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Potential theoretic methods for the analysis of singularly perturbed problems in linearized elasticity

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Introduction

This dissertation consists of two chapters. Chapter one is devoted to the investigation of some properties of the layer potentials of an elliptic partial differential operator with constant coefficients. In particular, we investigate the dependence upon perturbation of the density, the support and the coefficients of the operator. The main result is a real analyticity theorem for the single layer potential and its derivatives, which has been proved under quite general assumptions on the operator. This result is applied to the special case of Helmholtz and bi-Helmholtz operators, Lamé equations and Stokes system. A real analyticity theorem is proved for the corresponding single and double layer potential.

Chapter two deals with the boundary value problems for the Lamé equations, which describe some physical processes, in particular, the elastic properties of an isotropic homogeneous elastic body. Special attention is paid to the case of boundary value problems defined in a domain with a small hole which shrinks to a point. The aim is to describe the behavior of the solution and of the corresponding energy integral. This kind of problem is not new and has been investigated by the techniques of *asymptotic analysis* (see, *e.g.*, the works of Keller, Kozlov, Movchan, Maz'ya, Nazarov, Plamenewskii, Ozawa, Ward.) Let $\epsilon > 0$ be a parameter which is proportional to the diameter of the hole, so that the singularity of the domain appears when $\epsilon = 0$. By the approach of the asymptotic analysis, one can expect to obtain results which are expressed by means of known functions of ϵ plus an unknown term which is smaller than a positive known function of ϵ. The approach adopted here stems from the papers of Lanza de Cristoforis [20, 21, 22, 23, 25] and it is in some sense alternative to the approach of the asymptotic analysis. The aim is to express the dependence upon ϵ in terms of real analytic functions defined in a whole open neighborhood of $\epsilon = 0$ and in terms of possibly singular but completely known function of ϵ, such as ϵ^{2-n} or $\log \epsilon$. As a corollary, one could obtain asymptotic formulas which agree with those in the literature.

We now describe in details the content of each chapter.

Chapter 1. In subsection 1.1.1, we present the construction of a particular fundamental solution S of a given elliptic constant coefficient partial differential operator $\mathbf{L}$ of order $2k$ on $\mathbb{R}^n$, $k \geq 1$, $n \geq 2$. For this purpose, we exploit John [14, Chapter III]. In subsection 1.1.2, we investigate the dependence of such a fundamental solution upon perturbation of the coefficients of the operator. We verify that, if the coefficients of the operator are contained in a bounded set, then there exists a particular fundamental solution of the type introduced in subsection 1.1.1 which is a sum of functions which depend real analytically on the coefficients of the operator. Such a result resembles the results of Mantlik [29, 30] (see also Trèves [43]), where more general assumptions on the operator are considered. We observe that it is not a corollary. Indeed, the suitably detailed expression for the fundamental solution, which is obtained in subsection 1.1.1, cannot be deduced by [29, 30].

Section 1.2 deals with the single layer potential of the elliptic operator $\mathbf{L}$ corresponding to the fundamental solution S introduced in subsection 1.1.1. We fix an open and bounded subset Ω of $\mathbb{R}^n$ with Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ connected and we assume that the boundary $\partial\Omega$ is a compact sub-manifold of $\mathbb{R}^n$ of Hölder class $C^{m,\lambda}$, with $m \in \mathbb{N} \setminus \{0\}$ and $0 < \lambda < 1$. We regard Ω as a given fixed set and we consider open subsets of $\mathbb{R}^n$ whose boundary is parametrized by a diffeomorphism of class $C^{m,\lambda}$ defined on $\partial\Omega$. Clearly not all the functions of $\partial\Omega$ to $\mathbb{R}^n$ give rise to the boundary of an open subset of $\mathbb{R}^n$. So in subsection 1.2.1 we introduce a class $\mathcal{A}_{\partial\Omega}$ of admissible functions on $\partial\Omega$. We also recall some useful properties of the functions of $\mathcal{A}_{\partial\Omega}$ pointed out by Lanza de Cristoforis and Rossi in [27, 28]. Then for each $\phi \in \mathcal{A}_{\partial\Omega}$ of class $C^{m,\lambda}$ and for each density function μ of class $C^{m-1,\lambda}$ defined on $\partial\Omega$ we consider the single layer potential v which is the function defined on $\mathbb{R}^n$ by

$$v(\xi) \equiv \int_{\phi(\partial\Omega)} S(\xi - \eta)\mu \circ \phi^{(-1)}(\eta)\, d\sigma_\eta, \quad \forall\, \xi \in \mathbb{R}^n$$

where S is the fundamental solution of $\mathbf{L}$ introduced in subsection 1.1.2. Moreover, for each multi-index β with $|\beta| \leq 2k - 1$, we denote by v_β the function of $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$v_\beta(\xi) \equiv \int_{\phi(\partial\Omega)} (\partial^\beta S)(\xi - \eta)\mu \circ \phi^{(-1)}(\eta)\, d\sigma_\eta, \quad \forall\, \xi \in \mathbb{R}^n,$$

so that $v_\beta = \partial_\xi^\beta v$ on $\mathbb{R}^n \setminus \phi(\partial\Omega)$. We denote by V_β the function of $\partial\Omega$ to $\mathbb{R}$ which is the composition of v_β and ϕ. Our purpose is to investigate the dependence of V_β upon suitable perturbations of the function ϕ, of the density μ, and of the coefficients of the operator $\mathbf{L}$. We state in subsection 1.2.2 the main result of the present chapter, which is a real analyticity theorem in the frame of Schauder spaces for V_β , $|\beta| \leq 2k-1$ (see Theorem 1.7.) The rest of section 1.2 is dedicated to the proof of such a theorem. The main idea of the proof stems from the papers of Lanza de Cristoforis and Preciso and of Lanza de Cristoforis and Rossi and exploits the Implicit Function Theorem for the analytic functions. Indeed, Theorem 1.7 is in some sense a natural extension of Theorem 3.23 of [26] (where the Cauchy integral has been considered), of Theorem 3.25 of [27] (where the Laplace operator Δ has been considered) and of Theorem 3.45 of [28] (where the Helmholtz operator has been considered.) Here we confine our attention to elliptic operators $\mathbf{L}$ which can be factorized with operators of order 2 (cf. Theorem 1.7.) In order to prove our main Theorem 1.7, we need some preliminaries. In subsection 1.2.3 we summarize some regularity properties of the layer potentials. In subsection 1.2.4 we summarize the jumping properties of the corresponding derivatives. In subsection 1.2.5 we introduce an auxiliary boundary value problem and in subsection 1.2.5 we investigate some stability properties of such an auxiliary boundary value problem. In subsection 1.7 we prove our main Theorem 1.7.

In the last section of the chapter we focus our attention on single and double layer potentials which arise in the study of certain boundary value problems of physical interest, such as the basic boundary value problems for the Lamé equations and the Stokes system. First, we need some auxiliary results for the bi-Helmholtz single layer potential and its derivatives, which are obtained as a straightforward consequence of Theorem 1.7 in the subsection 1.3.1. Then in subsection 1.3.2 we give a suitable expression of the fundamental solution of the Lamé equations and we deduce that the corresponding single and double layer potentials depend real analytically upon perturbations of the domain, the density and the coefficients of the operator. By a similar argument we deduce also that the single and double layer potentials relative to the Stokes system depend real analytically upon perturbations of the domain and the density (see subsection 1.3.3.)

Chapter 2. In the second chapter we consider some boundary value problems for the operator $\mathbf{L}[b] \equiv \Delta + b\nabla\mathrm{div}$, where $b > 1 - 2/n$ is a constant, in a domain with a small hole. The behavior of the solution and of the corresponding energy integral as the hole shrinks to a point is investigated.

To explain our results, we now present in details the statement of one of the boundary value problems. First we introduce the domain. We fix a bounded open subset Ω^d of $\mathbb{R}^n$, such that Ω^d and $\mathbb{R}^n \setminus \mathrm{cl}\Omega^d$ are connected, and such that $\partial\Omega^d$ is submanifold of class $C^{m,\lambda}$, with $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0, 1[$. If ϕ^d is a $C^{m,\lambda}$ function defined on $\partial\Omega^d$ which belongs to the class of the admissible functions $\mathcal{A}_{\partial\Omega^d}$, then its image $\phi^d(\partial\Omega^d)$ splits $\mathbb{R}^n$ into two connected components. We denote by $\mathbb{I}[\phi^d]$ the bounded one. Then $\mathbb{I}[\phi^d]$ is a bounded open subset of $\mathbb{R}^n$ with boundary of class $C^{m,\lambda}$ parametrized by the function ϕ^d. Now, we make a hole in the domain $\mathbb{I}[\phi^d]$. We fix another bounded open subset Ω^h of $\mathbb{R}^n$, with Ω^h, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^h$ connected, and $\partial\Omega^h$ of class $C^{m,\lambda}$. The hole will be obtained by a suitable affine transformation of the domain $\mathbb{I}[\phi^h]$, with ϕ^h a $C^{m,\lambda}$ diffeomorphism on $\partial\Omega^h$ which belongs to $\mathcal{A}_{\partial\Omega^h}$. So, we take a point ω in the domain $\mathbb{I}[\phi^d]$, and we take a scalar $\epsilon \in \mathbb{R}$. Clearly, if ϵ is small enough, the closure of the set $\omega + \epsilon\mathbb{I}[\phi^h]$ is contained in $\mathbb{I}[\phi^d]$. If this is the case, $\omega + \epsilon\mathbb{I}[\phi^h]$ is our hole, and we obtain a perforated domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ by removing the closure of the hole $\omega + \epsilon\mathbb{I}[\phi^h]$ from the domain $\mathbb{I}[\phi^d]$. We note that $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ is a bounded open and connected subset of $\mathbb{R}^n$ with boundary made of two connected components, $\omega + \epsilon\phi^h(\partial\Omega^h)$ and $\phi^d(\partial\Omega^d)$. We denote by $\mathcal{E}^{m,\lambda}$ the set of all the admissible quadruples $(\omega, \epsilon, \phi^h, \phi^d)$ which give rise to a perforated domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ and we point out that $\mathcal{E}^{m,\lambda}$ is an open subset of the Banach space $\mathbb{R}^n \times \mathbb{R} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. In particular, $\mathbb{A}[\omega, 0, \phi^h, \phi^d] = \mathbb{I}[\phi^d] \setminus \{\omega\}$.

We now introduce a boundary value problem in the domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ with $\epsilon > 0$. So let g^h and g^d be two functions of $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ and $C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. We consider the following system of equations,

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \\ u = g^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ u = g^d \circ (\phi^h)^{(-1)} & \text{on } \phi^d(\partial\Omega^d). \end{cases} \tag{0.1}$$

Problem (0.1) has a unique solution u of $C^{m,\lambda}(\mathrm{cl}\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \mathbb{R}^n)$ for each given $b > 1 - 2/n$, $(\omega, \epsilon, \phi^h, \phi^d) \in \mathcal{E}^{m,\lambda}$ and $(g^h, g^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ (cf. subsection 2.2.2.) So it makes sense to consider such a solution as a function of the variables $(b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d)$ and to write $u[b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d]$ to denote it. Our purpose is to investigate the dependence of $u[b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d]$ upon the 7-tuple $(b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d)$. We notice that we investigate the dependence of the solution upon perturbation of the coefficient of the operator, and of the point where the hole is situated, and of the diameter of the hole, and of the shape of the hole, and of the shape of the outer domain, and of the boundary data on the boundary of the hole and on the outer boundary. In particular we want to investigate the behavior of $u[b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d]$ as $\epsilon \to 0^+$ and the hole shrinks to a point.

For fixed values of b, ω, ϕ^h, ϕ^d, g^h, g^d, the last problem is not new at all. Indeed it has been long investigated by the techniques of asymptotic analysis. It is perhaps difficult to provide a complete list of contributions. Here we mention the work of Kozlov, Maz'ya and Movchan [17], Maz'ya, Nazarov and Plamenewskii [31, 32], Ozawa [37], Ward and Keller [46]. To understand the kind of results that we can expect by asymptotic analysis, we consider a simpler situation. Let $n \geq 3$. Let Ω^d be the bounded open set introduced above. We assume that Ω^d contains the origin of $\mathbb{R}^n$ and we fix a function g^d of $C^{m,\lambda}(\partial\Omega^d)$. Then we denote by $\mathbb{B}_n$ the unit ball of $\mathbb{R}^n$, and we fix a function g^h of $C^{m,\lambda}(\partial\mathbb{B}_n)$. We consider the following boundary value problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^d \setminus \mathrm{cl}(\epsilon\mathbb{B}_n), \\ u = g^d & \text{on } \partial\Omega^d, \\ u(\xi) = g^h(\xi/\epsilon) & \text{for } \xi \in \partial(\epsilon\mathbb{B}_n), \end{cases}$$

where $\epsilon > 0$ and $\mathrm{cl}(\epsilon\mathbb{B}_n) \subset \Omega^d$. It is well known that such a boundary value problem has a unique solution $u[\epsilon]$. Now let $\xi_0 \neq 0$ be a point of Ω^d. Then, by the so-called compound asymptotic expansion method, we can deduce that the asymptotic behavior of the solution $u[\epsilon]$ evaluated at ξ_0 as $\epsilon \to 0^+$ is delivered by the following equation,

$$u[\epsilon](\xi_0) = \sum_{j=0}^{N} \epsilon^j \left(v_j(\xi_0) + w_j(\xi_0/\epsilon)\right) + O(\epsilon^{N+1}),$$

where the v_j are solutions of suitable boundary value problems in Ω^d, and the w_j are solutions of suitable boundary value problems in the exterior domain $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{B}_n$ (cf. Maz'ya, Nazarov and Plamenewskii [31, Theorem 2.1.1].)

As announced, we adopt the different approach proposed by Lanza de Cristoforis in [20, 21, 22, 23, 25]. The results that we obtain are expressed by means of real analytic functions and by completely known function of ϵ. With the notation introduced above, let $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d)$ be an admissible 7-tuple with $\epsilon = 0$. Let Ω be an open bounded subset of $\mathbb{R}^n$ with $\mathrm{cl}\Omega \subset \mathbb{I}[\phi_0^d] \setminus \{\omega_0\}$. We prove that the solution $u[\mathbf{e}]$ of (0.1) can be written in the form

$$u[\mathbf{e}](\xi) = U^{(1)}[\mathbf{e}](\xi) + \sum_{i,j=1}^{n+\binom{n}{2}} \left(\gamma_n(\epsilon)V^{(1)}[\mathbf{e}] + V^{(2)}[\mathbf{e}]\right)^{-1}_{ij} U^{(2)}_{ij}[\mathbf{e}](\xi) \qquad (0.2)$$

for all $\xi \in \mathrm{cl}\Omega$ and all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ with $\epsilon > 0$ in a open neighborhood $\mathcal{U}_0$ of $\mathbf{e}_0$, where $U^{(1)}$, $U^{(2)}_{ij}$, $V^{(1)}_{ij}$, $V^{(2)}_{ij}$ are real analytic operators defined on the whole open neighborhood $\mathcal{U}_0$ and γ_n is the function of ϵ defined by $\gamma_n(\epsilon) \equiv \log\epsilon$ for $n = 2$, $\gamma_n(\epsilon) \equiv \epsilon^{2-n}$ for $n \geq 3$ (cf. Theorem 2.53.) In particular, if $n \geq 3$, one sees that $u[\cdot]$ admits a real analytic continuation, while for $n = 2$, $u[\cdot]$ has a logarithm behavior around a degenerate 7-tuple. A similar result is obtained for the energy integral of the solution $u[\mathbf{e}]$ (cf. Theorem 2.55.)

Then we turn to consider the following Robin boundary value problem for $\epsilon > 0$,

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \\ -T(b, Du)\nu_{(\omega+\epsilon\phi^h)} = g^i \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ T(b, Du)\nu_{\phi^d} + \alpha \circ (\phi^d)^{(-1)}u = g^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d), \end{cases} \qquad (0.3)$$

where $T(b, Du) \equiv (b-1)(\mathrm{div}\, u) + (Du + Du^t)$, and $\nu_{(\omega+\epsilon\phi^h)}$, ν_{ϕ^d} are the unit outward normal to the boundary of $\omega + \epsilon\mathbb{I}[\phi^h]$ and $\mathbb{I}[\phi^d]$, respectively, and α is a matrix valued function on $\partial\Omega^d$. Under reasonable conditions one verifies that (0.3) has a unique solution. Then we prove a real analytic continuation theorem for the solution in terms of $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ in Theorem 2.69, and a real analytic continuation theorem for the corresponding energy integral in terms of $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ in Theorem 2.70. We also consider the following boundary value problem for $\epsilon > 0$,

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \\ -T(b, Du)\nu_{(\omega+\epsilon\phi^h)} = \epsilon^{1-n} f \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ T(b, Du)\nu_{\phi^d} + \alpha \circ (\phi^d)^{(-1)} u = 0 & \text{on } \phi^d(\partial\Omega^d), \end{cases} \qquad (0.4)$$

where both the domain and the boundary data display a singular behavior for $\epsilon \to 0^+$. We deduce also in this case a functional analytic representation formula for the solution and for the corresponding energy integral (cf. Theorems 2.71 and 2.72.) We notice that in this case we have real analytic continuation of the solution around $\epsilon = 0$, while the energy integral can have a singular behavior (cf. Remark 2.73.)

Finally we consider the following inhomogeneous Dirichlet boundary value problem for $\epsilon > 0$,

$$\begin{cases} \mathbf{L}[b]u = F & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^o], \\ u = g^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ u = g^d \circ \phi^{d(-1)} & \text{on } \phi^d(\partial\Omega^d). \end{cases} \qquad (0.5)$$

Under reasonable conditions we prove also in this case a formula similar to (0.1) both for the solution and for the energy integral (cf Theorems 2.80 and 2.82.)

We now briefly outline our general strategy.

Step 1. We show that the solution of the boundary value problem can be expressed in terms of layer potentials and elementary functions. The density of the layer potentials are determined by suitable boundary integral equations of Fredholm type defined on the boundary of the domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$.

Step 2. We recast such boundary integral equations into an equivalent system of boundary integral equations defined on the boundary of the fixed domains Ω^h and Ω^d. The new system will admit a unique solution for all admissible 7-tuple $(b, \omega, \epsilon, \phi^h, \phi^d, f^h, f^d)$ with $\epsilon \geq 0$.

Step 3. By exploiting the real analyticity results for the layer potentials obtained in the first chapter of the dissertation and by the Implicit Mapping Theorem we deduce a real analyticity theorem for the solution of the system on the boundary of the fixed domains.

Step 4. We compound the results of Chapter 1, of Step 1 and of Step 2 and we deduce a representation formula like (0.2) for the solution of original boundary value problem.

The chapter is organized as follows. In section 2.1 we consider the *basic* boundary value problems for the operator $\mathbf{L}[b]$. To each boundary value problem a suitable boundary integral equation is associated, and some properties of the corresponding boundary integral operator are investigated. The results of section 2.1 will enable us to produce solutions for the boundary value problems which are expressed by means of layer potentials and by elementary known functions. So section 2.1 accomplishes the first step of our general strategy. Then, in section 2.2, we consider the boundary value problem (0.1). In section 2.3, problems (0.3) and (0.4) are considered and finally, in section 2.4 problem (0.5) is considered.

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Notation

Let $\mathcal{X}$ and $\mathcal{Y}$ be real normed spaces. $\mathscr{L}(\mathcal{X},\mathcal{Y})$ denotes the normed space of the continuous linear maps of $\mathcal{X}$ into $\mathcal{Y}$ and is equipped with the topology of the uniform convergence on the unit sphere of $\mathcal{X}$. For standard definitions of calculus in normed spaces, we refer, *e.g.*, to Prodi and Ambrosetti [40]. We understand that a finite product of normed spaces is equipped with the sup-norm of the norm of the components, while we use the euclidean norm for $\mathbb{R}^n$. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N}\setminus\{0,1\}$. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a real-valued function g which is denoted g^{-1}. For all $x\in\mathbb{R}^n$, x_i denotes the i-th coordinate of x, $|x|$ denotes the euclidean modulus of x, and $\mathbb{B}_n$ denotes the unit ball $\{x\in\mathbb{R}^n : |x|<1\}$. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$. $\mathrm{M}_{n\times n}(\mathbb{R})$ is the set of the real $n\times n$-matrices. Let $A\in\mathrm{M}_{n\times n}(\mathbb{R})$. Then A^t denotes the transpose matrix of A and A_{ij} denotes the (i,j) entry of A. If A is invertible A^{-1} denotes the inverse matrix of A and we set $A^{-t}\equiv(A^{-1})^t$. Let $B\subset\mathbb{R}^n$. Then $\mathrm{cl}B$ denotes the closure of B, ∂B denotes the boundary of B, and $x+RB\equiv\{x+Ry : y\in B\}$ for all $x\in\mathbb{R}^n$, $R\in\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of the m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega)$. Let $f\in C^m(\Omega)$. The partial derivative of f with respect to x_i is denoted by $\partial_i f$, $\partial_{x_i} f$ or $\frac{\partial f}{\partial x_i}$. The space of the m times continuously differentiable vector-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R}^n)$. Let $f\in C^m(\Omega,\mathbb{R}^n)$. The i-th component of f is denoted by f_i and Df denotes the gradient matrix $(\partial_j f_i)_{i,j=1,\dots,n}$. Let $\alpha\equiv(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|\equiv\alpha_1+\cdots+\alpha_n$. Then $D^\alpha f$ denotes $\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n} f$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\alpha f$ of order $|\alpha|\le m$ can be extended with continuity to a bounded function of $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$. Then $C^{m,\lambda}(\mathrm{cl}\Omega)$ endowed with the norm $\|f\|_m\equiv\sum_{|\alpha|\le m}\sup_{\mathrm{cl}\Omega}|D^\alpha f|$ is a Banach space. The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\lambda\in]0,1]$ is denoted $C^{m,\lambda}(\mathrm{cl}\Omega)$ (see *e.g.* Gilbarg and Trudinger [11].) Let $B\subset\mathbb{R}^n$. Then $C^{m,\lambda}(\mathrm{cl}\Omega,B)$ denotes $\{f\in(C^{m,\lambda}(\mathrm{cl}\Omega))^n : f(\mathrm{cl}\Omega)\subset B\}$. If $f\in C^{0,\lambda}(\mathrm{cl}\Omega)$, then its Hölder quotient $|f|_\lambda$ is defined as $\sup\left\{|f(x)-f(y)||x-y|^{-\lambda} : x,y\in\mathrm{cl}\Omega, x\neq y\right\}$. The space $C^{m,\lambda}(\mathrm{cl}\Omega)$, equipped with the norm $\|f\|_{m,\lambda}=\|f\|_m+\sum_{|\alpha|=m}|D^\alpha f|_\lambda$, is well known to be a Banach space. We denote by $C^{-1,\lambda}(\mathrm{cl}\Omega)$ the space of distributions $\{\operatorname{div} g \,:\, g \in C^{0,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)\}$ endowed with the norm $\|f\|_{-1,\lambda} \equiv \inf\{\|g\|_{0,\lambda} \,:\, f = \operatorname{div} g, g \in C^{0,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)\}$. $C^{-1,\lambda}(\mathrm{cl}\Omega)$ is a Banach space (see *e.g.*, Lanza de Cristoforis and Rossi [28].) We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or $C^{m,\lambda}$, if its boundary $\partial\Omega$ is a submanifold of $\mathbb{R}^n$ of class C^m or $C^{m,\lambda}$, respectively (see *e.g.*, Gilbarg and Trudinger [11, §6.2].) We define the space $C^{k,\lambda}(\partial\Omega)$, with $0 \leq k \leq m$, by exploiting the local parametrizations.

Chapter 1

The layer potentials

In this chapter, we construct a particular single layer potential of a given constant coefficient elliptic partial differential operator of order $2k$. Then, in the frame of Schauder spaces, we prove a real analyticity result for the dependence of such a potential and its derivatives till order $2k-1$ upon suitable perturbations of the domain, the coefficients of the operator and of the density. Exploiting such a result, we study the dependence upon perturbations of the domain, the coefficients and the density of the single and double layer potentials which arise in certain boundary value problems, such as the Dirichlet and Neumann problems for the Lamé equations and the Stokes system. We deduce also in this case that the dependence is real analytic.

1.1 A particular fundamental solution

1.1.1 Construction of a particular fundamental solution

Let $P \in \mathbb{R}[\xi_1, \ldots, \xi_n]$ be a real polynomial of degree $2k$ ($n \geq 2$, $k \geq 1$) and denote by P_{2k} the homogeneous term of P of degree $2k$. We assume that the operator $\mathbf{L} = P(\partial_{x_1}, \ldots, \partial_{x_n})$ is elliptic on $\mathbb{R}^n$ (*i.e.* $P_{2k}(\xi) > 0$ for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $\xi \neq 0$) and then we investigate the structure of a particular fundamental solution S of $\mathbf{L}$ ($\mathbf{L}S(z) = \delta(z)$, where δ is the Dirac delta function.) For this purpose we exploit the construction of a fundamental solution given by John in [14, Chapter III].

Theorem 1.1. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 1$. Let $P \in \mathbb{R}[\xi_1, \ldots, \xi_n]$ be a real polynomial of degree $2k$ and let $\mathbf{L} = P(\partial_{x_1}, \ldots, \partial_{x_n})$ be elliptic. Then there exist real valued functions $A(\cdot,\cdot)$ defined on $\partial\mathbb{B}_n \times \mathbb{R}$, $B(\cdot)$ defined on $\mathbb{R}^n$ and $C(\cdot)$ defined on $\mathbb{R}^n$, such that the following statements hold.*

(i) There exists a sequence $\{f_j(\cdot)\}_{j\in\mathbb{N}}$ of continuous functions of $\partial\mathbb{B}_n$ to $\mathbb{R}$ such that

$$f_j(-\theta) = (-1)^j f_j(\theta), \quad \forall\, \theta \in \partial\mathbb{B}_n$$

and

$$A(\theta,r)=\sum_{j=0}^{\infty} f_j(\theta)r^j, \quad \forall\ (\theta,r)\in\partial\mathbb{B}_n\times\mathbb{R}. \tag{1.1}$$

The series in (1.1) *converges absolutely and uniformly in all compact subsets of* $\partial\mathbb{B}_n\times\mathbb{R}$.

(ii) B *is real analytic and there exists a family* $\{b_\alpha : \alpha\in\mathbb{N}^n, |\alpha|\geq 2k-n\}$ *of real numbers such that*

$$B(z)=\sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\geq 2k-n}} b_\alpha z^\alpha, \quad \forall\ z\in\mathbb{R}^n. \tag{1.2}$$

Furthermore B *can be chosen to be identically* 0 *if* n *is odd.*

(iii) C *is continuous and can be chosen to be identically* 0 *if* n *is odd.*

(iv) The function S *of* $\mathbb{R}^n\setminus\{0\}$ *defined by*

$$S(z)\equiv|z|^{2k-n}A(z/|z|,|z|)+B(z)\log|z|+C(z), \quad \forall\ z\in\mathbb{R}^n\setminus\{0\},$$

is a fundamental solution of $\mathbf{L}$.

Proof. In John [14, Chapter III] the existence of the functions $A(\cdot,\cdot)$, $B(\cdot)$ and $C(\cdot)$ satisfying (iii) and (iv) has been proved. We claim that such functions satisfy also (i) and (ii).

For all $\xi\in\partial\mathbb{B}_n$ and $t\in\mathbb{R}$, we denote by $v(z,\xi,t)$ the solution of the equation $\mathbf{L}v=1$ for which v and all its derivatives of order $\leq 2k-1$ vanish on the hyper-plane $z\cdot\xi=t$. Then we define, for every $z\in\mathbb{R}^n\setminus\{0\}$,

$$W_0(z) \equiv \frac{1}{4(2\pi i)^{n-1}}\int_{\partial\mathbb{B}_n}\int_0^{z\cdot\xi} v(z,\xi,t)\,\mathrm{sgn}\,t\ dt\,d\sigma_\xi\,, \tag{1.3}$$

$$W_1(z) \equiv -\frac{1}{(2\pi i)^n}\int_{\partial\mathbb{B}_n} v(z,\xi,0)\log|z\cdot\xi|\ d\sigma_\xi\,, \tag{1.4}$$

$$W_2(z) \equiv -\frac{1}{(2\pi i)^n}\int_{\partial\mathbb{B}_n}\int_0^{z\cdot\xi}\frac{v(z,\xi,t)-v(z,\xi,0)}{t}\ dt\,d\sigma_\xi\,, \tag{1.5}$$

and

$$S(z)\equiv\begin{cases}\Delta^{\frac{n+1}{2}}W_0(z) & \text{if } n \text{ is odd,}\\ \Delta^{\frac{n}{2}}(W_1+W_2)(z) & \text{if } n \text{ is even.}\end{cases} \tag{1.6}$$

By John [14, Chapter 3] we know that S is a fundamental solution of $\mathbf{L}$. We now investigate the structure of S.

By writing $P(\zeta\xi)=\zeta^{2k}P_{2k}(\xi)P(\zeta\xi)\left[\zeta^{2k}P_{2k}(\xi)\right]^{-1}$, for all $\zeta\in\mathbb{C}\setminus\{0\}$ and $\xi\in\partial\mathbb{B}_n$, one can easily recognize that there exists $R>0$ such that $P(\zeta\xi)$ has no complex zeros ζ outside of the ball $R\mathbb{B}_2$, for all $\xi\in\partial\mathbb{B}_n$. Now let γ be an arbitrary simple closed curve of class C^1 in the complex plane that describes the contour of the ball $R\mathbb{B}_2$. Since $\mathbf{L}$ has constant coefficients, the function $v(\cdot,\cdot,\cdot)$ is delivered by the formula

$$v(z,\xi,t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{(z\cdot\xi - t)\zeta}}{\zeta P(\zeta\xi)}\,d\zeta, \quad \forall\ (z,\xi,t)\in\mathbb{R}^n\times\partial\mathbb{B}_n\times\mathbb{R}$$

(see John [14, pp. 65–66].) Thus we have

$$v(z,\xi,t) = \sum_{j=0}^{\infty}\frac{a_j(\xi)}{j!}(z\cdot\xi - t)^j, \quad \forall\ (z,\xi,t)\in\mathbb{R}^n\times\partial\mathbb{B}_n\times\mathbb{R} \tag{1.7}$$

where

$$a_j(\xi) \equiv \frac{1}{2\pi i}\int_\gamma \frac{\zeta^{j-1}}{P(\zeta\xi)}\,d\zeta, \quad \forall\ \xi\in\partial\mathbb{B}_n, \tag{1.8}$$

for every $j\in\mathbb{N}$.

We now show that $a_0 = a_1 = \cdots = a_{2k-1} = 0$ and $a_{2k} = 1/P_{2k}$. Let $\xi\in\partial\mathbb{B}_n$ and $g_j(\zeta)\equiv\zeta^{j-1}/P(\zeta\xi)$. By (1.8), we have $a_j(\xi) = -\mathrm{Res}(g_j,\infty)$. Since g_j is holomorphic in a punctured neighborhood and $\lim_{\zeta\to\infty}\zeta g_j(\zeta) = 0$ for $j = 0,\ldots,2k-1$ and $\lim_{\zeta\to\infty}\zeta g_{2k}(\zeta) = 1/P_{2k}(\xi)$, we have $a_j(\xi) = -\mathrm{Res}(g_j,\infty) = 0$ for $j = 0,\ldots,2k-1$ and $a_{2k}(\xi) = -\mathrm{Res}(g_{2k},\infty) = -c_{2k,-1} = 1/P_{2k}(\xi)$.

Furthermore, one easily verifies that there exists $M > 0$ such that $|a_j(\xi)| \le MR^{(j-1)-2k}$ for all $\xi\in\partial\mathbb{B}_n$. Hence the series in (1.7) converges absolutely and uniformly in all compact subsets of $\mathbb{R}^n\times\partial\mathbb{B}_n\times\mathbb{R}$.

Now let n be odd and substitute v with its expression (1.7) in equation (1.3). Since we can integrate term by term, we obtain

$$W_0(z) = \sum_{j=2k}^{\infty} W_{0,j}(z), \tag{1.9}$$

with

$$W_{0,j}(z) = \frac{1}{4(2\pi i)^{n-1}}\int_{\partial\mathbb{B}_n}\frac{a_j(\xi)}{(j+1)!}\ (z\cdot\xi)^{j+1}\ \mathrm{sgn}(z\cdot\xi)\ d\sigma_\xi\,. \tag{1.10}$$

We observe that, for every $j\in\mathbb{N}$, $c>0$ and $z\in\mathbb{R}^n$,

$$W_{0,j}(cz) = c^{j+1}W_{0,j}(z) \quad\text{and}\quad W_{0,j}(-z) = (-1)^j W_{0,j}(z). \tag{1.11}$$

Moreover we can prove that $W_{0,j}\in C^\infty(\mathbb{R}^n\setminus\{0\})$. To do so we produce a convenient expression for $W_{0,j}$. By definition of a_j and of $W_{0,j}$, we have

$$W_{0,j}(z) = \frac{1}{4(2\pi i)^n}\int_{\partial\mathbb{B}_n}\int_\gamma \frac{|z|^{j+1}}{(j+1)!}P(\zeta\xi)^{-1}\,\zeta^{j-1}((z/|z|)\cdot\xi)^{j+1} \cdot\mathrm{sgn}((z/|z|)\cdot\xi)\ d\zeta\, d\sigma_\xi$$

for all $z \in \mathbb{R}^n$. Then we fix an arbitrary unit vector η and we restrict z to lay in the half space $z \cdot \eta > 0$. With such a restriction we introduce a new variable of integration ξ' instead of ξ on $\partial\mathbb{B}_n$, namely, for all $\theta \in \partial\mathbb{B}_n$, $\theta \cdot \eta > 0$, we set

$$\xi = T(\xi', \theta) \equiv \xi' + 2(\xi' \cdot \eta)\theta - \frac{\xi' \cdot (\theta + \eta)}{1 + \theta \cdot \eta}(\theta + \eta)\,. \tag{1.12}$$

We note that $T(\cdot, \theta)$ is an orthogonal transformation: $|T(\xi', \theta)| = |\xi'|$, and in addition $\theta \cdot \xi = \eta \cdot \xi'$. The expression for $W_{0,j}(z)$, for z in the half-space $z \cdot \eta > 0$, becomes

$$W_{0,j}(z) = \frac{1}{4(2\pi i)^n} \int_{\partial\mathbb{B}_n} \int_\gamma \frac{|z|^{j+1}}{(j+1)!} P(\zeta T(\xi', z/|z|))^{-1} \cdot \zeta^{j-1} (\eta \cdot \xi')^{j+1} \operatorname{sgn}(\eta \cdot \xi')\; d\zeta\, d\sigma_{\xi'}$$

and since the integrand depends real analytically on z for $z \cdot \eta > 0$, we have

$$D^\alpha W_{0,j}(z) = \frac{1}{4(2\pi i)^n} \int_{\partial\mathbb{B}_n} \int_\gamma \frac{\partial^{|\alpha|}}{\partial z^\alpha} \left[\frac{|z|^{j+1}}{(j+1)!} P(\zeta T(\xi', z/|z|))^{-1} \right] \cdot \zeta^{j-1} (\eta \cdot \xi')^{j+1} \operatorname{sgn}(\eta \cdot \xi')\; d\zeta\, d\sigma_{\xi'} \tag{1.13}$$

for all $z \in \mathbb{R}^n$ such that $z \cdot \eta > 0$. By equation (1.13) and by standard theorems on integral depending on parameters we deduce that $D^\alpha W_{0,j}(z)$ is a continuous function for $z \cdot \eta > 0$. Since η is an arbitrary unit vector it follows that $D^\alpha W_{0,j}$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ for all $\alpha \in \mathbb{N}^n$.

We now set

$$f_j(\theta) \equiv \left(\Delta^{\frac{n+1}{2}} W_{0,j+2k}\right)(\theta), \quad \forall \theta \in \partial\mathbb{B}_n,$$

for every $j \in \mathbb{N}$. Clearly $f_j(\theta)$ is a continuous function of $\theta \in \partial\mathbb{B}_n$, and by (1.11), $f_j(-\theta) = (-1)^j f_j(\theta)$ and

$$\left(\Delta^{\frac{n+1}{2}} W_{0,j+2k}\right)(z) = |z|^{j-n} f_j(z/|z|), \quad \forall z \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, substituting W_0 in the form (1.9) into equation (1.6) we obtain

$$S(z) = \Delta^{\frac{n+1}{2}} \sum_{j=2k}^\infty W_{0,j}(z) = \sum_{j=2k}^\infty \Delta^{\frac{n+1}{2}} W_{0,j}(z) = |z|^{2k-n} \sum_{j=0}^\infty f_j(z/|z|)|z|^j \tag{1.14}$$

which immediately implies statement (ii). Here we still have to justify that one can exchange the summation with $\Delta^{\frac{n+1}{2}}$.

To do so we estimate $|D^\alpha W_{0,j}(z)|$ by exploiting (1.13). Let

$$w_{0,0}(\zeta, \xi', \theta) \equiv P(\zeta T(\xi', \theta))^{-1}.$$

The function $w_{0,0}(\zeta,\xi',\theta)$ does not depend on j and is real analytic for (ζ,ξ',θ) in the set A_η of the triples which satisfy

$$\zeta \in R\partial\mathbb{B}_2, \quad \xi',\theta \in \partial\mathbb{B}_n, \quad \theta\cdot\eta > 0.$$

Moreover, for all multi-indexes $\iota \in \mathbb{N}^n$ with $|\iota| = 1$,

$$\frac{\partial^{|\iota|}}{\partial z^\iota}\left[\frac{|z|^{j+1}}{(j+1)!}P(\zeta T(\xi', z/|z|))^{-1}\right] = \frac{|z|^j}{j!}\left(w_{0,\iota}(\zeta,\xi',z/|z|) + \frac{1}{j+1}w_{1,\iota}(\zeta,\xi',z/|z|)\right)$$

where $w_{0,\iota}$ and $w_{1,\iota}$ are the functions of A_η defined by

$$\begin{aligned} w_{0,\iota}(\zeta,\xi',\theta) &\equiv \theta^\iota w_{0,0}(\zeta,\xi',\theta)\,, \\ w_{1,\iota}(\zeta,\xi',\theta) &\equiv \partial_\theta^\iota\, w_{0,0}(\zeta,\xi',\theta) - \theta^\iota \textstyle\sum_{i=1}^n \theta_i \partial_{\theta_i} w_{0,0}(\zeta,\xi',\theta). \end{aligned}$$

We note that $w_{0,\iota}$ and $w_{1,\iota}$ do not depend on j and are analytic for (ζ,ξ',θ) in the set A_η. By an inductive argument on $|\alpha|$, one can show that for each $\alpha\in\mathbb{N}^n$ with $|\alpha|\le j$ there exist a natural number $N(\alpha)$ and sequences $\{c_i(j) : j\in\mathbb{N}, j\ge|\alpha|\}$ of real numbers in $[0,1]$, for $i = 0,\ldots,N(\alpha)$, and functions $w_{i,\alpha}(\cdot,\cdot,\cdot)$ of A_η to $\mathbb{C}$ such that

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha}\left(\frac{|z|^{j+1}}{(j+1)!}P(\zeta T(\xi', z/|z|))^{-1}\right) = \frac{|z|^{j+1-|\alpha|}}{(j+1-|\alpha|)!}\sum_{i=0}^{N(\alpha)} c_i(j)\, w_{i,\alpha}(\zeta,\xi',z/|z|)$$

for all (ζ,ξ',z) such that $(\zeta,\xi',z/|z|)\in A_\eta$. By such an equality we deduce that, for all $\alpha\in\mathbb{N}^n$ there exists a positive constant M such that

$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha}\left(\frac{|z|^{j+1}}{(j+1)!}P(\zeta T(\xi', z/|z|))^{-1}\right)\right| \le M\frac{|z|^{j+1-|\alpha|}}{(j+1-|\alpha|)!}$$

for all $j\ge|\alpha|$ and for all $\zeta\in R\partial\mathbb{B}_n$, $\xi'\in\partial\mathbb{B}_n$ and $z\in\mathbb{R}^n$ such that $(z/|z|)\cdot\eta \ge 1/2$. Then by (1.13) there exist $M', M'' > 0$ such that

$$|D_z^\alpha W_{0,j}(z)| \le M'(M'')^{j-1}\,\frac{|z|^{j+1-|\alpha|}}{(j+1-|\alpha|)!}$$

for all $j\ge|\alpha|$ and z in the cone $(z/|z|)\cdot\eta\ge 1/2$.

Finally we recall that η is an arbitrary unit vector. So, possibly choosing larger constant M' and M'', the previous inequality holds for z in the whole of $\mathbb{R}^n\setminus\{0\}$. It follows that the series $\sum_{j=0}^\infty D^\alpha W_{0,j}(z)$ is dominated by a convergent power series and hence it converges absolutely and uniformly in all compact subsets of $\mathbb{R}^n \setminus \{0\}$. This implies that

$$D^\alpha \sum_{j=0}^{\infty} W_{0,j}(z) = \sum_{j=0}^{\infty} D^\alpha W_{0,j}(z)$$

for all $\alpha \in \mathbb{N}^n$. In particular we can deduce (1.14). Moreover, if we set for all $\theta \in \partial\mathbb{B}_n$ and $r \in \mathbb{R}$

$$A(\theta, r) \equiv \sum_{j=0}^{\infty} f_j(\theta) r^j,$$

then $A(\theta, r)$ satisfies statement (i).Then if we take $B \equiv 0$, $C \equiv 0$, statement (iv) for n odd follows by equation (1.14) and statements (ii) and (iii) for n odd are obvious.

Now we only sketch the proof for n even, the argument being very similar to the one developed for the case n odd. If n is even we have $S(z) = \Delta^{\frac{n}{2}}(W_1(z) + W_2(z))$. We observe that $W_2(z)$ is C^∞ function defined in the whole of $\mathbb{R}^n$ (even for $z = 0$), and so if we set $C(z) \equiv \Delta^{\frac{n}{2}} W_2(z)$, then C satisfies statement (iii). Now we consider $S_1(z) \equiv \Delta^{\frac{n}{2}} W_1(z)$. Substituting v with its expression (1.7) into equation (1.4), we obtain

$$W_1(z) = \sum_{j=2k}^{\infty} W_{1,j}(z), \quad \forall\, z \in \mathbb{R}^n,$$

with

$$W_{1,j}(z) = -\frac{1}{(2\pi i)^n} \int_{\partial\mathbb{B}_n} \frac{a_j(\xi)}{j!}\, (z \cdot \xi)^j \, \log|z \cdot \xi| \, d\sigma_\xi.$$

By arguing as above one can show that $W_{1,j} \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Moreover we have

$$\Delta^{\frac{n}{2}} W_{1,j}(z) = U_j(z) + V_j(z) \log|z|$$

where, for every $j \in \mathbb{N}$, $j \geq 2k$, V_j is a homogeneous polynomial of degree $j - n$ in $\mathbb{R}^n$ with $V_j = 0$ for $j < n$, U_j is continuous in $\mathbb{R}^n \setminus \{0\}$ and, $U_j(cz) = c^{j-n} U_j(z)$, $U_j(-z) = (-1)^j U_j(z)$ for all $c > 0$ and $z \in \mathbb{R}^n \setminus \{0\}$. We set, for every $j \in \mathbb{N}$ and $\theta \in \partial\mathbb{B}_n$, $f_j(\theta) \equiv U_{j+2k}(\theta)$. By arguing as above, we can show that we can exchange the order of differentiation and of summation and thus we obtain

$$\begin{aligned} S_1(z) = \Delta^{\frac{n}{2}} \sum_{j=2k}^{\infty} W_{1,j}(z) &= \sum_{j=2k}^{\infty} \Delta^{\frac{n}{2}} W_{1,j}(z) \\ &= |z|^{2k-n} \sum_{j=0}^{\infty} f_j(z/|z|) |z|^j + \log|z| \sum_{j=2k}^{\infty} V_j(z). \end{aligned}$$

Now we set $A(\theta, r) \equiv \sum_{j=0}^{\infty} f_j(\theta) r^j$ for all $(\theta, r) \in \partial\mathbb{B}_n \times \mathbb{R}$ and $B(z) \equiv \sum_{j=2k}^{\infty} V_j(z)$. By arguing as above we deduce that the series which defines

A converges absolutely and uniformly in all compact subsets of $\partial\mathbb{B}_n \times \mathbb{R}$ and the accordingly statement (i) holds. Similarly we deduce that the series which defines B converges absolutely in the compact subsets of $\mathbb{R}^n$. Since such series is a power series, B is real analytic. Hence statement (ii) follows. Then also statements (iv) for n even follows. □

Remark 1.2. *With the notation introduced in Theorem 1.1, we denote by $S_0(z)$ the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by*

$$S_0(z) \equiv |z|^{2k-n} f_0(z/|z|) + \log|z| \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=2k-n}} b_\alpha z^\alpha$$

where $f_0(\cdot)$ is the coefficient of the first term in (1.1) and b_α are the coefficients with $|\alpha| = 2k - n$ of (1.2). Hence $b_\alpha = 0$ if n is odd. Then $S_0(z)$ is a fundamental solution of the homogeneous operator $\mathbf{L}_0 \equiv P_{2k}(\partial_x)$.

Proof. In the proof of Theorem 1.1 we saw how to construct a particular fundamental solution of $P(\partial_x)$. Here we specialize such a construction to the case $P = P_{2k}$ where the function $v(z, \xi, t)$ defined as a solution of $\mathbf{L}_0 v = 1$ is a polynomial. Indeed a direct computation based on the definition of $v(z, \xi, t)$ shows that

$$v(z, \xi, t) = \frac{a_{2k}(\xi)}{(2k)!}(z \cdot \xi - t)^{2k} = \frac{1}{(2k)! P_{2k}(\xi)}(z \cdot \xi - t)^{2k},$$

for all $(z, \xi, t) \in \mathbb{R}^n \times \partial\mathbb{B}_n \times \mathbb{R}$. Moreover, for n even, $\mathbf{L}_0(\Delta^{\frac{n}{2}} W_2) = 0$, because $\Delta^{\frac{n}{2}} W_2$ is a polynomial function of degree $< 2k$, and thus the fundamental solution is given only by the term $\Delta^{\frac{n}{2}} W_1$. □

1.1.2 Dependence upon the coefficients of the operator

If $\mathcal{X}$ is a subset of a Banach space, we say that a function f defined on $\mathcal{X}$ is real analytic if f is the restriction to $\mathcal{X}$ of a real analytic function defined on an open neighborhood of $\mathcal{X}$. We need the following elementary result.

Lemma 1.3. *Let $n, m \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{X}$ be a subset of $\mathbb{R}^n$ and let $\mathcal{Y}$ be a compact subset of $\mathbb{R}^m$. Let τ be a finite measure on $\mathcal{Y}$ and let $f(\cdot, \cdot)$ be a real analytic function of $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$. Then the function $F(\cdot)$ of $\mathcal{X}$ to $\mathbb{R}$ defined by $F(x) \equiv \int_{\mathcal{Y}} f(x, y)\, d\tau_y$ for all $x \in \mathcal{X}$ is real analytic.*

Proof. In the sequel for each $x \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\epsilon > 0$ we denote by $Q(x, \epsilon)$ the rectangle

$$Q(x, \epsilon) \equiv \{x' \in \mathbb{R}^n : |x'_1 - x_1| < \epsilon, \ldots, |x'_n - x_n| < \epsilon\}.$$

Similarly, we define $Q(y, \epsilon)$ for all $y \equiv (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $\epsilon > 0$.

Now, let x_0 be a point of $\mathcal{X}$. Clearly the set $\{x_0\} \times \mathcal{Y}$ is compact. It follows that there exist $\epsilon > 0$ and $y_1, \dots, y_l \in \mathcal{Y}$ such that the set $\{x_0\} \times \mathcal{Y}$ is covered by the finite set of rectangles $\{Q(x_0, \epsilon) \times Q(y_i, \epsilon)\}_{i=1,\dots,l}$, and for all $i = 1, \dots, l$ there exist a family $\{a^{(i)}_{\alpha\beta} : \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m\}$ of real numbers such that

$$f(x,y) = \sum_{\alpha\in\mathbb{N}^n,\, \beta\in\mathbb{N}^m} a_{\alpha\beta}(x - x_0)^\alpha (y - y_i)^\beta \tag{1.15}$$

for all $(x,y) \in (Q(x_0,\epsilon)\cap\mathcal{X})\times(Q(y_i,\epsilon)\cap\mathcal{Y})$, and the series in (1.15) converges absolutely and uniformly for all $(x,y) \in \mathrm{cl}Q(x_0,\epsilon) \times \mathrm{cl}Q(y_i,\epsilon)$. Then the series in (1.15) equals

$$\sum_{\alpha\in\mathbb{N}^n} b^{(i)}_\alpha(y)(x - x_0)^\alpha, \quad \forall\ (x,y) \in Q(x_0,\epsilon) \times Q(y_i,\epsilon), \tag{1.16}$$

with

$$b^{(i)}_\alpha(y) \equiv \sum_{\beta\in\mathbb{N}^m} a^{(i)}_{\alpha\beta}(y - y_i)^\beta, \quad \forall\ y \in B(y_i,\epsilon) \tag{1.17}$$

for all $i = 1, \dots, l$. We note that the series in (1.16) and (1.17) converge absolutely and uniformly in $\mathrm{cl}Q(x_0,\epsilon)\times\mathrm{cl}Q(y_i,\epsilon)$ and $\mathrm{cl}Q(y_i,\epsilon)$, respectively. In particular, $b_\alpha(\cdot)$ is a continuous function on $\mathrm{cl}Q(y_i,\epsilon)$ and the series

$$\sum_{\alpha\in\mathbb{N}^n} \|b^{(i)}_\alpha(\cdot)\|_{C^0(\mathrm{cl}Q(y_i,\epsilon))}(x - x_0)^\alpha$$

converges absolutely and uniformly in $\mathrm{cl}Q(x_0,\epsilon)$. Since $b^{(i)}_\alpha(y) = b^{(j)}_\alpha(y)$ for all $y \in Q(y_i,\epsilon) \cap Q(y_j,\epsilon)$ and for all $\alpha \in \mathbb{N}^n$ and $i, j = 1, \dots, l$, we have

$$f(x,y) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha(y)(x - x_0)^\alpha, \quad \forall\ (x,y) \in (\mathcal{X} \cap Q(x_0,\epsilon)) \times \mathcal{Y},$$

where $b_\alpha(\cdot)$ is a continuous function on $\mathcal{Y}$ and the series

$$\sum_{\alpha\in\mathbb{N}^n} \|b_\alpha(\cdot)\|_{C^0(\mathcal{Y})}(x - x_0)^\alpha$$

converges absolutely in $\mathrm{cl}Q(x_0,\epsilon)$. Then we have

$$\begin{aligned} F(x) &= \int_{\mathcal{Y}} f(x,y)\, d\tau_y \\ &= \sum_{\alpha\in\mathbb{N}^n} \left(\int_{\mathcal{Y}} b_\alpha(y)\, d\tau_y\right)(x - x_0)^\alpha, \quad \forall\ x \in Q(x_0,\epsilon) \cap \mathcal{X}, \end{aligned}$$

where the series converges absolutely in $\mathrm{cl}Q(x_0,\epsilon)$. Since x_0 was an arbitrary point of $\mathcal{X}$ the proof of the lemma is completed. □

Now, for each $n, l \in \mathbb{N}$, $n > 0$, we denote by $N(n,l)$ the set of all multi-indexes α with $|\alpha| \leq l$. We say that $\mathbf{a}$ is a *vector of coefficients of order* l if $\mathbf{a} \equiv (a_\alpha)_{\alpha \in N(n,l)}$ is a real function on $N(n,l)$. We denote by $R(n,l)$ space of all vectors of coefficients of order l. Then, by ordering $N(n,l)$ on arbitrary way, we identify $R(n,l)$ with a finite dimension real vector space and we endow $R(n,l)$ with the corresponding Euclidean norm $|\cdot|$. For each $\mathbf{a} \in R(n,l)$ we denote by $P[\mathbf{a}](\xi) = P[\mathbf{a}](\xi_1, \ldots, \xi_n)$ the polynomial $\sum_{\alpha \in N(n,l)} a_\alpha \xi^\alpha$ and we set $\mathbf{L}[\mathbf{a}] \equiv P[\mathbf{a}](\partial_{x_1}, \ldots, \partial_{x_n})$. Then $\mathbf{L}[\mathbf{a}]$ is a partial differential operator with constant coefficients. If $\mathbf{L}[\mathbf{a}]$ is elliptic we can construct the corresponding functions $S(\mathbf{a}, z)$, $A(\mathbf{a}, \theta, r)$ and $B(\mathbf{a}, z)$, $C(\mathbf{a}, z)$ as in Theorem 1.1. We have the following.

Theorem 1.4. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 1$. Let $\mathscr{E}$ be a bounded open subset of $R(n, 2k)$ such that $\mathbf{L}[\mathbf{a}]$ is an elliptic operator of order $2k$ for all $\mathbf{a} \in \mathrm{cl}\mathscr{E}$. Then there exist a real analytic function $A(\cdot,\cdot,\cdot)$ defined on $\mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$ and real analytic functions $B(\cdot,\cdot)$, $C(\cdot,\cdot)$ defined on $\mathscr{E} \times \mathbb{R}^n$ such that the following statements hold.*

(i) There exists a sequence $\{f_j(\cdot,\cdot)\}_{j \in \mathbb{N}}$ of real analytic functions of $\mathscr{E} \times \partial\mathbb{B}_n$ such that

$$f_j(\mathbf{a}, -\theta) = (-1)^j f_j(\mathbf{a}, \theta), \quad \forall\ (\mathbf{a}, \theta) \in \mathscr{E} \times \partial\mathbb{B}_n$$

and

$$A(\mathbf{a}, \theta, r) = \sum_{j=0}^{\infty} f_j(\mathbf{a}, \theta) r^j, \quad \forall\ (\mathbf{a}, \theta, r) \in \mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R},$$

where the series converges absolutely and uniformly in all compact subsets of $\mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$.

(ii) There exists a family $\{b_\alpha(\cdot) : \alpha \in \mathbb{N}^n, |\alpha| \geq 2k - n\}$ of real analytic functions of $\mathscr{E}$ to $\mathbb{R}$ such that

$$B(\mathbf{a}, z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \geq 2k-n}} b_\alpha(\mathbf{a}) z^\alpha, \ \forall\ (\mathbf{a}, z) \in \mathscr{E} \times \mathbb{R}^n.$$

Furthermore B can be chosen to be identically 0 if n is odd.

(iii) If n is odd C can be chosen to be identically 0.

(iv) For all $\mathbf{a} \in \mathscr{E}$ the function $S(\mathbf{a}, \cdot)$ of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S(\mathbf{a}, z) \\ \equiv |z|^{2k-n} A(\mathbf{a}, z/|z|, |z|) + B(\mathbf{a}, z) \log |z| + C(\mathbf{a}, z), \ \forall\ z \in \mathbb{R}^n \setminus \{0\},$$

is a fundamental solution of $\mathbf{L}[\mathbf{a}]$.

Proof. The proof for n odd and n even is similar and so we confine our attention to the case n odd, and we begin by considering the construction of A (see Theorem 1.1.)

First we note that there exists a positive constant $\epsilon > 0$ such that $P_{2k}[\mathbf{a}](\xi) \geq \epsilon$ for all $\xi \in \partial\mathbb{B}_n$ and for all $\mathbf{a} \in \mathscr{E}$. Indeed, by a continuity argument, one verifies that, for each $\mathbf{a}_0 \in \mathrm{cl}\mathscr{E}$, there exist a neighborhood $\mathcal{V}(\mathbf{a}_0)$ and a constant $\epsilon(\mathbf{a}_0) > 0$ such that $P_{2k}[\mathbf{a}](\xi) \geq \epsilon(\mathbf{a}_0)$ for all $\xi \in \partial\mathbb{B}_n$ and all $\mathbf{a} \in \mathcal{V}(\mathbf{a}_0)$. Since $\mathrm{cl}\mathscr{E}$ is compact, it is covered by a finite number of such neighborhoods and taking as ϵ the minimum of the corresponding constants we deduce that $P_{2k}[\mathbf{a}](\xi) \geq \epsilon$ on their finite union. Moreover, since $\mathscr{E}$ is bounded, there exists $L > 0$ such that $|\mathbf{a}| \leq L$ for all $\mathbf{a} \in \mathscr{E}$.

As we have already noted (see the proof of Theorem 1.1), for all elliptic constant coefficients operators $\mathbf{L}[\mathbf{a}]$ the solution $v(\mathbf{a}, x, \xi, t)$ of the equation $\mathbf{L}[\mathbf{a}]v = 1$ for which v and all its derivatives of order $\leq 2k-1$ vanish on the hyper-plane $x \cdot \xi = t$ is delivered by the equation

$$v(\mathbf{a}, x, \xi, t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{(x\cdot\xi - t)\zeta}}{\zeta P[\mathbf{a}](\zeta\xi)}\, d\zeta\,, \tag{1.18}$$

where γ is a simple closed C^1 curve on the complex ζ-plane, which encloses all roots of denominator for all $\xi \in \partial\mathbb{B}_n$. Now we show that it is possible to choose the same curve γ for all $\mathbf{a} \in \mathscr{E}$. We observe that

$$\zeta P[\mathbf{a}](\zeta\xi) = \zeta^{2k+1}P_{2k}[\mathbf{a}](\xi)\left(1 + \sum_{\alpha\in N(n,2k-1)} \frac{a_\alpha \xi^\alpha}{\zeta^{2k-|\alpha|}P_{2k}[\mathbf{a}](\xi)}\right),$$

and thus, if we set $R \equiv \max\{2, 2L\epsilon^{-1}\mathrm{Card}N(n, 2k-1)\}$, then, for all $|\zeta| \geq R$, $\xi \in \partial\mathbb{B}_n$ and $\mathbf{a} \in \mathcal{V}$, we have

$$\left|\frac{a_\alpha \xi^\alpha}{\zeta^{2k-|\alpha|}P_{2k}[\mathbf{a}](\xi)}\right| \leq (2\,\mathrm{Card}N(n, 2k-1))^{-1}\,.$$

This immediately implies that $|\zeta P[\mathbf{a}](\zeta\xi)| \geq \epsilon$ and in particular, if γ describes the contour of a ball of radius R in the complex ζ-plane, then all the roots of $\zeta P[\mathbf{a}](\zeta\xi)$ are enclosed by γ for all $\xi \in \partial\mathbb{B}_n$ and all $\mathbf{a} \in \mathscr{E}$. So, for this particular choice of γ, expression (1.18) provides a solution of the equation $\mathbf{L}[\mathbf{a}]v = 1$, which vanishes together with all its derivatives of order $\leq 2k-1$ on the hyper-plane $x \cdot \xi = t$, for all $\mathbf{a} \in \mathscr{E}$. Moreover, since the integrand in (1.18) depends real analytically on $(\mathbf{a}, x, \xi, t, \zeta) \in \mathscr{E} \times \mathbb{R}^n \times \partial\mathbb{B}_n \times \mathbb{R} \times R\partial\mathbb{B}_2$ then, by Lemma 1.3, $v(\mathbf{a}, x, \xi, t)$ is a real analytic function of $(\mathbf{a}, x, \xi, t) \in \mathscr{E} \times \mathbb{R}^n \times \partial\mathbb{B}_n \times \mathbb{R}$.

We also note that the function $\widetilde{v}(\mathbf{a}, \xi, t)$ of $(\mathbf{a}, \xi, t) \in \mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$ defined by

$$\widetilde{v}(\mathbf{a}, \xi, t) \equiv v(\mathbf{a}, x, \xi, x\cdot\xi - t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{t\zeta}}{\zeta P[\mathbf{a}](\zeta\xi)}\, d\zeta$$

depends real analytically on $(\mathbf{a},\xi,t) \in \mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$ and vanishes for $t=0$ together with its derivatives with respect to t till order $2k-1$. Then it must be $\widetilde{v}(\mathbf{a},\xi,t) = t^{2k}w(\mathbf{a},\xi,t)$ where w is a real analytic function on $\mathscr{E}\times\partial\mathbb{B}_n\times\mathbb{R}$.

Now we go on with the construction of the fundamental solution and we consider the function W_0 introduced in the proof of Theorem 1.1. We have

$$W_0(\mathbf{a},z) = \frac{1}{4(2\pi i)^{n-1}} \int_{\partial\mathbb{B}_n} \int_0^{z\cdot\xi} v(\mathbf{a},z,\xi,t)\,\mathrm{sgn}\,t\;dt\,d\sigma_\xi$$

for all $(\mathbf{a},z) \in \mathscr{E}\times\mathbb{R}^n\setminus\{0\}$. We denote by A' the real function of $\mathscr{E}\times\partial\mathbb{B}_n\times\mathbb{R}$ defined by

$$A'(\mathbf{a},\theta,r) \equiv \frac{1}{4(2\pi i)^{n-1}} \int_{\partial\mathbb{B}_n} \int_0^{\theta\cdot\xi} (\theta\cdot\xi - s)^{2k}\, w\left(\mathbf{a},\xi,r\left(\theta\cdot\xi - s\right)\right) \mathrm{sgn}\,s\;ds\,d\sigma_\xi \tag{1.19}$$

for all $(\mathbf{a},\theta,r) \in \mathscr{E}\times\partial\mathbb{B}_n\times\mathbb{R}$. Thus $W_0(\mathbf{a},z) = |z|^{2k+1}A'(\mathbf{a},z/|z|,|z|)$ for all $(\mathbf{a},z) \in \mathscr{E}\times\mathbb{R}^n\setminus\{0\}$. We note that we can make the limits of integration in the inner integral in (1.19) locally independent of θ by a suitable orthogonal substitution. Let η be an arbitrary chosen unit vector and consider θ restricted to the half sphere $S_\eta^+ \equiv \{\theta\in\partial\mathbb{B}_n : \theta\cdot\eta > 0\}$. We introduce a new variable of integration ξ' instead of ξ by means of the formula $\xi = T(\xi', z/|z|)$ introduced in (1.12). We recall that, for any $\theta\in\partial\mathbb{B}_n$ with $\theta\cdot\eta>0$, the transformation $T(\cdot,\theta)$ is an orthogonal one. In particular, $|T(\xi',\theta)| = |\xi'|$. Furthermore $\theta\cdot\xi = \eta\cdot\xi'$. Thus the integral in (1.19) becomes

$$\int_{\partial\mathbb{B}_n}\int_0^{\eta\cdot\xi'} (\eta\cdot\xi'-s)^{2k} w\left(\mathbf{a},T\left(\xi',\theta\right),r(\eta\cdot\xi'-s)\right)\,\mathrm{sgn}\,s\;ds\,d\sigma_{\xi'}$$

where the limits of integration do not depend on θ, at least for $\theta\in S_\eta^+$. Moreover, if we set

$$\widetilde{w}\left(\mathbf{a},\theta,r,\xi',t\right) \equiv w\left(\mathbf{a},T\left(\xi',\theta\right),r(\eta\cdot\xi')(1-t)\right) - w\left(\mathbf{a},T\left(\xi',\theta\right),r(\eta\cdot\xi')(t-1)\right)$$

for all $(\mathbf{a},\theta,r,\xi',t)\in\mathscr{E}\times S_\eta^+\times\mathbb{R}\times\partial\mathbb{B}_n\times[0,1]$, then we have

$$A'(\mathbf{a},\theta,r) = \frac{1}{4(2\pi i)^{n-1}}\int_{\mathrm{cl}S_\eta^+}\int_0^1 (\eta\cdot\xi')^{2k}(1-t)^{2k}\widetilde{w}\left(\mathbf{a},\theta,r,\xi',t\right)\;dt\,d\sigma_{\xi'}$$

for all $(\mathbf{a},\theta,r)\in\mathscr{E}\times S_\eta^+\times\mathbb{R}$. We recall that, the function w is real analytic on $\mathscr{E}\times\partial\mathbb{B}_n\times\mathbb{R}$, the function $T(\xi',\theta)$ is real analytic in $(\xi',\theta)\in\partial\mathbb{B}_n\times S_\eta^+$ and satisfies $|T(\xi',\theta)|=1$ for $\xi'\in\partial\mathbb{B}_n$. It follows that the function $\widetilde{w}$ is real analytic on $\mathscr{E}\times S_\eta^+\times\mathbb{R}\times\mathrm{cl}S_\eta^+\times[0,1]$. Then the previous Lemma 1.3 implies that A' is a real analytic function on $\mathscr{E} \times S_\eta^+ \times \mathbb{R}$. Since η is an arbitrary unit vector it follows that A' is real analytic on $\mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$.

Now we recall that $W_0(\mathbf{a}, z) = |z|^{2k+1} A'(\mathbf{a}, z/|z|, |z|)$. Therefore, if we differentiate W_0 with respect to z_i, $i = 1, \ldots, n$, we find

$$\partial_{z_i} W_0(\mathbf{a}, z) = |z|^{2k} A''(\mathbf{a}, z/|z|, |z|)$$

where A'' is the function of $\mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$ defined by

$$A''(\mathbf{a}, \theta, r) \equiv (2k+1)\theta_i A'(\mathbf{a}, \theta, r) + \partial_{\theta_i} A'(\mathbf{a}, \theta, r) \\ -\theta_i \sum_{j=1}^{n} \theta_j \partial_{\theta_j} A'(\mathbf{a}, \theta, r) + \theta_i \, r \partial_r A'(\mathbf{a}, \theta, r)\,.$$

We observe that A'' have the same regularity properties of A'. By iterating we verify that

$$S(\mathbf{a}, z) = \Delta^{\frac{n+1}{2}} W_0(\mathbf{a}, z) = |z|^{2k-n} A(\mathbf{a}, z/|z|, |z|)$$

where A is a real analytic function on $\mathscr{E} \times \partial\mathbb{B}_n \times \mathbb{R}$.

Finally we exploit the proof of Theorem 1.1 and we obtain that for all fixed $\mathbf{a} \in \mathscr{E}$ the function $A(\mathbf{a}, \theta, r)$ has a power series representation as in statement (i). The coefficients $f_j(\mathbf{a}, \theta)$ are given by

$$f_j(\mathbf{a}, \theta) \equiv \left(\Delta_z^{\frac{n+1}{2}} W_{0,j+2k} \right)(\mathbf{a}, \theta)$$

where

$$W_{0,j}(\mathbf{a}, z) \equiv \frac{1}{4(2\pi i)^{n-1}} \int_{\partial\mathbb{B}_n} \frac{a_j(\mathbf{a}, \xi)}{(j+1)!} \, (z \cdot \xi)^{j+1} \, \mathrm{sgn}(z \cdot \xi) \, d\sigma_\xi$$

and

$$a_j(\mathbf{a}, \xi) \equiv \frac{1}{2\pi i} \int_\gamma \frac{\zeta^{j-1}}{P[\mathbf{a}](\zeta\xi)} \, d\zeta$$

With our choice of the curve γ such equations hold for all $\mathbf{a} \in \mathscr{E}$ and we can verify the real analyticity of $f_j(\mathbf{a}, \theta)$ with the argument developed in the proof of the real analyticity of $A(\mathbf{a}, \theta, r)$. So for n odd the theorem is proved. The proof for n even is similar. □

1.2 The single layer potential

1.2.1 Technical preliminaries and notation

We first recall some technical facts of Lanza de Cristoforis and Rossi [28].

Definition 1.5. *Let Ω be an open and connected subset of $\mathbb{R}^n$. We denote by $\mathcal{A}_{\partial\Omega}$ the set of all functions $\phi \in C^1(\partial\Omega, \mathbb{R}^n)$ which are injective and whose differential $d\phi(x)$ is injective for all $x \in \partial\Omega$. Similarly, we denote by $\mathcal{A}_{\mathrm{cl}\Omega}$ the set of all functions $\phi \in C^1(\mathrm{cl}\Omega, \mathbb{R}^n)$ which are injective and whose differential $d\phi(x)$ is injective for all $x \in \mathrm{cl}\Omega$.*

Now we fix two constants $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0,1[$ and an open and bounded subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$. We assume that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected and we consider a function $\phi \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$. By the Jordan's Separation Theorem (see *e.g.* Deimling [8, p. 26]), $\phi(\partial\Omega)$ separates $\mathbb{R}^n$ into two connected components. We denote by $\mathbb{E}[\phi]$ the unbounded one and by $\mathbb{I}[\phi]$ the bounded one, and we denote by ν_ϕ the unit outward normal to the boundary of $\mathbb{I}[\phi]$. Then we denote by ν_Ω the unit outward normal to the boundary of Ω, we select a vector field $\omega \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ such that $|\omega(x)| = 1$ and $\omega(x) \cdot \nu_\Omega(x) \geq 1/2$ for all $x \in \partial\Omega$. With this notation we have the following (see Lanza de Cristoforis and Rossi [28, §2].)

Proposition 1.6. *Let $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0,1[$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ be connected. Let $\omega \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ be a vector field such that $|\omega(x)| = 1$ and $\omega(x) \cdot \nu_\Omega(x) \geq 1/2$ for all $x \in \partial\Omega$. Then, for each given $\phi_0 \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, there exists a positive constant δ_0 such that, for all $\delta \in]0, \delta_0]$ the following statements hold.*

(i) The sets

$$\begin{aligned} \Omega_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in]-\delta, \delta[\}\,, \\ \Omega^+_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in]-\delta, 0[\}\,, \\ \Omega^-_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in]0, \delta[\}\,, \end{aligned}$$

are connected,

$$\begin{aligned} \partial\Omega_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in \{-\delta, \delta\}\}\,, \\ \partial\Omega^+_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in \{-\delta, 0\}\}\,, \\ \partial\Omega^-_{\omega,\delta} &\equiv \{x + t\omega(x) \,|\, x \in \partial\Omega\,,\, t \in \{0, \delta\}\}\,; \end{aligned}$$

$\Omega^+_{\omega,\delta} \subset \Omega$ and $\Omega^-_{\omega,\delta} \subset \mathbb{R}^n \setminus \mathrm{cl}\Omega$.

(ii) If $\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}}$, then $\phi \equiv \Phi|_{\partial\Omega} \in \mathcal{A}_{\partial\Omega}$.

(iii) The set $\mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}} \equiv \left\{\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}} \,:\, \Phi(\Omega^+_{\omega,\delta}) \subset \mathbb{I}[\Phi|_{\partial\Omega}]\right\}$ is an open subset of $\mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}}$ and $\Phi(\Omega^-_{\omega,\delta}) \subset \mathbb{E}[\Phi|_{\partial\Omega}]$ for all $\Phi \in \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$.

(iv) If $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, then both $\Phi(\Omega^+_{\omega,\delta})$ and $\Phi(\Omega^-_{\omega,\delta})$ are open sets of class $C^{m,\lambda}$, and $\partial\Phi(\Omega^+_{\omega,\delta}) = \Phi(\partial\Omega^+_{\omega,\delta})$, $\partial\Phi(\Omega^-_{\omega,\delta}) = \Phi(\partial\Omega^-_{\omega,\delta})$.

(v) *If* $\phi_0 \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ *then there exists* $\Phi_0 \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$ *such that* $\phi_0 \equiv \Phi_0|_{\partial\Omega}$.

(vi) *If* $\Phi_0 \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$ *and* $\phi_0 \equiv \Phi_0|_{\partial\Omega}$*, then there exist an open neighborhood* $\mathcal{W}_0$ *of* ϕ_0 *in* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ *and a real analytic operator* $\mathbf{E}_0[\cdot]$ *from* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ *to* $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n)$ *which maps* $\mathcal{W}_0$ *into* $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$ *and such that,* $\mathbf{E}_0[\phi_0] = \Phi_0$*, and* $\mathbf{E}_0[\phi]|_{\partial\Omega} = \phi$ *for all* $\phi \in \mathcal{W}_0$.

1.2.2 The analyticity theorem

Let $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 1$. Let $\mathscr{E}$ be an open bounded subset of $R(n, 2k)$ such that $\mathbf{L}[\mathbf{a}]$ is an elliptic operator of order $2k$ for all $\mathbf{a} \in \mathrm{cl}\mathscr{E}$. Thus, we can apply Theorem 1.4 to $\mathbf{L}[\mathbf{a}]$ with $\mathbf{a} \in \mathscr{E}$ and we can introduce the corresponding analytic functions $A(\mathbf{a}, \theta, r)$, $B(\mathbf{a}, z)$, $C(\mathbf{a}, z)$ and $S(\mathbf{a}, z)$. We now fix two constants $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0, 1[$ and we fix an open and bounded subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. If $\mathbf{a} \in \mathscr{E}$, $\phi \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, $\mu \in C^{m-1,\lambda}(\partial\Omega)$ and $\beta \in \mathbb{N}^n$, $|\beta| \leq 2k-1$, we set

$$v_\beta[\mathbf{a}, \phi, \mu](\xi) \equiv \int_{\phi(\partial\Omega)} (\partial_z^\beta S)(\mathbf{a}, \xi - \eta)\, \mu \circ \phi^{(-1)}(\eta)\; d\sigma_\eta, \quad \forall\, \xi \in \mathbb{R}^n, \quad (1.20)$$

where the integral is understood in the sense of singular integrals if $|\beta| = 2k - 1$ and $\xi \in \phi(\partial\Omega)$, namely

$$\begin{aligned} v_\beta[\mathbf{a}, \phi, \mu](\xi) &\equiv \int^*_{\phi(\partial\Omega)} (\partial_z^\beta S)(\mathbf{a}, \xi - \eta)\, \mu \circ \phi^{(-1)}(\eta)\; d\sigma_\eta \\ &\equiv \lim_{\epsilon \to 0^+} \int_{\phi(\partial\Omega) \setminus (\xi + \epsilon\mathbb{B}_n)} (\partial_z^\beta S)(\mathbf{a}, \xi - \eta)\, \mu \circ \phi^{(-1)}(\eta)\; d\sigma_\eta, \quad \forall\, \xi \in \phi(\partial\Omega). \end{aligned}$$

Then we set

$$V_\beta[\mathbf{a}, \phi, \mu](x) \equiv v_\beta[\mathbf{a}, \phi, \mu] \circ \phi(x), \quad \forall\, x \in \partial\Omega. \quad (1.21)$$

Our goal is to prove the following.

Theorem 1.7. *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $k \geq 1$. *Let* $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k$ *be bounded open subsets of* $R(n, 2)$ *such that* $\mathbf{L}[\mathbf{a}_i]$ *is an elliptic operator of order* 2 *for all* $\mathbf{a}_i \in \mathrm{cl}\mathscr{C}_i$ *and for all* $i = 1, \ldots, k$. *Let* $\mathbf{a}(\cdot)$ *be the map of* $\mathscr{C} \equiv \mathscr{C}_1 \times \mathscr{C}_2 \times \ldots \mathscr{C}_k$ *to* $R(n, 2k)$ *which takes a* k*-tuple* $b \equiv (\mathbf{a}_1, \cdots \mathbf{a}_k)$ *to the unique element* $\mathbf{a}(b)$ *of* $R(n, 2k)$ *such that*

$$P[\mathbf{a}(b)](\xi) = P[\mathbf{a}_1](\xi) \cdot P[\mathbf{a}_2](\xi) \cdot \cdots \cdot P[\mathbf{a}_k](\xi). \quad (1.22)$$

Then the following statements hold.

(i) $\mathbf{a}(\cdot)$ *is real analytic on* $\mathscr{C}$.

(ii) *There exists a bounded open neighborhood* $\mathscr{E}$ *of* $\mathrm{cl}\,\mathbf{a}(\mathscr{C})$ *in* $R(n, 2k)$ *such that* $\mathbf{L}[\mathbf{a}]$ *is an elliptic operator of order* $2k$ *for all* $\mathbf{a} \in \mathrm{cl}\mathscr{E}$.

Let $S(\mathbf{a}, z)$ *be a function defined for all* $\mathbf{a} \in \mathscr{E}$ *as in Theorem 1.4. Let* $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0, 1[$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$, *with* Ω *and* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ *connected. Let* $\beta, \iota \in \mathbb{N}^n$, $|\beta| \leq 2k-2$, $|\iota| = 1$. *Let* $V_\beta[\mathbf{a}, \phi, \mu]$ *and* $V_{\beta+\iota}[\mathbf{a}, \phi, \mu]$ *be as in* (1.21) *for all* $(\mathbf{a}, \phi, \mu) \in \mathscr{E} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega)$. *Then the following statements hold.*

(iii) *The map of* $\mathscr{C} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega)$ *to* $C^{m,\lambda}(\partial\Omega)$ *which takes a triple* (b, ϕ, μ) *to the function* $V_\beta[\mathbf{a}(b), \phi, \mu]$, *is real analytic.*

(iv) *The map of* $\mathscr{C} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega)$ *to* $C^{m-1,\lambda}(\partial\Omega)$ *which takes a triple* (b, ϕ, μ) *to the function* $V_{\beta+\iota}[\mathbf{a}(b), \phi, \mu]$, *is real analytic.*

Statements (i) and (ii) of Theorem 1.7 are elementary. The proof of the statements (iii) and (iv) is contained in subsection 1.2.7. Before giving such a proof we need some preliminaries. In subsection 1.2.3, we study the continuity and Hölder continuity property of $v_\beta[b, \phi, \mu]$. In subsection 1.2.4, we investigate the jump properties of $v_\beta[b, \phi, \mu]$, with $|\beta| = 2k - 1$, across the boundary $\phi(\partial\Omega)$. In subsections 1.2.5 and 1.2.6 we consider an auxiliary boundary value problem. Then we will be ready for the proof of statements (iii) and (iv) of Theorem 1.7.

1.2.3 Continuity and Hölder continuity

The purpose of this subsection is to prove the following.

Theorem 1.8. *Let* n, k, m, λ, Ω, $\mathscr{E}$, $S(\cdot, \cdot)$ *be as in Theorem 1.7, and let* $(\mathbf{a}, \phi, \mu) \in \mathscr{E} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega)$. *We denote by* $v_{(0,\ldots,0)}[\mathbf{a}, \phi, \mu]$ *the function in* (1.20) *with* $\beta = (0, \ldots, 0)$. *Then the following statements hold.*

(i) $v_{(0,\ldots,0)}[\mathbf{a}, \phi, \mu]$ *is an element of* $C^{2k-2}(\mathbb{R}^n)$.

(ii) *The map which takes* μ *to* $v_{(0,\ldots,0)}[\mathbf{a}, \phi, \mu]|_{\mathrm{cl}\mathbb{I}[\phi]}$ *is linear and continuous from the space* $C^{m-1,\lambda}(\partial\Omega)$ *to* $C^{m+2k-2,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$.

(iii) *For all positive constant* R *such that* $\mathbb{I}[\phi] \subset R\mathbb{B}_n$, *the map which takes* μ *to* $v_{(0,\ldots,0)}[\mathbf{a}, \phi, \mu]|_{R\mathbb{B}_n \setminus \mathbb{I}[\phi]}$ *is linear and continuous from* $C^{m-1,\lambda}(\partial\Omega)$ *to* $C^{m+2k-2,\lambda}(R\mathbb{B}_n \setminus \mathbb{I}[\phi])$.

For the sake of simplicity, we only consider in the proof the dependence on μ. So, we set $S(z) \equiv S(\mathbf{a}, z)$ for all $z \in \mathbb{R}^n \setminus \{0\}$, and $v[\mu] \equiv v_{(0,\dots,0)}[\mathbf{a}, \phi, \mu]$ for all $\mu \in C^{m-1,\lambda}(\partial\Omega)$. If follows that

$$v[\mu](\xi) \equiv \int_{\phi(\partial\Omega)} S(\xi - \eta)\mu \circ \phi^{(-1)}(\eta)\ d\sigma_\eta, \quad \forall\ \xi \in \mathbb{R}^n.$$

We split the proof into two parts. In the first we prove statement (i), but in the second one the statements (iii) and (iv).

Proof of statement (i). By Theorem 1.4, we have

$$\partial_z^\beta S(z) \\ = \widetilde{A}(z/|z|, |z|)|z|^{2k-n-|\beta|} + \widetilde{B}(z/|z|, |z|)|z|^{2k-n-|\beta|} \log|z| + \widetilde{C}(z/|z|, |z|)$$

for every $\beta \in \mathbb{N}^n$, where $\widetilde{A}(\theta, r)$, $\widetilde{B}(\theta, r)$ and $\widetilde{C}(\theta, r)$ are continuous function of $(\theta, r) \in \partial\mathbb{B}_n \times \mathbb{R}$. Thus, if $|\beta| \leq 2k - 2$, then $D^\beta S(z) = o(|z|^{1-n})$ as $|z| \to 0^+$. Then, by classical theorems on integrals depending on parameters and by Vitali convergence theorem, the function

$$f_\beta(\xi) \equiv \int_{\phi(\partial\Omega)} \partial_\xi^\beta S(\xi - \eta)\mu \circ \phi^{(-1)}(\eta)\ d\sigma_\eta, \quad \forall\ \xi \in \mathbb{R}^n,$$

is continuous on $\mathbb{R}^n$ and coincides with $D^\beta v[\mu]$ on $\mathbb{R}^n \setminus \phi(\partial\Omega)$ for all $|\beta| \leq 2k - 2$. Then by a classical argument based on the divergence theorem, we have $D^\beta v[\mu] = f_\beta$ in the sense of distributions. Hence, $v[\mu] \in C^{2k-2}(\mathbb{R}^n)$ and $D^\beta v[\mu] = f_\beta$ classically. □

Now we turn to the proof of statements (ii) and (iii). We exploit an idea of Miranda [35, §5]. To do so we first state a theorem by Miranda, cf. [35, Theorem 2.1], and we introduce the related definition.

Definition 1.9. *We denote by $\mathscr{K}_j$ the set of the positively homogeneous functions of degree $(1-n)$ of class $C^j(\mathbb{R}^n \setminus \{0\})$ and we denote by $\mathscr{K}_{0,j}$ the subset of $\mathscr{K}_j$ of the functions K such that $\int_{\Pi \cap \partial\mathbb{B}_n} K(\eta)\ d\sigma_\eta = 0$ for every hyper-plane Π of $\mathbb{R}^n$ which contains 0.*

Then we have the following (cf. Miranda [35, Theorem 2.1].)

Theorem 1.10. *Let $K \in \mathscr{K}_{0,2m}$. Let*

$$p[\mu](\xi) \equiv \int_{\phi(\partial\Omega)} K(\xi - \eta)\mu \circ \phi^{(-1)}(\eta)\ d\sigma_\eta, \quad \forall\ \xi \in \mathbb{R}^n \setminus \phi(\partial\Omega),$$

for every $\mu \in C^{m-1,\lambda}(\partial\Omega)$. Then $p[\mu]|_{\mathbb{I}[\phi]}$ extends uniquely to an element $p^+[\mu]$ of $C^{m-1,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$ and $p[\mu]|_{\mathbb{R}^n \setminus \mathrm{cl}\mathbb{I}[\phi]}$ extends uniquely to an element $p^-[\mu]$ of $C^{m-1,\lambda}(\mathbb{R}^n \setminus \mathbb{I}[\phi])$. The map which takes μ to $p^+[\mu]$ is linear and continuous from $C^{m-1,\lambda}(\partial\Omega)$ to $C^{m-1,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$. Let $R > 0$ be such that $\mathbb{I}[\phi] \subset R\mathbb{B}_n$. The map which takes μ to $p^-[\mu]|_{R\mathbb{B}_n \setminus \mathbb{I}[\phi]}$ is linear and continuous from $C^{m-1,\lambda}(\partial\Omega)$ to $C^{m-1,\lambda}(R\mathbb{B}_n \setminus \mathbb{I}[\phi])$.

Now we introduce the following three lemmas.

Lemma 1.11. *Let $j, q \in \mathbb{N}$, $q \geq 1$. Let f be a real analytic function of $\theta \in \partial\mathbb{B}_n$ such that $f(-\theta) = (-1)^j f(\theta)$. Let $h(z)$ be the function of $z \in \mathbb{R}^n \setminus \{0\}$ defined by $h(z) \equiv f(z/|z|)\,|z|^{q-n}$. If the sum $j+q$ is even, then $D^\alpha h \in \mathscr{K}_{0,\infty}$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = q-1$.*

Proof. The lemma is easily verified for $q = 1$ and j odd. If $q > 1$, then, for every $\iota \in \mathbb{N}^n$, $|\iota| = 1$, $D^\iota h$ is a function of the form $g_1(z/|z|)\,|z|^{q-n-1}$, with g_1 real analytic on $\partial\mathbb{B}_n$ and $g_1(-\theta) = (-1)^{j+1} g_1(\theta)$ for all $\theta \in \partial\mathbb{B}_n$. So that, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = q-1$, we have $D^\alpha h(z) = g_2(z/|z|)\,|z|^{1-n}$, with g_2 real analytic on $\partial\mathbb{B}_n$ and $g_2(-\theta) = -g_2(\theta)$ for all $\theta \in \partial\mathbb{B}_n$. □

Lemma 1.12. *Let $j \in \mathbb{N}$. Let $p(z)$ be a homogeneous polynomial function of $z \in \mathbb{R}^n$ of degree j. Let $h(z)$ be the function of $z \in \mathbb{R}^n \setminus \{0\}$ defined by $h(z) \equiv p(z) \log|z|$. If n is even, then $D^\beta h \in \mathscr{K}_{0,\infty}$ for all $\beta \in \mathbb{N}^n$ with $|\beta| = n+j-1$.*

Proof. We have

$$\begin{aligned} D^\beta h(z) &= \sum_{\substack{\beta' \in \mathbb{N}^n \\ \beta' \leq \beta}} \binom{\beta}{\beta'} \left(D^{\beta-\beta'} p(z)\right) \left(D^{\beta'} \log|z|\right) \\ &= \left(D^\beta p(z)\right) \log|z| + \sum_{\substack{\beta' \in \mathbb{N}^n \\ 0 < \beta' \leq \beta}} \binom{\beta}{\beta'} \left(D^{\beta-\beta'} p(z)\right) \left(D^{\beta'} \log|z|\right) \end{aligned}$$

and we note that, for $j - |\alpha| \geq 0$, $D^\alpha p(z)$ is a homogeneous polynomial function of degree $j - |\alpha|$ and is identically 0 if $j - |\alpha| < 0$. If $|\alpha| > 0$, so that $\alpha = \alpha' + \iota$, $|\iota| = 1$, $D^\alpha \log|z| = D^{\alpha'}(|z|^{-1}(z/|z|)^\iota)$ is a function of the form $f(z/|z|)|z|^{-|\alpha|}$, with f real analytic on $\partial\mathbb{B}_n$ and $f(-\theta) = (-1)^{|\alpha|} f(\theta)$ for all $\theta \in \partial\mathbb{B}_n$ (as in the proof of Lemma 1.11.) It follows that $D^\beta h(z) = q(z) \log|z| + g(z/|z|)\,|z|^{j-|\beta|}$ where $q(z)$ is a homogeneous polynomial function of degree $j - |\beta|$, g is real analytic on $\partial\mathbb{B}_n$ and $g(-\theta) = (-1)^{j-|\beta|} g(\theta)$. And thus by taking $|\beta| = n+j-1$ our conclusion follows. □

Lemma 1.13. *Let $j \in \mathbb{N} \setminus \{0\}$. Let A, B be real analytic functions on $\partial\mathbb{B}_n \times \mathbb{R}$. Then the function $M(z) \equiv A(z/|z|, |z|)|z|^j + B(z/|z|, |z|)|z|^j \log|z|$, for all $z \in \mathbb{R}^n \setminus \{0\}$, can be extended to an element of $C^{j-1}(\mathbb{R}^n)$.*

Proof. The lemma clearly holds for $j = 1$. Now we assume $j \geq 2$. If $\iota \in \mathbb{N}^n$, $|\iota| = 1$, then we have

$$D^\iota M(z) = A'(z/|z|, |z|)|z|^{j-1} + B'(z/|z|, |z|)|z|^{j-1} \log|z|$$

in $\mathbb{R}^n \setminus \{0\}$, where A' and B' are real analytic on $\partial\mathbb{B}_n \times \mathbb{R}$. Thus we can argue by induction on j. □

Finally we are ready to conclude the proof of Theorem 1.8.

Proof of statements (ii) and (iii). Let $m' \in \mathbb{N}$, $m' > m + 2k - 2$. By Theorem 1.4, we have

$$S(z) = \sum_{j=0}^{m'+(2k-n)} f_j(z/|z|)|z|^{2k-n+j} + \sum_{\substack{\alpha\in\mathbb{N}^n \\ 2k-n\le|\alpha|\le m'+(2k-n)}} b_\alpha z^\alpha \log|z| \quad (1.23)$$

$$+A'(z/|z|,|z|)|z|^{m'+1} + B'(z/|z|,|z|)|z|^{m'+1}\log|z| + C(z)\,,$$

where the functions f_j are real analytic on $\partial\mathbb{B}_n$ and $f_j(-\theta) = (-1)^j f_j(\theta)$, $b_\alpha = 0$ if n is odd, and A', B' are real analytic on $\partial\mathbb{B}_n \times \mathbb{R}$ and C is real analytic on $\mathbb{R}^n$.

Now, the terms $f_j(z/|z|)|z|^{2k-n+j}$ appearing in (1.23) are functions of the form considered in Lemma 1.11 (note that $j + 2k + j$ is even.) Accordingly, the map of $C^{m-1,\lambda}(\partial\Omega)$ to $C^{m-1,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$ which takes μ to the D^β derivative of the unique extension to $\mathrm{cl}\mathbb{I}[\phi]$ of

$$\int_{\phi(\partial\Omega)} f_j\left((\xi-\eta)/|\xi-\eta|\right)|\xi-\eta|^{2k-n+j}\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta, \qquad \forall\,\xi\in\mathbb{I}[\phi]$$

is continuous for all $|\beta| = 2k+j-1$. In particular, the same map with $|\beta| = 0$ maps continuously $C^{m-1,\lambda}(\partial\Omega)$ to $C^{m-2k-2+j,\lambda}(\mathrm{cl}\mathbb{I}[\phi]) \subset C^{m-2k-2,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$. Similar result we have if we replace $\mathrm{cl}\mathbb{I}[\phi]$ by $R\mathbb{B}_n \setminus \mathbb{I}[\phi]$.

The terms $b_\alpha z^\alpha \log|z|$ satisfy the assumption of Lemma 1.12. Accordingly, the map of $C^{m-1,\lambda}(\partial\Omega)$ to $C^{m-1,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$ which takes μ to the D^β derivative of the unique extension to $\mathrm{cl}\mathbb{I}[\phi]$ of

$$\int_{\phi(\partial\Omega)} b_\alpha(\xi-\eta)^\alpha\log|\xi-\eta|\,\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta, \qquad \forall\,\xi\in\mathbb{I}[\phi]$$

is continuous for all $|\beta| = n+|\alpha|-1$. In particular, the same map with $|\beta| = 0$ maps continuously $C^{m-1,\lambda}(\partial\Omega)$ to the subset $C^{m-1+(n+|\alpha|-1),\lambda}(\mathrm{cl}\mathbb{I}[\phi])$ of $C^{m-2k-2,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$. Similar result we have if we replace $\mathrm{cl}\mathbb{I}[\phi]$ by $R\mathbb{B}_n \setminus \mathbb{I}[\phi]$.

The term $A'(z/|z|,|z|)|z|^{m'+1} + B'(z/|z|,|z|)|z|^{m'+1}\log|z|$ satisfies the assumptions Lemma 1.13, hence is an element of $C^{m'}(\mathbb{R}^n)$. C is real analytic and thus C^∞. By means of derivation under integral sign, statements (ii) and (iii) follows. □

1.2.4 The jump across the boundary

In this subsection we assume that the assumptions of Theorem 1.7 hold, we fix two constants $m \in \mathbb{N}\setminus\{0\}$ and $\lambda \in]0,1[$ and an open bounded subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. Then we consider the functions $v_\beta[\mathbf{a},\phi,\mu]$ introduced in subsection 1.2.2. Our aim is to investigate the behavior of $v_\beta[\mathbf{a},\phi,\mu](\xi)$ when ξ approaches to the boundary $\phi(\partial\Omega)$. For $|\beta|\le 2k-2$, $v_\beta[\mathbf{a},\phi,\mu](\xi)$ is continuous on $\mathbb{R}^n$ and displays no particular behavior as ξ approaches to the boundary $\phi(\partial\Omega)$. So we assume now $|\beta|=2k-1$ and we prove the following.

Theorem 1.14. *Let n, k, m, λ, Ω, $\mathscr{E}$, $S(\cdot,\cdot)$ be as in Theorem 1.7. Let $\beta\in\mathbb{N}^n$ with $|\beta|=2k-1$. Let $v_\beta[\mathbf{a},\phi,\mu]$ be the function in* (1.20) *for all $(\mathbf{a},\phi,\mu)\in\mathscr{E}\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$. Then*

$$\lim_{t\to0^+}\Big(v_\beta[\mathbf{a},\phi,\mu](\xi_0-t\nu_\phi(\xi_0))-v_\beta[\mathbf{a},\phi,\mu](\xi_0+t\nu_\phi(\xi_0))\Big) \qquad (1.24)$$
$$=-\frac{\nu_\phi(\xi_0)^\beta}{P_{2k}[\mathbf{a}](\nu_\phi(\xi_0))}\;\mu\circ\phi^{(-1)}(\xi_0)$$

for all $\xi_0\in\phi(\partial\Omega)$ and all $(\mathbf{a},\phi,\mu)\in\mathscr{E}\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$.

Proof. If ξ is not on the boundary, *i.e.* $\xi\in\mathbb{R}^n\setminus\phi(\partial\Omega)$, we have

$$v_\beta[\mathbf{a},\phi,\mu](\xi)=\int_{\phi(\partial\Omega)}(\partial_z^\beta S)(\mathbf{a},\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\;d\sigma_\eta.$$

We introduce the function $S_0(\mathbf{a},z)$ defined by

$$S_0(\mathbf{a},z)\equiv|z|^{2k-n}f_0(\mathbf{a},z/|z|)+\log|z|\sum_{|\alpha|=2k-n}b_\alpha(\mathbf{a})z^\alpha,\quad\forall\ z\in\mathbb{R}^n\setminus\{0\},$$

for all $\mathbf{a}\in\mathscr{E}$, where the function $f_0(\mathbf{a},z/|z|)$ is the coefficient of the first term in the series introduced in statement (i) of Theorem 1.4, the functions $b_\alpha(\mathbf{a})$ are the coefficients with $|\alpha|=2k-n$ appearing in statement (ii) of Theorem 1.4 and where $b_\alpha=0$ if n is odd or larger that $2k$ (see Remark 1.2.) Then we set $S_\infty(\mathbf{a},z)\equiv S(\mathbf{a},z)-S_0(\mathbf{a},z)$ and we define

$$v_{0,\beta}[\mathbf{a},\phi,\mu](\xi)\equiv\int_{\phi(\partial\Omega)}(\partial_z^\beta S_0)(\mathbf{a},\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\;d\sigma_\eta$$

and

$$v_{\infty,\beta}[\mathbf{a},\phi,\mu](\xi)\equiv\int_{\phi(\partial\Omega)}(\partial_z^\beta S_\infty)(\mathbf{a},\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\;d\sigma_\eta$$

for all $\xi\in\mathbb{R}^n\setminus\phi(\partial\Omega)$, so that $v_\beta[\mathbf{a},\phi,\mu]=v_{0,\beta}[\mathbf{a},\phi,\mu]+v_{\infty,\beta}[\mathbf{a},\phi,\mu]$. We note that

$$\partial_z^\beta S_\infty(\mathbf{a},z)=\widetilde{A}(z/|z|,|z|)|z|^{2-n}+\widetilde{B}(z/|z|,|z|)r^{2-n}\log|z|+\widetilde{C}(z/|z|,|z|)$$

where $\widetilde{A}(\theta,r)$, $\widetilde{B}(\theta,r)$ and $\widetilde{C}(\theta,r)$ are continuous function of $(\theta,r)\in\partial\mathbb{B}_n\times\mathbb{R}$. Then, by arguing as in the proof of the statement (i) of Theorem 1.8, we deduce that $v_{\infty,\beta}[\mathbf{a},\phi,\mu]$ extend to a continuous function in $\mathbb{R}^n$. So the contribution of $v_{\infty,\beta}[\mathbf{a},\phi,\mu]$ to the limit (1.24) is zero and we are reduced to consider the limit

$$\lim_{t\to 0^+}\Big(v_{0,\beta}[\mathbf{a},\phi,\mu](\xi_0 - t\nu_\phi(\xi_0)) - v_{0,\beta}[\mathbf{a},\phi,\mu](\xi_0 + t\nu_\phi(\xi_0))\Big)$$

By Cialdea [6, Theorem 3] we deduce immediately the validity of the theorem. □

1.2.5 An auxiliary boundary value problem

Let $\mathrm{Sym}(n,\mathbb{R})$ be the linear subspace of $M_{n\times n}(\mathbb{R})$ of all symmetric matrices. We observe that there exists a unique triple $(a, a', a'') \in \mathbb{R}\times\mathbb{R}^n\times\mathrm{Sym}(n,\mathbb{R})$ such that

$$P[\mathbf{a}](\xi) = \xi \cdot a''\xi + a'\cdot\xi + a \tag{1.25}$$

for each vector of coefficient $\mathbf{a} \in R(n,2)$. Then we can introduce the following.

Definition 1.15. *Let $\mathbf{a} \in R(n,2)$ and let $(a, a', a'') \in \mathbb{R}\times\mathbb{R}^n\times\mathrm{Sym}(n,\mathbb{R})$ be as in (1.25). Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0,\delta_0]$, and let Φ be a function of $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta},\mathbb{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, and let $\phi \equiv \Phi|_{\partial\Omega}$. Then we define the bounded and linear operator $\mathbf{B}[\mathbf{a},\Phi]$ of $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ to $C^{m-1,\lambda}(\phi(\partial\Omega))$ by the equality*

$$\mathbf{B}[\mathbf{a},\Phi](v^+,v^-) \equiv (Dv^+)|_{\partial\Omega}\, a''\nu_\phi - (Dv^-)|_{\partial\Omega}\, a''\nu_\phi$$

for all $(v^+,v^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$.

Our purpose in this subsection is to prove the next Theorem 1.16.

Theorem 1.16. *Let $\mathbf{a} \in R(n,2)$ such that $\mathbf{L}[\mathbf{a}]$ is elliptic. Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let Φ_0 be a function of $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_0},\mathbb{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}}$. Then there exists $\delta_1 \in]0,\delta_0]$ such that the following boundary value problem,*

$$\begin{cases} \mathbf{L}[\mathbf{a}]v^+ = f^+ & \text{in } \mathrm{cl}\Phi(\Omega^+_{\omega,\delta}),\\ \mathbf{L}[\mathbf{a}]v^- = f^- & \text{in } \mathrm{cl}\Phi(\Omega^-_{\omega,\delta}),\\ v^+ - v^- = g & \text{on } \phi(\partial\Omega),\\ \mathbf{B}[\mathbf{a},\Phi](v^+,v^-) = \gamma & \text{on } \phi(\partial\Omega),\\ v^+ = h^+ & \text{on } \Phi(\partial\Omega^+_{\omega,\delta}\setminus\partial\Omega),\\ v^- = h^- & \text{on } \Phi(\partial\Omega^-_{\omega,\delta}\setminus\partial\Omega), \end{cases} \tag{1.26}$$

with $\Phi \equiv \Phi_0|_{\mathrm{cl}\Omega_{\omega,\delta}}$ and $\phi \equiv \Phi|_{\partial\Omega}$, admits a unique solution (v^+,v^-) in $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$, for each given $(f^+,f^-,g,\gamma,h^+,h^-)$ in the space

$$\begin{aligned}\mathcal{S}^\Phi \equiv\ & C^{m-2,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m-2,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))\times C^{m,\lambda}(\phi(\partial\Omega))\\ &\times C^{m-1,\lambda}(\phi(\partial\Omega))\times C^{m,\lambda}(\Phi(\partial\Omega^+_{\omega,\delta}\setminus\partial\Omega))\times C^{m,\lambda}(\Phi(\partial\Omega^-_{\omega,\delta}\setminus\partial\Omega)),\end{aligned}$$

and for all $\delta \in]0,\delta_1]$.

In order to prove Theorem 1.16 we need the following three lemmas.

Lemma 1.17. *Let* $\mathbf{a} \in R(n, 2)$ *such that* $\mathbf{L}[\mathbf{a}]$ *is elliptic. Let* m, λ, Ω, ω, δ_0 *be as in Proposition 1.6. Let* Φ_0 *be a function of* $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}}$. *Then the boundary value problem*

$$\begin{cases} \mathbf{L}[\mathbf{a}]u^+ = 0 & \textit{in } \mathrm{cl}\Phi(\Omega^+_{\omega,\delta}), \\ \mathbf{L}[\mathbf{a}]u^- = 0 & \textit{in } \mathrm{cl}\Phi(\Omega^-_{\omega,\delta}) \\ u^+ - u^- = 0 & \textit{on } \phi(\partial\Omega)\,, \\ \mathbf{B}[\mathbf{a}, \Phi](u^+, u^-) = \gamma & \textit{on } \phi(\partial\Omega)\,, \end{cases} \tag{1.27}$$

with $\Phi \equiv \Phi_0|_{\mathrm{cl}\Omega_{\omega,\delta}}$ *and* $\phi \equiv \Phi|_{\partial\Omega}$, *has a solution* (u^+, u^-) *which belongs to* $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ *for each given* $\gamma \in C^{m-1,\lambda}(\phi(\partial\Omega))$ *and for all* $\delta \in]0, \delta_0]$.

Proof. By an elementary topological argument one verifies that there exists a bounded open neighborhood $\mathscr{E}$ of $\mathbf{a}$ in $R(n, 2)$ such that $\mathbf{L}[\mathbf{b}]$ is elliptic of order 2 for all $\mathbf{b} \in \mathrm{cl}\mathscr{E}$. So we can introduce the function $v_{(0,\dots,0)}[\mathbf{a}, \phi, \mu]$ as in (1.20). We take $\delta \in]0, \delta_0]$ and we set $v^+[\mathbf{a}, \phi, \mu] \equiv v_{(0,\dots,0)}[\mathbf{a}, \phi, \mu]|_{\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}$ and $v^-[\mathbf{a}, \phi, \mu] \equiv v_{(0,\dots,0)}[\mathbf{a}, \phi, \mu]|_{\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})}$. Then by Theorem 1.8, $(v^+[\mathbf{a}, \phi, \mu], v^-[\mathbf{a}, \phi, \mu]) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ and satisfies the first, second and third equation of (1.27). Moreover, by Theorem 1.14, we have

$$\mathbf{B}[\mathbf{a}, \Phi](v^+[\mathbf{a}, \phi, \mu], v^-[\mathbf{a}, \phi, \mu]) = -\mu \circ \phi^{(-1)}$$

on $\phi(\partial\Omega)$. So, by taking $(u^+, u^-) \equiv (v^+[\mathbf{a}, \phi, -\gamma], v^-[\mathbf{a}, \phi, -\gamma])$ the validity of the lemma follows. □

Lemma 1.18. *Let* $\mathbf{a} \in R(n, 2)$ *such that* $\mathbf{L}[\mathbf{a}]$ *be elliptic. Then there exists a constant* $M[\mathbf{a}] > 0$ *such that the boundary value problem*

$$\mathbf{L}[\mathbf{a}]u = 0 \ \textit{in } \Omega', \quad u = 0 \ \textit{on } \partial\Omega' \tag{1.28}$$

has only trivial solution $u \in C^1(\mathrm{cl}\Omega')$ *for all bounded open subsets* Ω' *of* $\mathbb{R}^n$ *of class* C^1 *with* $|\Omega'| < M[\mathbf{a}]$.

Proof. Let $(a, a', a'') \in \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R})$ be as in (1.25). Then the operator norm $\|a''\|$ of the matrix a'' is positive and we have $\xi \cdot a''\xi \geq \|a''\|^2 |\xi|^2$ for all $\xi \in \mathbb{R}^n$. Now let Ω' be an open subset of $\mathbb{R}^n$ of class C^1 and let $u \in C^1(\mathrm{cl}\Omega)$ be a solution of (1.28). We consider the weak formulation of (1.28). We have

$$\int_{\Omega'} (Dv)\, a''(Du)^t + va'\,(Du)^t + auv \; dx = 0, \quad \forall\ v \in W^{1,2}_0(\Omega')$$

where $W_0^{1,2}(\Omega')$ denotes the Sobolev space of the functions $v \in L^2(\Omega')$ with $Dv \in L^2(\Omega', \mathbb{R}^n)$ and $v|_{\partial\Omega'} = 0$. By taking $u = v$ we obtain

$$\int_{\Omega'} (Du)\, a''(Du)^t + u \cdot a'(Du)^t + a|u|^2 \; dx = 0 \tag{1.29}$$

and we note that the left hand side is greater or equal than

$$\int_{\Omega'} \|a''\|^2 |Du|^2 - |a'||u||Du| - a|u|^2 \; dx$$

which is greater or equal than

$$\int_{\Omega'} \|a''\|^2 |Du|^2 - \epsilon |Du|^2 - (4\epsilon)^{-1} |a'|^2 |u|^2 - a|u|^2 \; dx$$

where ϵ is an arbitrary positive constant. Now, by the Poincaré inequality we have

$$\int_{\Omega'} |u|^2 \; dx \leq c_P(\Omega) \int_{\Omega'} |Du|^2 \; dx.$$

Moreover, by the Krann-Faber inequality and by known properties of the Poincaré constant on balls (see *e.g.*, Troianiello [44, Theorem 1.43]) there exists a positive constant $c_P(n)$ which depends only on the dimension n, and such that $c_P(\Omega) \leq c_P(n)|\Omega'|^{2/n}$. So by (1.29) we deduce that

$$\int_{\Omega'} \left\{ (\|a''\|^2 - \epsilon) - ((4\epsilon)^{-1}|a'|^2 + |a|) c_P(n) |\Omega'|^{2/n} \right\} |Du|^2 \; dx \leq 0. \tag{1.30}$$

If we set $\epsilon = \|a''\|^2/2$ and we assume that $|\Omega'| < \|a''\|^{2n} c_P(n)^{-n/2} (|a'|^2 + 2\|a''\|^2 |a|)^{-n/2}$, then the term in brackets in (1.30) is positive and we have $\int_{\Omega'} |Du|^2 \; dx = 0$, which implies $u = 0$. So, by taking

$$M[\mathbf{a}] \equiv \|a''\|^{2n} c_P(n)^{-n/2} (|a'|^2 + 2\|a''\|^2 |a|)^{-n/2},$$

the validity of the lemma follows. □

Lemma 1.19. *Let* $\mathbf{a} \in R(n, 2)$ *such that* $\mathbf{L}[\mathbf{a}]$ *be elliptic. Let* Ω' *be a bounded open subsets of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$ *with* $|\Omega'| < M[\mathbf{a}]$*, where* $M[\mathbf{a}]$ *is the constant introduced in Lemma 1.19. Let* $(f, g) \in C^{m-2,\lambda}(\mathrm{cl}\Omega') \times C^{m,\lambda}(\partial\Omega')$*. Then there exists a unique* $u \in C^{m,\lambda}(\mathrm{cl}\Omega')$ *such that*

$$\mathbf{L}[\mathbf{a}]u = f \text{ in } \Omega', \quad u = g \text{ on } \partial\Omega'. \tag{1.31}$$

Proof. As it is well known, there exists $\tilde{g} \in C^{m,\lambda}(\mathrm{cl}\Omega')$ such that $\tilde{g}|_{\partial\Omega'} = g$ (cf., *e.g.*, Troianiello [44, Theorem 1.3, Lemma 1.5].) So it will be enough to prove the lemma for $g = 0$. In this case the existence of a solution u belonging to the Sobolev space $W^{1,2}(\Omega)$ follows by Lax-Milgram Theorem and by noting that the sesquililear form associated to problem (1.31) is coercive (see the proof of Lemma 1.18.) Then we have $u \in C^{m,\lambda}(\mathrm{cl}\Omega')$ by Morrey [36, Theorem 6.4.8]. □

Proof of Theorem 1.16. Let $M[\mathbf{a}]$ be the constant introduced in the previous Lemma 1.18. We take $\delta_1 \in]0, \delta_0]$ such that $|\Phi_0(\Omega_{\omega,\delta})| < M[\mathbf{a}]$ for all $\delta \in]0, \delta_1]$. Then we fix $\delta \in]0, \delta_1]$ and we fix a point $(f^+, f^-, g, \gamma, h^+, h^-)$ of $\mathcal{S}^\Phi$. As a first step we wish to prove the existence of a solution $(v^+, v^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ of problem (1.26). To do so we note that $|\Phi(\Omega^+_{\omega,\delta})| < M[\mathbf{a}]$ and $|\Phi(\Omega^-_{\omega,\delta})| < M[\mathbf{a}]$. So by Lemma 1.19 there exist $\widetilde{v}^+ \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))$ and $\widetilde{v}^- \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ such that

$$\begin{cases} \mathbf{L}[\mathbf{a}]\widetilde{v}^+ = f^+ & \text{in } \mathrm{cl}\Phi(\Omega^+_{\omega,\delta}), \\ \widetilde{v}^+ = g & \text{on } \phi(\partial\Omega), \\ \widetilde{v}^+ = h^+ & \text{on } \partial\Phi(\Omega^+_{\omega,\delta} \setminus \partial\Omega), \end{cases}$$

and

$$\begin{cases} \mathbf{L}[\mathbf{a}]\widetilde{v}^- = f^- & \text{in } \mathrm{cl}\Phi(\Omega^-_{\omega,\delta}), \\ \widetilde{v}^+ = 0 & \text{on } \phi(\partial\Omega), \\ \widetilde{v}^- = h^- & \text{on } \partial\Phi(\Omega^-_{\omega,\delta} \setminus \partial\Omega). \end{cases}$$

Now let $(u^+, u^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ be as in Lemma 1.17 with γ replaced by $\gamma - \mathbf{B}[\mathbf{a}, \Phi](\widetilde{v}^+, \widetilde{v}^-)$. Then system (1.26) admits a solution $(v^+, v^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ if and only if system

$$\begin{cases} \mathbf{L}[\mathbf{a}]V^+ = 0 & \text{in } \mathrm{cl}\Phi(\Omega^+_{\omega,\delta}), \\ \mathbf{L}[\mathbf{a}]V^- = 0 & \text{in } \mathrm{cl}\Phi(\Omega^-_{\omega,\delta}) \\ V^+ - V^- = 0 & \text{on } \phi(\partial\Omega)\,, \\ \mathbf{B}[\mathbf{a}, \Phi](V^+, V^-) = 0 & \text{on } \phi(\partial\Omega)\,, \\ V^+ = u^+ & \text{on } \Phi(\partial\Omega^+_{\omega,\delta} \setminus \partial\Omega), \\ V^- = u^- & \text{on } \Phi(\partial\Omega^-_{\omega,\delta} \setminus \partial\Omega), \end{cases} \tag{1.32}$$

admits a solution $(V^+, V^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$, and in case of existence we have $V^+ = v^+ - \widetilde{v}^+ + u^+$ and $V^- = v^- - \widetilde{v}^- + u^-$. Thus we now show existence for system (1.32). By the third and fourth equation of (1.32), and by a standard argument based on the Divergence Theorem, system (1.32) is equivalent to the following system for $V \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega_{\omega,\delta}))$

$$\begin{cases} \mathbf{L}[\mathbf{a}]V = 0 & \text{in } \mathrm{cl}\Phi(\Omega_{\omega,\delta}), \\ V = u^+ & \text{on } \Phi(\partial\Omega^+_{\omega,\delta} \setminus \partial\Omega), \\ V = u^- & \text{on } \Phi(\partial\Omega^-_{\omega,\delta} \setminus \partial\Omega). \end{cases} \tag{1.33}$$

By Lemma 1.19 such system has a solution V.

We now show uniqueness for system (1.26). Let $(f^+, f^-, g, \gamma, h^+, h^-) = 0$ and let $(v^+, v^-) \in C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ be the corresponding solution of (1.26). We set $v \equiv v^+$ on $\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})$ and $v \equiv v^-$ on $\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})$. The function v satisfy the equation $\mathbf{L}[\mathbf{a}]v = 0$ in $\Phi(\Omega^+_{\omega,\delta}) \cup \Phi(\Omega^-_{\omega,\delta})$ and is continuous on $\mathrm{cl}\Phi(\Omega_{\omega,\delta})$. Thus by exploiting the third and the fourth equation of (1.26) with $g = \gamma = 0$ and by a standard argument based on the Divergence Theorem, v can be shown to satisfy equation $\mathbf{L}[\mathbf{a}]v = 0$ in $\Phi(\Omega_{\omega,\delta})$. By the fifth and sixth equation of (1.26) with $h^+ = 0$ and $h^- = 0$, v vanishes on $\partial\Phi(\Omega_{\omega,\delta})$. Then by our choice of $\delta \leq \delta_1$ and by Lemma 1.18 we conclude that $v = 0$, and (v^+, v^-) must be zero. □

1.2.6 An equivalent problem and a stability theorem

By following strategy of Lanza de Cristoforis and Preciso [26], Lanza de Cristoforis and Rossi [27, 28] we wish now to transform problem (1.26) into an equivalent problem defined on the fixed set $\Omega_{\omega,\delta}$. We also obtain a stability theorem for the result of Theorem 1.16, *i.e.* if $\mathbf{a}$, Φ and $\delta \in]0, \delta_1]$ are as in Theorem 1.16 we show that the property of having a unique solution $(v^+, v^-) \in C^{m,\lambda}(\mathrm{cl}\Phi_0(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi_0(\Omega^-_{\omega,\delta}))$ for all $(f^+, f^-, g, h^+, h^-) \in \mathcal{S}^\Phi$ is attained by (1.26) in a whole neighborhood of $(\mathbf{a}, \Phi)$ in $R(n,2) \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}})$. To state this proposition in a more convenient way we introduce the following.

Definition 1.20. *Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$ and let $(\mathbf{a}, \Phi) \in R(n,2) \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}})$. We denote by $\mathbf{D}[\mathbf{a}, \Phi]$ the continuous and linear operator of $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ to $\mathcal{S}^\Phi$ which takes a pair (v^+, v^-) to*

$$\Big(\mathbf{L}[\mathbf{a}]v^+, \mathbf{L}[\mathbf{a}]v^-, v^+ - v^-, \mathbf{B}[\mathbf{a}, \phi](v^+, v^-), v^+|_{\Phi(\partial\Omega^+_{\omega,\delta}\setminus\partial\Omega)}, v^-|_{\Phi(\partial\Omega^-_{\omega,\delta}\setminus\partial\Omega)}\Big).$$

Then, it is easy to verify that (1.26) is equivalent to the equation

$$\mathbf{D}[\mathbf{a}, \Phi](v^+, v^-) = (f^+, f^-, g, \gamma, h^+, h^-)$$

and thus, by the Open Mapping Theorem, the existence and uniqueness of a solution (v^+, v^-) to (1.26) for each sextuple $(f^+, f^-, g, \gamma, h^+, h^-)$ is equivalent to the fact that $\mathbf{D}[\mathbf{a}, \Phi]$ is an homeomorphism.

The following Lemmas 1.21, 1.22 and 1.23 are just slight modifications of Lanza de Cristoforis and Rossi [28, Lemmas 3.25 and 3.26].

Lemma 1.21. *Let $m, m' \in \mathbb{N}$, $m > 0$ and $m \geq m'$. Let Ω' be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Then the operator* div *from the space $C^{m',\lambda}(\mathrm{cl}\Omega', \mathbb{R}^n)$ to the space $C^{m'-1,\lambda}(\mathrm{cl}\Omega')$ is bounded linear open and surjective.*

Lemma 1.22. *Let $m \in \mathbb{N}\setminus\{0\}$, $\lambda \in]0, 1[$, and let Ω' be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. The set*

$$\mathcal{Y}^{m,\lambda}(\Omega') \equiv \left\{ w \in C^{m,\lambda}(\mathrm{cl}\Omega', \mathbb{R}^n) \;\Big|\; \int_\Omega (D\psi)\, w\, dx = 0 \textit{ for every } \psi \in \mathscr{D}(\Omega') \right\}$$

is a closed linear subspace of $C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$ *and the quotient*

$$\mathcal{Z}^{m,\lambda}(\Omega') \equiv C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)/\mathcal{Y}^{m,\lambda}(\Omega')$$

is a Banach space. Moreover, if we denote by $\Pi_{\Omega'}$ *the canonical projection of* $C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$ *onto* $\mathcal{Z}^{m,\lambda}(\Omega')$ *there exists a unique homeomorphism* $\widetilde{\mathrm{div}}$ *of* $\mathcal{Z}^{m,\lambda}$ *to* $C^{m-1,\lambda}(\mathrm{cl}\Omega')$ *such that* $\mathrm{div}\, u = \widetilde{\mathrm{div}}(\Pi_{\Omega'} u)$ *for each* $u \in C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$.

We recall the identification of $R(n,2)$ to $\mathbb{R}\times\mathbb{R}^n\times \mathrm{Sym}(n,\mathbb{R})$ introduced before Definition 1.15. We have the following.

Lemma 1.23. *Let* $m \in \mathbb{N}\setminus\{0\}$, $\lambda \in]0,1[$, *and let* Ω' *be an open and bounded subset of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$. *For all* $\mathbf{a} = (a,a',a'') \in R(n,2)$ *and all* $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)\cap \mathcal{A}'_{\mathrm{cl}\Omega'}$ *we denote by* $\mathfrak{L}[\mathbf{a},\Phi]$ *the operator of the space* $C^{m,\lambda}(\mathrm{cl}\Omega')$ *to* $\mathcal{Z}^{m-1,\lambda}(\Omega')$ *which takes* $u \in C^{m,\lambda}(\mathrm{cl}\Omega')$ *to*

$$\mathfrak{L}[\mathbf{a},\Phi]\, u \equiv \Pi_{\Omega} A[\mathbf{a},\Phi,u] + a\, \widetilde{\mathrm{div}}^{(-1)}\left(u|\mathrm{det}D\Phi|\right)$$

where $A[\mathbf{a},\Phi,u] \in C^{m-1,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$ *is defined by*

$$A[\mathbf{a},\Phi,u] \equiv \left\{(D\Phi)^{-1}a''(D\Phi)^{-t}(Du)^t + (D\Phi)^{-1}a'u\right\}|\mathrm{det}D\Phi|\,.$$

Then, for all $f \in C^{m-1,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$ *and* $u \in C^{m,\lambda}(\mathrm{cl}\Omega')$ *we have* $\mathfrak{L}[\mathbf{a},\Phi]\, u = \Pi_{\Omega'} f$ *if and only if*

$$\mathbf{L}[\mathbf{a}]\left(u\circ\Phi^{(-1)}\right) = \mathrm{div}\left\{\left((D\Phi)f\right)\circ\Phi^{(-1)}\left|\mathrm{det}D\Phi^{(-1)}\right|\right\}$$

in the sense of distributions in $\Phi(\Omega')$.

Proof. We recall that $\mathfrak{L}[\mathbf{a},\Phi]\, u = \Pi_{\Omega'}A[\mathbf{a},\Phi,u] + a\,\widetilde{\mathrm{div}}^{(-1)}(u|\mathrm{det}Du|)$ and we begin considering the second term on the right. Since $u|\mathrm{det}D\Phi|$ is in $C^{m-1,\lambda}(\mathrm{cl}\Omega')$, by Lemma 1.21 there exists $g \in C^{m,\lambda}(\mathrm{cl}\Omega',\mathbb{R}^n)$ such that

$$\widetilde{\mathrm{div}}^{(-1)}(u|\mathrm{det}Du|) = \Pi_{\Omega'}\, g\,.$$

So the equation $\mathfrak{L}[\mathbf{a},\Phi]\, u = \Pi_{\Omega'} f$ is equivalent to $\Pi_{\Omega'}A[\mathbf{a},\Phi,u] = \Pi_{\Omega'}(f - ag)$. Then, by an argument based on the convolution with a family of mollifiers, Divergence Theorem and the rules of change of variable under integral sign we deduce that $\Pi_{\Omega'}A[\mathbf{a},\Phi,u] = \Pi_{\Omega'}(f-ag)$ is equivalent to

$$\begin{aligned}\mathbf{L}[\mathbf{a}]\left(u\circ\Phi^{(-1)}\right) + a\,\mathrm{div}\left\{\left((D\Phi)g\right)\circ\Phi^{(-1)}\left|\mathrm{det}D\Phi^{(-1)}\right|\right\}\\ = \mathrm{div}\left\{\left((D\Phi)f\right)\circ\Phi^{(-1)}\left|\mathrm{det}D\Phi^{(-1)}\right|\right\}\end{aligned}$$

(cf. Lanza de Cristoforis and Rossi [27, Lemma 3.4].) By the Piola Identity, we have

$$\mathrm{div}\left\{\left((D\Phi)g\right)\circ\Phi^{(-1)}\left|\mathrm{det}D\Phi^{(-1)}\right|\right\} = (\mathrm{div}g)\circ\Phi^{(-1)}\left|\mathrm{det}D\Phi^{(-1)}\right|$$

and the lemma follows. □

We now turn to consider an open subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$ which satisfies the assumptions of Proposition 1.6. We take $(\mathbf{a}, \Phi) \in R(n,2) \times C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$ and we define the operator $\mathfrak{D}[\mathbf{a}, \Phi]$ which we obtain by transplanting $\mathbf{D}[\mathbf{a}, \Phi]$ on the fixed set $\Omega_{\omega,\delta}$. In Theorem 1.25 we show that equations for $\mathfrak{D}[\mathbf{a}, \Phi]$ and $\mathbf{D}[\mathbf{a}, \Phi]$ are equivalent, then an elementary lemma will immediately imply the validity of Theorem 1.27.

Definition 1.24. *Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$. For all $\mathbf{a} = (a, a', a'') \in R(n,2)$ and all $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$ we denote by $\mathfrak{B}[\mathbf{a}, \Phi]$ the bounded and linear operator from the space $C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta})$ to $C^{m-1,\lambda}(\partial\Omega_{\omega,\delta})$ which takes a pair (V^+, V^-) to*

$$\mathfrak{B}[\mathbf{a}, \Phi](V^+, V^-) \equiv \left(a'' D\Phi^{-1} DV^+\right) \cdot \mathfrak{n}[\Phi] - \left(a'' D\Phi^{-1} DV^-\right) \cdot \mathfrak{n}[\Phi]$$

where $\mathfrak{n}[\Phi]$ is the function of $x \in \partial\Omega$ given by

$$\mathfrak{n}[\Phi](x) \equiv \frac{(D\Phi(x))^{-t}\nu_\Omega(x)}{|(D\Phi(x))^{-t}\nu_\Omega(x)|}$$

(so that $\mathfrak{n}[\Phi] = \nu_\phi \circ \phi$, see Lanza de Cristoforis and Rossi [28, Lemma 3.22].) Then we denote by $\mathfrak{D}[\mathbf{a}, \Phi]$ the operator of $C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta})$ to the space

$$\mathcal{Z} \equiv \mathcal{Z}^{m-1,\lambda}(\Omega^+_{\omega,\delta}) \times \mathcal{Z}^{m-1,\lambda}(\Omega^-_{\omega,\delta}) \times C^{m,\lambda}(\partial\Omega) \times C^{m-1,\lambda}(\partial\Omega) \times \\ \times C^{m,\lambda}\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right) \times C^{m,\lambda}\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right)$$

which takes a pair (V^+, V^-) to

$$\Big(\mathfrak{L}[\mathbf{a}, \Phi]V^+, \mathfrak{L}[\mathbf{a}, \Phi]V^-, V^+ - V^-, \\ \mathfrak{B}[\mathbf{a}, \Phi](V^+, V^-), V^+|_{\partial\Omega^+_{\omega,\delta}\setminus\partial\Omega}, V^-|_{\partial\Omega^-_{\omega,\delta}\setminus\partial\Omega}\Big).$$

Theorem 1.25. *Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$. Let $\mathbf{a} \in R(n,2)$, and let $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$. Then $\mathbf{D}[\mathbf{a}, \Phi]$ is an homeomorphism if and only if $\mathfrak{D}[\mathbf{a}, \Phi]$ is an homeomorphism.*

Proof. We assume that $\mathbf{D}[\mathbf{a}, \Phi]$ is an homeomorphism and we prove that $\mathfrak{D}[\mathbf{a}, \Phi]$ is also an homeomorphism. The proof of the converse is similar and we omit it. By the Open Mapping Theorem it suffices to show that $\mathfrak{D}[\mathbf{a}, \Phi]$ is a bijection. So, let $(F^+, F^-, G, \Gamma, H^+, H^-)$ be a given sextuple of $\mathcal{Z}$ and consider the equation

$$\mathfrak{D}[\mathbf{a}, \Phi](V^+, V^-) = (F^+, F^-, G, \Gamma, H^+, H^-). \tag{1.34}$$

Since $\Pi_{\Omega^+_{\omega,\delta}}$ and $\Pi_{\Omega^-_{\omega,\delta}}$ are surjective there exist $\widetilde{f}^+ \in C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta}, \mathbb{R}^n)$ and $\widetilde{f}^- \in C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta}, \mathbb{R}^n)$ such that $\Pi_{\Omega^+_{\omega,\delta}} \widetilde{f}^+ = F^+$ and $\Pi_{\Omega^-_{\omega,\delta}} \widetilde{f}^+ = F^-$. We set

$$
\begin{aligned}
f^+ &\equiv \mathrm{div}\left\{\left((D\Phi)\widetilde{f}^+\right) \circ \Phi^{(-1)} \left|\mathrm{det} D\Phi^{(-1)}\right|\right\}, \\
f^- &\equiv \mathrm{div}\left\{\left((D\Phi)\widetilde{f}^-\right) \circ \Phi^{(-1)} \left|\mathrm{det} D\Phi^{(-1)}\right|\right\},
\end{aligned}
$$

and we note that, by Lemma 1.23, equation (1.34) is equivalent to the following one,

$$
\mathbf{D}[\mathbf{a}, \Phi](v^+, v^-) = (f^+, f^-, g, \gamma, h^+, h^-) \tag{1.35}
$$

where $v^+ \equiv V^+ \circ \Phi^{(-1)}|_{\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}$, $v^- \equiv V^- \circ \Phi^{(-1)}|_{\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}$, $g \equiv G \circ \phi^{(-1)}$, $\gamma \equiv \Gamma \circ \phi^{(-1)}$, $h^+ \equiv H^+ \circ \Phi^{(-1)}|_{\Phi(\partial\Omega^+_{\omega,\delta} \setminus \partial\Omega)}$ and $h^- \equiv H^- \circ \Phi^{(-1)}|_{\Phi(\partial\Omega^-_{\omega,\delta} \setminus \partial\Omega)}$. Now, if we assume $\mathbf{D}[\mathbf{a}, \Phi]$ to be an homeomorphism, equation (1.35) has a unique solution. It follows that also (1.34) has a unique solution and since $(F^+, F^-, G, \Gamma, H^+, H^-)$ is an arbitrary point of $\mathcal{Z}$, $\mathfrak{D}[\mathbf{a}, \Phi]$ is a bijection. □

Now, we recall that, for all Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, the space $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ of the continuous and linear maps of $\mathcal{X}$ into $\mathcal{Y}$ endowed with the operator norm is a Banach space. By the continuity of the pointwise product in Schauder spaces and standard calculus in Banach space we have the following.

Lemma 1.26. *With the notation introduced in Definition 1.24, $\mathfrak{D}[\cdot, \cdot]$ is a real analytic map from the Banach space $R(n, 2) \times \left(C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega}\right)$ to the Banach space $\mathscr{L}(C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta}), \mathcal{Z})$.*

The next theorem follows immediately.

Theorem 1.27. *Let m, λ, Ω, ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$. Let $\mathbf{a} \in R(n, 2)$, and let $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, and let $\mathbf{D}[\mathbf{a}, \Phi]$ be an homeomorphism. Then there exists a neighborhood $\mathcal{U}$ of $\mathbf{a}$ in $R(n, 2)$ and a neighborhood $\mathcal{V}$ of Φ in $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n)$, $\mathcal{V} \subset \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, such that $\mathbf{D}[\mathbf{b}, \Psi]$ is an homeomorphism for all $(\mathbf{b}, \Psi) \in \mathcal{U} \times \mathcal{V}$.*

Proof. Assume that $\mathbf{D}[\mathbf{a}, \Phi]$ is a homeomorphism, then $\mathfrak{D}[\mathbf{a}, \Phi]$ is a homeomorphism by Theorem 1.25. Moreover, for $(\mathbf{b}, \Psi)$ close to $(\mathbf{a}, \Phi)$ the operators $\mathfrak{D}[\mathbf{b}, \Psi]$ and $\mathfrak{D}[\mathbf{a}, \Phi]$ are close in operator norm by Lemma 1.26 (which in particular implies that $\mathfrak{D}[\cdot, \cdot]$ is continuous.) Thus, for $(\mathbf{b}, \Psi)$ close enough to $(\mathbf{a}, \Phi)$, $\mathfrak{D}[\mathbf{b}, \Psi]$ is still an homeomorphism (see Kato [15, Chap. IV, §5]) and, again by Theorem 1.25, $\mathbf{D}[\mathbf{b}, \Psi]$ is an homeomorphism too. □

1.2.7 Proof of the main Theorem 1.7

Here we adopt all the notation and assumptions introduced in the previous subsection 1.2.2 and in Theorem 1.7. Furthermore we select a multi-index

$\beta \in N(n, 2k-2)$ and multi-indexes $\beta_0, \beta_1, \ldots, \beta_{k-1} \in N(n, 2)$ such that $\beta_0 + \cdots + \beta_{k-1} = \beta$ and $\beta_0 \equiv 0$. The multi-indexes β, $\beta_0, \ldots, \beta_{k-1}$ will be considered as fixed in all the subsection. We begin with a definition.

Definition 1.28. *Let m, λ, Ω, $\mathscr{C}$, $\mathbf{a}(\cdot)$ be as in Theorem 1.7. Let ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$. We denote by $v_j[b, \Phi, \mu]$, with $j = 1, \ldots, k$, the functions defined by*

$$
\begin{aligned}
v_1[b, \Phi, \mu] &\equiv \mathbf{L}[\mathbf{a}_2] \ldots \mathbf{L}[\mathbf{a}_k]\, v_{\beta_0}[\mathbf{a}(b), \phi, \mu],\\
v_2[b, \Phi, \mu] &\equiv \mathbf{L}[\mathbf{a}_3] \ldots \mathbf{L}[\mathbf{a}_k]\, v_{\beta_0+\beta_1}[\mathbf{a}(b), \phi, \mu],\\
&\vdots\\
v_j[b, \Phi, \mu] &\equiv \mathbf{L}[\mathbf{a}_{j+1}] \ldots \mathbf{L}[\mathbf{a}_k]\, v_{\beta_0+\cdots+\beta_{j-1}}[\mathbf{a}(b), \phi, \mu],\\
&\vdots\\
v_k[b, \Phi, \mu] &\equiv v_\beta[\mathbf{a}(b), \phi, \mu]
\end{aligned}
$$

for all $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathscr{C}$ and for all $(\Phi, \mu) \in (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}) \times C^{m-1,\lambda}(\partial\Omega)$, where we understand $\phi \equiv \Phi|_{\partial\Omega}$.

Then by Theorem 1.8 we have the following.

Lemma 1.29. *Let m, λ, Ω, $\mathscr{C}$ be as in Theorem 1.7. Let ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$. Let $(b, \Phi, \mu) \in \mathscr{C} \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}) \times C^{m-1,\lambda}(\partial\Omega)$. Then $v_j[b, \Phi, \mu]$ is a continuous function on $\mathbb{R}^n$, and the restrictions*

$$
v_j^+[b, \Phi, \mu] \equiv v_j[b, \Phi, \mu]|_{\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}, \quad v_j^-[b, \Phi, \mu] \equiv v_j[b, \Phi, \mu]|_{\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})}
$$

are functions of $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))$ and of $C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$, respectively, for all $j = 1, \ldots, k$.

In the sequel we denote by $\mathbf{B}[\cdot,\cdot]$ and $\mathbf{D}[\cdot,\cdot]$ the operators introduced in Definition 1.15 and in Definition 1.20, respectively. By Theorem 1.14 we have the following.

Lemma 1.30. *Let m, λ, Ω, $\mathscr{C}$, $\mathbf{a}(\cdot)$ be as in Theorem 1.7. Let ω, δ_0 be as in Proposition 1.6. Let $\delta \in]0, \delta_0]$ and let $\Phi \in C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$. We denote by $J_j(\cdot,\cdot)$ the map of $\mathscr{C} \times \partial\mathbb{B}_n$ to $\mathbb{R}$ which takes (b, θ), with $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$, to*

$$
J_j[b, \theta] \equiv -\delta_j \frac{P_2[\mathbf{a}_{j+1}](\theta) \ldots P_2[\mathbf{a}_k](\theta)}{P_{2k}[\mathbf{a}(b)](\theta)}\, \theta^{\beta_0+\cdots+\beta_{j-1}}
$$

for all $j = 1, \ldots, k-1$, and to

$$
J_j[b, \theta] \equiv -\delta_j \frac{\theta^\beta}{P_{2k}[\mathbf{a}(b)](\theta)}
$$

if $j = k$, *where* $\delta_j \equiv 1$ *if* $|\beta_0 + \cdots + \beta_{j-1}| = 2j - 2$, *and* $\delta_j \equiv 0$ *if* $|\beta_0 + \cdots + \beta_{j-1}| < 2j - 2$. *Then we have*

$$\mathbf{B}[\mathbf{a}_j, \Phi](v_j^+[b, \Phi, \mu], v_j^-[b, \Phi, \mu]) = J_j(b, \nu_\phi)\, \mu \circ \phi^{(-1)},$$

for all $j = 1, \ldots, k$ *and for all* $(b, \mu) \in \mathscr{C} \times C^{m-1,\lambda}(\partial\Omega)$ *with* $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$.

We now focus our attention on a arbitrary given point (b_0, Φ_0) of $\mathscr{C} \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}})$.

Lemma 1.31. *Let* m, λ, Ω, $\mathscr{C}$ *be as in Theorem 1.7. Let* ω, δ_0 *be as in Proposition 1.6. Let* (b_0, Φ_0) *be a point of* $\mathscr{C} \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}})$. *Then there exist* $\delta_1 \in]0, \delta_0]$, *and a neighborhood* $\mathcal{U}$ *of* b_0 *in* $\mathscr{C}$, *and a neighborhood* $\mathcal{V}$ *of* $\Phi_1 \equiv \Phi_0|_{\mathrm{cl}\Omega_{\omega,\delta_1}}$ *in* $C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_1}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_1}}$, *such that* $\mathbf{D}[\mathbf{a}_j, \Phi]$ *is an homeomorphism for all* $j = 1, \ldots, k$ *and for all* $(b, \Phi) \in \mathcal{U} \times \mathcal{V}$ *with* $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$.

Proof. Let $b_0 \equiv (\mathbf{a}_{01}, \ldots, \mathbf{a}_{0k})$. Then $\mathbf{L}[\mathbf{a}_{0j}]$ is an elliptic operator of order 2 for all $j = 1, \ldots, k$. By exploiting Theorem 1.16 and Theorem 1.27 the assertion follows. □

Now we see that the $2k$-tuple

$$(v_1^+[b, \Phi, \mu], v_1^-[b, \Phi, \mu] \ldots, v_k^+[b, \Phi, \mu], v_k^-[b, \Phi, \mu]) \tag{1.36}$$

is the unique solution of a chain of coupled boundary value problems, at least for $b \in \mathcal{U}$ and $\Phi \in \mathcal{V}$.

Lemma 1.32. *Let* m, λ, Ω, ω, δ_1, $\mathcal{U}$, $\mathcal{V}$ *be as in Lemma 1.31. Let* (b, Φ, μ) *belong to* $\mathcal{U} \times \mathcal{V} \times C^{m-1,\lambda}(\partial\Omega)$ *and let* $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$. *Then there exists a unique* $2k$*-tuple*

$$(v_1^+, v_1^-, \ldots, v_k^+, v_k^-) \in \Big[C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta_1})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta_1}))\Big]^k$$

which is a solution of the following equations for all $j = 1, \ldots, k$,

$$\begin{aligned} &\mathbf{D}[\mathbf{a}_j, \Phi](v_j^+, v_j^-) \\ &= \Big(D^{\beta_{j-1}} v_{j-1}^+\,,\, D^{\beta_{j-1}} v_{j-1}^-\,,\, 0\,,\, J_j(b, \nu_\phi)\mu \circ \phi^{(-1)}\,,\, h_j^+[b, \Phi, \mu]\,,\, h_j^-[b, \Phi, \mu]\Big) \end{aligned} \tag{1.37}$$

where $v_0^+ \equiv 0$, $v_0^- \equiv 0$, *and* $\phi \equiv \Phi|_{\partial\Omega}$, *and the functions* $h_j^+[b, \Phi, \mu]$ *and* $h_j^-[b, \Phi, \mu]$ *are given by the restriction of* $v_j[b, \Phi, \mu]$ *to* $\Phi(\partial\Omega^+_{\omega,\delta} \setminus \partial\Omega)$ *and to* $\Phi(\partial\Omega^-_{\omega,\delta} \setminus \partial\Omega)$, *respectively. Moreover the unique solution of the* (1.37) *is given by the* $2k$*-tuple in* (1.36).

Proof. By the previous Lemmas 1.29 and 1.30 one proves that the $2k$-tuple in (1.36) is indeed a solution of (1.37). So we have to prove the uniqueness.

We note that, for $j = 1$ equation (1.37) involves only the two unknown functions v_1^+ and v_1^-. Then Lemma 1.31 implies that (v_1^+, v_1^-) is uniquely determined and equals $(v_1^+[b, \Phi, \mu], v_1^-[b, \Phi, \mu])$. Now we turn to consider (v_2^+, v_2^-) and we exploit (1.37) with $j = 2$. Since (v_1^+, v_1^-) is uniquely determined we can replace it by the pair $(v_1^+[b, \Phi, \mu], v_1^-[b, \Phi, \mu])$. So now (1.37) with $j = 2$ only involves the two unknown functions v_2^+ and v_2^- and again we deduce that (v_2^+, v_2^-) is uniquely determined by Lemma 1.31. Iterating this procedure till we get that also (v_k^+, v_k^-) is uniquely determined, we conclude the proof of the lemma. □

The problem with the (1.37) is that the right hand side of (1.37) belongs to the Φ-dependent space $\mathcal{S}^\Phi$. To write an equivalent chain of equations in the fixed space $\mathcal{Z}$ (see Definition 1.24), we need some more notation.

Definition 1.33. *Let m, λ, Ω, $\mathscr{C}$, $\mathbf{a}(\cdot)$ be as in Theorem 1.7. Let ω, δ_1 be as in Lemma 1.31. We set*

$$H_j[b, \Phi, \mu](x) \equiv \int_{\partial\Omega} K_j(b, \Phi(x) - \Phi(y))\mu(y)\sigma_n[\Phi](y)d\sigma_y, \quad \forall\ x \in \partial\Omega_{\omega,\delta_1},$$

for all $j = 1, \ldots, k$ and for all $(b, \Phi, \mu) \in \mathscr{C} \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_1}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_1}}) \times C^{m-1,\lambda}(\partial\Omega)$ with $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$, where

$$K_j(b, z) \equiv \mathbf{L}[\mathbf{a}_{j+1}] \ldots \mathbf{L}[\mathbf{a}_k]\, \partial_z^{\beta_0+\cdots+\beta_{j-1}} S(\mathbf{a}(b), z), \quad \forall\ z \in \mathbb{R}^n \setminus \{0\},$$

and $\sigma[\Phi]$ is given by $\sigma[\Phi] \equiv |\mathrm{det} D\Phi||(D\Phi)^{-t}\nu_\Omega|$, where ν_Ω is the exterior unit normal to the boundary of Ω. Moreover we set $H_j^+[b, \Phi, \mu] \equiv H_j[b, \Phi, \mu]|_{\partial\Omega^+_{\omega,\delta}\setminus\partial\Omega}$ and $H_j^-[b, \Phi, \mu] \equiv H_j[b, \Phi, \mu]|_{\partial\Omega^-_{\omega,\delta}\setminus\partial\Omega}$.

In particular, we note that $H_j[b, \Phi, \mu] = v_j[b, \Phi, \mu] \circ \Phi$ on $\partial\Omega_{\omega,\delta}$. Now, consider the fixed multi-indexes β_0, …,β_{k-1}. We recall that $R(n, 2)$ is the space of the real functions defined on the set $N(n, 2)$ of the multi-indexes $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$. So we can introduce the functions $\mathbf{b}_0$, …, $\mathbf{b}_{k-1}$ of $R(n, 2)$ which satisfy the following condition, $\mathbf{b}_j$ is the element of $R(n, 2)$ which attains value 1 on β_j and 0 on each other multi-index of $N(n, 2)$. The elements $\mathbf{b}_0$, …, $\mathbf{b}_{k-1}$ are clearly uniquely determined. Moreover we have $\mathbf{L}[\mathbf{b}_0] = 1$ and $\mathbf{L}[\mathbf{b}_j] = D^{\beta_j}$ for all $j = 1, \ldots, k - 1$. By Lemmas 1.22 and 1.23 we obtain the following.

Lemma 1.34. *Let m, λ, Ω, ω, δ_1, $\mathcal{U}$, $\mathcal{V}$ be as in Lemma 1.31. Let $(b, \Phi, \mu) \in \mathcal{U} \times \mathcal{V} \times C^{m-1,\lambda}(\partial\Omega)$ and let $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$. Then the $2k$-tuple*

$$(v_1^+, v_1^-, \ldots, v_k^+, v_k^-) \in \Big[C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta_1})) \times C^{m,\lambda}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta_1}))\Big]^k$$

satisfies (1.37) *for* $j = 1, \dots, k$ *if and only if the* $2k$*-tuple*

$$(V_1^+, V_1^-, \dots, V_k^+, V_k^-) \in \left[C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta_1}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta_1})\right]^k,$$

with $V_j^+ \equiv v_j^+ \circ \Phi$ *and* $V_j^- \equiv v_j^- \circ \Phi$ *(*$j = 1, \dots, k$*) satisfies the following equations for* $j = 1, \dots, k$*,*

$$\begin{aligned}\mathfrak{D}[\mathbf{a}_j, \Phi](V_j^+, V_j^-) \qquad\qquad\qquad\qquad (1.38)\\ = \Big(\mathfrak{L}[\mathbf{b}_{j-1}, \Phi]V_{j-1}^+,\ \mathfrak{L}[\mathbf{b}_{j-1}, \Phi]V_{j-1}^-,\ 0\,,\\ J_j(b, \mathfrak{n}[\Phi])\,\mu,\ H_j^+[b, \Phi, \mu],\ H_j^-[b, \Phi, \mu]\Big)\end{aligned}$$

where $\mathfrak{D}[\mathbf{a}_j, \Phi]$ *and* $\mathfrak{n}[\Phi]$ *have been introduced in Definition 1.24 and we set* $V_0^+ \equiv 0$ *and* $V_0^- \equiv 0$*.*

In the next Lemma 1.36 we recast the equations (1.38) into an equivalent equation of the form $\Lambda[b, \Phi, \mu, W] = 0$, where Λ is a suitable operator between Banach spaces and W is the $2k$-tuple of Lemma 1.34. Then we plan to analyze the equation $\Lambda[b, \Phi, \mu, W] = 0$ by means of the Implicit Mapping Theorem for real analytic operators, and accordingly we need to show that Λ is real analytic. To do so we introduce the following Lemma 1.35. We omit the proof which is just a straightforward modification of the proof of Lanza de Cristoforis and Rossi [28, Lemma 3.34] and is based on Böhme and Tomi [2, p. 10], Henry [13, p. 29] and Valent [45, Chapter 2, Theorem 5.2].

Lemma 1.35. *Let* m*,* λ*,* Ω*,* $\mathscr{C}$ *be as in Theorem 1.7. Let* ω*,* δ_1 *be as in Lemma 1.31. The map of* $\mathscr{C} \times (C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_1}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_1}}) \times C^{m-1,\lambda}(\partial\Omega)$ *to* $C^{m,\lambda}(\partial\Omega_{\omega,\delta_1})$ *which takes* (b, Φ, μ) *to* $H_j[b, \Phi, \mu]$ *is real analytic.*

We now set

$$V_j^+[b, \Phi, \mu] \equiv v_j^+[b, \Phi, \mu] \circ \Phi|_{\mathrm{cl}\Omega^+_{\omega,\delta_1}}, \quad V_j^-[b, \Phi, \mu] \equiv v_j^-[b, \Phi, \mu] \circ \Phi|_{\mathrm{cl}\Omega^-_{\omega,\delta_1}}$$

for all $j = 1, \dots, k$ and for all $(b, \Phi, \mu) \in \mathcal{U} \times \mathcal{V} \times C^{m-1,\lambda}(\partial\Omega)$. By Lemma 1.29 and by the properties of composition of functions in Schauder spaces, we have $V_j^+[b, \Phi, \mu] \in C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta_1})$ and $V_j^-[b, \Phi, \mu] \in C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta_1})$. Then we denote by $W[b, \Phi, \mu]$ the $2k$-tuple

$$W[b, \Phi, \mu] \equiv (V_1^+[b, \Phi, \mu], V_1^-[b, \Phi, \mu], \dots, V_k^+[b, \Phi, \mu], V_k^-[b, \Phi, \mu]) \quad (1.39)$$

of the space $\left[C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta_1}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta_1})\right]^k$. The following Lemma 1.36 will immediately imply the validity of the main Theorem 1.7.

Lemma 1.36. *Let* m*,* λ*,* Ω*,* ω*,* δ_1*,* $\mathcal{U}$*,* $\mathcal{V}$ *be as in Lemma 1.31. The map of* $\mathcal{U} \times \mathcal{V} \times C^{m-1,\lambda}(\partial\Omega)$ *to* $\left[C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta_1}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta_1})\right]^k$*, which takes a triple* (b, Φ, μ) *to the* $2k$*-tuple* $W[b, \Phi, \mu]$ *defined by* (1.39)*, is real analytic.*

Proof. Let $\mathcal{X}$ be the Banach space $R(n,2)^k \times C^{m,\lambda}(\mathrm{cl}\Omega_{\omega,\delta_1}, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega)$ and let $\mathcal{Y}$ be the Banach space $\left[C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta_1}) \times C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta_1})\right]^k$. Let $\mathcal{W} \equiv \mathcal{U} \times \mathcal{V} \times C^{m-1,\lambda}(\partial\Omega)$ and let Λ_j be the (nonlinear) operator of $\mathcal{W} \times \mathcal{Y}$ to the Banach space $\mathcal{Z}$ introduced in Definition 1.24, which takes each $(b, \Phi, \mu, W) \in \mathcal{W} \times \mathcal{Y}$ with $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ and $W \equiv (V_1^+, V_1^-, \ldots, V_k^+, V_k^-)$, to

$$
\begin{aligned}
\Lambda_j[b, \Phi, \mu, W] \equiv \mathfrak{D}[\mathbf{a}_j, \Phi](V_j^+, V_j^-) \\
-\Big(\mathfrak{L}[\mathbf{b}_{j-1}, \Phi]V_{j-1}^+,\ \mathfrak{L}[\mathbf{b}_{j-1}, \Phi]V_{j-1}^-,\ 0, \\
J_j(b, \mathfrak{n}[\Phi])\mu,\ H_j^+[b, \Phi, \mu],\ H_j^-[b, \Phi, \mu]\Big),
\end{aligned}
$$

for all $j = 1, \ldots, k$, where as usual $V_0^+ \equiv 0$ and $V_0^- \equiv 0$. $\mathfrak{D}[\mathbf{a}_j, \Phi]$ and $\mathfrak{n}[\Phi]$ have been introduced in Definition 1.24, $\mathfrak{L}[\mathbf{b}_{j-1}, \Phi]$ has been introduced in Lemma 1.23, $J_j(b, \mathfrak{n}[\Phi])$, $H_j^+[b, \Phi, \mu]$ and $H_j^-[b, \Phi, \mu]$ are the functions introduced in Lemmas 1.30 and Definition 1.33, respectively. We note that, by Lemmas 1.26 and 1.35, and by continuity of the pointwise product in Schauder spaces and by standard calculus in Banach space, Λ_j is a real analytic operator. Then we denote by Λ the operator $(\Lambda_1, \ldots, \Lambda_k)$ from $\mathcal{W} \times \mathcal{Y}$ to $\mathcal{Z}^k$, clearly Λ is real analytic too. By Lemmas 1.32 and 1.34 the graph of the map $W[\cdot,\cdot,\cdot]$ of $\mathcal{W}$ to $\mathcal{Y}$ coincides with the set of zeros of Λ in $\mathcal{W} \times \mathcal{Y}$. So if we prove that the differential

$$
d_W \Lambda[b, \Phi, \mu, W[b, \Phi, \mu]] \tag{1.40}
$$

is an homeomorphism of $\mathcal{Y}$ to $\mathcal{Z}^k$ for any point $(b, \Phi, \mu, W[b, \Phi, \mu])$ with $(b, \Phi, \mu) \in \mathcal{W}$, then the Implicit Mapping Theorem for real analytic operators (cf. *e.g.*, Prodi and Ambrosetti [40, Theorem 11.6]) implies that $W[\cdot,\cdot,\cdot]$ is real analytic on $\mathcal{W}$. By the Open Mapping Theorem, it suffices to show that the operator in (1.40) is a bijection. So, let

$$
\big((F_1^+, F_1^-, G_1, \Gamma_1, H_1^+, H_1^-), \ldots, (F_k^+, F_k^-, G_k, \Gamma_k, H_k^+, H_k^-)\big)
$$

be an arbitrary given point of $\mathcal{Z}^k$. We have to prove that there exists one and only one $2k$-tuple $(X_1^+, X_1^-, \ldots, X_k^+, X_k^-) \in \mathcal{Y}$ such that the following equations hold for all $j = 1, \ldots, k$,

$$
\begin{aligned}
&\mathfrak{D}[\mathbf{a}_j, \Phi_1]\left(X_j^+, X_j^-\right) \\
&\quad = \Big(\mathfrak{L}[\mathbf{b}_{j-1}, \Phi_1]X_{j-1}^+ + F_j^+,\ \mathfrak{L}[\mathbf{b}_{j-1}, \Phi_1]X_{j-1}^- + F_j^-,\ G_j,\ \Gamma_j,\ H_j^+,\ H_j^-\Big),
\end{aligned} \tag{1.41}
$$

where as usual $b \equiv (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ and $X_0^+ \equiv 0$, $X_0^- \equiv 0$. Now, by Lemma 1.31, the operator $\mathbf{D}[\mathbf{a}_j, \Phi_1]$ is an homeomorphism for all $j = 1, \ldots, k$. By Theorem 1.25 it follows that also $\mathfrak{D}[\mathbf{a}_j, \Phi_1]$ is an homeomorphism for all

$j = 1, \dots, k$. Then by an iteration argument, we deduce that the solution solution $(X_1^+, X_1^-, \dots, X_k^+, X_k^-)$ of (1.41) exists unique (see the proof of Lemma 1.32.) Thus the operator in (1.40) is a homeomorphism and the assertion of the lemma follows. □

Now we are ready for the proof of our main result.

Proof of statement (iii) of Theorem 1.7. It is clearly enough to show that if $(b_0, \phi_0, \mu_0) \in \mathscr{C} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega)$, then $V_\beta[\mathbf{a}(\cdot), \cdot, \cdot]$ is real analytic in a neighborhood of (b_0, ϕ_0, μ_0). Let ω and δ_0 be as in Proposition 1.6, and let $\Phi_0 \in C^{m,\lambda}(\partial\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega_{\omega,\delta_0}}$ be an extension of ϕ_0 (cf. statement (v) of Proposition 1.6.) Then we take δ_1, $\mathcal{U}$, $\mathcal{V}$ as in Lemma 1.31. By statement (vi) of Proposition 1.6 there exist a neighborhood $\mathcal{W}_0$ of ϕ_0 and a real analytic extension operator $\mathbf{E}_0[\cdot]$ which maps $\mathcal{W}_0$ into $C^{m,\lambda}(\partial\Omega_{\omega,\delta_1}, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega_{\omega,\delta_1}}$ and such that $\mathbf{E}_0[\phi_0] = \Phi_0|_{\mathrm{cl}\Omega_{\omega,\delta_1}}$. Furthermore we can assume $\mathbf{E}_0[\mathcal{W}_0] \subset \mathcal{V}$. With the notation introduced in the previous subsection 1.2.2 and in the remark before Lemma 1.36, we have

$$V_\beta[\mathbf{a}(b), \phi, \mu] = V_k^+[b, \mathbf{E}_0[\phi], \mu]|_{\partial\Omega}$$

for all $b \in \mathcal{U}$, $\phi \in \mathcal{W}_0$ and $\mu \in C^{m-1,\lambda}(\partial\Omega)$. So, by the previous Lemma 1.36 and by standard calculus in Banach space, statement (iii) follows. □

Proof of statement (iv) of Theorem 1.7. Now let $\iota \in \mathbb{N}^n$ with $|\iota| = 1$. By Theorem 1.8 one verifies that the restriction $v_{\beta+\iota}[\mathbf{a}(b), \phi, \mu]|_{\mathbb{I}[\phi]}$ extends to a function $v^+_{\beta+\iota}[\mathbf{a}(b), \phi, \mu]$ of $C^{m,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$. Moreover, on $\phi(\partial\Omega)$ we have

$$v^+_{\beta+\iota}[\mathbf{a}(b), \phi, \mu] = -\frac{1}{2}\frac{(\nu_\phi)^{\beta+\iota}}{P_{2k}[\mathbf{a}(b)](\nu_\phi)}\,\mu \circ \phi^{(-1)} + v_{\beta+\iota}[\mathbf{a}(b), \phi, \mu]$$

(see Theorem 1.14 and Cialdea [5, §2, IX].) Then, with the notation introduced above,

$$V_{\beta+\iota}[\mathbf{a}(b), \phi, \mu] = \frac{1}{2}\frac{(\mathfrak{n}[\mathbf{E}_0[\phi]])^{\beta+\iota}}{P_{2k}[\mathbf{a}(b)](\mathfrak{n}[\mathbf{E}_0[\phi]])}\,\mu + \left[(D\mathbf{E}_0[\phi])^{-t}\, DV_k^+[b, \mathbf{E}_0[\phi], \mu]\right]^\iota .$$

We deduce that the function $V_{\beta+\iota}[\mathbf{a}(b), \phi, \mu]$ belongs to $C^{m-1,\lambda}(\partial\Omega)$. Moreover, by the continuity of the pointwise product in Schauder spaces, and by standard calculus in Banach space, and by statement (iii) of Theorem 1.7, the statement (iv) follows. □

1.3 Some applications

1.3.1 The Helmholtz and bi-Helmholtz operator

We denote by $\mathbf{H}^2[b_1, b_2]$ the operator

$$\mathbf{H}^2[b_1, b_2] \equiv (\Delta + b_1)\,(\Delta + b_2)$$

where b_1 and b_2 are real coefficients. We denote by $\mathbf{a}_{H^2}(\cdot,\cdot)$ the real analytic map of $\mathbb{R}^2$ to $R(n,4)$ defined by the equality

$$P[\mathbf{a}_{H^2}(b_1,b_2)](\xi) \equiv \left(|\xi|^2+b_1\right)\left(|\xi|^2+b_2\right), \quad \forall\, b_1,b_2\in\mathbb{R}.$$

Then we fix two open and bounded subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\mathbb{R}$. By an elementary topological argument one can show that there exists a bounded open neighborhood $\mathscr{E}$ of $\mathbf{a}_{H^2}(\mathscr{B}_1,\mathscr{B}_2)$ in $R(n,4)$ such that $\mathbf{L}[\mathbf{a}]$ is an elliptic operator of order 4 for all $\mathbf{a}\in\mathrm{cl}\mathscr{E}$. Then by Theorem 1.4 there exists a real analytic function $S_{H^2}(\mathbf{a},z)$ of $\mathbf{a}\in\mathscr{E}$ and $z\in\mathbb{R}^n\setminus\{0\}$ such that $S_{H^2}(\mathbf{a},\cdot)$ is a fundamental solution of $\mathbf{L}[\mathbf{a}]$ for all $\mathbf{a}\in\mathscr{E}$. We fix $m\in\mathbb{N}\setminus\{0\}$, $\lambda\in]0,1[$ and a bounded and open subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n\setminus\mathrm{cl}\Omega$ are connected. We set

$$(v_{H^2})_\beta[\mathbf{a},\phi,\mu](\xi) \equiv \int_{\phi(\partial\Omega)} (\partial_z^\beta S_{H^2})(\mathbf{a},\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta, \quad \forall\,\xi\in\mathbb{R}^n,$$

and $(V_{H^2})_\beta[\mathbf{a},\phi,\mu] \equiv (v_{H^2})_\beta[\mathbf{a},\phi,\mu]\circ\phi$, for all the triple $(\mathbf{a},\phi,\mu)$ of $\mathscr{E}\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$ and all multi-indexes β with $|\beta|\le 3$, where the integral is understood in the sense of singular integrals if $|\beta|=3$ and $\xi\in\phi(\partial\Omega)$. Then the assumptions of Theorem 1.7 hold and by standard calculus in Banach space we deduce the following.

Proposition 1.37. *Let $\beta\in\mathbb{N}^n$. The map $(V_{H^2})_\beta[\mathbf{a}_{H^2}(\cdot,\cdot),\cdot,\cdot]$ is real analytic from $\mathscr{B}_1\times\mathscr{B}_2\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$ to $C^{m,\lambda}(\partial\Omega)$ and to $C^{m-1,\lambda}(\partial\Omega)$, if $|\beta|\le 2$ and $|\beta|=3$, respectively.*

Now, it is easily seen that the function

$$\widetilde{S}_{H^1}(b_1,b_2,z) \equiv (\Delta+b_2)S_{H^2}(\mathbf{a}_{H^2}(b_1,b_2),z)$$

is a fundamental solution of the operator $\mathbf{H}^1[b_1]\equiv(\Delta+b_1)$. We can define the corresponding single layer potential

$$\widetilde{v}_{H^1}[b_1,b_2,\phi,\mu](\xi) \equiv \int_{\phi(\partial\Omega)} \widetilde{S}_{H^1}(b_1,b_2,\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta, \quad \forall\,\xi\in\mathbb{R}^n,$$

and the function $\widetilde{V}_{H^1}[b_1,b_2,\phi,\mu]\equiv\widetilde{v}_{H^1}[b_1,b_2,\phi,\mu]\circ\phi$, for all $(b_1,b_2,\phi,\mu)\in\mathscr{B}_1\times\mathscr{B}_2\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$. By the previous Proposition 1.37 next Proposition follows immediately.

Proposition 1.38. *The map $\widetilde{V}_{H^1}[\cdot,\cdot,\cdot,\cdot]$ is real analytic from $\mathscr{B}_1\times\mathscr{B}_2\times(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m-1,\lambda}(\partial\Omega)$ to $C^{m,\lambda}(\partial\Omega)$.*

Then we denote by $\widetilde{w}_{H^1}[b_1,b_2,\phi,\mu]$ the double layer potential

$$\widetilde{w}_{H^1}[b_1,b_2,\phi,\mu](\xi) \equiv -\int_{\phi(\partial\Omega)} \nu_\phi(\eta)\cdot D_\xi\widetilde{S}_{H^1}(b_1,b_2,\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta,\ \forall\xi\in\mathbb{R}^n$$

and we set $\widetilde{W}_{H^1}[b_1, b_2, \phi, \mu] \equiv \widetilde{w}_{H^1}[b_1, b_2, \phi, \mu] \circ \phi$, for all $(b_1, b_2, \phi, \mu) \in \mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega)$. By arguing as in Lanza de Cristoforis and Rossi [28, Theorem 3.4] one verifies that the restriction $\widetilde{w}_{H^1}[b_1, b_2, \phi, \mu]|_{\mathbb{I}[\phi]}$ extends to a function $\widetilde{w}^+_{H^1}[b_1, b_2, \phi, \mu]$ of $C^{m,\lambda}(\mathrm{cl}\mathbb{I}[\phi])$ and $\widetilde{w}_{H^1}[b_1, b_2, \phi, \mu]|_{\mathbb{E}[\phi]}$ extends to a function $\widetilde{w}^-_{H^1}[b_1, b_2, \phi, \mu]$ of $C^{m,\lambda}(\mathrm{cl}\mathbb{E}[\phi])$. Moreover, on $\phi(\partial\Omega)$ we have $\widetilde{w}^+_{H^1}[b_1, b_2, \phi, \mu] = \frac{1}{2}\mu \circ \phi + \widetilde{w}_{H^1}[b_1, b_2, \phi, \mu]$. We deduce that $\widetilde{W}_{H^1}[b_1, b_2, \phi, \mu]$ belongs to $C^{m,\lambda}(\partial\Omega)$ and by a straightforward modification in the proof of Lanza de Cristoforis and Rossi [28, Theorem 3.45 statement (iii)], we have the following.

Proposition 1.39. *The map $\widetilde{W}_{H^1}[\cdot, \cdot, \cdot, \cdot]$ is real analytic from $\mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega)$ to $C^{m,\lambda}(\partial\Omega)$.*

1.3.2 The Lamé equations

We denote by $\mathbf{L}[b_1, b_2]$ the vector valued operator

$$\mathbf{L}[b_1, b_2] \equiv \Delta + b_1 \nabla \mathrm{div} + b_2$$

where b_1 and b_2 are real coefficients. For $b_1 > 1 - 2/n$ and $b_2 \geq 0$ such an operator is related to the equations describing the behavior of an isotropic and homogeneous elastic body, *i.e.* the Lamé equations (see *e.g.* Kupradze, Gegelia, Basheleĭshvili and Burchuladze [18].) For our purpose, we fix a bounded open subset $\mathscr{B}_1$ of $\mathbb{R} \setminus \{-1\}$ and a bounded open subset $\mathscr{B}_2$ of $\mathbb{R}$. Then for all $(b_1, b_2) \in \mathscr{B}_1 \times \mathscr{B}_2$ a fundamental solution $S_L(b_1, b_2, z)$ of the operator $\mathbf{L}[b_1, b_2]$ is given by the $n \times n$-matrix function defined by

$$\begin{aligned}(S_L(b_1, b_2, z))_{ij} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.42)\\ \equiv \left(\delta_{ij}\left(\Delta_z + \frac{b_2}{b_1+1}\right) - \frac{b_1}{b_1+1}\frac{\partial^2}{\partial z_i \partial z_j}\right) S_{H^2}\Big(\mathbf{a}_{H^2}(b_2, b_2/(b_1+1)), z\Big),\end{aligned}$$

for all $z \in \mathbb{R}^n \setminus \{0\}$ and for all $i, j = 1, \ldots, n$, where δ_{ij} denotes the Kronecker delta symbol and S_{H^2} is the function introduced in the previous subsection 1.3.1. Then we fix $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0, 1[$ and a bounded and open subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. We denote by $v_L[b_1, b_2, \phi, \mu]$ the single layer potential given by

$$v_L[b_1, b_2, \phi, \mu](\xi) \equiv \int_{\phi(\partial\Omega)} S_L(b_1, b_2, \xi - \eta)\, \mu \circ \phi^{(-1)}(\eta)\, d\sigma_\eta, \quad \forall\, \xi \in \mathbb{R}^n,$$

for all $(b_1, b_2, \phi, \mu) \in \mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, and as usual we set $V_L[b_1, b_2, \phi, \mu] \equiv v_L[b_1, b_2, \phi, \mu] \circ \phi$. Taking the derivatives out of the integral sign we deduce that

$$\begin{aligned}v_L[b_1, b_2, \phi, \mu] = \left(\left(\Delta + \frac{b_2}{b_1+1}\right) - \frac{b_1}{b_1+1}\nabla \mathrm{div}\right) \qquad\qquad (1.43)\\ \cdot \Big(v_{H^2}\big[\mathbf{a}_{H^2}(b_2, b_2/(b_1+1)), \phi, \mu_i\big]\Big)_{i=1,\ldots,n}\end{aligned}$$

where $v_{H^2} \equiv (v_{H^2})_{(0,\dots,0)}$ is defined as in the previous subsection. Moreover we set

$$(v_L)_\iota[b_1, b_2, \phi, \mu](\xi) \equiv \int_{\phi(\partial\Omega)} (\partial_z^\iota S_L)(b_1, b_2, \xi - \eta)\, \mu \circ \phi^{(-1)}(\eta)\; d\sigma_\eta, \quad \forall\, \xi \in \mathbb{R}^n,$$

for all $\iota \in \mathbb{N}^n$, $|\iota| \le 1$, and all $(b_1, b_2, \phi, \mu) \in \mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, where the integral is understood in the sense of singular integrals if $|\iota| = 1$ and $\xi \in \phi(\partial\Omega)$. Then we set $(V_L)_\iota[b_1, b_2, \phi, \mu] \equiv (v_L)_\iota[b_1, b_2, \phi, \mu] \circ \phi$. By equation (1.42) and by Proposition 1.37, and by standard calculus in Banach space, we have the following.

Proposition 1.40. *Let $\iota \in \mathbb{N}^n$, $|\iota| \le 1$. Then $(V_L)_\iota[\cdot,\cdot,\cdot,\cdot]$ is a real analytic map from $\mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ to $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ and to $C^{m-1,\lambda}(\partial\Omega, \mathrm{M}_{n\times n}(\mathbb{R}))$, if $|\iota| = 0$ or $|\iota| = 1$, respectively.*

Now we introduce the double layer potential $w_L[b_1, b_2, \phi, \mu]$. We denote by $S_L^{(i)}$ the vector valued function given by the i-th column of S_L for all $i = 1, \dots, n$. We denote by $T(b_1, A)$ the matrix

$$(b_1 - 1)(\mathrm{tr} A)\mathbf{1}_n + (A + A^t)$$

for all $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ and all $b_1 \in \mathbb{R}$. Here $\mathbf{1}_n$ denotes the unit matrix of $\mathrm{M}_{n\times n}(\mathbb{R})$. Then we set

$$w_L[b_1, b_2, \phi, \mu](\xi) \equiv -\int_{\phi(\partial\Omega)} \left[\left(T\left(b_1, D_z S_L^{(i)}(b_1, b_2, \xi - \eta)\right)\nu_\phi(\eta)\right) \cdot \mu \circ \phi^{(-1)}(\eta)\right]_{i=1,\dots,n} d\sigma_\eta$$

for all $\xi \in \mathbb{R}^n$, where the integral is understood in the sense of singular integrals if $\xi \in \phi(\partial\Omega)$, and we set $W_L[b_1, b_2, \phi, \mu] \equiv w_L[b_1, b_2, \phi, \mu] \circ \phi$, for all $(b_1, b_2, \phi, \mu) \in \mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$.

We wish to prove that the map $W_L[\cdot,\cdot,\cdot,\cdot]$ is real analytic from the space $\mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ to $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$. To do so we introduce the operator $\mathscr{M}_{ij}(\nu)$ which is defined by $\mathscr{M}_{ij}(\nu) \equiv \nu_j D_i - \nu_i D_j$ for all $i, j = 1, \dots, n$ and all vectors $\nu \in \mathbb{R}^n$. We note that, for any $\phi \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, we have $\mathscr{M}_{ij}(\nu_\phi) = (\nu_\phi)_j \mathscr{D}_i - (\nu_\phi)_i \mathscr{D}_j$, where $\mathscr{D}_i \equiv D_i - (\nu_\phi)_i \nu_\phi \cdot D^t$ is the Günter tangential derivative (see Günter [12, Chapter 1].) It follows that $\mathscr{M}_{ij}(\nu_\phi)$ is a tangential operator on $\phi(\partial\Omega)$. Moreover we have the following.

Lemma 1.41. *If* (b_1, b_2, ϕ, μ) *belongs to* $\mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega)$*, then*

$$\int_{\phi(\partial\Omega)} \Big[\mathscr{M}(\nu_\phi(\eta))_\xi\, S_L(b_1, b_2, \xi - \eta)\Big]\, \mu \circ \phi(\eta)\ d\sigma_\eta = \int_{\phi(\partial\Omega)} \Big[\mathscr{M}(\nu_\phi(\eta))_\eta\, \mu \circ \phi(\eta)\Big]\, S_L(b_1, b_2, \xi - \eta)\ d\sigma_\eta, \quad \forall\ \xi \in \mathbb{R}^n.$$

Proof. Clearly

$$\begin{aligned}\mathscr{M}(\nu_\phi(\eta))_\eta \left[S_L(b_1, b_2, \xi - \eta)\, \mu \circ \phi(\eta)\right] &= \left[\mathscr{M}(\nu_\phi(\eta))_\eta\, \mu \circ \phi(\eta)\right] S_L(b_1, b_2, \xi - \eta) \\ &\quad - \left[\mathscr{M}(\nu_\phi(\eta))_\xi\, S_L(b_1, b_2, \xi - \eta)\right] \mu \circ \phi(\eta).\end{aligned}$$

Thus it suffices to prove that

$$\int_{\phi(\partial\Omega)} \mathscr{M}(\nu_\phi(\eta))_\eta \Big[S_L(b_1, b_2, \xi - \eta)\, \mu \circ \phi(\eta)\Big] d\sigma_\eta = 0. \tag{1.44}$$

We note that, by the Divergence Theorem,

$$\int_{\phi(\partial\Omega)} \mathscr{M}_{ij}(\nu_\phi(\eta))\, \psi(\eta)\ d\sigma_\eta = 0$$

for any function $\psi \in C^\infty(\phi(\partial\Omega))$. It follows that (1.44) does not depend on the particular choice of the fundamental solutions of $\mathbf{L}[b_1, b_2]$. Indeed two different fundamental solutions differ by a real analytic function. So by Kupradze *et al.* [18, Chapter V, §1], we conclude the proof of the lemma. We note that in [18] only the case $n = 3$ has been considered, but that all the statements can be extended to case $n \geq 2$ with minor modifications. □

Now we are ready to prove the following.

Proposition 1.42. *The function* $W_L[\cdot, \cdot, \cdot, \cdot]$ *is real analytic from* $\mathscr{B}_1 \times \mathscr{B}_2 \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ *to* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$*.*

Proof. Let $\widetilde{S}_{H^1}(b_1, b_2, z)$ be the function introduced in the previous subsection. By a cumbersome but straightforward calculation, one verifies that

$$\begin{aligned}&\left\{T\left(b_1, D_z S_L^{(i)}(b_1, b_2, z)\right) \nu\right\}_j \\ &\quad = \delta_{ij} \nu \cdot D_z \widetilde{S}_{H^1}\left(b_2, b_2/(b_1+1), z\right) + \mathscr{M}_{ij}(\nu) \widetilde{S}_{H^1}\left(b_2, b_2/(b_1+1), z\right) \\ &\quad\quad -2\left[\mathscr{M}(\nu) S_L(b_1, b_2, z)\right]_{ij} + \frac{b_1 b_2}{b_1+1} \nu_j \frac{\partial}{\partial z_i} S_{H^2}(\mathbf{a}_{H^2}(b_1, b_2), z)\end{aligned} \tag{1.45}$$

for every vector $\nu \in \mathbb{R}^n$ and for every $i, j = 1, \ldots, n$. Moreover, one can prove that Lemma 1.41 still holds if we replace $S_L(b_1, b_2, z)$ by the function

$\widetilde{S}_{H^1}(b_1,b_2,z)$. So, given $\phi_0 \in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}$ and ω, δ, $\mathcal{W}_0$, $\mathbf{E}_0$ as in Proposition 1.6, we have

$$\begin{aligned}
W_L[b_1,b_2,\phi,\mu] = \widetilde{W}_{H^1}&\left[b_2,\frac{b_2}{b_1+1},\phi,\mu_i\right]_{i=1,\dots,n}\\
&+\widetilde{V}_{H^1}\left[b_2,\frac{b_2}{b_1+1},\phi,\textstyle\sum_{j=1}^n \mathfrak{M}_{ij}(\phi,\mathfrak{n}[\mathbf{E}_0[\phi]])\mu_j\right]_{i=1,\dots,n}\\
&-2\sum_{j=1}^n V_L[b_1,b_2,\phi,\mathfrak{M}_{ij}(\phi,\mathfrak{n}[\mathbf{E}_0[\phi]])\mu]_{i=1,\dots,n}\\
&-\frac{b_1b_2}{b_1+1}\left(D\mathbf{E}_0[\phi]\right)^{-t}DV_{H^2}[\mathbf{a}_{H^2}(b_1,b_2),\phi,\mathfrak{n}[\mathbf{E}_0[\phi]]\cdot\mu]
\end{aligned}$$

for all $(b_1,b_2,\phi,\mu)\in\mathscr{B}_1\times\mathscr{B}_2\times\mathcal{W}_0\times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$, where $\mathfrak{n}$ is the function introduced in Definition 1.24 and $\mathfrak{M}_{ij}(\phi,\nu)\equiv\nu_j\left[(D\mathbf{E}_0[\phi])^{-t}D^t\right]_i-\nu_i\left[(D\mathbf{E}_0[\phi])^{-t}D^t\right]_j$ for any $\phi\in\mathcal{W}_0$ and any vector $\nu\in\mathbb{R}^n$. Then by Propositions 1.37, 1.38, 1.39 and 1.40, by the continuity of the pointwise product in Schauder spaces and by standard calculus in Banach space, our proposition follows. □

1.3.3 The Stokes system

We say that $S_S\equiv(S_V,S_P)$ is a fundamental solution for the Stokes system in $\mathbb{R}^n$ if S_V is a real analytic $n\times n$-matrix valued function of $\mathbb{R}^n\setminus\{0\}$, S_P is a real analytic vector valued function of $\mathbb{R}^n\setminus\{0\}$ and

$$\Delta S_V(z)-\nabla S_P(z)=\delta(z)\mathbf{1}_n,\quad \operatorname{div}S_V(z)=0,\quad \forall\, z\in\mathbb{R}^n\setminus\{0\}$$

(cf. Ladyzhenskaya [19, Chapter 3].) We can verify that a suitable choice of the functions S_V and S_P is given by the following equalities,

$$(S_V(z))_{ij}\equiv\left(\delta_{ij}\Delta-\frac{\partial^2}{\partial z_i\partial z_j}\right)S_{\Delta^2}(z),\quad (S_P)_i(z)\equiv-\frac{\partial}{\partial z_i}S_\Delta(z),\qquad(1.46)$$

for all $z\in\mathbb{R}^n\setminus\{0\}$, where we understand $S_{\Delta^2}(z)\equiv S_{H^2}(\mathbf{a}_{H^2}(0,0),z)$ and $S_\Delta(z)\equiv\Delta S_{\Delta^2}(z)$, in accordance with the notation of subsection 1.3.1. Then we consider an open bounded subset Ω of $\mathbb{R}^n$ of class $C^{m,\lambda}$, $m\in\mathbb{N}\setminus\{0\}$, $\lambda\in]0,1[$, with Ω and $\mathbb{R}^n\setminus\mathrm{cl}\Omega$ connected. We introduce the single layer potentials $v_V[\phi,\mu]$ and $v_P[\phi,\mu]$ by the equalities

$$v_V[\phi,\mu](\xi)\equiv\int_{\phi(\partial\Omega)}S_V(\xi-\eta)\,\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta,\quad\forall\,\xi\in\mathbb{R}^n,\qquad(1.47)$$

$$v_P[\phi,\mu](\xi)\equiv\int_{\phi(\partial\Omega)}S_P(\xi-\eta)\cdot\mu\circ\phi^{(-1)}(\eta)\,d\sigma_\eta,\quad\forall\,\xi\in\mathbb{R}^n,\qquad(1.48)$$

for all $(\phi,\mu)\in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}\times C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$, where the integral in (1.48) is understood in the sense of singular integrals if $\xi\in\phi(\partial\Omega)$. As usual we set $V_V[\phi,\mu] \equiv v_V[\phi,\mu] \circ \phi$ and $V_P[\phi,\mu] \equiv v_P[\phi,\mu] \circ \phi$. Then, by equation (1.46) and by Proposition 1.37, we have the following.

Proposition 1.43. *$V_V[\cdot,\cdot]$ and $V_P[\cdot,\cdot]$ are real analytic function defined from the set $(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$ and to $C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$, respectively.*

Now for each scalar $b \in \mathbb{R}$ and each matrix $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ we set $\mathcal{T}(b,A) \equiv -b\mathbf{1}_n + (A + A^t)$. Then we denote by $S_V^{(i)}$ the vector valued function given by the i-th column of S_V for each $i = 1,\dots,n$. We define the double layer potential $w_V[\phi,\mu]$ by

$$w_V[\phi,\mu](\xi)$$
$$\equiv -\int_{\phi(\partial\Omega)} \Big[\Big(\mathcal{T}\left((S_P)_i(\xi-\eta), DS_V^{(i)}(\xi-\eta)\right)\nu_\phi(\eta)\Big)\cdot \mu\circ\phi^{(-1)}(\eta)\Big]_{i=1,\dots,n} d\sigma_\eta,$$

for all $\xi \in \mathbb{R}^n$ and for all $(\phi,\mu) \in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega} \times C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$. As usual, we set $W_V[\phi,\mu] \equiv w_V[\phi,\mu] \circ \phi$ and we have the following.

Proposition 1.44. *$W_V[\cdot,\cdot]$ is a real analytic map from $(C^{m,\lambda}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$.*

Proof. For each vector $\nu \in \mathbb{R}^n$ we have

$$\left\{\mathcal{T}\left((S_P)_i, DS_V^{(i)}\right)\nu\right\}_j = \delta_{ij}\,\nu\cdot DS_\Delta + \mathscr{M}_{ij}(\nu)S_\Delta - 2[\mathscr{M}(\nu)S_V]_{ij}.$$

Then one verifies that Lemma 1.41 still holds if we replace $S_L[b,\cdot]$ by $S_V(\cdot)$ or by $S_\Delta(\cdot)$. So, given $\phi_0 \in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ and ω, δ, $\mathcal{W}_0$, $\mathbf{E}_0$ as in Proposition 1.6, we have

$$\begin{aligned} W_V[\phi,\mu] = W_\Delta[\phi,\mu_i]_{i=1,\dots,n} &+ V_\Delta\big[\phi, \textstyle\sum_{j=1}^n \mathfrak{M}_{ij}(\phi,\mathfrak{n}[\mathbf{E}_0[\phi]])\mu_j\big]_{i=1,\dots,n} \\ &-2\sum_{j=1}^n V_V\big[\phi, \mathfrak{M}_{ij}(\phi,\mathfrak{n}[\mathbf{E}_0[\phi]])\mu\big]_{i=1,\dots,n} \end{aligned}$$

for all $(\phi,\mu) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$. By Propositions 1.38 and 1.43 we conclude the proof. □

We now consider the double layer potential $w_P[\phi,\mu]$ which is the function of $\mathbb{R}^n \setminus \phi(\partial\Omega)$ defined by

$$w_P[\phi,\mu](\xi) \equiv -2\,\mathrm{div}\left(\int_{\phi(\partial\Omega)} S_P(\xi-\eta)\cdot\mu\circ\phi^{(-1)}(\eta)\,\nu_\phi(\eta)\;d\sigma_\eta\right), \quad (1.49)$$

for all $\xi \in \mathbb{R}^n\setminus\phi(\partial\Omega)$ and all $(\phi,\mu) \in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}\times C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$. We note that the right hand side of (1.49) equals

$$-2\sum_{i,j=1}^n \frac{\partial}{\partial\xi_i}\int_{\phi(\partial\Omega)} \mathscr{M}_{ij}(\nu_\phi(\eta))_\xi S_\Delta(\xi-\eta)\,\mu_j\circ\phi^{(-1)}\;d\sigma_\eta, \quad (1.50)$$

for all $\xi \in \mathbb{R}^n \setminus \phi(\partial\Omega)$. Since Lemma 1.41 still holds if we replace S_L by S_Δ, the expression in (1.50) equals

$$-2 \sum_{i,j=1}^{n} \int_{\phi(\partial\Omega)} (\partial_i S_\Delta)(\xi - \eta) \left(\mathscr{M}_{ij}(\nu_\phi(\eta))_\eta \, \mu_j \circ \phi^{(-1)}(\eta)\right) d\sigma_\eta, \qquad (1.51)$$

for all $\xi \in \mathbb{R}^n \setminus \phi(\partial\Omega)$. We now observe that the expression in (1.51) defines a map from $\mathbb{R}^n$ to $\mathbb{R}$ for all $(\phi, \mu) \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega} \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$, where the integral is understood in the sense of singular integrals if $\xi \in \phi(\partial\Omega)$. So it makes sense to introduce the function $W_P[\phi, \mu]$ which is the composition of the function defined by (1.51) and the function ϕ, for all $(\phi, \mu) \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega} \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$. Then, by the previous Proposition 1.37, we deduce the following.

Proposition 1.45. *$W_P[\cdot, \cdot]$ is a real analytic map from $(C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ to $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$.*

Proof. By (1.51) we deduce that $W_P[\phi, \mu]$ is a sum of terms of the form $(V_\Delta)_\beta \, [\phi, \mathfrak{M}_{ij}(\phi, \mathfrak{n}[\mathbf{E}_0[\phi]])\mu_j]$, with $\beta \in \mathbb{N}^n$, $|\beta| = 1$. Then, by Proposition 1.37 and by standard calculus in Banach space, our statement follows. $\square$

Chapter 2

Elastic boundary value problems in a domain with a small hole

2.0.4 Basic boundary value problems

In this chapter we focus our attention on the vector valued partial differential operator $\mathbf{L}[b] \equiv \Delta + b\,\nabla\mathrm{div}$, where b is a real coefficient. One immediately recognizes that $\mathbf{L}[b] = \mathbf{L}[(b,0)]$, where $\mathbf{L}[(b,0)]$ is the operator introduced in subsection 1.3.2. Let $b \in \mathbb{R}$ and $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. We denote by $T(b,A)$ the matrix $(b-1)(\mathrm{tr}A)\mathbf{1}_n + (A+A^t)$. We also note that the matrix of polynomials

$$P[b](\xi_1,\dots,\xi_n) = (P_{ij}[b](\xi_1,\dots,\xi_n))_{i,j=1,\dots,n},$$

with

$$P_{ij}[b](\xi_1,\dots,\xi_n) = \delta_{ij}|\xi|^2 + b\,\xi_i\xi_j \quad \forall\ i,j = 1,\dots,n,$$

satisfies the equality $\mathbf{L}[b] = P[b](\partial_{x_1},\dots,\partial_{x_n})$. Moreover we have

$$\eta^t P[b](\xi)\eta = |\xi|^2|\eta|^2 + b(\xi\cdot\eta)^2 \geq |\xi|^2|\eta|^2 \quad \forall\ \xi,\eta \in \mathbb{R}^n,\ b \geq 0,$$

which implies that $\mathbf{L}[b]$ is a strictly elliptic operator for all $b \geq 0$. Now let Ω be an open bounded subset of $\mathbb{R}^n$ of class C^1, and let ν be the outward unit normal to $\partial\Omega$, and let f be a vector valued function defined on $\partial\Omega$. In the sequel we consider the following *basic* boundary value problems,

(i) $\mathbf{L}[b]u = 0$ in Ω and $u = f$ on $\partial\Omega$,

(ii) $\mathbf{L}[b]u = 0$ in $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ and $u = f$ on $\partial\Omega$,

(iii) $\mathbf{L}[b]u = 0$ in Ω and $T(b,Du)\nu = f$ on $\partial\Omega$,

(iv) $\mathbf{L}[b]u = 0$ in $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ and $T(b,Du)\nu = f$ on $\partial\Omega$.

We refer to problems (i), (ii), (iii) and (iv) as to the Dirichlet interior, the Dirichlet exterior, the Neumann interior and the Neumann exterior boundary value problem, respectively. One can verify that the boundary value problems (i), (ii), (iii) and (iv) satisfy the Shapiro-Lopatinskiĭ conditions for all $b \in \mathbb{R} \setminus \{0, -1, -2\}$ (see Kozhevnikov [16].) In particular, the spaces of the solutions of the problems with homogeneous data (namely with $f = 0$) have a completely explicit description for all $b > 1 - 2/n$ (cf. Theorem 2.4.) For this reason we focus our attention on this particular range of values of b and we denote by $\mathscr{B}$ the set $\{b \in \mathbb{R} : b > 1 - 2/n\}$.

2.1 Boundary integral equations

In this section, we investigate the basic boundary value problems for the operator $\mathbf{L}[b]$ in an open subset Ω of $\mathbb{R}^n$. We associate to such problems suitable boundary integral equations and we point out some properties of the corresponding integral operators.

2.1.1 Preliminaries

In the next Theorem 2.4, we describe the solutions of the first and second boundary value problems in Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ when homogeneous boundary data are considered. To do so, we first introduce the following definition.

Definition 2.1. *We denote by $\mathcal{R}$ the set of the vector valued function ρ on $\mathbb{R}^n$ such that $\rho(x) = Ax + b$ for every $x \in \mathbb{R}^n$, where A is a real $n \times n$ skew-symmetric matrix (briefly $A \in \mathrm{Skew}(n, \mathbb{R})$) and $b \in \mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{R}_\Omega$ the set of the functions on Ω which are restrictions of functions of $\mathcal{R}$. We denote by $\mathcal{R}_{\Omega,\mathrm{loc}}$ the set of the functions on Ω such that $\rho|_{\Omega'} \in \mathcal{R}_{\Omega'}$ for every connected component Ω' of Ω. We denote by $(\mathcal{R}_{\Omega,\mathrm{loc}})|_{\partial\Omega}$ the set of the functions on $\partial\Omega$ which are trace on $\partial\Omega$ of functions of $\mathcal{R}_{\Omega,\mathrm{loc}}$.*

Then we have the following proposition.

Proposition 2.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathbf{E}[b]$ be the symmetric bilinear operator of $[C^1(\Omega, \mathbb{R}^n)]^2$ to $C(\Omega)$ which takes a pair (u, v) to*

$$\begin{aligned} \mathbf{E}[b](u, v) \equiv & (b - 1 + 2/n) \operatorname{div} u \operatorname{div} v \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (\partial_i u_j + \partial_j u_i)(\partial_i v_j + \partial_i v_i) + \frac{1}{n} \sum_{i,j=1}^{n} (\partial_i u_i - \partial_j u_j)(\partial_i v_i - \partial_j v_j). \end{aligned} \tag{2.1}$$

If $b > 1 - 2/n$, then the following statements hold.

(i) There exists a real constant $c > 0$ such that $\mathbf{E}[b](u, u) \geq c|Du + (Du)^t|^2$ for all $u \in C^1(\Omega, \mathbb{R}^n)$.

(ii) *If* $u \in C^2(\Omega, \mathbb{R}^n)$*, then* $\mathbf{E}[b](u,u) = 0$ *if and only if* $u \in \mathcal{R}_{\Omega,\mathrm{loc}}$.

(iii) *If* $u \in C^1(\Omega, \mathbb{R}^n)$*, then* $\mathbf{E}[b](u,u) = 0$ *if and only if* $\mathbf{E}[b](u,v) = 0$ *for all* $v \in C^1(\Omega, \mathbb{R}^n)$.

Proof. *(i)* Let $u \in C^1(\Omega, \mathbb{R}^n)$. By a straightforward calculation we verify that

$$\mathbf{E}[b](u,u) \tag{2.2}$$
$$= (b+1)\sum_{i=1}^n (\partial_i u_i)^2 + (b-1)\sum_{\substack{i,j=1\\ i\neq j}}^n (\partial_i u_i)(\partial_j u_j) + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^n (\partial_i u_j + \partial_j u_i)^2.$$

We prove separately the statement for $b \geq 1$ and $b < 1$. Let $b \geq 1$. By (2.2), we deduce that $\mathbf{E}[b](u,u) = (b-1)(\operatorname{div} u)^2 + (1/2)|Du + (Du)^t|^2$, which immediately implies statement (i) with $c \equiv 1/2$. Now let $b < 1$. We observe that

$$\sum_{\substack{i,j=1\\ i\neq j}}^n (\partial_i u_i)(\partial_j u_j) \leq \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^n \left[(\partial_i u_i)^2 + (\partial_j u_j)^2\right] = (n-1)\sum_{i=1}^n (\partial_i u_i)^2.$$

Then, we deduce by (2.2) the following inequality,

$$\mathbf{E}[b](u,u) \geq n(b-1+2/n)\sum_{i=1}^n (\partial_i u_i)^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^n (\partial_i u_j + \partial_j u_i)^2.$$

If $b > 1 - 2/n$ the expression in the right hand side is greater or equal than $c|Du + (Du)^t|^2$, with $c \equiv (n/4)(b-1+2/n)$.

(ii) Let $b > 1 - 2/n$. Let $u \in C^2(\Omega, \mathbb{R}^n)$ such that $\mathbf{E}[b](u,u) = 0$. By statement (i) we deduce that $\partial_i u_i = 0$ and $\partial_i u_j = -\partial_j u_i$ for all $i,j = 1,\ldots,n$. It follows that $\partial^2_{ij} u_k = 0$ for all $i,j,k = 1,\ldots,n$. In particular, for every connected component Ω' of Ω there exist a skew symmetric matrix $A \in \mathrm{Skew}(n, \mathbb{R})$ and a constant vector $b \in \mathbb{R}^n$ such that $u(x) = Ax + b$ for all $x \in \Omega'$. The proof of statement (ii) can now be easily completed. The proof of (iii) is straightforward. □

In the following Theorem 2.3 we introduce a Green like formula for the operator $\mathbf{L}[b]$.

Theorem 2.3. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* C^1*, and let* ν *be the outward unit normal to* $\partial\Omega$*. Then the following statements hold.*

(i) *Let* $u \in C^1(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^2(\Omega, \mathbb{R}^n)$*, and let* $v \in C(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n)$*, and let* $\mathbf{L}[b]u \in L^2(\Omega, \mathbb{R}^n)$*. Then*

$$\int_{\partial\Omega} [T(b, Du)_{|\partial\Omega}\nu] \cdot v_{|\partial\Omega}\, d\sigma = \int_\Omega (\mathbf{L}[b]u) \cdot v + \mathbf{E}[b](u,v)\, dx. \tag{2.3}$$

(ii) *Let* $u \in C^1(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \mathrm{cl}\Omega, \mathbb{R}^n)$, *and let* $v \in C(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \mathrm{cl}\Omega, \mathbb{R}^n)$, *and let* $\mathbf{L}[b]u \in L^2(\mathbb{R}^n \setminus \mathrm{cl}\Omega, \mathbb{R}^n)$, *and let* $\nu_{R\mathbb{B}_n}$ *be the outward unit normal to the boundary of* $R\mathbb{B}_n$, *and assume that*

$$\lim_{R\to\infty} \int_{\partial R\mathbb{B}_n} [T(b, Du)|_{\partial R\mathbb{B}_n} \nu_{R\mathbb{B}_n}] \cdot v|_{\partial R\mathbb{B}_n} \, d\sigma = 0. \tag{2.4}$$

Then

$$\int_{\partial\Omega} [T(b, Du)|_{\partial\Omega}\nu] \cdot v|_{\partial\Omega} \, d\sigma = -\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} (\mathbf{L}[b]u) \cdot v + \mathbf{E}[b](u, v) \, dx. \tag{2.5}$$

Proof. *(i)* We note that we have $T(b, Du)v \in C(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n)$, and $\mathrm{div}\,(T(b, Du)v) = (\mathbf{L}[b]u)\cdot v + \mathbf{E}[b](u, v) \in L^1(\Omega)$. Then, by applying the Ostrogradski Formula, see Dautray and Lions [7, Chap. II §1.3], equation (2.3) follows.

(ii) Let $R > 0$ and let $\mathrm{cl}\Omega \subset R\mathbb{B}_n$. Equation (2.5) follows by applying statement (i) in the open bounded set $R\mathbb{B}_n \setminus \mathrm{cl}\Omega$ and by letting $R \to +\infty$. □

Now we can prove the following.

Theorem 2.4. *Let* $b > 1 - 2/n$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* C^1. *Then the following statements hold.*

(i) *Let* $u \in C^1(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^2(\Omega, \mathbb{R}^n)$ *such that* $\mathbf{L}[b]u = 0$ *in* Ω. *Then we have* $u|_{\partial\Omega} = 0$ *if and only if* $u = 0$, *and we have* $T(b, Du)|_{\partial\Omega}\nu = 0$ *if and only if* $u \in \mathcal{R}_{\Omega,\mathrm{loc}}$.

(ii) *Let* $u \in C^1(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \mathrm{cl}\Omega, \mathbb{R}^n)$ *such that* $\mathbf{L}[b]u = 0$ *in* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, *satisfy equation 2.4 with* $v = u$. *Then we have* $u|_{\partial\Omega} = 0$ *if and only if* $u = 0$, *and we have* $T(b, Du)|_{\partial\Omega}\nu = 0$ *if and only if* $u \in \mathcal{R}_{\mathbb{R}^n\setminus\mathrm{cl}\Omega,\mathrm{loc}}$ *and the restriction of* u *to the unbounded connected component of* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ *equals* 0.

Proof. The sufficiency is in each case a straightforward verification. To prove the necessity we exploit Proposition 2.2 and Theorem 2.3. Indeed if u satisfies the assumption of either statement (i) or statement (ii) then by means of Theorem 2.3 we deduce that $\mathbf{E}[b](u, u) = 0$. By Proposition 2.2 it follows that either $u \in \mathcal{R}_{\Omega,\mathrm{loc}}$ or $u \in \mathcal{R}_{\mathbb{R}^n\setminus\mathrm{cl}\Omega,\mathrm{loc}}$. Then by some elementary remarks we can conclude the proof. □

2.1.2 The elastic layer potentials

A fundamental solution of the operator $\mathbf{L}[b]$ is delivered by the function $S_L[(b, 0)]$ introduced in subsection 1.3.2. There, we have seen that $S_L[(b, 0)]$ can be expressed by means of a fundamental solution S_{Δ^2} of the bi-Laplacian operator Δ^2. Here, we find convenient to introduce an explicit expression for the fundamental solution of $\mathbf{L}[b]$. So we introduce an explicit expression for S_{Δ^2} by the following equation,

$$S_{\Delta^2}(z) \equiv \begin{cases} (-1)^{\frac{n-2}{2}}(4|\partial\mathbb{B}_n|)^{-1}|z|^{4-n}\log|z| & \text{if } n=2 \text{ or } n=4, \\ (2(n-2)(n-4)|\partial\mathbb{B}_n|)^{-1}\,|z|^{4-n} & \text{otherwise.} \end{cases}$$

We note that S_{Δ^2} satisfies is a fundamental solution in the form of Theorem 1.1. By exploiting equation (1.42) we deduce the following.

Proposition 2.5. *Let $\Gamma(\cdot,\cdot)$ be the matrix valued function of $(\mathbb{R}\setminus\{-1\})\times(\mathbb{R}^n\setminus\{0\})$ to $\mathrm{M}_{n\times n}(\mathbb{R})$ which takes a pair (b,z) to the matrix $\Gamma(b,z)$ defined by*

$$(\Gamma(b,z))_{ij} \equiv \begin{cases} \dfrac{b+2}{2(b+1)}\delta_{ij}\dfrac{\ln|z|}{2\pi} - \dfrac{b}{2(b+1)}\dfrac{1}{2\pi}\dfrac{z_i z_j}{|z|^n}\,, & \text{if } n=2\,, \\ \dfrac{b+2}{2(b+1)}\delta_{ij}\dfrac{|z|^{2-n}}{(2-n)|\partial\mathbb{B}_n|} - \dfrac{b}{2(b+1)}\dfrac{1}{|\partial\mathbb{B}_n|}\dfrac{z_i z_j}{|z|^n}\,, & \text{if } n\geq 3\,, \end{cases} \tag{2.6}$$

for all $i,j=1,\dots,n$. Then the function $\Gamma(b,\cdot)$ is a fundamental solution of the operator $\mathbf{L}[b]$ for all fixed $b\neq -1$.

Now let $m\in\mathbb{N}\setminus\{0\}$, and let $\lambda\in]0,1[$. Let Ω be a bounded and open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$, and let ν be the outward unit normal to $\partial\Omega$. Let $\mu\in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$. We denote by $v_{\partial\Omega}[b,\mu]$ and $w_{\partial\Omega}[b,\mu]$ the functions of $\mathbb{R}^n$ defined by

$$v_{\partial\Omega}[b,\mu](x) \equiv \int_{\partial\Omega}\Gamma(b,x-y)\,\mu(y)\;d\sigma_y$$

and

$$w_{\partial\Omega}[b,\mu](x) \equiv -\int_{\partial\Omega}\left[\Big(T(b,D_z\Gamma^{(i)}(b,x-y))\nu(y)\Big)\cdot\mu(y)\right]_{i=1,\dots,n} d\sigma_y \tag{2.7}$$

for all $x\in\mathbb{R}^n$, where $\Gamma^{(i)}$ is the vector valued function which coincides with the i-th column of Γ for each $i=1,\dots,n$. In the sequel we write $v[b,\mu]$ and $w[b,\mu]$ instead of $v_{\partial\Omega}[b,\mu]$ and $w_{\partial\Omega}[b,\mu]$ where no ambiguity can arise. $v[b,\mu]$, $w[b,\mu]$ are the elastic single and double layer potentials, respectively. Note that the definition of $v[b,\mu]$, $w[b,\mu]$ coincides with the definition of $v_L[b_1,b_2,\phi,\mu]$, $w_L[b_1,b_2,\phi,\mu]$ in subsection 1.3.2 if we take $b_1=b$, $b_2=0$, $\phi=\mathrm{id}_{\partial\Omega}$ and we replace the fundamental solution $S_L(b_1,b_2,\cdot)$ by $\Gamma(b,\cdot)$. In the following Propositions 2.6 and 2.7 we summarize some known facts on the layer potentials.

Proposition 2.6. *Let $b\in\mathbb{R}\setminus\{-1\}$, $m\in\mathbb{N}\setminus\{0\}$, $\lambda\in]0,1[$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mu\in C^{m-1,\lambda}(\partial\Omega,\mathbb{R}^n)$. Then the following statements hold.*

(i) $v[b,\mu]|_{\mathbb{R}^n\setminus\partial\Omega}$ *is a* C^∞ *function and* $\mathbf{L}[b]v[b,\mu]=0$ *in* $\mathbb{R}^n\setminus\partial\Omega$.

(ii) $v[b,\mu]$ *is a continuous function on* $\mathbb{R}^n$*, the restrictions* $v[b,\mu]|_{\mathrm{cl}\Omega}$ *and* $v[b,\mu]|_{\mathbb{R}^n\setminus\Omega}$ *belong to the sets* $C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{R}^n)$ *and* $C^{m,\lambda}(\mathbb{R}^n\setminus\Omega,\mathbb{R}^n)$*, respectively.*

(iii) We have

$$\begin{aligned}\lim_{t\to 0^-} T\left(b, Dv[b,\mu](x+t\nu(x))\right)\nu(x) &= -\frac{1}{2}\mu(x)+v^*_{\partial\Omega}[b,\mu](x),\\ \lim_{t\to 0^+} T\left(b, Dv[b,\mu](x+t\nu(x))\right)\nu(x) &= +\frac{1}{2}\mu(x)+v^*_{\partial\Omega}[b,\mu](x),\end{aligned}$$

for all $x\in\partial\Omega$*, where*

$$v^*_{\partial\Omega}[b,\mu](x)\equiv\int^*_{\partial\Omega}\sum_{i=1}^n\left[T(b,D_z\Gamma^{(i)}(b,x-y))\nu(x)\right]\mu_i(y)\,d\sigma_y. \tag{2.8}$$

(iv) If $n\geq 3$ *then the functions* $|x|^{n-2}v[b,\mu](x)$ *and* $|x|^{n-1}Dv[b,\mu](x)$ *of* $x\in\mathbb{R}^n$ *are bounded for* $|x|$ *in a neighborhood of* $+\infty$*. If* $n=2$ *and we assume that* $\int_{\partial\Omega}\mu\,d\sigma=0$ *then the functions* $|x|v[b,\mu](x)$ *and* $|x|^2Dv[b,\mu](x)$ *of* $x\in\mathbb{R}^n$ *are bounded for* $|x|$ *in a neighborhood of* $+\infty$*.*

Proof. Statement (i) is trivial. Statement (ii) follows by a slight modification of Theorem 1.8. To prove statement (iii) we note that, by Cialdea [6, Theorem 3] and [5, §2, IX],

$$\lim_{t\to 0^\mp} D^\beta v_{\Delta^2}[\mu](x+t\nu(x))=\mp\frac{1}{2}\nu(x)^\beta\mu+\int_{\partial\Omega}D^\beta S_{\Delta^2}(x-y)\mu(y)\,d\sigma_y,$$

for all $x\in\partial\Omega$ and for all multiindexes $\beta\in\mathbb{N}^n$ with $|\beta|=3$, where $v_{\Delta^2}[\mu]$ is the single later potential with density $\mu\in C^{m-1,\lambda}(\partial\Omega)$ corresponding to the bi-Laplace operator Δ^2. Then, by equations (1.42) and (1.43), we deduce the validity of statement (iii). Statement (iv) for $n\geq 3$ can be verified by a straightforward calculation. To prove statement (iv) for $n=2$ we note that

$$v[b,\mu](x)=\int_{\partial\Omega}(\Gamma(b,x-y)-\Gamma(b,x))\mu(y)\,d\sigma_y+\Gamma(b,x)\int_{\partial\Omega}\mu\,d\sigma\,.$$

Since $|\Gamma(b,x-y)-\Gamma(b,x)||x|$ is uniformly bounded for $y\in\partial\Omega$ and $|x|$ in a neighborhood of $+\infty$ and we have $\int_{\partial\Omega}\mu\,d\sigma=0$, we deduce that the function $|x||v[b,\mu](x)|$ is bounded for $|x|$ in a neighborhood of $+\infty$. The proof for $Dv[b,\mu]$ is similar. Indeed we have

$$|x|^2|Dv[b,\mu](x)|\leq|x|^2\int_{\partial\Omega}|D\Gamma(b,x-y)-D\Gamma(b,x)|\,|\mu(y)|\,d\sigma_y\,,$$

and we conclude by noting that $|x|^2|D\Gamma(b,x-y)-D\Gamma(b,x)|$ stays uniformly bounded for $y\in\partial\Omega$ and $|x|$ in a neighborhood of $+\infty$. □

In the sequel we find convenient the following notation. If Ω is an open and bounded subset of $\mathbb{R}^n$ of class C^1 and f and g are functions defined on Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, respectively, we denote by $f^+(x)$ and by $g^-(x)$ the limits $\lim_{t\to 0^-} f(x + t\nu(x))$ and $\lim_{t\to 0^+} g(x + t\nu(x))$, respectively, where x is a point of $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. Accordingly the limits in statement (iii) of the previous proposition can be denoted by $[T(b, Dv[b, \mu]))\nu]^+(x)$ and $[T(b, Dv[b, \mu]))\nu]^-(x)$, respectively. Now for the double layer potential we have the following.

Proposition 2.7. *Let $b \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0, 1[$, and let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$, and let $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. Then the following statements hold.*

(i) $w[b, \mu]|_{\mathbb{R}^n\setminus\partial\Omega}$ is a C^∞ function and $\mathbf{L}[b]w[b, \mu] = 0$ in $\mathbb{R}^n \setminus \partial\Omega$.

(ii) $w[b, \mu]|_\Omega$ extends to unique element $w^+[b, \mu]$ of $C^{m-1,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$ and $w[b, \mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}$ extends to unique element $w^-[b, \mu]$ of $C^{m-1,\lambda}(\mathbb{R}^n\setminus\Omega, \mathbb{R}^n)$.

(iii) On $\partial\Omega$ we have $w^+[b, \mu] = \frac{1}{2}\mu + w[b, \mu]$ and $w^-[b, \mu] = -\frac{1}{2}\mu + w[b, \mu]$.

(iv) The functions $|x|^{n-1}|w[b, \mu](x)|$ and $|x|^n|Dw[b, \mu](x)|$ of $x \in \mathbb{R}^n$ are bounded for $|x|$ in a neighborhood of $+\infty$.

(v) If $\mu \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ then $w[b, \mu]|_\Omega$ extends uniquely to an element $w^+[b, \mu]$ of $C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$ and $w[b, \mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}$ extends uniquely to an element $w^-[b, \mu]$ of $C^{m,\lambda}(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n)$.

Proof. The proof of statement (i) is trivial. Statement (ii) follows by a slight modification of Theorem 1.8. To prove statement (iii) we note that

$$\begin{aligned}
&\lim_{t\to 0^\pm} w_{\partial\Omega}[b, \mu](x + t\nu(x)) \\
&\quad = - \lim_{t\to 0^\pm} \int_{\partial\Omega} \Big[\Big(T(b, D\Gamma^{(i)}(b, x + t\nu(x) - y))\nu(x)\Big) \cdot \mu(y)\Big]_{i=1,\dots,n} d\sigma_y \\
&\qquad - \lim_{t\to 0^\pm} \int_{\partial\Omega} \Big[\Big(T(b, D\Gamma^{(i)}(b, x + t\nu(x) - y)) \\
&\qquad\qquad \cdot(\nu(y) - \nu(x))\Big) \cdot \mu(y)\Big]_{i=1,\dots,n} d\sigma_y,
\end{aligned}$$

for all $x \in \partial\Omega$. Then, we can investigate the first limit in the right hand side by arguing as in the proof of statement (iii) of Proposition 2.6, and we can show that the argument of the second limit is a continuous function of t by exploiting the Vitali Convergence Theorem. Statement (iii) follows. Statement (iv) can be verified by a straightforward calculation. To prove statements (v) we note that, by equation (1.45) and Lemma 1.41,

$$w[b, \mu] = w_\Delta[\mu] + \sum_{j=1}^{n} v_\Delta[\mathscr{M}_{ij}(\nu)\mu_j]_{i=1,\dots,n} - 2\sum_{j=1}^{n} v[b, \mathscr{M}_{ij}(\nu)\mu]_{i=1,\dots,n},$$

where v_Δ and w_Δ are the single and double layer potentials corresponding to the Laplace operator Δ. For v_Δ and w_Δ we have the equation $\partial_i w_\Delta[\mu_j] = -\sum_{k=1}^n \partial_k v_\Delta[\mathscr{M}_{ik}(\nu)\mu_j]$ for all $i, j = 1, \dots, n$ (see Kupradze, Gegelia, Basheleĭshvili and Burchuladze [18, Chapter V, §6] and Lanza de Cristoforis and Rossi [27, Theorem 3.1].) Then by exploiting Theorem 1.8 and by the previous Proposition 2.6 statements (v) follows immediately. □

Moreover we have the representation formula (2.9). For a proof in $\mathbb{R}^3$ see Kupradze *et al.* [18, Chapter V, Theorem 1.6], the proof for $n \geq 2$ can be deduced by a straightforward modification.

Theorem 2.8. *Let $b \in \mathbb{R} \setminus \{-1\}$, and let $\lambda \in]0,1[$, and let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{1,\lambda}$, and let $u \in C^{1,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^2(\Omega, \mathbb{R}^n)$ such that $\mathbf{L}[b]u = 0$. Then*

$$u(x) = w[b, u|_{\partial\Omega}](x) - v[b, T(b, Du)|_{\partial\Omega}\nu](x) \quad \forall\, x \in \Omega, \tag{2.9}$$

and the right hand side of (2.9) *vanishes for $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$.*

2.1.3 Boundary integral operators on $L^2(\partial\Omega, \mathbb{R}^n)$

Definition 2.9. *Let $\lambda \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\lambda}$. We set*

$$\mathbf{K}_{\partial\Omega}[b,\mu] \equiv \frac{1}{2}\mu + w_{\partial\Omega}[b,\mu]|_{\partial\Omega}, \qquad \mathbf{K}^*_{\partial\Omega}[b,\mu] \equiv \frac{1}{2}\mu + v^*_{\partial\Omega}[b,\mu]|_{\partial\Omega},$$
$$\mathbf{H}_{\partial\Omega}[b,\mu] \equiv -\frac{1}{2}\mu + v^*_{\partial\Omega}[b,\mu]|_{\partial\Omega}, \qquad \mathbf{H}^*_{\partial\Omega}[b,\mu] \equiv -\frac{1}{2}\mu + w_{\partial\Omega}[b,\mu]|_{\partial\Omega},$$

for all $b \in \mathbb{R} \setminus \{-1\}$ and all $\mu \in L^2(\partial\Omega, \mathbb{R}^n)$, where $w_{\partial\Omega}[b,\mu]$ and $v^*_{\partial\Omega}[b,\mu]$ are defined by (2.7) *and* (2.8)*, respectively. We write $\mathbf{K}$, $\mathbf{K}^*$, $\mathbf{H}$, $\mathbf{H}^*$ instead of $\mathbf{K}_{\partial\Omega}$, $\mathbf{K}^*_{\partial\Omega}$, $\mathbf{H}_{\partial\Omega}$, $\mathbf{H}^*_{\partial\Omega}$ where no ambiguity can arise.*

Proposition 2.10. *With the same notation of Definition 2.9, the operators $\mathbf{K}[b,\cdot]$, $\mathbf{K}^*[b,\cdot]$, $\mathbf{H}[b,\cdot]$, $\mathbf{H}^*[b,\cdot]$ are bounded on $L^2(\partial\Omega, \mathbb{R}^n)$.*

Proof. We note that $(\partial_{z_k}\Gamma(b,z))_{ij}$ is a singular integral kernel of the form $f(z)|z|^{1-n}$, where f is an odd, homogeneous of degree zero, real analytic function on $\mathbb{R}^n \setminus \{0\}$. Therefore the map which takes an element μ of $L^2(\partial\Omega)$ to $\int^*_{\partial\Omega}(\partial_{x_k}\Gamma(b, x-y))_{ij}\mu(y)\, d\sigma_y$ is a bounded operator on $L^2(\partial\Omega)$ (cf. Mikhlin [33, §27].) By equations (2.7) and (2.8) our proposition follows. □

Moreover we have the following.

Theorem 2.11. *With the same notation of Definition 2.9, $\mathbf{K}[b,\cdot]$, $\mathbf{K}^*[b,\cdot]$, $\mathbf{H}[b,\cdot]$, $\mathbf{H}^*[b,\cdot]$ are Fredholm operators of index 0 on $L^2(\partial\Omega, \mathbb{R}^n)$ for all fixed $b \in \mathbb{R} \setminus \{0, -1, -2\}$.*

Proof. For the sake of brevity we confine to consider $\mathbf{K}[b,\cdot]$, the proof for the other operators is very similar. We show that $\mathbf{K}[b,\cdot]$ have an Hermitian and invertible symbolic matrix. Then the theorem will follow by Mikhlin [33, Corollary to Theorem 4.40]. For the definition of the symbolic matrix of a singular integral operator defined on $\partial\Omega$ we refer to Mikhlin [33, §40] and Seeley [41] (see also Mikhlin and Prössdorf [34, Chapter XIII, §2].) Here we recall that the symbolic matrix of a singular integral operator defined on $\partial\Omega$ is a function of $\partial\Omega \times \partial\mathbb{B}_{n-1}$ to $M_{n\times n}(\mathbb{R})$. We denote such a function by $\sigma_{\mathbf{K}[b,\cdot]}(x,\theta)$, for all $x \in \partial\Omega$, $\theta \in \partial\mathbb{B}_{n-1}$. So, our claim is that $\sigma_{\mathbf{K}[b,\cdot]}(x,\theta)$ is an invertible Hermitian matrix for all $x \in \partial\Omega$ and $\theta \in \partial\mathbb{B}_{n-1}$.

By equation (2.6) and by a straightforward calculation we can see that

$$\begin{aligned}
&[T(b, D\Gamma^{(i)}(b,z))\nu]_j \qquad (2.10)\\
&\quad = \frac{1}{b+1}\frac{1}{|\partial\mathbb{B}_n|}\left(\frac{\nu_i z_j - \nu_j z_i}{|z|^n} + \left(\delta_{ij} + nb\frac{z_i z_j}{|z|^2}\right)\frac{\nu\cdot z}{|z|^n}\right)
\end{aligned}$$

for all $i,j = 1,\dots,n$, and all vectors $\nu \in \mathbb{R}^n$, and all $z \in \mathbb{R}^n \setminus \{0\}$. Now, let ν be the outward unit normal to the boundary of Ω and let $k_{ij}(x,z)$ and $a_{ij}(x,z)$ be defined by

$$k_{ij}(x,z) \equiv \frac{\nu_i(x) z_j - \nu_j(x) z_i}{|z|^n}, \quad a_{ij}(x,z) \equiv \left(\delta_{ij} + nb\frac{z_i z_j}{|z|^2}\right)\frac{\nu(x)\cdot z}{|z|^n}$$

for every $i,j = 1,\dots,n$, and all $x \in \partial\Omega$, and all $z \in \mathbb{R}^n \setminus \{0\}$. One easily verifies that $k_{ij}(x,x-y) - k_{ij}(y,x-y) = O(|x-y|^{n-1-\lambda})$ as $y \to x$, for all $x,y \in \partial\Omega$. Moreover $a_{ij}(y,x-y) = O(|x-y|^{n-1-\lambda})$ and thus, if we set $b_{ij}(x,y) \equiv k_{ij}(y,x-y) - k_{ij}(x,x-y) + a_{ij}(y,x-y)$, we have

$$\begin{aligned}
\mathbf{K}[b,\mu](x) = \frac{1}{2}\mu(x) - \frac{1}{|\partial\mathbb{B}_n|}\frac{1}{b+1}\int^*_{\partial\Omega} k(x,x-y)\mu(y)\, d\sigma_y \qquad (2.11)\\
-\frac{1}{|\partial\mathbb{B}_n|}\frac{1}{b+1}\int_{\partial\Omega} b(x,x-y)\mu(y)\, d\sigma_y, \quad \forall x \in \partial\Omega,
\end{aligned}$$

for all $\mu \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, where we intend

$$\int^*_{\partial\Omega} k(x,x-y)\mu(y)\, d\sigma_y \equiv \lim_{\epsilon\to 0^+}\int_{\partial\Omega\setminus(x+\epsilon\mathbb{B}_n)} k(x,x-y)\mu(y)\, d\sigma_y, \quad \forall\, x \in \partial\Omega.$$

We now fix a point x_0 of $\partial\Omega$. To calculate $\sigma_{\mathbf{K}[b,\cdot]}(x_0,\theta)$ we have to introduce a local parametrization of $\partial\Omega$ in a neighborhood of x_0. To do so we perform an orthogonal coordinate transformation in $\mathbb{R}^n$ such that $x_0 = (0,\dots,0)$, and all vectors of the form $(\xi,0)$, with $\xi \equiv (\xi_1,\dots,\xi_{n-1})$, are tangent to $\partial\Omega$ in x_0, and the outward normal to $\partial\Omega$ in x_0 is delivered by $(0,\dots,0,1)$. By a straightforward calculation, we verify that equation (2.11) still holds with respect to the new coordinate system.

Moreover, by our assumptions on Ω, there exist $r, \delta > 0$ and a function $\gamma \in C^{1,\lambda}(\mathrm{cl}(r\mathbb{B}_{n-1}))$ such that $-\delta < \gamma < \delta$ and the intersection $\Omega \cap (r\mathbb{B}_{n-1} \times]-\delta, \delta[)$ coincides with the subgraph of the function γ, namely with the set $\{(\xi, \xi_n) \in \mathbb{R}^n : |\xi| < r, \ -\delta < \xi_n < \gamma(\xi)\}$. In particular, the map of $r\mathbb{B}_{n-1}$ to $\partial\Omega \cap (r\mathbb{B}_{n-1} \times]-\delta, \delta[)$ which takes ξ to $(\xi, \gamma(\xi))$ is a local parametrization of $\partial\Omega$ in a neighborhood of x_0.

We recall that γ admits an extension to a function of class $C^{1,\lambda}$ defined on the whole of $\mathbb{R}^{n-1}$ (cf., *e.g.*, Troianiello [44, §1.2.2].) We still denote by γ such an extension. With this notation, we introduce the function $\widetilde{k}(\cdot,\cdot)$ of $\mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus \{0\})$ to $M_{n\times n}(\mathbb{R})$ which takes (ξ, ζ) to the $n \times n$ matrix

$$\frac{1}{|\zeta|^n} \begin{pmatrix} \zeta D\gamma(\xi) - (\zeta D\gamma(\xi))^t & -\zeta + (D\gamma(\xi))^t (D\gamma(\xi))\zeta \\ \zeta^t - (D\gamma(\xi))(D\gamma(\xi))\zeta & 0 \end{pmatrix},$$

where we intend

$$\zeta D\gamma(\xi) = (\zeta_i \partial_j \gamma(\xi))_{i,j=1,\dots,n-1}, \quad (D\gamma(\xi))\zeta = \sum_{i=1}^{n-1} (\partial_i \gamma(\xi))\zeta_i,$$

as accordingly to the fact that ζ is a column vector and $D\gamma(\xi)$ a row. By exploiting the equality $\nu(\xi, \gamma(\xi)) = (1 + |D\gamma(\xi)|^2)^{-1/2}(-D\gamma(\xi), 1)$, for all $\xi \in r\mathbb{B}_{n-1}$, we verify that

$$k_{ij}((\xi, \gamma(\xi)), (\xi - \eta, \gamma(\xi) - \gamma(\eta))) - \frac{\widetilde{k}_{ij}(\xi, \xi - \eta)}{(1 + |D\gamma(\xi)|^2)^{1/2}} = O(|\xi - \eta|^{n-1-\lambda}),$$

as $\eta \to \xi$, for all $\xi, \eta \in r\mathbb{B}_{n-1}$, $i, j = 1, \dots, n$.

Now, let μ be a function of $C^{0,\lambda}(\partial\Omega, \mathbb{R}^n)$ with support contained in $\partial\Omega \cap (r\mathbb{B}_{n-1} \times]-\delta, \delta[)$. We denote by $\widetilde{\mu}$ the function of $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$ defined by, $\widetilde{\mu}(\xi) \equiv \mu(\xi, \gamma(\xi))$, for all $\xi \in r\mathbb{B}_{n-1}$, $\widetilde{\mu}(\xi) \equiv 0$, for all $\xi \in \mathbb{R}^{n-1} \setminus r\mathbb{B}_{n-1}$. Then, with respect to the new coordinate system, we obtain by (2.11) the following equation,

$$\begin{aligned} &\mathbf{K}[b, \mu](\xi, \gamma(\xi)) \qquad (2.12) \\ &= \frac{1}{2}\widetilde{\mu}(\xi) - \frac{1}{|\partial\mathbb{B}_n|}\frac{1}{b+1} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^{n-1} \setminus B(\xi, \epsilon)} \frac{\widetilde{k}(\xi, \xi - \eta)\widetilde{\mu}(\eta)}{(1 + |D\gamma(\xi)|^2)^{1/2}} (1 + |D\gamma(\eta)|^2)^{1/2} \, d\eta \\ &\quad + \int_{\mathbb{R}^{n-1}} c(\xi, \eta)\widetilde{\mu}(\eta) \, d\eta, \end{aligned}$$

for all $\xi \in r\mathbb{B}_{n-1}$, where $c(\cdot,\cdot)$ is a weakly singular kernel and $B(\xi, \epsilon) \equiv \{\eta \in \mathbb{R}^{n-1} : |\xi - \eta|^2 + (\gamma(\xi) - \gamma(\eta))^2 < \epsilon^2\}$. Since we have $(1 + |D\gamma(\xi)|^2)^{1/2} - (1 + |D\gamma(\eta)|^2)^{1/2} = O(|\xi - \eta|^\lambda)$, the right hand side of (2.12) equals

$$\begin{aligned} &\frac{1}{2}\widetilde{\mu}(\xi) - \frac{1}{|\partial\mathbb{B}_n|}\frac{1}{b+1} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^{n-1} \setminus B(\xi, \epsilon)} \widetilde{k}(\xi, \xi - \eta)\widetilde{\mu}(\eta) \, d\eta \qquad (2.13) \\ &\quad + \int_{\mathbb{R}^{n-1}} d(\xi, \eta)\widetilde{\mu}(\eta) \, d\eta, \end{aligned}$$

for all $\xi \in \mathbb{R}^{n-1}$, where $d(\cdot,\cdot)$ is a weakly singular kernel. Moreover, the limit in (2.13) equals

$$\int^*_{\mathbb{R}^{n-1}} \widetilde{k}(\xi,\xi-\eta)\widetilde{\mu}(\eta)\, d\eta + \left(\int_{\partial\mathbb{B}_{n-1}} \widetilde{k}(\xi,\theta) \log \beta(\xi,\theta)\, d\sigma_\theta \right) \widetilde{\mu}(\xi), \tag{2.14}$$

where

$$\beta(\xi,\theta) \equiv \lim_{\epsilon\to 0^+} \frac{\epsilon}{(\epsilon^2 + (\gamma(\xi)-\gamma(\xi+\epsilon\theta))^2)^{1/2}} = \frac{1}{(1+((D\gamma(\xi))\theta)^2)^{1/2}},$$

for all $\xi \in \mathbb{R}^{n-1}$ and $\theta \in \partial\mathbb{B}_{n-1}$ (cf. Mikhlin and Prössdorf [34, Chapter IX, §1.3].) Then we easily verify that $\widetilde{k}(\xi,\theta)\log\beta(\xi,\theta)$ is a odd function of $\theta \in \partial\mathbb{B}_{n-1}$ for all fixed $\xi \in \mathbb{R}^{n-1}$. So, the second integral in (2.14) vanishes.

Now, by (2.12), we obtain

$$\mathbf{K}[b,\mu](\xi,\gamma(\xi)) = \widetilde{\mathbf{K}}[b,\widetilde{\mu}](\xi) + \widetilde{\mathbf{B}}[b,\widetilde{\mu}](\xi), \quad \forall\, \xi \in r\mathbb{B}_{n-1},$$

where $\widetilde{\mathbf{B}}[b,\cdot]$ is a weakly singular integral operator (cf. Mikhlin and Prössdorf [34, Chapter VIII, §3]) and $\widetilde{\mathbf{K}}[b,\cdot]$ is the singular integral operator on $L^2(\mathbb{R}^{n-1},\mathbb{R}^n)$ which takes ϕ to the function $\widetilde{\mathbf{K}}[b,\phi]$ defined by

$$\widetilde{\mathbf{K}}[b,\phi](\xi) \equiv \frac{1}{2}\phi(\xi) - \int^*_{\mathbb{R}^{n-1}} \widetilde{k}(\xi,\xi-\eta)\phi(\eta)\, d\eta, \quad \forall\, \xi \in \mathbb{R}^{n-1}.$$

Then, by definition, we have $\sigma_{\mathbf{K}[b,\cdot]}(x,\theta) \equiv \sigma_{\widetilde{\mathbf{K}}[b,\cdot]}(\xi,\theta)$, for all $\theta \in \partial\mathbb{B}_{n-1}$ and for all $x = (\xi,\gamma(\xi)) \in \partial\Omega \cap (r\mathbb{B}_{n-1}\times]-\delta,\delta[)$ (cf. Seeley [41], Mikhlin and Prössdorf [34, Chapter XIII, §2].) By exploiting Mikhlin and Prössdorf [34, Chapter X, §2.1], we verify that

$$\begin{aligned} &\sigma_{\widetilde{\mathbf{K}}[b,\cdot]}(\xi,\theta) \\ &\quad = \frac{1}{2}\mathbf{1}_n - \frac{i}{2(b+1)} \begin{pmatrix} \theta D\gamma(\xi) - (\theta D\gamma(\xi))^t & -\theta + (D\gamma(\xi))^t (D\gamma(\xi))\theta \\ \theta^t - (D\gamma(\xi))(D\gamma(\xi))\theta & 0 \end{pmatrix}, \end{aligned}$$

for all $\xi \in \mathbb{R}^{n-1}$, $\theta \in \partial\mathbb{B}_{n-1}$. In particular, for $\xi = 0$, we have

$$\sigma_{\widetilde{\mathbf{K}}[b,\cdot]}(0,\theta) = \frac{1}{2} \begin{pmatrix} \mathbf{1}_{n-1} & i(b+1)^{-1}\theta^t \\ -i(b+1)^{-1}\theta & 1 \end{pmatrix}.$$

One easily verifies that $\sigma_{\mathbf{K}[b,\cdot]}(x_0,\theta) = \sigma_{\widetilde{\mathbf{K}}[b,\cdot]}(0,\theta)$ is an Hermitian matrix and that $\det \sigma_{\mathbf{K}[b,\cdot]}(x_0,\theta) = (1/2^n)b(b+2)(b+1)^{-2}$, for all $\theta \in \partial\mathbb{B}_{n-1}$. So that $\det \sigma_{\mathbf{K}[b,\cdot]}(x_0,\theta) \neq 0$ for all $b \in \mathbb{R} \setminus \{0,-1,-2\}$. Since x_0 was an arbitrary point of $\partial\Omega$ and the symbolic matrix does not depend on the choice of the local parametrization (cf. Seeley [41]), we deduce that $\sigma_{\mathbf{K}[b,\cdot]}(x,\theta)$ is an invertible Hermitian matrix for all $x \in \partial\Omega$, $\theta \in \partial\mathbb{B}_{n-1}$. The proof is completed. □

Theorem 2.12. *With the same notation of Definition 2.9, the operator* $\mathbf{K}^*[b,\cdot]$ *is the adjoint to* $\mathbf{K}[b,\cdot]$ *and the operator* $\mathbf{H}^*[b,\cdot]$ *is the adjoint to* $\mathbf{H}[b,\cdot]$ *for all* $b \in \mathbb{R} \setminus \{-1\}$.

Proof. The proof follows by the properties of the composition of ordinary and singular integrals (see Mikhlin [33, §9]) and by equations (2.7) and (2.8). □

2.1.4 Kernels of the boundary integral operators

If the density μ is of class $C^{0,\lambda}$, then $\mathbf{K}[b,\mu]$, $\mathbf{K}^*[b,\mu]$, $\mathbf{H}[b,\mu]$, $\mathbf{H}^*[b,\mu]$ are related to the boundary values of the layer potentials $v[b,\mu]$ and $w[b,\mu]$ as it is stated in the following proposition, which we immediately deduce by Proposition 2.6 and Definition 2.9.

Proposition 2.13. *Let* $b \in \mathbb{R} \setminus \{-1\}$, $\lambda \in]0,1[$. *Let* Ω *be an open and bounded subset of* $\mathbb{R}^n$ *of class* $C^{1,\lambda}$. *Let* $\mu \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$. *Then* $\mathbf{K}[b,\mu] = w^+[b,\mu]|_{\partial\Omega}$, $\mathbf{K}^*[b,\mu] = [T(b,Dv[b,\mu]))\nu]^-$, $\mathbf{H}[b,\mu] = [T(b,Dv[b,\mu]))\nu]^+$, $\mathbf{H}^*[b,\mu] = w^-[b,\mu]|_{\partial\Omega}$.

Moreover, we have the following Lemma 2.14, which states that every $\mu \in L^2(\partial\Omega,\mathbb{R}^n)$ such that $\mathbf{K}[b,\mu] \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$ is indeed a function of $C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, and similarly for $\mathbf{K}^*$, $\mathbf{H}$ and $\mathbf{H}^*$.

Lemma 2.14. *Let* $\lambda \in]0,1[$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\lambda}$. *Let* μ *be a function of* $L^2(\partial\Omega,\mathbb{R}^n)$. *Let* $b \in \mathbb{R} \setminus \{0,-1,-2\}$. *If either* $\mathbf{K}[b,\mu] \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, *or* $\mathbf{K}^*[b,\mu] \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, *or* $\mathbf{H}[b,\mu] \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, *or* $\mathbf{H}^*[b,\mu] \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$, *then* $\mu \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$.

Proof. The lemma follows by Mikhlin and Prössdorf [34, Chapter XIII, Theorem 7.1]. Indeed, by arguing as in the proof of Theorem 2.11, we can verify that the assumptions of [34, Chapter XIII, Theorem 7.1] are satisfied. We note that [34, Chapter XIII, Theorem 7.1] is concerned with a singular integral operator of $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$, while $\mathbf{K}[b,\cdot]$, $\mathbf{K}^*[b,\cdot]$, $\mathbf{H}[b,\cdot]$, $\mathbf{H}^*[b,\cdot]$ are singular integral matrix operators of $L^2(\partial\Omega,\mathbb{R}^n)$ to $L^2(\partial\Omega,\mathbb{R}^n)$. By exploiting Ševčenko [42], one can verify that the statement of [34, Chapter XIII, Theorem 7.1] extends, with the obvious modifications, to the case of singular integral matrix operators. □

Now we are ready to give a detailed description of the kernels $\mathrm{Ker}\mathbf{H}[b,\cdot]$ and $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$ of the operators $\mathbf{H}[b,\cdot]$ and $\mathbf{H}^*[b,\cdot]$. This is the purpose of the next Theorem 2.16. In the following Theorem 2.17, we consider the kernels $\mathrm{Ker}\mathbf{K}[b,\cdot]$ and $\mathrm{Ker}\mathbf{K}^*[b,\cdot]$ of $\mathbf{K}[b,\cdot]$ and $\mathbf{K}^*[b,\cdot]$. Also in this case we give a detailed description. First we introduce some more notation.

Definition 2.15. *Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathbb{R}^n_\Omega$ the set of the vector valued constant functions on Ω. We denote by $\mathbb{R}^n_{\Omega,\mathrm{loc}}$ the set of the functions on Ω which are constant on the connected components of Ω. We denote by $(\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$ the set of the functions on $\partial\Omega$ which are trace on $\partial\Omega$ of functions of $\mathbb{R}^n_{\Omega,\mathrm{loc}}$.*

Theorem 2.16. *Let $b > 1 - 2/n$ and $\lambda \in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\lambda}$. Let $\mathbf{H}[b,\cdot]$ and $\mathbf{H}^*[b,\cdot]$ be the operators on $L^2(\partial\Omega,\mathbb{R}^n)$ introduced in Definition 2.9. We denote by $(\mathrm{Ker}\mathbf{H}[b,\cdot])_0$ the subspace of $\mathrm{Ker}\mathbf{H}[b,\cdot]$ of the functions μ such that $\int_{\partial\Omega'} \mu\, d\sigma = 0$ for every connected component Ω' of Ω. Then the following statements hold.*

(i) $v[b,\mu]|_{\partial\Omega} \in \mathrm{Ker}\mathbf{H}^*[b,\cdot]$ *for every* $\mu \in \mathrm{Ker}\mathbf{H}[b,\cdot]$.

(ii) The map which takes μ to $v[b,\mu]|_{\partial\Omega}$ is injective on $(\mathrm{Ker}\mathbf{H}[b,\cdot])_0$.

(iii) If $n \geq 3$ the map which takes μ to $v[b,\mu]|_{\partial\Omega}$ is an isomorphism from $\mathrm{Ker}\mathbf{H}[b,\cdot]$ *to* $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$.

(iv) $\mathrm{Ker}\mathbf{H}^*[b,\cdot] = v[b,(\mathrm{Ker}\mathbf{H})_0]|_{\partial\Omega} \oplus (\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$, *the sum being direct but not necessarily orthogonal.*

(v) $\mathrm{Ker}\mathbf{H}^*[b,\cdot] = (\mathcal{R}_{\Omega,\mathrm{loc}})|_{\partial\Omega}$.

Proof. (i) By Lemma 2.14 each $\mu \in \mathrm{Ker}(\mathbf{H})$ belongs to $C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$. Therefore we have $v[b,\mu] \in C^1(\mathrm{cl}\Omega,\mathbb{R}^n) \cap C^2(\Omega,\mathbb{R}^n)$ by Proposition 2.6. Then we apply Theorem 2.8 and we obtain that

$$\begin{aligned} w[b,v[b,\phi]|_{\partial\Omega}](x) &= v\left[b,[T(b,Dv[b,\phi])\nu]^+\right](x) \\ &= v\left[b,\mathbf{H}[b,\mu]\right](x) = v\left[b,0\right](x) = 0 \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$. We recall that, by Proposition 2.13, $\mathbf{H}^*\left[b, v[b,\mu]|_{\partial\Omega}\right] = w\left[b, v[b,\mu]|_{\partial\Omega}\right]^-$ and we deduce that $\mathbf{H}^*\left[b, v[b,\mu]|_{\partial\Omega}\right] = 0$.

(ii) Now, let $\mu \in (\mathrm{Ker}\mathbf{H})_0$ such that $v[b,\mu]|_{\partial\Omega} = 0$. Then $v[b,\mu] = 0$ by Theorem 2.4 and by Proposition 2.6. Since

$$\mu = [T(b,Dv[b,\mu])\nu]^- - [T(b,Dv[b,\mu])\nu]^+, \tag{2.15}$$

it follows that $\mu = 0$, which implies our claim.

(iii) We note that $\mathrm{Ker}\mathbf{H}[b,\cdot]$ and $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$ have the same finite dimension (cf. Theorem 2.11.) So it is enough to prove that the map which takes μ to $v[b,\mu]|_{\partial\Omega}$ is injective from $\mathrm{Ker}\mathbf{H}[b,\cdot]$ to $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$. This can be seen by arguing as in the proof of (ii).

(iv) First we prove that $v[b,(\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega}$ does not contain non-zero functions of $(\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$. Let $\mu \in (\mathrm{Ker}\mathbf{H}[b,\cdot])_0$ and let $v[b,\mu]|_{\partial\Omega} = \rho|_{\partial\Omega}$ with $\rho \in \mathbb{R}^n_{\Omega,\mathrm{loc}}$. We show that this implies $\mu = 0$ and thus $v[b,\mu]|_{\partial\Omega} = 0$. We

have $[T(b, Dv[b,\mu])\nu]^+ = \mathbf{H}[b,\mu] = 0$ and therefore $\mu = [T(b, v[b,\mu])\nu]^-$ by (2.15). Then, by Proposition 2.6 and Theorem 2.3, we get

$$\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} \mathbf{E}[b](v[b,\phi], v[b,\phi])\ dx = -\int_{\partial\Omega} v[b,\mu]|_{\partial\Omega} \cdot [T(b, Dv[b,\mu])\nu]^-\ d\sigma$$
$$= -\int_{\partial\Omega} \rho|_{\partial\Omega} \cdot \mu\ d\sigma = -\sum_{i=1}^{N} \rho|_{\partial\Omega_i} \cdot \int_{\partial\Omega_i} \mu\ d\sigma = 0,$$

where $\Omega_1, \dots, \Omega_N$ are the connected components of Ω. By Theorem 2.2, the restriction $v[b,\mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}$ belongs to $\mathcal{R}_{\mathbb{R}^n\setminus\mathrm{cl}\Omega,\mathrm{loc}}$. Hence $v[b,\mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} = \rho$ by the continuity of $v[b,\mu]$ in $\mathbb{R}^n$. In particular, we have $Dv[b,\mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} = 0$ and $[T(b, v[\mu])\nu]^- = 0$, which implies $\mu = 0$.

On the other hand $(\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$ is a subspace of $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$. In fact, by means of Theorem 2.8, we deduce that $w[b, \rho|_{\partial\Omega}] = 0$ on $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ for every $\rho \in \mathbb{R}^n_{\Omega,\mathrm{loc}}$. Thus $\mathbf{H}^*[\rho|_{\partial\Omega}] = w[b,\rho|_{\partial\Omega}]^- = 0$. Moreover, $\dim \mathrm{Ker}\mathbf{H}[b,\cdot] = \dim \mathrm{Ker}\mathbf{H}^*[b,\cdot]$, for $\mathbf{H}[b,\cdot]$ is a Fredholm operator of index 0, and

$$\mathrm{codim}\, v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega} = \mathrm{codim}\,(\mathrm{Ker}\mathbf{H}[b,\cdot])_0 \leq nN,$$

where N is the number of connected components of Ω, because $v[b,\cdot]|_{\partial\Omega}$ is injective on $(\mathrm{Ker}\mathbf{H}[b,\cdot])_0$ and $(\mathrm{Ker}\mathbf{H}[b,\cdot])_0$ is defined by the vanishing of the nN linear functionals which take $\mu \in \mathrm{Ker}\mathbf{H}[b,\cdot]$ to $\int_{\partial\Omega_j} \mu_i\, d\sigma$, with $i = 1, \dots, n$ and $j = 1, \dots, N$.

So summarizing we have

$$v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega} \subset \mathrm{Ker}\mathbf{H}^*[b,\cdot], \quad (\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega} \subset \mathrm{Ker}\mathbf{H}^*[b,\cdot],$$
$$\mathrm{codim}\, v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega} \leq nN, \quad \dim\,(\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega} = nN,$$

and $v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega} \cap (\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega} = \{0\}$. We deduce that the codimension of $v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega}$ in $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$ is exactly nN and $\mathrm{Ker}\mathbf{H}^*[b,\cdot] = v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0]|_{\partial\Omega} \oplus (\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$, the sum being direct but not necessarily orthogonal.

(v) $\mathrm{Ker}\mathbf{H}[b,\cdot]^* \subset (\mathcal{R}_{\Omega,\mathrm{loc}})\,|_{\partial\Omega}$ follows by statement (iii) and Theorem 2.4. To prove the converse consider a function $\rho \in \mathcal{R}_{\Omega,\mathrm{loc}}$. Then, by Theorem 2.4, $\mathbf{L}[b]\rho = 0$ in Ω and $[T(b, D\rho)\nu]^+ = 0$ on $\partial\Omega$. By Proposition 2.7 and Theorem 2.8 it follows that $w[b,\rho|_{\partial\Omega}] = \rho$ on $\mathrm{cl}\Omega$. Then $w[b,\rho|_{\partial\Omega}]^+ = \rho|_{\partial\Omega}$ and so $\mathbf{H}^*[b,\rho|_{\partial\Omega}] = w[b,\rho|_{\partial\Omega}]^- = w[b,\rho|_{\partial\Omega}]^+ - \rho|_{\partial\Omega} = 0$. □

For $\mathrm{Ker}\mathbf{K}[b,\cdot]$ and $\mathrm{Ker}\mathbf{K}^*[b,\cdot]$ we have the following.

Theorem 2.17. *Let $b > 1 - 2/n$ and $\lambda \in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\lambda}$. Let $\mathbf{K}[b,\cdot]$ and $\mathbf{K}^*[b,\cdot]$ be the operators on $L^2(\partial\Omega,\mathbb{R}^n)$ introduced in Definition 2.9. Then the following statements hold.*

(i) The map which takes μ to $v[b,\mu]|_{\partial\Omega}$ is an isomorphism from $\mathrm{Ker}\mathbf{K}^*[b,\cdot]$ *to* $\mathrm{Ker}\mathbf{K}[b,\cdot]$.

(ii) $\mathrm{Ker}\mathbf{K}[b,\cdot]$ *coincides with the set of the functions* $\rho \in (\mathcal{R}_{\mathbb{R}^n\setminus \mathrm{cl}\Omega,\mathrm{loc}})|_{\partial\Omega}$ *which vanish on the boundary of the unbounded connected component of* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$.

Proof. Since the proof is very similar to the proof of Theorem 2.16 we omit it. We only note that both the statement of the theorem and the proof are in this case simpler. The reason is the following. One can verify that $\int_{\partial\Omega} \mu\, d\sigma = \int_{\partial\Omega} \mathbf{K}^*[b,\mu]\, d\sigma$ for all $\mu \in L^2(\partial\Omega,\mathbb{R}^n)$. Indeed, by Definition 2.9, $\mathbf{K}^*[b,\mu] = \mu + \mathbf{H}[b,\mu]$, and by Theorems 2.11, 2.12, 2.16, $\mathbf{H}[b,\mu]$ is orthogonal to each constant function on $\partial\Omega$. In particular, $\int_{\partial\Omega} \mathbf{H}[b,\mu]\, d\sigma = 0$. Then $\mu \in \mathrm{Ker}\mathbf{K}^*[b,\cdot]$ implies $\int_{\partial\Omega} \mu\, d\sigma = 0$ and therefore we don't have to distinguish between $n = 2$ and $n \geq 3$. □

Moreover we have the following Proposition 2.18 which provides a direct decomposition of the space $L^2(\partial\Omega,\mathbb{R}^n)$ in terms of $\mathrm{Ker}\mathbf{K}[b,\cdot]$, $\mathrm{Ker}\mathbf{K}^*[b,\cdot]$, $\mathrm{Ker}\mathbf{H}[b,\cdot]$, $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$ and their orthogonal spaces.

Proposition 2.18. *Let* $b > 1 - 2/n$ *and* $\lambda \in]0,1[$. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *of class* $C^{1,\lambda}$. *We have* $L^2(\partial\Omega,\mathbb{R}^n) = (\mathrm{Ker}\mathbf{H}^*[b,\cdot])^\perp \oplus \mathrm{Ker}\mathbf{H}[b,\cdot] = (\mathrm{Ker}\mathbf{K}[b,\cdot])^\perp \oplus \mathrm{Ker}\mathbf{K}^*[b,\cdot]$, *the sum being direct but not necessarily orthogonal.*

Proof. By Theorems 2.11 and 2.12, $\mathbf{H}[b,\cdot]$ and $\mathbf{H}^*[b,\cdot]$ are adjoint Fredholm operators of index 0. So $\dim \mathrm{Ker}\mathbf{H}[b,\cdot] = \mathrm{codim}\,(\mathrm{Ker}\mathbf{H}^*[b,\cdot])^\perp < +\infty$. Then to prove the first equality it is enough to show that $(\mathrm{Ker}\mathbf{H}^*[b,\cdot])^\perp \cap \mathrm{Ker}\mathbf{H}[b,\cdot] = \{0\}$.

So let $\mu \in (\mathrm{Ker}\mathbf{H}^*[b,\cdot])^\perp \cap \mathrm{Ker}\mathbf{H}[b,\cdot]$. We claim that $\mu = 0$. We note that $\mathbf{H}[b,\mu] = 0$, and by Fredholm's Alternative Theorem, there exists $\psi \in L^2(\partial\Omega,\mathbb{R}^n)$ such that $\mu = \mathbf{H}[b,\psi]$. By Lemma 2.14 it follows that both μ and ψ belong to $C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$. Then by Proposition 2.13 we have

$$[T(b,Dv[b,\mu])\nu]^+ = 0$$

and

$$[T(b,Dv[b,\psi])\nu]^+ = \mu = \mu + \mathbf{H}[b,\mu] = \mathbf{K}^*[b,\mu] = [T(b,Dv[b,\mu])\nu]^- .$$

Multiplying the first equation by $v[b,\psi]$ and the second by $v[b,\mu]$, subtracting, and integrating over $\partial\Omega$, we obtain

$$\int_{\partial\Omega} [T(b,Dv[b,\mu])\nu]^+ \cdot v[b,\psi] - [T(b,Dv[b,\psi])\nu]^+ \cdot v[b,\mu]\, d\sigma \qquad (2.16)$$
$$= \int_{\partial\Omega} [T(b,Dv[b,\mu])\nu]^- \cdot v[b,\mu]\, d\sigma.$$

Since $\mu \in (\mathrm{Ker}\mathbf{H}^*[b,0])^\perp$, μ is orthogonal to each function of $(\mathcal{R}_{\Omega,\mathrm{loc}})|_{\partial\Omega}$. In particular $\int_{\partial\Omega} \mu\, d\sigma = 0$. So, by Proposition 2.6 and Theorem 2.3, the left hand side of (2.16) equals

$$\int_{\Omega} v[b,\mu] \cdot \mathbf{L}[b]v[b,\psi] - v[b,\psi] \cdot \mathbf{L}[b]v[b,\mu] \; dx = 0,$$

while the right hand side equals

$$-\int_{\mathbb{R}^n \setminus \mathrm{cl}\Omega} \mathbf{E}[b](v[b,\mu], v[b,\mu]) \; dx.$$

We deduce that $\mathbf{E}[b](v[b,\mu], v[b,\mu]) = 0$, and thus $v[b,\mu] \in \mathcal{R}_{\mathbb{R}^n \setminus \mathrm{cl}\Omega, \mathrm{loc}}$ by Proposition 2.2. Then we have $\mu = [T(b, Dv[b,\mu])\nu]^- = 0$, which is our claim. The proof that $L^2(\partial\Omega, \mathbb{R}^n) = (\mathrm{Ker}\mathbf{K}[b,\cdot])^\perp \oplus \mathrm{Ker}\mathbf{K}^*[b,\cdot]$ is very similar and we omit it. □

2.1.5 Boundary integral operators in Schauder spaces

Theorem 2.19. *Let $b > 1 - 2/n$, $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0,1[$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mu \in L^2(\partial\Omega, \mathbb{R}^n)$. If either $\mathbf{K}^*[b,\mu] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ or $\mathbf{H}[b,\mu] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, then $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$.*

Proof. For $m = 1$ the theorem follows immediately by Lemma 2.14. So let $m \geq 2$ and assume that $\mathbf{H}[b,\mu] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. By Lemma 2.14 we have $\mu \in C^{0,\lambda}(\partial\Omega, \mathbb{R}^n)$. So, by Proposition 2.13 we have $[T(b, Dv[b,\mu])\nu]^+ = \mathbf{H}[b,\mu]$, which implies that $[T(b, Dv[b,\mu])\nu]^+ \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. Therefore, by Proposition 2.6 and by exploiting Agmon, Douglis and Nirenberg [1, Theorem 9.3], we deduce that the restriction $v[b,\mu]|_{\mathrm{cl}\Omega}$ is a function of class $C^{m,\lambda}$. Now let $R > 0$ and let $\mathrm{cl}\Omega \subset R\mathbb{B}_n$. By the continuity of $v[b,\mu]$ on $\mathbb{R}^n$ we deduce that the restriction of $v[b,\mu]$ to the boundary of $R\mathbb{B}_n \setminus \Omega$ is a function of class $C^{m,\lambda}$. Thus, by [1, Theorem 9.3], the restriction $v[b,\mu]|_{(\mathrm{cl}R\mathbb{B}_n)\setminus\Omega}$ belongs to $C^{m,\lambda}((\mathrm{cl}R\mathbb{B}_n) \setminus \Omega, \mathbb{R}^n)$. In particular the limit $[T(b, Dv[b,\mu])\nu]^-$ exists and defines a function on $\partial\Omega$ of class $C^{m-1,\lambda}$. Now, by Definition 2.9 and by Proposition 2.13, we have

$$\mu = \mathbf{K}^*[b,\mu] - \mathbf{H}[b,\mu] = [T(b, Dv[b,\mu])\nu]^- - \mathbf{H}[b,\mu].$$

It follows that $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, which is our claim. The proof for $\mathbf{K}^*$ is very similar. □

In the following Theorem 2.19 we consider the operators $\mathbf{K}$ and $\mathbf{H}^*$ and prove a statement which is similar to the statement of Theorem 2.19. For this purpose we need the following lemma.

Lemma 2.20. *Let $b > 1 - 2/n$, $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mu \in C^{0,\lambda}(\partial\Omega, \mathbb{R}^n)$. Then the following statements are equivalent.*

(i) $w[b,\mu]|_{\Omega}$ *extends to a function* $w^+[b,\mu]$ *of* $C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$.

(ii) $w[b,\mu]|_{\mathbb{R}^n \setminus \mathrm{cl}\Omega}$ *extends to a function* $w^-[b,\mu]$ *of* $C^{m,\lambda}(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n)$.

Proof. We prove that (ii) imply (i). The proof of the other direction of the lemma is very similar. If we assume that (ii) holds than we clearly have that the limit $[T(b, Dw[b,\mu])\nu]^-(x)$ exists for all $x \in \partial\Omega$ and defines a function of class $C^{m-1,\lambda}$ on $\partial\Omega$. As a first step we prove that there exists $\psi \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ such that

$$\mathbf{H}[b,\psi] = [T(b, Dw[b,\mu])\nu]^- . \tag{2.17}$$

We note that, if $\psi \in L^2(\partial\Omega, \mathbb{R}^n)$ satisfies (2.17), then $\psi \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ by Theorem 2.19. So, by the Fredholm Alternative Theorem, it will be enough to show that $[T(b, Dw[b,\mu])\nu]^-$ is orthogonal to $\mathrm{Ker}\mathbf{H}^*[b,\cdot]$ (cf. Theorem 2.19.) Now, by Theorem 2.16, we have $\mathrm{Ker}\mathbf{H}^*[b,\cdot] = v[b, (\mathrm{Ker}\mathbf{H}[b,\cdot])_0] \oplus (\mathbb{R}^n_{\Omega,\mathrm{loc}})|_{\partial\Omega}$. So we are reduced the prove that

$$\int_{\partial\Omega} (v[b,\phi] + \rho)|_{\partial\Omega} \cdot [T(b, Dw[b,\mu])\nu]^- \, d\sigma = 0 \tag{2.18}$$

for all $\phi \in (\mathrm{Ker}\mathbf{H}[b,\cdot])_0$ and all $\rho \in \mathbb{R}^n_{\Omega,\mathrm{loc}}$. By Theorem 2.3 and Proposition 2.6 the integral in (2.18) equals

$$\int_{\partial\Omega} w^-[b,\mu] \cdot [T(b, Dv[b,\phi])\nu]^- \, d\sigma. \tag{2.19}$$

Since $[T(b, Dv[b,\phi])\nu]^+ = 0$ and $\phi = [T(b, Dv[b,\phi])\nu]^- - [T(b, Dv[b,\phi])\nu]^+$ we have $[T(b, Dv[b,\phi])\nu]^- = \phi$, which in turn implies that $[T(b, Dv[b,\phi])\nu]^-$ belongs to $\mathrm{Ker}\mathbf{H}[b,\cdot]$. Moreover we have $w^-[b,\mu]|_{\partial\Omega} = \mathbf{H}^*[b,\mu]$ by Proposition 2.13. Thus $w^-[b,\mu]|_{\partial\Omega}$ is orthogonal to $\mathrm{Ker}\mathbf{H}[b,\cdot]$ by the Fredholm Alternative Theorem. In particular $w^-[b,\mu]|_{\partial\Omega}$ is orthogonal to the function $[T(b, Dv[b,\phi])\nu]^-$ and therefore the integral in (2.19) vanishes. It follows that (2.18) holds and that ψ exists.

So, let ψ be a function of $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ which satisfies (2.17). Since $(\mathrm{codim}\,\mathrm{Ker}\mathbf{H}[b,\cdot])_0 \geq n$ (see the proof of Theorem 2.16) there exists $\widetilde{\psi} \in \mathrm{Ker}\mathbf{H}[b,\cdot]$ such that $\int_{\partial\Omega} \widetilde{\psi}\, d\sigma = \int_{\partial\Omega} \psi\, d\sigma$. We set $\psi_0 \equiv \psi - \widetilde{\psi}$ and we note that $\psi_0 \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ and satisfies (2.17). Now we set

$$u \equiv v[b,\psi_0] - w[b,\mu]. \tag{2.20}$$

By (ii) and by Proposition 2.6, the restriction $u|_{\mathbb{R}^n \setminus \mathrm{cl}\Omega}$ admits an extension $u^- \in C^{m,\lambda}(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n)$. We claim that

$$(u^-|_{\partial\Omega} - \mu) \in \mathrm{Ker}\mathbf{H}^*[b,\cdot]. \tag{2.21}$$

By Propositions 2.6 and 2.13 and by equation (2.17) we deduce that

$$\begin{aligned}\psi_0 &= [T(b, Dv[b,\psi_0])\nu]^- - \mathbf{H}[b,\psi_0] \\ &= [T(b, Dv[b,\psi_0])\nu]^- - [T(b, Dw[b,\mu])\nu]^- = \big[T(b, Du^-)\nu\big]^- .\end{aligned} \tag{2.22}$$

Then, by exploiting Theorem 2.8, we have

$$\begin{aligned} u(x) = &\int_{\partial\Omega} \Gamma(b, x-y)\ \big[T(b, Du^-)\nu\big]^-(y)\, d\sigma_y \\ &- \int_{\partial R\mathbb{B}_n} \Gamma(b, x-y)\ \big[T(b, Du^-(y))\nu_{R\mathbb{B}_n}(y)\big]\, d\sigma_y \\ &+ \int_{\partial\Omega} \Big\{ u^-|_{\partial\Omega}(y) \cdot \Big[T(b, D\Gamma^{(i)}(b, x-y))\nu(y)\Big]\Big\}_{i=1,\dots,n}\, d\sigma_y \\ &- \int_{\partial R\mathbb{B}_n} \Big\{ u(y) \cdot \Big[T(b, D\Gamma^{(i)}(b, x-y))\nu_{R\mathbb{B}_n}(y)\Big]\Big\}_{i=1,\dots,n}\, d\sigma_y, \end{aligned} \tag{2.23}$$

for every $x \in R\mathbb{B}_n \setminus \mathrm{cl}\Omega$ and every $R > 0$ such that $\mathrm{cl}\Omega \subset R\mathbb{B}_n$. By arguing as in the proof of Proposition 2.6, one can verify that the second and the fourth integral term in (2.22) vanish as $R \to +\infty$. Therefore we have $u(x) = v[[T(b, Du^-)\nu]^-](x) - w[b, u^-|_{\partial\Omega}](x)$ for all $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$, which implies

$$u(x) = v[b,\psi_0](x) - w[b, u^-|_{\partial\Omega}](x), \quad \forall\, x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega, \tag{2.24}$$

by equation (2.22). Taking the difference of (2.24) and (2.20) we obtain that $w[b, u^-|_{\partial\Omega} - \mu](x) = 0$ for all $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$. By Proposition 2.13 equation (2.21) immediately follows.

Now, by (2.21) and by Theorem 2.16, there exists $\rho \in \mathcal{R}_{\Omega,\mathrm{loc}}$ such that

$$u^-|_{\partial\Omega} = \mu + \rho|_{\partial\Omega}. \tag{2.25}$$

Moreover we note that the right hand side of (2.23) vanishes for $x \in \Omega$ by Theorem 2.3. So, by letting $R \to \infty$, we deduce that

$$v\big[b, \big[T(b, Du^-)\nu\big]^-\big](x) - w\left[b, u^-|_{\partial\Omega}\right](x) = 0, \quad \forall\, x \in \Omega. \tag{2.26}$$

Now, by exploiting (2.22), (2.25) and (2.26), we obtain

$$v[b,\psi_0](x) - w[b,\mu](x) = -\int_{\partial\Omega} [T(b, D\Gamma(b, x-y))\nu(y)] \cdot \rho|_{\partial\Omega}(y)\, d\sigma_y, \tag{2.27}$$

for all $x \in \Omega$. By Theorems 2.4 and 2.8 we deduce that the integral on the right hand side of (2.27) equals $\rho(x)$. So we have

$$w[b,\mu](x) = v[b,\psi_0](x) - \rho(x), \quad \forall\, x \in \Omega, \tag{2.28}$$

which immediately implies statement (i) of our Lemma 2.20.

We also note that, by (2.28), the limit $[T(b, Dw[b,\mu])\nu]^+$ exists and equals $[T(b, Dv[b,\psi])\nu]^+$. Indeed we have $[T(b, D\rho)\nu]^+ = 0$ by Theorem 2.4. Since, by (2.17), $[T(b, Dv[b,\psi_0])\nu]^+ = \mathbf{H}[b,\psi_0] = [T(b, Dw[b,\mu])\nu]^-$, we deduce that $[T(b, Dw[b,\mu])\nu]^+ = [T(b, Dw[b,\mu])\nu]^-$. □

Theorem 2.21. *Let $b > 1-2/n$, $m \in \mathbb{N}\setminus\{0\}$, $\lambda \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mu \in L^2(\partial\Omega, \mathbb{R}^n)$. If either $\mathbf{K}[b,\mu] \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ or $\mathbf{H}^*[b,\mu] \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$, then $\mu \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$.*

Proof. Assume that $\mathbf{K}[b,\mu] \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$. By Lemma 2.14 we have $\mu \in C^{0,\lambda}(\partial\Omega, \mathbb{R}^n)$. Then, by Proposition 2.7, the restriction $w[b,\mu]|_\Omega$ extends to a function $w^+[b,\mu]$ of $C^{0,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$. Moreover $w^+[b,\mu]|_{\partial\Omega} = \mathbf{K}[b,\mu]$ by Proposition 2.13. So we have $w^+[b,\mu]|_{\partial\Omega} \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ and we deduce by Giaquinta [10, §3.4] for $m = 1$, and by Agmon, Douglis and Nirenberg [1, Theorem 9.3] for $m \geq 2$, that $w^+[b,\mu] \in C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$. Now we exploit Lemma 2.20 which implies that the restriction $w[b,\mu]|_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}$ has an extension $w^-[b,\mu] \in C^{m,\lambda}(\mathbb{R}^n \setminus \Omega, \mathbb{R}^n)$. Since we have $\mu(x) = w^+[b,\mu](x) - w^-[b,\mu](x)$ for all $x \in \partial\Omega$, we conclude that $\mu \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$. The proof for $\mathbf{H}^*$ is very similar. □

2.2 Dirichlet boundary value problem

2.2.1 Description of the problem

In the sequel we consider an open connected subset $\mathbb{A}$ of $\mathbb{R}^n$ such that $\mathbb{R}^n\setminus\mathrm{cl}\mathbb{A}$ consists of two connected components. We let the unbounded connected component of $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{A}$ be perturbed in a "regular" way, whereas the other one will display a "singular" behavior parametrized by a real coefficient ϵ. To do so, we fix a constant $m \in \mathbb{N} \setminus \{0\}$, and a constant $\lambda \in]0,1[$, and we we fix a pair of open and bounded subsets Ω^h and Ω^d of $\mathbb{R}^n$ of class $C^{m,\lambda}$, such that Ω^h, Ω^d, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^h$ and $\mathbb{R}^n \setminus \mathrm{cl}\Omega^d$ are connected. Here "h" stays for "hole" and "d" for "domain". Then we consider two functions ϕ^h and ϕ^d belonging to $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega^h}$ and to $C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega^d}$, respectively, where the sets $\mathcal{A}_{\partial\Omega^h}$ and $\mathcal{A}_{\partial\Omega^d}$ are defined as in subsection 1.5. We denote by $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ the subset of $\mathbb{R}^n$ defined by

$$\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d] \equiv \mathbb{I}[\phi^d] \setminus (\omega + \mathrm{cl}(\epsilon\mathbb{I}[\phi^s]))$$

where ω is a point of $\mathbb{R}^n$ contained in $\mathbb{I}[\phi^d]$, ϵ is a real parameter, and we assume that $(\omega + \mathrm{cl}(\epsilon\mathbb{I}[\phi^s])) \subset \mathbb{I}[\phi^d]$. We note that, with this notation, $\mathbb{A}[\omega, 0, \phi^h, \phi^d]$ is the punctured domain $\mathbb{I}[\phi^d] \setminus \{\omega\}$.

As we have just seen ω, ϵ, ϕ^h and ϕ^d are subjected to certain conditions. We denote by $\mathcal{E}^{m,\lambda}$ the set of all quadruples $(\omega, \epsilon, \phi^h, \phi^d)$ which we retain as admissible. So $\mathcal{E}^{m,\lambda}$ is the set of all quadruples $(\omega, \epsilon, \phi^h, \phi^d) \in \mathbb{R}^n \times \mathbb{R} \times$

$(C^{m,\lambda}(\partial\Omega^s,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega^h})\times(C^{m,\lambda}(\partial\Omega^r,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega^d})$ such that $\omega+\mathrm{cl}(\epsilon\mathbb{I}[\phi^h])$ is contained in $\mathbb{I}[\phi^d]$. We also find convenient to denote by $\mathcal{E}^{m,\lambda}_+$ the subset of $\mathcal{E}^{m,\lambda}$ of all quadruples $(\omega,\epsilon,\phi^h,\phi^d)$ with $\epsilon>0$. Then one verifies that the sets $\mathcal{E}^{m,\lambda}$ and $\mathcal{E}^{m,\lambda}_+$ are open subsets of the Banach space $\mathbb{R}^n\times\mathbb{R}\times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. To simplify our notation we sometimes write $\mathbf{a}$ instead of $(\omega,\epsilon,\phi^d,\phi^h)$. By applying the Jordan-Leray Separation Theorem it is easily seen that

$$\partial\mathbb{A}[\mathbf{a}]=(\omega+\epsilon\phi^h(\partial\Omega^h))\cup\phi^d(\partial\Omega^d)$$

for all $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}^{m,\lambda}_+$. Moreover $\omega+\epsilon\phi^h(\partial\Omega^h)$ and $\phi^d(\partial\Omega^d)$ are manifolds of class $C^{m,\lambda}$. It follows that $\mathbb{A}[\mathbf{a}]$ is a bounded open and connected subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ and that $\mathbb{R}^n\setminus\mathrm{cl}\mathbb{A}[\mathbf{a}]$ has two connected components, for all $\mathbf{a}\in\mathcal{E}^{m,\lambda}_+$.

Now we introduce a Dirichlet boundary value problem for the operator $\mathbf{L}[b]$ in the domain $\mathbb{A}[\mathbf{a}]$, with $\mathbf{a}\in\mathcal{E}^{m,\lambda}_+$. Let $b>1-2/n$ and let $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}^{m,\lambda}_+$. Let (g^h,g^d) belong to $C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. We consider the following system of equations,

$$\begin{cases}\mathbf{L}[b]u=0 & \text{in } \mathbb{A}[\mathbf{a}],\\ u=g^h\circ(\omega+\epsilon\phi^h)^{(-1)} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ u=g^d\circ\phi^{d(-1)} & \text{on } \phi^d(\partial\Omega^d),\end{cases}\tag{2.29}$$

It is well known that there exists a unique $u\in C^{m,\lambda}(\mathrm{cl}\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$ which satisfies system (2.29) (cf., *e.g.*, Kupradze, Gegelia, Basheleĭshvili and Burchuladze [18].) We denote by $u[b,\mathbf{a},g^h,g^d]$ such a solution. Then we investigate the behavior of $u[b,\mathbf{a},g^h,g^d]$ upon perturbation of $(b,\mathbf{a},g^h,g^d)$ around a given degenerate 7-tuple $(b_0,\omega_0,0,\phi^h_0,\phi^d_0,g^h_0,g^d_0)$ of $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$, where $\mathscr{B}\equiv\{b\in\mathbb{R}\,:\,b>1-2/n\}$.

2.2.2 Solution by means of layer potentials

In the following Theorem 2.25 we show that the solution u of problem (2.29) is delivered by a linear combination of layer potentials. In order to prove such a result we need some more notation. So, if $\mathcal{X}$ is a measure space, we consider on $L^2(\mathcal{X},\mathbb{R}^n)$ the natural structure of Hilbert space and we denote by $\langle\cdot|\cdot\rangle_{\mathcal{X}}$ the natural product on $L^2(\mathcal{X},\mathbb{R}^n)$. We write $\langle\cdot|\cdot\rangle$ instead of $\langle\cdot|\cdot\rangle_{\mathcal{X}}$ if no ambiguity can arise. Moreover, if $\mathcal{Y}$ is a closed subspace of $L^2(\mathcal{X},\mathbb{R}^n)$ we denote by $\mathbf{P}[\mathcal{Y}]\mu$ the orthogonal projection on $\mathcal{Y}$ of μ, for all $\mu\in L^2(\mathcal{X},\mathbb{R}^n)$ (cf. *e.g.* Brezis [3, Ch. V, §1].) Then we have the following Lemmas 2.22, 2.23 and 2.24.

Lemma 2.22. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}^{m,\lambda}_+$ *and* $b\in\mathscr{B}$. *Let* $F\in C^{m,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)\cap(\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot])^\perp$

and $\beta \in \mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. *Then the following boundary value problem,*

$$\begin{cases} \mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\tau] = F, \\ \mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]\tau = \beta, \end{cases} \tag{2.30}$$

has a unique solution $\tau \in C^{m-1,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$.

Proof. The existence of a solution $\tau \in L^2(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$ which satisfies the first equation of (2.30) follows by Fredholm Alternative Theorem. Moreover, by Theorem 2.19, $\tau \in C^{m-1,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$. To conclude the proof we show that $\mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]$ is an isomorphism of $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$ to $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. Since $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$ and $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$ have the same finite dimension, it is enough to show that $\mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]$ is injective. We recall that, by Proposition 2.18, $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot] \cap (\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot])^\perp = \{0\}$. It follows that $\mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]\tau_0 = 0$ implies $\tau_0 = 0$, for all $\tau_0 \in \mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. □

Lemma 2.23. *Let the notation of subsection 2.2.1 hold. Let* $b \in \mathscr{B}$ *and* $\mathbf{a} \equiv (\omega,\epsilon,\phi^h,\phi^d) \in \mathcal{E}^{m,\lambda}_+$. *Let* $\widetilde{G} \in C^{m,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n) \cap (\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}(\mathbf{a})}[b,\cdot])^\perp$. *Then the boundary value problem,*

$$\begin{cases} \mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\mu] = \widetilde{G}, \\ \mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]\mu = 0, \end{cases} \tag{2.31}$$

has a unique solution $\mu \in C^{m,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$.

Proof. The existence of $\mu \in L^2(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$ follows by Fredholm Alternative Theorem and Theorem 2.11. The uniqueness is trivial. By Theorem 2.21 we have that $\mu \in C^{m,\lambda}(\partial\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$. □

By Lemma 2.22 we deduce the following.

Lemma 2.24. *Let the notation of subsection 2.2.1 hold. Let* $b \in \mathscr{B}$ *and* $\mathbf{a} \in \mathcal{E}^{m,\lambda}_+$. *Let* $\bar{n} \equiv \dim \mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. *Let* $\{\beta^{(i)}\}_{i=1,\dots,\bar{n}}$ *be orthonormal basis of* $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. *Then there exists a basis* $\{\alpha^{(i)}\}_{i=1,\dots,\bar{n}}$ *of* $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$ *such that* $\langle\alpha^{(i)}|\beta^{(j)}\rangle = \delta_{ij}$ *for all* $i,j = 1,\dots,\bar{n}$.

Proof. Let $\alpha^{(i)}$ be the unique solution of (2.30) with $F = 0$ and $\beta = \beta^{(i)}$, for all $i = 1,\dots,\bar{n}$. Then, by Proposition 2.18, $\{\alpha^{(i)}\}_{i=1,\dots,\bar{n}}$ is a basis of $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$. Moreover, $\langle\alpha^{(i)}|\beta^{(j)}\rangle = \langle\mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]\alpha^{(i)}|\beta^{(j)}\rangle = \langle\beta^{(i)}|\beta^{(j)}\rangle = \delta_{ij}$ for all $i,j = 1,\dots,\bar{n}$. □

Now we are ready to prove the following.

Theorem 2.25. *Let the notation introduced in subsection 2.2.1 hold. Let* $(b,\mathbf{a},g^h,g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}_+ \times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. *Let* G *be the function of* $\partial\mathbb{A}[\mathbf{a}]$ *to* $\mathbb{R}^n$ *defined by* $G(x) \equiv g^h \circ (\omega + \epsilon\phi^h)^{(-1)}(x)$ *for all* $x \in (\omega + \epsilon\phi^h(\partial\Omega^h))$ *and* $G(x) \equiv g^d \circ \phi^{d(-1)}(x)$ *for all* $x \in \phi^d(\partial\Omega^d)$. *Let*

$\{\alpha^{(i)}\}_{i=1,\dots,\bar{n}}$ and $\{\beta^{(i)}\}_{i=1,\dots,\bar{n}}$ be as in Lemma 2.24. Then the unique solution $u[b,\mathbf{a},g^h,g^d] \in C^{m,\lambda}(\mathrm{cl}\mathbb{A}[\mathbf{a}],\mathbb{R}^n)$ of problem (2.29) is delivered by the following equation,

$$u[b,\mathbf{a},g^h,g^d] \equiv w_{\partial\mathbb{A}[\mathbf{a}]}[b,\mu] + \sum_{i,j=1}^{\bar{n}} \langle G|\alpha^{(i)}\rangle_{\partial\mathbb{A}[\mathbf{a}]}\,(V^{-1})_{ij}\, v_{\partial\mathbb{A}[\mathbf{a}]}[b,\alpha^{(j)}], \quad (2.32)$$

where μ is the only solution of (2.31) with

$$\widetilde{G} \equiv G - \sum_{i=1}^{\bar{n}} \langle G|\alpha^{(i)}\rangle_{\partial\mathbb{A}[\mathbf{a}]}\,\beta^{(i)}, \quad (2.33)$$

and V is the $n\times n$ real matrix defined by

$$V_{ij} \equiv \langle v_{\partial\mathbb{A}[\mathbf{a}]}[b,\alpha^{(i)}]|\beta^{(j)}\rangle_{\partial\mathbb{A}[\mathbf{a}]},$$

for all $i,j=1,\dots,\bar{n}$.

Proof. The uniqueness of the solution follows by Theorem 2.4. The existence can be proved by the previous Lemma 2.23 and by exploiting equation 2.32. We just note here that the matrix V is invertible by Theorem 2.17. □

2.2.3 Auxiliary boundary value problems

The purpose of this subsection is to investigate the Dirichlet interior boundary value problem in $\mathbb{I}[\phi^d]$ and the Dirichlet exterior boundary value problem in $\mathbb{E}[\phi^h]$ and to provide an expression for the solutions of both by means of layer potentials. This will be done in the Theorems 2.27 and 2.30. First we state the following Lemma 2.26, which can be verified by arguing as in the proof of Lemmas 2.22 and 2.23.

Lemma 2.26. *Let the notation of subsection 2.2.1 hold. Let $b \in \mathscr{B}$ and $\mathbf{a} \equiv (\omega,\epsilon,\phi^h,\phi^d) \in \mathcal{E}_+^{m,\lambda}$. Then the following statements hold.*

(i) Let $F \in C^{m-1,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$. Then the equation

$$\mathbf{K}^*_{\phi^d(\partial\Omega^d)}[b,\tau] = F, \quad (2.34)$$

has one and only one solution $\tau \in C^{m-1,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$.

(ii) Let $G^d \in C^{m,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$. Then the equation

$$\mathbf{K}_{\phi^d(\partial\Omega^d)}[b,\mu] = G^d, \quad (2.35)$$

has a unique solution $\mu \in C^{m,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$.

(iii) *Let* F *be a function of* $C^{m-1,\lambda}(\phi^h(\partial\Omega^h),\mathbb{R}^n)\cap(\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot])^\perp$ *and* $\beta\in\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$. *Then the following boundary value problem,*

$$\begin{cases} \mathbf{H}_{\phi^h(\partial\Omega^h)}[b,\tau]=F, \\ \mathbf{P}[\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]]\tau=\beta, \end{cases} \tag{2.36}$$

has a unique solution $\tau\in C^{m-1,\lambda}(\phi^h(\partial\Omega^h),\mathbb{R}^n)$.

(iv) *Let* $\widetilde{G}^h$ *be a function of* $C^{m,\lambda}(\phi^h(\partial\Omega^h),\mathbb{R}^n)\cap(\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b,\cdot])^\perp$ *and* $\beta\in\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$. *Then the following boundary value problem,*

$$\begin{cases} \mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\mu]=\widetilde{G}^h, \\ \mathbf{P}[\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]]\mu=\beta, \end{cases} \tag{2.37}$$

has a unique solution $\mu\in C^{m,\lambda}(\phi^h(\partial\Omega^h),\mathbb{R}^n)$.

By means of Lemma 2.26 and by Proposition 2.13 we immediately deduce the following.

Theorem 2.27. *Let the notation of subsection 2.2.1 hold. Let* $b\in\mathscr{B}$ *and* $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}_+^{m,\lambda}$. *Let* $G^d\in C^{m,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$. *Let* $\mu\in C^{m,\lambda}(\phi^d(\partial\Omega^d),\mathbb{R}^n)$ *be the solution of* (2.35). *Then the boundary value problem*

$$\begin{cases} \mathbf{L}[b]u=0 & \text{in } \mathbb{I}[\phi^d]\,, \\ u=G^d & \text{on } \phi^d(\partial\Omega^d)\,, \end{cases} \tag{2.38}$$

has a unique solution $u\in C^{m,\lambda}(\mathrm{cl}\mathbb{I}[\phi^d],\mathbb{R}^n)$, *which is delivered by the double layer potential* $w_{\phi^d(\partial\Omega^d)}[b,\mu]$.

Now we turn to consider the first exterior boundary value problem in $\mathbb{E}[\phi^h]$. We need the following lemma, which can be proved by arguing as in the proof of Lemma 2.24.

Lemma 2.28. *Let the notation of subsection 2.2.1 hold. Let* $b\in\mathscr{B}$ *and* $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}_+^{m,\lambda}$. *Let* $\bar{n}\equiv\dim\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$. *Let* $\{b^{(i)}\}_{i=1,\dots,\bar{n}}$ *be an orthonormal basis of* $\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$. *Then there exists a basis* $\{a^{(i)}\}_{i=1,\dots,\bar{n}}$ *of* $\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b,\cdot]$ *such that* $\langle a^{(i)}|b^{(j)}\rangle=\delta_{ij}$, *for all* $i,j=1,\dots,\bar{n}$.

In the following Theorem 2.30 we distinguish between the case $n=2$ and $n\geq 3$. To treat the case $n=2$ we need to introduce an explicit basis for $\mathrm{Ker}\mathbf{H}^*_{\phi^d(\partial\Omega^d))}[b,\cdot]$. So, by exploiting Theorem 2.16 we deduce the following.

Lemma 2.29. *Let the notation of subsection 2.2.1 hold. Let* $n=2$, *and* $b\in\mathscr{B}$, *and* $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{E}_+^{m,\lambda}$. *Let* $b^{(1)}[\phi^h]$, $b^{(2)}[\phi^h]$ *and* $b^{(3)}[\phi^h]$ *be* *the functions of* $\phi^h(\partial\Omega^h)$ *to* $\mathbb{R}^2$ *defined by*

$$\begin{aligned} b^{(1)}[\phi^h](x) &\equiv |\phi^h(\partial\Omega^h)|^{-1/2}(1,0), \\ b^{(2)}[\phi^h](x) &\equiv |\phi^h(\partial\Omega^h)|^{-1/2}(0,1), \\ b^{(3)}[\phi^h](x) &\equiv \widetilde{b}[\phi^h](x) - \langle \widetilde{b}[\phi^h] | b^{(1)}[\phi^h]\rangle_{\phi^h(\partial\Omega^h)} b^{(1)}[\phi^h](x) \\ &\qquad - \langle \widetilde{b}[\phi^h] | b^{(2)}[\phi^h]\rangle_{\phi^h(\partial\Omega^h)} b^{(2)}[\phi^h](x), \end{aligned}$$

for all $x \in \phi^h(\partial\Omega^h)$, *where*

$$\widetilde{b}[\phi^h](x) \equiv c^{-1/2} |\phi^h(\partial\Omega^h)|^{-1/2}(-x_2, x_1)$$

for all $x \equiv (x_1, x_2) \in \phi^h(\partial\Omega^h)$, *with*

$$c \equiv \fint_{\phi^h(\partial\Omega^h)} |y|^2 \, d\sigma_y - \Big(\fint_{\phi^h(\partial\Omega^h)} y \, d\sigma_y \Big)^2,$$

where $\fint$ *denotes the integral average. Then* $\{b^{(1)}[\phi^h], b^{(2)}[\phi^h], b^{(3)}[\phi^h]\}$ *is an orthonormal basis of* $\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$. *Moreover there exists a unique* $(\widetilde{\alpha}[b,\phi^h], \widetilde{c}[b,\phi^h])$ *of* $(\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b,\cdot])^0 \times (\mathbb{R}^2_{\mathbb{I}[\phi^h],\mathrm{loc}})|_{\phi^h(\partial\Omega^h)}$ *such that*

$$v_{\phi^h(\partial\Omega^h)}[b, \widetilde{\alpha}[b,\phi^h]] + \widetilde{c}[b,\phi^h] = \widetilde{b}[\phi^h].$$

We are now ready to prove Theorem 2.30.

Theorem 2.30. *Let the notation of subsection 2.2.1 hold. Let* $b \in \mathscr{B}$ *and* $\mathbf{a} \equiv (\omega, \epsilon, \phi^h, \phi^d) \in \mathcal{E}^{m,\lambda}_+$. *Let* $G^h \in C^{m,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$. *Then the boundary value problem*

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \mathbb{E}[\phi^h], \\ u = G^h & \text{on } \phi^h(\partial\Omega^h), \\ \sup_{x\in\mathbb{E}[\phi^h]} |u(x)||x|^{n-2} < +\infty, \\ \sup_{x\in\mathbb{E}[\phi^h]} |Du(x)||x|^{n-1} < +\infty, \end{cases} \tag{2.39}$$

has a unique solution $u \in C^{m,\lambda}(\mathrm{cl}\mathbb{E}[\phi^h], \mathbb{R}^n)$. *Moreover the following statements hold.*

(i) Let $n \geq 3$. *Let* $\{a^{(i)}\}_{i=1,\dots,\bar{n}}$ *and* $\{b^{(i)}\}_{i=1,\dots,\bar{n}}$ *be as in Lemma 2.28. Then the solution* u *of* (2.39) *is delivered by the following equation,*

$$u \equiv w_{\phi^h(\partial\Omega^h)}[b,\mu] + \sum_{i,j=1,\dots,\bar{n}} \langle G^h | a^{(i)}\rangle \, (V^{-1})_{ij} \, v_{\phi^h(\partial\Omega^h)}[b, a^{(j)}], \tag{2.40}$$

where μ *is the only solution of* (2.37) *with* $\beta = 0$ *and*

$$\widetilde{G}^h = G^h - \sum_{i=1,\dots,\bar{n}} \langle G^h | a^{(i)}\rangle \, b^{(i)},$$

and where $V \in M_{\bar{n}\times\bar{n}}(\mathbb{R})$ *is defined by*

$$V_{ij} \equiv \langle v_{\phi^h(\partial\Omega^h)}[b, a^{(i)}] | b^{(j)}\rangle$$

for every $i, j = 1, \dots, \bar{n}$.

(ii) *Let $n = 2$. Let $\{b^{(i)}[\phi^h]\}_{i=1,\dots,3}$, $\widetilde{b}[\phi^h]$ and $(\widetilde{\alpha}[b,\phi^h], \widetilde{c}[b,\phi^h])$ be as in Lemma 2.29. Let $\{a^{(i)}[\phi^h]\}_{i=1,\dots,3}$ be defined as in Lemma 2.29 with $b^{(i)} \equiv b^{(i)}[\phi^h]$ for all $i = 1,\dots,3$. Then the solution of (2.39) is delivered by the following equation,*

$$u \equiv w_{\phi^h(\partial\Omega^h)}[b,\mu] + \langle G^h|a^{(3)}\rangle \left(v_{\phi^h(\partial\Omega^h)}[b,\widetilde{\alpha}[b,\phi^h]] + \widetilde{c}[b,\phi^h] \right) \quad (2.41)$$
$$+ \sum_{i=1,2} \left(\langle G^h|a^{(i)}\rangle - \langle G^h|a^{(3)}\rangle\langle \widetilde{b}[\phi^h]|b^{(i)}[\phi^h]\rangle \right) \bar{b}^{(i)}[\phi^h],$$

where μ is the only solution of (2.37) with $\beta = 0$, and

$$\widetilde{G}^h = G^h - \sum_{i=1,\dots,3} \langle G^h|a^{(i)}\rangle b^{(i)}[\phi^h],$$

and $\bar{b}^{(i)}[\phi^h]$ is the constant function on $\mathrm{cl}\mathbb{E}[\phi^h]$ which extends $b^{(i)}[\phi^h]$, for $i = 1, 2$.

By Theorem 2.27 and 2.30 we can introduce the following notation.

Definition 2.31. *Let the notation of subsection 2.2.1 hold. Let $(b, \mathbf{a}, g^h, g^d)$ be a point of $\mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. We denote by $u^d[\phi^d, g^d]$ the unique solution of (2.38) with $G^d = g^d \circ (\phi^d)^{(-1)}$, and we denote by $u^h[b,\phi^h,g^h]$ the unique solution of (2.39) with $G^h = g^h \circ (\phi^h)^{(-1)}$, and we denote by $u_r^h[b,\phi^h,g^h]$ the function $w_{\phi^h(\partial\Omega^h)}[b,\mu](x)$ of the variable $x \in \mathbb{E}[\phi^h]$ with μ as in Theorem 2.30, and we denote by $u_s^h[b,\phi^h,g^h]$ the difference $u^h[b,\phi^h,g^h] - u_r^h[b,\phi^h,g^h]$.*

2.2.4 Fixed basis for the kernels of the integral operators

In order to make the expressions (2.32) and (2.40) more explicit, we fix in the following Theorem 2.34 explicit expression for the orthonormal basis $\{\beta^{(i)}\}_{i=1,\dots,\bar{n}}$ and $\{b^{(i)}\}_{i=1,\dots,\bar{n}}$ of $\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$ and $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$, respectively. To do so, we need the following technical lemma, which can be verified by a standard calculus (see also Lanza de Cristoforis and Rossi [27, Lemma 3.13].)

Lemma 2.32. *Let $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. Let $\phi \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$. Then there exists a positive function $\tilde{\sigma}[\phi] \in C^{m-1,\lambda}(\partial\Omega)$ such that*

$$\int_{\phi(\partial\Omega)} f(\xi)\, d\sigma_\xi = \int_{\partial\Omega} f \circ \phi(x) \tilde{\sigma}[\phi](x)\, d\sigma_x,$$

for every $f \in L^1(\phi(\partial\Omega))$.

Definition 2.33. *We denote by* $\{e^{(i)}\}_{i=1,\dots,n}$ *the canonical basis of* $\mathbb{R}^n$ *and we denote by* $\{s^{(i)}\}_{i=1,\dots,\binom{n}{2}}$ *the canonical basis of* $\mathrm{Skew}(n,\mathbb{R})$. *Let* $m \in \mathbb{N}\setminus\{0\}$ *and* $\lambda \in]0,1[$. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$ *such that* Ω *and* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ *are connected. Let* $\phi \in C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}$. *We denote by* $\{b^{(i)}[\phi]\}_{1,\dots,\bar{n}}$, $\bar{n}\equiv n+\binom{n}{2}$, *the orthonormal basis of* $(\mathcal{R}_{\mathbb{I}[\phi]})|_{\phi(\partial\Omega)}$ *defined by*

$$b^{(i)}[\phi](\xi) \equiv \left(\int_{\partial\Omega} \tilde{\sigma}[\phi]\, d\sigma\right)^{-\frac{1}{2}} e^{(i)} \qquad \forall\, \xi \in \phi(\partial\Omega),\ i=1,\dots,n,$$

and

$$\begin{aligned} b^{(i)}[\phi](\xi) \equiv\ & \\ &\left(s^{(i-n)}\xi - \sum_{j=1}^{i-1}\int_{\partial\Omega}(s^{(i-n)}\phi(y))\cdot b^{(j)}[\phi]\circ\phi(y)\,\tilde{\sigma}[\phi](y)\, d\sigma_y\ b^{(j)}[\phi](\xi)\right) \\ &\cdot\Bigg(\int_{\partial\Omega}|(s^{(i-n)}\phi(y))|^2\tilde{\sigma}[\phi](y)\, d\sigma_y \\ &\qquad -\sum_{j=1}^{i-1}\left(\int_{\partial\Omega}(s^{(i-n)}\phi(y))\cdot b^{(j)}[\phi]\circ\phi(y)\,\tilde{\sigma}[\phi](y)\, d\sigma_y\right)^2\Bigg)^{-\frac{1}{2}} \end{aligned}$$

for all $\xi \in \phi(\partial\Omega)$ *and for all* $i = n+1,\dots,n+\binom{n}{2}$.

Then by Theorems 2.16 and 2.17 we have the following.

Theorem 2.34. *Let the notation of subsection 2.2.1 hold. Let* $b \in \mathscr{B}$ *and* $\mathbf{a} \equiv (\omega,\epsilon,\phi^h,\phi^d) \in \mathcal{E}_+^{m,\lambda}$. *Let* $b^{(i)}[\phi^h]$, $i=1,\dots,\bar{n}$, *be as in the previous definition. Let* $\beta^{(i)}[\mathbf{a}]$ *be the function on* $\partial\mathbb{A}[\mathbf{a}]$ *defined by*

$$\beta^{(i)}[\mathbf{a}](\xi) \equiv \begin{cases} \epsilon^{\frac{1-n}{2}} b^{(i)}[\phi^h]\left(\frac{\xi-\omega}{\epsilon}\right) & \text{for } \xi \in \omega+\epsilon\phi^h(\partial\Omega^h), \\ 0 & \text{for } \xi \in \phi^d(\partial\Omega^d), \end{cases}$$

for all $i=1,\dots,\bar{n}$. *Then the following statements hold.*

(i) $\{b^{(i)}[\phi^h]\}_{i=1,\dots,\bar{n}}$ *is an orthonormal basis of* $\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$.

(ii) $\{\beta^{(i)}[\mathbf{a}]\}_{i=1,\dots,\bar{n}}$ *is an orthonormal basis of* $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]$.

We note that, for $n=2$, the two bases $\{b^{(1)}[\phi^h], b^{(2)}[\phi^h], b^{(3)}[\phi^h]\}$ of $\mathrm{Ker}\mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b,\cdot]$ introduced in Lemma 2.29 and in Theorem 2.34, respectively, coincide.

2.2.5 A real analyticity theorem for the solutions of (2.30)

We start our analysis of the auxiliary problem (2.30), which is defined on the $\mathbf{a}$ dependent domain $\mathbb{A}[\mathbf{a}]$, by transforming it into a system of equations on the boundaries of the fixed domains Ω^h and Ω^d. This is done in the following Theorem 2.36. In order to abbreviate our notation we find convenient to introduce the following.

Definition 2.35. *Let $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. We set*

$$\mathbf{K}[b,\phi,\mu] \equiv \mathbf{K}_{\phi(\partial\Omega)}[b,\mu\circ\phi^{(-1)}]\circ\phi, \quad \mathbf{K}^*[b,\phi,\mu] \equiv \mathbf{K}^*_{\phi(\partial\Omega)}[b,\mu\circ\phi^{(-1)}]\circ\phi,$$
$$\mathbf{H}[b,\phi,\mu] \equiv \mathbf{H}_{\phi(\partial\Omega)}[b,\mu\circ\phi^{(-1)}]\circ\phi, \quad \mathbf{H}^*[b,\phi,\mu] \equiv \mathbf{H}^*_{\phi(\partial\Omega)}[b,\mu\circ\phi^{(-1)}]\circ\phi$$

for all $(b,\phi,\mu) \in \mathscr{B} \times (C^{m,\lambda}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times L^2(\partial\Omega,\mathbb{R}^n)$.

Theorem 2.36. *Let the notation of subsection 2.2.1 hold. We denote by $T \equiv (T^1,T^2,T^3)$ the map of the set $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ to $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ defined by*

$$T^1[b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d](x) \equiv -\mathbf{H}[b,\phi^h,\tau^h](x) \tag{2.42}$$
$$-\epsilon^{n-1}\int_{\partial\Omega^d}\sum_{i=1}^n\Big[T(b,D\Gamma^{(i)}(b,\omega+\epsilon\phi^h(x)-\phi^d(y)))$$
$$\cdot\nu_{\phi^h}\circ\phi^h(x)\Big]\tau_i^d(y)\tilde{\sigma}[\phi^d](y)\,d\sigma_y, \quad \forall\, x \in \partial\Omega^h,$$
$$T^2[b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d](x) \equiv \mathbf{K}^*[b,\phi^d,\tau^d](x) \tag{2.43}$$
$$+\int_{\partial\Omega^h}\sum_{i=1}^n\Big[T(b,D\Gamma^{(i)}(b,\phi^d(x)-\omega-\epsilon\phi^h(y))$$
$$\cdot\nu_{\phi^d}\circ\phi^d(x)\Big]\tau_i^h(y)\tilde{\sigma}[\phi^h](y)\,d\sigma_y, \quad \forall\, x \in \partial\Omega^d,$$
$$T^3[b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d] \tag{2.44}$$
$$\equiv \left(\int_{\partial\Omega^h}\tau^h(y)\cdot\Big(b^{(i)}[\phi^h]\circ\phi^h(y)\Big)\,\tilde{\sigma}[\phi^h](y)\,d\sigma_y - c_i\right)_{i=1,\dots,\bar{n}},$$

for all elements $(b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d)$ of the product $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$, where ν_{ϕ^h} and ν_{ϕ^d} denote the outward unit normal to the boundary of $\mathbb{I}[\phi^h]$ and $\mathbb{I}[\phi^d]$, respectively.

Let $(b,\omega,\epsilon,\phi^h,\phi^d,c) \in \mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times \mathbb{R}^{\bar{n}}$ be fixed. Then the pair of functions (τ^h,τ^d) of $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ satisfies the equation

$$T[b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d] = 0 \tag{2.45}$$

if and only if the function $\tau \in C^{m-1,\lambda}(\partial\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d],\mathbb{R}^n)$ defined by

$$\tau \equiv \begin{cases} \epsilon^{1-n}\,\tau^h\circ(\omega+\epsilon\phi^h)^{(-1)} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ \tau^d\circ(\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d),\end{cases}$$

satisfies (2.30) *with* $T = 0$ *and* $\beta = \epsilon^{\frac{1-n}{2}} \sum_{i=1}^{\bar{n}} c_i \beta^{(i)}[\omega, \epsilon, \phi^h, \phi^d]$. *In particular, equation* (2.45) *has exactly one solution* $(\tau^h, \tau^d) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *for each fixed* $(b, \omega, \epsilon, \phi^h, \phi^d, c) \in \mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times \mathbb{R}^{\bar{n}}$.

Let $(b, \omega, 0, \phi^h, \phi^d, c) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times R^{\bar{n}}$ *be fixed. Then the pair of functions* (τ^h, τ^d) *of* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *satisfies equation*

$$T[b, \omega, 0, \phi^h, \phi^d, c, \tau^h, \tau^d] = 0, \tag{2.46}$$

if and only if both the following conditions are fulfilled.

(i) The function $\tau \equiv \tau^d \circ (\phi^d)^{(-1)}$ *of* $C^{m-1,\lambda}(\phi^d(\partial\Omega^d), \mathbb{R}^n)$ *satisfies* (2.34) *with*

$$F(\xi) = -|\phi^h(\partial\Omega^h)|^{1/2} \sum_{i=1}^{n} c_i\, T(b, D\Gamma^{(i)}(b, \xi - \omega))\, \nu_{\phi^d}(\xi), \quad \forall \xi \in \phi^d(\partial\Omega^d). \tag{2.47}$$

(ii) The function $\tau \equiv \tau^h \circ (\phi^h)^{(-1)}$ *of* $C^{m-1,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$ *satisfies* (2.36) *with* $F = 0$ *and* $\beta = \sum_{i=1}^{\bar{n}} c_i\, b^{(i)}[\phi^h]$.

In particular, equation (2.46) *admits exactly one solution* (τ^h, τ^d) *belonging to* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *for each fixed* $(b, \omega, 0, \phi^h, \phi^d, c) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$.

Proof. The statement follows by a straightforward verification based on the theorem of change of variables in integrals and by the previous Lemmas 2.22, 2.26 and by Theorem 2.34. We only note that, if $(b, \omega, 0, \phi^h, \phi^d, c) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$ is fixed, then by statement (iv) of Lemma 2.26 the first and third component of (2.46) admit a unique solution $\tau^h \in C^{m-1,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$ which satisfies condition (ii) of Theorem 2.36. Then, by statement (i) of Lemma 2.26, the second component of equation (2.46) has a unique solution $\tau^d \in C^{m-1,\lambda}(\phi^d(\partial\Omega^d), \mathbb{R}^n)$. In particular, the function $\tau \equiv \tau^d \circ (\phi^d)^{(-1)}$ satisfies (2.34) with

$$F(\xi) = -\sum_{i=1}^{n} \left(\int_{\partial\Omega^h} \tau_i^h \tilde{\sigma}[\phi^h]\, d\sigma \right) T(b, D\Gamma^{(i)}(b, \xi - \omega)) \nu_{\phi^d}(\xi), \quad \forall\, \xi \in \phi^d(\partial\Omega^d).$$

The integral in the right hand side equals

$$|\phi^h(\partial\Omega^h)|^{1/2} \int_{\partial\Omega^h} \tau^h \cdot \left(b^i[\phi^h] \circ \phi^h \right) \tilde{\sigma}[\phi^h]\, d\sigma = |\phi^h(\partial\Omega^h)|^{1/2} c_i,$$

for all $i = 1, \dots, n$. Therefore τ^d satisfies condition (i). □

By Theorem 2.36, it makes sense to introduce the following.

Definition 2.37. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, c) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$ *with* $\epsilon > 0$ *or* $\epsilon = 0$. *We denote by* $(\hat{\tau}^h[\mathbf{c}], \hat{\tau}^d[\mathbf{c}])$ *the couple of functions* $(\tau^h, \tau^d) \in C^{m-1,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n) \times C^{m-1,\lambda}(\phi^d(\partial\Omega^d), \mathbb{R}^n)$ *which satisfies equation* (2.45) *or equation* (2.46), *respectively.*

Our goal is now to show that $\hat{\tau}^h[\cdot]$, $\hat{\tau}^d[\cdot]$ admit a real analytic continuation around a "degenerate" point $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, c_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$. By Theorem 2.36, it suffices to show that locally around $(\mathbf{c}_0, \hat{\tau}^h[\mathbf{c}_0], \hat{\tau}^d[\mathbf{c}_0])$ the set of zeros of T is the graph of a real analytic operator. We plan to do so by exploiting the following corollary of the Implicit Mapping Theorem in Banach space (for a proof see Lanza de Cristoforis [25, Appendix B].)

Proposition 2.38. *Let* $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Z}_1$ *be Banach spaces. Let* $\mathcal{O}$ *be an open set of* $\mathcal{X} \times \mathcal{Y}$ *such that* $(x_0, y_0) \in \mathcal{O}$. *Let* F *be a real analytic map of* $\mathcal{O}$ *to* $\mathcal{Z}$ *such that* $F(x_0, y_0) = 0$. *Let the partial differential* $\partial_y F(x_0, y_0)$ *with respect to the variable* y *be an homeomorphism of* $\mathcal{Y}$ *onto its image* $V \equiv \mathrm{Ran}(\partial_y F(x_0, y_0))$. *Assume that there exists a closed subspace* V_1 *of* $\mathcal{Z}$ *such that* $\mathcal{Z} = V \oplus V_1$ *algebraically. Let* $\mathcal{O}_1$ *be an open subset of* $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ *containig* $(x_0, y_0, 0)$ *such that* $\mathcal{O}_1 \supset \{(x, y, F(x, y)) : (x, y) \in \mathcal{O}\}$, $\mathcal{O}_1 \supset \{(x, y, 0) : (x, y) \in \mathcal{O}\}$. *Let* G *be a real analytic map of* $\mathcal{O}_1$ *to* $\mathcal{Z}_1$ *such that* $G(x, y, F(x, y)) = 0$ *for all* $(x, y) \in \mathcal{O}$, $G(x, y, 0) = 0$ *for all* $(x, y) \in \mathcal{O}$, *and such that the partial differential* $\partial_z G(x_0, y_0, 0)$ *is surjective onto* $\mathcal{Z}_1$ *and has kernel equal to* V. *Then there exists an open neighborhood* $\mathcal{U}$ *of* x_0 *in* $\mathcal{X}$ *and an open neighborhood* $\mathcal{V}$ *of* y_0 *in* $\mathcal{Y}$ *with* $\mathcal{U} \times \mathcal{V} \subset \mathcal{O}$ *and such that the set of zeros of* F *in* $\mathcal{U} \times \mathcal{V}$ *coincides with the graph of a real analytic function of* $\mathcal{U}$ *to* $\mathcal{V}$.

So, in order to apply the previous proposition to the operator T in a neighborhood of a point $\mathbf{c}_0 \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$ we have to understand the real analyticity properties of T. The definition of T involves the operators $\mathbf{H}[\cdot, \cdot, \cdot]$ and $\mathbf{K}^*[\cdot, \cdot, \cdot]$ and also integral operators which display no singularities. To analyze their regularity we need the following Propositions 2.39, where we summarize some known and some easily verifiable real analyticity results.

Proposition 2.39. *Let* $m \in \mathbb{N} \setminus \{0\}$ *and* $\lambda \in]0, 1[$. *Let* Ω *be an open bounded subsets of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$ *such that* Ω *and* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ *are connected. Let* F *be a real analytic map of* $\mathscr{B} \times (\mathbb{R}^n \setminus \{0\})$ *to* $\mathbb{R}$. *Then the following statements hold.*

(i) The map of $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ *to* $C^{m-1,\lambda}(\partial\Omega)$ *which takes* ϕ *to* $\tilde{\sigma}[\phi]$ *is real analytic.*

(ii) The map of $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ *to* $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ *which takes* ϕ *to* $\nu_\phi \circ \phi$ *is real analytic.*

(iii) *The map of* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ *to* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ *which takes* ϕ *to* $b^{(i)}[\phi] \circ \phi$ *is real analytic for all* $i = 1, \dots, \bar{n}$.

(iv) *Let* Ω' *be an open bounded subsets of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$ *such that* Ω' *and* $\mathbb{R}^n \setminus \mathrm{cl}\Omega'$ *are connected. Then the map* H_1 *of*

$$\Big\{(b, \phi, \psi, f) \in \mathscr{B} \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)) \times C^{m,\lambda}(\partial\Omega', \mathbb{R}^n) \times L^1(\partial\Omega') \\ : \ \phi(\partial\Omega) \cap \psi(\partial\Omega') = \emptyset\Big\}$$

to $C^{m,\lambda}(\partial\Omega)$ *which takes* (b, ϕ, ψ, f) *to the function* $H_1[b, \phi, \psi, f]$ *defined by*

$$H_1[b, \phi, \psi, f](x) \equiv \int_{\partial\Omega'} F(b, \phi(x) - \psi(y)) f(y)\, d\sigma_y, \quad \forall\, x \in \partial\Omega,$$

is real analytic.

(v) *Let* Ω' *be a bounded open subset of* $\mathbb{R}^n$. *Then the map* H_2 *of*

$$\big\{(\phi, f) \in \mathscr{B} \times C^0(\partial\Omega, \mathbb{R}^n) \times L^1(\partial\Omega) \, : \, \phi(\partial\Omega) \cap \mathrm{cl}\Omega' = \emptyset\big\}$$

to $C^0(\mathrm{cl}\Omega')$ *which takes* (b, ϕ, f) *to the function* $H_2[b, \phi, f]$ *defined by*

$$H_2[b, \phi, f](x) \equiv \int_{\partial\Omega} F(b, x - \phi(y)) f(y)\, d\sigma_y, \quad \forall\, x \in \mathrm{cl}\Omega',$$

is real analytic.

(vi) *Let* Ω' *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* C^1. *Then the map* H_3 *of*

$$\Big\{(b, \phi, \Phi, f) \in \mathscr{B} \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \times C^{m,\lambda}(\mathrm{cl}\Omega', \mathbb{R}^n) \times L^1(\partial\mathbb{B}_n) \\ : \ \phi(\Omega) \cap \Phi(\mathrm{cl}\Omega') = \emptyset\Big\}$$

to $C^{m,\lambda}(\mathrm{cl}\Omega')$ *which takes* (b, ϕ, Φ, f) *to the function* $H_3[b, \phi, \Phi, f]$ *defined by*

$$H_3[b, \phi, \Phi, f](x) \equiv \int_{\partial\Omega} F(b, \Phi(x) - \phi(y)) f(y)\, d\sigma_y, \quad \forall\, x \in \mathrm{cl}\Omega',$$

is real analytic.

(vii) $\mathbf{K}[\cdot, \cdot, \cdot]$ *and* $\mathbf{H}^*[\cdot, \cdot, \cdot]$ *are real analytic from* $\mathscr{B} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$ *to* $C^{m,\lambda}(\partial\Omega, \mathbb{R}^n)$.

(viii) $\mathbf{K}^*[\cdot, \cdot, \cdot]$ *and* $\mathbf{H}[\cdot, \cdot, \cdot]$ *are real analytic from* $\mathscr{B} \times (C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ *to* $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$.

Proof. A proof of (i) and (ii) can be found in Lanza de Cristoforis and Rossi [27, Lemma 3.13]. Statement (iii) follows by standard calculus in Banach space. Statement (iv) is a corollary of a known result for composition operators (cf. Böhme and Tomi [2, p. 10], Henry [13, p. 29], Valent [45, Theorem 5.2, p. 44]), its proof can be found in Lanza de Cristoforis [22, Proposition 3.7] and is just a straightforward modification of Lanza de Cristoforis and Rossi [27, Lemma 3.9]. The proof of (v) and (vi) is similar. Statement (vii) and (viii) follows by Proposition 1.40. □

Then we immdiately deduce the following.

Proposition 2.40. *With the notation of subsection 2.2.1 hold, the set* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *is an open subset of the Banach space* $\mathbb{R}^{n+2} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *and the operator T is real analytic.*

Moreover we need the following two lemmas.

Lemma 2.41. *With the notation introduced in subsection 2.2.1, let* $\mathbf{d}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, c_0, \tau_0^h, \tau_0^d)$ *be a point of* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *such that* $T[\mathbf{d}_0] = 0$. *Then the differential*

$$\partial_{(\tau^h,\tau^d)} T[\mathbf{d}_0] = \left(\partial_{(\tau^h,\tau^d)} T^1[\mathbf{d}_0], \partial_{(\tau^h,\tau^d)} T^2[\mathbf{d}_0], \partial_{(\tau^h,\tau^d)} T^3[\mathbf{d}_0]\right)$$

of T with respect to the variable (τ^h, τ^d) at $\mathbf{d}_0$ is delivered by the linear operators which takes a couple $(\bar{\tau}^h, \bar{\tau}^d)$ of $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ to the functions defined by

$$\partial_{(\tau^h,\tau^d)} T^1[\mathbf{d}_0](\bar{\tau}^h, \bar{\tau}^d)(x) = -\mathbf{H}[b_0, \phi_0^h, \bar{\tau}^h](x), \qquad \forall\, x \in \partial\Omega^h \tag{2.48}$$

$$\begin{aligned} &\partial_{(\tau^h,\tau^d)} T^2[\mathbf{d}_0](\bar{\tau}^h, \bar{\tau}^d)(x) \\ &\quad = \mathbf{K}^*[b_0, \phi_0^d, \bar{\tau}^d](x) + \int_{\partial\Omega^h} \sum_{i=1}^{n} \Big[T(b_0, D\Gamma^{(i)}(b_0, \phi_0^d(x) - \omega_0)) \\ &\qquad\qquad \cdot \nu_{\phi_0^d} \circ \phi_0^d(x) \Big]\, \bar{\tau}_i^h(y) \tilde{\sigma}[\phi_0^h](y)\; d\sigma_y, \quad \forall\, x \in \partial\Omega^h, \end{aligned} \tag{2.49}$$

$$\begin{aligned} &\partial_{(\tau^h,\tau^d)} T^3[\mathbf{d}_0](\bar{\tau}^h, \bar{\tau}^d) \\ &\qquad = \left(\int_{\partial\Omega^h} \bar{\tau}^h(y) \cdot \left(b^{(i)}[\phi_0^h] \circ \phi_0^h(y) \right) \tilde{\sigma}[\phi_0^h](y)\; d\sigma_y \right)_{i=1,\dots,\bar{n}}. \end{aligned} \tag{2.50}$$

Let $V_0^{m,\lambda}$ be the subspace of $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ of the triple (f^h, f^d, d) such that

$$\int_{\partial\Omega^h} f^h(y) \cdot \left(b^{(i)}[\phi_0^h] \circ \phi_0^h(y) \right) \tilde{\sigma}[\phi_0^h](y)\; d\sigma_y = 0$$

for all $i = 1, \dots, \bar{n}$. Then $\partial_{(\tau^h,\tau^d)} T[\mathbf{d}_0]$ is a linear homeomorphism of $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ to $V_0^{m,\lambda}$.

Proof. Expressions (2.48), (2.49), (2.50) follow by standard calculus in Banach space. Exploiting such expressions we recognize that $\partial_{(\tau^h,\tau^d)}T[\mathbf{d}_0]$ is a linear and bounded operator from $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ to $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^{\bar{n}}$. Moreover, by the Fredholm Alternative Theorem, the range of $\partial_{(\tau^h,\tau^d)}T[\mathbf{d}_0]$ is contained in $V_0^{m,\lambda}$. Indeed, the range of $\partial_{(\tau^h,\tau^d)}T^1[\mathbf{d}_0]$ is contained in the range of $\mathbf{H}[b_0,\phi_0^h,\cdot]$, which is orthogonal to the kernel of $\mathbf{H}^*[b_0,\phi_0^h,\cdot]$. Now, it remains to prove that $\partial_{(\tau^h,\tau^d)}T[\mathbf{d}_0]$ is a homeomorphism of $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ to $V_0^{m,\lambda}$. By the Open Mapping Theorem, it suffices to show that it is bijective. So, we fix $(f^h,f^d,c)\in V_0^{m,\lambda}$ and we verify that there exists a unique couple $(\bar{\tau}^h,\bar{\tau}^d)\in C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ such that

$$\begin{cases} \partial_{(\tau^h,\tau^d)}T^1[\mathbf{d}_0](\bar{\tau}^h,\bar{\tau}^d)=f^h,\\ \partial_{(\tau^h,\tau^d)}T^2[\mathbf{d}_0](\bar{\tau}^h,\bar{\tau}^d)=f^d,\\ \partial_{(\tau^h,\tau^d)}T^3[\mathbf{d}_0](\bar{\tau}^h,\bar{\tau}^d)=c. \end{cases} \tag{2.51}$$

Let $F^h\equiv f^h\circ(\phi_0^h)^{(-1)}$. Then, by changing the variable with the function ϕ_0^h in the definition of $V_0^{m,\lambda}$ and by Theorem 2.34, we deduce that F^h is an element of $(\mathrm{Ker}\mathbf{H}^*_{\phi_0^h(\partial\Omega^h)}[b_0,\cdot])^{\perp}$. Then, by changing the variable with the function ϕ_0^h if the first and third equation of (2.48) and by exploiting statement (iii) of Lemma 2.26, we deduce that the the system of the first and third equation of (2.51) has a unique solution $\bar{\tau}^h\in C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)$. Now let F^d be defined by

$$\begin{aligned} F^d(\xi)&\equiv f^d\circ(\phi_0^d)^{(-1)}(\xi)\\ &-\int_{\partial\Omega^h}\sum_{i=1}^n\Big[T(b_0,D\Gamma^{(i)}(b_0,\xi-\omega_0))\nu_{\phi_0^d}(x)\Big]\ \bar{\tau}_i^h(y)\tilde{\sigma}[\phi_0^h](y)\,d\sigma_y, \end{aligned}$$

for all $\xi\in\phi_0^d(\partial\Omega^d)$. Then $F^d\in C^{m-1,\lambda}(\phi_0^d(\partial\Omega^d),\mathbb{R}^n)$ and by statement (i) of the Lemma 2.26 the second equation of (2.51) has a unique solution $\bar{\tau}^d\in C^{m-1,\lambda}(\phi_0^d(\partial\Omega^d),\mathbb{R}^n)$. □

Lemma 2.42. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{d}\equiv(b,\omega,\epsilon,\phi^h,\phi^d,c,\tau^h,\tau^d)$ *be a point of* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times\mathbb{R}^{\bar{n}}\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)$. *Then*

$$\int_{\partial\Omega^h}T^1[\mathbf{d}](y)\cdot\Big(b^{(i)}[\phi^h]\circ\phi^h(y)\Big)\,\tilde{\sigma}[\phi^h](y)\,d\sigma_y=0, \tag{2.52}$$

for all $i=1,\dots,\bar{n}$.

Proof. Let $\widetilde{T}^1$ denote the second term on the right hand side of (2.42). By the Fredholm Alternative Theorem, equation (2.52) holds with $T^1[\mathbf{d}]$ replaced by $T^1[\mathbf{d}]-\widetilde{T}^1$. Thus, to conclude the proof, it is enough to show that (2.52) holds with $T^1[\mathbf{d}]$ replaced by $\widetilde{T}^1$. For $\epsilon = 0$, $\widetilde{T}^1 = 0$ and there is nothing to prove. Let $\epsilon \neq 0$. We note that

$$\widetilde{T}^1(x) = -\epsilon^{n-1} \int_{\partial\Omega^d} \sum_{i=1}^{n} \Big[T(b, D\Gamma^{(i)}(b, \omega + \epsilon\phi^h(x) - \phi^d(y))) \cdot \nu_{\phi^h} \circ \phi^h(x) \Big] \cdot \tau_i^d(y) \tilde{\sigma}[\phi^d](y)\, d\sigma_y, \quad \forall\, x \in \partial\Omega^h.$$

So, if we set

$$\widetilde{\mu}(\xi) \equiv -\int_{\partial\Omega^d} \sum_{i=1}^{n} \Gamma^{(i)}(b, \xi - \phi^d(y)) \tau_i^d(y) \tilde{\sigma}[\phi^d](y)\, d\sigma_y, \quad \forall\, \xi \in \mathbb{I}[\phi^d],$$

then we have

$$\widetilde{T}^1(x) = \epsilon^{n-1} T(b, D\widetilde{\mu}(\omega + \epsilon\phi^h(x)))\nu_{\omega+\epsilon\phi^h}(\omega + \epsilon\phi^h(x)), \quad \forall\, x \in \partial\Omega^h.$$

Now let ρ be an element of $\mathcal{R}_{\mathbb{I}[\omega+\epsilon\phi^h]}$ and let $\bar{\rho}$ be the continuous extension of ρ to $\mathrm{cl}\mathbb{I}[\omega + \epsilon\phi^h]$. Then, by Lemma 2.2, we have $\mathbf{E}[b](\rho, \widetilde{\mu}) = 0$. By Theorem 2.3 we deduce that

$$\begin{aligned} &\int_{\partial\Omega^h} \widetilde{T}^1(y) \cdot \bar{\rho} \circ (\omega + \epsilon\phi^h(y))\, \tilde{\sigma}[\phi^h](y)\, d\sigma_y \\ &\quad = (\mathrm{sgn}\epsilon)^{n-1} \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} \big[T(b, D\widetilde{\mu}(\xi))\nu_{\omega+\epsilon\phi^h}(\xi) \big] \cdot \bar{\rho}(\xi)\, d\sigma_\xi \\ &\quad = (\mathrm{sgn}\epsilon)^{n-1} \int_{\mathbb{I}[\omega+\epsilon\phi^h]} (\mathbf{L}[b]\widetilde{\mu}) \cdot \rho + \mathbf{E}[b](\widetilde{\mu}, \rho)\, d\xi = 0. \end{aligned}$$

Therefore, by Theorem 2.34, we deduce that $\widetilde{T}^1$ satisfies (2.52). □

We are now ready to prove a real analyticity result for $\hat{\tau}^h[\cdot]$ and $\hat{\tau}^d[\cdot]$.

Theorem 2.43. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, c_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$. *Then there exist an open neighborhood* $\mathcal{U}_0$ *of* $\mathbf{c}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$, *and an open neighborhood* $\mathcal{V}_0$ *of* $(\hat{\tau}^h[\mathbf{c}_0], \hat{\tau}^d[\mathbf{c}_0])$ *in* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, *and a real analytic operator* $(\mathcal{T}^h, \mathcal{T}^d)$ *of* $\mathcal{U}_0$ *to* $\mathcal{V}_0$ *such that*

$$(\mathcal{T}^h[\mathbf{c}], \mathcal{T}^d[\mathbf{c}]) = (\hat{\tau}^h[\mathbf{c}], \hat{\tau}^d[\mathbf{c}]) \tag{2.53}$$

for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, c) \in \mathcal{U}_0$ *with* $\epsilon \geq 0$. *Moreover, the graph of* $(\mathcal{T}^h, \mathcal{T}^d)$ *coincides with the set of zero of* T *in* $\mathcal{U}_0 \times \mathcal{V}_0$.

Proof. Let $\mathcal{H} \equiv \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^{\bar{n}}$, and let G be the function of $\mathcal{H}$ to $\mathbb{R}^{\bar{n}}$ defined by

$$G[\mathbf{d}, f^h, f^d, d] \equiv \int_{\partial\Omega^h} \Big(f^h(y) \cdot b^{(i)}[\phi^h] \circ \phi^h(y) \Big)_{i=1,\ldots,\bar{n}} \tilde{\sigma}[\phi^h](y)\, d\sigma_y,$$

for all $(\mathbf{d}, f^h, f^d, d) \in \mathcal{H}$, with $\mathbf{d} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, c, \tau^h, \tau^d)$. Then, by Proposition 2.39, G is real analytic. Moreover $G[\mathbf{d}, 0, 0, 0] = 0$ and $G[\mathbf{d}, T[\mathbf{d}]] = 0$ for all $\mathbf{d} \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ (see the previous Lemma 2.42.) Now, let $\mathbf{d}_0 \equiv (\mathbf{c}_0, \hat{\tau}^h[\mathbf{c}_0], \hat{\tau}^d[\mathbf{c}_0])$. The partial differential $\partial_{(f^h,f^d,d)} G[\mathbf{d}_0, 0, 0, 0]$ coincides with the linear map which takes $(\bar{f}^h, \bar{f}^d, \bar{d})$ to

$$\int_{\partial\Omega^h} \left(\bar{f}^h(y) \cdot b^{(i)}[\phi^h] \circ \phi^h(y) \right)_{i=1,\ldots,\bar{n}} \tilde{\sigma}[\phi^h](y) \, d\sigma_y.$$

We immediately recognize that $\partial_{(f^h,f^d,d)} G[\mathbf{d}_0, 0, 0, 0]$ is surjective onto $\mathbb{R}^{\bar{n}}$ and has kernel equal to $V_0^{m,\lambda}$. We note that $V_0^{m,\lambda}$ is a closed subspace of $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ of codimension $\bar{n}$, therefore it admits a closed topological supplement of dimension $\bar{n}$. Then, by Proposition 2.38 and Lemma 2.41, the statement of Theorem 2.43 follows. □

2.2.6 A real analyticity theorem for the solutions of (2.31)

By exploiting the results of the previous subsection we deduce the following.

Proposition 2.44. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^s, \phi_0^r)$ *be a point of* $\mathscr{B} \times \mathcal{E}^{m,\lambda}$. *Let* $\{e^{(i)}\}_{i=1,\ldots,\bar{n}}$ *be the canonical basis of* $\mathbb{R}^{\bar{n}}$. *We denote by* $\mathcal{U}_0^{(i)}$ *the neighborhood* $\mathcal{U}_0$ *of* $\mathbf{c}_0 \equiv (\mathbf{b}_0, e^{(i)})$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times \mathbb{R}^{\bar{n}}$ *introduced in Theorem 2.43, and we denote by* $\mathcal{W}_0^{(i)}$ *the projection of* $\mathcal{U}_0^{(i)}$ *to* $\mathscr{B} \times \mathcal{E}^{m,\lambda}$, *for all* $i = 1, \ldots, \bar{n}$. *Let* $\mathcal{W}_0$ *be an open neighborhood of* $\mathbf{b}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda}$ *contained in the intersection* $\cap_{i=1}^{\bar{n}} \mathcal{W}_0^{(i)}$. *Then there exist real analytic operators* $\mathcal{T}^{(i)} \equiv (\mathcal{T}_h^{(i)}, \mathcal{T}_d^{(i)})$ *from* $\mathcal{W}_0$ *to* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, *such that* $(\hat{\tau}^h[(\mathbf{b}, e^{(i)})], \hat{\tau}^d[(\mathbf{b}, e^{(i)})]) = \mathcal{T}^{(i)}[\mathbf{b}]$ *for all* $\mathbf{b} \in \mathcal{W}_0$ *with* $\epsilon \geq 0$, *and all* $i = 1, \ldots, \bar{n}$.

Now, let $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$ *with* $\epsilon > 0$. *We denote by* $\alpha^{(i)}[\mathbf{b}]$ *the function of* $\partial\mathbb{A}[\mathbf{a}]$, $\mathbf{a} \equiv (\omega, \epsilon, \phi^h, \phi^d)$, *defined by*

$$\alpha^{(i)}[\mathbf{b}] \equiv \begin{cases} \epsilon^{\frac{1-n}{2}} \mathcal{T}_h^{(i)}[\mathbf{b}] \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ \epsilon^{\frac{n-1}{2}} \mathcal{T}_d^{(i)}[\mathbf{b}] \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d), \end{cases}$$

for all $i = 1, \ldots, \bar{n}$. *Then* $\{\alpha^{(i)}[\mathbf{b}]\}_{i=1,\ldots,\bar{n}}$ *is a basis of* $\mathrm{Ker}\mathbf{K}^*_{\partial\mathbb{A}[\mathbf{a}]}[b, \cdot]$ *and we have* $\langle \alpha^{(i)}[\mathbf{b}], \beta^{(j)}[\mathbf{a}] \rangle = \delta_{ij}$ *for all* $i, j = 1, \ldots, \bar{n}$, *where* $\{\beta^{(i)}[\mathbf{a}]\}_{i=1,\ldots,\bar{n}}$ *is the basis of* $\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b, \cdot]$ *introduced in Theorem 2.34.*

Moreover, if $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d) \in \mathcal{W}_0$, *then* $\{\mathcal{T}_h^{(i)}[\mathbf{b}] \circ (\phi^h)^{(-1)}\}_{i=1,\ldots,\bar{n}}$ *is a basis of* $\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \cdot]$ *and we have* $\langle \mathcal{T}_h^{(i)}[\mathbf{b}] \circ (\phi^h)^{(-1)}, b^{(j)}[\phi] \rangle = \delta_{ij}$ *for all* $j = 1, \ldots, \bar{n}$.

In the next Theorem 2.45 we transform the problem

$$\begin{cases} \mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\mu] = G - \sum_{i=1}^{\bar{n}} \langle G, \alpha^{(i)}[\mathbf{b}]\rangle\, \beta^{(i)}[\mathbf{a}], \\ \mathbf{P}[\mathrm{Ker}\mathbf{K}_{\partial\mathbb{A}[\mathbf{a}]}[b,\cdot]]\mu = 0, \end{cases} \tag{2.54}$$

which is defined on the boundary of the $\mathbf{a}$-dependent domain $\mathbb{A}[\mathbf{a}]$, into a system of equations on the boundary of the fixed domains Ω^h and Ω^d. So, Theorem 2.45 is in some sense the corresponding of Theorem 2.36 for problem (2.31).

Theorem 2.45. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}$, *and let* $\mathcal{W}_0$ *and* $\mathcal{T}^{(i)}$, $i = 1, \dots, \bar{n}$, *be as in the previous proposition. We denote by* $M \equiv (M^1, M^2, M^3)$ *the map of* $\mathcal{O}_0 \equiv \mathcal{W}_0 \times (C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2$ *to* $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ *defined by*

$$\begin{aligned} M^1[b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d,\mu^h,\mu^d](x) &\equiv -\mathbf{H}^*[b,\phi^h,\mu^h](x) \\ &- \int_{\partial\Omega^d} \big(\big[T(b, D\Gamma^{(i)}(b, \omega + \epsilon\phi^h(x) - \phi^d(y))) \\ &\qquad \cdot\nu_{\phi^d} \circ \phi^d(y)\big] \cdot \mu^d(y)\big)_{i=1,\dots,n} \tilde{\sigma}[\phi^d](y)\, d\sigma_y - g^h(x) \\ &+ \sum_{i=1}^{\bar{n}} \bigg(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\, d\sigma \\ &\qquad + \int_{\partial\Omega^d} g^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma \bigg)\, b^{(i)}[\phi^h] \circ \phi^h(x), \quad \forall\, x \in \partial\Omega^h, \end{aligned} \tag{2.55}$$

$$\begin{aligned} M^2[b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d,\mu^h,\mu^d](x) &\equiv \mathbf{K}[b,\phi^d,\mu^d](x) \\ &+\epsilon^{n-1} \int_{\partial\Omega^h} \big(\big[T(b, D\Gamma^{(i)}(b, \phi^d(x) - \omega - \epsilon\phi^h(y))\nu_{\phi^h} \circ \phi^h(y)\big] \\ &\qquad \cdot\mu^h(y)\big)_{i=1,\dots,n} \tilde{\sigma}[\phi^h](y)\, d\sigma_y - g^d(x), \qquad \forall\, x \in \partial\Omega^d, \end{aligned} \tag{2.56}$$

$$\begin{aligned} M^3[b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d,\mu^h,\mu^d] & \\ \equiv \bigg(\int_{\partial\Omega^h} \mu^h(y) \cdot \Big(b^{(i)}[\phi^h] \circ \phi^h(y)\Big)\, \tilde{\sigma}[\phi^h](y)\, d\sigma_y \bigg)_{i=1,\dots,\bar{n}}, & \end{aligned} \tag{2.57}$$

for all $(b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d,\mu^h,\mu^d) \in \mathcal{O}_0$, *where* $\mathbf{b} \equiv (b,\omega,\epsilon,\phi^h,\phi^d)$.

Let $(b,\omega,\epsilon,\phi^h,\phi^d,g^i,g^o) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$, *with* $\epsilon > 0$, *be fixed. Then the pair of functions* $(\mu^h,\mu^d) \in C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ *satisfies equation*

$$M[b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d,\mu^h,\mu^d] = 0, \tag{2.58}$$

if and only if, the function $\mu \in C^{m,\lambda}(\partial\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d],\mathbb{R}^n)$ *defined by*

$$\mu \equiv \begin{cases} \mu^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ \mu^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega), \end{cases} \tag{2.59}$$

satisfies equation (2.54) *with*

$$G \equiv \begin{cases} g^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ g^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d). \end{cases}$$

In particular, equation (2.58) *has one and only one solution* $(\mu^h, \mu^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *for each fixed* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *with* $\epsilon > 0$.

Let $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *be fixed. Then the pair of functions* $(\mu^h, \mu^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *satisfies equation*

$$M[b, \omega, 0, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d] = 0 \tag{2.60}$$

if and only if the following two conditions are fulfilled.

(i) The function $\mu \equiv \mu^d \circ (\phi^d)^{(-1)}$ *of* $C^{m,\lambda}(\phi^d(\partial\Omega^d), \mathbb{R}^n)$ *satisfies* (2.35) *with* $\widetilde{G} = g^d \circ (\phi^d)^{-1}$.

(ii) The function $\mu \equiv \mu^h \circ (\phi^h)^{(-1)}$ *of* $C^{m,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$ *satisfies* (2.37) *with* $\beta = 0$ *and*

$$\widetilde{G} = g^h \circ (\phi^h)^{-1} - \sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\, d\sigma \right) b^{(i)}[\phi^h] \circ (\phi^h)^{(-1)},$$

where we abbreviated $(b, \omega, 0, \phi^h, \phi^d)$ *as* $\mathbf{b}$.

In particular, equation (2.60) *has one and only one solution* $(\mu^h, \mu^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *for each fixed* $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$.

Proof. The first part of the theorem follows by a straightforward verification based on the theorem of change of variable in integrals. So we consider only the last part relative to the case $\epsilon = 0$. Let $(b, \omega, 0, \phi^s, \phi^r, g^s, g^r) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ and let $(\mu^h, \mu^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ satisfy equation (2.60). Then, by the second equation in (2.60), we have $\mathbf{K}[b, \phi^d, \mu^d] = g^d$. By changing variable by means of the function ϕ^h, the validity of condition (i) follows. We now show that condition (ii) holds as well. By the first equation in (2.60) we have

$$\begin{aligned} &\mathbf{H}^*[b, \phi^h, \mu^h](x) \\ &= g^h(x) - \sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\, d\sigma \right) b^{(i)}[\phi^h] \circ (\phi^h)^{(-1)} \\ &+ \int_{\partial\Omega^d} \big(\big[T(b, D\Gamma^{(i)}(b, \omega + \epsilon\phi^h(x) - \phi^d(y))) \\ &\qquad \cdot \nu_{\phi^d} \circ \phi^d(y)\big] \cdot \mu^d(y) \big)_{i=1,\dots,n} \tilde{\sigma}[\phi^d](y)\, d\sigma_y \\ &- \sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^d} g^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma \right) b^{(i)}[\phi^h] \circ \phi^h(x), \qquad \forall\, x \in \partial\Omega^h. \end{aligned} \tag{2.61}$$

So, if we prove that the sum of the third and fourth term in the right hand side of (2.61) vanishes, the validity of statement (ii) follows by changing variable by means of the function ϕ^h in the first and third equation of (2.60). Now, let $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d)$. By condition (i) of Theorem 2.36 and by the previous Proposition 2.44, we have

$$\mathbf{K}^*[b, \phi^d, \mathcal{T}_d^{(i)}[\mathbf{b}]](x) = |\phi^h(\partial\Omega^h)|^{1/2}\, T(b, D\Gamma^{(i)}(b, \omega - \phi^d(x)))\nu_{\phi^d} \circ \phi^d(x),$$

for all $x \in \partial\Omega^h$, and all $i = 1, \dots, n$, and $\mathbf{K}^*[b, \phi^o, \mathcal{T}_d^{(i)}[\mathbf{b}]] = 0$ for all $i = n+1, \dots, \bar{n}$. We deduce that

$$\begin{aligned}
&\int_{\partial\Omega^d} \mathbf{K}[b, \phi^d, \mu^d] \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma \\
&\quad = \int_{\partial\Omega^d} \mathbf{K}^*[b, \phi^d, \mathcal{T}_d^{(i)}[\mathbf{b}]] \cdot \mu^d\, \tilde{\sigma}[\phi^d]\, d\sigma \\
&\quad = |\phi^h(\partial\Omega^h)|^{1/2} \int_{\partial\Omega^d} \Big[T(b, D\Gamma^{(i)}(b, \omega - \phi^d(y))) \\
&\qquad\qquad \cdot\nu_{\phi^d} \circ \phi^d(y)\Big] \cdot \mu^d(y)\, \tilde{\sigma}[\phi^d](y)\, d\sigma_y,
\end{aligned}$$

for all $i = 1, \dots, n$, and similarly,

$$\int_{\partial\Omega^d} \mathbf{K}[b, \phi^d, \mu^d] \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma = 0,$$

for all $i = n+1, \dots, \bar{n}$. Then, by exploiting Definition 2.33, we have the following equality,

$$\begin{aligned}
&\sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^d} \mathbf{K}[b, \phi^d, \mu^d] \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma \right) b^{(i)}[\phi^h] \circ \phi^h(x) \qquad (2.62) \\
&\quad = \int_{\partial\Omega^d} \Big\{ \Big[T(b, D\Gamma^{(i)}(b, \omega - \phi^d(y))) \\
&\qquad\qquad \cdot\nu_{\phi^d} \circ \phi^d(y)\Big] \cdot \mu^d(y) \Big\}_{i=1,\dots,n} \tilde{\sigma}[\phi^d](y)\, d\sigma_y,
\end{aligned}$$

for all $x \in \partial\Omega^h$. Since $\mathbf{K}[b, \phi^d, \mu^d] = g^d$, it follows that the sum of the third and fourth term in the right hand side of (2.61) vanishes.

Similarly we can verify that (i) and (ii) imply (2.60). The existence and uniqueness of the solution (μ^h, μ^d) follows by Lemmas 2.23 and 2.26. □

By Theorem 2.45, it makes sense to introduce the following.

Definition 2.46. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}$. *Let* $\mathcal{W}_0$ *be the open neighborhood of* $\mathbf{b}_0$ *of Theorem 2.45. Let* $\mathbf{e} \equiv (\omega, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *with* $\epsilon > 0$ *or* $\epsilon = 0$. *We denote by* $(\hat{\mu}^h[\mathbf{e}], \hat{\mu}^d[\mathbf{e}])$ *the unique solution* $(\mu^h, \mu^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *of equation* (2.58) *or equation* (2.60)*, respectively.*

We now prove that $\hat{\mu}^h[\cdot]$, $\hat{\mu}^d[\cdot]$ have a real analytic continuation in whole open neighborhood of a fixed point $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d)$ of $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. By Theorem 2.45, if suffices to show that locally around $(\mathbf{e}_0, \hat{\mu}^h[\mathbf{e}_0], \hat{\mu}^d[\mathbf{e}_0])$ the set of zero of M is the graph of a real analytic function. We plan to prove such a fact by means of Proposition 2.38. For this reason we prove the following Lemmas 2.48 and 2.49 and we state Proposition 2.47, which can be deduced by Proposition 2.39.

Proposition 2.47. *With the notation introduced in subsection 2.2.1 and in Theorem 2.45, the set $\mathcal{O}_0$ is an open subset of the Banach space $\mathbb{R}^{n+2} \times (C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n))^3$ and the operator M is real analytic.*

Lemma 2.48. *Let the notation of subsection 2.2.1 and of Theorem 2.45 hold. Let $\mathbf{e}_0 \equiv (\mathbf{b}_0, g_0^h, g_0^d)$. Let $(\mu_0^h, \mu_0^d) \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ be a solution on (2.60). Then the partial differential*

$$\partial_{(\mu^h,\mu^d)} M[\mathbf{e}_0, \mu_0^h, \mu_0^d] = \left(\partial_{(\mu^h,\mu^d)} M^i[\mathbf{e}_0, \mu_0^h, \mu_0^d]\right)_{i=1,\dots,3}$$

of M with respect to the variable (μ^h, μ^d) at the point $(\mathbf{e}_0, \mu_0^h, \mu_0^d)$ is delivered by the linear operator which takes a couple $(\bar{\mu}^h, \bar{\mu}^d)$ of $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ to the functions defined by

$$\begin{aligned} \partial_{(\mu^d,\mu^h)} M^1[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d)(x) &= -\mathbf{H}^*[b_0, \phi_0^h, \bar{\mu}^h](x) \\ - \int_{\partial\Omega^d} \Big\{ \Big[T(b_0, D\Gamma^{(i)}(b_0, \omega_0 - \phi_0^d(y)) \\ \cdot \nu_{\phi_0^d} \circ \phi_0^d(y) \Big] \cdot \bar{\mu}^d(y) \Big\}_{i=1,\dots,n} \tilde{\sigma}[\phi_0^d](y)\, d\sigma_y, \quad \forall\, x \in \partial\Omega^h, \end{aligned} \tag{2.63}$$

$$\partial_{(\mu^d,\mu^h)} M^2[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d)(x) = \mathbf{K}[b_0, \phi_0^d, \bar{\mu}^d](x), \quad \forall\, x \in \partial\Omega^d, \tag{2.64}$$

$$\begin{aligned} \partial_{(\mu^d,\mu^h)} M^3[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d) \\ = \left(\int_{\partial\Omega^h} \bar{\mu}^h \cdot \left(b^{(i)}[\phi_0^h] \circ \phi_0^h \right) \tilde{\sigma}[\phi_0^h]\, d\sigma \right)_{i=1,\dots,\bar{n}} . \end{aligned} \tag{2.65}$$

Let $W_0^{m,\lambda}$ be the subspace of $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ of all the triple (f^h, f^d, d) such that

$$\int_{\partial\Omega^h} f^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^h]\, d\sigma + \int_{\partial\Omega^d} f^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^d]\, d\sigma = 0 \tag{2.66}$$

for all $i = 1, \dots, \bar{n}$. Then $\partial_{(\mu^h,\mu^d)} M[\mathbf{e}_0, \mu_0^h, \mu_0^d]$ is a linear homeomorphism of the space $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ onto the space $W_0^{m,\lambda}$.

Proof. Expression (2.63), (2.64), (2.65) follow by standard calculus in Banach space. By such expressions we recognize that $\partial_{(\mu^h,\mu^d)} M[\mathbf{e}_0, \mu_0^h, \mu_0^d]$ is a linear and bounded operator from $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ to

$C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$. We now show that the range of $\partial_{(\mu^h,\mu^d)} M[\mathbf{e}_0, \mu_0^h, \mu_0^d]$ is contained in $W_0^{m,\lambda}$. To do so, we prove that (2.66) holds with f^h and f^d replaced by the right hand side of (2.63) and (2.64), respectively. By the Fredholm Alternative Theorem we have

$$\int_{\partial\Omega^h} \mathbf{H}^*[b_0, \phi_0^h, \bar{\mu}^h] \cdot \mathcal{T}_h^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^h]\; d\sigma = 0,$$

for all $i = 1, \ldots, \bar{n}$. So the first term in the right hand side of (2.63) give no contribution to left hand side of the corresponding equation (2.66). We now consider the contribution of the second term in the right hand side of (2.63). By arguing as in the proof of Theorem 2.45, we see that

$$\begin{aligned} \sum_{j=1}^{\bar{n}} \left(\int_{\partial\Omega^d} \mathbf{K}[b_0, \phi_0^d, \bar{\mu}^d] \cdot \mathcal{T}_d^{(j)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^d]\; d\sigma \right) b^{(j)}[\phi_0^h] \circ \phi_0^h(x) \qquad (2.67) \\ = \int_{\partial\Omega^d} \Big\{ \Big[T(b_0, D\Gamma^{(j)}(b_0, \omega_0 - \phi_0^d(y))) \\ \cdot \nu_{\phi_0^d} \circ \phi_0^d(y) \Big] \cdot \bar{\mu}^d(y) \Big\}_{j=1,\ldots,n} \tilde{\sigma}[\phi_0^d](y)\; d\sigma_y, \end{aligned}$$

for all $x \in \partial\Omega^h$. Since, by Proposition (2.44),

$$\int_{\partial\Omega^h} \left(b^{(j)}[\phi_0^h] \circ \phi_0^h \right) \cdot \mathcal{T}_h^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^h]\; d\sigma = \delta_{ij}, \qquad \forall\, i,j = 1, \ldots, \bar{n},$$

we deduce by (2.67) that the contribution of the second term of (2.63) is

$$-\int_{\partial\Omega^d} \mathbf{K}[b_0, \phi_0^d, \bar{\mu}^d] \cdot \mathcal{T}_d^{(j)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^d]\; d\sigma,$$

which is clearly opposite to the contribution of the right hand side of (2.64). Thus (2.66) holds with f^h and f^d replaced by the right hand side of (2.63) and (2.64), respectively.

We now prove that $\partial_{(\mu^h,\mu^d)} M[\mathbf{e}_0, \mu_0^h, \mu_0^d]$ is an homeomorphism. By the Open Mapping Theorem it suffices to show that it is bijective. So, let (f^h, f^d, d) be a given point of $W_0^{m,\lambda}$. We verify that there exists a unique $(\bar{\mu}^h, \bar{\mu}^d) \in C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ such that

$$\begin{cases} \partial_{(\mu^d,\mu^h)} M^1[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d) = f^h, \\ \partial_{(\mu^d,\mu^h)} M^2[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d) = f^d, \\ \partial_{(\mu^d,\mu^h)} M^3[\mathbf{e}_0, \mu_0^h, \mu_0^d](\bar{\mu}^h, \bar{\mu}^d) = d. \end{cases} \qquad (2.68)$$

The second equation of (2.68) is equivalent to $\mathbf{K}[b_0, \phi_0^d, \bar{\mu}^d] = f^d$. By statement (ii) of Lemma 2.26 such an equation has a unique solution $\bar{\mu}^d \in C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. Then, by (2.67), the first equation of (2.68) is equivalent to

$$-\mathbf{H}^*[b_0, \phi_0^h, \bar{\mu}^h] = f^h + \sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^d} f^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^d]\; d\sigma \right) b^{(i)}[\phi_0^h] \circ \phi_0^h.$$

By (2.54) the right hand side equals

$$f^h - \sum_{i=1}^{\bar{n}} \left(\int_{\partial\Omega^d} f^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}_0]\, \tilde{\sigma}[\phi_0^h]\, d\sigma \right) b^{(i)}[\phi_0^h] \circ \phi_0^h.$$

Hence, by statement (iv) of Lemma 2.26, we deduce that the first and third equation of (2.58) has a unique solution $\bar{\mu}^h \in C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n)$. □

Lemma 2.49. *Let the notation of subsection 2.2.1 and Theorem 2.45 hold. Let* $(\omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d) \in \mathcal{O}_0$. *Then we have*

$$\int_{\partial\Omega^h} M^1[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d] \cdot \mathcal{T}_h^{(i)}[\omega, \epsilon, \phi^h, \phi^d]\, \tilde{\sigma}[\phi^d]\, d\sigma \qquad (2.69)$$

$$+ \int_{\partial\Omega^d} M^2[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d] \cdot \mathcal{T}_d^{(i)}[\omega, \epsilon, \phi^h, \phi^d]\, \tilde{\sigma}[\phi^d]\, d\sigma = 0$$

for all $i = 1, \dots, \bar{n}$.

Proof. Let $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d)$. By linearity we have $M[\mathbf{b}, g^h, g^d, \mu^h, \mu^d] = M[\mathbf{b}, g^h, g^d, 0, 0] + M[\mathbf{b}, 0, 0, \mu^h, \mu^d]$. So, we can prove the lemma by proving it for $\mu^h = \mu^d = 0$ and for $g^h = g^d = 0$ separately.

Let $\mu^h = \mu^d = 0$. By the third equation in (2.45) and by the definition of $\mathcal{T}_h^{(i)}$ in Proposition 2.44, we have

$$\int_{\partial\Omega^h} \left(b^{(j)}[\phi^h] \circ \phi^h \right) \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\, d\sigma = \delta_{ij}, \qquad \forall\, i, j = 1, \dots, \bar{n}.$$

We deduce that the first integral in (2.69) equals

$$\int_{\partial\Omega^d} g^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\, \tilde{\sigma}[\phi^d]\, d\sigma,$$

which clearly opposite to the second integral in equation (2.69).

Now, let $g^h = g^d = 0$. If $\epsilon = 0$ the statement can be proved by arguing as in the proof of Lemma 2.48, so we consider $\epsilon \neq 0$. By exploiting the adjointness of $\mathbf{K}$ and $\mathbf{K}^*$, $\mathbf{H}$ and $\mathbf{H}^*$, and by straightforward application of the Fubini Theorem, we find that the left hand side of equation 2.69 equals

$$\int_{\partial\Omega^h} \mu^h \cdot T^1\left[\mathbf{b}, e^{(i)}, \mathcal{T}_h^{(i)}[\mathbf{b}], \mathcal{T}_d^{(i)}[\mathbf{b}]\right] \tilde{\sigma}[\phi^d] d\sigma$$

$$+ \int_{\partial\Omega^d} \mu^d \cdot T^2\left[\mathbf{b}, e^{(i)}, \mathcal{T}_h^{(i)}[\mathbf{b}], \mathcal{T}_d^{(i)}[\mathbf{b}]\right] \tilde{\sigma}[\phi^d]\, d\sigma,$$

where T^1 and T^2 are the first two components of the operator T introduced in Theorem 2.36. By Proposition 2.44 and Theorem 2.43, the point $(\mathbf{b}, e^{(i)}, \mathcal{T}_h^{(i)}[\mathbf{b}], \mathcal{T}_d^{(i)}[\mathbf{b}])$ is contained in the set of zeros of T. Hence, both the integrand functions in the previous expression are 0. The validity of the statement of the lemma follows. □

We are now ready to prove the main result of this subsection.

Theorem 2.50. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. *Let* $\mathcal{O}_0$ *be as in Theorem 2.45. Let* $\bar{\mu}^h[\cdot]$, $\bar{\mu}^d[\cdot]$ *be the functions introduced in Definition 2.46. Then there exist an open neighborhood* $\mathcal{U}_1$ *of* $\mathbf{e}_0$ *in* $\mathbb{R} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, *and an open neighborhood* $\mathcal{V}_1$ *of* $(\hat{\mu}^h[\mathbf{e}_0], \hat{\mu}^d[\mathbf{e}_0])$ *in* $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, *and a real analytic operator* $(\mathcal{M}^h, \mathcal{M}^d)$ *of* $\mathcal{U}_1$ *to* $\mathcal{V}_1$, *such that* $\mathcal{U}_1 \times \mathcal{V}_1 \subset \mathcal{O}_0$ *and*

$$(\mathcal{M}^h[\mathbf{e}], \mathcal{M}^d[\mathbf{e}]) = (\hat{\mu}^h[\mathbf{e}], \hat{\mu}^d[\mathbf{e}]), \tag{2.70}$$

for all $\mathbf{e} \equiv (\omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_1$ *with* $\epsilon \geq 0$. *Moreover, the graph of* $(\mathcal{M}^h, \mathcal{M}^d)$ *coincides with the set of zeros of* M *in* $\mathcal{U}_1 \times \mathcal{V}_1$.

Proof. Let $\mathcal{H} \equiv \mathcal{O}_0 \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^{\bar{n}}$ and let G be the function of $\mathcal{H}$ to $\mathbb{R}^{\bar{n}}$ defined by

$$G[\mathbf{b}, g^h, g^d, f^h, f^d, d] \equiv \left(\int_{\partial\Omega^h} f^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} f^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^d]\, d\sigma \right)_{i=1,\dots,\bar{n}}$$

for all $(\mathbf{b}, g^h, g^d, f^h, f^d, d) \in \mathcal{H}$. Then, by Propositions 2.39 and 2.44, G is real analytic. Moreover $G[\mathbf{b}, g^h, g^d, 0, 0, 0] = 0$ and, by Lemma 2.49, $G[\mathbf{b}, g^h, g^d, M[\mathbf{b}, g^h, g^d]] = 0$ for all $(\mathbf{b}, g^h, g^d) \in \mathcal{O}_0$. Now consider the partial differential $\partial_{(f^h, f^d, d)} G[\mathbf{e}_0, 0, 0, 0]$. One easily verifies that the range of $\partial_{(f^h, f^d, d)} G[\mathbf{e}_0, 0, 0, 0]$ equals $\mathbb{R}^{\bar{n}}$ and the kernel coincides with the space $W_0^{m,\lambda}$ introduced in Lemma 2.48. Clearly $W_0^{m,\lambda}$ is a closed subspace of $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ of codimension $\bar{n}$, therefore it admits a closed topological complement of dimension $\bar{n}$. Then, by Proposition 2.38 and Lemma 2.48, the statement of the theorem follows. □

2.2.7 Solution of the singularly perturbed problem

In this subsection we finally investigate the behavior of the solution u of problem (2.29) around a given degenerate 7-tuple $(b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d)$, as we have announced in subsection 2.2.1. In the following Theorem 2.53 we provide a representation formula for u in terms of real analytic operators and singular, but completely known, functions of ϵ. By equation 2.32 we first deduce a more explicit representation formula for the solution u of problem (2.29) with $\epsilon > 0$. Indeed by means of Theorems 2.43, 2.50 and Proposition 2.44, we obtain the following.

Lemma 2.51. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. *Let* $\mathcal{T}_h^{(i)}[\cdot]$, $\mathcal{T}_d^{(i)}[\cdot]$, $\alpha^{(i)}[\cdot]$, $i = 1, \dots, \bar{n}$, *be as in Proposition 2.44. Let* $\mathcal{U}_1$, $\mathcal{M}^h[\cdot]$,

$\mathcal{M}^d[\cdot]$ be as in Theorem 2.50. Let $\mathbf{e}$ be a point $(b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d)\in\mathcal{U}_1$ with $\epsilon>0$. Let $u[\mathbf{e}]$ be defined as in subsection 2.2.1. Then we have

$$u[\mathbf{e}] = u_r[\mathbf{e}] + u_s[\mathbf{e}],$$

where

$$\begin{aligned} u_r[\mathbf{e}](\xi) \equiv \epsilon^{n-1}\int_{\partial\Omega^h} &\Big\{\Big[T(b, D\Gamma^{(i)}(b,\xi-\omega-\epsilon\phi^h(y)))\\ &\cdot\nu_{\phi^h}\circ\phi^h(y)\Big]\cdot\mathcal{M}^h[\mathbf{e}]\Big\}_{i=1,\dots,n}\tilde{\sigma}[\phi^h](y)\,d\sigma_y\\ -\int_{\partial\Omega^d} &\Big\{\Big[T(b, D\Gamma^{(i)}(b,\xi-\phi^d(y)))\\ &\cdot\nu_{\phi^d}\circ\phi^d(y)\Big]\cdot\mathcal{M}^d[\mathbf{e}]\Big\}_{i=1,\dots,n}\tilde{\sigma}[\phi^d](y)\,d\sigma_y, \end{aligned} \tag{2.71}$$

for all $\xi\in\mathbb{A}[\mathbf{a}]$, $\mathbf{a}\equiv(\omega,\epsilon,\phi^h,\phi^o)$, and

$$u_s[\mathbf{e}](\xi) \equiv \sum_{i,j=1}^{\bar{n}} \langle G|\alpha^{(i)}[\mathbf{b}]\rangle_{\partial\mathbb{A}[\mathbf{a}]}\,(V[\mathbf{b}]^{-1})_{ij}\,v_{\partial\mathbb{A}[\mathbf{a}]}[b,\alpha^{(j)}[\mathbf{b}]](\xi), \tag{2.72}$$

for all $\xi\in\mathbb{A}[\mathbf{a}]$, where G is defined as in Theorem 2.25, and we abbreviated $(b,\omega,\epsilon,\phi^h,\phi^o)$ as $\mathbf{b}$, and

$$V[\mathbf{b}] \equiv \Big(\big\langle v_{\partial\mathbb{A}[\mathbf{a}]}[b,\alpha^{(i)}[\mathbf{b}]]\,\big|\,\beta^{(j)}[\mathbf{a}]\big\rangle_{\partial\mathbb{A}[\mathbf{a}]}\Big)_{i,j=1,\dots,\bar{n}}. \tag{2.73}$$

Moreover we have

$$\begin{aligned}&\langle G|\alpha^{(i)}[\mathbf{b}]\rangle_{\partial\mathbb{A}[\mathbf{a}]}\\ &\quad= \epsilon^{\frac{n-1}{2}}\Big(\int_{\partial\Omega^h} g^h\cdot\mathcal{T}_h^{(i)}[\mathbf{b}]\,\tilde{\sigma}[\phi^h]\,d\sigma + \int_{\partial\Omega^d} g^d\cdot\mathcal{T}_d^{(i)}[\mathbf{b}]\,\tilde{\sigma}[\phi^d]\,d\sigma\Big),\end{aligned}$$

and

$$\begin{aligned} &V_{ij}[\mathbf{b}]\\ &\quad= \epsilon^{n-1}\Big(\int_{\partial\Omega^h}\int_{\partial\Omega^h}\Gamma\big(b,\epsilon(\phi^h(x)-\phi^h(y)\big)\mathcal{T}_h^{(i)}[\mathbf{b}](y)\,\tilde{\sigma}[\phi^h](y)\,d\sigma_y\\ &\qquad\qquad\cdot\big(b^{(j)}[\phi^h]\circ\phi^h(x)\big)\,\tilde{\sigma}[\phi^h](x)\,d\sigma_x\\ &\qquad+\int_{\partial\Omega^h}\int_{\partial\Omega^d}\Gamma(b,\omega+\epsilon\phi^h(x)-\phi^d(y))\mathcal{T}_d^{(i)}[\mathbf{b}](y)\,\tilde{\sigma}[\phi^d](y)\,d\sigma_y\\ &\qquad\qquad\cdot\big(b^{(j)}[\phi^h]\circ\phi^h(x)\big)\,\tilde{\sigma}[\phi^h](x)\,d\sigma_x\Big), \end{aligned} \tag{2.74}$$

for all $i, j = 1, \dots, \bar{n}$, and

$$
\begin{aligned}
&v_{\partial\mathbb{A}[\mathbf{a}]}[b, \alpha^{(j)}[\mathbf{b}]](\xi) \\
&\quad = \epsilon^{\frac{n-1}{2}} \Big(\int_{\partial\Omega^h} \Gamma(b, \xi - \omega - \epsilon\phi^h(y)) \mathcal{T}_h^{(j)}[\mathbf{b}](y)\, \tilde{\sigma}[\phi^h](y)\ d\sigma_y \\
&\qquad\qquad + \int_{\partial\Omega^d} \Gamma(b, \xi - \phi^d(y)) \mathcal{T}_d^{(j)}[\mathbf{b}](y)\, \tilde{\sigma}[\phi^d](y)\ d\sigma_y \Big),
\end{aligned}
$$

for all $\xi \in \mathbb{R}^n$ and for all $j = 1, \dots, \bar{n}$.

We also need the following technical lemma.

Lemma 2.52. *With the notation introduced in subsection 2.2.1, let $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}$, and let $\mathcal{W}_0$ be the neighborhood of $\mathbf{b}_0$ introduced in Proposition 2.44, and let $V[\mathbf{b}]$ be defined by (2.73) for all $\mathbf{b} \in \mathcal{W}_0$. Then there exist real analytic operators $V^{(1)}$, $V^{(2)}$ of $\mathcal{W}_0$ to $M_{\bar{n}\times\bar{n}}(\mathbb{R})$ such that the following statements hold.*

(i) We have

$$
\epsilon^{1-n} V[\mathbf{b}] = \begin{cases} (\log\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}] & \text{if } n = 2, \\ \epsilon^{2-n} V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}] & \text{if } n \geq 3, \end{cases}
$$

for all $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$ with $\epsilon > 0$.

(ii) For $n = 2$ we have

$$
V^{(1)}[\mathbf{b}] = \frac{|\phi^h(\partial\Omega^h)|}{2\pi} \frac{b+2}{2(b+1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.75}
$$

for all $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$. While for $n \geq 3$, $V^{(1)}[\mathbf{b}]$ is an invertible matrix for all $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d) \in \mathcal{W}_0$ with $\epsilon = 0$.

(iii) Let $n = 2$. Let $\mathcal{W}_0' \equiv \{(b, \omega, \phi^h, \phi^d) : (b, \omega, 0, \phi^h, \phi^d) \in \mathcal{W}_0\}$. Then there exists a real analytic operator $\lambda[\cdot]$ of $\mathcal{W}_0'$ to $\mathbb{R} \setminus \{0\}$ such that

$$
\begin{aligned}
&\lim_{\epsilon \to 0^+} \Big((\log\epsilon) V^{(1)}[b, \omega, \epsilon, \phi^h, \phi^d] + V^{(2)}[b, \omega, \epsilon, \phi^h, \phi^d] \Big)^{-1} \\
&\qquad\qquad = \lambda[b, \omega, \phi^h, \phi^d]^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

for all $(b, \omega, \phi^h, \phi^d) \in \mathcal{W}_0'$.

If $n \geq 3$, then the expression $(\epsilon^{2-n} V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}])^{-1}$ has a real analytic continuation in the whole of $\mathcal{W}_0$ which vanishes for $\epsilon = 0$.

Proof. It is convenient to prove separately the lemma for $n = 2$ and $n \geq 3$.

Let $n = 2$. By the definition (2.6) of the fundamental solution $\Gamma(b, \cdot)$, we have $\Gamma(b, \epsilon z) = \frac{1}{2\pi} \frac{b+2}{2(b+1)} \log \epsilon + \Gamma(b, z)$, for all $b \neq -1$, $\epsilon > 0$, $z \in \mathbb{R}^n \setminus \{0\}$. So, if we set

$$
\begin{aligned}
V_{ij}^{(1)}[\mathbf{b}] &\equiv \frac{1}{2\pi} \frac{b+2}{2(b+1)} \int_{\partial\Omega^h} \mathcal{T}_h^{(i)}[\mathbf{b}] \tilde{\sigma}[\phi^h] \, d\sigma \cdot \int_{\partial\Omega^h} b^{(j)}[\phi^h] \circ \phi^h \tilde{\sigma}[\phi^h] \, d\sigma, \\
V_{ij}^{(2)}[\mathbf{b}] &\equiv \int_{\partial\Omega^h} \Big(\int_{\partial\Omega^h} \Gamma\big(b, \phi^h(x) - \phi^h(y)\big) \mathcal{T}_h^{(i)}[\mathbf{b}](y) \, \tilde{\sigma}[\phi^h](y) \, d\sigma_y \Big) \\
&\qquad \cdot \Big(b^{(j)}[\phi^h] \circ \phi^h(x) \Big) \, \tilde{\sigma}[\phi^h](x) \, d\sigma_x \\
&\quad + \int_{\partial\Omega^h} \Big(\int_{\partial\Omega^d} \Gamma(b, \omega + \epsilon\phi^h(x) - \phi^d(y)) \mathcal{T}_d^{(i)}[\mathbf{b}](y) \, \tilde{\sigma}[\phi^d](y) \, d\sigma_y \Big) \\
&\qquad \cdot \Big(b^{(j)}[\phi^h] \circ \phi^h(x) \Big) \, \tilde{\sigma}[\phi^h](x) \, d\sigma_x,
\end{aligned}
$$

for all $i, j = 1, \ldots, 3$ and for all $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$, then, by (2.74), $V^{(1)}$ and $V^{(2)}$ satisfy statement (i) of the lemma. Moreover, by Propositions 1.40, 2.39, 2.44, $V^{(1)}$ and $V^{(2)}$ are real analytic operators of $\mathcal{W}_0$ to $M_{\bar{n}\times\bar{n}}(\mathbb{R})$. To prove statement (iii) we note that, by Theorem 2.34, $\int_{\partial\Omega^h} b^{(j)}[\phi^h] \circ \phi^h \tilde{\sigma}[\phi^h] \, d\sigma = (1 - \delta_{j3}) |\phi^h(\partial\Omega^h)|^{1/2} e^{(j)}$, $j = 1, \ldots, 3$, and, by the definition of $\mathcal{T}_h^{(i)}$ in Proposition 2.44 and by Theorems 2.43 and 2.36, $\int_{\partial\Omega^h} \mathcal{T}_h^{(i)}[\mathbf{b}] \tilde{\sigma}[\phi^h] \, d\sigma = (1 - \delta_{i3}) |\phi^h(\partial\Omega^h)|^{1/2} e^{(i)}$, $i = 1, \ldots, 3$.

We now turn to prove statement (iii) for $n = 2$. Let $\mathbf{b}' \equiv (b, \omega, \phi^h, \phi^d) \in \mathcal{W}_0'$ and let $\mathbf{b} \equiv (b, 0, \omega, \phi^h, \phi^d)$, so that $\mathbf{b} \in \mathcal{W}_0$. By Proposition 2.44, the function $\mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}(x)$ of $x \in \phi^h(\partial\Omega^h)$ belongs to $(\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \cdot])_0$. So, by Theorem 2.16, we have

$$
v[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]|_{\phi^h(\partial\Omega^h)} \in (\mathcal{R}_{\mathbb{I}[\phi^h], \mathrm{loc}})|_{\phi^h(\partial\Omega^h)},
$$

and we deduce that $v[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]|_{\mathbb{I}[\phi^h]}$ is a function of $\mathcal{R}_{\mathbb{I}[\phi^h]}$. Therefore, there exists a unique couple $(s[\mathbf{b}'], c[\mathbf{b}']) \in \mathrm{Skew}(2, \mathbb{R}) \times \mathbb{R}^2$ such that

$$
v[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](\xi) = s[\mathbf{b}']\xi + c[\mathbf{b}'], \quad \forall \, \xi \in \mathbb{I}[\phi^h].
$$

We now prove that the map which takes $\mathbf{b}'$ to $(s[\mathbf{b}'], c[\mathbf{b}'])$ is real analytic from $\mathcal{W}_0'$ to $\mathrm{Skew}(2, \mathbb{R}) \times \mathbb{R}^2$. We fix a point $\mathbf{b}_1' \equiv (b_1, \omega_1, \phi_1^h, \phi_1^d)$ of $\mathcal{W}_0'$. Then there exist a point $\xi_1 \in \mathbb{R}^2$ and positive constant $r > 0$ such that the points ξ_1, $\xi_1 + r(1, 0)$, $\xi_1 + r(0, 1)$ are contained in $\mathbb{I}[\phi_1^h]$. Furthermore there exists an open neighborhood $\mathcal{W}_1'$ of $\mathbf{b}_1'$ in $\mathcal{W}_0'$ such that $\xi_1, \xi_1 + r(1, 0), \xi_1 + r(0, 1) \in \mathbb{I}[\phi^h]$ for all $\mathbf{b}' \equiv (b, \omega, \phi^h, \phi^d) \in \mathcal{W}_1'$. Now, the entry $s[\mathbf{b}']_{ij}$ of the matrix $s[\mathbf{b}']$ equals $(s[\mathbf{b}']e^{(j)})_i$, which in turn equals $r^{-1}(s[\mathbf{b}'](\xi_1 + r\, e^{(j)}) - s[\mathbf{b}']\xi_1)_i$

by linearity. We deduce that

$$
\begin{aligned}
s[\mathbf{b}'] = r^{-1}\Big(& v_i[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](\xi_1 + r\, e^{(j)}) \\
& -v_i[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](\xi_1)\Big)_{i,j=1,2}, \\
c[\mathbf{b}'] = v[b, & \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](x_1) - s[\mathbf{b}']x_1,
\end{aligned}
$$

for all $\mathbf{b}' \equiv (b, \omega, \phi^h, \phi^d) \in \mathcal{W}_1'$, where $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d)$. Then, by Proposition 2.39, the map which takes $\mathbf{b}' \in \mathcal{W}_1'$ to $(s[\mathbf{b}'], c[\mathbf{b}']) \in \mathrm{Skew}(2, \mathbb{R}) \times \mathbb{R}^2$ is real analytic. Since $\mathbf{b}_1'$ was an arbitrary point of $\mathcal{W}_0'$ we have our claim.

Moreover, we can prove that $s[\mathbf{b}'] \neq 0$ for all $\mathbf{b}' \in \mathcal{W}_0'$. Indeed if we assume that $s[\mathbf{b}'] = 0$ then $v[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}] = c[\mathbf{b}']$ on $\mathbb{I}[\phi^h]$. By Proposition 2.6 and Theorem 2.4, we deduce that $v[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]$ is constant on the whole of $\mathbb{R}^2$. Thus, by Proposition 2.6, we have

$$
\begin{aligned}
&\mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}(\xi) \\
&\quad = \lim_{t\to 0^+} T\big(b, Dv[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](\xi + t\nu_{\phi^h}(\xi))\big)\nu_{\phi^h}(\xi) \\
&\qquad - \lim_{t\to 0^+} T\big(b, Dv[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}](\xi - t\nu_{\phi^h}(\xi))\big)\nu_{\phi^h}(\xi) = 0,
\end{aligned}
$$

for all $\xi \in \phi^h(\partial\Omega^h)$. Such an equality is in contradiction with the statement of Proposition 2.44 and thus it must be $s[\mathbf{b}'] \neq 0$ for all $\mathbf{b}' \in \mathcal{W}_0'$. So, if we set

$$
\lambda[\mathbf{b}'] \equiv \int_{\partial\Omega^h} (s[\mathbf{b}']\phi^h) \cdot (b^{(3)}[\phi^h] \circ \phi^h)\, \tilde{\sigma}[\phi^h]\, d\sigma, \quad \forall\, \mathbf{b}' \in \mathcal{W}_0',
$$

then we have $\lambda[\mathbf{b}'] \neq 0$ and the map which takes $\mathbf{b}'$ to $\lambda[\mathbf{b}']$ is real analytic from $\mathcal{W}_0'$ to $\mathbb{R}$.

We now show that $V_{33}^{(2)}[\mathbf{b}] = \lambda[\mathbf{b}']$ for all $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d) \in \mathcal{W}_0$ with $\epsilon = 0$, where $\mathbf{b}' \equiv (b, \omega, \phi^h, \phi^d)$. First we note that $\mathcal{T}_d^{(3)}[\mathbf{b}] = 0$. Indeed, by statement Proposition 2.44, $\mathcal{T}_d^{(3)}[\mathbf{b}] \circ (\phi^d)^{(-1)} \in \mathrm{Ker}\mathbf{K}^*_{\phi^d(\partial\Omega^d)}[b, \cdot]$, and by Theorem 2.17, $\mathrm{Ker}\mathbf{K}^*_{\phi^d(\partial\Omega^d)}[b, \cdot] = \{0\}$. So, by the definition, we have

$$
\begin{aligned}
V_{33}^{(2)}[\mathbf{b}] &= \big\langle v_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}] \,\big|\, b^{(3)}[\phi^h] \big\rangle_{\phi^h(\partial\Omega^h)} \\
&= \big\langle s[\mathbf{b}'] + c[\mathbf{b}'] \,\big|\, b^{(3)}[\phi^h] \big\rangle_{\phi^h(\partial\Omega^h)} = \lambda[\mathbf{b}'] + \big\langle c[\mathbf{b}'] \,\big|\, b^{(3)}[\phi^h] \big\rangle_{\phi^h(\partial\Omega^h)} = \lambda[\mathbf{b}'],
\end{aligned}
\tag{2.76}
$$

where $\mathbf{b}' \equiv (b, \omega, \phi^h, \phi^d)$.

Finally we are ready to calculate the limit value as $\epsilon \to 0^+$ of the inverse of the matrix $\epsilon^{1-n}V[b,\omega,\epsilon,\phi^h,\phi^d]$. We set

$$\begin{aligned} A[\mathbf{b}] &\equiv (\log\epsilon)\frac{|\phi^h(\partial\Omega^h)|}{2\pi}\frac{b+2}{2(b+1)}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} V^{(2)}_{11}[\mathbf{b}] & V^{(2)}_{12}[\mathbf{b}] \\ V^{(2)}_{21}[\mathbf{b}] & V^{(2)}_{22}[\mathbf{b}]\end{pmatrix}, \\ B[\mathbf{b}] &\equiv \begin{pmatrix} V^{(2)}_{13}[\mathbf{b}] \\ V^{(2)}_{23}[\mathbf{b}]\end{pmatrix}, \\ C[\mathbf{b}] &\equiv \begin{pmatrix} V^{(2)}_{31}[\mathbf{b}] & V^{(2)}_{32}[\mathbf{b}]\end{pmatrix}, \\ D[\mathbf{b}] &\equiv V^{(2)}_{33}[\mathbf{b}], \end{aligned}$$

for all $\mathbf{b}\in\mathcal{W}_0$. So that

$$\epsilon^{1-n}V[\mathbf{b}] = \begin{pmatrix} A[\mathbf{b}] & B[\mathbf{b}] \\ C[\mathbf{b}] & D[\mathbf{b}]\end{pmatrix}.$$

Then we fix a point $\mathbf{b}\equiv(b,\omega,0,\phi^h,\phi^d)\in\mathcal{W}_0$ with $\epsilon=0$. We denote by $\mathbf{b}^\epsilon$ the point $\mathbf{b}^\epsilon\equiv(b,\omega,\epsilon,\phi^h,\phi^d)$ and we note that, for $\epsilon>0$ close to 0, $\mathbf{b}^\epsilon\in\mathcal{W}_0$ and $A[\mathbf{b}^\epsilon]$ is an invertible matrix. Then we consider the Schur complement $S_A[\mathbf{b}^\epsilon]$ of $A[\mathbf{b}^\epsilon]$ (cf., *e.g.*, Carlson [4, §2].) $S_A[\mathbf{b}^\epsilon]$ is defined by

$$S_A[\mathbf{b}^\epsilon]\equiv D[\mathbf{b}^\epsilon]-C[\mathbf{b}^\epsilon]A[\mathbf{b}^\epsilon]^{-1}B[\mathbf{b}^\epsilon].$$

It is easily seen that

$$\lim_{\epsilon\to0^+}A[\mathbf{b}^\epsilon]^{-1}=0 \quad\text{and}\quad \lim_{\epsilon\to0^+}S_A[\mathbf{b}^\epsilon]=\lambda[\mathbf{b}].$$

Thus, $S_A[\mathbf{b}^\epsilon]$ does not vanish for $\epsilon>0$ in a neighborhood of 0 and the inverse of $\epsilon^{1-n}V[\mathbf{b}^\epsilon]$ is delivered by the following matrix,

$$\begin{pmatrix} A[\mathbf{b}^\epsilon]^{-1}+A[\mathbf{b}^\epsilon]^{-1}B[\mathbf{b}^\epsilon]S_A[\mathbf{b}^\epsilon]^{-1}C[\mathbf{b}^\epsilon]A[\mathbf{b}^\epsilon]^{-1} & -A[\mathbf{b}^\epsilon]^{-1}B[\mathbf{b}^\epsilon]S_A[\mathbf{b}^\epsilon]^{-1} \\ -S_A[\mathbf{b}^\epsilon]^{-1}C[\mathbf{b}^\epsilon]A[\mathbf{b}^\epsilon]^{-1} & S_A[\mathbf{b}^\epsilon]^{-1}\end{pmatrix}.$$

Statement (iii) immediately follows.

For $n\geq3$ the proof is simpler. We note that $\Gamma(b,\epsilon z)=\epsilon^{2-n}\Gamma(b,z)$, for all $b\neq-1$, $\epsilon>0$, $z\in\mathbb{R}^n\setminus\{0\}$. Then we set

$$\begin{aligned} V^{(1)}_{ij}[\mathbf{b}] \equiv \int_{\partial\Omega^h}&\left(\int_{\partial\Omega^h}\Gamma\big(b,\phi^h(x)-\phi^h(y)\big)\mathcal{T}^{(i)}_h[\mathbf{b}](y)\,\tilde\sigma[\phi^h](y)\,d\sigma_y\right) \\ &\cdot\Big(b^{(j)}[\phi^h]\circ\phi^h(x)\Big)\,\tilde\sigma[\phi^h](x)\,d\sigma_x \\ V^{(2)}_{ij}[\mathbf{b}] \equiv \int_{\partial\Omega^h}&\left(\int_{\partial\Omega^d}\Gamma(b,\omega+\epsilon\phi^h(x)-\phi^d(y))\mathcal{T}^{(i)}_d[\mathbf{b}](y)\,\tilde\sigma[\phi^d](y)\,d\sigma_y\right) \\ &\cdot\Big(b^{(j)}[\phi^h]\circ\phi^h(x)\Big)\,\tilde\sigma[\phi^h](x)\,d\sigma_x, \end{aligned}$$

for all $\mathbf{b}\in\mathcal{W}_0$ and for all $i,j=1,\dots,\bar n$. By equation (2.74), statement (i) immediately follows. Moreover, by Proposition 2.39, $V^{(1)}$ and $V^{(2)}$ are real analytic from $\mathcal{W}_0$ to $M_{\bar{n}\times\bar{n}}(\mathbb{R})$. Let $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d) \in \mathcal{W}_0$ with $\epsilon = 0$. To verify that $V^{(1)}[\mathbf{b}]$ is an invertible matrix, we note that

$$V^{(1)}[\mathbf{b}] = \Big(\big\langle v_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(i)}[\mathbf{b}] \circ (\phi^h)^{(-1)}] \,\big|\, b^{(j)}[\phi^h]\big\rangle_{\phi^h(\partial\Omega^h)}\Big)_{i,j=1,\dots,\bar{n}}.$$

The matrix in the right hand side is invertible by Theorem 2.16 and Proposition 2.44. Finally, we deduce statement (iii) by a straightforward calculation. □

Now we are ready to draw out conclusions from Theorems 2.43, 2.50 and Lemmas 2.51 and 2.52.

Theorem 2.53. *Let the notation introduced in subsection 2.2.1 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. *Let* $\mathcal{W}_0$ *be the neighborhood of* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d)$ *introduced in Proposition 2.44. Let* $V^{(1)}$, $V^{(2)}$ *be as in Lemma 2.52. Let* Ω *be an open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\Omega \subset \mathbb{I}[\phi_0^d] \setminus \{\omega_0\}$. *Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{e}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *and real analytic operators* $U^{(1)}$ *and* $U^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, *of* $\mathcal{U}$ *to* $C(\mathrm{cl}\Omega, \mathbb{R}^n)$, *endowed with the norm of the uniform convergence, such that the following conditions hold.*

(i) $\mathrm{cl}\Omega \subset \mathbb{A}[\omega, \epsilon, \phi^h, \phi^o]$ *for all* $(b, \omega, \epsilon, \phi^h, \phi^o, g^h, g^o) \in \mathcal{U}$.

(ii) $(b, \omega, \epsilon, \phi^h, \phi^o) \in \mathcal{W}_0$ *for all* $(b, \omega, \epsilon, \phi^h, \phi^o, g^h, g^d) \in \mathcal{U}$.

(iii) We have

$$u[\mathbf{e}](\xi) = U^{(1)}[\mathbf{e}](\xi) + \sum_{i,j=1}^{\bar{n}} \Big(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij} U^{(2)}_{ij}[\mathbf{e}](\xi), \tag{2.77}$$

for all $\xi \in \mathrm{cl}\Omega$, *and all* $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$ *with* $\epsilon > 0$, *where* $\gamma_n(\epsilon) \equiv \log\epsilon$ *if* $n = 2$, *and* $\gamma_n(\epsilon) \equiv \epsilon^{2-n}$ *if* $n \geq 3$. *Here we abbreviated* $(b, \omega, \epsilon, \phi^h, \phi^d)$ *as* $\mathbf{b}$.

(iv)

$$U^{(1)}[b, \omega, 0, \phi^h, \phi^d, g^h, g^d](\xi) = u^d[b, \phi^d, g^d](\xi)$$

for all $\xi \in \mathrm{cl}\Omega$ *and for all* $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$, *where* $u^d[b, \phi^d, g^d]$ *is the solution of the first interior boundary value problem in* $\mathbb{I}[\phi^d]$ *with boundary data* $g^d \circ (\phi^d)^{(-1)}$ *(see Definition 2.31.)*

(v) Let $\mathbf{e} \equiv (b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$, *we denote by* $\mathbf{e}^\epsilon$, $\mathbf{b}^\epsilon$ *the points* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ *and* $(b, \omega, \epsilon, \phi^h, \phi^d)$, *respectively, for all* $\epsilon > 0$. *Then*

$$\lim_{\epsilon\to 0^+} \sum_{i,j=1}^{\bar{n}} \Big(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}^\epsilon] + V^{(2)}[\mathbf{b}^\epsilon]\Big)^{-1}_{ij} U^{(2)}_{ij}[\mathbf{e}^\epsilon](\xi) = 0 \tag{2.78}$$

uniformly for $\xi \in \mathrm{cl}\Omega$.

Proof. Possibly shrinking the neighborhood $\mathcal{U}_1$ of Theorem 2.50, we can assume that condition (i) holds and that $(b, \omega, \epsilon, \phi^h, \phi^d)$ belongs to the domain $\mathcal{W}_0$ of $V^{(1)}$, $V^{(2)}$ for all $\mathbf{e} \in \mathcal{U}$ (cf. Lemma 2.52.) Then we denote by $U^{(1)}$ the operator which takes $\mathbf{e} \in \mathcal{U}$ to the function of $C(\mathrm{cl}\Omega, \mathbb{R}^n)$ defined by the right hand side of (2.71), and we denote by $U^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, the operator which takes $\mathbf{e} \in \mathcal{U}$ to the function $U^{(2)}_{ij}[\mathbf{e}]$ of $C(\mathrm{cl}\Omega, \mathbb{R}^n)$ defined by

$$
\begin{aligned}
U^{(2)}_{ij}[\mathbf{e}](\xi) \equiv \Big(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}^{(i)}_h[\mathbf{b}]\,\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} g^d \cdot \mathcal{T}^{(i)}_d[\mathbf{b}]\,\tilde{\sigma}[\phi^d]\, d\sigma \Big) \\
\cdot \Big(\int_{\partial\Omega^h} \Gamma(b, \xi - \omega - \epsilon\phi^h(y)) \mathcal{T}^{(j)}_h[\mathbf{b}](y)\,\tilde{\sigma}[\phi^h](y)\, d\sigma_y \\
+ \int_{\partial\Omega^d} \Gamma(b, \xi - \phi^d(y)) \mathcal{T}^{(j)}_d[\mathbf{b}](y)\,\tilde{\sigma}[\phi^d](y)\, d\sigma_y \Big),
\end{aligned}
$$

for all $\xi \in \mathrm{cl}\Omega$, where G is defined as in Theorem 2.25 and as usual $\mathbf{a} \equiv (\omega, \epsilon, \phi^h, \phi^d)$ and $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d)$. Then by Propositions 2.39 and 2.44 and by Theorem 2.50, $U^{(1)}$ and $U^{(2)}_{ij}$ are real analytic for all $i, j = 1, \dots, \bar{n}$. Moreover, statement (iii) follows by Lemmas 2.51 and 2.52, statement (iv) follows by Theorems 2.27 and 2.36.

We now prove statement (v). First let $n = 2$. By Lemma 2.52, the limit in (2.78) equals $\lambda[b, \omega, \phi^h, \phi^d]^{-1} U^{(2)}_{33}[\mathbf{e}](\xi)$. Let $\mathbf{b} \equiv (b, \omega, 0, \phi^h, \phi^d)$. By Theorems 2.36 and 2.17, we have $\mathcal{T}^{(3)}_d[\mathbf{b}] = 0$. Therefore $U^{(2)}_{33}[\mathbf{e}](\xi)$ equals

$$
\Big(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}^{(3)}_h[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma \Big) \Big(\Gamma(b, \xi - \omega) \int_{\partial\Omega^h} \mathcal{T}^{(3)}_h[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma \Big),
$$

for all $\xi \in \mathrm{cl}\Omega$, and the last integral vanishes by the definition of $\mathcal{T}^{(3)}_h$ (see Proposition 2.44.) Hence statement (iv) for $n = 2$ follows. For $n \geq 3$, it is an immediate consequence of Lemma 2.52. □

We conclude this subsection by noting that, if $n \geq 3$, then the right hand side of (2.77) can be continued real analytically in the whole of $\mathcal{U}$ (cf. statement (iii) of Lemma 2.52.)

2.2.8 The corresponding energy integral

In this subsection we show that the energy integral $\int_{\mathbb{A}[\mathbf{a}]} \mathbf{E}[b](u, u) d\xi$ of the solution u of problem 2.29 can be expressed by means of real analytic operators defined in a whole open neighborhood of a given point $(b_0, \omega_0, 0, \phi^h_0, \phi^d_0, g^h, g^d)$ and by completely known functions of ϵ. We also investigate the behavior of the energy integral as $\epsilon \to 0^+$. To do so we need the following technical Lemma 2.54, which is an immediate consequence of Lemma 1.36, and can be proved by arguing as in subsection 1.3.2 of the previous chapter.

Lemma 2.54. *Let $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ with Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ connected. Let $\phi_0 \in C^{m,\lambda}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$. Let ω, δ, $\Omega_{\omega,\delta}$, $\Omega^+_{\omega,\delta}$, $\Omega^-_{\omega,\delta}$, $\mathcal{W}_0$ be as in Proposition 1.6. Let $w^+_{\phi(\partial\Omega)}[b,\mu]$ and $w^-_{\phi(\partial\Omega)}[b,\mu]$ be as in Proposition 2.7 for all $(b,\phi,\mu) \in \mathscr{B} \times \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$. Then the map of $\mathscr{B} \times \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\lambda}(\mathrm{cl}\Omega^+_{\omega,\delta},\mathbb{R}^n)$, which takes a triple (b,ϕ,μ) to the function $w^+_{\phi(\partial\Omega)}[b,\mu] \circ \mathbf{E}_0[\phi](x)$ of $x \in \mathrm{cl}\Omega^+_{\omega,\delta}$, is real analytic. Similarly, the map of $\mathscr{B} \times \mathcal{W}_0 \times C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\lambda}(\mathrm{cl}\Omega^-_{\omega,\delta},\mathbb{R}^n)$, which takes a triple (b,ϕ,μ) to the function $w^-_{\phi(\partial\Omega)}[b,\mu] \circ \mathbf{E}_0[\phi](x)$ of $x \in \mathrm{cl}\Omega^-_{\omega,\delta}$, is real analytic.*

Theorem 2.55. *Let the notation introduced in subsection 2.2.1 hold. Let $\mathbf{e}_0 \equiv (b_0,\omega_0,0,\phi^h_0,\phi^d_0,g^h,g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. Let $\mathcal{W}_0$ be the neighborhood of $\mathbf{b}_0 \equiv (b_0,\omega_0,0,\phi^h_0,\phi^d_0)$ introduced in Proposition 2.44. Let $V^{(1)}$, $V^{(2)}$ be as in Lemma 2.52. Then there exist an open neighborhood $\mathcal{U}$ of $\mathbf{e}_0$ in $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ and real analytic operators $E^{(1)}$ and $E^{(2)}_{ij}$, $i,j = 1,\dots,\bar{n}$, of $\mathcal{U}$ to $\mathbb{R}$, such that $(b,\omega,\epsilon,\phi^h,\phi^d) \in \mathcal{W}_0$ for all $\mathbf{e} \equiv (\omega,\epsilon,\phi^h,\phi^d,g^h,g^d) \in \mathcal{U}$ and*

$$\int_{\mathbb{A}[\mathbf{a}]} \mathbf{E}[b](u[\mathbf{e}],u[\mathbf{e}])\,d\xi \tag{2.79}$$
$$= E^{(1)}[\mathbf{e}] + \sum_{i,j=1}^{\bar{n}} \Big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij} E^{(2)}_{ij}[\mathbf{e}],$$

for all $\mathbf{e} \equiv (b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d) \in \mathcal{U}$ with $\epsilon > 0$, where $\mathbf{a} \equiv (\omega,\epsilon,\phi^h,\phi^d)$, and $\mathbf{b} \equiv (b,\omega,\epsilon,\phi^h,\phi^d)$, and $\gamma_n(\epsilon)$ is defined as in Theorem 2.53.

Moreover, if $\mathbf{e} \equiv (b,\omega,0,\phi^h,\phi^d,g^h,g^d)$ belongs to $\mathcal{U}$ and we set $\mathbf{e}^\epsilon \equiv (b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d)$, $\mathbf{b}^\epsilon \equiv (b,\omega,\epsilon,\phi^h,\phi^d)$ for all $\epsilon > 0$, then

$$\lim_{\epsilon\to 0^+} E^{(1)}[\mathbf{e}^\epsilon] + \sum_{i,j=1}^{\bar{n}} \Big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}^\epsilon] + V^{(2)}[\mathbf{b}^\epsilon]\Big)^{-1}_{ij} E^{(2)}_{ij}[\mathbf{e}^\epsilon] \tag{2.80}$$
$$= \int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u^d[b,\phi^d,g^d],u^d[b,\phi^d,g^d])\,d\xi$$
$$+\delta_{2,n}\int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u^h[b,\phi^h,g^h],u^h[b,\phi^h,g^h])\,d\xi,$$

where $\delta_{2,n} = 1$ if $n = 2$, and $\delta_{2,n} = 0$ if $n \neq 2$, and $u^d[b,\phi^d,g^d]$ is the solution of the first interior boundary value problem in $\mathbb{I}[\phi^d]$ with boundary data $g^d \circ (\phi^d)^{(-1)}$, and $u^h[b,\phi^h,g^h]$ is the solution of the first exterior boundary value problem in $\mathbb{E}[\phi^h]$ with boundary data $g^h \circ (\phi^h)^{(-1)}$ (see Definition 2.31.).

Proof. We now exploit the results that we have summarized in subsection 1.2.1 of the previous chapter. By Proposition 1.6, there exist a neighborhood $\mathcal{W}_0^h$ of ϕ_0^h in $C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega^h}$, and a neighborhood $\mathcal{W}_0^d$ of ϕ_0^d in $C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega^d}$, and real analytic extension operators

$$\mathbf{E}_0^h:\mathcal{W}_0^h\to C^{m,\lambda}(\mathrm{cl}\Omega^h_{\omega^h,\delta^h},\mathbb{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\Omega^h_{\omega^h,\delta^h}},$$
$$\mathbf{E}_0^d:\mathcal{W}_0^d\to C^{m,\lambda}(\mathrm{cl}\Omega^d_{\omega^d,\delta^d},\mathbb{R}^d)\cap\mathcal{A}'_{\mathrm{cl}\Omega^d_{\omega^d,\delta^d}},$$

where ω^h, δ^h, $\Omega^h_{\omega^h,\delta^h}$, ω^d, δ^d, $\Omega^d_{\omega^d,\delta^d}$ are defined as in subsection 1.2.1, with the obvious modification. Possibly choosing a smaller δ^d, we can also assume that $\omega_0\notin\mathbf{E}_0^d[\phi_0^d](\mathrm{cl}\Omega^d_{\omega^d,\delta^d})$. Then there exists an open neighborhood $\mathcal{U}$ of $\mathbf{e}_0$ in $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times C^{m,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m,\lambda}(\partial\Omega^d,\mathbb{R}^n)$ such that, $\mathcal{U}$ is contained in the neighborhood $\mathcal{O}_1$ of Theorem 2.50, and we have $\phi^h\in\mathcal{W}_0^h$, and $\phi^d\in\mathcal{W}_0^d$, and $(b,\omega,\epsilon,\phi^h,\phi^d)\in\mathcal{W}_0$, and

$$(\omega+\epsilon\mathbf{E}_0^h[\phi^h](\mathrm{cl}\Omega^h_{\omega^h,\delta^h}))\cap\phi^d(\partial\Omega^d)=\emptyset,$$

and

$$(\omega+\epsilon\phi^h(\partial\Omega^h))\cap\mathbf{E}_0^d[\phi^d](\mathrm{cl}\Omega^d_{\omega^d,\delta^d})=\emptyset,$$

for all $\mathbf{e}\equiv(b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d)\in\mathcal{U}$. Then, by Theorem 2.3, we have

$$\begin{aligned}
&\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]}\mathbf{E}[b](u[\mathbf{e}],u[\mathbf{e}])\,d\xi \qquad (2.81)\\
&\quad=-\epsilon^{n-2}\int_{\partial\Omega^h}\Big\{T\Big(b,D\Big(u[\mathbf{e}]\circ(\omega+\epsilon\mathbf{E}_0^h[\phi^h])\Big)(x)\,\Big(D\mathbf{E}_0^h[\phi^h](x)\Big)^{-1}\Big)\\
&\qquad\qquad\cdot\nu_{\phi^h}\circ\phi^h(x)\Big\}\cdot g^h(x)\,\tilde\sigma[\phi^h](x)\,d\sigma_x\\
&\quad+\int_{\partial\Omega^d}\Big\{T\Big(b,D\Big(u[\mathbf{e}]\circ\mathbf{E}_0^d[\phi^d]\Big)(x)\,\Big(D\mathbf{E}_0^d[\phi^d](x)\Big)^{-1}\Big)\\
&\qquad\qquad\cdot\nu_{\phi^d}\circ\phi^d(x)\Big\}\cdot g^d(x)\,\tilde\sigma[\phi^d](x)\,d\sigma_x,
\end{aligned}$$

for all $\mathbf{e}\equiv(b,\omega,\epsilon,\phi^h,\phi^d,g^h,g^d)\in\mathcal{U}$ with $\epsilon>0$.

We now consider the first integral in the right hand side of (2.81). By equations (2.71) and (2.72), we have

$$\begin{aligned}
&u_r[\mathbf{e}]\circ(\omega+\epsilon\mathbf{E}_0^h[\phi^h])(x) \qquad (2.82)\\
&\quad=-w^-_{\phi^h(\partial\Omega^h)}[b,\mathcal{M}^h[\mathbf{e}]\circ(\phi^h)^{(-1)}]\circ\mathbf{E}_0^h[\phi^h](x)\\
&\qquad-\int_{\partial\Omega^d}\Big\{\Big[T(b,D\Gamma^{(i)}(b,\omega+\epsilon\mathbf{E}_0^h[\phi^h](x)-\phi^d(y)))\\
&\qquad\qquad\cdot\nu_{\phi^d}\circ\phi^d(y)\Big]\cdot\mathcal{M}^d[\mathbf{e}]\Big\}_{i=1,\dots,n}\tilde\sigma[\phi^d](y)\,d\sigma_y,
\end{aligned}$$

and

$$
\begin{aligned}
&u_s[\mathbf{e}] \circ (\omega + \epsilon \mathbf{E}^h_0[\phi^h])(x) \qquad (2.83)\\
&\quad = \sum_{i,j=1}^{\bar{n}} \Big(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}^{(i)}_h[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} g^d \cdot \mathcal{T}^{(i)}_d[\mathbf{b}]\tilde{\sigma}[\phi^d]\, d\sigma \Big) \\
&\qquad\quad \cdot \Big(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij} \\
&\qquad\quad \cdot \Big(\int_{\partial\Omega^h} \Gamma\big(b, \epsilon(\mathbf{E}^h_0[\phi^h](x) - \phi^h(y))\big) \mathcal{T}^{(j)}_h[\mathbf{b}](y)\, \tilde{\sigma}[\phi^h](y)\; d\sigma_y \\
&\qquad\qquad + \int_{\partial\Omega^d} \Gamma(b, \omega + \epsilon \mathbf{E}^h_0[\phi^h](x) - \phi^d(y)) \mathcal{T}^{(j)}_d[\mathbf{b}](y)\, \tilde{\sigma}[\phi^d](y)\; d\sigma_y \Big),
\end{aligned}
$$

for all $x \in (\Omega^h)^-_{\omega^h,\delta^h}$ (cf. Proposition 1.6) and all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ with $\epsilon > 0$, where as usual we abbreviated $(b, \omega, \epsilon, \phi^h, \phi^d)$ as $\mathbf{b}$. Then, by Propositions 1.6, 2.39, 2.44, and by Theorem 2.50, and by Lemma 2.54, and by equations (2.82) and (2.83), we deduce that there exist real analytic operators $G^{(1)}$, $G^{(3)}_j$, $G^{(4)}_j$, $j = 1, \dots, \bar{n}$, of $\mathcal{U}$ to $C^{m,\lambda}(\mathrm{cl}(\Omega^h)^-_{\omega^h,\delta^h}, \mathbb{R}^n)$, and real analytic operators $G^{(2)}_i$, $i = 1, \dots, \bar{n}$, of $\mathcal{U}$ to $\mathbb{R}$, such that

$$
\begin{aligned}
&u[\mathbf{e}] \circ (\omega + \epsilon \mathbf{E}^h_0[\phi^h])(x) = G^{(1)}[\mathbf{e}](x) \\
&\quad + \sum_{i,j=1}^{\bar{n}} G^{(2)}_i[\mathbf{e}]\big(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\big)^{-1}_{ij} \big(\gamma_n(\epsilon) G^{(3)}_j[\mathbf{e}](x) + G^{(4)}_j[\mathbf{e}](x)\big),
\end{aligned}
$$

for all $x \in (\Omega^h)^-_{\omega^h,\delta^h}$, and all $\mathbf{e} \in \mathcal{U}$ with $\epsilon > 0$. Indeed, we can take as $G^{(1)}[\mathbf{e}]$ the function of $\mathrm{cl}(\Omega^h)^-_{\omega^h,\delta^h}$ defined by the right hand side of (2.82) and we can take as $G^{(2)}_i[\mathbf{p}]$ the first factor in parentheses in the right hand side of (2.83), for all $i = 1, \dots, \bar{n}$. To define $G^{(3)}_j[\mathbf{p}]$ we consider two different cases. If $n = 2$ we take

$$
G^{(3)}_j[\mathbf{e}](x) \equiv \frac{|\phi^h(\partial\Omega^h)|^{1/2}}{2\pi} \frac{b+2}{2(b+1)} (\delta_{ij})_{i=1,2}, \quad \forall\, x \in \mathrm{cl}(\Omega^h)^-_{\omega^h,\delta^h},\ j = 1, 2, 3.
$$

In particular, we have $G^{(3)}_3[\mathbf{e}](x) = 0$ identically. If $n \geq 3$ we take as $G^{(3)}_j[\mathbf{e}](x)$ the function of $x \in \mathrm{cl}(\Omega^h)^-_{\omega^h,\delta^h}$ given by

$$
\int_{\partial\Omega^h} \Gamma(\mathbf{E}^h_0[\phi^h](x) - \phi^h(y)) \mathcal{T}^{(j)}_h[\mathbf{b}](y) \tilde{\sigma}[\phi^h](y)\, d\sigma_y, \quad \forall\, x \in \mathrm{cl}(\Omega^h)^-_{\omega^h,\delta^h}.
$$

Then we take as $G^{(4)}_j[\mathbf{e}]$ the difference between the last factor in parentheses in the right hand side of (2.83) and $\gamma_n(\epsilon) G^{(3)}_j[\mathbf{e}]$, both for $n = 2$ and $n \geq 3$.

Clearly

$$\partial_k\big(u[\mathbf{e}]\circ(\omega+\epsilon\mathbf{E}_0^h[\phi^h])\big)=\partial_k G^{(1)}[\mathbf{e}]$$
$$+\sum_{i,j=1}^{\bar{n}} G_i^{(2)}[\mathbf{e}]\big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}]+V^{(2)}[\mathbf{b}]\big)^{-1}_{ij}\big(\gamma_n(\epsilon)\partial_k G_j^{(3)}[\mathbf{e}]+\partial_k G_j^{(4)}[\mathbf{e}]\big),$$

in $(\Omega^h)^-_{\omega^h,\delta^h}$, for all $k=1,\dots,n$ and all $\mathbf{e}\in\mathcal{U}$ with $\epsilon>0$. Hence the first integral in the right hand side of (2.81) equals

$$\int_{\partial\Omega^h}\Big\{T\Big(b,(DG^{(1)}[\mathbf{e}])(D\mathbf{E}_0^h[\phi^h])^{-1}\Big)\nu_{\phi^h}\circ\phi^h\Big\}\cdot g^h\,\tilde{\sigma}[\phi^h]\,d\sigma \tag{2.84}$$
$$+\sum_{i,j=1}^{\bar{n}} G_i^{(2)}[\mathbf{e}]\Big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}]+V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij}$$
$$\cdot\int_{\partial\Omega^h}\Big\{T\Big(b,\big(\gamma_n(\epsilon)DG_j^{(3)}[\mathbf{e}]+DG_j^{(4)}[\mathbf{e}]\big)\big(D\mathbf{E}_0^h[\phi^h]\big)^{-1}\Big)$$
$$\cdot\nu_{\phi^h}\circ\phi^h\Big\}\cdot g^h\,\tilde{\sigma}[\phi^h]\,d\sigma,$$

for all $\mathbf{e}\in\mathcal{U}$ with $\epsilon>0$.

Now let $\epsilon=0$. By Theorems 2.45, 2.50, we have

$$\begin{aligned} G^{(1)}[\mathbf{e}] &= u^d[b,\phi^d,g^d](\omega)+u_r^h[b,\phi,g^d]\circ\mathbf{E}_0^h[\phi^h],\\ DG^{(1)}[\mathbf{e}] &= D(u_r^h[b,\phi^h,g^h]\circ\mathbf{E}_0^h[\phi^h]),\end{aligned}$$

in $(\Omega^h)^-_{\omega^h,\delta^h}$, for all $\mathbf{e}\equiv(b,\omega,0,\phi^h,\phi^d,g^h,g^d)\in\mathcal{U}$, where $u_r^h[b,\phi^h,g^h]$ is the function introduced in Definition 2.31. Therefore, if $\epsilon=0$ the first integral in (2.84) equals

$$\int_{\phi^h(\partial\Omega^h)}\Big[T(b,Du_r^h[b,\phi^h,g^h])\nu_{\phi^h}\Big]^-\cdot g^h\circ(\phi^h)^{(-1)}\,d\sigma,$$

which in turn equals

$$-\int_{\mathbb{E}[\phi^h]}\mathbf{E}[b](u_r^h[b,\phi^h,g^h],u_r^h[b,\phi^h,g^h])\,d\xi$$

by Theorem 2.3. Moreover, if $n=2$, then $DG_j^{(3)}[\mathbf{e}]=0$ for all $j=1,\dots,3$. So, for $n=2$ and $\epsilon=0$, the last integral in (2.84) equals

$$\int_{\phi^h(\partial\Omega^h)}\Big[T\Big(b,Dv_{\phi^h(\partial\Omega^h)}[b,\mathcal{T}_h^{(j)}[\mathbf{b}]\circ(\phi^h)^{(-1)}]\Big)\nu_{\phi^h}\Big]^-\cdot g^h\circ(\phi^h)^{(-1)}\,d\sigma$$
$$=\int_{\phi^h(\partial\Omega^h)}\mathbf{K}^*_{\phi^h(\partial\Omega^h)}[b,\mathcal{T}_h^{(j)}[\mathbf{b}]\circ(\phi^h)^{(-1)}]\cdot g^h\circ(\phi^h)^{(-1)}\,d\sigma.$$

Since $\mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)} \in \mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b,\cdot]$ (cf. Proposition 2.44) we have $\mathbf{K}^*_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}] = \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}$. We conclude that, for $n = 2$ and $\epsilon = 0$ the last integral in (2.84) equals

$$\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(j)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\; d\sigma.$$

We now consider the second integral in the right hand side of (2.81). By equations (2.71) and (2.72), we have

$$\begin{aligned}
&u_r[\mathbf{e}] \circ \mathbf{E}_0^d[\phi^d](x) \qquad (2.85)\\
&\quad = \epsilon^{n-1} \int_{\partial\Omega^h} \Big\{ \Big[T(b, D\Gamma^{(i)}(b, \mathbf{E}_0^d[\phi^d](x) - \omega - \epsilon\phi^h(y))) \\
&\qquad\qquad \cdot \nu_{\phi^h} \circ \phi^h(y) \Big] \cdot \mathcal{M}^h[\mathbf{e}] \Big\}_{i=1,\ldots,n} \tilde{\sigma}[\phi^h](y)\; d\sigma_y, \\
&\qquad + w^+_{\phi^d(\partial\Omega^d)} \big[b, \mathcal{M}^d[\mathbf{e}] \circ (\phi^d)^{(-1)} \big] \circ \mathbf{E}_0^d[\phi^d](x),
\end{aligned}$$

and

$$\begin{aligned}
&u_s[\mathbf{e}] \circ \mathbf{E}_0^d[\phi^d](x) \qquad (2.86)\\
&\quad = \sum_{i,j=1}^{\bar{n}} \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} g^d \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^d]\, d\sigma \right) \\
&\qquad \cdot \Big(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}] \Big)^{-1}_{ij} \\
&\qquad \cdot \bigg(\int_{\partial\Omega^h} \Gamma\big(\mathbf{E}_0^d[\phi^d](x) - \omega - \epsilon\phi^h(y)\big)\big) \mathcal{T}_h^{(j)}[\mathbf{b}](y)\, \tilde{\sigma}[\phi^h](y)\; d\sigma_y \\
&\qquad\quad + \int_{\partial\Omega^d} \Gamma(\mathbf{E}_0^d[\phi^d](x) - \phi^d(y)) \mathcal{T}_d^{(j)}[\mathbf{b}](y)\, \tilde{\sigma}[\phi^d](y)\; d\sigma_y \bigg),
\end{aligned}$$

for all $x \in (\Omega^d)^+_{\omega^d,\delta^d}$ (cf. Proposition 1.6) and all $\mathbf{e} \in \mathcal{U}$ with $\epsilon > 0$. Then, by Propositions 1.6, 2.39, 2.44, and by Theorem 2.50, and by Lemma 2.54, and by equations (2.85) and (2.86), there exist real analytic operators $G^{(5)}$, $G_j^{(6)}$, $j = 1, \ldots, \bar{n}$, of $\mathcal{U}$ to $C^{m,\lambda}(\mathrm{cl}\mathbb{A}^+_\delta, \mathbb{R}^n)$, such that

$$u[\mathbf{e}] \circ \mathbf{E}_0^d[\phi^d] = G^{(5)}[\mathbf{e}] + \sum_{i,j=1}^{\bar{n}} G_i^{(2)}[\mathbf{e}](\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}])^{-1}_{ij} G_j^{(6)}[\mathbf{e}]$$

in $(\Omega^d)^+_{\omega^d,\delta^d}$, for all $\mathbf{e} \in \mathcal{U}$ with $\epsilon > 0$. Indeed, we can take as $G^{(5)}[\mathbf{e}](x)$ the function defined by the right hand side of (2.85), and we can take as $G_j^{(6)}[\mathbf{e}](x)$ the function defined by the last factor in parentheses in the right hand side of (2.86), and we note that $G_i^{(2)}[\mathbf{e}]$ coincides with the first factor in parentheses in the right hand side of (2.86). Clearly

$$\partial_k\big(u[\mathbf{e}]\circ \mathbf{E}_0^d[\phi^d]\big) = \partial_k G^{(5)}[\mathbf{e}] + \sum_{i,j=1}^{\bar{n}} G_i^{(2)}[\mathbf{e}](\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}])_{ij}^{-1}\partial_k G_j^{(6)}[\mathbf{e}]$$

in $(\Omega^d)^+_{\omega^d,\delta^d}$, for all $k = 1,\ldots,n$, and all $\mathbf{e}\in\mathcal{U}$ with $\epsilon > 0$. Then the second integral in the right hand side of (2.81) equals

$$\begin{aligned}&\int_{\partial\Omega^d}\Big\{T\Big(b,(DG^{(5)}[\mathbf{e}])(D\mathbf{E}_0^d[\phi^d])^{-1}\Big)\nu_{\phi^d}\circ\phi^d\Big\}\cdot g^d\,\tilde{\sigma}[\phi^d]\,d\sigma \qquad (2.87)\\ &+\sum_{i,j=1}^{\bar{n}} G_i^{(2)}[\mathbf{e}]\,\Big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}]+V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij}\\ &\quad\cdot\int_{\partial\Omega^d}\Big\{T\big(b,(DG_j^{(6)}[\mathbf{e}])(D\mathbf{E}_0^h[\phi^h])^{-1}\big)\nu_{\phi^d}\circ\phi^d\Big\}\cdot g^d\,\tilde{\sigma}[\phi^d]\,d\sigma,\end{aligned}$$

for all $\mathbf{e}\in\mathcal{U}$ with $\epsilon > 0$.

Now let $\epsilon = 0$. We note that

$$\begin{aligned}G^{(5)}[b,\omega,0,\phi^h,\phi^d,g^h,g^d] &= u^d[b,\phi^d,g^d]\circ\mathbf{E}_0^d[\phi^d],\\ DG^{(5)}[b,\omega,0,\phi^h,\phi^d,g^h,g^d] &= D(u^d[b,\phi^d,g^d]\circ\mathbf{E}_0^d[\phi^d]),\end{aligned}$$

in $(\Omega^d)^+_{\omega^d,\delta^d}$, for all $(b,\omega,0,\phi^h,\phi^d,g^h,g^d)\in\mathcal{U}$. Therefore the first integral in (2.87) equals

$$\int_{\phi^d(\partial\Omega^d)}\Big[T(b,Du^d[b,\phi^d,g^d])\nu_{\phi^d}\Big]^+\cdot g^d\circ(\phi^d)^{(-1)}\,d\sigma,$$

which is equal to

$$\int_{\mathbb{I}[\phi^d]}\mathbf{E}[b](u^d[b,\phi^d,g^d],u^d[b,\phi^d,g^d])\,d\xi,$$

by Theorem 2.3. Moreover, for $n = 2$ and $\epsilon = 0$, the last integral in (2.87) equals

$$\int_{\phi^d(\partial\Omega^d)}\Big[T\Big(b,Dv_{\phi^d(\partial\Omega^d)}[b,\mathcal{T}_d^{(j)}[\mathbf{b}]\circ(\phi^d)^{(-1)}]\Big)\,\nu_{\phi^d}\Big]^+\cdot g^d\circ(\phi^d)^{(-1)}\,d\sigma.$$

In particular, for $j = 3$ such an expression vanishes because, by Theorem 2.36 and by Proposition 2.44, $\mathcal{T}_d^{(3)}[\mathbf{b}] = 0$.

Now, by (2.81), (2.84) and (2.87), we immediately deduce the existence of $E^{(1)}$, $E^{(2)}_{ij}$, $i,j = 1,\ldots,\bar{n}$, and the validity of (2.79). For $n \geq 3$ the validity of (2.80) follows by the above computation of the first integrals of

(2.84), (2.87) at $\epsilon = 0$ and by Lemma 2.52. Similarly one verifies that, for $n = 2$, the limit in (2.79) converges to

$$\int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u^d[b, \phi^d, g^d], u^d[b, \phi^d, g^d])\ d\xi \qquad (2.88)$$
$$+ \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u_r^h[b, \phi^h, g^h], u_r^h[b, \phi^h, g^h])\ d\xi$$
$$-\lambda[b, \omega, \phi^h, \phi^d]^{-1} \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(3)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\ d\sigma \right)^2 .$$

To recognize that such an expression coincides with the right hand side of (2.80) we have to do some more calculations.

We note that, by Definition 2.31 and Proposition 2.44, $u_r^h[b, \phi^h, g^h] = w_{\phi^h(\partial\Omega^h)}[b, \mu]$, with $\mu \in C^{m,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$, and

$$u_s^h[b, \phi^h, g^h] = \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(3)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\ d\sigma \right) v_{\phi^h(\partial\Omega)}[b, \tilde{\alpha}] + c$$

where $\tilde{\alpha}$ is an element of $(\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \cdot])_0$ and c is a constant vector.

Then we have

$$\int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u_r^h[b, \phi^h, g^h], u_s^h[b, \phi^h, g^h])\ d\xi \qquad (2.89)$$
$$= \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(3)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\ d\sigma \right)$$
$$\cdot \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](w_{\phi^h(\partial\Omega^h)}[b, \mu], v_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}])\ d\xi,$$

and, by Theorem 2.3 and 2.12, the second integral in the right hand side equals

$$\int_{\phi^h(\partial\Omega^h)} w_{\phi^h(\partial\Omega^h)}[b, \mu]^- \cdot \left[T(b, Dv_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}])\nu_{\phi^h} \right]^-\ d\sigma \qquad (2.90)$$
$$= \int_{\phi^h(\partial\Omega^h)} \mathbf{H}^*_{\phi^h(\partial\Omega^h)}[b, \mu] \cdot \tilde{\alpha}\ d\sigma = \int_{\phi^h(\partial\Omega^h)} \mu \cdot \mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}]\ d\sigma = 0.$$

Furthermore, we have

$$\int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u_s^h[b, \phi^h, g^h], u_s^h[b, \phi^h, g^h])\ d\xi \qquad (2.91)$$
$$= \left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(3)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\ d\sigma \right)^2$$
$$\cdot \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](v_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}], v_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}])\ d\xi,$$

and, by Theorem 2.3, the second integral in the right hand side equals

$$\int_{\phi^h(\partial\Omega^h)} \Big[T(b, Dv_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}])\nu_{\phi^h}\Big]^- \cdot v_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}]\; d\sigma \qquad (2.92)$$
$$= \int_{\phi^h(\partial\Omega^h)} \mathbf{K}^*_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}] \cdot v_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}]\; d\sigma$$
$$= \int_{\phi^h(\partial\Omega^h)} \tilde{\alpha} \cdot v_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}]\; d\sigma.$$

Now, by definition we have $v_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}] = b^{(3)}[\phi^h] + c'$ on $\phi^h(\partial\Omega^h)$, where c' is a constant vector. Then, by exploiting equation (2.76), it follows that $v_{\phi^h(\partial\Omega^h)}[b,\tilde{\alpha}] = \lambda[b,\omega,\phi^h,\phi^d]^{-1} v_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]$, which implies that $\tilde{\alpha} = \lambda[b,\omega,\phi^h,\phi^d]^{-1}\mathcal{T}_h^{(3)}[\mathbf{b}] \circ (\phi^h)^{(-1)}$. So the last integral in (2.92) equals

$$\lambda[b,\omega,\phi^h,\phi^d]^{-1} \int_{\partial\Omega^h} \mathcal{T}_h^{(3)}[\mathbf{b}] \cdot \left(b^{(3)}[\phi^h] \circ \phi^h\right) \tilde{\sigma}[\phi^h]\; d\sigma = \lambda[b,\omega,\phi^h,\phi^d]^{-1}. \qquad (2.93)$$

Finally, by exploiting (2.89), (2.90), (2.91), (2.92) and (2.93) we deduce that

$$\int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u^h[b,\phi^h,g^h], u^h[b,\phi^h,g^h])\; d\xi$$
$$= \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b]\left(u_r^h[b,\phi^h,g^h] + u_s^h[b,\phi^h,g^h], u_r^h[b,\phi^h,g^h] + u_s^h[b,\phi^h,g^h]\right)\; d\xi$$
$$= \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u_r^h[b,\phi^h,g^h], u_r^h[b,\phi^h,g^h])\; d\xi$$
$$-\lambda[b,\omega,\phi^h,\phi^d]^{-1}\left(\int_{\partial\Omega^h} g^h \cdot \mathcal{T}_h^{(3)}[\mathbf{b}]\, \tilde{\sigma}[\phi^h]\; d\sigma\right)^2,$$

which immediately implies that expression (2.88) coincides with the right hand side of equation (2.80). □

2.3 Robin boundary value problem

2.3.1 Preliminaries

Let Ω be a bounded open and connected subset of $\mathbb{R}^n$ of class C^1 and let a be a continuous matrix valued function on $\partial\Omega$ which satisfies the following conditions.

(a1) $\det a$ is not identically equal to zero on $\partial\Omega$.

(a2) $\xi \cdot a(x)\xi \geq 0$ for all $\xi \in \mathbb{R}^n$ and for all $x \in \partial\Omega$.

In this section we consider the following Robin boundary value problem,

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \Omega, \\ T(b, Du)\nu + au = g & \text{on } \partial\Omega, \end{cases} \tag{2.94}$$

where g is a given function on $\partial\Omega$ and $b > 1 - 2/n$. The conditions (a1) and (a2) on the matrix function a are motivated by the following Theorem.

Theorem 2.56. *Let $b > 1 - 2/n$. Let Ω be a bounded open and connected subset of $\mathbb{R}^n$ of class C^1. Let $a \in C(\partial\Omega, \mathrm{M}_{n\times n}(\mathbb{R}))$ satisfy (a1) and (a2). If $u \in C^1(\mathrm{cl}\Omega, \mathbb{R}^n) \cap C^2(\Omega, \mathbb{R}^n)$ is a solution of problem* (2.94) *with $g = 0$, then $u = 0$.*

Proof. By Theorem 2.3, we have

$$\int_\Omega \mathbf{E}[b](u, u)\, dx = \int_{\partial\Omega} u_{|\partial\Omega} \cdot T(b, Du)_{|\partial\Omega}\nu\, d\sigma = -\int_{\partial\Omega} u_{|\partial\Omega} \cdot au_{|\partial\Omega}\, d\sigma.$$

By condition (a2) the last integral is ≤ 0. By Proposition 2.2, we have $\mathbf{E}[b](u, u) \geq 0$. Thus $\mathbf{E}[b](u, u) = 0$ and we deduce that $u \in \mathcal{R}_\Omega$. Then, by Theorem 2.4, $T(b, Du)_{|\partial\Omega}\nu = 0$. Since $T(b, Du)_{|\partial\Omega}\nu + au_{|\partial\Omega} = 0$ by assumption, it follows that $au_{|\partial\Omega} = 0$. So, to conclude the proof we have to show that $au_{|\partial\Omega} = 0$ implies $u = 0$.

Since $u \in \mathcal{R}_\Omega$, there exist $A \in \mathrm{Skew}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ such that $u(x) = Ax + b$ for all $x \in \Omega$. We now prove that $A = 0$. To do so we assume, by contradiction, that $A \neq 0$. Then A has a non zero minor of rank at least 2. It follows that the affine subspace $\mathcal{Z} \equiv \{x \in \mathbb{R}^n : Ax + b = 0\}$ of $\mathbb{R}^n$ has codimension larger than 2. Therefore, the intersection $\mathcal{Z} \cap \partial\Omega$ cannot be open and not empty in $\partial\Omega$. Conversely, by equation $au_{|\partial\Omega} = 0$ and by condition (a1), the set of the points $x \in \partial\Omega$ where $u(x) = 0$ is open and not empty in $\partial\Omega$. So we have a contradiction, because $\mathcal{Z} \cap \partial\Omega$ coincides with the set where $u(x) = 0$. It follows that $A = 0$ and $u = b$. Moreover, since $\det a$ is not identically zero, $ab = 0$ implies $b = 0$. □

Now let $m \in \mathbb{N} \setminus \{0\}$, and let $\lambda \in]0, 1[$, and let Ω be a bounded open and connected subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$, and let $a \in C^{m-1,\lambda}(\partial\Omega, M_{n\times n}(\mathbb{R}))$ satisfy conditions (a1) and (a2), and let $g \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. With these assumptions we associate to the boundary value problem (2.94) a boundary integral equation of Fredholm type and we prove that problem (2.94) has a unique solution $u \in C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$, which is expressed by means of suitable layer potentials. It will be necessary to distinguish between the case $n = 2$ and the case $n \geq 3$. Indeed for $n = 2$ we look for solutions in the form $v_{\partial\Omega}[b, \mu] + c$, where the density μ belongs to $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ and satisfies $\int_{\partial\Omega} \mu\, d\sigma = 0$, and c is a constant function. If $n \geq 3$ we look for solutions in the form of a single layer potential $v_{\partial\Omega}[b, \mu]$, with density $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. We introduce the following.

Definition 2.57. *Let* $b > 1-2/n$, $m \in \mathbb{N}\setminus\{0\}$, $\lambda \in]0,1[$. *Let* Ω *be a bounded open connected subset of* $\mathbb{R}^n$ *of class* $C^{m,\lambda}$. *Let* $a \in C^{m-1,\lambda}(\partial\Omega, M_{n\times n}(\mathbb{R}))$ *satisfy (a1) and (a2). We denote by* $\mathbf{J}_{\partial\Omega}[b,\cdot]$ *the operator on* $L^2(\partial\Omega,\mathbb{R}^n)$ *which takes a function* μ *to*

$$\mathbf{J}_{\partial\Omega}[b,\mu] \equiv \mathbf{H}_{\partial\Omega}[b,\mu] + a v_{\partial\Omega}[b,\mu]|_{\partial\Omega}.$$

We denote by $L^2(\partial\Omega,\mathbb{R}^n)_0$ *the closed subspace of* $L^2(\partial\Omega,\mathbb{R}^n)$ *of the function* μ *such that* $\int_{\partial\Omega}\mu\, d\sigma = 0$, *and we denote by* $\widetilde{\mathbf{J}}_{\partial\Omega}[b,\cdot,\cdot]$ *the operator of* $L^2(\partial\Omega,\mathbb{R}^n)_0 \times \mathbb{R}^n_\Omega$ *to* $L^2(\partial\Omega,\mathbb{R}^n)$ *which takes a couple* (μ,c) *to*

$$\widetilde{\mathbf{J}}_{\partial\Omega}[b,\mu,c] \equiv \mathbf{H}_{\partial\Omega}[b,\mu] + a\,(v_{\partial\Omega}[b,\mu]+c)\,|_{\partial\Omega}.$$

We write $\mathbf{J}$ *and* $\widetilde{\mathbf{J}}$ *instead of* $\mathbf{J}_{\partial\Omega}$ *and* $\widetilde{\mathbf{J}}_{\partial\Omega}$ *where no ambiguity can arise.*

We have the following.

Proposition 2.58. *With the notation introduced in Definition 2.57, the operator* $\mathbf{J}[b,\cdot]$ *is a Fredholm operator of index* 0 *on* $L^2(\partial\Omega,\mathbb{R}^n)$, *and the operator* $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$ *is a Fredholm operator of index* 0 *from* $L^2(\partial\Omega,\mathbb{R}^n)_0 \times \mathbb{R}^n_\Omega$ *to* $L^2(\partial\Omega,\mathbb{R}^n)$.

Proof. The proof that $\mathbf{J}[b,\cdot]$ is a Fredholm operator of index 0 follows by a slight modification in the proof of Theorem 2.11. We now consider the operator $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$. We have $\widetilde{\mathbf{J}}[b,\cdot,\cdot] = \mathbf{J}_1 \circ \mathbf{J}_2 \circ \mathbf{J}_3$, where $\mathbf{J}_1$ is the operator of $L^2(\partial\Omega,\mathbb{R}^n)\times\mathbb{R}^n_\Omega$ to $L^2(\partial\Omega,\mathbb{R}^n)$ which takes a couple (f,c) to the function $f + ac|_{\partial\Omega}$, and $\mathbf{J}_2$ is the operator from $L^2(\partial\Omega,\mathbb{R}^n)\times\mathbb{R}^n_\Omega$ to $L^2(\partial\Omega,\mathbb{R}^n)\times\mathbb{R}^n_\Omega$ which takes a couple (μ,c) to the couple $(\mathbf{J}[b,\mu],c)$, and $\mathbf{J}_3$ is the immersion of $L^2(\partial\Omega,\mathbb{R}^n)_0\times\mathbb{R}^n_\Omega$ into $L^2(\partial\Omega,\mathbb{R}^n)\times\mathbb{R}^n_\Omega$. Then we easily verify that $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are Fredholm operators of index n, 0 and $-n$, respectively. Since the composition of Fredholm operators is a Fredholm operator of index equal to the sum of the indexes of the components, we deduce that $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$ is a Fredholm operator of index 0. □

Now, by Proposition 2.58, we deduce the following Theorem 2.59 which motivates the distinction between the case $n = 2$ and the case $n \geq 3$.

Theorem 2.59. *Let the notation of Definition 2.57 hold. If* $n \geq 3$, *then the operator* $\mathbf{J}[b,\cdot]$ *is a linear homeomorphism of* $L^2(\partial\Omega,\mathbb{R}^n)$ *onto* $L^2(\partial\Omega,\mathbb{R}^n)$. *If* $n = 2$, *then the operator* $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$ *is a linear homeomorphism of* $L^2(\partial\Omega,\mathbb{R}^2)_0\times\mathbb{R}^2_\Omega$ *onto* $L^2(\partial\Omega,\mathbb{R}^2)$.

Proof. Since $\mathbf{J}[b,\cdot]$ and $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$ are Fredholm operators of index 0 it will be enough to prove that they have trivial kernel. So let $n \geq 3$, and let $\mathbf{J}[b,\mu] = 0$. By arguing as in the proof of Lemma 2.14, we deduce that $\mu \in C^{0,\lambda}(\partial\Omega,\mathbb{R}^n)$. Then, by Proposition 2.13, the function $v_{\partial\Omega}[b,\mu]|_{\mathrm{cl}\Omega}$ is a solution of problem (2.94) with $g = 0$. Then $v_{\partial\Omega}[b,\mu]|_{\mathrm{cl}\Omega} = 0$ by

Theorem 2.56. It follows that $v_{\partial\Omega}[b,\mu] = 0$ on $\partial\Omega$, which implies that $v_{\partial\Omega}[b,\mu] = 0$ in the whole of $\mathbb{R}^n$ (cf. Theorem 2.4 and Proposition 2.6.) Then $\mu = [T(b, Dv_{\partial\Omega}[b,\mu])\nu]^- - [T(b, Dv_{\partial\Omega}[b,\mu])\nu]^+ = 0$. The proof that $\widetilde{\mathbf{J}}[b,\cdot,\cdot]$ is an homeomorphism for $n = 2$ is very similar and we omit it. □

Moreover, by Theorem 2.19 and Proposition 2.6, we have the following.

Proposition 2.60. *With the notation introduced in Definition 2.57, if either $\mathbf{J}[b,\mu] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ or $\widetilde{\mathbf{J}}[b,\mu,c] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, then $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$.*

Proof. Let $\mathbf{J}[b,\mu] \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. If $m = 1$ the statement follows by a slight modification of the proof of Lemma 2.14. So let $m \geq 2$. If $\mu \in C^{m'-1,\lambda}(\partial\Omega, \mathbb{R}^n)$, with $m' < m$, then $v_{\partial\Omega}[b,\mu]|_{\partial\Omega} \in C^{m',\lambda}(\partial\Omega, \mathbb{R}^n)$ by Proposition 2.6. Then, by exploiting the definition of $\mathbf{J}[b,\mu]$, we have $\mathbf{H}[b,\mu] \in C^{m',\lambda}(\partial\Omega, \mathbb{R}^n)$. By Theorem 2.19, it follows that $\mu \in C^{m',\lambda}(\partial\Omega, \mathbb{R}^n)$. By induction on m' we deduce that $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. The proof for $\widetilde{\mathbf{J}}$ is very similar ad we omit it. □

Now, by the previous Theorems 2.56 and 2.59 and by Proposition 2.60, we are ready to deduce the following.

Theorem 2.61. *Let $b > 1 - 2/n$, $m \in \mathbb{N} \setminus \{0\}$, $\lambda \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $a \in C^{m-1,\lambda}(\partial\Omega, M_{n\times n}(\mathbb{R}))$ satisfy (a1) and (a2). Then problem (2.94) admits a unique solution $u \in C^{m,\lambda}(\mathrm{cl}\Omega, \mathbb{R}^n)$ for each given $g \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$. If $n \geq 3$, then the solution u is delivered by the function $v_{\partial\Omega}[b,\mu]|_{\mathrm{cl}\Omega}$, where $\mu \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n)$ is the unique solution of*

$$\mathbf{J}[b,\mu] = g. \tag{2.95}$$

If $n = 2$, then the solution u is delivered by the function $v_{\partial\Omega}[b,\mu]|_{\mathrm{cl}\Omega} + c$, where $(\mu, c) \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^2) \times \mathbb{R}^2_\Omega$ is the unique solution of

$$\begin{cases} \widetilde{\mathbf{J}}[b,\mu,c] = g, \\ \int_{\partial\Omega} \mu \, d\sigma = 0. \end{cases} \tag{2.96}$$

We now present a technical remark which is needed in the sequel, the proof can be easily deduced by the previous Theorem 2.59 and Proposition 2.60 by linearity.

Remark 2.62. *Let the notation of Definition 2.57 hold. If $n = 2$, then for each given $(g, d) \in C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n) \times \mathbb{R}^n_\Omega$ there exists a unique pair (μ, c) of $C^{m-1,\lambda}(\partial\Omega, \mathbb{R}^n) \times \mathbb{R}^n_\Omega$ such that*

$$\begin{cases} \mathbf{J}[b,\mu] + c = g, \\ \int_{\partial\Omega} \mu \, d\sigma = d. \end{cases} \tag{2.97}$$

2.3.2 Robin problem in a singularly perturbed domain

We now introduce a Robin problem on a singularly perturbed domain. We fix a constant $m \in \mathbb{N} \setminus \{0\}$, and a constant $\lambda \in]0,1[$, and two bounded open subsets Ω^h and Ω^d of $\mathbb{R}^n$ with Ω^h, Ω^d, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^h$, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^d$ connected, and a matrix valued function $\alpha \in C^{m-1,\lambda}(\partial\Omega^d, \mathrm{M}_{n\times n}(\mathbb{R}))$ which satisfies conditions (a1) and (a2). Then, for each $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ (cf. subsection 2.2.1), we consider the following Robin boundary value problem in the domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d] \equiv \mathbb{I}[\phi^d] \setminus \mathrm{cl}\mathbb{I}[\omega + \epsilon\phi^h]$,

$$\begin{cases} \mathbf{L}[b]u = 0 & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \\ -T(b, Du)\nu_{(\omega+\epsilon\phi^h)} = g^i \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ T(b, Du)\nu_{\phi^d} + \alpha \circ (\phi^d)^{(-1)}u = g^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d), \end{cases} \tag{2.98}$$

In the previous subsection 2.3.1 we have proved that the system of equations (2.98) has a unique solution $u[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d] \in C^{m,\lambda}(\mathrm{cl}\mathbb{A}[\mathbf{a}], \mathbb{R}^n)$. We shall investigate the behavior of the solution $u[b, \omega, \epsilon, \phi^h, \phi^d, g^i, g^o]$ upon perturbations of $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ around a given degenerate sextuple $(b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d)$.

2.3.3 A real analyticity theorem for the solutions of (2.95) and (2.96)

We start our analysis of the Robin problem (2.98), which is defined in the $\mathbf{a} \equiv (\omega, \epsilon, \phi^h, \phi^d)$ dependent domain $\mathbb{A}[\mathbf{a}]$. As a first step we transform the corresponding equation (2.95) and system (2.96) into a system of equations on the boundary of the fixed domains Ω^h and Ω^d. This is done in the following Theorem 2.63.

Theorem 2.63. *With the notation of subsections 2.3.2, we denote by* $N \equiv (N^1, N^2, N^3)$ *the map of* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$ *to* $(C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$ *defined by*

$$\begin{aligned} N^1[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c](x^h) \equiv \mathbf{K}^*[b, \phi^h, \mu^h](x) \\ +\epsilon^{n-1} \int_{\partial\Omega^d} \sum_{i=1}^n \Big[T(b, D\Gamma^{(i)}(b, \omega + \epsilon\phi^h(x) - \phi^d(y)) \\ \cdot\nu_{\phi^h} \circ \phi^h(x)\Big] \mu_i^d(y)\, \tilde{\sigma}[\phi^d](y)\, d\sigma_y + \epsilon^{n-1} g^h(x), \\ \forall\, x \in \partial\Omega^h, \end{aligned} \tag{2.99}$$

$$
\begin{aligned}
N^2[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c](x) \qquad\qquad\qquad\qquad (2.100)\\
\equiv \mathbf{H}[b, \phi^d, \mu^d](x) + \alpha(x)\big(V[b, \phi^d, \mu^d](x) + c\big)\\
+ \int_{\partial\Omega^h} \sum_{i=1}^{n} \Big[T(b, D\Gamma^{(i)}(b, \phi^d(x) - \omega - \epsilon\phi^h(y)))\\
\cdot\nu_{\phi^d} \circ \phi^d(x)\Big] \mu_i^h(y)\, \tilde{\sigma}[\phi^h](y)\ d\sigma_y\\
+\alpha(x) \int_{\partial\Omega^h} \Gamma(b, \phi^d(x) - \omega - \epsilon\phi^h(y))\mu^h(y)\, \tilde{\sigma}[\phi^h](y)\ d\sigma_y - g^d(x),\\
\forall\ x \in \partial\Omega^d,
\end{aligned}
$$

$$
\begin{aligned}
N^3[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c] \equiv (1 - \delta_{2,n})\, c \qquad\qquad (2.101)\\
+\delta_{2,n} \left(\int_{\partial\Omega^h} \mu^h\, \tilde{\sigma}[\phi^h]\ d\sigma + \int_{\partial\Omega^d} \mu^d\, \tilde{\sigma}[\phi^d]\ d\sigma \right),
\end{aligned}
$$

for all $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$*, where* $V[b, \phi^d, \mu^d] \equiv v[b, \phi^d, \mu^d] \circ \phi^d$.

Let $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c) \in \mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$*, and let*

$$
\mu \equiv \begin{cases} \epsilon^{1-n}\, \mu^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ \mu^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d), \end{cases} \tag{2.102}
$$

$$
\Omega \equiv \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \tag{2.103}
$$

$$
a \equiv \begin{cases} \mathbf{0}_{n\times n} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ \alpha \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d), \end{cases} \tag{2.104}
$$

$$
g \equiv \begin{cases} g^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ g^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d). \end{cases} \tag{2.105}
$$

Then, we have

$$
N[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c] = 0, \tag{2.106}
$$

if and only if either one of the following two conditions is satisfied.

(i) $n = 2$ *and the pair* (μ, c)*, with* μ *defined by* (2.102)*, satisfies* (2.96) *with* Ω*,* a *and* g *defined by* (2.103)*,* (2.104) *and* (2.105)*, respectively.*

(ii) $n \geq 3$*, and* $c = 0$*, and the function* μ *defined by* (2.102) *satisfies* (2.95) *with* Ω*,* a *and* g *defined by* (2.103)*,* (2.104) *and* (2.105)*, respectively.*

In particular, for each given 7-tuple $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d)$ *in* $\mathscr{B} \times \mathcal{E}_+^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$*, there exists a unique triple* (μ^h, μ^d, c) *of* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *such that equation 2.106 hold.*

Now, let $\epsilon = 0$*. Let* $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *be fixed. Then the triple* (μ^h, μ^d, c) *of* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *satisfies equation*

$$
N[\omega, 0, \phi^h, \phi^d, g^h, g^d, \mu^h, \mu^d, c] = 0, \tag{2.107}
$$

if and only if $\mu^h = 0$ and either one of the following two conditions hold.

(iii) $n = 2$ *and the pair* (μ, d)*, with* $\mu \equiv \mu^d \circ (\phi^d)^{(-1)}$*, is a solution of* (2.96) *with* $\Omega \equiv \mathbb{I}[\phi^d]$*,* $a \equiv \alpha \circ (\phi^d)^{(-1)}$ *and* $g \equiv g^d \circ (\phi^d)^{(-1)}$*.*

(iv) $n \geq 3$*, and* $c = 0$*, and the function* $\mu \equiv \mu^d \circ (\phi^d)^{(-1)}$ *is a solution of* (2.95) *with* $\Omega \equiv \mathbb{I}[\phi^d]$*,* $a \equiv \alpha \circ (\phi^d)^{(-1)}$ *and* $g \equiv g^d \circ (\phi^d)^{(-1)}$*.*

In particular, for each $(\omega, 0, \phi^h, \phi^d, g^h, g^d)$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$*, equation* (2.106) *has exactly one solution* (μ^h, μ^d, c) *in* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$*.*

Proof. The statement follows by a straightforward verification based on the theorem of change of variables in integrals and by the previous Theorem 2.61. We only note that, if $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ is fixed, then the first component of equation (2.107) admits the unique solution $\mu^h = 0$ (cf. statement (i) of Lemma 2.26.) Then, by the second and third component of (2.107), μ^d satisfies either condition (iii) or condition (iv) of the theorem. □

By Theorem 2.63, it makes sense to introduce the following.

Definition 2.64. *With the notation introduced in subsection 2.3.2, let* $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ *with* $\epsilon > 0$ *or* $\epsilon = 0$*. We denote by* $(\hat{\mu}^h[\mathbf{e}], \hat{\mu}^d[\mathbf{e}], \hat{c}[\mathbf{e}])$ *the unique solution* $(\mu^h, \mu^d, c) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *of equation* (2.106) *or* (2.107)*, respectively.*

Our goal is to show that $\hat{\mu}^h[\cdot]$, $\hat{\mu}^d[\cdot]$ and $\hat{c}[\cdot]$ admit a real analytic continuation around a "degenerate" point $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. By Propositions 1.40 and 2.39 we get the following.

Proposition 2.65. *With the notation of subsection 2.3.2, the set* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$ *is an open subset of the Banach space* $\mathbb{R}^{n+2} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$*, and the operator* N *is real analytic.*

Moreover, we need the following.

Lemma 2.66. *With the notation introduced in subsection 2.3.2, let* $\mathbf{f}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d, \mu_0^h, \mu_0^d, c_0)$ *belong to* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n))^2 \times \mathbb{R}^n$*, and let* $N[\mathbf{f}_0] = 0$*. Then the differential*

$$\partial_{(\mu^h,\mu^d,c)} N[\mathbf{f}_0] = \left(\partial_{(\mu^h,\mu^d,c)} N^1[\mathbf{f}_0], \partial_{(\mu^h,\mu^d,c)} N^2[\mathbf{f}_0], \partial_{(\mu^h,\mu^d,c)} N^3[\mathbf{f}_0]\right)$$

of N with respect to the variable (μ^h, μ^d, c) at $\mathbf{f}_0$ is delivered by the linear operators which takes $(\bar{\mu}^h, \bar{\mu}^d, \bar{c}) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times \mathbb{R}^n$ to the functions defined by

$$
\begin{aligned}
&\partial_{(\mu^h,\mu^d,c)} N^1[\mathbf{f}_0](\bar{\mu}^h, \bar{\mu}^d, \bar{c})(x) = \mathbf{K}^*[b_0, \phi_0^h, \bar{\mu}^h](x), \qquad \forall\, x \in \partial\Omega^h, \\
&\partial_{(\mu^h,\mu^d,c)} N^2[\mathbf{f}_0](\bar{\mu}^h, \bar{\mu}^d, \bar{c})(x) \\
&\quad = \mathbf{H}[b_0, \phi_0^d, \bar{\mu}^d](x) + \alpha(x) \Big(V[b_0, \phi_0^d, \bar{\mu}^d](x) + \bar{c} \Big) \\
&\qquad + \sum_{i=1}^n \Big[T(b_0, D\Gamma^{(i)}(b_0, \phi_0^d(x) - \omega_0)) \nu_{\phi_0^d} \circ \phi_0^d(x) \Big] \int_{\partial\Omega^h} \bar{\mu}_i^h \, \tilde{\sigma}[\phi_0^h] \, d\sigma \\
&\qquad + \alpha(x) \Gamma(b_0, \phi_0^d(x) - \omega_0) \int_{\partial\Omega^h} \bar{\mu}^h \, \tilde{\sigma}[\phi_0^h] \, d\sigma, \qquad \forall\, x \in \partial\Omega^d, \\
&\partial_{(\mu^h,\mu^d,c)} N^3[\mathbf{f}_0](\bar{\mu}^h, \bar{\mu}^d, \bar{c}) = (1 - \delta_{2,n}) \, \bar{c} \\
&\qquad + \delta_{2,n} \left(\int_{\partial\Omega^h} \bar{\mu}^h \, \tilde{\sigma}[\phi_0^h] \, d\sigma + \int_{\partial\Omega^d} \bar{\mu}^d \, \tilde{\sigma}[\phi_0^d] \, d\sigma \right).
\end{aligned}
\tag{2.108}
$$

Moreover, the partial differential $\partial_{(\mu^h,\mu^d,c)} N[\mathbf{f}_0]$ is a linear homeomorphism of $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ to $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$.

Proof. Expression (2.108) follows by standard calculus in Banach space. Then we recognize that the differential $\partial_{(\mu^h,\mu^d,c)} N[\mathbf{f}_0]$ is a bounded linear operator on $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$. By the Open Mapping Theorem, it follows that $\partial_{(\mu^h,\mu^d,c)} N[\mathbf{f}_0]$ is an homeomorphism if it is an isomorphism. So we fix $(g^h, g^d, d) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ and we conclude the proof of the lemma by showing that there exists a unique triple $(\bar{\mu}^h, \bar{\mu}^d, \bar{c}) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ such that

$$
\partial_{(\mu^h,\mu^d,c)} N[\mathbf{f}_0](\bar{\mu}^h, \bar{\mu}^d, \bar{c}) = (g^h, g^d, d). \tag{2.109}
$$

By the Fredholm Alternative Theorem, and by Theorem 2.17, and by Theorem 2.19, we deduce that there exists a unique $\bar{\mu}^h \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ such that $\mathbf{K}^*[b_0, \phi_0^h, \bar{\mu}^h] = g^h$. So, the first equation of system (2.109) admits a unique solution $\bar{\mu}^h$ and to conclude the proof we have to show that the system of the second and third equation of (2.109) admits a unique solution $(\bar{\mu}^d, \bar{c}) \in C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$. The second and third equation are equivalent to the following system,

$$
\begin{cases}
\mathbf{H}[b_0, \phi_0^d, \bar{\mu}^d] + \alpha \left(V[b_0, \phi_0^d, \bar{\mu}^d] + \bar{c} \right) = G^d & \text{on } \partial\Omega^d, \\
(1 - \delta_{2,n}) \bar{c} + \delta_{2,n} \int_{\partial\Omega^d} \bar{\mu}^d \, \tilde{\sigma}[\phi_0^d] \, d\sigma = C,
\end{cases}
$$

where

$$
\begin{aligned}
G^d(x) \equiv -\sum_{i=1}^{n} \Big[T\big(b_0, D\Gamma^{(i)}(b_0, \phi_0^d(x)-\omega_0)\big)\nu_{\phi_0^d}\circ\phi_0^d(x)\Big]\int_{\partial\Omega^h} \bar{\mu}_i^h\,\tilde{\sigma}[\phi_0^h]\,d\sigma \\
-\alpha(x)\Gamma(b_0, \phi_0^d(x)-\omega_0)\int_{\partial\Omega^h} \bar{\mu}^h\,\tilde{\sigma}[\phi_0^h]\,d\sigma + g^d(x), \quad \forall\, x\in\partial\Omega^d,
\end{aligned}
$$

and $B \equiv -\delta_{2,n}\int_{\partial\Omega^h}\bar{\mu}^h\tilde{\sigma}[\phi_0^h]\,d\sigma + d$. Then, the existence and the uniqueness of the solution $(\bar{\mu}^d, \bar{c}) \in C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)\times\mathbb{R}^n$ follows by Theorem 2.59, and by Proposition 2.60, and by Remark 2.62. □

Now,by exploiting Proposition 2.65, and Lemma 2.66, and the Implicit Mapping Theorem, we deduce the following.

Theorem 2.67. *Let the notation introduced in subsection 2.3.2 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d)$ *be an element of* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$. *Then there exist an open neighborhood* $\mathcal{U}_0$ *of* $\mathbf{e}_0$ *in* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)$, *and an open neighborhood* $\mathcal{V}_0$ *of* $(\hat{\mu}^h[\mathbf{e}_0], \hat{\mu}^d[\mathbf{e}_0], \hat{c}[\mathbf{e}_0])$ *in* $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$, *and a real analytic operator* $(\mathcal{N}^h, \mathcal{N}^d, \mathcal{C})$ *of* $\mathcal{U}_0$ *to* $\mathcal{V}_0$ *such that*

$$
(\mathcal{N}^h[\mathbf{e}], \mathcal{N}^d[\mathbf{e}], \mathcal{C}[\mathbf{e}]) = (\hat{\mu}^h[\mathbf{e}], \hat{\mu}^d[\mathbf{e}], \hat{c}[\mathbf{e}]) \tag{2.110}
$$

for all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_0$ *with* $\epsilon > 0$. *Moreover, the graph of* $(\mathcal{N}^h, \mathcal{N}^d, \mathcal{C})$ *coincides with the set of zeros of* N *in* $\mathcal{U}_0\times\mathcal{V}_0$.

Remark 2.68. *With the same notation of Theorem 2.67, there exists a real analytic operator* $\widetilde{\mathcal{N}}^h$ *of* $\mathcal{U}_0$ *to* $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)$ *such that*

$$
\mathcal{N}^h[\mathbf{e}] = \epsilon^{n-1}\widetilde{\mathcal{N}}^h[\mathbf{e}], \tag{2.111}
$$

for all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_0$.

Moreover, $\widetilde{\mathcal{N}}^h[b, \omega, 0, \phi^h, \phi^d, g^h, g^d] = 0$ *if and only if*

$$
g^h = -T(b, Du^d[b, \phi^d, g^d](\omega))\nu_{\phi^h}\circ(\phi^h)^{(-1)}, \tag{2.112}
$$

for all $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_0$, *where* $u^d[b, \phi^d, g^d]$ *is the unique solution of the Robin boundary value problem* (2.94) *with* $\Omega = \mathbb{I}[\phi^d]$, $a = \alpha\circ(\phi^d)^{(-1)}$ *and* $g = g^o\circ(\phi^d)^{(-1)}$.

Proof. $\widetilde{\mathcal{N}}^h[\mathbf{e}]$ is the unique solution of the following equation,

$$
\begin{aligned}
\mathbf{K}^*\big[\phi^h, \widetilde{\mathcal{N}}^h[\mathbf{e}]\big](x) = -g^h(x) \qquad\qquad (2.113)\\
-\int_{\partial\Omega^d}\sum_{i=1}^{n}\Big[T\big(b, D\Gamma^{(i)}(b, \omega+\epsilon\phi^h(x)-\phi^d(y))\big) \\
\cdot\nu_{\phi^h}\circ\phi^h(x)\Big]\mathcal{N}_i^d[\mathbf{e}](y)\tilde{\sigma}[\phi^d](y)\,d\sigma_y, \qquad \forall\, x\in\partial\Omega^d.
\end{aligned}
$$

By Proposition 2.39 and by Theorem 2.67, the right hand side of (2.113) depends real analytically on $\mathbf{e} \in \mathcal{U}_0$. Then, by the Implicit Mapping Theorem, $\widetilde{\mathcal{N}}^h$ is real analytic on $\mathcal{U}_0$. Equation (2.111) follows by linearity. Let $\mathbf{e} \equiv (b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_0$ with $\epsilon = 0$ such that $\widetilde{\mathcal{N}}^h[\mathbf{e}] = 0$. Then the left hand side of (2.113) vanishes. We deduce (2.112) by Theorems 2.61, 2.63 and by straightforward calculation. The proof of the converse is similar. □

2.3.4 Solution of the singularly perturbed problem

Theorem 2.69. *Let the notation of subsection 2.3.2 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\Omega \subset \mathbb{I}[\phi_0^d] \setminus \{\omega_0\}$. *Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{e}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, *and a real analytic operator* $U[\cdot]$ *of* $\mathcal{U}$ *to* $C(\mathrm{cl}\Omega, \mathbb{R}^n)$, *endowed with the norm of the uniform convergence, such that the following conditions hold.*

(i) $\mathrm{cl}\Omega \subset \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ *for all* $(\omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$.

(ii) $u[\mathbf{e}](\xi) = U[\mathbf{e}](\xi)$ *for all* $\xi \in \mathrm{cl}\Omega$ *and all* $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$ *with* $\epsilon > 0$.

(iii)
$$U[b, \omega, 0, \phi^h, \phi^d, g^h, g^d](\xi) = u^d[b, \phi^d, g^d](\xi), \quad \forall\, \xi \in \mathrm{cl}\Omega,$$
for all $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$, *where* $u^d[b, \phi^d, g^d]$ *is the unique solution of the Robin boundary value problem* (2.94) *with* $\Omega = \mathbb{I}[\phi^d]$, $a = \alpha \circ (\phi^d)^{(-1)}$ *and* $g = g^o \circ (\phi^d)^{(-1)}$.

Proof. Let $\mathcal{U}_0$ be the open neighborhood of $\mathbf{e}_0$ introduced in Theorem 2.67. We set

$$U^{(1)}[\mathbf{e}](\xi) \equiv \int_{\partial\Omega^d} \Gamma(\xi - \phi^d(y))\mathcal{N}^d[\mathbf{e}](y)\tilde{\sigma}[\phi^d](y)\, d\sigma_y + \mathcal{C}[\mathbf{e}],$$
$$U^{(2)}[\mathbf{e}](\xi) \equiv \int_{\partial\Omega^h} \Gamma(\xi - \omega - \epsilon\phi^h(y))\widetilde{\mathcal{N}}^h[\mathbf{e}](y)\tilde{\sigma}[\phi^h](y)\, d\sigma_y,$$

for all $\xi \in \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ and for all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}_0$. Then let $\mathcal{U}$ be an open neighborhood of $\mathbf{e}_0$ contained in $\mathcal{U}_0$ and such that condition (i) holds. By Proposition 2.39, $U^{(1)}[\cdot]|_{\mathrm{cl}\Omega}$, $U^{(2)}[\cdot]|_{\mathrm{cl}\Omega}$ are real analytic operators of $\mathcal{U}$ to $C(\mathrm{cl}\Omega, \mathbb{R}^n)$. Moreover we have

$$u[\mathbf{e}](\xi) = U^{(1)}[\mathbf{e}](\xi) + \epsilon^{n-1}U^{(2)}[\mathbf{e}](\xi), \quad \forall\, \xi \in \mathrm{cl}\Omega,$$

for all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$ with $\epsilon > 0$. So, by taking $U[\mathbf{e}] \equiv U^{(1)}[\mathbf{e}]|_{\mathrm{cl}\Omega} + \epsilon^{n-1}U^{(2)}[\mathbf{e}]|_{\mathrm{cl}\Omega}$, the theorem follows. □

2.3.5 The corresponding energy integral

Theorem 2.70. *Let the notation of subsection 2.3.2 hold. Let* $\mathbf{e}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$. *Let* $\mathcal{U}$ *be the open neighborhood of* $\mathbf{e}_0$ *introduced in Theorem 2.69. Then there exist a real analytic operator* E *of* $\mathcal{U}$ *to* $\mathbb{R}$ *such that*

$$E[\mathbf{e}] = \int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u[\mathbf{e}], u[\mathbf{e}])\, d\xi$$

for all $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$ *with* $\epsilon > 0$. *Moreover, we have*

$$E[b, \omega, 0, \xi, \phi^o, g^i, g^o] = \int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u^d[b, \phi^d, g^d], u^d[b, \phi^d, g^d])\, d\xi,$$

for all $(b, \omega, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$.

Proof. Let $\mathbf{e} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$ and let $\epsilon > 0$. Then, we have

$$\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u[\mathbf{e}], u[\mathbf{e}])\, d\xi = \int_{\phi^d(\partial\Omega^d)} u[\mathbf{e}] \cdot \left[T(b, Du[\mathbf{e}])\nu_{\phi^d}\right]^+ d\sigma$$
$$- \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} u[\mathbf{e}] \cdot \left[T(b, Du[\mathbf{e}])\nu_{\omega+\epsilon\phi^h}\right]^- d\sigma.$$

We denote by $\mathcal{I}_1[\mathbf{e}]$ and by $\mathcal{I}_2[\mathbf{e}]$ the first and the second term in the right hand side, respectively. Then

$$\mathcal{I}_1[\mathbf{e}] = \int_{\phi^d(\partial\Omega^d)} \left(U^{(1)}[\mathbf{e}] + \epsilon^{n-1}U^{(2)}[\mathbf{e}]\right)$$
$$\cdot \left[-\alpha \circ (\phi^d)^{(-1)} \left(U^{(1)}[\mathbf{e}] + \epsilon^{n-1}U^{(2)}[\mathbf{e}]\right) + g^d \circ (\phi^d)^{(-1)}\right] d\sigma$$

and

$$\mathcal{I}_2[\mathbf{e}] \equiv - \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} \left(U^{(1)}[\mathbf{e}] + \epsilon^{n-1}U^{(2)}[\mathbf{e}]\right) \cdot g^h \circ (\omega + \epsilon\phi^h)^{(-1)}\, d\sigma,$$

where $U^{(1)}[\mathbf{e}]$ and $U^{(2)}[\mathbf{e}]$ are defined as in the proof of Theorem 2.69. By a straightforward calculation we verify that

$$\mathcal{I}_1[\mathbf{e}] = \int_{\partial\Omega^d} \left(U^{(1)}[\mathbf{e}] \circ \phi^d\right) \cdot \left(-\alpha U^{(1)}[\mathbf{e}] \circ \phi^d + g^d\right) \tilde{\sigma}[\phi^d]\, d\sigma \qquad (2.114)$$
$$+\epsilon^{n-1} \int_{\partial\Omega^d} \left(U^{(2)}[\mathbf{e}] \circ \phi^d\right) \cdot g^d\, \tilde{\sigma}[\phi^d]\, d\sigma$$
$$-\epsilon^{n-1} \int_{\partial\Omega^d} \left(U^{(1)}[\mathbf{e}] \circ \phi^d\right) \cdot \left[(\alpha + \alpha^t)U^{(2)}[\mathbf{e}] \circ \phi^d\right] \tilde{\sigma}[\phi^d]\, d\sigma$$
$$-\epsilon^{2n-2} \int_{\partial\Omega^d} \left(U^{(2)}[\mathbf{e}] \circ \phi^d\right) \cdot \left(\alpha U^{(2)}[\mathbf{e}] \circ \phi^o\right) \tilde{\sigma}[\phi^d]\, d\sigma$$

and

$$\begin{aligned}\mathcal{I}_2[\mathbf{e}] &= -\epsilon^{n-1}\int_{\partial\Omega^h}\Big(U^1[\mathbf{e}]\circ(\omega+\epsilon\phi^h)\Big)\cdot g^h\,\tilde\sigma[\phi^h]\,d\sigma \\ &\quad -\epsilon^{2n-2}\int_{\partial\Omega^h}\Big(U^2[\mathbf{e}]\circ(\omega+\epsilon\phi^h)\Big)\cdot g^h\,\tilde\sigma[\phi^h]\,d\sigma.\end{aligned} \tag{2.115}$$

Moreover, we note that

$$U^{(1)}[\mathbf{e}]\circ\phi^d(x) = v_{\phi^d(\partial\Omega^d)}[b,\mathcal{N}^d[\mathbf{e}]\circ(\phi^d)^{(-1)}]\circ\phi^d(x)+\mathcal{C}[\mathbf{e}],$$
$$U^{(2)}[\mathbf{e}]\circ\phi^d(x) = \int_{\partial\Omega^h}\Gamma(\phi^d(x)-\omega-\epsilon\phi^h(y))\widetilde{\mathcal{N}}^h[\mathbf{e}](y)\,\tilde\sigma[\phi^h](y)\,d\sigma_y,$$

for all $x\in\partial\Omega^d$, and

$$\begin{aligned}&U^{(1)}[\mathbf{e}]\circ(\omega+\epsilon\phi^h(x))\\ &\qquad=\int_{\partial\Omega^d}\Gamma(b,\omega+\epsilon\phi^h(x)-\phi^d(y))\mathcal{N}^d[\mathbf{e}](y)\,\tilde\sigma[\phi^d](y)\,d\sigma_y+\mathcal{C}[\mathbf{b}],\\ &U^{(2)}[\mathbf{e}]\circ(\omega+\epsilon\phi^h(x)) = \epsilon^{n-1}v_{\phi^h(\partial\Omega^h)}[b,\widetilde{\mathcal{N}}^h[\mathbf{e}]\circ(\phi^h)^{(-1)}](x),\end{aligned}$$

for all $x\in\partial\Omega^h$. By Propositions 1.40 and 2.39, we deduce that each term on the right hand side of (2.114) and (2.115) depends real analytically on $\mathbf{e}\in\mathcal{U}$. Now, we denote by $E[\mathbf{e}]$ the sum of the right hand side of (2.114) and (2.115) and we conclude the proof by a straightforward calculation. □

2.3.6 Robin problem in a singularly perturbed domain with singularly perturbed data on the boundary

We now investigate a slightly different problem. With the notation introduced in the previous subsection 2.3.2, let $\mathbf{c}\equiv(b,\omega,\epsilon,\phi^h,\phi^d,f)\in\mathscr{B}\times\mathcal{E}_+^{m,\lambda}\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)$. We consider the following system of equations,

$$\begin{cases}\mathbf{L}[b]u=0 & \text{in } \mathbb{A}[\omega,\epsilon,\phi^h,\phi^d],\\ -T(b,Du)\nu_{(\omega+\epsilon\phi^h)}=\epsilon^{1-n}f\circ(\omega+\epsilon\phi^h)^{(-1)} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ T(b,Du)\nu_{\phi^d}+\alpha\circ(\phi^d)^{(-1)}u=0 & \text{on } \phi^d(\partial\Omega^d).\end{cases} \tag{2.116}$$

In subsection 2.3.1 we have proved that system (2.116) has a unique solution $u[\mathbf{c}]\in C^{m,\lambda}(\mathrm{cl}\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d],\mathbb{R}^n)$. We investigate the behavior of $u[\mathbf{c}]$ and of the energy integral

$$\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]}\mathbf{E}[b](u[\mathbf{c}],u[\mathbf{c}])\,d\xi$$

upon perturbations of $\mathbf{c}$ around a given point $\mathbf{c}_0\equiv(b_0,\omega_0,0,\phi_0^h,\phi_0^d,f_0)$ of $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)$. In the following Theorems 2.71 and 2.72

we draw out our conclusions. In the next subsection 2.3.7 we prove the validity of such theorems by adapting to the present situation the machinery exploited in subsection 2.3.4. So, for the solution of (2.116) we have the following.

Theorem 2.71. *Let the notation introduced in subsection 2.3.2 hold. Let* $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, f_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\Omega \subset \mathbb{I}[\phi_0^d] \setminus \{\omega_0\}$. *Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{c}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ *and a real analytic operator* $U[\cdot]$ *of* $\mathcal{U}$ *to* $C(\mathrm{cl}\Omega, \mathbb{R}^n)$, *endowed with the norm of the uniform convergence, such that the following conditions hold.*

(i) $\mathrm{cl}\Omega \subset \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ *for all* $(b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$.

(ii) $u[b, \omega, \epsilon, \phi^h, \phi^d, f](\xi) = U[b, \omega, \epsilon, \phi^h, \phi^d, f](\xi)$ *for all* $\xi \in \mathrm{cl}\Omega$ *and all* $(b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$ *with* $\epsilon > 0$.

(iii)

$$\begin{aligned} &U[b, \omega, 0, \phi^h, \phi^d, f](\xi) \\ &\qquad = u^d[b, \omega, \phi^h, \phi^d, f](\xi) - \Gamma(b, \xi - \omega) \int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\; d\sigma, \quad \forall\, \xi \in \mathrm{cl}\Omega, \end{aligned}$$

for all $(b, \omega, 0, \phi^h, \phi^d, f) \in \mathcal{U}$, *where* $u^d[b, \omega, \phi^h, \phi^d, f]$ *is the unique solution of the Robin boundary value problem* (2.94) *with* $\Omega = \mathbb{I}[\phi^d]$, $a = \alpha \circ (\phi^d)^{(-1)}$ *and*

$$\begin{aligned} g(\xi) = \left(\int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\; d\sigma \right) \Big(& T(b, D\Gamma(b, \xi - \omega))|_{\phi^d(\partial\Omega^d)} \nu_{\phi^d} \\ & + \alpha \circ (\phi^d)^{(-1)}(\xi) \Gamma(b, \xi - \omega)|_{\phi^d(\partial\Omega^d)} \Big), \qquad \forall \xi \in \partial\Omega^d. \end{aligned}$$

In particular, $U[b, \omega, 0, \phi^h, \phi^d, f] = 0$ *if and only if*

$$\int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\; d\sigma = 0.$$

For the energy integral we have the following.

Theorem 2.72. *Let the notation of subsection 2.3.2 hold. Let* $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, f_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$. *Let* $\mathcal{U}$ *be the open neighborhood of* $\mathbf{c}_0$ *introduced in Theorem 2.71. Then there exist real analytic operators* $E^{(1)}$, $E^{(2)}$ *and* $E^{(3)}$ *of* $\mathcal{U}$ *to* $\mathbb{R}$ *such that*

$$\int_{\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]} \mathbf{E}[b](u[\mathbf{c}], u[\mathbf{c}])\, d\xi = E^{(1)}[\mathbf{c}] + \delta_{2,n} (\log \epsilon) E^{(2)}[\mathbf{c}] + \epsilon^{2-n} E^{(3)}[\mathbf{c}],$$

for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$ *with* $\epsilon > 0$. *Moreover, the following two statements hold.*

(i) *Let* $(b,\omega,0,\phi^h,\phi^d,f)\in\mathcal{U}$*. Then* $E^{(2)}[b,\omega,0,\phi^h,\phi^d,f]=0$ *if and only if* $\int_{\partial\Omega^h} f\,\tilde\sigma[\phi^h]\,d\sigma=0$*. If this is the case, then* $E^{(1)}[b,\omega,0,\phi^h,\phi^d,f]=0$ *and*

$$E^{(3)}[b,\omega,0,\phi^h,\phi^d,f]=\int_{\mathbb{E}[\phi^h]}\mathbf{E}[b](u^h[b,\phi^h,f],u^h[b,\phi^h,f])\;d\xi,$$

where $u^h[b,\phi^h,f]$ *is the unique solution of the exterior Neumann boundary value problem in* $\mathbb{E}[\phi^h]$ *with boundary data* $-f\circ(\phi^h)^{(-1)}$ *and with* $|x|^{n-2}|u^h[b,\phi^h,f](x)|$ *and* $|x|^{n-1}|Du^h[b,\phi^h,f](x)|$ *bounded for* $|x|$ *in a neighborhood of* $+\infty$*.*

(ii) *Let* $n\geq 3$ *and* $(b,\omega,0,\phi^h,\phi^d,f)\in\mathcal{U}$*. Then* $E^{(3)}[b,\omega,0,\phi^h,\phi^d,f]=0$ *if and only if* $f=0$*. If this is the case, then* $E^{(1)}[b,\omega,0,\phi^h,\phi^d,f]=E^{(2)}[b,\omega,0,\phi^h,\phi^d,f]=0$*.*

We summarize in the following Remark 2.73 some easily verifiable considerations which can be deduced by Theorems 2.71 and 2.72.

Remark 2.73. *With the same notations of Theorems 2.72 and 2.74, let* $(b,\omega,0,\phi^h,\phi^d,f)\in\mathcal{U}$*. Then the following statements hold.*

(i) *If* $U[b,\omega,0,\phi^h,\phi^d,f]\neq 0$*, then the energy integral of* $u[b,\omega,\epsilon,\phi^h,\phi^d,f]$ *diverges as* $\epsilon\to 0^+$*.*

(ii) *If* $n=2$ *and* $U[b,\omega,0,\phi^h,\phi^d,f]=0$*, then the energy integral of* $u[b,\omega,\epsilon,\phi^h,\phi^d,f]$ *converges to the energy integral of* $u^h[b,\phi^h,f]$ *as* $\epsilon\to 0^+$*, and therefore its limit value as* $\epsilon\to 0^+$ *vanishes only if* $f=0$*.*

(iii) *If* $n\geq 3$*, and* $U[b,\omega,0,\phi^h,\phi^d,f]=0$*, and* $f\neq 0$*, then the energy integral of* $u[b,\omega,\epsilon,\phi^h,\phi^d,f]$ *diverges as* $\epsilon\to 0^+$*.*

(iv) *If* $f=0$ *then the energy integral of* $u[b,\omega,\epsilon,\phi^h,\phi^d,f]$ *is identically equal to* 0 *for all* $\epsilon>0$ *in the right neighborhood of* $\epsilon=0$ *where* $u[b,\omega,\epsilon,\phi^h,\phi^d,f]$ *is defined.*

2.3.7 Proof of Theorems 2.71 and 2.72

Theorem 2.74. *With the notation of the previous subsection 2.3.2, we denote by* $P\equiv(P^1,P^2,P^3)$ *the map of* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times(C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n))^2\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$ *to* $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$ *defined by*

$$\begin{aligned}P^1[b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c](x^h)&\equiv\mathbf{K}^*[b,\phi^h,\mu^h](x)\\&+\epsilon^{n-1}\int_{\partial\Omega^d}\sum_{i=1}^n\Big[T(b,D\Gamma^{(i)}(b,\omega+\epsilon\phi^h(x)-\phi^d(y)))\\&\qquad\cdot\nu_{\phi^h}\circ\phi^h(x)\Big]\mu_i^d(y)\tilde\sigma[\phi^d](y)\;d\sigma_y+f(x),\quad\forall\,x\in\partial\Omega^h,\end{aligned}\tag{2.117}$$

$$\begin{aligned}
&P^2[b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c](x) \qquad (2.118)\\
&\quad \equiv \mathbf{H}[b,\phi^d,\mu^d](x)+\alpha(x)(V[b,\phi^d,\mu^d](x)+c)\\
&\qquad +\int_{\partial\Omega^h}\sum_{i=1}^{n}\Big[T(b,D\Gamma^{(i)}(b,\phi^d(x)-\omega-\epsilon\phi^h(y)))\\
&\qquad\qquad \cdot\nu_{\phi^d}\circ\phi^d(x)\Big]\mu_i^h(y)\tilde\sigma[\phi^h](y)\;d\sigma_y\\
&\qquad +\alpha(x)\int_{\partial\Omega^h}\Gamma(\phi^d(x)-\omega-\epsilon\phi^h(y))\mu^h(y)\,\tilde\sigma[\phi^h](y)\;d\sigma_y,\quad \forall\, x\in\partial\Omega^d,\\
&P^3[b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c]\equiv(1-\delta_{2,n})\,c \qquad (2.119)\\
&\qquad +\delta_{2,n}\left(\int_{\partial\Omega^h}\mu^h\,\tilde\sigma[\phi^h]\;d\sigma+\int_{\partial\Omega^d}\mu^d\,\tilde\sigma[\phi^d]\;d\sigma\right),
\end{aligned}$$

for each $(b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c)$ *in* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times(C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n))^2\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$*, where* $V[b,\phi^d,\mu^d]\equiv v_{\phi^d(\partial\Omega^d)}[b,\mu^d]\circ\phi^d$ *(cf. subsection 1.3.2 of the previous chapter.)*

Let $(b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c)$ *belong to* $\mathscr{B}\times\mathcal{E}_+^{m,\lambda}\times(C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n))^2\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$*. We set*

$$\mu\equiv\begin{cases}\epsilon^{1-n}\,\mu^h\circ(\omega+\epsilon\phi^h)^{(-1)} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ \mu^d\circ(\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d),\end{cases} \qquad (2.120)$$

$$\Omega\equiv\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d], \qquad (2.121)$$

$$a\equiv\begin{cases}\mathbf{0}_{n\times n} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ \alpha\circ(\phi^o)^{(-1)} & \text{on } \phi^d(\partial\Omega^d),\end{cases} \qquad (2.122)$$

$$g\equiv\begin{cases}\epsilon^{1-n}f\circ(\omega+\epsilon\phi^h)^{(-1)} & \text{on } \omega+\epsilon\phi^h(\partial\Omega^h),\\ 0 & \text{on } \phi^d(\partial\Omega^d).\end{cases} \qquad (2.123)$$

Then, we have

$$P[b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c]=0 \qquad (2.124)$$

if and only if either one of the following two conditions is satisfied.

(i) $n=2$ *and the pair* (μ,c)*, with* μ *defined by* (2.120)*, satisfies* (2.96) *with* Ω*,* a *and* g *defined by* (2.121)*,* (2.122) *and* (2.123)*, respectively.*

(ii) $n\geq 3$*, and* $c=0$*, and the function* μ *defined by* (2.120) *satisfies* (2.95) *with* Ω*,* a *and* g *defined by* (2.121)*,* (2.122) *and* (2.123)*, respectively.*

In particular, there exists a unique triple (μ^h,μ^d,c) *of* $C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$ *which satisfies equation* (2.124) *for each given 9-tuple* $(b,\omega,\epsilon,\phi^h,\phi^d,f,\mu^h,\mu^d,c)$ *of the set* $\mathscr{B}\times\mathcal{E}_+^{m,\lambda}\times(C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n))^2\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$*.*

Let $(b,\omega,0,\phi^h,\phi^d,f,\mu^h,\mu^d,c)$ *belong to* $\mathscr{B}\times\mathcal{E}^{m,\lambda}\times(C^{m-1,\lambda}(\partial\Omega^h,\mathbb{R}^n))^2\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$*. Then the triple* (μ^h,μ^d,c) *of* $C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times C^{m-1,\lambda}(\partial\Omega^d,\mathbb{R}^n)\times\mathbb{R}^n$ *satisfies equation*

$$P[b,\omega,0,\phi^h,\phi^d,f,\mu^h,\mu^d,c]=0, \qquad (2.125)$$

if and only if the function $\mu^h \circ (\phi^h)^{(-1)}$ *is the unique solution of*

$$\mathbf{K}^*_{\mathbb{I}[\phi^h]}[b, \mu^h \circ (\phi^h)^{(-1)}] = -f \circ (\phi^h)^{(-1)} \tag{2.126}$$

and either one of the following two conditions is satisfied.

(iii) $n = 2$ *and the pair* (μ, c), *with* $\mu \equiv \mu^d \circ (\phi^d)^{(-1)}$, *is a solution of* (2.97) *with* $\Omega \equiv \mathbb{I}[\phi^d]$, $a \equiv \alpha \circ (\phi^d)^{(-1)}$,

$$\begin{aligned} g(\xi) \equiv \left(\int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\, d\sigma\right) \Big(& T(b, D\Gamma[b, \xi - \omega])|_{\phi^d(\partial\Omega^d)}\nu_{\phi^d} \\ & + \alpha \circ (\phi^d)^{(-1)}(\xi)\Gamma[b, \xi - \omega]|_{\phi^d(\partial\Omega^d)}\Big), \qquad \forall \xi \in \partial\Omega^d, \end{aligned} \tag{2.127}$$

and $d \equiv \int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\, d\sigma$.

(iv) $n \geq 3$, *and* $c = 0$, *and the function* $\mu^d \circ (\phi^d)^{(-1)}$ *is a solution of* (2.95) *with* $\Omega \equiv \mathbb{I}[\phi^d]$, $a \equiv \alpha \circ (\phi^d)^{(-1)}$, *and* g *defined by* (2.127).

In particular, for each fixed $(b, \omega, 0, \phi^h, \phi^d, f, \mu^h, \mu^d, c)$ *in the set* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n))^2 \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *there exists a unique triple* (μ^h, μ^d, c) *of* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *which satisfies equation* (2.125).

Proof. The statement follows by a straightforward verification based on the theorem of change of variables in integrals and by the previous Theorem 2.61. We only note that, if $(b, \omega, 0, \phi^h, \phi^d, f) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ is fixed, then the first component of equation (2.107) is equivalent to (2.126). By statement (i) of Lemma 2.26 equation 2.126 admits a unique solution $\mu^h \circ (\phi^h)^{(-1)} \in C^{m-1,\lambda}(\phi^h(\partial\Omega^h), \mathbb{R}^n)$. Moreover, by Theorems 2.11, 2.12, 2.16, $\mathbf{H}_{\mathbb{I}[\phi^h]}[b, \mu^h \circ (\phi^h)^{(-1)}]$ is orthogonal to each constant function defined on $\phi^h(\partial\Omega^h)$. In particular, $\int_{\phi^h(\partial\Omega^h)} \mathbf{H}_{\mathbb{I}[\phi^h]}[b, \mu^h \circ (\phi^h)^{(-1)}]\, d\sigma = 0$. So we have

$$\begin{aligned} \int_{\partial\Omega^h} \mu^h\, \tilde{\sigma}[\phi^h]\, d\sigma &= \int_{\phi^h(\partial\Omega^h)} \mu^h \circ (\phi^h)^{(-1)} d\sigma \\ &= \int_{\phi^h(\partial\Omega^h)} \mu^h \circ (\phi^h)^{(-1)} + \mathbf{H}_{\mathbb{I}[\phi^h]}[b, \mu^h \circ (\phi^h)^{(-1)}]\, d\sigma \\ &= \int_{\phi^h(\partial\Omega^h)} \mathbf{K}^*_{\mathbb{I}[\phi^h]}[b, \mu^h \circ (\phi^h)^{(-1)}] d\sigma = -\int_{\partial\Omega^h} f\, \tilde{\sigma}[\phi^h]\, d\sigma. \end{aligned}$$

Then, by the second and third component of (2.107), μ^d satisfies either condition (iii) or condition (iv) of the theorem. □

By Theorem 2.74 it makes sense to introduce the following.

Definition 2.75. *With the notation of the previous subsection 2.3.2, let* $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ *with* $\epsilon > 0$ *or* $\epsilon = 0$. *We denote by* $(\hat{\mu}^h[\mathbf{c}], \hat{\mu}^d[\mathbf{c}], \hat{c}[\mathbf{c}])$ *the unique triple* $(\mu^h, \mu^d, c) \in C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *which satisfies* (2.124) *or* (2.125), *respectively.*

We shall show that $(\hat{\mu}^h[\cdot], \hat{\mu}^d[\cdot], \hat{c}[\cdot])$ admit a real analytic continuation around a "degenerate" sextuple $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, f_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$. To do so we need the following Proposition 2.76 and Lemma 2.77, which can be proved by a slight modification in the proof of Proposition 2.65 and Lemma 2.66, respectively.

Proposition 2.76. *With the notation of subsection 2.3.2, the set* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n))^2 \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *is an open subset of the Banach space* $\mathbb{R}^{n+2} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n))^2 \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$, *and the operator* P *is real analytic.*

Lemma 2.77. *With the notation introduced in subsection 2.3.2, let* $\mathbf{d}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, f_0, \mu_0^h, \mu_0^d, c_0)$ *belong to* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times (C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n))^2 \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *and let* $P[\mathbf{d}_0] = 0$. *Then the partial differential* $\partial_{(\mu^h,\mu^d,c)} P[\mathbf{d}_0]$ *of* P *with respect to the variable* (μ^h, μ^d, c) *at* $\mathbf{d}_0$ *is a linear homeomorphism of the Banach space* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times \mathbb{R}^n$ *onto* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times \mathbb{R}^n$.

Now, by the Implicit Mapping Theorem, we deduce the following.

Theorem 2.78. *With the notation introduced in subsection 2.3.2, let* $\mathbf{c}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d, f_0)$ *be an element of* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$. *Then there exist an open neighborhood* $\mathcal{U}_0$ *of* $\mathbf{c}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$, *and an open neighborhood* $\mathcal{V}_0$ *of* $(\hat{\mu}^h[\mathbf{c}_0], \hat{\mu}^d[\mathbf{c}_0], \hat{c}[\mathbf{c}_0])$ *in* $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times \mathbb{R}^n$ *and a real analytic operator* $(\mathcal{P}^h, \mathcal{P}^d, \mathcal{D})$ *of* $\mathcal{U}_0$ *to* $\mathcal{V}_0$ *such that*

$$(\mathcal{P}^h[\mathbf{c}], \mathcal{P}^d[\mathbf{c}], \mathcal{D}[\mathbf{c}]) = (\hat{\mu}^h[\mathbf{c}], \hat{\mu}^d[\mathbf{c}], \hat{c}[\mathbf{c}]) \tag{2.128}$$

for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}_0$ *with* $\epsilon > 0$. *Moreover, the graph of* $(\mathcal{P}^h, \mathcal{P}^d, \mathcal{D})$ *coincides with the set of zeros of* P *in* $\mathcal{U}_0 \times \mathcal{V}_0$.

We are now ready for the proof of Theorems 2.71 and 2.72.

Proof of Theorem 2.71. Let $\mathcal{U}_0$ be the open neighborhood of $\mathbf{c}_0$ introduced in Theorem 2.78. We set

$$U^{(1)}[\mathbf{c}](\xi) \equiv \int_{\partial\Omega^d} \Gamma(\xi - \phi^d(y)) \mathcal{P}^d[\mathbf{c}](y)\, \tilde{\sigma}[\phi^d](y)\, d\sigma_y + \mathcal{D}[\mathbf{c}] \tag{2.129}$$

and

$$U^{(2)}[\mathbf{c}](\xi) \equiv \int_{\partial\Omega^h} \Gamma(\xi - \omega - \epsilon\phi^h(y)) \mathcal{P}^h[\mathbf{c}](y)\, \tilde{\sigma}[\phi^h](y)\, d\sigma_y, \tag{2.130}$$

for all $\xi \in \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ and for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}_0$. Then let $\mathcal{U}$ be an open neighborhood of $\mathbf{c}_0$ contained in $\mathcal{U}_0$ and such that condition (i) of Theorem 2.71 holds. By Proposition 2.39, $U^{(1)}[\cdot]|_{\mathrm{cl}\Omega}$, $U^{(2)}[\cdot]|_{\mathrm{cl}\Omega}$ are real analytic operators of $\mathcal{U}$ to $C(\mathrm{cl}\Omega, \mathbb{R}^n)$. Moreover we have

$$u[\mathbf{c}](\xi) = U^{(1)}[\mathbf{c}](\xi) + U^{(2)}[\mathbf{c}](\xi), \quad \forall\, \xi \in \mathrm{cl}\Omega,$$

for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$ with $\epsilon > 0$. So, by taking $U[\mathbf{c}] \equiv U^{(1)}[\mathbf{c}]|_{\mathrm{cl}\Omega} + U^{(2)}[\mathbf{c}]|_{\mathrm{cl}\Omega}$, statement (ii) of the Theorem follows. Statement (iii) is an immediate consequence of Theorem 2.74. □

Proof of Theorem 2.72. Let $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$ with $\epsilon > 0$. Then, we have

$$\begin{aligned}\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u[\mathbf{c}], u[\mathbf{c}])\, d\xi = &\int_{\phi^d(\partial\Omega^d)} u[\mathbf{c}] \cdot \left[T(b, Du[\mathbf{c}])\nu_{\phi^h}\right]^+ d\sigma \\ &- \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} u[\mathbf{c}] \cdot \left[T(b, Du[\mathbf{c}])\nu_{\omega+\epsilon\phi^h}\right]^- d\sigma.\end{aligned}$$

We denote by $\mathcal{I}_1[\mathbf{c}]$ and by $\mathcal{I}_2[\mathbf{c}]$ the first and the second term in the right hand side, respectively. Then

$$\begin{aligned}\mathcal{I}_1[\mathbf{c}] = -\int_{\phi^d(\partial\Omega^d)} &\left(U^{(1)}[\mathbf{c}] + U^{(2)}[\mathbf{c}]\right) \\ &\cdot \alpha \circ (\phi^o)^{(-1)} \left(U^{(1)}[\mathbf{c}] + U^{(2)}[\mathbf{c}]\right)\, d\sigma\end{aligned} \tag{2.131}$$

and

$$\begin{aligned}\mathcal{I}_2[\mathbf{c}] = &-\epsilon^{1-n} \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} U^{(1)}[\mathbf{c}] \cdot f \circ (\omega + \epsilon\phi^h)^{(-1)}\, d\sigma \\ &-\epsilon^{1-n} \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} U^{(2)}[\mathbf{c}] \cdot f \circ (\omega + \epsilon\phi^h)^{(-1)}\, d\sigma,\end{aligned} \tag{2.132}$$

where $U^{(1)}[\mathbf{c}]$ and $U^{(2)}[\mathbf{c}]$ are defined as in the proof of Theorem 2.71. By arguing as in the proof of Theorem 2.70 one verifies that right hand side of (2.131) and the first term in the right hand side of (2.132) have a real analytic continuation in the whole of $\mathcal{U}$. So, we denote by $E^{(1)}[\cdot]$ the real analytic operator on $\mathcal{U}$ which is defined by

$$E^{(1)}[\mathbf{c}] \equiv \mathcal{I}_1[\mathbf{c}] - \epsilon^{1-n} \int_{\omega+\epsilon\phi^h(\partial\Omega^h)} U^{(1)}[\mathbf{c}] \cdot f \circ (\omega + \epsilon\phi^h)^{(-1)}\, d\sigma,$$

for all $\mathbf{c} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, f) \in \mathcal{U}$ with $\epsilon > 0$.

We now recall that, $\Gamma(b, \epsilon z) = \delta_{2,n}\frac{1}{2\pi}\frac{b+2}{2(b+1)}(\log\epsilon) + \epsilon^{2-n}\Gamma(b,z)$, for all $b \in \mathbb{B}$, $\epsilon > 0$, $z \in \mathbb{R}^n \setminus \{0\}$ (cf. definition (2.6).) So, by (2.129) and (2.130), the second term in the right hand side of (2.132) equals

$$-\frac{\delta_{2,n}}{2\pi}\frac{b+2}{2(b+1)}(\log\epsilon)\int_{\partial\Omega^h}\mathcal{P}^h[\mathbf{c}]\,\tilde{\sigma}[\phi^h]\,d\sigma\cdot\int_{\partial\Omega^h} f\,\tilde{\sigma}[\phi^h]\,d\sigma$$
$$-\epsilon^{2-n}\int_{\partial\Omega^h} V[b,\phi^h,\mathcal{P}^h[\mathbf{c}]]\cdot f\,\tilde{\sigma}[\phi^h]\,d\sigma. \qquad (2.133)$$

We set,

$$E^{(2)}[\mathbf{c}] \equiv -\frac{1}{2\pi}\frac{b+2}{2(b+1)}\int_{\partial\Omega^h}\mathcal{P}^h[\mathbf{c}]\,\tilde{\sigma}[\phi^h]\,d\sigma\cdot\int_{\partial\Omega^h} f\,\tilde{\sigma}[\phi^h]\,d\sigma,$$
$$E^{(3)}[\mathbf{c}] \equiv -\int_{\partial\Omega^h} V[b,\phi^h,\mathcal{P}^h[\mathbf{c}]]\cdot f\,\tilde{\sigma}[\phi^h]\,d\sigma,$$

for all $\mathbf{c} \in \mathcal{U}$. Then, by Propositions 1.40 and 2.39 and by Theorems 2.71, $E^{(2)}[\mathbf{c}]$ and $E^{(3)}[\mathbf{c}]$ depend real analytically on $\mathbf{c} \in \mathcal{U}$.

Now, to conclude the proof it remains to verify statements (i) and (ii). To do so, we exploit Theorem 2.74. By equation (2.126) we deduce that

$$E^{(2)}[b,\omega,0,\phi^h,\phi^d,f] = -\frac{1}{2\pi}\frac{b+2}{2(b+1)}\left(\int_{\partial\Omega^h} f\,\tilde{\sigma}[\phi^h]\,d\sigma\right)^2,$$

for all $(b,\omega,0,\phi^h,\phi^d,f) \in \mathcal{U}$. So, $E^{(2)}[b,\omega,0,\phi^h,\phi^d,f] = 0$ if and only if $\int_{\partial\Omega^h} f\tilde{\sigma}[\phi^h]\,d\sigma = 0$. Moreover, if this is the case, then both the functions $U^{(1)}[b,\omega,0,\phi^h,\phi^d,f]$ and $U^{(2)}[b,\omega,0,\phi^h,\phi^d,f]$ are identically equal to 0, which implies that $E^{(1)}[b,\omega,0,\phi^h,\phi^d,f] = 0$. Besides we have

$$E^{(3)}[b,\omega,0,\phi^h,\phi^d,f] = \int_{\mathbb{E}[\phi^h]}\mathbf{E}[b](u^h[b,\phi^h,f],u^h[b,\phi^h,f])\,d\xi$$

for all $(b,\omega,0,\phi^h,\phi^d,f) \in \mathcal{U}$, where

$$u^h[b,\phi^h,f] \equiv v_{\phi^h(\partial\Omega^h)}[b,\mathcal{P}^h[b,\omega,0,\phi^h,\phi^d,f]\circ(\phi^h)^{(-1)}]$$

is the unique solution of the exterior Neumann boundary value problem in $\mathbb{E}[\phi^h]$ with boundary data $-f\circ(\phi^h)^{(-1)}$ (note that, if $n = 2$ the condition $\int_{\partial\Omega^h} f\tilde{\sigma}[\phi^h]\,d\sigma = 0$ is necessary.) Therefore $E^{(3)}[b,\omega,0,\phi^h,\phi^d,f] = 0$ if and only if $\mathbf{E}[b](u^h[b,\phi^h,f],u^h[b,\phi^h,f]) = 0$ in $\mathbb{E}[\phi^h]$, if and only if $u^h[b,\phi^h,f] \in \mathcal{R}_{\mathbb{E}[\phi^h]}$ (cf. Lemma 2.2), if and only if $[T(b,Du^h[b,\phi^h,f])\nu_{\phi^h}]^-$ equals 0 (cf. Theorem 2.4.) Summarizing, we have $E^{(3)}[b,\omega,0,\phi^h,\phi^d,f] = 0$ if and only if $f = 0$. □

2.4 Inhomogeneous interior data

2.4.1 Description of the problem

In this subsection we introduce a Dirichlet boundary value problem in a perforated domain with a non-homogeneous data in the interior and we investigate the behavior of the solution and of the corresponding energy integral as the hole shrinks to a point. Our approach to the problem stems from Lanza de Cristoforis [24]. Let Ω_0 be a fixed bounded open connected subset of $\mathbb{R}^n$. Let $m \in \mathbb{N} \setminus \{0\}$ and $\lambda \in]0,1[$. Let Ω^h and Ω^d be bounded open subsets of $\mathbb{R}^n$ of class $C^{m,\lambda}$ with Ω^h, Ω^d, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^h$, $\mathbb{R}^n \setminus \mathrm{cl}\Omega^d$ connected. Let $\mathcal{E}^{m,\lambda}$ be as in subsection 2.2.1. We denote by $\mathcal{E}^{m,\lambda}_{\Omega_0}$ the subset of $\mathcal{E}^{m,\lambda}$ of the quadruples $(\omega, \epsilon, \phi^h, \phi^d)$ with $\mathrm{cl}\mathbb{I}[\phi^d] \subset \Omega_0$. Then, for each $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ with $\epsilon > 0$ and for each vector valued function F defined on Ω_0, we consider the following Dirichlet boundary value problem in the domain $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d] \equiv \mathbb{I}[\phi^d] \setminus \omega + \mathrm{cl}(\epsilon\mathbb{I}[\phi^h])$,

$$\begin{cases} \mathbf{L}[b]u = F & \text{in } \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d], \\ u = g^h \circ (\omega + \epsilon\phi^h)^{(-1)} & \text{on } \omega + \epsilon\phi^h(\partial\Omega^h), \\ u = g^d \circ (\phi^d)^{(-1)} & \text{on } \phi^d(\partial\Omega^d). \end{cases} \tag{2.134}$$

Under reasonable conditions on F, problem (2.134) has a unique solution $u = u[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F]$ and such a solution can be written in the form

$$\begin{aligned} &u[b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F] \\ &\quad = P[b, F] + u[b, \omega, \epsilon, \phi^h, \phi^d, g^h - P[b, F] \circ (\omega + \epsilon\phi^h), g^d - P[b, F] \circ \phi^d, 0], \end{aligned}$$

where

$$P[b, F](\xi) \equiv \int_\Omega \Gamma(b, \xi - \eta) F(\eta) \, d\eta, \quad \forall\, \xi \in \Omega, \tag{2.135}$$

is the Newtonian potential of F in Ω.

Our purpose is to investigate the behavior the solution of (2.134) and of its energy integral for $\epsilon \to 0^+$. By Theorem 2.53, we know that we can represent $u[b, \omega, \epsilon, \phi^h, \phi^o, g^h, g^d, 0]$ in terms of the function $\gamma_n(\epsilon)$ and in terms of real analytic operators of the variable $(b, \omega, \epsilon, \phi^h, \phi^o, g^h, g^d)$. Thus, what remains to be done here is to choose an appropriate Banach space for F so that $P[b, F]$, $P[b, F] \circ (\omega + \epsilon\phi^h)$, $P[b, F] \circ \phi^d$ depend real analytically on $(b, \omega, \epsilon, \phi^h, \phi^d, F)$. Now, for a large variety of choices of function spaces for F, $P[b, F]$ depends real analytically on (b, F), and this is so in particular for the Schauder spaces $C^{m,\lambda}$. Less clear instead is the choice for the function spaces for F, $P[b, F] \circ \phi$ in order that $P[b, F] \circ \phi$ depends real analytically on (b, ϕ, F) when ϕ is in a Schauder space. Then we resort to results on composition operators of Preciso [38], [39], which indicate that the right choice for the space for F, $P[b,F]$ is a Romieu class, and thus the corresponding real analyticity results for $P[b,F]\circ\phi$ of Lanza de Cristoforis [23, Lemma 2.15], where a regular perturbation problem for the Poisson equation has been treated (such results concern the Newtonian potential relative to the Laplace operator Δ but the proof given there applies with only minor modifications to the present situation, where the Newtonian potential relative to the operator $\mathbf{L}[b]$ is considered.) Then we can prove Theorems 2.80 for the behavior of the solution of (2.134), and Theorem 2.82 for the behavior of the corresponding energy integral, which extend the corresponding Theorems 2.53 and 2.55 for $F=0$.

2.4.2 Introduction of the Romieu classes

For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho>0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)\equiv\left\{u\in C^\infty(\mathrm{cl}\Omega,\mathbb{R}^n)\,\Big|\,\sup_{\beta\in\mathbb{N}^n}\frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}\Omega,\mathbb{R}^n)}<+\infty\right\}$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)}\equiv\sup_{\beta\in\mathbb{N}^n}\frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}\Omega,\mathbb{R}^n)},\quad\forall\, u\in C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)\,.$$

As is well known, the Romieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n),\|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)}\right)$ is a Banach space.

Then we have the following technical lemma.

Lemma 2.79. *Let $m\in\mathbb{N}^n$, $\lambda\in]0,1[$ and $\rho>0$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$. Let Ω_1 be an open connected subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\Omega_1\subset\Omega$. Then the following statements hold.*

(i) If Ω_2 is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_2\subset\Omega_1$, then there exists $\rho_1\in]0,\rho]$ such that the map of $\mathscr{B}\times C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)$ to $C^0_{\omega,\rho_1}(\mathrm{cl}\Omega_2,\mathbb{R}^n)$ which takes (b,F) to $P[b,F|_{\mathrm{cl}\Omega_1}]|_{\mathrm{cl}\Omega_2}$ is real analytic.

(ii) The map of $\{(b,F,\xi)\in\mathscr{B}\times C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)\times\mathbb{R}^n\,|\,\xi\in\Omega_1\}$ to $\mathbb{R}^n$ which take (b,F,ξ) to $P[b,F|_{\mathrm{cl}\Omega_1}](\xi)$ is real analytic.

(iii) If Ω_2 is a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$, the map of $\mathscr{B}\times C^0_{\omega,\rho}(\mathrm{cl}\Omega,\mathbb{R}^n)\times C^{m,\lambda}(\partial\Omega_2,\Omega_1)$ to $C^{m,\lambda}(\partial\Omega_2,\mathbb{R}^n)$ which takes (b,F,ϕ) to $P[b,F|_{\mathrm{cl}\Omega_1}]\circ\phi$ is real analytic.

Proof. Let $F\equiv(F_i)_{i=1,\dots,n}$ be a vector valued function on a bounded open subset Ω of $\mathbb{R}^n$. By (2.6) we have

$$P[b,F]=\frac{b+2}{2(b+1)}P_\Delta[F_i]_{i=1,\dots,n}-\frac{b}{2(b+1)}Q[F],$$

where $P_\Delta[F_i]$ denotes the Newtonian potential corresponding to the Laplace operator and $Q[F]$ is defined by

$$Q[F](\xi) \equiv \frac{1}{|\partial\mathbb{B}_n|}\int_\Omega \frac{(\xi-\eta)}{|\xi-\eta|^n}(\xi-\eta)\cdot F(\eta)\, d\eta, \quad \forall\, \xi \in \mathrm{cl}\Omega.$$

By Lanza de Cristoforis [23, Lemma 2.15], statements (i), (ii) and (iii) hold with $P[b,F]$ replaced by $P_\Delta[F_i]$, $i=1,\dots,n$. Moreover, by a straightforward modification of the proof of Lanza de Cristoforis [23, Lemma 2.15], we can also verify that statements (i), (ii) and (iii) hold with $P[b,F]$ replaced by $Q[F]$. Then the proof of the Lemma can be easily completed. □

2.4.3 Solution of the singularly perturbed problem

By Therem 2.53 and by Lemma 2.79, we deduce the following.

Theorem 2.80. *Let the notation of subsection 2.4.1 hold. Let* $\rho > 0$. *Let* $\mathbf{f}_0 \equiv (b_0, \omega, 0, \phi^h_0, \phi^d_0, g^h_0, g^d_0, F_0)$ *belong to* $\mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$. *Let* $\mathcal{W}_0$ *be the neighborhood of the point* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi^h_0, \phi^d_0)$ *introduced in Proposition 2.44, and let* $V^{(1)}$, $V^{(2)}$ *be as in Lemma 2.52. Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\Omega \subset \mathbb{I}[\phi^d_0] \setminus \{\omega_0\}$. *Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{f}_0$ *in* $\mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$, *and real analytic operators* $U^{(1)}$ *and* $U^{(2)}_{ij}$, $i,j=1,\dots,\bar{n}$, *of* $\mathcal{U}$ *to* $C(\mathrm{cl}\Omega, \mathbb{R}^n)$ *such that the following conditions hold.*

(i) $\mathrm{cl}\Omega \subset \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ *for all* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$.

(ii) $(b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$ *for all* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$.

(iii) We have

$$u[\mathbf{f}](\xi) = U^{(1)}[\mathbf{f}](\xi) + \sum_{i,j=1}^{\bar{n}} \Big(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\Big)^{-1}_{ij} U^{(2)}_{ij}[\mathbf{f}](\xi), \tag{2.136}$$

for all $\xi \in \mathrm{cl}\Omega$, *and all* $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$ *with* $\epsilon > 0$, *where* $\gamma_n(\epsilon) \equiv \log\epsilon$ *if* $n = 2$, *and* $\gamma_n(\epsilon) \equiv \epsilon^{2-n}$ *if* $n \geq 3$, *and we abbreviated* $(b, \omega, \epsilon, \phi^h, \phi^d)$ *as* $\mathbf{b}$.

(iv)

$$U^{(1)}[b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F](\xi) = u^d[b, \phi^d, g^d, F](\xi),$$

for all $\xi \in \mathrm{cl}\Omega$ *and for all* $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d) \in \mathcal{U}$, *where* $u^d[b, \phi^d, g^d]$ *is the solution of the Dirichlet boundary value problem in* $\mathbb{I}[\phi^d]$ *with boundary data* $g^d \circ (\phi^d)^{(-1)}$ *and interior data* $F|_{\mathbb{I}[\phi^d]}$.

(v) Let $\mathbf{f} \equiv (b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$, *we denote by* $\mathbf{f}^\epsilon$, $\mathbf{b}^\epsilon$ *the points* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F)$ *and* $(b, \omega, \epsilon, \phi^h, \phi^d)$, *respectively, for all* $\epsilon > 0$. *Then*

$$\lim_{\epsilon \to 0^+} \sum_{i,j=1}^{\bar{n}} \left(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}^\epsilon] + V^{(2)}[\mathbf{b}^\epsilon] \right)^{-1}_{ij} U^{(2)}_{ij}[\mathbf{f}^\epsilon](\xi) = 0$$

uniformly for $\xi \in \mathrm{cl}\Omega$.

Proof. Let Ω_1 be an open connected subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\mathbb{I}[\phi^d_0] \subset \Omega_1$, $\mathrm{cl}\Omega_1 \subset \Omega_0$. Let $\mathcal{U}$ be an open neighborhood of $\mathbf{f}_0$ in $\mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$ such that $(b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$, and $\mathrm{cl}\Omega \subset \mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$, and $\mathrm{cl}\mathbb{I}[\phi^d] \subset \Omega_1$, for all $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$. Thus we have

$$u[\mathbf{f}] = P[b, F|_{\mathrm{cl}\Omega_1}] + u[b, \omega, \epsilon, \phi^h, \gamma^h[\mathbf{f}], \gamma^d[\mathbf{f}], 0], \qquad (2.137)$$

for all $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$, where $\gamma^h[\mathbf{f}] \equiv g^h - P[b, F|_{\mathrm{cl}\Omega_1}] \circ (\omega + \epsilon\phi^h)$ and $\gamma^d[\mathbf{f}] \equiv g^d - P[b, F|_{\mathrm{cl}\Omega_1}] \circ \phi^d$ for all $\mathbf{f} \in \mathcal{U}$ with $\epsilon > 0$. By statement (iii) of Lemma 2.79, the maps which take $\mathbf{f}$ to $\gamma^h[\mathbf{f}]$ and $\gamma^d[\mathbf{f}]$ are real analytic from $\mathcal{U}$ to $C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ and to $C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n)$, respectively. Then, by Theorem 2.53, possibly shrinking the neighborhood $\mathcal{U}$ of $\mathbf{f}_0$, the second term in the right hand side of (2.137) admits a functional analytic representation as in the right hand side of (2.136). Let $\widetilde{U}^{(1)}$ and $\widetilde{U}^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, be the corresponding real analytic operators. We denote by $U^{(1)}$ and $U^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, the operators which take $\mathbf{f} \in \mathcal{U}$ to $P[b, F|_{\mathrm{cl}\Omega_2}]|_{\mathrm{cl}\Omega} + \widetilde{U}^{(1)}[\mathbf{f}]$ and $\widetilde{U}^{(2)}_{ij}[\mathbf{f}]$, $i, j = 1, \dots, \bar{n}$, respectively. Then, by Lemma 2.79 (i) and Theorem 2.53, $U^{(1)}$ and $U^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, are real real analytic operators of $\mathcal{U}$ to the space $C(\mathrm{cl}\Omega, \mathbb{R}^n)$ and satisfies conditions (iii), (iv) and (v) of the Theorem. □

We note that, if $n \geq 3$, the right hand side of (2.136) admit a real analytic continuation in the whole of $\mathcal{U}$, while for $n = 2$, the right hand side of (2.136) displays a logarithmic behavior.

2.4.4 The corresponding energy integral

In the proof of Theorem 2.82 we need the following technical lemma, which can be verified by a straightforward modification of the proof of Lanza de Cristoforis [24, Proposition 2.2].

Lemma 2.81. *Let the notations introduced in subsection 2.4.1 hold. Let* Ω_1 *be a bounded open connected subset of* $\mathbb{R}^n$ *of class* C^1 *such that* $\mathrm{cl}\Omega_1 \subset \Omega_0$. *Let* $(b_0, \omega, 0, \phi^h_0, \phi^d_0, F_0) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_1} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$. *Then there exist an* *open neighborhood* $\mathcal{V}_1$ *of* $(b_0, \omega, 0, \phi_0^h, \phi_0^d, F_0)$ *in* $\mathscr{B} \times \mathcal{E}_{\Omega_0}^{m,\lambda} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$ *and a real analytic operator* Π *of* $\mathcal{V}_1$ *to* $\mathbb{R}$ *such that*

$$\Pi[b, \omega, \epsilon, \phi^h, \phi^d, F] = \int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^o]} \mathbf{E}[b](P[F|_{\mathrm{cl}\Omega_1}], P[F|_{\mathrm{cl}\Omega_1}])\, d\xi,$$

for all $(b, \omega, \epsilon, \phi^h, \phi^d, F) \in \mathcal{V}_1$ *with* $\epsilon > 0$.

Theorem 2.82. *Let the notation of subsection 2.4.1 hold. Let* $\rho > 0$. *Let* $\mathbf{f}_0 \equiv (b_0, \omega, 0, \phi_0^h, \phi_0^d, g_0^h, g_0^d, F_0)$ *belong to the set* $\mathscr{B} \times \mathcal{E}_{\Omega_0}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$. *Let* $\mathcal{W}_0$ *be the neighborhood of the point* $\mathbf{b}_0 \equiv (b_0, \omega_0, 0, \phi_0^h, \phi_0^d)$ *introduced in Proposition 2.44, and let* $V^{(1)}$, $V^{(2)}$ *be as in Lemma 2.52. Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{f}_0$ *in* $\mathscr{B} \times \mathcal{E}_{\Omega_0}^{m,\lambda} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$ *and real analytic operators* $E^{(1)}$ *and* $E^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, *of* $\mathcal{U}$ *to* $\mathbb{R}$, *such that* $(b, \omega, \epsilon, \phi^h, \phi^d) \in \mathcal{W}_0$ *if* $(b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$ *and*

$$\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u[\mathbf{f}], u[\mathbf{f}])\, d\xi \qquad (2.138)$$

$$= E^{(1)}[\mathbf{f}] + \sum_{i,j=1}^{\bar{n}} \left(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\right)^{-1}_{ij} E^{(2)}_{ij}[\mathbf{q}],$$

for all $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$ *with* $\epsilon > 0$, *where we abbreviated* $(b, \omega, \epsilon, \phi^h, \phi^d)$ *as* $\mathbf{b}$.

Moreover, if $\mathbf{f} \equiv (b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F)$ *belongs to* $\mathcal{U}$ *and we set* $\mathbf{f}^\epsilon \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F)$, $\mathbf{b}^\epsilon \equiv (b, \omega, \epsilon, \phi^h, \phi^d)$ *for all* $\epsilon > 0$, *then*

$$\lim_{\epsilon \to 0^+} E^{(1)}[\mathbf{f}^\epsilon] + \sum_{i,j=1}^{\bar{n}} \left(\gamma_n(\epsilon) V^{(1)}[\mathbf{b}^\epsilon] + V^{(2)}[\mathbf{b}^\epsilon]\right)^{-1}_{ij} E^{(2)}_{ij}[\mathbf{f}^\epsilon] \qquad (2.139)$$

$$= \int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u^d[b, \phi^d, g^d, F], u^d[b, \phi^d, g^d, F])\, d\xi$$

$$+ \delta_{2,n} \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u^h[b, \phi^h, g^h], u^h[b, \phi^h, g^h])\, d\xi,$$

where $u^d[b, \phi^d, g^d, F]$ *is the solution of the Dirichlet boundary value problem in* $\mathbb{I}[\phi^d]$ *with boundary data* $g^d \circ (\phi^d)^{(-1)}$ *and interior data* $F|_{\mathbb{I}[\phi^d]}$, *and* $u^h[b, \phi^h, g^h]$ *is the solution of the Dirichlet exterior boundary value problem in* $\mathbb{E}[\phi^h]$ *with boundary data* $g^h \circ (\phi^h)^{(-1)}$ *(cf. Definition 2.31.)*

Proof. Let Ω_1 be an open bounded open connected subset of $\mathbb{R}^n$ of class C^∞ such that $\mathrm{cl}\mathbb{I}[\phi_0^d] \subset \Omega_1$ and $\mathrm{cl}\Omega_1 \subset \Omega_0$. Then we have

$$u[\mathbf{f}] = u_1[\mathbf{f}] + P[F|_{\mathrm{cl}\Omega_1}]$$

for all $\mathbf{f} \in \mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$ with $\epsilon > 0$, where

$$u_1[\mathbf{f}] \equiv u[b, \omega, \epsilon, \phi^h, \phi^d, g^h - P[F|_{\mathrm{cl}\Omega_1}] \circ (\omega + \epsilon\phi^h), g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d, 0]$$

is the solution of the homogeneous Dirichlet boundary value problem in $\mathbb{A}[\omega, \epsilon, \phi^h, \phi^d]$ with boundary data $g^h \circ (\omega + \epsilon\phi^h)^{(-1)} - P[F|_{\mathrm{cl}\Omega_1}]|_{\omega+\epsilon\phi^h(\partial\Omega^h)}$ and $g^d \circ (\phi^d)^{(-1)} - P[F|_{\mathrm{cl}\Omega_1}]|_{\phi^d(\partial\Omega^d)}$. Then, by Theorem 2.3, we have

$$\begin{aligned}
&\int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u[\mathbf{f}], u[\mathbf{f}])\, d\xi \qquad (2.140)\\
&\quad = \int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](u_1[\mathbf{f}], u_1[\mathbf{f}])\, d\xi \\
&\qquad + \int_{\mathbb{A}[\omega,\epsilon,\phi^h,\phi^d]} \mathbf{E}[b](P[F|_{\mathrm{cl}\Omega_1}], P[F|_{\mathrm{cl}\Omega_1}])\, d\xi \\
&\qquad -2\epsilon^{n-1} \int_{\partial\Omega^h} (P[F|_{\mathrm{cl}\Omega_1}] \circ (\omega + \epsilon\phi^h)) \\
&\qquad\qquad \cdot \left[T(b, Du_1[\mathbf{f}])\nu_{\omega+\epsilon\phi^h}\right] \circ (\omega + \epsilon\phi^h)\, \tilde{\sigma}[\phi^h]\, d\sigma \\
&\qquad +2 \int_{\partial\Omega^d} (P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d) \cdot \left[T(b, Du_1[\mathbf{f}])\nu_{\phi^d}\right] \circ \phi^d\, \tilde{\sigma}[\phi^d]\, d\sigma
\end{aligned}$$

for all $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F)$ in the set $\mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$ with $\epsilon > 0$.

By Theorem 2.55 and by statement (iii) of Lemma 2.79, the first integral in the right hand side of (2.140) admits a representation as in the right hand side of (2.138). Moreover, the corresponding limit (2.139) converges to

$$\int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u^d_0[\mathbf{f}], u^d_0[\mathbf{f}])\, d\xi + \delta_{2,n} \int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u^h_0[\mathbf{f}], u^h_0[\mathbf{f}])\, d\xi, \qquad (2.141)$$

where $u^d_0[\mathbf{f}] \equiv u^d[b, \phi^d, g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d, 0]$ and $u^h_0[\mathbf{f}] \equiv u^h[b, \phi^h, g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)]$ are the solution of the homogeneous Dirichlet boundary value problem in $\mathbb{I}[\phi^d]$ with boundary data $g^d \circ \phi^d - P[F|_{\mathrm{cl}\Omega_1}]|_{\phi^d(\partial\Omega^d)}$ and the solution of the homogeneous Dirichlet exterior boundary value problem in $\mathbb{E}[\phi^h]$ with boundary data $g^h \circ \phi^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)$, respectively. In particular, for $n = 2$, $u^h[b, \phi^h, g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)]$ differs from $u^h[b, \phi^h, g^h]$ by a constant function, and the second integral in (2.141) equals

$$\int_{\mathbb{E}[\phi^h]} \mathbf{E}[b](u^h[b, \phi^h, g^h], u^h[b, \phi^h, g^h])\, d\xi.$$

By Lemma 2.81, the second integral in the right hand side of (2.140) admits a real analytic continuation in the variable $\mathbf{f}$ around $\mathbf{f}_0$, and accordingly it admits a representation as in the right hand side of (2.138) and the corresponding $E^{(1)}$, $E^{(2)}$ equal Π and 0, respectively.

We now consider the third and fourth integrals in the right hand side of (2.140). By statement (iii) of Lemma 2.79, both the functions $g^h - P[F|_{\mathrm{cl}\Omega_1}] \circ (\omega + \epsilon\phi^h)$ and $g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d$ depend real analytically on $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathscr{B} \times \mathcal{E}^{m,\lambda}_{\Omega_0} \times C^{m,\lambda}(\partial\Omega^h, \mathbb{R}^n) \times C^{m,\lambda}(\partial\Omega^d, \mathbb{R}^n) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega_0, \mathbb{R}^n)$. So, by arguing as in the proof of Theorem 2.55, we verify that, possibly shrinking the neighborhood $\mathcal{U}$ of $\mathbf{f}_0$, there exist real analytic operators $H^{(1)}$, $H^{(2)}_{ij}$, $i, j = 1, \dots, \bar{n}$, of $\mathcal{U}$ to $C^{m-1,\lambda}(\partial\Omega^h, \mathbb{R}^n)$ and real analytic operators $H^{(3)}$, $H^{(4)}_{ij}$, $i, j = 1, \dots, n'$, of $\mathcal{U}$ to $C^{m-1,\lambda}(\partial\Omega^d, \mathbb{R}^n)$ such that

$$\epsilon^{n-1}\left[T(b, Du_1[\mathbf{f}])\nu_{\omega+\epsilon\phi^h}\right] \circ (\omega + \epsilon\phi^h) = H^{(1)}[\mathbf{f}] + \sum_{i,j=1}^{\bar{n}} \left(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\right)^{-1}_{ij} H^{(2)}_{ij}[\mathbf{f}], \tag{2.142}$$

$$\left[T(b, Du_1[\mathbf{f}])\nu_{\phi^d}\right] \circ \phi^d = H^{(3)}[\mathbf{f}] + \sum_{i,j=1}^{n'} \left(\gamma_n(\epsilon)V^{(1)}[\mathbf{b}] + V^{(2)}[\mathbf{b}]\right)^{-1}_{ij} H^{(4)}_{ij}[\mathbf{f}], \tag{2.143}$$

for all $\mathbf{f} \equiv (b, \omega, \epsilon, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$ with $\epsilon > 0$, where as usual $\mathbf{b} \equiv (b, \omega, \epsilon, \phi^h, \phi^d)$. By such equations, and by Proposition 2.39, and by statement (iii) of the previous Lemma 2.79, and by standard calculus in Banach space, one easily deduces the existence of $E^{(1)}$, $E^{(2)}$ for the third and fourth integral in the right hand side of (2.140).

To complete the proof we have to verify equation (2.139). To do so, we compute the limit as $\epsilon \to 0^+$ for the right hand sides of (2.142) and (2.143). First we note that

$$H^{(1)}[b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F] = \delta_{2,n}\Big[T\big(b, Du^h_r[b, \phi^h, g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)]\big)\nu_{\phi^h}\Big] \circ \phi^h,$$

$$H^{(3)}[b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F] = \Big[T\big(b, Du^d[b, \phi^d, g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d, 0]\big)\nu_{\phi^d}\Big] \circ \phi^d,$$

for all $(b, \omega, 0, \phi^h, \phi^d, g^h, g^d, F) \in \mathcal{U}$, where u^h_r is the function introduced in Definition 2.31 (cf. proof of Theorem 2.55.) In particular, for $n = 2$, we have

$$\begin{aligned} &Du^h_r[b, \phi^h, g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)] \\ &\quad = Du^h[b, \phi^h, g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)] + c_1[b, \omega, \phi^h, g^h, F]\, Dv_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}[b, \phi^h]] \\ &\quad = Du^h[b, \phi^h, g^h] + c_1[b, \omega, \phi^h, g^h, F]\, Dv_{\phi^h(\partial\Omega^h)}[b, \tilde{\alpha}[b, \phi^h]], \end{aligned}$$

where $\tilde{\alpha}[b, \phi^h] \in (\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \cdot])_0$ is defined as in Lemma 2.29, and

$c_1[b, \omega, \phi^h, g^h, F]$ is a real constant. Now consider $H_{ij}^{(2)}$. We have

$$H_{ij}^{(2)}[\mathbf{f}](x) = \Big(\int_{\partial\Omega^h} (g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)) \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} (g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d) \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^d]\, d\sigma \Big) \cdot \Big(\Big[T\Big(b, Dv_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]\Big) \nu_{\phi^h} \Big]^- \circ \phi^h(x) + \epsilon^{n-1} \sum_{k=1}^n \int_{\partial\Omega^d} \Big[T\big(b, D\Gamma^{(k)}\big(b, \omega + \epsilon\phi^h(x) - \phi^d(y)\big)\big) \cdot \nu_{\phi^h} \circ \phi^h(x) \Big] \Big(\mathcal{T}_d^{(j)}[\mathbf{b}]\Big)_k (y)\, \tilde{\sigma}[\phi^d](y)\, d\sigma_y \Big),$$

for all $x \in \partial\Omega^h$ and for all $\mathbf{f} \in \mathcal{U}$ with $\epsilon > 0$, where $\mathcal{T}_h^{(i)}[\mathbf{b}]$ and $\mathcal{T}_d^{(i)}[\mathbf{b}]$ are defined as in Proposition 2.44 (cf. equation (2.83).) Clearly, for $\epsilon \to 0^+$, the last factor in parentheses converges to

$$\Big[T\Big(b, Dv_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}]\Big) \nu_{\phi^h} \Big]^- \circ \phi^h. \qquad (2.144)$$

We recall that, for $\epsilon = 0$, $\mathcal{T}_h^{(i)}[\mathbf{b}] \circ (\phi^h)^{(-1)}$ is an element of $\mathrm{Ker}\mathbf{H}_{\phi^h(\partial\Omega^h)}[b, \cdot]$. Thus $\mathbf{K}^*_{\phi^h(\partial\Omega^h)}[b, \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}] = \mathcal{T}_h^{(j)}[\mathbf{b}] \circ (\phi^h)^{(-1)}$. So the expression in (2.144) equals $\mathcal{T}_h^{(j)}[\mathbf{b}]$. Now consider $H_{ij}^{(4)}$. We can verify that

$$H_{ij}^{(4)}[\mathbf{f}](x) = \Big(\int_{\partial\Omega^h} (g^h - P[F|_{\mathrm{cl}\Omega_1}](\omega)) \cdot \mathcal{T}_h^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^h]\, d\sigma + \int_{\partial\Omega^d} (g^d - P[F|_{\mathrm{cl}\Omega_1}] \circ \phi^d) \cdot \mathcal{T}_d^{(i)}[\mathbf{b}]\tilde{\sigma}[\phi^d]\, d\sigma \Big) \cdot \Big(\sum_{k=1}^n \int_{\partial\Omega^h} \Big[T\big(b, D\Gamma^{(k)}\big(b, \phi^d(x) - \omega + \epsilon\phi^h(y)\big)\big) \cdot \nu_{\phi^d} \circ \phi^d(x) \Big] \Big(\mathcal{T}_h^{(j)}[\mathbf{b}]\Big)_k (y)\, \tilde{\sigma}[\phi^h](y)\, d\sigma_y + \Big[T\Big(b, v_{\phi^d(\partial\Omega^d)}[b, \mathcal{T}_d^{(j)}[\mathbf{b}] \circ (\phi^d)^{(-1)}]\Big) \nu_{\phi^d} \Big]^+ \circ \phi^d(x) \Big),$$

for all $x \in \partial\Omega^d$ and for all $\mathbf{f} \in \mathcal{U}$ with $\epsilon > 0$ (cf. equation 2.86.) For $\epsilon \to 0^+$, the last factor in parentheses converges to

$$\sum_{k=1}^n \Big[T\big(b, D\Gamma^{(k)}\big(b, \phi^d(x) - \omega\big)\big)\nu_{\phi^d} \circ \phi^d(x) \Big] \int_{\partial\Omega^d} \Big(\mathcal{T}_h^{(j)}[\mathbf{b}]\Big)_k \tilde{\sigma}[\phi^h]\, d\sigma + \Big[T\Big(b, Dv_{\phi^d(\partial\Omega^d)}[b, \mathcal{T}_d^{(j)}[\mathbf{b}] \circ (\phi^d)^{(-1)}]\Big) \nu_{\phi^d} \Big]^+ \circ \phi^d(x),$$

which is equal to

$$|\phi^h(\partial\Omega^h)|^{1/2}\sum_{k=1}^n \delta_{jk} T\big(b, D\Gamma^{(k)}\big(b,\phi^d(x)-\omega\big)\big)\nu_{\phi^d}\circ\phi^d(x)$$
$$+\mathbf{H}[b,\phi^d,\mathcal{T}_d^{(j)}[\mathbf{b}]](x).$$

We recall that, for $n=2$ and $\epsilon=0$, $\mathcal{T}_d^{(3)}[\mathbf{b}]=0$ (see proof of Lemma 2.52.) Thus, for $n=2$ and $\epsilon=0$, $H_{i3}^{(4)}[\mathbf{f}]=0$ for all $i=1,\dots,3$.

We now summarize what we have seen for the limit value as $\epsilon\to 0^+$ of the terms on the right hand side of (2.140). By exploiting Lemma 2.52, we find that the limit in (2.139) converges to

$$\begin{aligned}
&\int_{\mathbb{I}[\phi^d]} \mathbf{E}[b](u_0^d[\mathbf{f}],u_0^d[\mathbf{f}])\;d\xi+\delta_{2,n}\int_{\mathbb{E}[\phi^h]}\mathbf{E}[b](u^h[b,\phi^h,g^h],u^h[b,\phi^h,g^h])\;d\xi\\
&+\int_{\mathbb{I}[\phi^d]}\mathbf{E}[b](P[F|_{\mathrm{cl}\Omega_1}],P[F|_{\mathrm{cl}\Omega_1}])\;d\xi\\
&-2\delta_{2,n}\int_{\partial\Omega^h}\big(P[F|_{\mathrm{cl}\Omega_1}](\omega)\big)\cdot\Big(T(b,Du^h[b,\phi^h,g^h])\nu_{\phi^h}\Big)\circ\phi^h\;\tilde\sigma[\phi^h]\;d\sigma\\
&-2\delta_{2,n}c[\mathbf{f}]\lambda[\mathbf{b}']^{-1}\int_{\partial\Omega^h}\big(P[F|_{\mathrm{cl}\Omega_1}](\omega)\big)\cdot\mathcal{T}_h^{(3)}[\mathbf{b}]\circ\phi^h\;\tilde\sigma[\phi^h]\;d\sigma\\
&+2\delta_{2,n}\left(\int_{\partial\Omega^h} g^h\cdot\mathcal{T}_h^{(3)}[\mathbf{b}]\,\tilde\sigma[\phi^h]\;d\sigma\right)\big(P[F|_{\mathrm{cl}\Omega_1}](\omega)\big)\\
&\qquad\qquad\cdot\int_{\partial\Omega^h}\Big[T(b,Dv_{\phi^h(\partial\Omega^h)}[b,\widetilde\alpha[b,\phi^h]])\nu_{\phi^h}\Big]^-\circ\phi^h\,\tilde\sigma[\phi^h]\;d\sigma\\
&+2\int_{\partial\Omega^h}\Big(P[F|_{\mathrm{cl}\Omega_1}]\circ\phi^d\Big)\cdot\Big(T(b,Du_0^d[\mathbf{f}])\nu_{\phi^d}\Big)\circ\phi^d\;\tilde\sigma[\phi^d]\;d\sigma\\
&-2\delta_{2,n}\lambda[\mathbf{b}']^{-1}\int_{\partial\Omega^h}\Big(P[F|_{\mathrm{cl}\Omega_1}]\circ\phi^d\Big)\cdot H_{33}^{(4)}[\mathbf{f}]\;\tilde\sigma[\phi^d]\;d\sigma,
\end{aligned}\tag{2.145}$$

where $c[\mathbf{f}]$ is a real constant and $\lambda[\mathbf{b}']$ is defined as in Lemma 2.52. To conclude the proof we show that third, fourth, fifth and sixth term in (2.145) vanish. In fact, the third term vanishes because, by Theorem 2.3,

$$\begin{aligned}
&\int_{\partial\Omega^h}\big(P[F|_{\mathrm{cl}\Omega_1}](\omega)\big)\cdot\Big(T(b,Du^h[b,\phi^h,g^h])\nu_{\phi^h}\Big)\circ\phi^h\;\tilde\sigma[\phi^h]\;d\sigma\\
&\qquad=\int_{\phi^h(\partial\Omega^h)}\big(P[F|_{\mathrm{cl}\Omega_1}](\omega)\big)\cdot T(b,Du^h[b,\phi^h,g^h])\nu_{\phi^h}d\sigma\\
&\qquad=\int_{\mathbb{E}[\phi^h]}\mathbf{E}[b](P[F|_{\mathrm{cl}\Omega_1}](\omega),u^h[b,\phi^h,g^h])\;dt=0,
\end{aligned}$$

and the fourth term vanishes because, for $n=2$ and $\epsilon=0$, $\mathcal{T}_h^{(3)}[\mathbf{b}]=0$

(cf. proof of Lemma 2.52), and the fifth term vanishes because, by Theorem 2.13,

$$\int_{\partial\Omega^h} \Big[T(b, Dv_{\phi^h(\partial\Omega^h)}[b, \widetilde{\alpha}[b, \phi^h]])\nu_{\phi^h}\Big]^- \circ \phi^h \, \tilde{\sigma}[\phi^h] \, d\sigma$$
$$= \int_{\phi^h(\partial\Omega^h)} \mathbf{K}^*_{\phi^h(\partial\Omega^h)}[b, \widetilde{\alpha}[b, \phi^h]] \, d\sigma = \int_{\phi^h(\partial\Omega^h)} \widetilde{\alpha}[b, \phi^h] \, d\sigma = 0,$$

(cf. proof of Theorem 2.74) and finally the sixth term vanishes because, for $n = 2$ and $\epsilon = 0$, $H^{(4)}_{33}[\mathbf{f}] = 0$. Then, by Theorem 2.3, we deduce the validity of formula (2.139). □

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