{aligned} z_1'(t) &= m(t)\left[p_1 + (q_1+\delta)\frac{z_1(t)}{m(t)} + q_1\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_1(t)\frac{m'(t)}{m(t)} \\ z_2'(t) &= m(t)\left[p_2 + (q_2-\delta)\frac{z_1(t)}{m(t)} + q_2\frac{z_2(t)}{m(t)}\right]\left[1 - \frac{z(t)}{m(t)}\right] + z_2(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{4}$$

In Equation (4), we may observe innovators' effects (parameters p_1 and p_2) and word-of-mouth effects (parameters q_1, q_2 and δ). These parameters may be different for the two competitors in order to describe products with different strengths in the market. Observe that this structure is similar to the model used in Guseo and Mortarino (2014b), with within-brand word-of-mouth ($q_1 + \delta$ and q_2 for the two brands, respectively) that may be different from the cross-brand word-of-mouth (q_1 and $q_2 - \delta$). In other words, this model may deal with situations when word-of-mouth acts asymmetrically for the two products. In Guseo and Mortarino (2014b), however, $m(t)$ was supposed to be a common market potential invariant throughout the life cycle, $m(t) = m \ \forall t$.

The final additive terms in Equation (4) (which would obviously vanish for a constant $m(t)$) represent a *self-reinforcing* component as described in Guseo and Guidolin (2009). The sales of the products are accelerated when $m(t)$ grows faster, i.e., when the awareness about the product spreads rapidly. Notice that this model can also be used with the expression for $m(t)$ different from Equation (3). In that case, $m(t)$ could also be a non-monotonic function, and the self-reinforcing term could be negative in time periods when the market potential undergoes a contraction.

Let us define $p_s = p_1 + p_2$ and $q_s = q_1 + q_2$. Through the introduction of the functions

$$w(t) = \frac{1 - e^{-(p_s+q_s)t}}{1 + \frac{q_s}{p_s}e^{-(p_s+q_s)t}} \tag{5}$$

and

$$y(t) = 1 + \frac{q_s}{p_s} w(t) = \frac{1 + \frac{q_s}{p_s}}{1 + \frac{q_s}{p_s} e^{-(p_s+q_s)t}}, \tag{6}$$

in Appendix 1 it is proven that, for any $m(t)$, the closed-form solution of the system (4) is

$$\begin{aligned} \frac{z_1(t)}{m(t)} &= \frac{q_1}{q_s - \delta} w(t) + \left[\frac{p_s}{\delta} \left(\frac{p_1}{p_s} - \frac{q_1}{q_s - \delta} \right) \right] \left[y(t)^{\frac{\delta}{q_s}} - 1 \right] \\ \frac{z_2(t)}{m(t)} &= \left(\frac{q_2 - \delta}{q_s - \delta} \right) w(t) + \left[\frac{p_s}{\delta} \left(\frac{p_2}{p_s} - \frac{q_2 - \delta}{q_s - \delta} \right) \right] \left[y(t)^{\frac{\delta}{q_s}} - 1 \right], \end{aligned} \tag{7}$$

when $\delta \neq 0$ and $\delta \neq q_s$.

When $\delta = q_s$, the solution reduces to

$$\begin{aligned} \frac{z_1(t)}{m(t)} &= \left(\frac{p_1}{p_s} - \frac{q_1}{q_s} \right) w(t) + \frac{q_1 p_s}{q_s^2} y(t) \ln y(t) \\ \frac{z_2(t)}{m(t)} &= \left(1 - \frac{p_1}{p_s} + \frac{q_1}{q_s} \right) w(t) - \frac{q_1 p_s}{q_s^2} y(t) \ln y(t), \end{aligned} \tag{8}$$

while, in the special case $\delta = 0$, we obtain

$$\begin{aligned} \frac{z_1(t)}{m(t)} &= \frac{q_1}{q_s} w(t) + \frac{p_s}{q_s} \left(\frac{p_1}{p_s} - \frac{q_1}{q_s} \right) \ln y(t) \\ \frac{z_2(t)}{m(t)} &= \frac{q_2}{q_s} w(t) + \frac{p_s}{q_s} \left(\frac{p_2}{p_s} - \frac{q_2}{q_s} \right) \ln y(t). \end{aligned} \tag{9}$$

The solutions allow us to use a nonlinear regression model with dependent variables given by the observed cumulative sales of the two brands. A reasonable and robust inferential methodology for estimating and testing the performance of this structure may be implemented through the model

$$v_i(t) = \eta_i(\beta, t) + \varepsilon_i(t), \quad i = 1, 2, \tag{10}$$

where $v_i(t)$ represents the observed cumulative sales data for each of the two products and $\eta_i(\beta, t)$ denotes the cumulative distribution functions (7) depending on the vector of parameters $\beta = \{K, p_c, q_c, p_1, q_1, p_2, q_2, \delta\}$ and on time t. Here, we assume that $m(t)$ is modeled as in (3). The residual term $\varepsilon_i(t)$ is usually a white noise or a more complex stationary process if seasonality and/or autoregressive aspects are included as stochastic components. The joint estimate of β is obtained with a single model where $v_1(t)$ and $v_2(t)$ are stacked. This estimate could be generated using the Beauchamp and Cornell technique (Beauchamp and Cornell, 1966). Recent results, however, show that it is advisable to use ordinary nonlinear least squares (Guseo and Mortarino, 2014a).

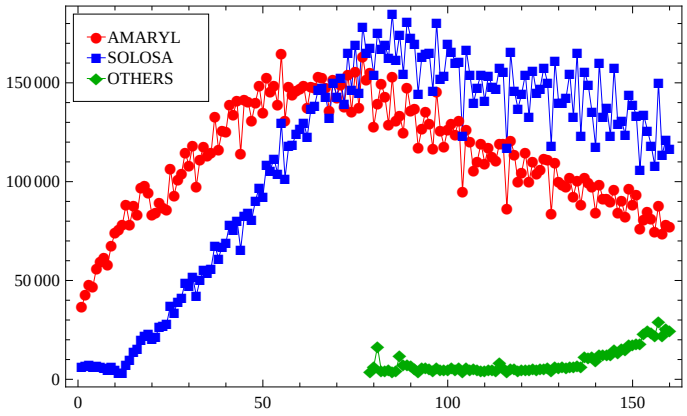

Figure 1: Monthly sales data for Amaryl 2 mg and Solosa 2 mg. The series of the sum of all the sales of alternative products is also shown (source: IMS Health Italy).

4 An application

Amaryl (Sanofi-Aventis) and Solosa (Lab. Guidotti) are two drugs based upon glimepiride, used by people with type 2 diabetes. Glimepiride belongs to the class of drugs known as sulfonylureas. It lowers hyperglycemia by causing the release of the body's natural insulin. These drugs, at a dose of 2 mg, were launched in the Italian market in January 1999 and were for many years duopolists in the glimepiride market. Figure 1 shows monthly sales data (available until April 2013) for the two drugs separately. In addition, the series of the sum of all the sales of alternative products (12 generic drugs) commercialized since 2006 is shown. The more recent products have never represented a real threat to the two oldest brands.

These two drugs are perfect substitutes from the medical point of view, and thus a model with a common market potential appears as an adequate solution. Moreover, in 1999 glimepiride represented a radical novelty in the Italian market, since it was the first type of sulfonylurea available. Other dosages of the same drugs were launched later in 2006. These considerations suggest that awareness of the properties and the efficacy of these drugs perhaps were not not widespread among Italian physicians in 1999. A dynamic market potential appears a reasonable approach for the description of these data. The complete unobservability of the communication network that since 1999 spread knowledge about glimepiride finally suggests that model (3) could be an appropriate tentative solution. Of course, only good agreement between available data and model (7), which incorporates these features, could confirm or lead to rejection of these assumptions.

Joint nonlinear regression of the two main competitors' cumulative sales on the functions (7) gives rise to the parameters' estimates shown in Table 1.

Table 1: Estimation results for model (7).

	Estimate	Standard Error	95% Confidence Interval
K	$9.09385{*}10^{8}$	$3.02298{*}10^{-17}$	$(9.09385{*}10^{8}, 9.09385{*}10^{8})$
p_c	$9.62807{*}10^{-6}$	$3.95248{*}10^{-7}$	$(8.85038{*}10^{-6}, 0.0000104)$
q_c	0.0017675	0.0004215	(0.0009382, 0.0025968)
p_1	0.0049407	0.0000671	(0.0048087, 0.0050727)
q_1	-0.0037018	0.0009179	(-0.005508, -0.001896)
p_2	-0.0000204	0.0000412	(-0.000101, 0.0000606)
q_2	0.0261077	0.0011429	(0.0238590, 0.0283565)
δ	0.0122482	0.0016330	(0.0090352, 0.0154612)
$R^2 = 0.999899$			

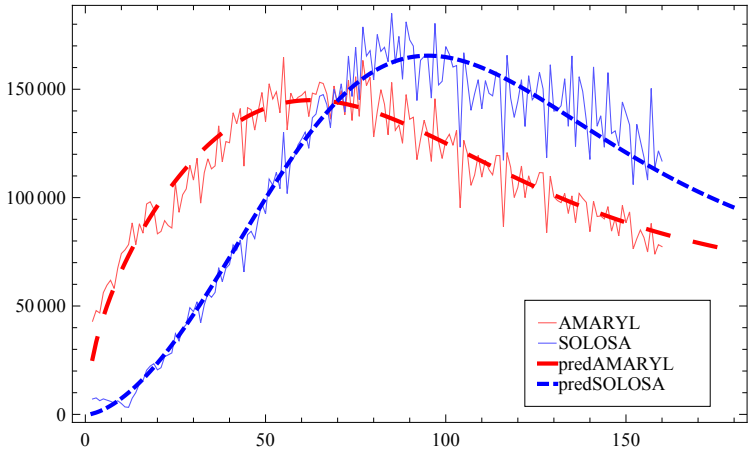

Figure 2: Comparison of the observed and fitted values, instantaneous sales, model 7.

In that case, the huge value of R^2=0.999899 is unsurprising since we work with cumulative data and any S-shaped fitting produces high determination indexes. A standard approach advises the use of the R^2 measure only for comparative purposes, as will be described in the sequel. In addition, the evaluation of the squared linear correlation coefficient between observed *instantaneous* sales and fitted *instantaneous* sales gives a value of 0.999899.

The agreement between the observed and fitted values can also be assessed by inspecting Figure 2. The two estimated profiles follow very well the observations and discrepancies (essentially due to seasonal effects) could be easily modeled through a SARMAX approach for the second step refinement (see Appendix 2). The residuals' analysis is proposed in Figure 3.

Since we deal with consumables (i.e., repeatedly purchased goods), $\hat{K}$ (909 million) represents an estimate of the total number of packages that could be sold by the two drugs. Figure 4 shows the estimated evolution of the common dynamic market potential, $m(t)$. It is very far from a fixed m pattern, since knowledge about these

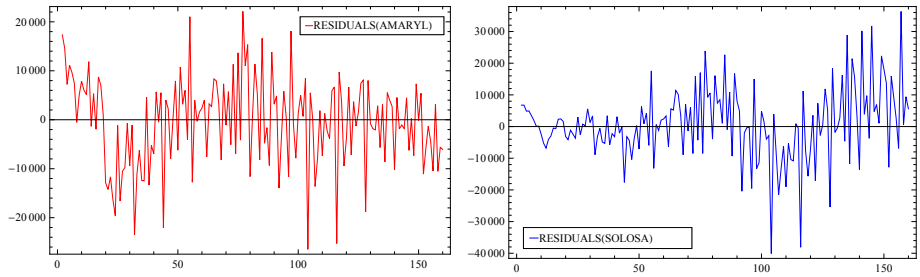

Figure 3: Residuals for the two products (instantaneous sales scale).

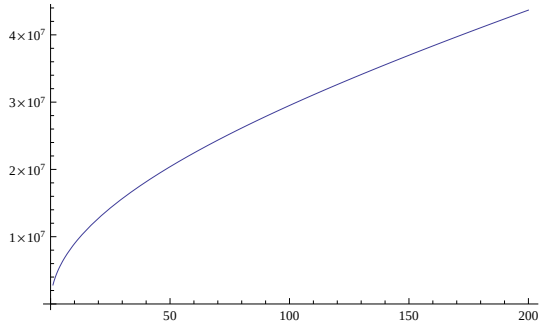

Figure 4: Plot of the estimated market potential function, $\hat{m}(t)$.

drugs seems to have spread among physicians slowly. This could be explained by observing that a new active compound (as it was glimepiride in the Italian market in 1999) is accepted with caution until side effects are not totally disclosed.

If we focus on innovation parameters, we see that this component did not play a significant role for Solosa, and this may explain its slow start. Lab. Guidotti, which launched Solosa, is a big Italian company, but it was not able to compete, in terms of commercial pressure, with the strength of the international company Sanofi-Aventis, which promoted Amaryl.

Imitative parameters have to be interpreted with reference to the proposed model. If we substitute the estimates in model (4), we have the following:

$$\begin{aligned} z_1'(t) - z_1(t)\frac{m'(t)}{m(t)} &\propto 0.0049 + 0.0085\frac{z_1(t)}{m(t)} - 0.0037\frac{z_2(t)}{m(t)} \\ z_2'(t) - z_2(t)\frac{m'(t)}{m(t)} &\propto + 0.0139\frac{z_1(t)}{m(t)} + 0.0261\frac{z_2(t)}{m(t)}. \end{aligned}$$

Amaryl experienced a weaker within-brand word-of-mouth effect than Solosa (0.0085 vs. 0.0261) and suffered from the competition due to negative cross-brand word-of-mouth by Solosa adoption. Conversely, Solosa added to the stronger within-brand word-of-mouth effect positive cross-brand word-of-mouth arising from the adoption of its competitor. This ultimately led Solosa to outsell Amaryl. Both drugs now appear to be in a declining phase of their life cycle (due to the appearance of other

active compounds in the type 2 diabetes market).

The efficacy of the proposed model in this application has to be proved with reference to a simpler model with constant market potential. The model proposed in Guseo and Mortarino (2014b) fits this purpose since it can be obtained by (4) with the only restriction $m(t) = m$. In addition, other models available in the literature are nested within the Guseo and Mortarino (2014b) model. The R^2 for the Guseo and Mortarino (2014b) model equals 0.999259. Since this model is nested within model (4), we calculate an F test to detect whether the gain from the simpler model to the more complex model is significant. In detail, as the first step, the squared multiple partial correlation coefficient

$$\widetilde{R}^2 = (R^2_{M1} - R^2_{M2})/(1 - R^2_{M2}) \tag{11}$$

is calculated (here, R^2_{M2} denotes the determination index of the reduced model that has to be compared to model $M1$). A possible test that verifies the significance of the s parameters of the $M1$ model that are not included in model $M2$ may be given by

$$F = [\widetilde{R}^2(N - k)]/[(1 - \widetilde{R}^2)s], \tag{12}$$

where N denotes the number of observations used to fit the models and k is the number of parameters included in model $M1$. Under the null hypothesis of equivalence between models $M1$ and $M2$, (12) is distributed as a Snedecor's F with $(s, N - k)$ degrees of freedom, if the stochastic component of the regression model is normal i.i.d. This is not our case. Nevertheless, the F ratio (12) can be used as an approximate robust criterion for comparing model $M2$ nested in $M1$, by considering the well-known common threshold 4 (Guseo et al., 2007). Here, the test comparing model (4) with the Guseo and Mortarino (2014b) model assigns the huge value of F=991.967 ($\widetilde{R}^2$=0.864108), denoting the relevance of the extended (4) model.

In Figure 5, the fitted values of model (4) and the Guseo and Mortarino (2014b) model are compared. The rigidity of a fixed market potential make the latter model inadequate to describe these data and, even worse, for larger t values, it shows a heavy underestimation that make forecasts unreliable.

5 Concluding remarks

Diffusion of innovation methodologies have faced and are facing new challenges in order to incorporate, in parsimonious model building the major effects that can modify their evolutionary shapes over time.

The aim of this paper is to highlight the key features of the competition between Amaryl and Solosa. These two drugs differ essentially in the persuasion effects exerted by the two companies that launched the drugs and their acceptance by the physicians' community spreading word-of-mouth about their efficacy.

The novelty represented by the active compound of the drugs suggested we should enrich the existing competing models' family with the introduction of dynamic market potential. This extension rests on the statement that awareness is a fundamental prerequisite for adoption. We can imagine that, at the individual level, awareness and adoption are two subsequent states that subjects (physicians) may go through.

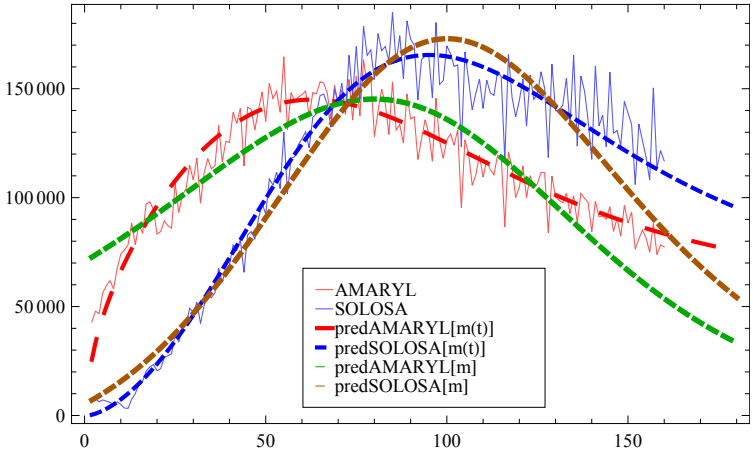

Figure 5: Comparison of the fitted values for model (4) and the model used in Guseo and Mortarino (2014b).

The first state, awareness, is latent. In addition, since individual data are in this case unavailable, the description is aggregated (as a mean profile) leading to Equation (3).

Notice that the Guseo and Guidolin (2009) paper, in a completely different context, inspired the approach followed by Furlan and Mortarino (2012) to describe and predict the death toll due to pleural mesothelioma contracted through the exposure to asbestos fibers in a residential area close to a big plant. In that case, contamination (state 1), i.e., contact with lethal asbestos fibers, was the latent prerequisite for developing the disease (state 2).

As a final remark, we emphasize that our proposed model is useful for analyzing competition between two products. The tractability of the model, in terms of the estimate of the parameters involved, allows us to deal with a higher number of competitors only if they entered the market simultaneously. Diachronic competition, i.e., competitors launched in the market at different time points, requires, in general, model structures with multiple regimes (a change-point in the evolution of existing products occurs whenever a new one appears). In that case, for more than three products, the parameter cardinality becomes to high too obtain reliable estimates, unless each regime is covered by an adequate observation period.

Appendix 1. Proof

Let $z(t) = z_1(t) + z_2(t)$ denote the sum of the cumulative sales of the two products and by $z'(t)$ the total instantaneous sales. If we sum the equations of system (4),

we obtain

$$\begin{aligned} z'(t) &= z_1'(t) + z_2'(t) \\ &= m(t)\left[(p_1+p_2)+(q_1+q_2)\frac{z_1(t)}{m(t)}+(q_1+q_2)\frac{z_2(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right] \\ &\quad +[z_1(t)+z_2(t)]\frac{m'(t)}{m(t)} \\ &= m(t)\left[(p_1+p_2)+(q_1+q_2)\frac{z(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right]+z(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{13}$$

Equation (13) defines a coevolutive model (Guseo and Guidolin, 2009) with unspecified market potential $m(t)$ and adoption parameters $p_s = p_1 + p_2$ and $q_s = q_1 + q_2$. It follows that the solution of the differential equation (13), with initial condition $z(0) = 0$, is the following:

$$\frac{z(t)}{m(t)} = w(t) = \frac{1-e^{-(p_s+q_s)t}}{1+\frac{q_s}{p_s}e^{-(p_s+q_s)t}}. \tag{14}$$

In order to find a solution for $z_1(t)$, the first equation in system (4) should be rearranged:

$$\begin{aligned} z_1'(t) - z_1(t)\frac{m'(t)}{m(t)} &= m(t)\left[p_1+(q_1+\delta)\frac{z_1(t)}{m(t)}+q_1\frac{z_2(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right] \\ \frac{z_1'(t)\,m(t)-z_1(t)\,m'(t)}{m^2(t)} &= \left[p_1+q_1\frac{z(t)}{m(t)}+\delta\frac{z_1(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right] \\ \left[\frac{z_1(t)}{m(t)}\right]' &= \left[p_1+q_1\frac{z(t)}{m(t)}+\delta\frac{z_1(t)}{m(t)}\right]\left[1-\frac{z(t)}{m(t)}\right] \\ \left[\frac{z_1(t)}{m(t)}\right]' &= \left[p_1+q_1w(t)+\delta\frac{z_1(t)}{m(t)}\right][1-w(t)]. \end{aligned} \tag{15}$$

Equation (15) perfectly matches the differential equation in Guseo and Mortarino (2014b) (p. 308, between (A.1) and (A.2)), where m_c has to be replaced by $m(t)$ and in the expression for $w(t)$ we have to add the further condition $z_s = 0$. In our case, differently from Guseo and Mortarino (2014b), we examine competition between two products entering simultaneously into the market. If we add this two conditions to the solution to the differential equation, we obtain exactly (7), (8), and (9).

Appendix 2. SARMAX refinement

As mentioned in Section 3, for estimation purposes, we use a two-phase procedure. First, we apply to model (10) a robust nonlinear least squares algorithm (NLS), which ignores the stochastic structure of $\varepsilon_i(t)$, under the well-known Levenberg-Marquardt correction of the Gauss-Newton recursive procedure; see, for instance, Seber and Wild (2003). Second, the prediction $\eta_i(\hat{\beta}, t)$ based on a NLS solution, $\hat{\beta}$, may be used in a model based on a seasonal, autoregressive, moving average process with an input X (SARMAX) in order to improve short-term prediction,

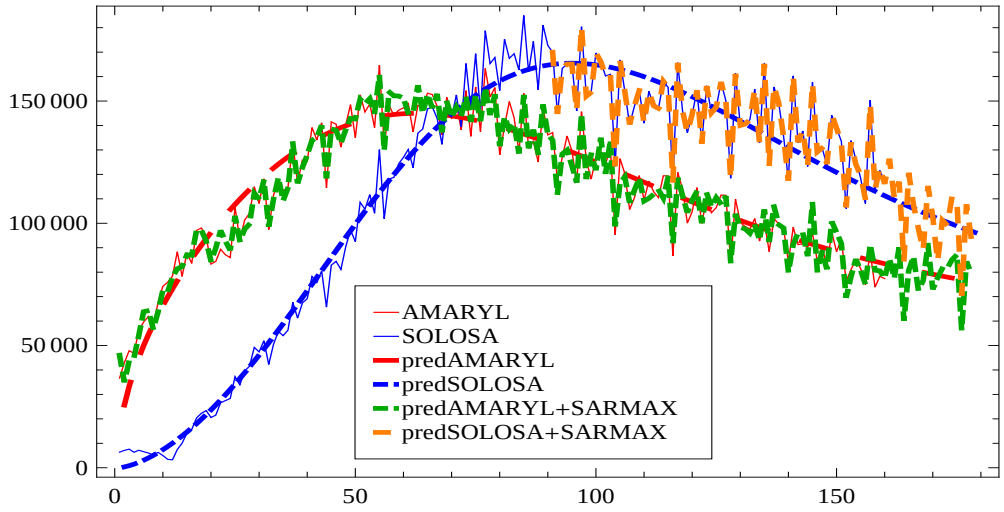

Figure 6: Comparison of the fitted values for model (4) and the SARMAX refinement.

which is relevant for managerial applications. This second stage is implemented if the residuals of the first stage do not follow a standard white noise pattern. The Durbin-Watson statistic may be used as an exploratory test to diagnose whether this second step is necessary. In this case, the Durbin-Watson statistic equals 0.03317, distinctly detecting a positive autocorrelation.

The SARMAX improvement for short-term predictions rests on the following equation based on a polynomial function of backward operators, namely,

$$\Psi(B)\Phi(B^s)[z_i'(t) - c_i\eta_i(\hat{\beta}, t)] = \Omega(B)\Theta(B^s)a_i(t) \tag{16}$$

with $a_i(t)$ a White Noise process; B and B^s the standard and seasonal backward operators; and $\Psi(B)$, $\Phi(B^s)$, $\Omega(B)$, and $\Theta(B^s)$ the usual backward polynomials of order g, G, h, and H, respectively. The calibration parameters c_i allow a global assessment of the stability of the predicted regressive values stemming from the estimated models $\eta_i(\hat{\beta}, t)$.

In this situation, we observe that the residuals for the Solosa series (Figure 3) denote a relevant change-point essentially around the middle of the series. Since SARMAX is meant as an improvement for predictive purposes, we chose to apply it only to the second part of the series ($t > 90$), whose data are more relevant for future evolution.

The estimates of the parameters involved in Equation (16) applied to instantaneous data are proposed in Table 2. The agreement between the observed data and the fitted values with the SARMAX refinement is almost perfect (see Figure

Table 2: Parameter estimates for the SARMAX refinement. [ ] t statistic. $*$: significant, 95%. $**$: strongly significant, 99%. prM denotes the fitted values with model (4). The subscripts of $\tilde{R}^2$ and F define the involved nested models; in particular, $M|GG$ denotes the comparison of model (4) and the SARMAX model.

Model	Parameter	**Amaryl**	**Solosa**
SARMAX + prM	AR1	0.793562**	1.020960**
	AR2	0.710546	-1.564090**
	AR3	-0.406301	1.332160**
	AR4	0.250634	-1.156470**
	AR5	-0.375593	0.664820**
	MA1	0.796107**	-0.203328
	MA2	0.564276	0.983558**
	MA3	-0.593492**	0.381771
	MA4	0.270996	-0.273830
	MA5	-0.037560	-0.137799
	SAR1	-0.239924**	-0.048891**
	SAR2	0.696152**	0.727976**
	SAR3	-0.619403**	-0.728324**
	SAR4	0.162768*	0.049397**
	SAR5	1.080770**	0.999809**
	SMA1	-0.465237**	0.002783
	SMA2	0.563276**	0.000324
	SMA3	-0.360688**	-0.000017
	SMA4	0.043359	-0.000235
	SMA5	0.630513**	-0.000104
prM	c	1.0094**	0.866576**
$[t_M]$		[30.7978]	[255.083]
RSS		$2.915695*10^9$	$1.623998 *10^7$
RSS_M		$1.349733*10^{10}$	$1.515860*10^{10}$
$\tilde{R}^2_{M\|S}$		0.783980	0.998929
$F_{M\|S}$		21.775193	2712.418722

6). This confirms that the discrepancies between the observed data and the fitted values with model (4) were essentially due only to an autoregressive component and a seasonal effect.

The overlapping of the fitted trajectory on the observed data strongly supports the choice of the model (4) for the description of our data.

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