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CICLO XXVII

Shape sensitivity analysis of the eigenvalues of polyharmonic operators and elliptic systems

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A mia nonna Elide

Riassunto

In questa tesi, studiamo la dipendenza degli autovalori di operatori differenziali ellittici da perturbazioni del dominio nello spazio N-dimensionale. In particolare, proviamo risultati di analiticità degli autovalori di operatori poliarmonici e sistemi ellittici di equazioni alle derivate parziali del secondo ordine, e li applichiamo a problemi di ottimizzazione di forma; d'altro canto, otteniamo anche stime di stabilità spettrale per sistemi ellittici generali di equazioni alle derivate parziali di ordine superiore. Per dimostrare l'analiticità usiamo una tecnica generale sviluppata da Lamberti e Lanza de Cristoforis, e otteniamo delle formule alla Hadamard che ci permettono di fornire una caratterizzazione dei domini critici sotto il vincolo di volume. Per quanto riguarda le stime di stabilità degli autovalori, dimostriamo risultati di lipschitzianità rispetto alla distanza d'atlante, alla distanza di Hausdorff e alla misura di Lebesgue, adattando gli argomenti utilizzati da Burenkov e Lamberti per operatori ellittici al caso di sistemi ellittici generali di equazioni alle derivate parziali.

La tesi è organizzata come segue. Il Capitolo 1 è dedicato ad alcuni preliminari. Nel Capitolo 2 consideriamo l'operatore biarmonico con diverse condizioni al contorno, ovvero di Dirichlet, di Neumann, intermedie e di Steklov. Per tutti questi casi mostriamo la dipendenza analitica degli autovalori dal dominio e calcoliamo formule alla Hadamard, che vengono usate per formire una caratterizzazione dei domini critici per le funzioni elementari simmetriche degli autovalori sotto il vincolo di volume; a seguire proviamo che le palle sono domini critici per tali funzioni degli autovalori di tutti questi problemi sotto il vincolo di volume. Riguardo al problema di Steklov, mostriamo anche che la palla è un massimizzatore del tono fondamentale tra tutti gli aperti limitati di misura fissata. Nel Capitolo 3 consideriamo il problema agli autovalori con condizioni di Dirichlet per gli operatori poliarmonici. Come nel Capitolo 2, dimostriamo l'analiticità delle funzioni elementari simmetriche degli autovalori fornendo formule alla Hadamard, e diamo una caratterizzazione dei domini critici sotto il vincolo di volume; a

seguire mostriamo che per tutti gli operatori poliarmonici la palla è un dominio critico. Il Capitolo 4 è dedicato alle stime di stabilità degli autovalori dei sistemi ellittici di equazioni alle derivate parziali con condizioni al bordo di Dirichlet e di Neumann. Adattando gli argomenti usati da Burenkov e Lamberti per operatori ellittici siamo in grado di provare stime con la distanza d'atlante, con la deviazione inferiore di Hausdorff-Pompeiu e con la misura di Lebesgue. Nel Capitolo 5 dimostriamo analiticità, formule alla Hadamard e condizioni di criticità per sistemi ellittici del secondo ordine con condizioni al bordo di Dirichlet e di Neumann. Mostriamo anche che, se il sistema è invariante per rotazioni, allora le palle sono domini critici sotto il vincolo di volume. Infine, nel Capitolo 6 consideriamo il problema di Reissner-Mindlin per la vibrazione di una piastra incastrata. Prima dimostriamo stime simili a quelle del Capitolo 4, che non dipendono dallo spessore della piastra; poi dimostriamo l'analiticità e formule alla Hadamard per le funzioni elementari simmetriche degli autovalori, che vengono usate per fornire una caratterizzazione di criticità; a seguire, dopo aver provato che il sistema di Reissner-Mindlin è invariante per rotazioni, mostriamo che le palle sono domini critici sotto il vincolo di volume.

Abstract

In this thesis, we study the dependence of the eigenvalues of elliptic partial differential operators upon domain perturbations in the N-dimensional space. Namely, we prove analyticity results for the eigenvalues of polyharmonic operators and elliptic systems of second order partial differential equations, and we apply them to certain shape optimization problems. On the other hand, we also prove spectral stability estimates for general elliptic systems of partial differential equations of higher order. In order to prove analyticity, we use a general technique developed by Lamberti and Lanza de Cristoforis, and we obtain Hadamard-type formulas which are used to provide a characterization of critical domains under volume constraint. As for stability estimates of the eigenvalues, we prove indeed Lipschitz continuity results with respect to the atlas distance, the Hausdorff distance and the Lebesgue measure. We adapt the arguments used by Burenkov and Lamberti for elliptic operators to the case of general elliptic systems of partial differential equations.

The thesis is organized as follows. Chapter 1 is dedicated to some preliminaries. In Chapter 2 we consider the biharmonic operator under different boundary conditions, namely Dirichlet, Neumann, intermediate and Steklov. For all these cases we show analytic dependence of the eigenvalues upon the domain and compute Hadamard-type formulas, which will be used to provide a characterization of critical domains for the elementary symmetric functions of the eigenvalues under volume constraint. Then we prove that balls are critical domains for such functions of the eigenvalues of all these problems under volume constraint. Regarding the Steklov problem, we also prove that the ball is a maximizer of the fundamental tone among all bounded open sets of given measure. In Chapter 3 we consider the Dirichlet eigenvalue problem for general polyharmonic operators. As in Chapter 2, we prove analyticity of the elementary symmetric functions of the eigenvalues providing Hadamard-type formulas, and we give a characterization of critical domains under volume constraint. Then we show that for all the polyharmonic operators the ball is a critical domain. Chapter 4 is devoted to the stability estimates of the eigenvalues of elliptic systems of partial differential equations with Dirichlet and Neumann boundary conditions. Adapting the arguments used by Burenkov and Lamberti for elliptic operators, we can prove estimates via the atlas distance, the lower Hausdorff-Pompeiu deviation and the Lebesgue measure. In Chapter 5 we prove analyticity, Hadamard-type formulas and criticality conditions for second order elliptic systems under Dirichlet and Neumann boundary conditions. We also show that, if the system is rotation invariant, then balls are critical domains under volume constraint. Finally, in Chapter 6 we consider the Reissner-Mindlin problem for the vibration of a clamped plate. We first prove estimates similar to those of Chapter 4, which are independent of the thickness of the plate. Then we prove analyticity and Hadamard-type formulas for the elementary symmetric functions of the eigenvalues, which are used to provide a characterization of criticality. Then, after proving that the Reissner-Mindlin system is rotation invariant, we show that balls are critical domains under volume constraint.

Introduction

The study of polyharmonic operators started long ago. It was already known at the beginning of the nineteenth century that the study of the bending of a clamped plate leads to the following problem

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ represents the midplane of the plate, and f represents the applied load (see e.g., [74, §223] for historical information). Problem (1) clearly resembles the well known Poisson problem for the Laplace operator

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

which is instead related to the study of the deformation of a fixed membrane of shape $\Omega \subset \mathbb{R}^2$. The similarity between these two problems naturally leads to the study of more general equations involving the polyharmonic operators $(-\Delta)^n$, $n \in \mathbb{N}$ under different types of boundary conditions. After the innovative papers of Almansi [10, 11] and the book of Nicolesco [71], the interest for polyharmonic operators has developed, so that several papers and books on the subject appeared. Among the most relevant works of the last decades on polyharmonic operators, we mention the partial solution of the celebrated Rayleigh's conjecture for the clamped plate [12, 70], and a book [51] devoted to an extensive study of boundary value problems for such operators.

In this thesis we are mainly interested in eigenvalue problems, which in the case of the biharmonic operator subject to Dirichlet boundary conditions can be written as

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We also consider the eigenvalue problem for the biharmonic operator with other boundary conditions, such as Neumann boundary conditions

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = \frac{\partial \Delta u}{\partial \nu} + \mathrm{div}_{\partial\Omega}(\nu \cdot D^2 u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

and intermediate boundary conditions

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

We remark that problems (2), (3), and (4) are related to the study of the vibrations of an elastic plate which is clamped, free, and hinged, respectively.

As for higher order operators, we consider the following eigenvalue problem

$$\begin{cases} (-\Delta)^n u = \lambda (-\Delta)^m u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{n-1} u}{\partial \nu^{n-1}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

for any n, m non-negative integers with $n > m$. We observe that, for $n = 1, m = 0$, problem (5) gives the Helmholtz problem, i.e., the eigenvalue problem for the Dirichlet Laplacian, while for $n = 2, m = 0$ it gives problem (1). Moreover, for $n = 2, m = 1$, problem (5) gives the well known buckling problem for the plate.

We note that, in the case of a clamped plate, problem (2) arises within the so-called Kirchhoff-Love model. If we consider instead the Reissner-Mindlin model, we get the following system of partial differential equations

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla \mathrm{div}\beta - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \mathrm{div}\beta) = \lambda w, & \text{in } \Omega, \\ \beta = 0, \quad w = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^2$ represents the midplane of the plate, and t is the thickness. Here w is the transverse displacement of the midplane, $\beta = (\beta_1, \beta_2)$ the fiber rotation, λt^2 the vibration frequency, μ_1 and μ_2 are the Lamé constants and $k > 0$ is a correction factor. This problem has been studied as an alternative to problem (2) because it is of the second order, and therefore easier to treat numerically using finite element methods (see e.g., [20]). However, as pointed out for instance in [21], when the parameter t is very small the finite element method leads to poor results, and this is known as shear locking phenomenon.

We also remark that problems (2) and (6) are strictly related. In fact, as is proved in [47], we have that the eigenvalues $\lambda_{n,t}[\Omega]$ of (6) converge to the eigenvalues $\lambda_{n,0}[\Omega]$ of problem

$$\begin{cases} \frac{2\mu_1+\mu_2}{12}\Delta^2 w = \lambda w, & \text{in } \Omega, \\ w = \nabla w = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

as $t \to 0$.

Problem (6) motivates our interest to even more general eigenvalue problems for systems of partial differential equations of the type

$$\sum_{|\alpha|,|\beta|\leq l}\sum_{i=1}^{m}(-1)^{|\beta|}D^{\beta}(A_{\alpha\beta}^{ij}D^{\alpha}u_i) = \lambda u_j,\ j = 1,\dots,m,$$

subject to Dirichlet or Neumann boundary conditions.

In this thesis, we study the dependence of the eigenvalues of elliptic partial differential operators upon domain perturbations in the N-dimensional space, with special attention to the above mentioned eigenvalue problems. The study of domain perturbation problems for partial differential operators represents a vast area of investigation which provides a large variety of results. One of the fundamental problems concerns the study of the qualitative behavior of the eigenvalues when the domain is perturbed and the corresponding results give information such as continuity, differentiability and even analyticity. This problem is also closely related to shape optimization problems where typically one has to identify the shape of the domain which minimizes or maximizes certain functionals of the eigenvalues when the domain is subject to suitable constraints, such as volume or perimeter constraint. Another important problem concerns the quantitative analysis of the variation of the eigenvalues, aiming at estimates for the deviation of the eigenvalues expressed in terms of certain measures of vicinity of sets, such as the Hausdorff distance. We refer to the extensive monographs [13, 23, 55, 56, 59, 75, 76] and to the survey papers [40, 54] for an introduction to this subject. We also refer to the recent works [2, 3, 4, 5, 6, 42, 43, 63, 64, 68] for qualitative results on domain perturbation problems, and to [7, 14, 15, 16, 35, 36, 37, 38, 39, 41, 65, 66] for quantitative estimates.

In this thesis, we face both problems. Namely, we prove analyticity results for the dependence of the eigenvalues of polyharmonic operators and elliptic systems of second order partial differential equations, and we apply them to certain shape optimization problems. On the other hand, we also prove spectral stability estimates for general elliptic systems of partial differential equations of higher order.

In order to prove analyticity, we use the general technique developed by Lamberti and Lanza de Cristoforis in [64] for compact selfadjoint operators in Hilbert space. We remark that in general one cannot expect to prove analytic dependence of the eigenvalues themselves upon the domain, when the eigenvalues are not simple. This is due to well known bifurcation phenomena of eigenvalues splitting from a multiple eigenvalue. Note that this is not in contrast with either the continuity results of e.g., [38], or the celebrated Rellich-Nagy Theorem [75, Theorem 1] which deals with families of operators parametrized by one real variable. Hence, in order to avoid such a situation, in the case of multiple eigenvalues we consider the elementary symmetric functions of the eigenvalues. The use of these functions has the effect of bypassing the splitting phenomenon, in fact we can prove that they are real analytic. Then we compute the shape derivatives, getting Hadamard-type formulas for the elementary symmetric functions of the eigenvalues. Note that such formulas were already obtained by Lamberti and Lanza de Cristoforis for the Dirichlet Laplacian in [64], and for the Neumann Laplacian in [68].

Then we address the problem of isovolumetric perturbation. This is related to the problem of shape optimization for the eigenvalues under volume constraint, in the spirit of Rayleigh's conjecture. We recall that Lord Rayleigh in [74] formulated the conjecture that, among all open sets of finite fixed area, the ball is the minimizer of the fundamental tones of the fixed membrane and of the clamped plate, i.e.,

$$\lambda_1(B) \leq \lambda_1(\Omega), \tag{8}$$

where B is a ball having the same measure of Ω and λ_1 is the lowest positive eigenvalue of the Laplace operator and of the biharmonic operator respectively, with Dirichlet boundary conditions. Inequality (8) for the Dirichlet Laplacian in $\mathbb{R}^N$ with $N \geq 2$ has been proved by Faber [48] and Krahn [60] via Schwartz rearrangement techniques in the twenties of the last century, while the analogue for problem (2) has shown to be a much more difficult task. In fact, it has been proved only twenty years ago by Nadirashvili [70] and Ashbaugh and Benguria [12], and only up to three dimensions, the general case remaining an open problem (see also [78]).

Regarding the Laplace operator, there is a number of results available in the literature showing that the ball is an extremizer, i.e., either a minimizer (for Dirichlet and Robin boundary conditions) or a maximizer (for Neumann and Steklov boundary conditions) of the first positive eigenvalue (we refer to [55] for an extensive discussion of the subject). In particular, those results show that the ball is a critical domain under volume constraint for the fundamental tone. Using Lagrange Multipliers Theorem and our Hadamard-type formulas, we provide a characterization of critical domains, which is valid for any elementary symmetric function of the eigenvalues. Then we show that, for polyharmonic operators and other rotation invariant operators, the ball is a critical domain for all the symmetric functions of the eigenvalues under volume constraint. Note that our criticality result does not say whether the ball is an extremizer or not, since criticality is a more general property. However, if we consider the following Steklov-type problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - \mathrm{div}_{\partial\Omega}(\nu \cdot D^2 u)_{\partial\Omega} = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

we can actually prove that the ball is a maximizer of the first positive eigenvalue among all bounded open sets of given volume, for any constant $\tau > 0$. We do it by following the approach of [79] (see also [22]). Note that problem (9) arises in the study of the vibrations of a free plate whose mass is concentrated at the boundary, and therefore is a natural generalization to the biharmonic operator of the classical Steklov problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases}$$

which is related to the study of the vibrations of a free membrane whose mass is concentrated at the boundary (see [77] for the physical derivation of the problem). The results concerning problem (9) have been obtained in collaboration with Luigi Provenzano (see also [31, 32, 73]). Note that problem (9) should not be confused with another important Steklov-type problem already discussed in the literature, namely

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases} \tag{10}$$

which has a rather different nature. In fact, for the first postitive eigenvalue of problem (10) the minimization is the interesting problem (rather than maximization), and explicit examples show that, surprisingly, the ball is not a minimizer (see e.g., [24, 25] and the references therein).

As for stability estimates of the eigenvalues, we prove indeed Lipschitz continuity results with respect to the atlas distance, the Hausdorff distance and the Lebesgue measure. We adapt the arguments used by Burenkov and Lamberti for elliptic operators (see [38, 39]) to the case of general elliptic systems of partial differential equations. Then we consider the special case of the Reissner-Mindlin eigenvalue system (6). Note that, as the parameter t goes to zero, the coefficients of the problem diverge, possibly spoiling stability estimates for small values of t. This can be explained with the above mentioned shear locking phenomenon. However, we know that the eigenvalues of problem (6) converge to those of problem (7) as $t \to 0$, for which we already have stability estimates. Nevertheless, using a particular pull-back operator we can prove stability estimates for the eigenvalues of problem (6) which are indepentent of t.

The thesis is organized as follows. Chapter 1 is dedicated to some preliminaries. In Chapter 2 we consider the biharmonic operator under different boundary conditions, namely Dirichlet, Neumann, intermediate and Steklov. For all these cases we show analyticity results in the spirit of [64] and compute Hadamard-type formulas, which will be used to provide a characterization of critical domains for the elementary symmetric functions of the eigenvalues under volume constraint. Then we prove that balls are critical domains for such functions of the eigenvalues of all these problems under volume constraint. Regarding the Steklov problem (9), we also prove that the ball is a maximizer of the fundamental tone among all bounded open sets of given measure. In Chapter 3 we consider the Dirichlet eigenvalue problem for general polyharmonic operators. As in Chapter 2, we prove analyticity of the elementary symmetric functions of the eigenvalues providing Hadamard-type formulas, and we give a characterization of critical domains under volume constraint. Then we show that for all the polyharmonic operators the ball is a critical domain. Chapter 4 is devoted to the stability estimates of the eigenvalues of elliptic systems of partial differential equations with Dirichlet and Neumann boundary conditions. Adapting the arguments used in [38, 39] we can prove estimates via the atlas distance, the lower Hausdorff-Pompeiu deviation and the Lebesgue measure. In Chapter 5 we prove analyticity, Hadamard-type formulas and criticality conditions for second order elliptic systems under Dirichlet and Neumann boundary conditions. We also show that, if the system is rotation invariant, then balls are critical domains under volume constraint. Finally, in Chapter 6 we consider the Reissner-Mindlin problem for the vibration of a clamped plate. We first prove estimates similar to those of Chapter 4, which are independent of the thickness of the plate. Then we prove analyticity and Hadamard-type formulas for the elementary symmetric functions of the eigenvalues, which are used to provide a characterization of criticality. Then, after proving that the Reissner-Mindlin system is rotation invariant, we show that balls are critical domains under volume constraint.

Part of the results in this thesis have been published or accepted for publication. The discussion for the hinged plate problem in Chapter 2 has been partially published in [28]. The discussion in Chapter 3 has been published in [27]. The discussion in Chapter 6 has been published in [29]. A survey paper on Hadamard-type formulas and critical domains for the problems considered in this thesis has been accepted for publication in [30]. Moreover, the discussion in Chapter 5 has been submitted as the paper [26]. The discussion on the Neumann problem and on the Steklov problem contained in Chapter 2 is part of the submitted papers [31, 32].

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Contents

Chapter 1

Preliminaries

In this chapter we introduce some basic results which will be used in the sequel, and we set the notation.

1.1 The atlas class and the atlas distance

We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$. Let $N \in \mathbb{N}$. In the sequel, we shall always assume $N \geq 2$. For any set V in $\mathbb{R}^N$ and $\delta > 0$ we denote by V_δ the set $\{x \in V : \ d(x, \partial V) > \delta\}$. We shall also denote by V^δ the set $\{x \in \mathbb{R}^N : \ d(x, V) < \delta\}$. Here $d(x, A)$ denotes the Euclidean distance from x to a set A. We recall the following definition from [38], where by cuboid we mean a set which is the isometric image of a set of the form $\Pi_{i=1}^N]a_i, b_i[$.

Definition 1.1. *Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of isometries in $\mathbb{R}^N$. We say that that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.*

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ *and* $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ *for* $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ *for* $s' < j \leq s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{ x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij}, \, i = 1, ..., N\},$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}), \ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij}, \, i = 1, ..., N-1\}$ *and* g_j *is a continuous function defined on* $\overline{W}_j$ *(it is meant that if* $s' < j \leq s$ *then* $g_j(\bar{x}) = b_{Nj}$ *for all* $\bar{x} \in \overline{W}_j$*); moreover for* $j = 1, \dots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$.

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set with a continuous boundary if* Ω *is of class* $C(\mathcal{A})$ *for some atlas* $\mathcal{A}$.

Let $\omega : [0, \infty[\to [0, \infty[$ *be a modulus of continuity, i.e., a continuous non-decreasing function such that* $\omega(0) = 0$ *and, for some* $k > 0$, $\omega(t) \geq kt$ *for all* $0 \leq t \leq 1$. *Let* $M > 0$. *We denote by* $C_M^{\omega(\cdot)}(\mathcal{A})$ *the family of all open sets* Ω *in* $\mathbb{R}^N$ *belonging to* $C(\mathcal{A})$ *and such that all the functions* g_j *in (iii) satisfy the condition*

$$|g_j(\bar{x}) - g_j(\bar{y})| \leq M\omega(|x - \bar{y}|),$$

for all $\bar{x}, y \in \overline{W}_j$. *We also say that an open set is of class* $C^{\omega(\cdot)}$ *if there exists an atlas* $\mathcal{A}$ *and* $M > 0$ *such that* $\Omega \in C_M^{\omega(\cdot)}(\mathcal{A})$.

Let $l \in \mathbb{N}, M > 0$. *We say that an open set* Ω *is of class* $C_M^l(\mathcal{A})$, $C_M^{l,1}(\mathcal{A})$ *if* Ω *is of class* $C(\mathcal{A})$ *and all the functions* g_j *in (iii) are of class* $C^l(\overline{W}_j)$ *with*

$$|g_j|_{c^l(\overline{W}_j)} = \sum_{1 \leq |\alpha| \leq l} \|D^\alpha g_j\|_{L^\infty(\overline{W}_j)} \leq M,$$

$$|g_j|_{c^{l,1}(\overline{W}_j)} = |g_j|_{c^l(\overline{W}_j)} + \sum_{|\alpha| = l} \sup_{\substack{\bar{x}, \bar{y} \in \overline{W}_j \\ x \neq y}} \frac{|D^\alpha g_j(\bar{x}) - D^\alpha g_j(\bar{y})|}{|\bar{x} - \bar{y}|} \leq M$$

respectively[1].

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set of class* C^l, $C^{l,1}$ *if* Ω *is of class* $C_M^l(\mathcal{A})$, $C_M^{l,1}(\mathcal{A})$ *respectively, for some atlas* $\mathcal{A}$ *and some* $M > 0$.

The family of open sets of class $C(\mathcal{A})$ can be thought as a metric space endowed with the so-called Atlas Distance. We recall the definition introduced in [38].

[1]Note that as customary $\|g_j\|_{C^l(\overline{W}_j)} = \|g_j\|_{L^\infty(W_j)} + |g_j|_{c^l(\overline{W}_j)}$ and $\|g_j\|_{C^{l,1}(\overline{W}_j)} = \|g_j\|_{L^\infty(W_j)} + |g_j|_{c^{l,1}(\overline{W}_j)}$.

Definition 1.2. *Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas in $\mathbb{R}^N$. For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ we define the atlas distance $d_{\mathcal{A}}$ by*

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1,\dots,s} \sup_{(\bar{x}, x_N) \in r_j(V_j)} |g_{1j}(\bar{x}) - g_{2j}(\bar{x})|, \tag{1.1}$$

where g_{1j}, g_{2j} are the functions describing the boundaries of Ω_1, Ω_2 respectively, as in Definition 1.1 (iii).

Moreover, if $\Omega \in C(\mathcal{A})$ we set

$$d_j(x, \partial\Omega) = |g_j(\,\overline{(r_j(x))}\,) - (r_j(x))_N|, \tag{1.2}$$

for all $j = 1, \dots, s$ and $x \in V_j$, where g_j is as in Definition 1.1.

Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas in $\mathbb{R}^N$. For all $x \in V' = \cup_{j=1}^{s'} V_j$ we set $J'(x) = \{j \in \{1, \dots, s'\} : \ x \in V_j\}$. Let $\Omega \in C(\mathcal{A})$. Then we set

$$d_{\mathcal{A}}(x, \partial\Omega) = \max_{j \in J'(x)} d_j(x, \partial\Omega),$$

for all $x \in V'$, where $d_j(x, \partial\Omega)$ is defined in (1.2). Observe that if $\Omega \in C(\mathcal{A})$ then $\partial\Omega \subset V'$. Therefore if $\Omega_1, \Omega_2 \in C(\mathcal{A})$ then

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) = \sup_{x \in \partial\Omega_1} d_{\mathcal{A}}(x, \partial\Omega_2) = \sup_{x \in \partial\Omega_2} d_{\mathcal{A}}(x, \partial\Omega_1).$$

For all $\epsilon > 0$ we set

$$\Omega_{\epsilon,\mathcal{A}} = \Omega \setminus \{x \in V' : \ d_{\mathcal{A}}(x, \partial\Omega) \le \epsilon\},$$

$$\Omega^{\epsilon,\mathcal{A}} = \Omega \cup \{x \in V' : \ d_{\mathcal{A}}(x, \partial\Omega) < \epsilon\}.$$

We recall the following lemma from [38].

Lemma 1.3. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and $\epsilon > 0$. If Ω_1 and Ω_2 are two open sets in $C(\mathcal{A})$ satisfying the inclusion*

$$(\Omega_1)_{\epsilon,\mathcal{A}} \subset \Omega_2 \subset (\Omega_1)^{\epsilon,\mathcal{A}}, \tag{1.3}$$

or

$$(\Omega_2)_{\epsilon,\mathcal{A}} \subset \Omega_1 \subset (\Omega_2)^{\epsilon,\mathcal{A}}, \tag{1.4}$$

then

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) \le \epsilon. \tag{1.5}$$

The atlas distance depends on the chosen atlas but has the advantage of being easily computable. Moreover, we observe that it can be controlled via the Hausdorf distance. Indeed, we have the following theorem where, for the sake of completeness, we collect also other relevant properties of the atlas distance proved in [38].

Given two sets A, B in $\mathbb{R}^N$ the lower Hausdorff-Pompeiu deviation of A from B is defined in [38] by

$$d_{\mathcal{HP}}(A,B) = \min\left\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right\}.$$

Note that the standard Hausdorff-Pompeiu distance of A and B is

$$d^{\mathcal{HP}}(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right\}.$$

Theorem 1.4. *Let $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ be an atlas, ω a modulus of continuity as in Definition 1.1 and $M > 0$. Let $\tilde{\mathcal{A}} = (\rho/2,\ s, s', \{(V_j)_{\rho/2}\}_{j=1}^s,\ \{r_j\}_{j=1}^s)$. Then the following statements hold:*

(i) $(C(\mathcal{A}), d_{\mathcal{A}})$ *is a complete metric space;*

(ii) $C_M^{\omega(\cdot)}(\mathcal{A})$ *is a compact subset of* $C(\mathcal{A})$*;*

(iii) There exists $c > 0$ depending only on $N, \mathcal{A}, \omega, M$ such that

$$d^{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) \le d_{\tilde{\mathcal{A}}}(\Omega_1,\Omega_2) \le c\,\omega(d_{\mathcal{HP}}(\partial\Omega_1,\partial\Omega_2)), \tag{1.6}$$

for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$.

We also recall the following lemma from [38].

Lemma 1.5. *If Ω_1 and Ω_2 are two open sets satisfying the inclusions*

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon \tag{1.7}$$

or

$$(\Omega_2)_\epsilon \subset \Omega_1 \subset (\Omega_2)^\epsilon, \tag{1.8}$$

then

$$d_{\mathcal{HP}}(\partial\Omega_2, \partial\Omega_1) \le \epsilon. \tag{1.9}$$

Observe that if Ω_1 and Ω_2 are two open sets satisfying inclusion (1.7) then it may happen that they do not satisfy inclusion (1.8), and

$$\sup_{x\in\partial\Omega_1} d(x,\partial\Omega_2) > \epsilon.$$

We refer to [38, Appendix] for some counterexamples.

1.2 Sobolev spaces and elliptic operators

Let $N, l \in \mathbb{N}$, $1 \le p \le \infty$ and Ω be an open set in $\mathbb{R}^N$. We denote by $W^{l,p}(\Omega)$ the Sobolev space of real-valued functions in $L^p(\Omega)$, which have all distributional derivatives up to order l in $L^p(\Omega)$, endowed with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \sum_{|\alpha| \le l} \|D^\alpha u\|_{L^p(\Omega)}. \tag{1.10}$$

We denote by $W_0^{l,p}(\Omega)$ the closure in $W^{l,p}(\Omega)$ of the space of the C^∞-functions with compact support in Ω. We shall also use the notation $H^l(\Omega)$, $H_0^l(\Omega)$ for the spaces $W^{l,2}(\Omega)$, $W_0^{l,2}(\Omega)$ respectively.

Let $m \in \mathbb{N}$. We endow the product space $W^{l,p}(\Omega)^m$ with the norm

$$\|u\|_{W^{l,p}(\Omega)^m} = \sum_{j=1}^m \|u_j\|_{W^{l,p}(\Omega)}, \tag{1.11}$$

where by u_j we mean the j-th component of the vector-valued function u.

We have the following result.

Lemma 1.6. *Let Ω be an open set in $\mathbb{R}^N$, $1 \le p \le \infty$. Let $V(\Omega)$ be a subspace of $W^{l,p}(\Omega)$ such that the embedding $V(\Omega) \subset W^{l-1,p}(\Omega)$ is compact. Then there exists $c > 0$ such that*

$$\|u\|_{W^{l,p}(\Omega)} \le c\Bigg(\|u\|_{L^p(\Omega)} + \sum_{|\alpha| = l} \|D^\alpha u\|_{L^p(\Omega)} \Bigg), \tag{1.12}$$

for all $u \in V(\Omega)$.

Proof. The case $p = 2$ can be found in [38, Lemma 2.2]. The general case can be treated in the same way. □

We have the following Gagliardo-Nirenberg-type inequality for intermediate derivatives of vector-valued functions (see also [18, 33]).

Theorem 1.7. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class C^1. Let $1 \le p_1, p_2 \le \infty$, $l, r \in \mathbb{N}_0$ with $0 \le r < l$ and let $\frac{r}{l} \le \theta \le 1$. Suppose that if $1 < p_2 < \infty$ and $\theta = 1$, then $l - r - \frac{N}{p_2}$ is not a non-negative integer. Let $1 \le p \le \infty$ be such that*

$$\frac{N}{p} - r = (1-\theta)\frac{N}{p_1} + \theta\left(\frac{N}{p_2} - l\right).$$

Let $m \in \mathbb{N}$. Then, for any $u \in L^{p_1}(\Omega)^m \cup W^{l,p_2}(\Omega)^m$, there exists $C = C(N, p_1, p_2, l, r, \theta)$ such that

$$\sum_{|\alpha|=r} ||D^\alpha u||_{L^p(\Omega)^m} \leq C m^2 ||u||^{1-\theta}_{L^{p_1}(\Omega)^m} ||u||^{\theta}_{W^{l,p_2}(\Omega)^m}.$$

Proof. For the case $m = 1$ we refer to [50, 72]. As for the general case, we have

$$\sum_{|\alpha|=r} ||D^\alpha u_j||_{L^p(\Omega)} \leq C ||u_j||^{1-\theta}_{L^{p_1}(\Omega)} ||u_j||^{\theta}_{W^{l,p_2}(\Omega)},$$

for all $j = 1, \dots, m$, where $C > 0$ is independent of $u = (u_1, \dots, u_m)$. Using the fact that, if $a_1, \dots, a_m \geq 0$ and $t > 0$, then

$$\sum_{j=1}^m a_j^t \leq m \left(\sum_{j=1}^m a_j \right)^t,$$

summing on j we get

$$\begin{aligned}
\sum_{|\alpha|=r} ||D^\alpha u||_{L^p(\Omega)^m} &\leq C \sum_{j=1}^m \left(||u_j||^{1-\theta}_{L^{p_1}(\Omega)} ||u_j||^{\theta}_{W^{l,p_2}(\Omega)} \right) \\
&\leq C \left(\sum_{j=1}^m ||u_j||^{1-\theta}_{L^{p_1}(\Omega)} \right) \left(\sum_{j=1}^m ||u_j||^{\theta}_{W^{l,p_2}(\Omega)} \right) \\
&\leq C m^2 ||u||^{1-\theta}_{L^{p_1}(\Omega)^m} ||u||^{\theta}_{W^{l,p_2}(\Omega)^m}.
\end{aligned}$$

□

Let $V(\Omega)$ be a closed subspace of $W^{l,2}(\Omega)$ containing $W^{l,2}_0(\Omega)$, and $m \in \mathbb{N}$. We consider the following eigenvalue problem

$$\int_\Omega \sum_{|\alpha|,|\beta| \leq l} \sum_{i,j=1}^m A^{ij}_{\alpha\beta} D^\alpha u_i D^\beta v_j dx = \lambda \int_\Omega u \cdot v dx, \tag{1.13}$$

for all functions $v \in V(\Omega)^m$, in the unknowns $u \in V(\Omega)^m$ (the eigenfunctions) and $\lambda \in \mathbb{R}$ (the eigenvalues), where for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, the coefficients $A^{ij}_{\alpha\beta}$ are bounded measurable real-valued functions defined on Ω. Here and in the sequel we shall denote by $u \cdot \phi$ the standard scalar product in $\mathbb{R}^m$. We set

$$Q_\Omega(u, v) = \int_\Omega \sum_{|\alpha|,|\beta| \leq l} \sum_{i,j=1}^m A^{ij}_{\alpha\beta} D^\alpha u_i D^\beta v_j dx,$$

and $Q_\Omega(u) = Q_\Omega(u,u)$, for all $u, v \in W^{l,2}(\Omega)^m$.

We make the following assumptions on the coefficients.

- Symmetry: for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \le l$ and for all $i, j \in \mathbb{N}$ with $i, j \le m$,
$$A^{ij}_{\alpha\beta} = A^{ji}_{\beta\alpha}. \tag{1.14}$$

- Positivity: for any $u \in W^{l,1}_{\mathrm{loc}}(\Omega)^m$,
$$\sum_{|\alpha|,|\beta|\le l} \sum_{i,j=1}^{m} A^{ij}_{\alpha\beta} D^\alpha u_i D^\beta u_j \ge 0, \ \text{a.e. in } \Omega. \tag{1.15}$$

- Coercivity: there exist two constants $a, b > 0$ such that, for all $u \in W^{l,2}(\Omega)^m$,
$$a||u||^2_{W^{l,2}(\Omega)} \le Q_\Omega(u) + b||u||^2_{L^2(\Omega)}. \tag{1.16}$$

Remark 1.8. *We note that conditions (1.14)-(1.16) are not very restrictive. For instance, the biharmonic operator considered in Chapter 2 and the polyharmonic operators $\mathcal{P}_{n0}$ considered in Chapter 3 satisfy conditions (1.14)-(1.16). Also the Lamé system $-\Delta - k\nabla\mathrm{div}$ (under Dirichlet or Neumann boundary conditions) for any $k \ge 1 - \frac{2}{N}$ and the Reissner-Mindlin problem (6.1) are in this class of operators.*

If the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, then the eigenvalues of equation (1.13) coincide with the eigenvalues of a suitable operator $H_{V(\Omega)}$ canonically associated with the restriction of the quadratic form Q_Ω to $V(\Omega)$. In fact, we have the following theorem.

Theorem 1.9. *Let Ω be an open set in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \le l$ and for all $i, j \in \mathbb{N}_0$ with $i, j \le m$, let $A^{ij}_{\alpha\beta}$ be bounded measurable real-valued functions defined on Ω, satisfying conditions (1.14)-(1.16).*

Let $V(\Omega)$ be a closed subspace of $W^{l,2}(\Omega)$ containing $W^{l,2}_0(\Omega)$ and such that the embedding $V(\Omega) \subset L^2(\Omega)$ is compact.

Then there exists a non-negative selfadjoint linear operator $H_{V(\Omega)}$ on $L^2(\Omega)^m$ with compact resolvent, such that $\mathrm{Dom}(H^{1/2}_{V(\Omega)}) = V(\Omega)^m$ and

$$< H^{1/2}_{V(\Omega)} u, H^{1/2}_{V(\Omega)} v >_{L^2(\Omega)^m} = Q_\Omega(u,v),$$

for all $u, v \in V(\Omega)^m$. Moreover, the eigenvalues of equation (1.13) coincide with the eigenvalues $\lambda_n[H_{V(\Omega)}]$ of $H_{V(\Omega)}$ and

$$\lambda_n[H_{V(\Omega)}] = \inf_{\substack{\mathcal{L} \leq V(\Omega)^m \\ \dim \mathcal{L} = n}} \sup_{\substack{u \in \mathcal{L} \\ u \neq 0}} \frac{Q_\Omega(u)}{\|u\|^2_{L^2(\Omega)^m}}. \tag{1.17}$$

Proof. The proof is similar to that of [38, Theorem 2.8], and is based on the variational characterization of the spectrum (see e.g., [44, Chapter 4]). □

Definition 1.10. *Let Ω be an open set in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta}$ be bounded measurable real-valued functions defined on Ω, satisfying conditions (1.14)-(1.16).*

If the embedding $W^{l,2}_0(\Omega) \subset L^2(\Omega)$ is compact, we set

$$\lambda_{n,\mathcal{D}}[\Omega] = \lambda_n[H_{W^{l,2}_0(\Omega)}].$$

If the embedding $W^{l,2}(\Omega) \subset L^2(\Omega)$ is compact, we set

$$\lambda_{n,\mathcal{N}}[\Omega] = \lambda_n[H_{W^{l,2}(\Omega)}].$$

The numbers $\lambda_{n,\mathcal{D}}[\Omega]$, $\lambda_{n,\mathcal{N}}[\Omega]$ are called the Dirichlet eigenvalues, Neumann eigenvalues respectively, of operator (3.1).

When we refer to both Dirichlet and Neumann boundary conditions we write just $\lambda_n[\Omega]$ instead of $\lambda_{n,\mathcal{D}}[\Omega]$ and $\lambda_{n,\mathcal{N}}[\Omega]$.

We note that, for an open set Ω of class $C(\mathcal{A})$ (see Definition 1.1), inequality (1.12) holds for all $u \in W^{l,2}(\Omega)$ with a constant c depending only on $\mathcal{A}$. More precisely, we denote by $\mathcal{D}_\Omega$ the best constant for which inequality (1.12) is satisfied for $V(\Omega) = W^{l,2}_0(\Omega)$. We denote by $\mathcal{N}_\Omega$ the best constant for which inequality (1.12) is satisfied for $V(\Omega) = W^{l,2}(\Omega)$. Then we have the following (for a proof we refer to [33, Theorem 6, p. 160]).

Lemma 1.11. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$, $m \in \mathbb{N}$. There exists $c > 0$ depending only on $N, \mathcal{A}$ and m such that*

$$1 \leq \mathcal{D}_\Omega \leq \mathcal{N}_\Omega \leq c,$$

for all open sets $\Omega \in C(\mathcal{A})$.

Lemma 1.12. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $L, a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta} \in L^\infty(\cup_{h=1}^s V_h)$ satisfy $\|A^{ij}_{\alpha\beta}\|_{L^\infty(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C(\mathcal{A})$.*

Then for each $n \in \mathbb{N}$ there exists $\Lambda_n > 0$ depending only on $n, N, \mathcal{A}, l, m$ and L such that

$$\lambda_{n,\mathcal{N}}[\Omega] \leq \lambda_{n,\mathcal{D}}[\Omega] \leq \Lambda_n,$$

for all open sets $\Omega \in C(\mathcal{A})$.

Proof. The proof is similar to that of [38, Lemma 3.4]. □

1.3 Domain perturbations and pull-back of operators

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problems of the type of (1.13) in a family of open sets parameterized by suitable diffeomorphisms ϕ defined on Ω. Namely, we set

$$\mathcal{A}^l_\Omega = \left\{ \phi \in C^l_b(\Omega\,; \mathbb{R}^N) : \inf_{\substack{x_1, x_2 \in \Omega \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} > 0 \right\},$$

where $C^l_b(\Omega\,; \mathbb{R}^N)$ denotes the space of all functions from Ω to $\mathbb{R}^N$ of class C^l, with bounded derivatives up to order l. Note that if $\phi \in \mathcal{A}^l_\Omega$ then ϕ is injective, Lipschitz continuous and $\inf_\Omega |\mathrm{det}\nabla\phi| > 0$. Moreover, $\phi(\Omega)$ is a bounded open set and the inverse map $\phi^{(-1)}$ belongs to $\mathcal{A}^l_{\phi(\Omega)}$.

Let $V(\Omega)$ be a closed subspace of $W^{l,2}(\Omega)$ containing $W^{l,2}_0(\Omega)$. We observe that, if the embedding $\mathcal{E}_\Omega : V(\Omega) \subset L^2(\Omega)$ is compact, then also the embedding $\mathcal{E}_{\phi(\Omega)} : V(\phi(\Omega)) \subset L^2(\phi(\Omega))$ is compact, for any $\phi \in \mathcal{A}^l_\Omega$. Here and in the sequel, we denote by $V(\phi(\Omega))$ the space of all functions u such that $u \circ \phi \in V(\Omega)$. In fact, the map i_ϕ from $W^{l,2}(\phi(\Omega))$ to $W^{l,2}(\Omega)$ which takes $u \in W^{l,2}(\phi(\Omega))$ to $u \circ \phi$ is a linear homeomorphism, and $i_\phi^{-1} = i_{\phi^{-1}}$. Therefore

$$\mathcal{E}_{\phi(\Omega)} = i_\phi \circ \mathcal{E}_\Omega \circ i_{\phi^{-1}},$$

hence $\mathcal{E}_{\phi(\Omega)}$ is compact.

Thanks to these observations, it is natural to consider problem (1.13) on $\phi(\Omega)$ and study the dependence of $\lambda_n[\phi(\Omega)]$ on $\phi \in \mathcal{A}^l_\Omega$. We shall endow the space $C^l_b(\Omega\,; \mathbb{R}^N)$ with its usual norm $\|f\|_{C^l_b(\Omega\,; \mathbb{R}^N)} = \sup_{|\alpha| \leq l,\ x \in \Omega} |D^\alpha f(x)|$. Note that $\mathcal{A}^l_\Omega$ is an open set in $C^l_b(\Omega\,; \mathbb{R}^N)$, see [64, Lemma 3.11]. Thus, it makes sense to study differentiability and analyticity properties of the maps $\phi \mapsto \lambda_n[\phi(\Omega)]$ defined for $\phi \in \mathcal{A}^l_\Omega$. For simplicity, we write $\lambda_n[\phi]$ instead of $\lambda_n[\phi(\Omega)]$. To do so, we shall consider problem (1.13) on $\phi(\Omega)$ and pull it back to Ω.

Let $V(\Omega)$ be a closed subspace of $W^{l,2}(\Omega)$ containing $W^{l,2}_0(\Omega)$, $m \in \mathbb{N}$. Let T be an operator from $V(\phi(\Omega))^m$ to its dual, for any $\phi \in \mathcal{A}^l_\Omega$. We recall that the pull-back of T is defined by

$$T_\phi[u^{(1)}][u^{(2)}] = T[u^{(1)} \circ \phi^{-1}][u^{(2)} \circ \phi^{-1}], \ \forall u^{(1)}, u^{(2)} \in V(\Omega)^m. \tag{1.18}$$

Using the pull-back of operators as defined by (1.18), we will be able to study differentiability and analyticity properties of the eigenvalues.

Since bifurcation phenomena may occur when dealing with multiple eigenvalues, we shall consider the elementary symmentric functions of the eigenvalues of (1.13) (or other operators). To do so, as in [64], we fix a finite set of indexes $F \subset \mathbb{N}$ and we consider those maps $\phi \in \mathcal{A}^l_\Omega$ for which the eigenvalues with index in F do not coincide with eigenvalues with index not in F; namely, we set

$$\mathcal{A}_{F,\Omega} = \left\{ \phi \in \mathcal{A}^l_\Omega : \ \lambda_n[\phi] \neq \lambda_k[\phi], \ \forall\, n \in F, \, k \in \mathbb{N} \setminus F \right\}.$$

It is also convenient to consider those maps $\phi \in \mathcal{A}_{F,\Omega}$ such that all the eigenvalues with index in F coincide and set

$$\Theta_{F,\Omega} = \left\{ \phi \in \mathcal{A}_{F,\Omega} : \ \lambda_{n_1}[\phi] = \lambda_{n_2}[\phi], \ \forall\, n_1, n_2 \in F \right\}.$$

For $\phi \in \mathcal{A}_{F,\Omega}$, the elementary symmetric functions of the eigenvalues with index in F are defined by

$$\Lambda_{F,s}[\phi] = \sum_{\substack{n_1,\dots,n_s \in F \\ n_1 < \cdots < n_s}} \lambda_{n_1}[\phi] \cdots \lambda_{n_s}[\phi], \quad s = 1, \dots, |F|.$$

When the meaning will be clear from the context, we shall use the notation $\mathcal{A}_{F,\Omega}$, $\Theta_{F,\Omega}$, $\Lambda_{F,h}$ for all the problems we consider in the sequel without any additional specification.

Chapter 2

Biharmonic operator and plate problems

In this chapter we discuss the eigenvalue problem for the biharmonic operator Δ^2 subject to different types of boundary conditions. This operator is related to the study of the bending of a plate via the Kirchhoff-Love model; we refer to [51, 74] for the physical derivation of the problem (see also [42]). In particular, the problem of a vibrating plate leads to the equation

$$\Delta^2 u - \tau \Delta u = \lambda u, \tag{2.1}$$

on a bounded open set Ω in $\mathbb{R}^2$. Here τ is a non-negative constant related to the lateral tension of the plate.

Since the dimension does not play any relevant role in our discussion, we consider from the beginning bounded open sets in $\mathbb{R}^N$. The weak formulation of problem (2.1) is

$$\int_\Omega (1-\alpha) D^2 u : D^2 \varphi + \alpha \Delta u \Delta \varphi + \tau \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega u \varphi dx, \tag{2.2}$$

for any $\varphi \in C_c^\infty(\Omega)$, where α is a parameter depending on the material, typically $0 \leq \alpha < 1$. However, for some particular material, the parameter α happens to be negative (cf. [51, §1.1.2]). We note that, thanks to the inequality

$$|D^2 u|^2 \geq \frac{1}{N} (\Delta u)^2, \ \forall u \in H^2(\Omega),$$

the quadratic form associated with problem (2.2) turns out to be positive for $-\frac{1}{N-1} < \alpha < 1$.

As we have said, we shall consider equation (2.1) subject to different boundary conditions. These conditions obviously depend on the choice of the energy space $V(\Omega)$ in which we study the problem (2.2). If we choose $V(\Omega) = H^2_0(\Omega)$ as the energy space, we have Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

which are related to a clamped plate. Note that in this case the problem does not depend on the parameter α. The choice of $V(\Omega) = H^2(\Omega)$ as the energy space leads to Neumann boundary conditions

$$(1-\alpha)\frac{\partial^2 u}{\partial \nu^2} + \alpha\Delta u = \tau\frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1-\alpha)\mathrm{div}_{\partial\Omega}(\nu \cdot D^2 u)_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

which are related to a free plate. Here $\mathrm{div}_{\partial\Omega}$ is the tangential divergence operator and, for any vector field f, $f_{\partial\Omega} = f - (f \cdot \nu)\nu$ is the tangential component of f.

Since the problem is of the fourth order, we also have the possibility of choosing the energy space $V(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$, which gives the so-called intermediate boundary conditions

$$u = (1-\alpha)\frac{\partial^2 u}{\partial \nu^2} + \alpha\Delta u = 0 \text{ on } \partial\Omega,$$

which are related to a hinged plate.

We remark that the operator P defined by the left-hand side of (2.2) (cf. (2.6)) satisfies conditions (1.14)-(1.16), and therefore Theorem 1.9 applies to all these problems.

We shall also consider the so-called Steklov boundary value problem for the biharmonic operator, namely

$$\int_\Omega (1-\alpha)D^2u : D^2\varphi + \alpha\Delta u\Delta\varphi + \tau\nabla u \cdot \nabla\varphi dx = \lambda\int_{\partial\Omega} u\varphi d\sigma, \tag{2.3}$$

with $u, \varphi \in H^2(\Omega)$, which is related to the vibration of a plate whose mass is concentrated at the boundary. This problem is a generalization of the classical Steklov problem (see [77]), and we refer to [31, 73] for the physical derivation of the problem.

2.1 Dirichlet boundary conditions (clamped plates)

Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. The Dirichlet problem for the biharmonic operator reads

$$\begin{cases} \Delta^2 u - \tau\Delta u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ of class C^1 and ν is the outer unit normal. We observe that the eigenvalues of problem (2.4) are strictly positive.

We consider on $H_0^2(\Omega)$ the bilinear form

$$< u, v >= \int_\Omega (1-\alpha)D^2u : D^2v + \alpha\Delta u\Delta v + \tau\nabla u \cdot \nabla v dx, \tag{2.5}$$

for any $u, v \in H_0^2(\Omega)$. One can prove that the bilinear form (2.5) defines on $H_0^2(\Omega)$ a scalar product whose induced norm is equivalent to the standard one defined by (1.10). In this section we shall consider the space $H_0^2(\Omega)$ endowed with the scalar product (2.5).

We consider the operator P from $H_0^2(\Omega)$ to its dual defined by

$$P[u][v] = \int_\Omega (1-\alpha)D^2u : D^2v + \alpha\Delta u\Delta v + \tau\nabla u \cdot \nabla v dx, \tag{2.6}$$

for any $u, v \in H_0^2(\Omega)$. The operator P is easily seen to be a linear homeomorphism of $H_0^2(\Omega)$ onto its dual. We also denote by $\mathcal{J}$ the continuous embedding of $H_0^2(\Omega)$ into its dual, defined by

$$\mathcal{J}[u][v] := \int_\Omega uvdx, \ \forall u, v \in H_0^2(\Omega).$$

Note that problem (2.4) can be written in the following weak formulation

$$P[u][v] = \lambda\mathcal{J}[u][v], \ \forall v \in H_0^2(\Omega). \tag{2.7}$$

We define the operator $T := P^{(-1)} \circ \mathcal{J}$ from $H_0^2(\Omega)$ to itself. We have the following

Lemma 2.1. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T is a non-negative compact selfadjoint operator in the Hilbert space $H_0^2(\Omega)$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $Tu = \mu u$ is satisfied for some $u \in H_0^2(\Omega)$, $\mu > 0$ if and only if equation (2.2) is satisfied with $0 \neq \lambda = \mu^{-1}$ for any $\varphi \in H_0^2(\Omega)$.*

Proof. For the selfadjointness, it suffices to observe that

$$< Tu, v >=< P^{-1} \circ \mathcal{J}u, v >= P[P^{-1} \circ \mathcal{J}u][v] = \mathcal{J}[u][v],$$

for any $u, v \in H_0^2(\Omega)$. For the compactness, just observe that the operator $\mathcal{J}$ is compact. The remaining statements can be deduced by Theorem 1.9. □

2.1.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (2.4) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^2_\Omega$ and study the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ.

The main result of this section is the following analogue for the biharmonic operator of [64, Theorem 3.38] concerning the Dirichlet Laplacian.

Theorem 2.2. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1 and F be a finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $\mathcal{A}^2_\Omega$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^4 then the Frechét differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_l}{\partial\nu^2}\right)^2 \zeta\cdot\nu d\sigma, \tag{2.8}$$

for all $\psi \in C^2_b(\Omega;\mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ and $\{v_l\}_{l\in F}$ is an orthonormal basis in $H^2_0(\tilde{\phi}(\Omega))$ (with respect to the scalar product (2.5)) of the eigenspace associated with $\lambda_F[\tilde{\phi}]$.

In order to prove Theorem 2.2 we consider equation (2.7) in $\phi(\Omega)$ and pull it back to Ω. We note that

$$P = (1-\alpha)H^2 + \alpha\Delta^2 - \tau\Delta,$$

where the operators H^2, Δ^2, Δ are defined by

$$H^2[u][v] = \int_\Omega D^2u : D^2v dx,$$

$$\Delta^2[u][v] = \int_\Omega \Delta u \Delta v dx,$$

and

$$\Delta[u][v] = \int_\Omega \nabla u \cdot \nabla v dx,$$

for all $u, v \in H^2(\Omega)$. We consider the equation

$$P[v][\psi] = \lambda\mathcal{J}[v][\psi], \quad \forall\ \psi \in H^2_0(\phi(\Omega)), \tag{2.9}$$

in the unknowns $v \in H_0^2(\phi(\Omega))$, $\lambda \in]0, \infty[$. Recall that the pull-back to Ω of the operator H^2 on $\phi(\Omega)$ is defined by

$$H_\phi^2[u][\varphi] = \int_\Omega \Big(D^2(u \circ \phi^{(-1)}) : D^2(\varphi \circ \phi^{(-1)}) \Big) \circ \phi |\det \nabla \phi| dx,$$

for all $u, \varphi \in H^2(\Omega)$, and similarly for $\Delta_\phi^2, \Delta_\phi$. We have

$$P_\phi = (1-\alpha)H_\phi^2 + \alpha \Delta_\phi^2 - \tau \Delta_\phi.$$

We will denote by $H_{0,\phi}^2(\Omega)$ the space $H_0^2(\Omega)$ endowed with the form

$$< u, v >_\phi = P_\phi[u][v], \ \forall u, v \in H_0^2(\Omega).$$

We also recall that

$$\mathcal{J}_\phi[u][w] = \int_\Omega uw |\det \nabla \phi| dx, \ \forall u, w \in H^2(\Omega).$$

Note that the map from $H^2(\Omega)$ to $H^2(\phi(\Omega))$ which maps u to $u \circ \phi^{(-1)}$ for all $u \in H^2(\Omega)$ is a linear homeomorphism. Hence, equation (2.7) is equivalent to

$$P_\phi[u][\varphi] = \lambda \mathcal{J}_\phi[u][\varphi], \ \ \forall \ \varphi \in H_{0,\phi}^2(\Omega),$$

where $u = v \circ \phi$. It turns out that the operator T defined in Lemma 2.1 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator T_ϕ defined on $H_{0,\phi}^2(\Omega)$ by

$$T_\phi := P_\phi^{(-1)} \circ \mathcal{J}_\phi. \tag{2.10}$$

Thus we can prove the following lemma where $\mathcal{L}(H_0^2(\Omega))$ denotes the space of linear bounded operators from $H_0^2(\Omega)$ to itself and and $\mathcal{B}_s(H_0^2(\Omega))$ denotes the space of bilinear forms on $H_0^2(\Omega)$ (both spaces are equipped with their usual norms).

Lemma 2.3. *Let* $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1. *The operator* T_ϕ *defined in (2.10) is non-negative selfadjoint and compact on the Hilbert space* $H_{0,\phi}^2(\Omega)$. *The equation (2.9) is satisfied for some* $v \in H_0^2(\phi(\Omega))$ *if and only if the equation* $T_\phi u = \mu u$ *is satisfied with* $u = v \circ \phi$ *and* $\mu = \lambda^{-1}$. *Moreover, the map from* $\mathcal{A}_\Omega^2$ *to* $\mathcal{L}(H_0^2(\Omega)) \times \mathcal{B}_s(H_0^2(\Omega))$ *which takes* $\phi \in \mathcal{A}_\Omega^2$ *to* $(T_\phi, < \cdot, \cdot >_\phi)$ *is real-analytic.*

Proof. Since the operator T_ϕ is unitarily equivalent to the operator T, the first part of the lemma immediately follows by Lemma 2.1. In order to prove the real-analytic dependence of T_ϕ upon ϕ, we note that by standard calculus

$$(H_\phi u)_{ij} = (\sigma^t \cdot D^2 u \cdot \sigma)_{ij} + \left(\sum_{r,s=1}^{N} \frac{\partial u}{\partial x_r} \frac{\partial \sigma_{ri}}{\partial x_s} \sigma_{sj} \right)_{ij}, \tag{2.11}$$

for all $u \in H^2(\Omega)$, where $H_\phi u$ is the classical pull-back of the Hessian matrix $D^2 u$, and $\sigma = (\nabla \phi)^{-1}$ (cf. [28, p. 240]). Moreover

$$\Delta_\phi u = \sum_{r,s,i=1}^{N} \left(\frac{\partial^2 u}{\partial x_r \partial x_s} \sigma_{ri} \sigma_{si} + \frac{\partial u}{\partial x_r} \frac{\partial \sigma_{ri}}{\partial x_s} \sigma_{si} \right),$$

for all $u \in H^2(\Omega)$, where Δ_ϕ is the classical pull-back of the Laplace operator Δ (see also [64, Proposition 3.1]), and

$$\nabla_\phi u = \nabla u \cdot \sigma,$$

for all $u \in H^1(\Omega)$, where ∇_ϕ is the classical pull-back of the gradient ∇.

By formula (2.11), it follows that the map from $\mathcal{A}^2_\Omega \times H^2(\Omega)$ to $L^2(\Omega)^{N^2}$ which takes $(\phi, u) \in \mathcal{A}^2_\Omega$ to $H_\phi u$ is real-analytic, and similarly for $\Delta_\phi u$, $\nabla_\phi u$. Thus also the map from $\mathcal{A}^2_\Omega \times H^2(\Omega)$ to $L^2(\Omega)$ which takes $(\phi, u) \in \mathcal{A}^2_\Omega$ to $P_\phi u$ is real-analytic since it is composition of real-analytic maps. This implies the real-analytic dependence of T_ϕ and $< \cdot, \cdot >_\phi$ upon ϕ. □

Proof of Theorem 2.2. We denote by $\mu_j[\phi]$, $j \in \mathbb{N}$, the eigenvalues of T_ϕ. By Lemma 2.3, $\mu_j[\phi] = \lambda_j^{-1}[\phi]$ for all $j \in \mathbb{N}$, hence the set $\mathcal{A}_{F,\Omega}$ coincides with the set $\{\phi \in \mathcal{A}^2_\Omega : \ \mu_j[\phi] \neq \mu_l[\phi],\ \forall j \in F,\ l \in \mathbb{N} \setminus F\}$. By Lemma 2.3, T_ϕ is selfadjoint with respect to the scalar product $< \cdot, \cdot >_\phi$ and both T_ϕ and $< \cdot, \cdot >_\phi$ depend real-analytically on ϕ. Thus, by applying [64, Thm. 2.30], it follows that $\mathcal{A}_{F,\Omega}$ is an open set in $C^2_b(\Omega\,; \mathbb{R}^N)$ and the functions which take $\phi \in \mathcal{A}_{F,\Omega}$ to

$$M_{F,s}[\phi] = \sum_{\substack{j_1,\dots,j_s \in F \\ j_1 < \cdots < j_s}} \mu_{j_1}[\phi] \cdots \mu_{j_s}[\phi]$$

are real-analytic for all $s = 1, \dots, |F|$. Since

$$\Lambda_{F,s}[\phi] = \frac{M_{F,|F|-h}[\phi]}{M_{F,|F|}[\phi]},$$

for all $s = 1, \dots, |F|$, where $M_{F,0}[\phi] = 1$, it follows that $\Lambda_{F,s}[\phi]$ depends real-analytically on $\phi \in \mathcal{A}_{F,\Omega}$ (see also [64, Theorem 3.21]).

It remains to prove formula (2.8). Let $\tilde{\phi} \in \Theta_{F,\Omega}$, $\lambda_F[\tilde{\phi}]$ and $\{v_l\}_{l\in F}$ be as in the statement. We set $u_l = v_l \circ \tilde{\phi}$ for all $l \in F$ and we note that $\{u_l\}_{l\in F}$ is an orthonormal basis in $H^2_{0,\tilde{\phi}}(\Omega)$ for the eigenspace corresponding to the eigenvalue $\lambda_F^{-1}[\tilde{\phi}]$ of the operator $T_{\tilde{\phi}}$. By [64, Thm. 2.30], it follows that

$$\mathrm{d}|_{\phi=\tilde{\phi}}\Gamma_{F,s}[\psi] = \lambda_F^{1-s}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l\in F} < \mathrm{d}|_{\phi=\tilde{\phi}}T_\phi[\psi][u_l], u_l >_{\tilde{\phi}}$$

for all $\psi \in C^2_b(\Omega\,;\mathbb{R}^N)$. Note that by standard regularity theory (see e.g., [51, Thm. 2.20]) $v_l \in H^4(\tilde{\phi}(\Omega))$ for all $l \in F$.

We have

$$\begin{aligned} < \mathrm{d}|_{\phi=\tilde{\phi}}T_\phi[\psi][u_l], u_l >_{\tilde{\phi}} = d|_{\phi=\tilde{\phi}}\mathcal{J}_\phi[\psi][u_l][u_l] \\ - \lambda_F^{-1}[\tilde{\phi}]d|_{\phi=\tilde{\phi}}P_\phi[\psi][u_l][u_l]. \end{aligned}$$

Moreover, by standard calculus,

$$\left[\left(\mathrm{d}|_{\phi=\tilde{\phi}}\left(\det\nabla\phi\right)[\psi]\right)\circ\tilde{\phi}^{(-1)}\right]\det\nabla\tilde{\phi}^{(-1)} = \mathrm{div}\left(\psi\circ\tilde{\phi}^{(-1)}\right), \tag{2.12}$$

and therefore

$$d|_{\phi=\tilde{\phi}}\mathcal{J}_\phi[\psi][u_l][u_l] = \int_{\tilde{\phi}(\Omega)} v_l^2 \mathrm{div}\zeta dy. \tag{2.13}$$

Using Lemmas 2.4, 2.5 and 2.6 below, and the fact that $v_l = |\nabla v_l| = 0$ on $\partial\tilde{\phi}(\Omega)$ we obtain

$$\begin{aligned} &d|_{\phi=\tilde{\phi}}P_\phi[\psi][u_l][u_l] \\ &\quad= \int_{\partial\tilde{\phi}(\Omega)}\left((1-\alpha)|D^2v_l|^2 + \alpha(\Delta v_l)^2\right)\zeta\cdot\nu d\sigma - \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu dy \\ &\qquad= \int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_l}{\partial\nu^2}\right)^2\zeta\cdot\nu d\sigma - \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\mu dy. \end{aligned}$$

To conclude, just observe that

$$\int_{\tilde{\phi}(\Omega)}\nabla(v_l^2)\cdot\zeta dy = \int_{\partial\tilde{\phi}(\Omega)} v_l^2\zeta\cdot\nu d\sigma - \int_{\tilde{\phi}(\Omega)} v_l^2 \mathrm{div}\zeta dy. \tag{2.14}$$

$\square$

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde\phi \in \mathcal{A}^2_\Omega$ be such that $\tilde\phi(\Omega)$ is of class C^2. Let $u_1, u_2 \in H^2(\Omega)$ be such that $v_1 = u_1 \circ \tilde\phi^{-1}, v_2 = u_2 \circ \tilde\phi^{-1} \in H^4(\tilde\phi(\Omega))$. Then*

$$
\begin{aligned}
d|_{\phi=\tilde\phi} H^2_\phi[\psi][u_1][u_2] &= \int_{\partial\tilde\phi(\Omega)} (D^2v_1 : D^2v_2)\zeta\cdot\nu d\sigma \\
&+ \int_{\partial\tilde\phi(\Omega)} \Big(\mathrm{div}_{\partial\tilde\phi(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde\phi(\Omega)}\nabla v_2 + \mathrm{div}_{\partial\tilde\phi(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde\phi(\Omega)}\nabla v_1\Big)\cdot\zeta d\sigma \\
&+ \int_{\partial\tilde\phi(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma - \int_{\tilde\phi(\Omega)} \left(\Delta^2 v_1\nabla v_2 + \Delta^2 v_2\nabla v_1\right)\cdot\zeta d\sigma \\
&- \int_{\partial\tilde\phi(\Omega)} \left(\frac{\partial^2 v_1}{\partial\nu^2}\nabla v_2 + \frac{\partial^2 v_2}{\partial\nu^2}\nabla v_1\right)\cdot\frac{\partial\zeta}{\partial\nu} d\sigma \\
&- \int_{\partial\tilde\phi(\Omega)} \left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2 + \frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma, \quad (2.15)
\end{aligned}
$$

for all $\psi \in C^2_b(\Omega;\mathbb{R}^N)$, where $\zeta = \psi\circ\tilde\phi^{-1}$.

Proof. We have

$$
\begin{aligned}
&d|_{\phi=\tilde\phi} H^2_\phi[\psi][u_1][u_2] \\
&\quad = \int_\Omega (d|_{\phi=\tilde\phi} D^2(u_1\circ\phi^{-1})\circ\phi)[\psi] : (D^2(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)|\det D\tilde\phi| dx \\
&\quad + \int_\Omega (D^2(u_1\circ\tilde\phi^{-1})\circ\tilde\phi) : (d|_{\phi=\tilde\phi} D^2(u_2\circ\phi^{-1})\circ\phi)[\psi]|\det D\tilde\phi| dx \\
&\quad + \int_\Omega (D^2(u_1\circ\tilde\phi^{-1})\circ\tilde\phi) : (D^2(u_2\circ\tilde\phi^{-1})\circ\tilde\phi) d|_{\phi=\tilde\phi}|\det D\phi|[\psi] dx, \quad (2.16)
\end{aligned}
$$

and we note that, by (2.12), the last summand in (2.16) equals

$$
\int_{\tilde\phi(\Omega)} \left(D^2v_1 : D^2v_2\right)\mathrm{div}\zeta dy.
$$

By standard calculus we have (see [28, formula (2.15)])

$$
D^2(u\circ\phi^{-1})\circ\phi = (\nabla\phi)^{-t}D^2u(\nabla\phi)^{-1} + \left(\sum_{k,l=1}^N \frac{\partial u}{\partial x_k}\frac{\partial\sigma_{k,i}}{\partial x_l}\sigma_{l,j}\right)_{i,j},
$$

where $\sigma = (\nabla\phi)^{-1}$. This yields the following formula

$$
d|_{\phi=\tilde\phi}(D^2(u\circ\phi^{-1})\circ\phi)[\psi]\circ\phi^{-1} = -D^2v\nabla\zeta - \nabla\zeta^t D^2v - \sum_{r=1}^N \frac{\partial v}{\partial y_r} D^2\zeta_r, \quad (2.17)
$$

where $\zeta=\psi\circ\phi^{-1}$ and $v=u\circ\phi^{-1}$. By rewriting formula (2.17) componentwise we get

$$\begin{aligned}\Big(d|_{\phi=\tilde\phi}(D^2(u\circ\phi^{-1})\circ\phi)[\psi]\circ\phi^{-1}\Big)_{i,j}\\ =-\sum_{r=1}^{N}\left(\frac{\partial^2 v}{\partial y_i\partial y_r}\frac{\partial\zeta_r}{\partial y_j}+\frac{\partial^2 v}{\partial y_j\partial y_r}\frac{\partial\zeta_r}{\partial y_i}+\frac{\partial^2\zeta_r}{\partial y_i\partial y_j}\frac{\partial v}{\partial y_r}\right).\end{aligned}$$

Now we use Einstein notation, dropping all the summation symbols. The first summand of the right-hand side of (2.16) equals

$$-\int_{\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial\zeta_r}{\partial y_j}+\frac{\partial^2 v_1}{\partial y_j\partial y_r}\frac{\partial\zeta_r}{\partial y_i}+\frac{\partial^2\zeta_r}{\partial y_i\partial y_j}\frac{\partial v_1}{\partial y_r}\right)\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy. \tag{2.18}$$

In order to compute (2.18), integrating by parts, we have

$$\begin{aligned}\int_{\tilde\phi(\Omega)}\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial\zeta_r}{\partial y_j}\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy&=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial\zeta_r}{\partial y_j}\nu_r\frac{\partial^2 v_2}{\partial y_i\partial y_j}d\sigma\\ &\quad-\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial \mathrm{div}\zeta}{\partial y_j}\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy-\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial\zeta_r}{\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}dy\\ &=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial\zeta_r}{\partial y_j}\nu_r\frac{\partial^2 v_2}{\partial y_i\partial y_j}d\sigma-\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial\zeta_r}{\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}dy\\ &\quad-\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\mathrm{div}\zeta\frac{\partial^2 v_2}{\partial y_i\partial y_j}\nu_j d\sigma+\int_{\tilde\phi(\Omega)}D^2v_1:D^2v_2\mathrm{div}\zeta dy\\ &\qquad+\int_{\tilde\phi(\Omega)}\mathrm{div}\zeta\nabla v_1\cdot\nabla\Delta v_2 dy,\end{aligned}$$

and

$$\begin{aligned}\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial^2\zeta_r}{\partial y_i\partial y_j}\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy&=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\nu_j\frac{\partial^2 v_2}{\partial y_i\partial y_j}d\sigma\\ &\quad-\int_{\tilde\phi(\Omega)}\frac{\partial^2 v_1}{\partial y_r\partial y_j}\frac{\partial\zeta_r}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy-\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\frac{\partial\Delta v_2}{\partial y_i}dy\\ &=\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\nu_j\frac{\partial^2 v_2}{\partial y_i\partial y_j}d\sigma-\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\frac{\partial\Delta v_2}{\partial y_i}dy\\ &\quad-\int_{\partial\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_j}\frac{\partial\zeta_r}{\partial y_i}\nu_r\frac{\partial^2 v_2}{\partial y_i\partial y_j}d\sigma+\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_j}\frac{\partial\mathrm{div}\zeta}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_j}dy\\ &\qquad+\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_j}\frac{\partial\zeta_r}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}dy\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r}\frac{\partial \zeta_r}{\partial y_i}\nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r}\frac{\partial \zeta_r}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_i} dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j}\frac{\partial \zeta_r}{\partial y_i}\nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j}\frac{\partial \zeta_r}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j}\mathrm{div}\zeta \frac{\partial^2 v_2}{\partial y_i \partial y_j}\nu_i d\sigma - \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mathrm{div}\zeta dy \\
&\qquad - \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\zeta dy.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} H^2_{\phi}[\psi][u_1][u_2] &= -\int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mathrm{div}\zeta dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \frac{\partial \zeta_r}{\partial y_j}\nu_r d\sigma \\
&+ \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \frac{\partial \zeta_r}{\partial y_j} dy \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \mathrm{div}\zeta d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \left(\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1 \right) \mathrm{div}\zeta dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \frac{\partial \zeta_r}{\partial y_i} d\sigma \\
&\qquad + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r}\frac{\partial \Delta v_2}{\partial y_i} + \frac{\partial v_2}{\partial y_r}\frac{\partial \Delta v_1}{\partial y_i} \right) \frac{\partial \zeta_r}{\partial y_i} dy \qquad (2.19)
\end{aligned}
$$

We now recall that, since $\partial\tilde{\phi}(\Omega)$ is of class C^4, we have

$$
\mathrm{div}\zeta = \mathrm{div}_{\partial\tilde{\phi}(\Omega)}\zeta + \frac{\partial \zeta}{\partial \nu}\cdot \nu \quad \text{on} \quad \partial\tilde{\phi}(\Omega),
$$

and

$$
\Delta f = \Delta_{\partial\tilde{\phi}(\Omega)} f + K\frac{\partial f}{\partial \nu} + \frac{\partial^2 f}{\partial \nu^2} \text{ on } \quad \partial\tilde{\phi}(\Omega),
$$

for any function f smooth enough in a neighborhood of $\partial\tilde{\phi}(\Omega)$ (see also [46, §8.5]). Moreover, since $\nu = \nabla b$, where b is the distance from the boundary defined in an appropriate tubular neighborhood of the boundary, then $\nabla\nu = (\nabla\nu)^t$ and $\frac{\partial \nu}{\partial \nu} = 0$, from which it follows that

$$
\nabla_{\partial\tilde{\phi}(\Omega)}\nu = (\nabla_{\partial\tilde{\phi}(\Omega)}\nu)^t \quad \text{on} \quad \partial\tilde{\phi}(\Omega). \qquad (2.20)
$$

We will use these identities throughout all the following computations.

We get that the sixth summand in the right-hand side of (2.19) equals

$$
\begin{aligned}
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}+\frac{\partial v_2}{\partial y_r}(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\right)\cdot\nabla_{\partial\tilde{\phi}(\Omega)}\zeta_r d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial\nu^2}+\frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial\nu^2}\right)\frac{\partial\zeta_r}{\partial\nu}d\sigma\\
&=\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\right)(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}\right.\\
&\qquad\left.+\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_2}{\partial y_r}\right)(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\right)\zeta_r d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\right)\cdot\zeta d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial\nu^2}+\frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial\nu^2}\right)\frac{\partial\zeta_r}{\partial\nu}d\sigma\\
&=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\zeta_r d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\right)\cdot\zeta d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial^2 v_2}{\partial\nu^2}+\frac{\partial v_2}{\partial y_r}\frac{\partial^2 v_1}{\partial\nu^2}\right)\frac{\partial\zeta_r}{\partial\nu}d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma. \qquad (2.21)
\end{aligned}
$$

The seventh summand in the right-hand side of (2.19) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\right)\cdot\zeta dy\\
&\qquad-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial\Delta v_2}{\partial y_i}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial\Delta v_1}{\partial y_i}\right)\zeta_r dy\\
&=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\right)\cdot\zeta d\sigma\\
&\qquad-\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\zeta\cdot\nu d\sigma\\
&\qquad+\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\zeta_r dy
\end{aligned}
$$

$$+\int_{\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mathrm{div}\zeta dy. \quad (2.22)$$

The second summand in the right-hand side of (2.19) equals

$$\begin{aligned}&-\int_{\partial\tilde{\phi}(\Omega)}\nabla(\nabla v_1\cdot\nabla v_2)\nabla(\zeta_r)\nu_r d\sigma\\ &\qquad=-\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde{\phi}(\Omega)}(\zeta_r)\nu_r d\sigma\\ &\qquad\qquad-\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\frac{\partial\zeta_r}{\partial\nu}\nu_r d\sigma. \quad (2.23)\end{aligned}$$

The third summand in the right-hand side of (2.19) equals

$$\begin{aligned}&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\zeta_r d\sigma\\ &\qquad-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\zeta_r dy\\ &\quad-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial^2 v_2}{\partial y_i\partial y_j}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\zeta_r dy\\ &\quad=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\zeta_r d\sigma\\ &\qquad-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2\Delta v_2}{\partial y_i\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2\Delta v_1}{\partial y_i\partial y_r}\right)\zeta_r dy\\ &\quad-\int_{\partial\tilde{\phi}(\Omega)}D^2v_1:D^2v_2\zeta\cdot\nu d\sigma+\int_{\tilde{\phi}(\Omega)}D^2v_1:D^2v_2\mathrm{div}\zeta dy. \quad (2.24)\end{aligned}$$

From (2.19)-(2.24), it follows that

$$\begin{aligned}&d|_{\phi=\tilde{\phi}}H^2_{\phi}[\psi][u_1][u_2]=-\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde{\phi}(\Omega)}(\zeta_r)\nu_r d\sigma\\ &\quad-\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\frac{\partial\zeta_r}{\partial\nu}\nu_r d\sigma+\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mathrm{div}\zeta d\sigma\\ &+\int_{\partial\tilde{\phi}(\Omega)}\Big(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\Big)\cdot\zeta d\sigma\\ &\qquad+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial^2 v_2}{\partial y_i\partial y_j}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial^2 v_1}{\partial y_i\partial y_j}\right)\nu_j\zeta_r d\sigma\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_i}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_j\zeta_r d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)} D^2v_1 : D^2v_2\zeta\cdot\nu d\sigma-\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\zeta\cdot\nu d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2 v_1\nabla v_2+\Delta^2 v_2\nabla v_1\right)\cdot\zeta d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\nabla v_1\right)\cdot\frac{\partial\zeta}{\partial\nu}d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}}H^2_{\phi}[\psi][u_1][u_2] &= -\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\nabla_{\partial\tilde{\phi}(\Omega)}(\zeta_r)\nu_r d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}(\nabla v_1\cdot\nabla v_2)\mathrm{div}_{\partial\tilde{\phi}(\Omega)}\zeta d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde{\phi}(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde{\phi}(\Omega)}\nabla v_1\right)\cdot\zeta d\sigma\\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma+\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial}{\partial\nu}\left(\frac{\partial}{\partial y_r}(\nabla v_1\cdot\nabla v_2)\right)\zeta_r d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)} D^2v_1 : D^2v_2\zeta\cdot\nu d\sigma-\int_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\zeta\cdot\nu d\sigma\\
&-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2 v_1\nabla v_2+\Delta^2 v_2\nabla v_1\right)\cdot\zeta d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\nabla v_1\right)\cdot\frac{\partial\zeta}{\partial\nu}d\sigma\\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma. \qquad (2.25)
\end{aligned}
$$

The first summand on the right-hand side of (2.25) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\Delta_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\zeta\cdot\nu d\sigma\\
&\qquad+\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla v_2)\cdot(\nabla_{\partial\tilde{\phi}(\Omega)}\nu_r)\zeta_r d\sigma,
\end{aligned}
$$

while the fifth one equals

$$\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial^2}{\partial\nu^2}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial}{\partial\nu}\left(\frac{\partial}{\partial y_r}(\nabla v_1 \cdot \nabla v_2)\right)\right)_{\partial\tilde{\phi}(\Omega)} \zeta_r d\sigma$$
$$= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial^2}{\partial\nu^2}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2)\right) \cdot \zeta d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1 \cdot \nabla v_2) \cdot (\nabla_{\partial\tilde{\phi}(\Omega)}\nu_r)\zeta_r d\sigma,$$

where in the last term we have used equality (2.20). Using the fact that

$$\int_{\partial\tilde{\phi}(\Omega)} \mathrm{div}_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2) \cdot \zeta\right) d\sigma = \int_{\partial\tilde{\phi}(\Omega)} K \frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma,$$

where K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$ (see [46, §8.5]), we finally obtain

$$d|_{\phi=\tilde{\phi}} H^2_\phi[\psi][u_1][u_2] = \int_{\partial\tilde{\phi}(\Omega)} \Delta_{\partial\tilde{\phi}(\Omega)}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma$$
$$+ \int_{\partial\tilde{\phi}(\Omega)} K \frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial^2}{\partial\nu^2}(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \zeta \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla\Delta v_2 + \nabla v_2 \cdot \nabla\Delta v_1)\zeta \cdot \nu d\sigma$$
$$+ \int_{\partial\tilde{\phi}(\Omega)} \left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_1)_{\partial\tilde{\phi}(\Omega)} \nabla v_2 + \mathrm{div}_{\partial\tilde{\phi}(\Omega)}(nu \cdot D^2 v_2)_{\partial\tilde{\phi}(\Omega)} \nabla v_1\right) \cdot \zeta d\sigma$$
$$+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right) \cdot \zeta d\sigma$$
$$- \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1\right) \cdot \zeta d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial\nu^2}\nabla v_2 + \frac{\partial^2 v_2}{\partial\nu^2}\nabla v_1\right) \cdot \frac{\partial\zeta}{\partial\nu} d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2 + \frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right) \cdot \zeta d\sigma$$
$$= \int_{\partial\tilde{\phi}(\Omega)} \Delta(\nabla v_1 \cdot \nabla v_2)\zeta \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \zeta \cdot \nu d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla\Delta v_2 + \nabla v_2 \cdot \nabla\Delta v_1)\zeta \cdot \nu d\sigma$$

$$+\int_{\partial\tilde\phi(\Omega)}\Big(\mathrm{div}_{\partial\tilde\phi(\Omega)}(\nu\cdot D^2v_1)_{\partial\tilde\phi(\Omega)}\nabla v_2+\mathrm{div}_{\partial\tilde\phi(\Omega)}(\nu\cdot D^2v_2)_{\partial\tilde\phi(\Omega)}\nabla v_1\Big)\cdot\zeta d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma$$
$$-\int_{\tilde\phi(\Omega)}\big(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\big)\cdot\zeta d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\nabla v_1\right)\cdot\frac{\partial\zeta}{\partial\nu}d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_2+\frac{\partial^2 v_2}{\partial\nu^2}\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma.$$

Using the equality

$$\Delta(\nabla v_1\cdot\nabla v_2)=\nabla\Delta v_1\cdot\nabla v_2+\nabla v_1\cdot\nabla\Delta v_2+2D^2v_1:D^2v_2$$

we finally get formula (2.15). □

Lemma 2.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde\phi\in\mathcal{A}^2_\Omega$ be such that $\tilde\phi(\Omega)$ is of class C^2. Let $u_1,u_2\in H^2(\Omega)$ be such that $v_1=u_1\circ\tilde\phi^{-1}, v_2=u_2\circ\tilde\phi^{-1}\in H^4(\tilde\phi(\Omega))$. Then*

$$d|_{\phi=\tilde\phi}\Delta^2_\phi[\psi][u_1][u_2]=\int_{\partial\tilde\phi(\Omega)}\Delta v_1\Delta v_2\zeta\cdot\nu d\sigma$$
$$+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial\Delta v_1}{\partial\nu}\nabla v_2+\frac{\partial\Delta v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde\phi(\Omega)}\big(\Delta^2v_1\nabla v_2+\Delta^2v_2\nabla v_1\big)\cdot\zeta d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}(\Delta v_1\nabla v_2+\Delta v_2\nabla v_1)\cdot\frac{\partial\zeta}{\partial\nu}d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}\left(\Delta v_1\frac{\partial}{\partial\nu}\nabla v_2+\Delta v_2\frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma,\quad(2.26)$$

for all $\psi\in C^2_b(\Omega;\mathbb{R}^N)$, where $\zeta=\psi\circ\tilde\phi^{-1}$.

Proof. We have

$$d|_{\phi=\tilde\phi}\Delta^2_\phi[\psi][u_1][u_2]$$
$$=\int_\Omega(d|_{\phi=\tilde\phi}\Delta(u_1\circ\phi^{-1})\circ\phi)[\psi](\Delta(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)|\det D\tilde\phi|dx$$
$$+\int_\Omega(\Delta(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)(d|_{\phi=\tilde\phi}\Delta(u_2\circ\phi^{-1})\circ\phi)[\psi]|\det D\tilde\phi|dx$$
$$+\int_\Omega(\Delta(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)(\Delta(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)d|_{\phi=\tilde\phi}|\det D\phi|[\psi]dx,\quad(2.27)$$

and we note that, by (2.12), the last summand in (2.27) equals

$$\int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy.$$

We have

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_r \partial y_s}\frac{\partial \zeta_r}{\partial y_s}\Delta v_2 dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_r}{\partial y_s}\nu_r \Delta v_2 d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \mathrm{div}\zeta}{\partial y_s}\Delta v_2 dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_r}{\partial y_s}\frac{\partial \Delta v_2}{\partial y_r} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_r}{\partial y_s}\nu_r \Delta v_2 d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_r}{\partial y_s}\frac{\partial \Delta v_2}{\partial y_r} dy \\
&\quad - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu}\Delta v_2 \mathrm{div}\zeta d\sigma + \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy \\
&\qquad + \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\zeta dy, \qquad (2.28)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\Delta \zeta_s \Delta v_2 dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial \nu}\Delta v_2 d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_i \partial y_s}\frac{\partial \zeta_s}{\partial y_i}\Delta v_2 dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_i} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial \nu}\Delta v_2 d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_i} dy \\
&\quad - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i}\frac{\partial \zeta_s}{\partial y_i}\nu_s \Delta v_2 d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i}\frac{\partial \mathrm{div}\zeta}{\partial y_i}\Delta v_2 dy \\
&\quad + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i}\frac{\partial \zeta_s}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_s} dy = \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial \nu}\Delta v_2 d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s}\frac{\partial \zeta_s}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_i} dy - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i}\frac{\partial \zeta_s}{\partial y_i}\nu_s \Delta v_2 d\sigma \\
&\quad + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i}\frac{\partial \zeta_s}{\partial y_i}\frac{\partial \Delta v_2}{\partial y_s} dy + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu}\Delta v_2 \mathrm{div}\zeta d\sigma \\
&\qquad - \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy - \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\zeta dy. \qquad (2.29)
\end{aligned}$$

Using (2.27)–(2.29) we get

$$
\begin{aligned}
d|_{\phi=\tilde\phi}\Delta^2_{\phi}[\psi][u_1][u_2] = &-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\Delta v_2+\frac{\partial v_2}{\partial y_s}\Delta v_1\right)\frac{\partial \zeta_r}{\partial y_s}\nu_r d\sigma\\
&+\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\frac{\partial \Delta v_2}{\partial y_r}+\frac{\partial v_2}{\partial y_s}\frac{\partial \Delta v_1}{\partial y_r}\right)\frac{\partial \zeta_r}{\partial y_s}dy\\
&+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\Delta v_2+\frac{\partial v_2}{\partial \nu}\Delta v_1\right)\mathrm{div}\zeta d\sigma\\
&-\int_{\tilde\phi(\Omega)}\Delta v_1\Delta v_2\mathrm{div}\zeta dy-\int_{\tilde\phi(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mathrm{div}\zeta dy\\
&-\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\Delta v_2+\frac{\partial v_2}{\partial y_s}\Delta v_1\right)\frac{\partial \zeta_s}{\partial \nu}d\sigma\\
&+\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\frac{\partial \Delta v_2}{\partial y_i}+\frac{\partial v_2}{\partial y_s}\frac{\partial \Delta v_1}{\partial y_i}\right)\frac{\partial \zeta_s}{\partial y_i}dy. \qquad (2.30)
\end{aligned}
$$

The last summand in the right-hand side of (2.30) equals

$$
\begin{aligned}
&\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial \Delta v_1}{\partial \nu}\nabla v_2+\frac{\partial \Delta v_2}{\partial \nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde\phi(\Omega)}\left(\Delta^2 v_1\nabla v_2+\Delta^2 v_2\nabla v_1\right)\cdot\zeta dy\\
&\quad-\int_{\tilde\phi(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_s}\frac{\partial \Delta v_2}{\partial y_i}+\frac{\partial^2 v_2}{\partial y_i\partial y_s}\frac{\partial \Delta v_1}{\partial y_i}\right)\zeta_s dy\\
&=\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial \Delta v_1}{\partial \nu}\nabla v_2+\frac{\partial \Delta v_2}{\partial \nu}\nabla v_1\right)\cdot\zeta d\sigma-\int_{\tilde\phi(\Omega)}\left(\Delta^2 v_1\nabla v_2+\Delta^2 v_2\nabla v_1\right)\cdot\zeta dy\\
&\quad-\int_{\partial\tilde\phi(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\zeta\cdot\nu d\sigma\\
&\quad+\int_{\tilde\phi(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\mathrm{div}\zeta dy\\
&\quad+\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_i}\frac{\partial^2 \Delta v_2}{\partial y_i\partial y_s}+\frac{\partial v_2}{\partial y_i}\frac{\partial^2 \Delta v_1}{\partial y_i\partial y_s}\right)\zeta_s dy,
\end{aligned}
$$

while the second one equals

$$
\begin{aligned}
&\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\nabla\Delta v_2+\frac{\partial v_2}{\partial \nu}\nabla\Delta v_1\right)\cdot\zeta d\sigma\\
&\quad-\int_{\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\frac{\partial^2 \Delta v_2}{\partial y_r\partial y_s}+\frac{\partial v_2}{\partial y_s}\frac{\partial^2 \Delta v_1}{\partial y_r\partial y_s}\right)\zeta_r dy\\
&\quad-\int_{\partial\tilde\phi(\Omega)}\Delta v_1\Delta v_2\zeta\cdot\nu d\sigma+\int_{\tilde\phi(\Omega)}\Delta v_1\Delta v_2\mathrm{div}\zeta dy.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}}\Delta^2_{\phi}[\psi][u_1][u_2] = &-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\Delta v_2+\frac{\partial v_2}{\partial y_s}\Delta v_1\right)\frac{\partial \zeta_r}{\partial y_s}\nu_r d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\nabla\Delta v_2+\frac{\partial v_2}{\partial \nu}\nabla\Delta v_1\right)\cdot\zeta d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\Delta v_2+\frac{\partial v_2}{\partial \nu}\Delta v_1\right)\mathrm{div}\zeta d\sigma \\
&-\int_{\partial\tilde{\phi}(\Omega)}\Delta v_1\Delta v_2\zeta\cdot\nu d\sigma-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_s}\Delta v_2+\frac{\partial v_2}{\partial y_s}\Delta v_1\right)\frac{\partial \zeta_s}{\partial \nu}d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial \Delta v_1}{\partial \nu}\nabla v_2+\frac{\partial \Delta v_2}{\partial \nu}\nabla v_1\right)\cdot\zeta d\sigma \\
&-\int_{\tilde{\phi}(\Omega)}\left(\Delta^2 v_1\nabla v_2+\Delta^2 v_2\nabla v_1\right)\cdot\zeta dy \\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1\right)\zeta\cdot\nu d\sigma \qquad (2.31)
\end{aligned}
$$

The first summand in (2.31) equals

$$
\begin{aligned}
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\Delta v_2+\frac{\partial v_2}{\partial \nu}\Delta v_1\right)\frac{\partial \zeta_r}{\partial \nu}\nu_r d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\nabla_{\partial\tilde{\phi}(\Omega)}v_1\cdot\nabla_{\partial\tilde{\phi}(\Omega)}\Delta v_2+\nabla_{\partial\tilde{\phi}(\Omega)}v_2\cdot\nabla_{\partial\tilde{\phi}(\Omega)}\Delta v_1\right)\zeta\cdot\nu d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\Delta v_1\Delta_{\partial\tilde{\phi}(\Omega)}v_2+\Delta v_2\Delta_{\partial\tilde{\phi}(\Omega)}v_1\right)\zeta\cdot\nu d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}\left(\Delta v_1\nabla_{\partial\tilde{\phi}(\Omega)}v_2+\Delta v_1\nabla_{\partial\tilde{\phi}(\Omega)}v_1\right)\cdot(\nabla_{\partial\tilde{\phi}(\Omega)}\nu_r)\zeta_r d\sigma,
\end{aligned}
$$

while the second one equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\frac{\partial \Delta v_2}{\partial \nu}+\frac{\partial v_2}{\partial \nu}\frac{\partial \Delta v_1}{\partial \nu}\right)\zeta\cdot\nu d\sigma \\
&+\int_{\partial\tilde{\phi}(\Omega)}K\left(\frac{\partial v_1}{\partial \nu}\Delta v_2+\frac{\partial v_2}{\partial \nu}\Delta v_1\right)\zeta\cdot\nu d\sigma \\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\Delta v_2+\frac{\partial v_2}{\partial \nu}\Delta v_1\right)\mathrm{div}_{\partial\tilde{\phi}(\Omega)}\zeta d\sigma \\
&-\int_{\partial\tilde{\phi}(\Omega)}\left(\Delta v_1\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_2}{\partial \nu}+\Delta v_2\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_1}{\partial \nu}\right)\cdot\zeta d\sigma,
\end{aligned}
$$

where K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$. Therefore the first three terms in the right-hand side of (2.31) equal

$$\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla\Delta v_2 + \nabla v_2 \cdot \nabla\Delta v_1)\,\zeta\cdot\nu d\sigma \\
&\quad - 2\int_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \zeta\cdot\nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \frac{\partial^2 v_2}{\partial\nu^2} + \Delta v_2 \frac{\partial^2 v_1}{\partial\nu^2}\right)\zeta\cdot\nu d\sigma \\
&\qquad + \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_1\right)\cdot(\nabla_{\partial\tilde{\phi}(\Omega)}\nu_r)\zeta_r d\sigma \\
&\qquad\quad - \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial\nu} + \Delta v_2 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\right)\cdot\zeta d\sigma. \qquad (2.32)
\end{aligned}$$

Now note that summing the third and the fifth terms in (2.32) and using (2.20) we get

$$\begin{aligned}
&-\int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla\frac{\partial v_2}{\partial\nu} + \Delta v_2 \nabla\frac{\partial v_1}{\partial\nu}\right)\cdot\zeta d\sigma \\
&\qquad = -\int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \frac{\partial}{\partial\nu}\nabla v_2 + \Delta v_2 \frac{\partial}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma \\
&\quad - \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_2 \nabla_{\partial\tilde{\phi}(\Omega)} v_1\right)\cdot(\nabla_{\partial\tilde{\phi}(\Omega)}\nu_r)\zeta_r d\sigma. \qquad (2.33)
\end{aligned}$$

Using (2.31), (2.32) and (2.33), we finally get formula (2.26). □

Lemma 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde{\phi} \in \mathcal{A}^2_\Omega$. Let $u_1, u_2 \in H^2(\Omega)$ and let $v_1 = u_1 \circ \tilde{\phi}^{-1}, v_2 = u_2 \circ \tilde{\phi}^{-1}$. Then*

$$\begin{aligned}
&d|_{\phi=\tilde{\phi}}\Delta_\phi[\psi][u_1][u_2] = -\int_{\partial\tilde{\phi}(\Omega)} \nabla v_1\cdot\nabla v_2 \zeta\cdot\nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu}\nabla v_2 + \frac{\partial v_2}{\partial\nu}\nabla v_1\right)\cdot\zeta d\sigma - \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1)\cdot\zeta d\sigma, \qquad (2.34)
\end{aligned}$$

for all $\psi \in C^2_b(\Omega;\mathbb{R}^N)$, where $\zeta = \psi\circ\tilde{\phi}^{-1}$.

Proof. We have

$$
\begin{aligned}
&d|_{\phi=\tilde\phi}\Delta_\phi[\psi][u_1][u_2]\\
&\quad= -\int_\Omega (d|_{\phi=\tilde\phi}\nabla(u_1\circ\phi^{-1})\circ\phi)[\psi]\cdot(\nabla(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)|\det D\tilde\phi|dx\\
&\qquad -\int_\Omega (\nabla(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)\cdot(d|_{\phi=\tilde\phi}\nabla(u_2\circ\phi^{-1})\circ\phi)[\psi]|\det D\tilde\phi|dx\\
&\qquad -\int_\Omega (\nabla(u_1\circ\tilde\phi^{-1})\circ\tilde\phi)\cdot(\nabla(u_2\circ\tilde\phi^{-1})\circ\tilde\phi)d|_{\phi=\tilde\phi}|\det D\phi|[\psi]dx, \qquad (2.35)
\end{aligned}
$$

and we note that, by (2.12), the last summand in (2.35) equals

$$
-\int_{\tilde\phi(\Omega)} \nabla v_1\cdot\nabla v_2 \mathrm{div}\zeta dy.
$$

Using the fact that

$$
\left(d|_{\phi=\tilde\phi}(\nabla(u\circ\phi^{-1})\circ\phi)[\psi]\circ\tilde\phi^{-1}\right)_i = -\sum_{r=1}^N \frac{\partial v}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i},
$$

where $v=u\circ\tilde\phi^{-1}$, and

$$
\begin{aligned}
&\int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\frac{\partial v_2}{\partial y_i}dy = \int_{\partial\tilde\phi(\Omega)}\frac{\partial v_2}{\partial\nu}\nabla v_1\cdot\zeta d\sigma - \int_{\tilde\phi(\Omega)}\Delta v_2\nabla v_1\cdot\zeta dy\\
&\quad -\int_{\tilde\phi(\Omega)}\frac{\partial v_2}{\partial y_i}\frac{\partial^2 v_1}{\partial y_i\partial y_r}\zeta_r dy = \int_{\partial\tilde\phi(\Omega)}\frac{\partial v_2}{\partial\nu}\nabla v_1\cdot\zeta d\sigma - \int_{\tilde\phi(\Omega)}\Delta v_2\nabla v_1\cdot\zeta dy\\
&-\int_{\partial\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\zeta\cdot\nu d\sigma + \int_{\tilde\phi(\Omega)}\frac{\partial v_1}{\partial y_i}\frac{\partial^2 v_2}{\partial y_i\partial y_r}\zeta_r dy + \int_{\tilde\phi(\Omega)}\nabla v_1\cdot\nabla v_2\mathrm{div}\zeta dy,
\end{aligned}
$$

we easily get formula (2.34). □

2.1.2 Isovolumetric perturbations

We consider the following extremum problems for the symmetric functions of the eigenvalues

$$
\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi], \qquad (2.36)
$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. Note that if $\tilde\phi\in\mathcal{A}^2_\Omega$ is a minimizer or maximizer in (2.36) then $\tilde\phi$ is a critical domain transformation for the map $\phi\mapsto\Lambda_{F,s}[\phi]$ subject to volume constraint, i.e.,

$$
\mathrm{Ker}\ \mathrm{d}|_{\phi=\tilde\phi}V \subset \mathrm{Ker}\ \mathrm{d}|_{\phi=\tilde\phi}\Lambda_{F,s}, \qquad (2.37)
$$

where V is the real valued function defined on $\mathcal{A}^2_\Omega$ which takes $\phi \in \mathcal{A}^2_\Omega$ to $V[\phi]$.

The following theorem provides a characterization of all critical domain transformations ϕ. We refer to [67] for the case of the Dirichlet and Neumann Laplacians.

Theorem 2.7. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Let $\tilde{\phi} \in \Theta_{F,\Omega}$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}] = \lambda_F[\tilde{\phi}]$ for all $j \in F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constrain if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (2.7) in $H^2_0(\tilde{\phi}(\Omega))$ (with respect to the scalar product (2.5)), and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\frac{\partial^2 v_l}{\partial \nu} \right)^2 = c, \tag{2.38}$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$.

Proof. Note that $V[\phi] = \int_\Omega |\det \nabla\phi| dx$, hence by formula (2.12) it follows that

$$\mathrm{d}|_{\phi=\tilde{\phi}} V[\psi] = \int_{\partial\tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{(-1)}) \cdot \nu d\sigma, \tag{2.39}$$

for all $\psi \in C^2_b(\Omega\,; \mathbb{R}^N)$. The proof of (2.38) follows immediately by formulas (2.8) and (2.39), and by observing that condition (2.37) is satisfied if and only if there exists $c \in \mathbb{R}$ (a Lagrange multiplier) such that

$$\mathrm{d}|_{\phi=\tilde{\phi}} \Lambda_{F,h} = c\, \mathrm{d}|_{\phi=\tilde{\phi}} V.$$

□

Then, we are led to the following

Theorem 2.8. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let B be a ball in $\mathbb{R}^N$ centered at zero, and let λ be an eigenvalue of problem (2.7) in B. Let F be the subset of $\mathbb{N}$ of all j such that the j-th eigenvalue of problem (2.7) in B coincides with λ. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ, where the orthonormality is taken with respect to the scalar product in $H^2_0(B)$. Then*

$$\sum_{j=1}^{|F|} v_j^2,\ \sum_{j=1}^{|F|} |\nabla v_j|^2,\ \sum_{j=1}^{|F|} |\Delta v_j|^2,\ \sum_{j=1}^{|F|} |D^2 v_j|^2$$

are radial functions.

Proof. Let $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in $\mathbb{R}^N$. Since the Laplace operator is invariant under rotations, then $v_k \circ A$, where $A \in O_N(\mathbb{R})$, is still an eigenfunction with eigenvalue λ; moreover, $\{v_j \circ A : j = 1, \dots, |F|\}$ is another orthonormal basis for the eigenspace associate with λ. Since both $\{v_j : j = 1, \dots, |F|\}$ and $\{v_j \circ A : j = 1, \dots, |F|\}$ are orthonormal bases, then there exists $R[A] \in O_N(\mathbb{R})$ with matrix $(R_{ij}[A])_{i,j=1,\dots,|F|}$ such that

$$v_j = \sum_{l=1}^{|F|} R_{jl}[A] v_l \circ A. \tag{2.40}$$

This implies that

$$\sum_{j=1}^{|F|} v_j^2 = \sum_{j=1}^{|F|} (v_j \circ A)^2,$$

from which we get that $\sum_{j=1}^{|F|} v_j^2$ is radial. Moreover, using standard calculus and (2.40), we get

$$\sum_{j=1}^{|F|} |\nabla v_j|^2 = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] \left(\nabla v_{l_1} \circ A\right) \cdot \left(\nabla v_{l_2} \circ A\right) = \sum_{l=1}^{|F|} |\nabla v_l \circ A|^2.$$

Similarly,

$$\sum_{j=1}^{|F|} |\Delta v_j|^2 = \sum_{j=1}^{|F|} |\Delta v_j \circ A|^2.$$

On the other hand,

$$\begin{aligned} &D^2 v_j \cdot D^2 v_j \\ &\quad = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot A \cdot A^t \cdot (D^2 v_{l_2} \circ A) \cdot A \\ &\qquad\qquad = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot (D^2 v_{l_2} \circ A) \cdot A, \end{aligned}$$

therefore

$$|D^2 v_j|^2 = \operatorname{tr}(D^2 v_j \cdot D^2 v_j) = \sum_{l_1,l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] (D^2 v_{l_1} \circ A) : (D^2 v_{l_2} \circ A),$$

from which we get

$$\sum_{j=1}^{|F|} |D^2 v_j|^2 = \sum_{j=1}^{|F|} |D^2 v_j \circ A|^2.$$

□

Remark 2.9. *We observe that in the proof of Theorem 2.8 we have never used the fact that the operator P is acting on $H_0^2(B)$, but only its rotation invariance. In fact, the same arguments allow to prove similar results also for problems (2.41), (2.46) and (2.62).*

Observing that

$$|D^2 v_l|^2 = (\Delta v_l)^2 = \left(\frac{\partial^2 v_l}{\partial \nu^2}\right)^2$$

on $\partial\tilde{\phi}(\Omega)$ for any l, we get the following

Corollary 2.10. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi} \in \mathcal{A}_\Omega^2$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (2.7) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ with volume constraint, for all $s = 1, \dots, |F|$.*

2.2 Neumann boundary conditions (free plates)

Let $-\frac{1}{N-1} < \alpha < 1$ and $\tau \geq 0$. The Neumann problem for the biharmonic operator reads

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \Omega, \\ (1-\alpha)\frac{\partial^2 u}{\partial \nu^2} + \alpha \Delta u = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1-\alpha)\mathrm{div}_{\partial\Omega}(\nu \cdot D^2 u)_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.41}$$

where Ω is a bounded domain (i.e., a bounded connected open set) in $\mathbb{R}^N$ of class C^1 and ν is the outer unit normal to $\partial\Omega$. We refer to [42] for the physical derivation of problem (2.41). Note that we need Ω to be at least of class C^2 for the classical formulation to make sense, since we need the normal ν to be differentiable, as can be easily seen from the boundary conditions; however, we shall interpret problem (2.41) in the weak sense of (2.2), in the energy space $V(\Omega) = H^2(\Omega)$. Note also that, differently from Dirichlet boundary conditions, in this case the partial differential operator associated with problem (2.41) has a is a nontrivial kernel. In fact, if $\tau > 0$, it is easy to see that the kernel is one dimensional and is given by the constants. On the other hand, when $\tau = 0$ the kernel enlarges including all the coordinate functions x_i, $i = 1, \dots, N$. Since we shall use suitable projections in order to get rid of the kernel, for the sake of simplicity in this section we will consider only the case $\tau > 0$, but the same arguments allow to treat the case $\tau = 0$ as well.

We set
$$H^{2,0}(\Omega) := \left\{ u \in H^2(\Omega) : \int_\Omega u dx = 0 \right\}.$$
We consider on $H^2(\Omega)$ the bilinear form (2.5) for any $u, v \in H^2(\Omega)$. One can prove that it defines on $H^{2,0}(\Omega)$ a scalar product whose induced norm is equivalent to the standard one defined by (1.10). We shall consider the space $H^{2,0}(\Omega)$ endowed with the scalar product (2.5). We denote by π the map from $H^2(\Omega)$ to $H^{2,0}(\Omega)$ defined by
$$\pi[u] = u - \frac{\int_\Omega u dx}{|\Omega|},$$
for all $u \in H^2(\Omega)$, where $|\Omega|$ denotes the Lebesgue measure of Ω. We denote by $\pi^\sharp$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H^{2,0}(\Omega)$ defined by the equality $\pi = \pi^\sharp \circ p$, where p is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$.

We consider the operator P defined by (2.6) as a map from $H^{2,0}(\Omega)$ to its dual. Note that, thanks to the Poincaré-Wirtinger Inequality, the norm induced from the quadratic form associated with the operator P is equivalent to the standard one of $H^{2,0}(\Omega)$ (as a closed subspace of $H^2(\Omega)$), and therefore it turns out that P is a linear homeomorphism of $H^{2,0}(\Omega)$ onto its dual.

We denote by $\mathcal{J}$ the continuous embedding of $H^2(\Omega)$ into its dual, defined by
$$\mathcal{J}[u][v] := \int_\Omega uv dx, \ \forall u, v \in H^2(\Omega).$$
Note that problem (2.41) can be written in the following weak formulation
$$P[u][v] = \lambda \mathcal{J}[u][v], \ \forall v \in H^{2,0}(\Omega). \tag{2.42}$$
We define the operator $T := (\pi^\sharp)^{(-1)} \circ P^{(-1)} \circ \mathcal{J} \circ \pi^\sharp$ from $H^2(\Omega)/\mathbb{R}$ to itself. We have the following result, whose proof is analogous to that of Lemma 2.1.

Lemma 2.11. *Let* $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* C^1. *The operator* T *is a non-negative compact selfadjoint operator in the Hilbert space* $H^2(\Omega)/\mathbb{R}$. *Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation* $Tu = \mu u$ *is satisfied for some* $u \in H^{2,0}(\Omega)$, $\mu > 0$ *if and only if equation (2.2) is satisfied with* $0 \neq \lambda = \mu^{-1}$ *for any* $\varphi \in H^{2,0}(\Omega)$.

We observe that the whole spectrum of problem (2.41) is given by the non-decreasing sequence $\{\lambda_j[\Omega]\}_{j\in\mathbb{N}}$, where $\lambda_1[\Omega] = 0$ and the other eigenvalues are given by Lemma 2.11 (if $\tau = 0$, then $\lambda_1[\Omega] = \cdots = \lambda_{N+1}[\Omega] = 0$).

2.2.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (2.41) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^2_\Omega$ and study the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ.

The main result of this section is the following analogue for the biharmonic operator of the results [68, Theorems 2.2 and 2.5] concerning the Neumann Laplacian.

Theorem 2.12. *Let* $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* C^1 *and* F *be a finite set in* $\mathbb{N}$. *The set* $\mathcal{A}_{F,\Omega}$ *is open in* $C^2_b(\Omega;\mathbb{R}^N)$ *and the real-valued maps which take* $\phi \in \mathcal{A}_{F,\Omega}$ *to* $\Lambda_{F,s}[\phi]$ *are real-analytic on* $\mathcal{A}_{F,\Omega}$ *for all* $s = 1, \dots, |F|$. *Moreover, if* $\tilde{\phi} \in \Theta_{F,\Omega}$ *is such that the eigenvalues* $\lambda_j[\tilde{\phi}]$ *assume the common value* $\lambda_F[\tilde{\phi}]$ *for all* $j \in F$, *and* $\tilde{\phi}(\Omega)$ *is of class* C^4 *then the Frechét differential of the map* $\Lambda_{F,s}$ *at the point* $\tilde{\phi}$ *is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}$$

$$\int_{\partial\tilde{\phi}(\Omega)} \left(\lambda_F v_l^2 - \tau|\nabla v_l|^2 - (1-\alpha)|D^2 v_l|^2 - \alpha(\Delta v_l)^2\right)\zeta\cdot\nu d\sigma, \quad (2.43)$$

for all $\psi \in C^2_b(\Omega;\mathbb{R}^N)$, *where* $\zeta = \psi\circ\tilde{\phi}^{(-1)}$ *and* $\{v_l\}_{l\in F}$ *is an orthonormal basis in* $H^{2,0}(\tilde{\phi}(\Omega))$ *(with respect to the scalar product (2.5)) of the eigenspace associated with* $\lambda_F[\tilde{\phi}]$.

As we have done for Theorem 2.2, in order to prove Theorem 2.12 we consider equation (2.42) on $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$P[v][\psi] = \lambda \mathcal{J}[v][\psi], \quad \forall\ \psi \in H^{2,0}(\phi(\Omega)), \tag{2.44}$$

in the unknowns $v \in H^{2,0}(\phi(\Omega))$, $\lambda \in]0, \infty[$. We consider the operator P_ϕ as an operator acting from $H^{2,0}_\phi(\Omega)$ to its dual, where

$$H^{2,0}_\phi(\Omega) = \left\{ u \in H^2(\Omega) : \int_\Omega u|\mathrm{det}\nabla\phi| dx = 0 \right\}.$$

We will endow the space $H^{2,0}_\phi(\Omega)$ with the form

$$< u, v >_\phi = P_\phi[u][v],\ \forall u, v \in H^{2,0}_\phi(\Omega).$$

Moreover, we denote by π_ϕ the map from $H^2(\Omega)$ to $H^{2,0}_\phi(\Omega)$ defined by

$$\pi_\phi[u] = u - \frac{\int_\Omega u|\mathrm{det}\nabla\phi|dx}{\int_\Omega |\mathrm{det}\nabla\phi|dx},$$

and by $\pi^\sharp_\phi$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H^{2,0}_\phi(\Omega)$ defined by the equality $\pi_\phi = \pi^\sharp_\phi \circ p$. Note that the map from $H^2(\Omega)$ to $H^2(\phi(\Omega))$ which maps u to $u \circ \phi^{(-1)}$ for all $u \in H^2(\Omega)$ is a linear homeomorphism. We also recall that

$$\mathcal{J}_\phi[u][w] = \int_\Omega uw|\mathrm{det}\nabla\phi|dx,\ \forall u, w \in H^2(\Omega).$$

Hence, equation (2.42) is equivalent to

$$P_\phi[u][\varphi] = \lambda \mathcal{J}_\phi[u][\varphi], \quad \forall\ \varphi \in H^{2,0}_\phi(\Omega)$$

where $u = v \circ \phi$. It turns out that the operator T defined in Lemma 2.11 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator T_ϕ defined on $H^{2,0}_\phi(\Omega)/\mathbb{R}$ by

$$T_\phi := (\pi^\sharp_\phi)^{(-1)} \circ P^{(-1)}_\phi \circ \mathcal{J}_\phi \circ i \circ \pi^\sharp_\phi \tag{2.45}$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3.

Lemma 2.13. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T_ϕ defined in (2.45) is non-negative selfadjoint and compact on the Hilbert space $H^{2,0}_\phi(\Omega)/\mathbb{R}$. The equation (2.44) is satisfied for some $v \in H^{2,0}(\phi(\Omega))$ if and only if the equation $T_\phi u = \mu u$ is satisfied with $u = v \circ \phi$ and $\mu = \lambda^{-1}$. Moreover, the map from $\mathcal{A}^2_\Omega$ to $\mathcal{L}(H^{2,0}(\Omega)) \times \mathcal{B}_s(H^{2,0}(\Omega))$ which takes $\phi \in \mathcal{A}^2_\Omega$ to $(T_\phi, < \cdot, \cdot >_\phi)$ is real-analytic.*

Proof of Theorem 2.12. First of all, we note that by standard regularity theory (see e.g., [51, Thm. 2.20]) $v_l \in H^4(\tilde{\phi}(\Omega))$ for all $l \in F$. We observe that the proof is very similar to that of Teorem 2.2. It only remains to compute

$$
\begin{aligned}
< \mathrm{d}|_{\phi=\tilde{\phi}} T_\phi[\psi][u_l], u_l >_{\tilde{\phi}} = d|_{\phi=\tilde{\phi}} \mathcal{J}_\phi \circ \pi_\phi[\psi][u_l][\pi_{\tilde{\phi}}(u_l)] \\
- \lambda_F^{-1}[\tilde{\phi}] d|_{\phi=\tilde{\phi}} P_\phi \circ \pi_\phi[\psi][u_l][\pi_{\tilde{\phi}}(u_l)].
\end{aligned}
$$

By (2.12) we have

$$
d|_{\phi=\tilde{\phi}} \mathcal{J}_\phi \circ \pi_\phi[\psi][u_l][\pi_{\tilde{\phi}}(u_l)] = \int_{\tilde{\phi}(\Omega)} v_l^2 \mathrm{div}\zeta dy,
$$

see also (2.13). Using Lemmas 2.4, 2.5 and 2.6 we obtain

$$
\begin{aligned}
& d|_{\phi=\tilde{\phi}} P_\phi \circ \pi_\phi[\psi][u_l][\pi_{\tilde{\phi}}(u_l)] \\
&\qquad = \int_{\partial\tilde{\phi}(\Omega)} \left((1-\alpha)|D^2 v_l|^2 + \alpha(\Delta v_l)^2 + \tau|\nabla v_l|^2\right) \zeta \cdot \nu d\sigma \\
&\qquad\qquad - \lambda_F[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \zeta dy.
\end{aligned}
$$

Using formula (2.14) we get formula (2.43). □

2.2.2 Isovolumetric perturbations

As in the previous section, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$
\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi],
$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 2.14. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Let $\tilde{\phi} \in \Theta_{F,\Omega}$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}] = \lambda_F[\tilde{\phi}]$ for all $j \in F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding* *to the eigenvalue $\lambda_F[\tilde\phi]$ of problem (2.42) in $H^{2,0}(\tilde\phi(\Omega))$ (with respect to the scalar product (2.5)), and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\lambda_F v_l^2 - \tau|\nabla v_l|^2 - (1-\alpha)|D^2 v_l|^2 - \alpha(\Delta v_l)^2\right) = c,$$

almost everywhere in $\partial\tilde\phi(\Omega)$.

Using Theorem 2.8 and Remark 2.9 we easily get the following

Corollary 2.15. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $\tilde\phi \in \mathcal{A}^2_\Omega$ be such that $\tilde\phi(\Omega)$ is a ball. Let $\tilde\lambda$ be an eigenvalue of problem (2.42) in $\tilde\phi(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde\phi] = \tilde\lambda$. Then $\Lambda_{F,s}$ has a critical point at $\tilde\phi$ with volume constraint, for all $s = 1, \dots, |F|$.*

2.3 Intermediate boundary conditions (hinged plates)

Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. The intermediate boundary value problem for the biharmonic operator reads

$$\begin{cases} \Delta^2 u - \tau\Delta u = \lambda u, & \text{in } \Omega, \\ (1-\alpha)\frac{\partial^2 u}{\partial\nu^2} + \alpha\Delta u = 0, & \text{on } \partial\Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.46}$$

where Ω is a bounded open set in $\mathbb{R}^N$ of class C^1. Note that in this case, as for Dirichlet boundary conditions, the kernel is trivial, so the eigenvalues are strictly positive.

Remark 2.16. *We observe that the limiting case $\alpha = 1$ gives the so-called Navier problem*

$$\begin{cases} \Delta^2 u - \tau\Delta u = \lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

Note also that, if Ω is either of class C^2 or convex, it is possible to prove coercivity of the associated operator, hence Theorem 1.9 applies. However, if Ω is neither of class C^2 nor convex, coercivity does not hold in general (see e.g., [51] and the references therein).

We set $V(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$ endowed with the form (2.5), for any $u, v \in V(\Omega)$. We observe that, thanks to the Poincaré inequality, such a form is indeed a scalar product in $V(\Omega)$ equivalent to the standard one. Then it is easy to see that the operator P defined in (2.6), for any $u, v \in V(\Omega)$, is a linear homeomorphism from $V(\Omega)$ to its dual. We denote by $\mathcal{J}$ the continuous embedding of $V(\Omega)$ to its dual defined by

$$\mathcal{J}[u][v] = \int_\Omega uvdx, \ \forall u, v \in V(\Omega).$$

Note that problem (2.46) can be written in the following weak formulation

$$P[u][v] = \lambda \mathcal{J}[u][v], \ \forall v \in V(\Omega).$$

We define the operator $T = P^{-1} \circ \mathcal{J}$ from $V(\Omega)$ to itself. We have the following result, whose proof is analogous to that of Lemma 2.1.

Lemma 2.17. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. The operator T is a non-negative selfadjoint compact operator in the Hilbert space $V(\Omega)$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $Tu = \mu u$ is satisfied for some $u \in V(\Omega)$, $\mu > 0$ if and only if equation (2.2) is satisfied with $\lambda = \mu^{-1}$ for any $\varphi \in V(\Omega)$.*

2.3.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (2.46) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^2_\Omega$ and study the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ.

The main result of this section is the following theorem (see also Theorems 2.2 and 2.12).

Theorem 2.18. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1 and F be a finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $C^2_b(\Omega; \mathbb{R}^N)$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^4 then the Frechét differential of the map $\Lambda_{F,s}$ at the* *point $\tilde\phi$ is delivered by the formula*

$$d|_{\phi=\tilde\phi}(\Lambda_{F,s})[\psi] = \lambda_F^s[\tilde\phi]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde\phi(\Omega)}\Big((1-\alpha)|D^2v_l|^2+\alpha(\Delta v_l)^2$$
$$+2\frac{\partial v_l}{\partial\nu}\frac{\partial\Delta v_l}{\partial\nu}+2(1-\alpha)\frac{\partial v_l}{\partial\nu}\mathrm{div}_{\partial\tilde\phi(\Omega)}(\nu\cdot D^2v_l)-\tau\left(\frac{\partial v_l}{\partial\nu}\right)^2\Big)\zeta\cdot\nu d\sigma, \quad (2.47)$$

for all $\psi\in C_b^2(\Omega;\mathbb{R}^N)$, where $\zeta=\psi\circ\phi^{(-1)}$, and $\{v_l\}_{l\in F}$ is an orthonormal basis in $V(\tilde\phi(\Omega))$ of the eigenspace associated with $\lambda_F[\tilde\phi]$.

Moreover, in the case $\alpha=0,\tau=0$ we also have

$$d|_{\phi=\tilde\phi}\Lambda_{F,s}[\psi] = \lambda_F^s[\tilde\phi]\binom{|F|-1}{s-1}$$
$$\sum_{l\in F}\int_{\partial\tilde\phi(\Omega)}\left(2\Delta_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2-|D^2v_l|^2+2\frac{\partial v_l}{\partial\nu}\frac{\partial^3v_l}{\partial\nu^3}\right)\zeta\cdot\nu d\sigma, \quad (2.48)$$

for all $\psi\in C_b^2(\Omega;\mathbb{R}^N)$, where $\Delta_{\partial\tilde\phi(\Omega)}$ denotes the Laplace-Beltrami operator on $\partial\tilde\phi(\Omega)$.

Proof. Note that by standard regularity theory (see e.g., [51, Thm. 2.20]) $v_l\in H^4(\tilde\phi(\Omega))$ for all $l\in F$.

The first part of the theorem can be proved by adapting that of Theorem 2.2 and using Lemmas 2.4, 2.5 and 2.6. In order to prove formula (2.48), we have to show that, in the case $\alpha=0$, $\tau=0$ we have

$$d|_{\phi=\tilde\phi}H_\phi^2[\psi][u_1][u_2] = 2\int_{\partial\tilde\phi(\Omega)}\Delta_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\frac{\partial v_2}{\partial\nu}\right)\zeta\cdot\nu d\sigma$$
$$-\int_{\partial\tilde\phi(\Omega)}D^2v_1:D^2v_2\zeta\cdot\nu d\sigma+\int_{\partial\tilde\phi(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\frac{\partial^3v_2}{\partial\nu^3}+\frac{\partial v_2}{\partial\nu}\frac{\partial^3v_1}{\partial\nu^3}\right)\zeta\cdot\nu d\sigma$$
$$+\lambda_F[\tilde\phi]\int_{\tilde\phi(\Omega)}v_1v_2\mathrm{div}\zeta dy. \quad (2.49)$$

We recall that the eigenfunctions v_l satisfy the boundary conditions $v_l=\frac{\partial^2v_l}{\partial\nu^2}=0$ on $\partial\tilde\phi(\Omega)$, in particular $\nabla v_l=\frac{\partial v_l}{\partial\nu}\nu$ on $\partial\tilde\phi(\Omega)$, for all $l\in F$. Therefore, we can rewrite (2.19) in the following form

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}}H^2_{\phi}[\psi][u_1][u_2] &= \int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_r}\frac{\partial \Delta v_2}{\partial y_i}+\frac{\partial v_2}{\partial y_r}\frac{\partial \Delta v_1}{\partial y_i}\right)\frac{\partial \zeta_r}{\partial y_i}dy \\
&\quad + \int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_j}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\frac{\partial \zeta_r}{\partial y_i}dy \\
-2\int_{\partial\tilde{\phi}(\Omega)}&\left(\frac{\partial v_1}{\partial \nu}\frac{\partial}{\partial \nu}\nabla v_2+\frac{\partial v_2}{\partial \nu}\frac{\partial}{\partial \nu}\nabla v_1\right)\cdot\nabla\zeta_r\nu_r d\sigma-\int_{\tilde{\phi}(\Omega)}D^2v_1:D^2v_2\mathrm{div}\zeta dy \\
&\quad - \int_{\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\mathrm{div}\zeta dy. \qquad (2.50)
\end{aligned}
$$

The first summand in (2.50) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\frac{\partial \Delta v_2}{\partial \nu}+\frac{\partial v_2}{\partial \nu}\frac{\partial \Delta v_1}{\partial \nu}\right)\zeta\cdot\nu d\sigma \\
&-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial \Delta v_2}{\partial y_i}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial \Delta v_1}{\partial y_i}\right)\zeta_r dy-\int_{\tilde{\phi}(\Omega)}(\Delta^2 v_2\nabla v_1+\Delta^2 v_1\nabla v_2)\zeta dy \\
&\qquad = \int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\frac{\partial \Delta v_2}{\partial \nu}+\frac{\partial v_2}{\partial \nu}\frac{\partial \Delta v_1}{\partial \nu}\right)\zeta\cdot\nu d\sigma \\
&\quad -\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_1}{\partial y_i\partial y_r}\frac{\partial \Delta v_2}{\partial y_i}+\frac{\partial^2 v_2}{\partial y_i\partial y_r}\frac{\partial \Delta v_1}{\partial y_i}\right)\zeta_r dy+\lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}v_1v_2\mathrm{div}\zeta dy \\
&\quad = \int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial \nu}\frac{\partial \Delta v_2}{\partial \nu}+\frac{\partial v_2}{\partial \nu}\frac{\partial \Delta v_1}{\partial \nu}\right)\zeta\cdot\nu d\sigma+\lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}v_1v_2\mathrm{div}\zeta dy \\
&\qquad -\int_{\partial\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\zeta\cdot\nu d\sigma \\
&\qquad +\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2\Delta v_1}{\partial y_r\partial y_i}\frac{\partial v_2}{\partial y_i}+\frac{\partial^2\Delta v_2}{\partial y_r\partial y_i}\frac{\partial v_1}{\partial y_i}\right)\zeta_r dy \\
&\qquad +\int_{\tilde{\phi}(\Omega)}(\nabla v_1\cdot\nabla\Delta v_2+\nabla v_2\cdot\nabla\Delta v_1)\mathrm{div}\zeta dy. \qquad (2.51)
\end{aligned}
$$

The second summand in (2.50) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_j}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_j}\frac{\partial^3 v_1}{\partial y_i\partial y_j\partial y_r}\right)\nu_i\zeta_r d\sigma \\
&\qquad -\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial y_j}\frac{\partial^2\Delta v_2}{\partial y_j\partial y_r}+\frac{\partial v_2}{\partial y_j}\frac{\partial^2\Delta v_1}{\partial y_j\partial y_r}\right)\zeta_r dy
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial^2 v_2}{\partial y_i \partial y_j} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \zeta_r dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_j} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \nu_i \zeta_r d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_j} \frac{\partial^2 \Delta v_2}{\partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_j} \frac{\partial^2 \Delta v_1}{\partial y_j \partial y_r} \right) \zeta_r dy \\
&+ \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mathrm{div}\zeta dy - \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \zeta \cdot \nu d\sigma. \qquad (2.52)
\end{aligned}
$$

By combining (2.50)-(2.52), we get that

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} H^2_{\phi}[\psi][u_1][u_2] &= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial \nu} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \nu_i \nu_j \zeta_r d\sigma \\
&- 2 \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} (\nu \cdot D^2 v_2) + \frac{\partial v_2}{\partial \nu} (\nu \cdot D^2 v_1) \right) \cdot \nabla \zeta_r \nu_r d\sigma \\
&- \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \zeta \cdot \nu d\sigma + \lambda_F[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} v_1 v_2 \mathrm{div}\zeta dy. \qquad (2.53)
\end{aligned}
$$

Now we claim that

$$
\nu \cdot D^2 v_m = \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_m}{\partial \nu} \text{ on } \partial\tilde{\phi}(\Omega), \qquad (2.54)
$$

for all $m \in F$, where $\nabla_{\partial\tilde{\phi}(\Omega)}$ denotes the tangential gradient to $\partial\tilde{\phi}(\Omega)$. Here and in the sequel it is understood that the normal vector field ν is extended to a neighborhood of $\partial\tilde{\phi}(\Omega)$ as a unitary vector field. We have

$$
\nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_m}{\partial \nu} = \nabla_{\partial\tilde{\phi}(\Omega)} (\nabla v_m \cdot \nu) = \nabla(\nabla v_m \cdot \nu) - (\nabla(\nabla v_m \cdot \nu) \cdot \nu)\nu
$$

Clearly,

$$
(\nabla(\nabla v_m \cdot \nu))_j = \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_i + \frac{\partial v_m}{\partial y_i} \frac{\partial \nu_i}{\partial y_j} = \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_i + \frac{1}{2} \frac{\partial v_m}{\partial \nu} \frac{\partial (\nu_i)^2}{\partial y_j} = \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_i,
$$

on $\partial\tilde{\phi}(\Omega)$. Thus

$$
\nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_m}{\partial \nu} = \nu \cdot D^2 v_m - (\nu \cdot D^2 v_m \cdot \nu)\nu = \nu \cdot D^2 v_m - \frac{\partial^2 v_m}{\partial \nu^2} \nu = \nu \cdot D^2 v_m, \qquad (2.55)
$$

and (2.54) is proved. Now we note that

$$
\nabla(\zeta \cdot \nu) = \nu_r \nabla \zeta_r + \zeta_r \nabla \nu_r \quad \text{hence} \quad \nu_r \nabla \zeta_r = \nabla(\zeta \cdot \nu) - \nabla \nu_r \zeta_r. \qquad (2.56)
$$

By observing that $|\nu|^2 = 1$ implies that $\nu_r \nabla \nu_r = 0$, by (2.54) and (2.56) we get

$$\begin{aligned}\frac{\partial v_1}{\partial \nu}(\nu \cdot D^2 v_2) \cdot \nabla \zeta_r \nu_r &= \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla(\zeta \cdot \nu) - \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla \nu_r \zeta_r \\ &= \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla_{\partial \tilde{\phi}(\Omega)}(\zeta \cdot \nu) - \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla \nu_r (\zeta_{\nu,r} + \zeta_{\partial \tilde{\phi}(\Omega),r}) \\ &= \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla_{\partial \tilde{\phi}(\Omega)}(\zeta \cdot \nu) - \frac{\partial v_1}{\partial \nu} \nabla_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \cdot \nabla \nu_r \zeta_{\partial \tilde{\phi}(\Omega),r},\end{aligned}$$

where $\zeta = \zeta_\nu + \zeta_{\partial \tilde{\phi}(\Omega)}$, ζ_ν is the normal component of ζ and $\zeta_{\partial \tilde{\phi}(\Omega)}$ the tangential one. Hence the second integral in (2.53) equals

$$\begin{aligned}2 \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu} \right) \cdot \nabla \nu_r \zeta_{\partial \tilde{\phi}(\Omega),r} d\sigma \\ - 2 \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu} \right) \cdot \nabla_{\partial \tilde{\phi}(\Omega)}(\zeta \cdot \nu) d\sigma. \quad (2.57)\end{aligned}$$

Now we consider the first integral in (2.53), and we recall that

$$\frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_i \nu_j = 0\,, \quad \text{on} \quad \partial \tilde{\phi}(\Omega). \quad (2.58)$$

By differentiating (2.58) with respect to any tangential direction t to $\partial \tilde{\phi}(\Omega)$ we obtain

$$\frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} \nu_i \nu_j t_r + 2 \frac{\partial^2 v_m}{\partial y_i \partial y_j} \frac{\partial \nu_i}{\partial y_r} \nu_j t_r = 0,$$

hence

$$\frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} \nu_i \nu_j (\zeta \cdot t) t_r = -2 \frac{\partial^2 v_m}{\partial y_i \partial y_j} \frac{\partial \nu_i}{\partial y_r} \nu_j (\zeta \cdot t) t_r. \quad (2.59)$$

By taking in (2.59) vectors t belonging to a basis of the tangent hyperplane to $\partial \tilde{\phi}(\Omega)$ and using (2.55), we easily get

$$\begin{aligned}\frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} \nu_i \nu_j \zeta_{\partial \tilde{\phi}(\Omega),r} &= -2(\nu \cdot D^2 v_m) \cdot \nabla \nu_r \zeta_{\partial \tilde{\phi}(\Omega),r} \\ &= -2 \nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_m}{\partial \nu} \right) \cdot \nabla \nu_r \zeta_{\partial \tilde{\phi}(\Omega),r}.\end{aligned}$$

Thus

$$\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\zeta_r d\sigma = \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j(\zeta_{\nu,r}+\zeta_{\partial\tilde{\phi}(\Omega),r})d\sigma$$
$$= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\nu_r\zeta\cdot\nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\zeta_{\partial\tilde{\phi}(\Omega),r}d\sigma$$
$$= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial\nu^3}\zeta\cdot\nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_2}{\partial\nu}\cdot\nabla\nu_r\zeta_{\partial\tilde{\phi}(\Omega),r}d\sigma.$$

Hence the first integral in (2.53) is equal to

$$\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial\nu^3} + \frac{\partial v_2}{\partial\nu}\frac{\partial^3 v_1}{\partial\nu^3}\right)\zeta\cdot\nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\frac{\partial v_2}{\partial\nu}\right)\cdot\nabla\nu_r\zeta_{\partial\tilde{\phi}(\Omega),r}d\sigma. \quad (2.60)$$

Finally, by (2.57), (2.60) and by the tangential Green formula (see [46, § 5.5]), we get that the right-hand side of (2.53) equals

$$\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial\nu^3} + \frac{\partial v_2}{\partial\nu}\frac{\partial^3 v_1}{\partial\nu^3}\right)\zeta\cdot\nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} D^2v_1 : D^2v_2\zeta\cdot\nu d\sigma$$
$$- 2\int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\frac{\partial v_2}{\partial\nu}\right)\cdot\nabla_{\partial\tilde{\phi}(\Omega)}(\zeta\cdot\nu)d\sigma + \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)} v_1v_2\mathrm{div}\zeta dy$$
$$= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu}\frac{\partial^3 v_2}{\partial\nu^3} + \frac{\partial v_2}{\partial\nu}\frac{\partial^3 v_1}{\partial\nu^3}\right)\zeta\cdot\nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} D^2v_1 : D^2v_2\zeta\cdot\nu d\sigma$$
$$+ \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)} v_1v_2\mathrm{div}\zeta dy + 2\int_{\partial\tilde{\phi}(\Omega)} \Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_1}{\partial\nu}\frac{\partial v_2}{\partial\nu}\right)\zeta\cdot\nu d\sigma.$$

This proves formula (2.49). □

We observe that formula (2.47) and formula (2.48) are actually equivalent. In fact, we can get a more general result. We first observe that (2.54) is valid for any function $f \in V(\tilde{\phi}(\Omega))$. Hence, we get

$$\Delta_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu} = \mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_l),$$

and therefore

$$\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2 = 2\frac{\partial v_l}{\partial\nu}\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_l) + 2\left|\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\right|^2.$$

On the other hand,

$$\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2=\Delta_{\partial\tilde{\phi}(\Omega)}|\nabla v_l|^2=\Delta|\nabla v_l|^2-\frac{\partial^2}{\partial\nu^2}|\nabla v_l|^2-K\frac{\partial}{\partial\nu}|\nabla v_l|^2$$

$$=2\nabla v_l\cdot\nabla\Delta v_l+2|D^2v_l|^2-2\left|\nu\cdot D^2v_l\right|^2-2\nabla v_l\cdot\frac{\partial^2}{\partial\nu^2}\nabla v_l-2K\nabla v_l\cdot\frac{\partial}{\partial\nu}\nabla v_l$$

$$=2\frac{\partial v_l}{\partial\nu}\frac{\partial\Delta v_l}{\partial\nu}+2|D^2v_l|^2-2\left|\nu\cdot D^2v_l\right|^2-2\frac{\partial v_l}{\partial\nu}\frac{\partial^3v_l}{\partial\nu^3}-2K\frac{\partial v_l}{\partial\nu}\frac{\partial^2v_l}{\partial\nu^2},$$

where K denotes the mean curvature on $\partial\tilde{\phi}(\Omega)$. By observing that

$$\left|\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\right|^2=\left|\nu\cdot D^2v_l\right|^2,$$

we have finally proved the following

Theorem 2.19. *Under the same assumptions of Theorem 2.18, formula (2.47) is equivalent to the following*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi]=\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Bigg(\alpha(\Delta v_l)^2+2\alpha\frac{\partial v_l}{\partial\nu}\frac{\partial\Delta v_l}{\partial\nu}$$

$$+2(1-\alpha)\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2+2(1-\alpha)\frac{\partial v_l}{\partial\nu}\frac{\partial^3v_l}{\partial\nu^3}-2(1-\alpha)|D^2v_l|^2$$

$$+2K(1-\alpha)\frac{\partial v_l}{\partial\nu}\frac{\partial^2v_l}{\partial\nu^2}-\tau\left(\frac{\partial v_l}{\partial\nu}\right)^2\Bigg)\zeta\cdot\nu d\sigma,$$

for all $\psi\in C_b^2(\Omega;\mathbb{R}^N)$, where $\zeta=\psi\circ\phi^{(-1)}$, and $\{v_l\}_{l\in F}$ is an orthonormal basis in $V(\tilde{\phi}(\Omega))$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$.

2.3.2 Isovolumetric perturbations

As we have done in the previous sections, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\text{const}}\Lambda_{F,s}[\phi]\quad\text{or}\quad\max_{V[\phi]=\text{const}}\Lambda_{F,s}[\phi],$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 2.20. *Let* $-\frac{1}{N-1}<\alpha<1$, $\tau\geq 0$. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* C^1. *Let* F *be a non-empty finite subset of* $\mathbb{N}$. *Let* $\tilde{\phi}\in\Theta_{F,\Omega}$ *be such that* $\partial\tilde{\phi}(\Omega)\in C^4$ *and* $\lambda_j[\tilde{\phi}]=\lambda_F[\tilde{\phi}]$ *for all* $j\in F$. *For* $s=1,\dots,|F|$, *the function* $\tilde{\phi}$ *is a critical point for* $\Lambda_{F,s}$ *with volume constraint if and only if there exists an orthonormal basis* $v_1,\dots,v_{|F|}$ *of the eigenspace corresponding to the eigenvalue* $\lambda_F[\tilde{\phi}]$ *of problem (2.46) in* $V(\tilde{\phi}(\Omega))$, *and a constant* $c\in\mathbb{R}$ *such that*

$$\sum_{l=1}^{|F|}\Bigg((1-\alpha)|D^2v_l|^2+\alpha(\Delta v_l)^2+2\frac{\partial v_l}{\partial\nu}\frac{\partial\Delta v_l}{\partial\nu}+2(1-\alpha)\frac{\partial v_l}{\partial\nu}\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu\cdot D^2v_l)-\tau\left(\frac{\partial v_l}{\partial\nu}\right)^2\Bigg)=c,$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$, *or equivalently*

$$\sum_{l=1}^{|F|}\Bigg(\alpha(\Delta v_l)^2+2\alpha\frac{\partial v_l}{\partial\nu}\frac{\partial\Delta v_l}{\partial\nu}+2(1-\alpha)\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2+2(1-\alpha)\frac{\partial v_l}{\partial\nu}\frac{\partial^3v_l}{\partial\nu^3}-2(1-\alpha)|D^2v_l|^2+2K(1-\alpha)\frac{\partial v_l}{\partial\nu}\frac{\partial^2v_l}{\partial\nu^2}-\tau\left(\frac{\partial v_l}{\partial\nu}\right)^2\Bigg)=c,\quad(2.61)$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$.

Then we can prove the following

Theorem 2.21. *Let the same assumptions of Theorem 2.18 hold. If* $\tilde{\phi}(\Omega)$ *is a ball then condition (2.61) is satisfied.*

Proof. Without loss of generality, let us assume that $\tilde{\phi}(\Omega)$ is a ball B of radius R centered at zero. By Theorem 2.8 and Remark 2.9, we have that that $\sum_{l\in F}v_l^2$, $\sum_{l\in F}|\nabla v_l|^2$, $\sum_{l\in F}(\Delta v_l)^2$ and $\sum_{l\in F}|D^2v_l|^2$ are radial functions. In particular we get that $\sum_{l\in F}|\frac{\partial v_l}{\partial\nu}|^2$ is constant on ∂B, hence

$$\sum_{l\in F}\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2=0,\quad\text{on}\quad\partial B.$$

The function

$$\frac{\partial^4}{\partial r^4}\sum_{l\in F}v_l^2=\sum_{l\in F}\left(6\left(\frac{\partial^2v_l}{\partial r^2}\right)^2+8\frac{\partial v_l}{\partial r}\frac{\partial^3v_l}{\partial r^3}+2v_l\frac{\partial^4v_l}{\partial r^4}\right),$$

where r is the radial coordinate, is clearly radial, and using the fact that $\frac{\partial^2 v_l}{\partial \nu^2} = \frac{\alpha}{\alpha-1}\Delta v_l$ on ∂B for any $l \in F$, we obtain that

$$\sum_{l\in F} \frac{\partial v_l}{\partial \nu}\frac{\partial^3 v_l}{\partial \nu^3} \quad \text{is constant on} \quad \partial B.$$

Moreover, the function

$$\frac{\partial}{\partial r}\sum_{l\in F} |\nabla v_l|^2 = \sum_{l\in F} \nabla v_l \cdot \frac{\partial}{\partial r}\nabla v_l,$$

is radial, hence

$$\sum_{l\in F} \frac{\partial v_l}{\partial \nu}\frac{\partial^2 v_l}{\partial \nu^2} \quad \text{is constant on} \quad \partial B.$$

Finally, note that

$$\Delta^2 \sum_{l\in F} v_l^2 = \sum_{l\in F}\Big(2\lambda_F[\tilde{\phi}]v_l^2 + 2(\Delta v_l)^2 + 4|D^2 v_l|^2 + 6\nabla v_l \cdot \nabla \Delta v_l\Big)$$

is radial, thus

$$\sum_{l\in F} \frac{\partial v_l}{\partial \nu}\frac{\partial \Delta v_l}{\partial \nu} \quad \text{is constant on} \quad \partial B.$$

This concludes the proof. □

2.4 Steklov boundary conditions

Let $-\frac{1}{N-1} < \alpha < 1$, $\tau \geq 0$. The Steklov problem for the biharmonic operator reads

$$\begin{cases} \Delta^2 u - \tau\Delta u = 0, & \text{in } \Omega, \\ (1-\alpha)\frac{\partial^2 u}{\partial \nu^2} + \alpha \Delta u = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1-\alpha)\mathrm{div}_{\partial\Omega}(\nu \cdot D^2 u)_{\partial\Omega} = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{2.62}$$

where Ω is a bounded domain in $\mathbb{R}^N$ of class C^1 and ν is the outer unit normal to $\partial\Omega$. We refer to [31] for the physical derivation of problem (2.62). Note that, as for the Neumann problem, we need Ω to be at least of class C^2 for the classical formulation to make sense, since we need the normal ν to be differentiable, as can easily be seen from the boundary conditions; however, we shall interpret problem (2.62) in the weak sense of (2.3). Note also that the kernel is the same of the Neumann problem. In fact, if $\tau > 0$, it is easy to see that the kernel is one dimensional and is given by the constants. On the other hand, when $\tau = 0$ the kernel enlarges including all the coordinate functions $x_i, i = 1, \dots, N$. Since we shall use suitable projections in order to get rid of the kernels, for the sake of simplicity in the sequel we will consider only the case $\tau > 0$, but the same arguments allow to treat the case $\tau = 0$ as well.

We set
$$\tilde{H}^{2,0}(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} u d\sigma = 0 \right\},$$
and we endow this space with the form defined in (2.5). One can prove that the bilinear form (2.5) defines on $\tilde{H}^{2,0}(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We denote by $\tilde{\pi}$ the map of $H^2(\Omega)$ to $\tilde{H}^{2,0}(\Omega)$ defined by
$$\tilde{\pi}[u] = u - \frac{\int_{\partial\Omega} u d\sigma}{|\partial\Omega|},$$
for all $u \in \mathcal{H}^2(\Omega)$, where by $|\partial\Omega|$ we mean the $N-1$ dimensional measure of $\partial\Omega$. We denote by $\tilde{\pi}^\sharp$ the map of $H^2(\Omega)/\mathbb{R}$ onto $\tilde{H}^{2,0}(\Omega)$ defined by the equality $\tilde{\pi} = \tilde{\pi}^\sharp \circ p$, where p is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$. The operator P defined in (2.6) considered as an operator acting from $\tilde{H}^{2,0}(\Omega)$ to its dualis a linear homeomorphism.

We denote by $\tilde{\mathcal{J}}$ the continuous embedding of $H^2(\Omega)$ into its dual defined by
$$\tilde{\mathcal{J}}[u][v] := \int_{\partial\Omega} uv d\sigma, \ \forall u, v \in H^2(\Omega).$$
Note that problem (2.62) can be written in the following weak form
$$P[u][v] = \lambda \tilde{\mathcal{J}}[u][v], \ \forall v \in \tilde{H}^{2,0}(\Omega). \tag{2.63}$$
We define the operator $T := (\tilde{\pi}^\sharp)^{(-1)} \circ P^{(-1)} \circ \tilde{\mathcal{J}} \circ \tilde{\pi}^\sharp$ from $\mathcal{H}^2(\Omega)/\mathbb{R}$ to itself. We have the following result (see also Lemma 2.11).

Lemma 2.22. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T is a non-negative compact selfadjoint operator in the Hilbert space $H^2(\Omega)/\mathbb{R}$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $Tu = \mu u$ is satisfied for some $u \in \tilde{H}^{2,0}(\Omega)$, $\mu > 0$ if and only if equation (2.2) is satisfied with $0 \neq \lambda = \mu^{-1}$ for any $\varphi \in \tilde{H}^{2,0}(\Omega)$.*

We observe that, as for the Neumann problem, the whole spectrum of problem (2.62) is given by the non-decreasing sequence $\{\lambda_j[\Omega]\}_{j\in\mathbb{N}}$, where $\lambda_1[\Omega]=0$ and the other eigenvalues are given by Lemma 2.22 (if $\tau=0$, then $\lambda_1[\Omega]=\cdots=\lambda_{N+1}[\Omega]=0$). We have the following result (cf. Theorem 1.9).

Theorem 2.23. *Let $-\frac{1}{N-1}<\alpha<1$, $\tau>0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. The eigenvalues of problem (2.62) are non-negative, have finite multiplicity and can be represented as a non-decreasing divergent sequence $\lambda_j[\Omega]$, $j\in\mathbb{N}$ where each eigenvalue is repeated according to its multiplicity. Moreover,*

$$\lambda_j[\Omega]=\min_{\substack{E\subset H^2(\Omega)\\ \dim E=j}}\max_{\substack{u\in E\\ u\neq 0}} R[u],$$

for all $j\in\mathbb{N}$, where $R[u]$ is the Rayleigh quotient defined by

$$R[u]=\frac{\int_\Omega(1-\alpha)|D^2u|^2+\alpha|\Delta u|^2+\tau|\nabla u|^2dx}{\int_{\partial\Omega}|u|^2d\sigma}. \tag{2.64}$$

As can be inferred, the Steklov problem (2.62) and the Neumann problem (2.41) share several spectral properties. In fact, they are strictly related. Consider the following problem

$$\begin{cases}\Delta^2u-\tau\Delta u=\lambda\rho_\epsilon u, & \text{in } \Omega,\\ (1-\alpha)\frac{\partial^2u}{\partial\nu^2}+\alpha\Delta u=0, & \text{on } \partial\Omega,\\ \tau\frac{\partial u}{\partial\nu}-\frac{\partial\Delta u}{\partial\nu}-(1-\alpha)\mathrm{div}_{\partial\Omega}(\nu\cdot D^2u)_{\partial\Omega}=0, & \text{on } \partial\Omega,\end{cases} \tag{2.65}$$

where

$$\rho_\epsilon(x)=\begin{cases}\epsilon, & \text{if } x\in\Omega_\epsilon,\\ \frac{|\partial\Omega|-\epsilon|\Omega_\epsilon|}{|\Omega\setminus\overline{\Omega}_\epsilon|}, & \text{otherwise.}\end{cases} \tag{2.66}$$

Here ρ_ϵ plays the role of a mass density. The weak formulation of problem (2.65) reads

$$\int_\Omega(1-\alpha)D^2u:D^2v+\alpha\Delta u\Delta v+\tau\nabla u\cdot\nabla v dx=\lambda\int_\Omega uv\rho_\epsilon dx,$$

for any $v\in H^2(\Omega)$. We have the following

Theorem 2.24. *Let $-\frac{1}{N-1}<\alpha<1$, $\tau>0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let ρ_ε be defined as (2.66). Let $\lambda_j(\rho_\varepsilon)$ be the eigenvalues of problem (2.65) on Ω for all $j\in\mathbb{N}$. Let λ_j, $j\in\mathbb{N}$ denote the eigenvalues of problem (2.62). Then $\lim_{\varepsilon\to0}\lambda_j(\rho_\varepsilon)=\lambda_j$ for all $j\in\mathbb{N}$.*

Proof. For a proof, we refer to [31], where the authors discuss the case $\alpha = 0$ only (see also [32, 69, 73]). However, the argument can be easily adapted to the general case. $\square$

Roughly speaking, we may think of the Steklov problem (2.62) as a limiting Neumann problem where the mass is concentrated at the boundary.

2.4.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, and let Tr_Ω be the trace operator from $H^2(\Omega)$ to $L^2(\partial\Omega)$. It is well known that Tr_Ω is a compact map. We note that any $\phi \in \mathcal{A}^2_\Omega$ is in particular Lipschitz continuous in Ω together with its gradient, and since Ω is of class C^1, it follows that $\phi \in C^{1,1}(\bar{\Omega})$, in the sense that there exists $\varepsilon = \varepsilon(\phi)$ such that $\phi \in C^{1,1}(\Omega^\varepsilon)$. Thus $\phi(\Omega)$ is of class C^1, so a trace operator $\mathrm{Tr}_{\phi(\Omega)}$ is well defined and compact. Moreover, since the map i_ϕ from $H^2(\phi(\Omega))$ to $H^2(\Omega)$ which takes $u \in H^2(\phi(\Omega))$ to $u \circ \phi$ is a linear homeomorphism, and $i_\phi^{-1} = i_{\phi^{-1}}$, we have that $\mathrm{Tr}_{\phi(\Omega)} = i_\phi \circ \mathrm{Tr}_\Omega \circ i_{\phi^{-1}}$. Therefore, it is natural to consider problem (2.62) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^2_\Omega$ and study the dipendence of $\lambda_j[\phi(\Omega)]$ on ϕ, as we have done in the previous sections.

The main result of this section is the following

Theorem 2.25. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1 and F be a finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $C^2_b(\Omega;\mathbb{R}^N)$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1,\dots,|F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^4 then the Frechét differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}$$
$$\int_{\partial\tilde{\phi}(\Omega)}\left(\lambda_F K v_l^2 + \lambda_F\frac{\partial(v_l)^2}{\partial\nu} - \tau|\nabla v_l|^2 - (1-\alpha)|D^2 v_l|^2 - \alpha(\Delta v_l)^2\right)\zeta\cdot\nu d\sigma,$$

for all $\psi \in C^2(\Omega;\mathbb{R}^N)$, where $\zeta = \psi\circ\tilde{\phi}^{(-1)}$ and $\{v_l\}_{l\in F}$ is an orthonormal basis in $\tilde{H}^{2,0}(\tilde{\phi}(\Omega))$ (with respect to the scalar product (2.5)) of the eigenspace associated with $\lambda_F[\tilde{\phi}]$.

In order to prove Theorem 2.25 we consider equation (2.63) on $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$P[v][\psi] = \lambda\tilde{\mathcal{J}}[v][\psi], \quad \forall\ \psi \in \tilde{H}^{2,0}(\phi(\Omega)), \tag{2.67}$$

in the unknowns $v \in \tilde{H}^{2,0}(\phi(\Omega))$, $\lambda \in]0,\infty[$. We consider P_ϕ as an operator acting from $\tilde{H}^{2,0}_\phi(\Omega)$ to its dual, where

$$\tilde{H}^{2,0}_\phi(\Omega) = \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} u|\nu(\nabla\phi)^{-1}||\mathrm{det}\nabla\phi| d\sigma = 0 \right\}.$$

We will endow the space $\tilde{H}^{2,0}_\phi(\Omega)$ with the form

$$< u, v >_\phi = P_\phi[u][v], \ \forall u, v \in \tilde{H}^{2,0}_\phi(\Omega).$$

Moreover, we denote by $\tilde{\pi}_\phi$ the map from $H^2(\Omega)$ to $\tilde{H}^{2,0}_\phi(\Omega)$ defined by

$$\tilde{\pi}_\phi[u] = u - \frac{\int_{\partial\Omega} u|\nu(\nabla\phi)^{-1}||\mathrm{det}\nabla\phi| d\sigma}{\int_{\partial\Omega} |\nu(\nabla\phi)^{-1}||\mathrm{det}\nabla\phi| d\sigma},$$

and by $\tilde{\pi}^\sharp_\phi$ the map from $H^2(\Omega)/\mathbb{R}$ onto $\tilde{H}^{2,0}_\phi(\Omega)$ defined by the equality $\tilde{\pi}_\phi = \tilde{\pi}^\sharp_\phi \circ p$. We also recall that

$$\tilde{\mathcal{J}}_\phi[u][w] = \int_{\partial\Omega} uw|\nu(\nabla\phi)^{-1}||\mathrm{det}\nabla\phi| d\sigma, \ \forall u, w \in H^2(\Omega).$$

Hence, equation (2.67) is equivalent to

$$P_\phi[u][\varphi] = \lambda\tilde{\mathcal{J}}_\phi[u][\varphi], \quad \forall\ \varphi \in \tilde{H}^{2,0}_\phi(\Omega)$$

where $u = v \circ \phi$. It turns out that the operator T defined in Lemma 2.22 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator T_ϕ defined on $\tilde{H}^{2,0}_\phi(\Omega)/\mathbb{R}$ by

$$T_\phi := (\tilde{\pi}^\sharp_\phi)^{(-1)} \circ P^{(-1)}_\phi \circ \tilde{\mathcal{J}}_\phi \circ \tilde{\pi}^\sharp_\phi \tag{2.68}$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3.

Lemma 2.26. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. The operator T_ϕ defined in (2.68) is non-negative selfadjoint and compact on the Hilbert space $H^2(\Omega)/\mathbb{R}$. The equation (2.67) is satisfied for some $v \in \tilde{H}^{2,0}(\phi(\Omega))$ if and only if the equation $T_\phi u = \mu u$ is satisfied with $u = v \circ \phi$ and $\mu = \lambda^{-1}$. Moreover, the map from $\mathcal{A}^2_\Omega$ to $\mathcal{L}(\tilde{H}^{2,0}(\Omega)) \times \mathcal{B}_s(\tilde{H}^{2,0}(\Omega))$ which takes $\phi \in \mathcal{A}^2_\Omega$ to $(T_\phi, < \cdot, \cdot >_\phi)$ is real-analytic.*

Proof of Theorem 2.25. First of all, we note that by standard regularity theory (see e.g., [51, Thm. 2.20]) $v_l \in H^4(\tilde{\phi}(\Omega))$ for all $l \in F$. We observe that the proof is very similar to that of Teorem 2.12. It only remains to compute

$$< \mathrm{d}|_{\phi=\tilde{\phi}} T_\phi[\psi][u_l], u_l >_{\tilde{\phi}} = d|_{\phi=\tilde{\phi}} \tilde{\mathcal{J}}_\phi \circ \tilde{\pi}_\phi[\psi][u_l][\tilde{\pi}_{\tilde{\phi}}(u_l)] \\ - \lambda_F^{-1}[\tilde{\phi}] d|_{\phi=\tilde{\phi}} P_\phi \circ \tilde{\pi}_\phi[\psi][u_l][\tilde{\pi}_{\tilde{\phi}}(u_l)].$$

By [62, Lemma 3.3] we have

$$d|_{\phi=\tilde{\phi}} \tilde{\mathcal{J}}_\phi \circ \tilde{\pi}_\phi[\psi][u_l][\tilde{\pi}_{\tilde{\phi}}(u_l)] \\ = \int_{\partial\tilde{\phi}(\Omega)} \left(K v_l^2 + \frac{\partial(v_l^2)}{\partial\nu} \right) \zeta \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \zeta d\sigma.$$

Using Lemmas 2.4, 2.5 and 2.6 we obtain

$$d|_{\phi=\tilde{\phi}} P_\phi \circ \tilde{\pi}_\phi[\psi][u_l][\tilde{\pi}_{\tilde{\phi}}(u_l)] \\ = \int_{\partial\tilde{\phi}(\Omega)} \left((1-\alpha)|D^2 v_l|^2 + \alpha(\Delta v_l)^2 + \tau|\nabla v_l|^2 \right) \zeta \cdot \nu d\sigma \\ - \lambda_F[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \mu d\sigma.$$

This concludes the proof. □

2.4.2 Isovolumetric perturbations

As we have done in the previous sections, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi],$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 2.27. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Let $\tilde{\phi} \in \Theta_{F,\Omega}$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}] = \lambda_F[\tilde{\phi}]$ for all $j \in F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding* *to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (2.63) in $\tilde{H}^{2,0}(\tilde{\phi}(\Omega))$ (with respect to the scalar product (2.5)), and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\lambda_F K v_l^2 + \lambda_F \frac{\partial (v_l^2)}{\partial \nu} - \tau |\nabla v_l|^2 - (1-\alpha)|D^2 v_l|^2 - \alpha(\Delta v_l)^2 \right) = c,$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$.

Using Theorem 2.8 and Remark 2.9, we easily get the following

Corollary 2.28. *Let $-\frac{1}{N-1} < \alpha < 1$, $\tau > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi} \in \mathcal{A}^2_{\Omega}$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (2.63) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ with volume constraint, for all $s = 1, \dots, |F|$.*

2.4.3 An isoperimetric inequality for the fundamental tone

In the previous section we have shown that the ball is a critical point for all the elementary symmetric functions of the eigenvalues of problem (2.62). In this section we prove that, if $\alpha = 0$ and $\tau > 0$, the ball is actually a maximizer for the fundamental tone, that is

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*), \tag{2.69}$$

where Ω^* is a ball such that $|\Omega| = |\Omega^*|$. We recall that inequality (2.69) has been proved for the Neumann problem (2.41) in [42], with $\alpha = 0$ and $\tau > 0$.

In the rest of this section we shall consider only the case $\alpha = 0$, and we shall think of τ as a fixed positive constant.

Eigenvalues and eigenfunctions on the ball

We characterize the eigenvalues and the eigenfunctions of (2.62) when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at the origin. It is convenient to use spherical coordinates (r, θ), where $\theta = (\theta_1, ..., \theta_{N-1})$. The corresponding trasformation of coordinates is

$$\begin{aligned}
x_1 &= r\cos(\theta_1),\\
x_2 &= r\sin(\theta_1)\cos(\theta_2),\\
&\vdots\\
x_{N-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\cos(\theta_{N-1}),\\
x_N &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\sin(\theta_{N-1}),
\end{aligned}$$

with $\theta_1, ..., \theta_{N-2} \in [0,\pi]$, $\theta_{N-1} \in [0, 2\pi[$ (here it is understood that $\theta_1 \in [0, 2\pi[$ if $N = 2$).

The boundary conditions of (2.62) in this case are written as

$$\begin{cases} \frac{\partial^2 u}{\partial r^2}_{|r=1} = 0, \\ \tau \frac{\partial u}{\partial r} - \frac{1}{r^2}\Delta_S\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) - \frac{\partial \Delta u}{\partial r}_{|r=1} = \lambda u_{|r=1}, \end{cases}$$

where Δ_S is the angular part of the Laplacian. It is well known that the eigenfunctions can be written as a product of a radial part and an angular part (see [42] for details). The radial part is given in terms of ultraspherical modified Bessel functions and powertype functions. The ultraspherical modified Bessel functions $i_l(z)$ and $k_l(z)$ are defined as follows

$$i_l(z) = z^{1-\frac{N}{2}} I_{\frac{N}{2}-1+l}(z),$$
$$k_l(z) = z^{1-\frac{N}{2}} K_{\frac{N}{2}-1+l}(z),$$

for $l \in \mathbb{N}$, where $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of first and second kind respectively. We recall that $i_l(z)$ and all its derivatives are positive on $]0, +\infty[$ (see [1, §9.6]). We recall that the Bessel functions J_ν and N_ν solve the Bessel equation

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0,$$

while the modified Bessel functions I_ν and K_ν solve the modified Bessel equation

$$z^2 y''(z) + z y'(z) + (z^2 + \nu^2) y(z) = 0.$$

We have the following

Theorem 2.29. *Let Ω be the unit ball in $\mathbb{R}^N$ centered at the origin. Any eigenfunction u_l of problem (2.62) is of the form $u_l(r,\theta) = R_l(r) Y_l(\theta)$ where $Y_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}$ and*

$$R_l(r) = A_l r^l + B_l i_l(\sqrt{\tau} r),$$

where A_l and B_l are suitable constants such that

$$B_l = \frac{l(1-l)}{\tau i_l''(\sqrt{\tau})} A_l.$$

Moreover, the eigenvalue $\lambda_{(l)}$ associated with the eigenfunction u_l is delivered by formula

$$\begin{aligned}\lambda_{(l)} = l\Big((1-l)l i_l(\sqrt{\tau}) + \tau i_l''(\sqrt{\tau})\Big)^{-1}\Big[&3(l-1)l(l+N-2)i_l(\sqrt{\tau})\\ &-(l-1)\sqrt{\tau}\big(N-1+2Nl+2l(l-2)l+\tau\big)i_l'(\sqrt{\tau})\\ &+\tau\big((l-1)(l+2N-3)+\tau\big)i_l''(\sqrt{\tau})\\ &+(l-1)\tau\sqrt{\tau}i_l'''(\sqrt{\tau})\Big],\end{aligned} \tag{2.70}$$

for any $l \in \mathbb{N}$.

Proof. Solutions of problem (2.62) in the unit ball are smooth (see e.g., [51, Theorem 2.20]). We consider two cases: $\Delta u = 0$ and $\Delta u \neq 0$.

Let u be such that $\Delta u = 0$. The Laplacian can be written in spherical coordinates as

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_S.$$

Separating variables so that $u = R(r)Y(\theta)$ we obtain the equations

$$R'' + \frac{N-1}{r}R' - \frac{l(l+N-2)}{r^2}R = 0 \tag{2.71}$$

and

$$\Delta_S Y = -l(l+N-2)Y. \tag{2.72}$$

The solutions of equation (2.71) are given by $R(r) = ar^l + br^{2-N-l}$ if $l > 0, N \geq 2$, and by $R(r) = a + b\log(r)$ if $l = 0, N = 2$. Since the solutions cannot blow up at $r = 0$, we must impose $b = 0$. The solutions of the second equation are the spherical harmonics of order l. Then u can be written as

$$u(r,\theta) = a_l r^l Y_l(\theta)$$

for some $l \in \mathbb{N}$.

Let us consider now the case $\Delta u \neq 0$. We set $v = \Delta u$ and solve the equation

$$\Delta v = \tau v.$$

By writing $v = R(r)Y(\theta)$ we obtain that R solves the equation

$$R'' + \frac{N-1}{r}R' - \frac{l(l+N-2)}{r^2}R = \tau R, \tag{2.73}$$

while Y solves equation (2.72). Equation (2.73) is the modified ultraspherical Bessel equation that is solved by the modified ultraspherical Bessel functions of first and second kind $i_l(\sqrt{\tau}r)$ and $k_l(\sqrt{\tau}r)$. Since the solutions cannot blow up at $r=0$, we must choose only $i_l(z)$ since $k_l(z)$ has a singularity at $z=0$. Then

$$v(r,\theta)=b_{l_1}i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta)$$

for some $l_1\in\mathbb{N}$. Now $v=\frac{\Delta v}{\tau}=\Delta u$, that is $\Delta(v/\tau-u)=0$. This means that

$$u(r,\theta)=\frac{b_{l_1}}{\tau}i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta)-c_{l_2}r^{l_2}Y_{l_2}(\theta) \tag{2.74}$$

for some $l_2\in\mathbb{N}$.

Now we prove that the indexes l_1 and l_2 in (2.74) must coincide. This can be shown by imposing the boundary condition $\frac{\partial^2 u}{\partial r^2}_{|r=1}=0$, which can be written as

$$b_{l_1}i''_{l_1}(\sqrt{\tau})Y_{l_1}(\theta)-c_{l_2}l_2(l_2-1)Y_{l_2}(\theta)=0. \tag{2.75}$$

If the two indexes do not agree, the coefficients of $Y_{l_i}, i=1,2$ must vanish since spherical harmonics with different indexes are linearly independent on $\partial\Omega$. Since $i''_{l_1}(\sqrt{\tau})>0$, this implies $b_{l_1}=0$ and therefore $l_2=0$ or $l_2=1$. Then we have

$$u_l(r,\theta)=\Big(A_lr^l+B_li_l(\sqrt{\tau}r)\Big)Y_l(\theta), \tag{2.76}$$

with suitable constants A_l, B_l. In the case $l\neq 0,1$, again from the boundary condition (2.75) we have

$$l(l-1)A_l+\tau i''_l(\sqrt{\tau})B_l=0, \tag{2.77}$$

then $B_l=\frac{l(1-l)}{\tau i''_l(\sqrt{\tau})}A_l$. Note that the formula holds also in the case $l=0,1$ since these indexes correspond to $B_l=0$.

Finally, let us consider the boundary condition

$$\tau\frac{\partial u}{\partial r}-\frac{1}{r^2}\Delta_S\Big(\frac{\partial u}{\partial r}-\frac{u}{r}\Big)-\frac{\partial\Delta u}{\partial r}_{|r=1}=\lambda u_{|r=1}. \tag{2.78}$$

Using in (2.78) the representation of u_l provided by formula (2.76), we get

$$\begin{aligned}\Bigg[\Big(-\lambda+l\big((l-1)(l+N-2)+\tau\big)\Big)A_l+\Big(-\big(3l(l+N-2)+\lambda\big)i_l(\sqrt{\tau})\\ -\sqrt{\tau}\big((N-1-2Nl-2(l-2)l-\tau)i'_l(\sqrt{\tau})+(N-1)\sqrt{\tau}i''_l(\sqrt{\tau})\\ +\tau i'''_l(\sqrt{\tau})\big)\Big)B_l\Bigg]Y_l(\theta)=\lambda\left(A_l+B_li_l(\sqrt{\tau})\right)Y_l(\theta).\end{aligned}$$

Using equality (2.77) we get that u_l given by (2.76) is an eigenfunction of (2.62) on the unit ball. Moreover, as a consequence, we also get formula (2.70) for the associated eigenvalue. This concludes the proof. $\square$

We are ready to state and prove the following theorem concerning the first positive eigenvalue.

Theorem 2.30. *Let Ω be the unit ball in $\mathbb{R}^N$ centered at the origin. The first positive eigenvalue of (2.62) is $\lambda_2 = \lambda_{(1)} = \tau$. The corresponding eigenspace is generated by the coordinate functions $\{x_1, x_2, \dots, x_N\}$.*

Proof. By Theorem 2.29, $0 = \lambda_{(0)} < \tau = \lambda_{(1)}$. We consider formula (2.70) with $l = 2$. We have

$$\lambda_{(2)} = 2\Big(\tau i_2''(\sqrt{\tau}) - 2i_2(\sqrt{\tau})\Big)^{-1}\Big[6Ni_2(\sqrt{\tau}) - \sqrt{\tau}(5N-1+\tau)i_2'(\sqrt{\tau}) \\ + \tau(2N-1+\tau)i_2''(\sqrt{\tau}) + \tau\sqrt{\tau}i_2'''(\sqrt{\tau})\Big]. \quad (2.79)$$

In order to prove that $\lambda_{(2)} > \tau$, we use some well known recurrence relations between ultraspherical Bessel functions (see [1, p. 376]),

$$\begin{aligned} i_l'(\sqrt{\tau}) &= \frac{l}{\sqrt{\tau}} i_l(\sqrt{\tau}) + i_{l+1}(\sqrt{\tau}), \\ i_l''(\sqrt{\tau}) &= \frac{l(l-1)}{\tau} i_l(\sqrt{\tau}) + \frac{l+2}{\tau} i_{l+1}(\sqrt{\tau}) + i_{l+2}(\sqrt{\tau}), \\ i_l'''(\sqrt{\tau}) &= \frac{l(l-1)(l-2)}{\tau\sqrt{\tau}} i_l(\sqrt{\tau}) + \frac{l(2l+1)}{\tau} i_{l+1}(\sqrt{\tau}) + \frac{2(l+2)}{\sqrt{\tau}} i_{l+2}(\sqrt{\tau}) \\ &\quad + i_{l+3}(\sqrt{\tau}). \end{aligned}$$

Using these relations in (2.79), we obtain an equivalent formula for $\lambda_{(2)}$,

$$\lambda_{(2)} = 2\Big(5\sqrt{\tau}i_3(\sqrt{\tau}) + \tau i_4(\sqrt{\tau})\Big)^{-1}\Big[(10N-2+2\tau)i_2(\sqrt{\tau}) \\ + \big(2-10N+(7+10N)\sqrt{\tau}-2\tau+5\tau\sqrt{\tau}\big)i_3(\sqrt{\tau}) \\ + \tau(8+2N+\tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau})\Big].$$

By well known properties of the functions I_ν (see [1, §9]), it follows that $i_l \geq i_{l+1}$ for all $l \in \mathbb{N}$. This implies

$$(10N-2+2\tau)i_2(\sqrt{\tau}) + \big(2-10N+(7+10N)\sqrt{\tau}-2\tau+5\tau\sqrt{\tau}\big)i_3(\sqrt{\tau}) \\ + \tau(8+2N+\tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau}) \geq \Big(5\tau\sqrt{\tau}i_3(\sqrt{\tau}) + \tau^2 i_4(\sqrt{\tau})\Big),$$

then

$$\lambda_{(2)} \geq 2\tau > \tau = \lambda_{(1)}.$$

Now it remains to prove that $\lambda_{(l)}$ is an increasing function of l for $l \geq 2$. We adapt the method used in [42, Theorem 3]. We claim that for any smooth radial function $R(r)$ the Rayleigh quotient

$$\mathcal{Q}(R(r)Y_l(\theta)) = \frac{\int_B |D^2(R(r)Y_l(\theta))|^2 + \tau|\nabla(R(r)Y_l(\theta))|^2 dx}{\int_{\partial B} R(r)^2 Y_l(\theta)^2 d\sigma}$$

is an increasing function of l for $l \geq 2$. We consider the spherical harmonics to be normalized with respect to the $L^2(\partial B)$ scalar product. In particular, we have that the denominator $D[R(r)Y_l(\theta)]$ of $\mathcal{Q}(R(r)Y_l(\theta))$ is $R^2(1)$. For the numerator $N[R(r)Y_l(\theta)]$ of the Rayleigh quotient we have

$$\begin{aligned} N&[R(r)Y_l(\theta)] \\ &= \int_0^1 \left(\frac{2k}{r^4}\left(rR' - \frac{3}{2}R\right)^2 + \frac{k(k-N-1/2)}{r^4}R^2 + \tau\frac{kR^2}{r^2}\right) r^{N-1} dr \\ &\qquad + \int_0^1 \left((R''^2) + \frac{N-1}{r^2}(R')^2 + \tau(R')^2\right) r^{N-1} dr, \end{aligned}$$

where $k = l(l+N-2)$. The above expression is increasing in k for $k \geq N + 1/2$ and since k is an increasing function of l, we easily get that each term involving l is an increasing function of l for $l \geq 2$. Thus the claim above is proved.

For each $l \in \mathbb{N}$,

$$\lambda_{(l)} = \inf \mathcal{Q}(u) = \inf \frac{\int_B |D^2 u|^2 + \tau|\nabla u|^2 dx}{\int_{\partial B} u^2 d\sigma}, \tag{2.80}$$

where the infimum is taken among all functions u that are $L^2(\partial B)$-orthogonal to the first $m-1$ eigenfunctions u_i and $m \in \mathbb{N}$ is such that $\lambda_{(l)} = \lambda_m$ is the $m-$th eigenvalue of problem (2.62). The eigenfunctions u_l are of the form $u_l = R_l(r)Y_l(\theta)$, and u_l realizes the infimum in (2.80). Then

$$\lambda_{(l)} = \mathcal{Q}(R_l(r)Y_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)Y_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)Y_{l+1}(\theta)) = \lambda_{(l+1)},$$

where the first inequality follows from the fact that $R_{l+1}(r)Y_l(\theta)$ is also orthogonal with respect to the $L^2(\partial B)$ scalar product to the first $m-1$ eigenfunctions $R_i(r)Y_i(\theta)$ for $i = 1, ...m-1$, and then it is a suitable trial function in (2.80). The second inequality follows from the fact that the quotient $\mathcal{Q}(R(r)Y_l(\theta))$ is an increasing function of l, for $l \geq 2$. This concludes the proof. $\square$

The isoperimetric inequality

In this section we prove the isoperimetric inequality (2.69). Throughout this section Ω is a bounded domain of class C^1.

We recall the following lemma from [19].

Lemma 2.31. *Let Ω be an open set and let f be a continuous, non-negative, non-decreasing function defined on $[0,+\infty)$. Let us assume that the function*

$$t \mapsto \big(f(t^{1/N}) - f(0)\big)t^{1-(1/N)} \tag{2.81}$$

is convex. Then

$$\int_{\partial\Omega} f(|x|)d\sigma \geq \int_{\partial\Omega^*} f(|x|)d\sigma,$$

where Ω^* is the ball centered at zero with the same measure as Ω.

We observe that (2.81) is satisfied for functions of the type t^p, with $p \geq 1$. We need a characterization of the inverse of the eigenvalues of (2.62).

Lemma 2.32. *Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$. Then the eigenvalues of problem (2.62) on Ω satisfy,*

$$\sum_{l=k+1}^{k+N} \frac{1}{\lambda_l(\Omega)} = \max\left\{\sum_{l=k+1}^{k+N} \int_{\partial\Omega} v_l^2 d\sigma\right\}, \tag{2.82}$$

where the maximum is taken over the families $\{v_l\}_{l=k+1}^{k+N}$ in $H^2(\Omega)$ satisfying $\int_\Omega D^2 v_i : D^2 v_j + \tau\nabla v_i \cdot \nabla v_j dx = \delta_{ij}$, and $\int_{\partial\Omega} v_i u_j d\sigma = 0$ for all $i = k+1, ..., k+N$ and $j = 1, 2, ..., k$, where $u_1, u_2, ..., u_k$ are the first k eigenfunctions of problem (2.62).

For a proof of this result we refer to [57] (see also [13]). Now we are ready to prove the isoperimetric inequality.

Theorem 2.33. *Among all bounded domains of class C^1 with fixed measure, the ball maximizes the first non-negative eigenvalue of problem (2.62), that is $\lambda_2(\Omega) \leq \lambda_2(\Omega^*)$, where $\lambda_2(\Omega)$ has been defined in (2.64) and Ω^* is a ball with the same measure as Ω.*

Proof. Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$ with the same measure as the unit ball B. We consider in (2.82) $l = 2, ..., N+1$ and $v_l = (\tau|\Omega|)^{-1/2}x_l$ as trial functions. The trial functions must have zero integral mean over $\partial\Omega$. This can be obtained by a change of coordinates $x = y - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} y d\sigma$.

Moreover, the functions v_l satisfy the normalization condition of Lemma 2.32. Then v_l are suitable trial functions to test in formula (2.82). We get

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} \geq \frac{1}{\tau|\Omega|} \int_{\partial\Omega} |x|^2 d\sigma.$$

We use Lemma 2.31 with $f(t) = t^2$. This yields

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} \geq \frac{1}{\tau|\Omega|} \int_{\partial B} |x|^2 d\sigma = \frac{N|B|}{\tau|B|} = \frac{N}{\tau} = \sum_{l=2}^{N+1} \frac{1}{\lambda_l(B)}.$$

This concludes the proof in the case Ω has the same measure as the unit ball. The proof for general finite values of $|\Omega|$ relies on the well known scaling properties of the eigenvalues. Namely, for all $\alpha > 0$, if we write an eigenvalue of problem (2.62) as $\lambda(\tau, \Omega)$, we have

$$\lambda(\tau, \Omega) = \alpha^4 \lambda(\alpha^{-2}\tau, \alpha\Omega).$$

This is easy to prove by looking at the variational characterization of $\lambda(\tau, \Omega)$ and $\lambda(\alpha^{-2}\tau, \alpha\Omega)$ and performing a change of variable $x \mapsto x/\alpha$ in the Rayleigh quotient (2.64). This last observation concludes the proof of the theorem. $\square$

Chapter 3

The Dirichlet problem for the polyharmonic operators

In this chapter we consider the following eigenvalue problem

$$\mathcal{P}_{nm}: \begin{cases} (-\Delta)^n u = \lambda(-\Delta)^m u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{n-1} u}{\partial \nu^{n-1}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $n, m \in \mathbb{N}_0$ with $0 \leq m < n$, Ω is a bounded open set in $\mathbb{R}^N$ of class C^1 and ν denotes the outer unit normal to $\partial\Omega$. The case $m = 0$ corresponds to the well known eigenvalue problem for the polyharmonic operator $(-\Delta)^n$ subject to Dirichlet boundary conditions, while the case $m > 0$ represents a buckling-type problem. These cases include important problems in linear elasticity. For instance, for $N = 2$, $\mathcal{P}_{10}$ arises in the study of a vibrating membrane stretched in a fixed frame, $\mathcal{P}_{20}$ corresponds to the case of a vibrating clamped plate and $\mathcal{P}_{21}$ is related to plate buckling.

We consider the weak formulation of problem (3.1). To do so, for any $m \in \mathbb{N}_0$ with $0 \leq m \leq n$, we consider the polyharmonic operator Δ^m as the operator from $H^n_0(\Omega)$ to its dual which takes any $u \in H^n_0(\Omega)$ to the functional $\Delta^m[u]$ defined by

$$\Delta^{2s}[u][\varphi] = \int_\Omega \Delta^s u \Delta^s \varphi dx, \quad \forall \varphi \in H^n_0(\Omega),$$

if $m = 2s$ and

$$\Delta^{2s+1}[u][\varphi] = -\int_\Omega \nabla(\Delta^s u) \cdot \nabla(\Delta^s \varphi) dx, \quad \forall \varphi \in H^n_0(\Omega),$$

if $m = 2s + 1$, where $s \in \mathbb{N}_0$. Thus, the weak formulation of the classic problem (3.1) reads

$$(-\Delta)^n[u][\varphi] = \lambda(-\Delta)^m[u][\varphi], \quad \forall \varphi \in H_0^n(\Omega). \tag{3.2}$$

By the Poincaré inequality, it follows that the quadratic form defined by $(-\Delta)^n[u][u]$ for all $u \in H_0^n(\Omega)$ is coercive in $H_0^n(\Omega)$, hence the operator $(-\Delta)^n$ is a linear homeomorphism from $H_0^n(\Omega)$ onto $(H_0^n(\Omega))'$. Thus equation (3.2) is equivalent to the equation $(-\Delta)^{-n} \circ (-\Delta)^m[u] = \lambda^{-1}u$, where $(-\Delta)^{-n}$ denotes the inverse of $(-\Delta)^n$. It is convenient to endow the space $H_0^n(\Omega)$ with the scalar product defined by

$$< u_1, u_2 >_n = (-\Delta)^n[u_1][u_2], \tag{3.3}$$

for all $u_1, u_2 \in H_0^n(\Omega)$. The norm induced by this scalar product is equivalent to the standard norm (1.10). In this chapter, unless otherwise indicated, we shall think of $H_0^n(\Omega)$ as a Hilbert space equipped with the scalar product (3.3). This allows to give a straightforward proof of the following

Lemma 3.1. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1 and $m, n \in \mathbb{N}_0$ with $0 \le m < n$. The operator $S_\Omega \equiv (-\Delta)^{-n} \circ (-\Delta)^m$ is a non-negative selfadjoint compact operator in the Hilbert space $H_0^n(\Omega)$. The spectrum of S_Ω is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $S_\Omega u = \mu u$ is satisfied for some $u \in H_0^n(\Omega)$, $\mu > 0$ if and only if equation (3.2) is satisfied with $\lambda = \mu^{-1}$.*

Proof. The equality $< S_\Omega u_1, u_2 >_n = (-\Delta)^m[u_1][u_2]$, for all $u_1, u_2 \in H_0^n(\Omega)$ and the symmetry of the operator $(-\Delta)^m$ implies that S_Ω is a selfadjoint operator. Since Ω is bounded and $m < n$, the space $H_0^n(\Omega)$ is compactly embedded into $H_0^m(\Omega)$. This implies that the operator $(-\Delta)^m$ is a compact operator from the space $H_0^n(\Omega)$ to its dual. The rest of the proof is trivial. □

By Lemma 3.1 and standard spectral theory we deduce the following variational characterization of the eigenvalues of problem (3.1) (see also Theorems 1.9, 2.64).

Corollary 3.2. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1 and $m, n \in \mathbb{N}$ with $0 \le m < n$. The eigenvalues of problem (3.2) are positive, have finite multiplicity and can be represented as a non-decreasing divergent sequence*

$\lambda_j[\Omega]$, $j \in \mathbb{N}$ where each eigenvalue is repeated according to its multiplicity. Moreover,

$$\lambda_j[\Omega] \equiv \lambda_j^{n,m}[\Omega] = \min_{\substack{E\subset H_0^n(\Omega)\\ \dim E=j}} \max_{\substack{u\in E\\ u\neq 0}} R_{nm}[u],$$

for all $j \in \mathbb{N}$, where $R_{nm}[u]$ is the Rayleigh quotient defined by

$$R_{nm}[u] = \begin{cases} \frac{\int_\Omega |\Delta^r u|^2 dx}{\int_\Omega |\Delta^s u|^2 dx}, & \text{if } n = 2r, \quad m = 2s, \\ \frac{\int_\Omega |\Delta^r u|^2 dx}{\int_\Omega |\nabla\Delta^s u|^2 dx}, & \text{if } n = 2r, \quad m = 2s+1, \\ \frac{\int_\Omega |\nabla\Delta^r u|^2 dx}{\int_\Omega |\Delta^s u|^2 dx}, & \text{if } n = 2r+1, \quad m = 2s, \\ \frac{\int_\Omega |\nabla\Delta^r u|^2 dx}{\int_\Omega |\nabla\Delta^s u|^2 dx}, & \text{if } n = 2r+1, \quad m = 2s+1. \end{cases}$$

Clearly, the eigenvalues $\lambda_j^{n,m}[\Omega]$ depend on n, m. However, for the sake of simplicity, we shall write $\lambda_j[\Omega]$ instead of $\lambda_j^{n,m}[\Omega]$.

3.1 Analyticity results

The main result of this section is the following generalization to polyharmonic operators on smooth domains of the results in [64, §3] concerning the Dirichlet Laplacian (see also Theorem 2.2).

Theorem 3.3. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $n, m \in \mathbb{N}_0$ with $0 \leq m < n$, and F be a finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $C_b^n(\Omega\,;\mathbb{R}^N)$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^{2n} then the Frechét differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}\Lambda_{F,s}[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial^n v_l}{\partial\nu^n}\right)^2 \zeta\cdot\nu d\sigma, \qquad (3.4)$$

for all $\psi \in C_b^n(\Omega;\mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ and $\{v_l\}_{l\in F}$ is an orthonormal basis in $H_0^n(\tilde{\phi}(\Omega))$ (with respect to the scalar product (3.3)) of the eigenspace associated with $\lambda_F[\tilde{\phi}]$.

Note that formula (3.4) is a generalization of the celebrated Hadamard formula. We refer to [53] for a recent paper on this topic.

In order to prove Theorem 3.3 we consider equation (3.2) on $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$(-\Delta)^n[v][\psi] = \lambda(-\Delta)^m[v][\psi], \quad \forall\, \psi \in H^n_0(\phi(\Omega)), \tag{3.5}$$

in the unknowns $v \in H^n_0(\phi(\Omega))$, $\lambda \in]0,\infty[$. Recall that the pull-back to Ω of the classic Laplace operator on $\phi(\Omega)$ is defined by

$$\Delta_\phi u = (\Delta(u \circ \phi^{(-1)})) \circ \phi$$

for all $u \in W^{2,1}_{loc}(\Omega)$, $\phi \in \mathcal{A}^2_\Omega$. The operator Δ_ϕ is in fact the Laplace-Beltrami operator associated with the change of variables defined by ϕ. Note that

$$\Delta^s_\phi u = (\Delta^s(u \circ \phi^{(-1)})) \circ \phi$$

for all $u \in W^{2s,1}_{loc}(\Omega)$, $\phi \in \mathcal{A}^{2s}_\Omega$. For any $0 \le m \le n$, the operator Δ^m_ϕ can be considered as the operator acting from $H^n_0(\Omega)$ to its dual, which takes any $u \in H^n_0(\Omega)$ to the functional $\Delta^n_\phi[u]$ defined by

$$\Delta^m_\phi[u][\varphi] = \Delta^m[u \circ \phi^{(-1)}][\varphi \circ \phi^{(-1)}],$$

for all $\varphi \in H^n_0(\Omega)$. More precisely, if $m = 2s$, $s \in \mathbb{N}_0$ then

$$\Delta^{2s}_\phi[u][\varphi] = \int_\Omega \Delta^s_\phi u \Delta^s_\phi \varphi |\mathrm{det}\nabla\phi| dx, \tag{3.6}$$

for all $\varphi \in H^n_0(\Omega)$. If $m = 2s+1$, $s \in \mathbb{N}_0$ then

$$-\Delta^{2s+1}_\phi[u][\varphi] = \int_\Omega \nabla(\Delta^s_\phi u)\ (\nabla\phi)^{-1}(\nabla\phi)^{-t}\nabla^t(\Delta^s_\phi\varphi)|\mathrm{det}\nabla\phi| dx, \tag{3.7}$$

for all $\varphi \in H^n_0(\Omega)$, where $(\nabla\phi)^{-1}$ denotes the inverse of the Jacobian matrix of ϕ and $(\nabla\phi)^{-t}$ the transpose of $(\nabla\phi)^{-1}$. Note that the map from $H^n_0(\Omega)$ to $H^n_0(\phi(\Omega))$ which maps u to $u \circ \phi^{(-1)}$ for all $u \in H^n_0(\Omega)$ is a linear homeomorphism. Hence, equation (3.5) is equivalent to

$$(-\Delta_\phi)^n[u][\varphi] = \lambda(-\Delta_\phi)^m[u][\varphi], \quad \forall\, \varphi \in H^n_0(\Omega)$$

where $u = v \circ \phi$. It is also natural to pull-back the scalar product of $H^n_0(\phi(\Omega))$ to Ω by setting

$$< u_1, u_2 >_{n,\phi} = < u_1 \circ \phi^{(-1)}, u_2 \circ \phi^{(-1)} >_n$$

for all $u_1, u_2 \in H^n_0(\Omega)$, where $< \cdot, \cdot >_n$ is the scalar product in $H^n_0(\phi(\Omega))$ defined by (3.3). By $H^n_{0,\phi}(\Omega)$ we denote the Hilbert space $H^n_0(\Omega)$ endowed with the scalar product $< \cdot, \cdot >_{n,\phi}$. It turns out that the operator $S_{\phi(\Omega)}$ defined in Lemma 3.1 is unitarily equivalent to the operator T_ϕ defined on $H^n_{0,\phi}(\Omega)$ by

$$T_\phi = (-\Delta_\phi)^{-n} \circ (-\Delta_\phi)^m. \tag{3.8}$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3.

Lemma 3.4. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $n, m \in \mathbb{N}_0$, $0 \leq m < n$. The operator T_ϕ defined in (3.8) is non-negative selfadjoint and compact on the Hilbert space $H^n_{0,\phi}(\Omega)$. Equation (3.5) is satisfied for some $v \in H^n_0(\phi(\Omega))$ if and only if the equation $T_\phi u = \mu u$ is satisfied with $u = v \circ \phi$ and $\mu = \lambda^{-1}$. Moreover, the map from $\mathcal{A}^n_\Omega$ to $\mathcal{L}(H^n_0(\Omega)) \times \mathcal{B}_s(H^n_0(\Omega))$ which takes $\phi \in \mathcal{A}^n_\Omega$ to $(T_\phi, < \cdot, \cdot >_{n,\phi})$ is real-analytic.*

Proof of Theorem 3.3. First of all, we note that by standard regularity theory (see e.g., [8, Thm. 9.8]) $v_l \in H^{2n}(\tilde{\phi}(\Omega))$ for all $l \in F$. We observe that the proof is very similar to that of Teorem 2.2. It only remains to compute

$$< \mathrm{d}|_{\phi=\tilde{\phi}} T_\phi[\psi][u_l], u_l >_{n,\tilde{\phi}} .$$

By standard calculus and Theorem 3.8 below we have

$$\begin{aligned}
< \mathrm{d}|_{\phi=\tilde{\phi}} & T_\phi[\psi][u_l], u_l >_{n,\tilde{\phi}} \\
&= (\mathrm{d}|_{\phi=\tilde{\phi}} (-\Delta_\phi)^m[\psi])[u_l][u_l] - \lambda_F^{-1}[\tilde{\phi}](\mathrm{d}|_{\phi=\tilde{\phi}}(-\Delta_\phi)^n[\psi])[u_l][u_l] \\
&= -\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^m v_l}{\partial\nu^m}\right)^2 \zeta \cdot \nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)} (-\Delta)^m v_l \nabla v_l \cdot \zeta d\sigma \\
&+ \lambda_F^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^n v_l}{\partial\nu^n}\right)^2 \zeta \cdot \nu d\sigma + 2\lambda_F^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} (-\Delta)^n v_l \nabla v_l \cdot \zeta d\sigma \\
&= \lambda_F^{-1}[\tilde{\phi}] \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^n v_l}{\partial\nu^n}\right)^2 \zeta \cdot \nu d\sigma,
\end{aligned}$$

where we have set $\zeta = \psi \circ \tilde{\phi}^{(-1)}$. This concludes the proof. $\square$

3.2 Isovolumetric perturbations

As we have done in the previous chapter, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi],$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 3.5. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $n, m \in \mathbb{N}_0$ with $0 \leq m < n$, and F be a finite subset of $\mathbb{N}$. Assume that $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that $\tilde{\phi}(\Omega)$ is of class C^{2n} and that the eigenvalues $\lambda_j[\tilde{\phi}]$ have the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$. Let $\{v_l\}_{l\in F}$ be an orthornormal basis in $H^n_0(\tilde{\phi}(\Omega))$ of the eigenspace corresponding to $\lambda_F[\tilde{\phi}]$. Then $\tilde{\phi}$ is a critical domain transformation for any of the functions $\Lambda_{F,h}$, $h = 1, \dots, |F|$, with volume constraint if and only if there exists $c \in \mathbb{R}$ such that*

$$\sum_{l\in F} \left| \frac{\partial^n v_l}{\partial \nu^n} \right|^2 = c, \quad \text{on } \partial\tilde{\phi}(\Omega)\,. \tag{3.9}$$

Finally, we can prove the following

Theorem 3.6. *Let the same assumptions of Theorem 3.5 hold. If $\tilde{\phi}(\Omega)$ is a ball then condition (3.9) is satisfied.*

Proof. Without any loss of generality, we assume that $\tilde{\phi}(\Omega)$ is a ball with radius R centered at zero. By the rotation invariance of the Laplace operator, $\{v_l \circ A\}_{l\in F}$ is an orthonormal basis of the eigenspace corresponding to $\lambda_F[\tilde{\phi}]$ for all $A \in O_N(\mathbb{R})$ where $O_N(\mathbb{R})$ denotes the group of orthogonal linear transformations in $\mathbb{R}^N$. Since both $\{v_l\}_{l\in F}$ and $\{v_l \circ A\}_{l\in F}$ are orthonormal bases of the same space, it follows that $\sum_{l=1}^{|F|} v_l^2 \circ A = \sum_{l=1}^{|F|} v_l^2$, for all $A \in O_n(\mathbb{R})$. Thus $\sum_{l=1}^{|F|} v_l^2$ is a radial function. By differentiating with respect to the radial coordinate r, by Leibniz's formula and by recalling that all derivatives up to order $n-1$ of the eigenfunctions vanish at the boundary of $\tilde{\phi}(\Omega)$, we obtain that

$$\left.\frac{\partial^{2n} v_l^2}{\partial r^{2n}}\right|_{r=R} = \sum_{k=0}^{2n} \binom{2n}{k} \left.\left(\frac{\partial^k v_l}{\partial r^k} \frac{\partial^{2n-k} v_l}{\partial r^{2n-k}} \right)\right|_{r=R} = \binom{2n}{n} \left.\left(\frac{\partial^n v_l}{\partial r^n} \right)^2 \right|_{r=R}. \tag{3.10}$$

Since $\sum_{l\in F} \frac{\partial^{2n} v_l^2}{\partial r^{2n}}$ is a radial function, then by formula (3.10) we conclude that $\sum_{l\in F} (\frac{\partial^n v_l}{\partial \nu^n})^2$ is constant on $\partial\tilde{\phi}(\Omega)$. □

3.3 A formula for the Frechét differential of the 'poly-Laplace-Beltrami' operator

In this section we prove Theorem 3.8 which has its own interest since it provides an explicit formula for the Frechét differential with respect to ϕ of the weak 'poly-Laplace-Beltrami' operator Δ_ϕ^n defined by (3.6), (3.7). That formula has been used in the proof of (3.4).

Lemma 3.7. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $s \in \mathbb{N}$, $u_1 \in L^2(\Omega)$, $u_2 \in H_0^{2s}(\Omega)$. Let $\tilde{\phi} \in \mathcal{A}_\Omega^{2s}$ and $v_i = u_i \circ \tilde{\phi}^{(-1)}$, $i = 1, 2$. Assume that $\tilde{\phi}(\Omega)$ is of class C^1 and that $v_1 \in H^{2s}(\tilde{\phi}(\Omega))$, $v_2 \in H^{2s+1}(\tilde{\phi}(\Omega))$. Then*

$$\int_\Omega u_1 \mathrm{d}|_{\phi=\tilde{\phi}} \Delta_\phi^s[\psi] u_2 |\det \nabla \tilde{\phi}| dx = \int_{\tilde{\phi}(\Omega)} (v_1 \nabla \Delta^s v_2 - \Delta^s v_1 \nabla v_2) \cdot \zeta dy - \int_{\partial \tilde{\phi}(\Omega)} (v_1 \Delta^s v_2) \zeta \cdot \nu d\sigma, \quad (3.11)$$

for all $\psi \in C_b^{2s}(\Omega; \mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$.

Proof. First, we recall the following formula from [63, Lemma 3.42] which holds for any $u \in H^2(\Omega)$:

$$(\mathrm{d}|_{\phi=\tilde{\phi}} \Delta_\phi[\psi] u) \circ \tilde{\phi}^{(-1)} = -2 \sum_{i,j=1}^N \frac{\partial^2 (u \circ \tilde{\phi}^{(-1)})}{\partial y_i \partial y_j} \frac{\partial \zeta_j}{\partial y_i} - \sum_{j=1}^N \frac{\partial (u \circ \tilde{\phi}^{(-1)})}{\partial y_j} \Delta \zeta_j. \quad (3.12)$$

We also observe that

$$d|_{\phi=\tilde{\phi}} \Delta_\phi^s[\psi] = \sum_{\substack{h,k=0 \\ h+k=s-1}}^{s-1} \Delta_{\tilde{\phi}}^h \circ (d|_{\phi=\tilde{\phi}} \Delta_\phi[\psi]) \circ \Delta_{\tilde{\phi}}^k. \quad (3.13)$$

By formulas (3.12) and (3.13), by changing variables in integrals and integrating by parts, we obtain

$$\begin{aligned}
&\int_\Omega u_1 \mathrm{d}|_{\phi=\tilde{\phi}} \Delta_\phi^s[\psi] u_2 |\det \nabla \tilde{\phi}| dx \\
&\quad = -\sum_{\substack{h,k=0 \\ h+k=s-1}}^{s-1} \int_{\tilde{\phi}(\Omega)} \Delta^h v_1 \left(2 \sum_{i,j=1}^N \frac{\partial^2 \Delta^k v_2}{\partial y_i \partial y_j} \frac{\partial \zeta_j}{\partial y_i} + \sum_{j=1}^N \frac{\partial \Delta^k v_2}{\partial y_j} \Delta \zeta_j \right) dy \\
&\quad = \sum_{\substack{h,k=0 \\ h+k=s-1}}^{s-1} \int_{\tilde{\phi}(\Omega)} \sum_{i,j=1}^N \left[\frac{\partial \Delta^h v_1}{\partial y_i} \frac{\partial \Delta^k v_2}{\partial y_j} \left(\frac{\partial \zeta_j}{\partial y_i} + \frac{\partial \zeta_i}{\partial y_j} \right) \right] dy
\end{aligned}$$

$$- \sum_{\substack{h,k=0 \\ h+k=s-1}}^{s-1} \int_{\tilde{\phi}(\Omega)} (\Delta^h v_1 \Delta^{k+1} v_2 + \nabla\Delta^h v_1 \nabla\Delta^k v_2) \mathrm{div}\zeta \, dy, \tag{3.14}$$

see also [63, Formula (3.45)]. Moreover, integrating by parts yields

$$\begin{aligned} \int_{\tilde{\phi}(\Omega)} \frac{\partial\Delta^h v_1}{\partial y_i} \frac{\partial\Delta^k v_2}{\partial y_j} \left(\frac{\partial\zeta_j}{\partial y_i} + \frac{\partial\zeta_i}{\partial y_j}\right) dy &= - \int_{\partial\tilde{\phi}(\Omega)} \Delta^h v_1 \Delta^{k+1} v_2 \zeta \cdot \nu d\sigma \\ &+ \int_{\tilde{\phi}(\Omega)} \Delta^h v_1 \nabla\Delta^{k+1} v_2 \cdot \zeta dy + \int_{\tilde{\phi}(\Omega)} \Delta^h v_1 \Delta^{k+1} v_2 \mathrm{div}\zeta dy \\ &+ \int_{\tilde{\phi}(\Omega)} \nabla\Delta^h v_1 \cdot \nabla\Delta^k v_2 \mathrm{div}\zeta dy - \int_{\tilde{\phi}(\Omega)} \Delta^{h+1} v_1 \nabla\Delta^k v_2 \cdot \zeta dy. \end{aligned} \tag{3.15}$$

By observing that the first summand in the right-hand side of (3.15) vanish if $k < s-1$, and by combining (3.14) and (3.15), we obtain a telescopic sum and we deduce the validity of (3.11). □

Theorem 3.8. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $n \in \mathbb{N}$, $u_1, u_2 \in H^n_0(\Omega)$. Let $\tilde{\phi} \in \mathcal{A}^n_\Omega$ and $v_i = u_i \circ \tilde{\phi}^{(-1)}$, $i = 1, 2$. Assume that $\tilde{\phi}(\Omega)$ is of class C^1 and that $v_1, v_2 \in H^{2n}(\tilde{\phi}(\Omega))$. Then*

$$\begin{aligned} (\mathrm{d}|_{\phi=\tilde{\phi}}(-\Delta_\phi)^n[\psi])[u_1][u_2] = &- \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial^n v_1}{\partial\nu^n} \frac{\partial^n v_2}{\partial\nu^n} \zeta \cdot \nu d\sigma \\ &- \int_{\tilde{\phi}(\Omega)} ((-\Delta)^n v_1 \nabla v_2 + (-\Delta)^n v_2 \nabla v_1) \cdot \zeta dy, \end{aligned} \tag{3.16}$$

for all $\psi \in C^n_b(\Omega\,;\mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$.

Proof. First, we consider the case where n is an even number of the form $n = 2s$ with $s \in \mathbb{N}_0$. By formula (3.6) and standard calculus we have

$$\begin{aligned} d|_{\phi=\tilde{\phi}} \Delta^{2s}_\phi[\psi][u_1][u_2] = &\int_\Omega d|_{\phi=\tilde{\phi}} \Delta^s_\phi[\psi] u_1 \Delta^s_{\tilde{\phi}} u_2 |\det D\tilde{\phi}| dx \\ &+ \int_\Omega \Delta^s_{\tilde{\phi}} u_1 d|_{\phi=\tilde{\phi}} \Delta^s_\phi[\psi] u_2 |\det D\tilde{\phi}| dx \\ &+ \int_\Omega \Delta^s_{\tilde{\phi}} u_1 \Delta^s_{\tilde{\phi}} u_2 d|_{\phi=\tilde{\phi}} |\det D\phi|[\psi] dx. \end{aligned} \tag{3.17}$$

Moreover, by (2.12) we have

$$\int_\Omega \Delta^s_{\tilde{\phi}} u_1 \Delta^s_{\tilde{\phi}} u_2 d|_{\phi=\tilde{\phi}} |\det D\phi|[\psi] dx = \int_{\tilde{\phi}(\Omega)} \Delta^s \tilde{v}_1 \Delta^s \tilde{v}_2 \mathrm{div}\zeta dy. \tag{3.18}$$

Formula (3.16) easily follows by combining formulas (3.11), (3.17), (3.18), by integrating by parts and by observing that $\Delta^s v_i = \frac{\partial^{2s} v_i}{\partial \nu^{2s}}$ on $\partial\tilde{\phi}(\Omega)$ since $v_i \in H^{2s}_0(\tilde{\phi}(\Omega))$.

Now, we consider the case where n is an odd number of the form $n = 2s+1$ with $s \in \mathbb{N}_0$. By formula (3.7) and standard calculus we have

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} \Delta^{2s+1}_{\phi}[\psi][u_1][u_2] &= \int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_1 (\nabla\zeta + \nabla^t\zeta)\nabla^t\Delta^s v_2 dy \\
- \int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_1 \nabla\Delta^s v_2 \mathrm{div}\zeta dy &- \int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_1 \nabla\left((d|_{\phi=\tilde{\phi}} \Delta^s_{\phi}[\psi]u_2) \circ \tilde{\phi}^{(-1)} \right) dy \\
&- \int_{\tilde{\phi}(\Omega)} \nabla\left((d|_{\phi=\tilde{\phi}} \Delta^s_{\phi}[\psi]u_1) \circ \tilde{\phi}^{(-1)} \right) \nabla\Delta^s v_2 dy. \quad (3.19)
\end{aligned}
$$

Moreover, integrating by parts yields

$$
\begin{aligned}
&\int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_1 (\nabla\zeta + \nabla^t\zeta)\nabla^t\Delta^s v_2 dy \\
&\quad = \sum_{h,k=1}^{N} \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial \zeta_h}{\partial y_k} \frac{\partial \Delta^s v_1}{\partial y_h} \frac{\partial \Delta^s v_2}{\partial y_k} + \frac{\partial \zeta_k}{\partial y_h} \frac{\partial \Delta^s v_1}{\partial y_h} \frac{\partial \Delta^s v_2}{\partial y_k} \right) dx \\
&\quad = 2\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial \Delta^s v_1}{\partial \nu} \frac{\partial \Delta^s v_2}{\partial \nu} \zeta \cdot \nu d\sigma \\
&\quad - \sum_{h,k=1}^{N} \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 \Delta^s v_1}{\partial y_h \partial y_k} \frac{\partial \Delta^s v_2}{\partial y_k} \zeta_h + \frac{\partial \Delta^s v_1}{\partial y_h} \frac{\partial^2 \Delta^s v_2}{y_k^2} \zeta_h \right) dy \\
&\quad - \sum_{h,k=1}^{N} \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 \Delta^s v_1}{\partial y_h^2} \frac{\partial \Delta^s v_2}{\partial y_k} \zeta_k + \frac{\partial \Delta^s v_1}{\partial y_h} \frac{\partial^2 \Delta^s v_2}{\partial y_h \partial y_k} \zeta_k \right) dy \\
&\quad = \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial \Delta^s v_1}{\partial \nu} \frac{\partial \Delta^s v_2}{\partial \nu} \zeta \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_1 \nabla\Delta^s v_2 \mathrm{div}\zeta dy \\
&\quad - \int_{\tilde{\phi}(\Omega)} (\Delta^{s+1} v_1 \nabla\Delta^s v_2 + \Delta^{s+1} v_2 \nabla\Delta^s v_1) \cdot \zeta dy.
\end{aligned}
$$

By integrating by parts, changing variables in integrals and using formula (3.11), we obtain

$$
\begin{aligned}
&\int_{\tilde{\phi}(\Omega)} \nabla\Delta^s v_i \nabla\left((d|_{\phi=\tilde{\phi}} \Delta^s_{\phi}[\psi]u_j) \circ \tilde{\phi}^{(-1)} \right) dy \\
&\qquad = -\int_{\Omega} \Delta^{s+1}_{\tilde{\phi}} u_i d|_{\phi=\tilde{\phi}} \Delta^s_{\phi}[\psi]u_j |\mathrm{det}\nabla\tilde{\phi}| dx
\end{aligned}
$$

$$= -\int_{\tilde{\phi}(\Omega)} (\Delta^{s+1} v_i \nabla \Delta^s v_j - \Delta^{2s+1} v_i \nabla v_j) \cdot \zeta dy \quad (3.20)$$

for all $i, j \in \{1, 2\}$. Finally, formula (3.16) easily follows by combining formulas (3.19)-(3.20) and by observing that $\frac{\partial \Delta^s v_i}{\partial \nu} = \frac{\partial^{2s+1} v_i}{\partial \nu^{2s}}$ on $\partial\tilde{\phi}(\Omega)$ since $v_i \in H_0^{2s+1}(\tilde{\phi}(\Omega))$. $\square$

Chapter 4

Quantitative estimates for systems of partial differential equations

In this chapter we consider the eigenvalue problem (1.13) and prove quantitative estimates for the eigenvalues. In particular, we show that the arguments used in [38, 39] can be easily adapted to the general case of systems of partial differential equations.

Note that the classical formulation of problem (1.13) is

$$\sum_{|\alpha|,|\beta|\leq l}\sum_{i=1}^{m}(-1)^{|\beta|}D^{\beta}(A^{ij}_{\alpha\beta}D^{\alpha}u_i)=\lambda u_j,\ j=1,\dots,m,$$

with suitable homogeneous boundary conditions depending on the choice of the space $V(\Omega)\subset W^{l,2}(\Omega)$.

We recall that problem (1.13) includes several important problems in linear elasticity, e.g., the Lamé system (5.3), and more generally problem (5.1) discussed in Chapter 5. Also the Reissner-Mindlin system (6.1) is a special case of problem (1.13). Clearly, also scalar problems are included: in fact, this is the case $m=1$. Therefore, problem (2.2) discussed in Chapter 2 and problem $\mathcal{P}_{n0}$ (3.1) can be considered as well.

In this chapter, we shall think the coefficients $A^{ij}_{\alpha\beta}$ as fixed and satisfying some or all the conditions (1.14)-(1.16). Moreover, we shall consider problem (1.13) under Dirichlet and Neumann boundary conditions only, i.e., the space $V(\Omega)$ will be either $W^{l,2}_0(\Omega)$ or $W^{l,2}(\Omega)$. In particular, by $\varphi_{n,\mathcal{D}}[\Omega]$, $\varphi_{n,\mathcal{N}}[\Omega]$ we shall denote an orthonormal sequence of eigenfunctions associated with $\lambda_{n,\mathcal{D}}[\Omega]$, $\lambda_{n,\mathcal{N}}[\Omega]$ respectively. We shall also denote by $H_{W^{l,2}_0(\Omega)}$,

$H_{W^{l,2}(\Omega)}$ respectively the corresponding operators. When no distinction between the Dirichlet and the Neumann case is required and we refer to both problems, we shall simply write $\lambda_n[\Omega]$, $\varphi_n[\Omega]$, H_Ω to indicate the eigenvalues and the corresponding eigenfunctions and operators.

4.1 Estimates via diffeomorphisms

Let Ω be an open set in $\mathbb{R}^N$, and let ϕ be a diffeomorphism of Ω into $\phi(\Omega)$. In this section we provide a few estimates for the difference $|\lambda_n[\Omega]-\lambda_n[\phi(\Omega)]|$, which will be useful in the sequel.

We have the following lemma (cf. [38, Lemma 4.1]).

Lemma 4.1. *Let Ω be an open set in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $B_1, B_2 > 0$ and ϕ be a diffeomorphism of Ω onto $\phi(\Omega)$ of class C^l such that*

$$\max_{|\alpha|\leq l} |D^\alpha \phi(x)| \leq B_1, \qquad |\det \nabla\phi(x)| \geq B_2, \tag{4.1}$$

for all $x \in \Omega$. Let $B_3 > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta}$ be measurable real-valued functions defined on $\Omega \cup \phi(\Omega)$ satisfying

$$\max_{|\alpha|,|\beta|\leq l,\ i,j\leq m} |A^{ij}_{\alpha\beta}(x)| \leq B_3, \tag{4.2}$$

for almost all $x \in \Omega \cup \phi(\Omega)$. Then there exists a positive constant $c = c(N, l, m, B_1, B_2, B_3)$ such that

$$\left| Q_{\phi(\Omega)}(u \circ \phi^{(-1)}) - Q_\Omega(u) \right| \leq c\mathcal{L}(\phi) \int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^2 dx, \tag{4.3}$$

for all $u \in W^{l,2}(\Omega)^m$, where

$$\mathcal{L}(\phi) = \max_{|\alpha|\leq l} \|D^\alpha(\phi - \mathrm{Id})\|_{L^\infty(\Omega)} + \max_{|\alpha|,|\beta|\leq l,\ i,j\leq m} \|A^{ij}_{\alpha\beta} \circ \phi - A^{ij}_{\alpha\beta}\|_{L^\infty(\Omega)}.$$

Proof. By changing variables and using a known formula for high derivatives of composite functions (cf. e.g., [49, Formula B]), we have that

$$
\begin{aligned}
Q_{\phi(\Omega)}(u\circ\phi^{(-1)}) &= \int_{\phi(\Omega)} \sum_{\substack{|\alpha|,|\beta|\le l\\ i,j\le m}} A^{ij}_{\alpha\beta} D^{\alpha}(u_i\circ\phi^{(-1)})\, D^{\beta}(u_j\circ\phi^{(-1)})dy \\
&= \int_{\Omega} \sum_{\substack{|\alpha|,|\beta|\le l\\ i,j\le m}} \Big(A^{ij}_{\alpha\beta} D^{\alpha}(u_i\circ\phi^{(-1)}) D^{\beta}(u_j\circ\phi^{(-1)})\Big)\circ\phi|\mathrm{det}\nabla\phi|dx \\
&= \int_{\Omega} \sum_{\substack{|\alpha|,|\beta|\le l\\ i,j\le m}} A^{ij}_{\alpha\beta}\circ\phi \sum_{\substack{|\eta|\le|\alpha|\\ |\xi|\le|\beta|}} D^{\eta}u_i D^{\xi}u_j\ (p_{\alpha\eta}(\phi^{(-1)})p_{\beta\xi}(\phi^{(-1)}))\circ\phi|\mathrm{det}\nabla\phi|dx \\
&= \sum_{\substack{|\alpha|,|\beta|\le l\\ i,j\le m}} \sum_{\substack{|\eta|\le|\alpha|\\ |\xi|\le|\beta|}} \int_{\Omega} (A^{ij}_{\alpha\beta} p_{\alpha\eta}(\phi^{(-1)})p_{\beta\xi}(\phi^{(-1)}))\circ\phi\, D^{\eta}u_i D^{\xi}u_j\,|\mathrm{det}\nabla\phi|dx,
\end{aligned}
\tag{4.4}
$$

for all $u\in W^{l,2}(\Omega)^m$, where for all α,η with $|\eta|\le|\alpha|\le l$, $p_{\alpha\eta}(\phi^{(-1)})$ denotes a polynomial of degree $|\eta|$ in derivatives of $\phi^{(-1)}$ of order between 1 and $|\alpha|$, with coefficients depending only on N,α,η.

We recall that for each α with $|\alpha|\le l$ there exists a polynomial $p_{\alpha}(\phi)$ in derivatives of ϕ of order between 1 and $|\alpha|$, with coefficients depending only on N,α, such that

$$
(D^{\alpha}\phi^{(-1)})\circ\phi = \frac{p_{\alpha}(\phi)}{(\mathrm{det}\nabla\phi)^{2|\alpha|-1}}. \tag{4.5}
$$

Using (4.4), in order to get inequality (4.3) it is enough to estimate the expressions

$$
\begin{aligned}
&(A^{ij}_{\alpha\beta} p_{\alpha\eta}(\phi^{(-1)})p_{\beta\xi}(\phi^{(-1)}))\circ\phi\ |\mathrm{det}\nabla\phi| \\
&\qquad\qquad - (A^{ij}_{\alpha\beta} p_{\alpha\eta}(\tilde\phi^{(-1)})p_{\beta\xi}(\tilde\phi^{(-1)}))\circ\tilde\phi\ |\mathrm{det}\nabla\tilde\phi|,
\end{aligned}
$$

where $\tilde\phi=\mathrm{Id}$. This can be done by means of the triangle inequality and by observing that (4.5) implies that

$$
|(D^{\alpha}\phi^{(-1)})\circ\phi-(D^{\alpha}\tilde\phi^{(-1)})\circ\tilde\phi|\le C\max_{|\beta|\le|\alpha|}\|D^{\beta}(\phi-\tilde\phi)\|_{L^{\infty}(\Omega)},
$$

for some $C=C(N,\alpha,B_1,B_2)$. □

We observe that, if condition (1.16) is satisfied in Ω, and if ϕ is a diffeomorphism of Ω onto $\phi(\Omega)$ smooth enough, then condition (1.16) is satisfied in $\phi(\Omega)$ as well. Note also that, if condition (1.16) is satisfied in Ω with constants a_1, b_1, and in $\phi(\Omega)$ with constants a_2, b_2, then the condition is satisfied in $\Omega \cup \phi(\Omega)$ with constants $a = \min\{a_1, a_2\}, b = \max\{b_1, b_2\}$.

Theorem 4.2. *Let U be an open set in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $B_1, B_2, B_3, a, b > 0$. For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A_{\alpha\beta}^{ij}$ be measurable real-valued functions defined on U, and satisfying conditions (1.14), (1.16), and (4.2) in U. Then the following statements hold.*

(i) There exists a positive constant $c_1 = c_1(N, l, m, B_1, B_2, B_3, a, b)$ such that, for all $n \in \mathbb{N}$, for all open sets $\Omega \subset U$ for which the embedding $W_0^{l,2}(\Omega) \subset L^2(\Omega)$ is compact, and for all diffeomorphisms of Ω onto $\phi(\Omega)$ of class C^l satisfying $\phi(\Omega) \subset U$ and condition (4.1), if $\mathcal{L}(\phi) < (c_1 \mathcal{D}_\Omega^2)^{-1}$, we have

$$|\lambda_{n,\mathcal{D}}[\Omega] - \lambda_{n,\mathcal{D}}[\phi(\Omega)]| \leq c_1 \mathcal{D}_\Omega^2 (1 + \lambda_{n,\mathcal{D}}[\Omega]) \mathcal{L}(\phi).$$

(ii) There exists a positive constant $c_2 = c_2(N, l, m, B_1, B_2, B_3, a, b)$ such that, for all $n \in \mathbb{N}$, for all open sets $\Omega \subset U$ for which (1.16) is satisfied, and for which the embedding $W^{l,2}(\Omega) \subset L^2(\Omega)$ is compact, and for all diffeomorphisms of Ω onto $\phi(\Omega)$ of class C^l satisfying $\phi(\Omega) \subset U$ and condition (4.1), if $\mathcal{L}(\phi) < (c_2 \mathcal{N}_\Omega^2)^{-1}$, we have

$$|\lambda_{n,\mathcal{N}}[\Omega] - \lambda_{n,\mathcal{N}}[\phi(\Omega)]| \leq c_2 \mathcal{N}_\Omega^2 (1 + \lambda_{n,\mathcal{N}}[\Omega]) \mathcal{L}(\phi).$$

Proof. The proof can be done adapting that of [38, Theorem 4.8], and using Lemma 4.1. □

Applying Lemma 1.11 and Theorem 4.2, we get the following

Corollary 4.3. *Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $B_1, B_2, L, a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A_{\alpha\beta}^{ij} \in C^{0,1}(\cup_{h=1}^s V_h)$ satisfy $\|A_{\alpha\beta}^{ij}\|_{C^{0,1}(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C(\mathcal{A})$ with the same constants a, b.*

Then there exists a positive constant $c = c(N, \mathcal{A}, l, m, B_1, B_2, L, a, b)$ such that, for all $n \in \mathbb{N}$, for all open sets $\Omega \in C(\mathcal{A})$, and for all diffeomorphisms of Ω onto $\phi(\Omega)$ of class C^l satisfying $\phi(\Omega) \subset \cup_{h=1}^s V_h$ and condition (4.1), if $\max_{0 \leq |\alpha| \leq l} \|D^\alpha(\phi - \mathrm{Id})\|_{L^\infty(\Omega)} < c^{-1}$, we have

$$|\lambda_n[\Omega] - \lambda_n[\phi(\Omega)]| \leq c(1 + \lambda_n[\Omega]) \max_{0 \leq |\alpha| \leq l} \|D^\alpha(\phi - \mathrm{Id})\|_{L^\infty(\Omega)}.$$

4.2 Estimates for Dirichlet eigenvalues via the atlas distance

In general, even if two open sets Ω_1 and Ω_2 are known to be diffeomorphic, it is not easy to construct a diffeomorphism ϕ such that $\phi(\Omega_1) = \Omega_2$ and provide information on the measure of vicinity $\max_{0\le|\alpha|\le l} \|D^\alpha(\phi - \mathrm{Id})\|_{L^\infty(\Omega)}$ in terms of explicit geometric quantities. However, if Ω_1, Ω_2 belong to the same class $C(\mathcal{A})$, then it is possible to construct a suitable diffeomorphism ϕ such that $\phi(\Omega_1) \subset \Omega_2$ and estimate the measure of vicinity via the atlas distance (1.1). Such a construction was first used in [34] and then implemented in [38]. We briefly recall it.

Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$ and let $\{\psi_h\}_{h=1}^s$ be a partition of unity such that $\psi_h \in C_c^\infty(\mathbb{R}^N)$, $\operatorname{supp}\psi_h \subset (V_h)_{\frac{3}{4}\rho}$, $0 \le \psi_h \le 1$ and $\sum_{h=1}^s \psi_h(x) = 1$ for all $x \in \cup_{h=1}^s (V_h)_\rho$. For $\epsilon \ge 0$ we consider the following transformation

$$\phi_\epsilon(x) = x - \epsilon \sum_{h=1}^s \xi_h \psi_h(x)\,, \quad x \in \mathbb{R}^N, \tag{4.6}$$

where $\xi_h = r_h^{(-1)}((0, \dots, 1))$. We recall the following technical lemma from [38].

Lemma 4.4. *Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$. There exist $A_1, A_2, E > 0$ depending only on N and $\mathcal{A}$ such that*

$$\max_{0\le|\alpha|\le l} \left\| D^\alpha(\phi_\epsilon - \mathrm{Id}) \right\|_{L^\infty(\mathbb{R}^N)} \le A_1\epsilon, \tag{4.7}$$

and such that

$$\frac{1}{2} \le 1 - A_2\epsilon \le \det \nabla\phi_\epsilon \le 1 + A_2\epsilon, \tag{4.8}$$

for all $0 \le \epsilon < E$. Furthermore,

$$\phi_\epsilon(\Omega_1) \subset \Omega_2$$

for all $0 < \epsilon < E$, for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ such that $\Omega_2 \subset \Omega_1$ and

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) < \frac{\epsilon}{s}.$$

Theorem 4.5. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $L, a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \le l$ and for any $i, j \in \mathbb{N}$ with $i, j \le m$, let*

$A^{ij}_{\alpha\beta} \in C^{0,1}(\cup_{h=1}^s V_h)$ satisfy $\|A^{ij}_{\alpha\beta}\|_{C^{0,1}(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C(\mathcal{A})$ with the same constants a, b.

Then for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, a, b$ such that

$$|\lambda_{n,\mathcal{D}}[\Omega_1] - \lambda_{n,\mathcal{D}}[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2), \tag{4.9}$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon_n$.

Proof. Let $0 < \epsilon < E$ where $E > 0$ is as in Lemma 4.4, and let $\Omega_1, \Omega_2 \in C(\mathcal{A})$ be such that $d_{\mathcal{A}}(\Omega_1, \Omega_2) \leq \epsilon/s$. We set $\Omega_3 = \Omega_1 \cap \Omega_2$. Clearly, $\Omega_3 \in C(\mathcal{A})$ and $d_{\mathcal{A}}(\Omega_3, \Omega_1)$, $d_{\mathcal{A}}(\Omega_3, \Omega_2) < \epsilon/s$. By Lemma 4.4 applied to the couples of open sets Ω_k, Ω_3 it follows that $\phi_\epsilon(\Omega_k) \subset \Omega_3$, $k = 1, 2$, where ϕ_ϵ is defined in (4.6). By the monotonicity of the eigenvalues with respect to inclusion it follows that

$$\lambda_{n,\mathcal{D}}[\Omega_k] \leq \lambda_{n,\mathcal{D}}[\Omega_3] \leq \lambda_{n,\mathcal{D}}[\phi_\epsilon(\Omega_k)],$$

for $k = 1, 2$. Using Lemma 1.12, Corollary 4.3, and Lemma 4.4, it follows that there exist $\tilde{c}_n, \tilde{\epsilon}_n > 0$ such that

$$\lambda_{n,\mathcal{D}}[\Omega_3] - \lambda_{n,\mathcal{D}}[\Omega_k] \leq \lambda_{n,\mathcal{D}}[\phi_\epsilon(\Omega_k)] - \lambda_{n,\mathcal{D}}[\Omega_k] \leq \tilde{c}_n \epsilon,$$

for $k = 1, 2$, if $0 < \epsilon < \tilde{\epsilon}_n$. Hence

$$|\lambda_{n,\mathcal{D}}[\Omega_1] - \lambda_{n,\mathcal{D}}[\Omega_2]| \leq \max_{k=1,2} \{\lambda_{n,\mathcal{D}}[\Omega_3] - \lambda_{n,\mathcal{D}}[\Omega_k]\} \leq \tilde{c}_n \epsilon.$$

By choosing $\epsilon = 2s d_{\mathcal{A}}(\Omega_1, \Omega_2)$, we get that inequality (4.9) holds with $c_n = 2s\tilde{c}_n$ if $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon_n = \tilde{\epsilon}_n/(2s)$. □

Using Lemma 1.3 we get the following

Corollary 4.6. *Under the same assumptions of Theorem 4.5, for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m,$ L, a, b such that*

$$|\lambda_{n,\mathcal{D}}[\Omega_1] - \lambda_{n,\mathcal{D}}[\Omega_2]| \leq c_n \epsilon, \tag{4.10}$$

for all $0 < \epsilon < \epsilon_n$ and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying (1.3) or (1.4).

Proof. Inequality (4.10) follows by inequality (1.5) and inequality (4.9). □

4.3 Estimates for Neumann eigenvalues via the atlas distance

Using the variational characterization (1.17), it is not difficult to see that, if $\Omega_1 \subset \Omega_2$, then $\lambda_{n,\mathcal{D}}[\Omega_1] \geq \lambda_{n,\mathcal{D}}[\Omega_2]$ for all $n \in \mathbb{N}$. This was used in the proof of Theorem 4.5. Unfortunately, Neumann eigenvalues do not enjoy monotonicity properties, and therefore we have to use different arguments.

We start by recalling the following definition from [38].

Definition 4.7. *Let U be an open set in $\mathbb{R}^N$ and ρ an isometry. We say that U is a ρ-patch if there exist an open set $G_U \subset \mathbb{R}^{N-1}$ and functions $\eta_U, \psi_U : G_U \to \mathbb{R}$ such that*

$$\rho(U) = \left\{(\bar{x}, x_N) \in \mathbb{R}^N : \ \psi_U(\bar{x}) < x_N < \eta_U(\bar{x}), \ x \in G_U \right\}.$$

The thickness of the ρ-patch is defined by

$$R_U = \inf_{\bar{x} \in G_U} (\eta_U(\bar{x}) - \psi_U(\bar{x})).$$

The thinness of the ρ-patch is defined by

$$S_U = \sup_{\bar{x} \in G_U} (\eta_U(\bar{x}) - \psi_U(\bar{x})).$$

If $\Omega_2 \subset \Omega_1$ and $\Omega_1 \setminus \Omega_2$ is covered by a finite number of ρ-patches contained in Ω_1, then we can estimate $\lambda_{n,\mathcal{N}}[\Omega_2] - \lambda_{n,\mathcal{N}}[\Omega_1]$ via the thickness of the patches.

Lemma 4.8. *Let $l, m \in \mathbb{N}$ and Ω_1 be an open set in $\mathbb{R}^N$ such that the embedding $W^{l,2}(\Omega_1) \subset W^{l-1,2}(\Omega_1)$ is compact. For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$, and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta}$ be bounded measurable real-valued functions defined on Ω_1, satisfying conditions (1.14)-(1.16) in Ω_1. Let $\sigma \in \mathbb{N}$, $R > 0$.*

Assume that $\Omega_2 \subset \Omega_1$ is such that the embedding $W^{l,2}(\Omega_2) \subset L^2(\Omega_2)$ is compact, that (1.16) is satisfied and that there exist isometries $\{\rho_h\}_{h=1}^{\sigma}$ and two sets $\{U_h\}_{h=1}^{\sigma}$, $\{\tilde{U}_h\}_{h=1}^{\sigma}$ of ρ_h-patches U_h and $\tilde{U}_h$ satisfying the following properties

- *(a) $U_h \subset \tilde{U}_h \subset \Omega_1$, for all $h = 1, \ldots, \sigma$;*
- *(b) $G_{U_h} = G_{\tilde{U}_h}$, $\eta_{U_h} = \eta_{\tilde{U}_h}$, for all $h = 1, \ldots, \sigma$;*
- *(c) $R_{\tilde{U}_h} > R$, for all $h = 1, \ldots, \sigma$;*

(d) $\Omega_1 \setminus \Omega_2 \subset \cup_{h=1}^{\sigma} U_h$.

Then there exists a positive constant $d = d(N, l, m, R)$ *such that, for all* $n \in \mathbb{N}$,

$$\lambda_{n,\mathcal{N}}[\Omega_2] \leq \lambda_{n,\mathcal{N}}[\Omega_1](1 + d_n \max_{h=1,\dots,\sigma} S_{U_h}),$$

if $\max_{h=1,\dots,\sigma} S_{U_h} < d_n^{-1}$, *where*

$$d_n = 2\sigma d a^{-1} \left(b + \lambda_{n,\mathcal{N}}[\Omega_1]\right). \tag{4.11}$$

Proof. By (a) and (b) it follows that $\psi_{\tilde{U}_h} \leq \psi_{U_h}$ for all $h = 1, \dots, \sigma$. Let $u \in W^{l,2}(\Omega_1)^m$. By (d) we have

$$\int_{\Omega_1 \setminus \Omega_2} |u|^2 dy \leq \sum_{h=1}^{\sigma} \int_{U_h} |u|^2 dy = \sum_{h=1}^{\sigma} \int_{\rho_h(U_h)} |u \circ \rho_h^{(-1)}|^2 dx. \tag{4.12}$$

Let us set $v_{j,h} = u_j \circ \rho_h^{(-1)}$. Clearly,

$$\int_{\rho_h(U_h)} |u \circ \rho_h^{-1}|^2 = \sum_{j=1}^{m} \int_{G_{U_h}} \int_{\psi_{U_h}(\bar{x})}^{\eta_{U_h}(\bar{x})} |v_{j,h}(\bar{x}, x_N)|^2 dx_N d\bar{x}. \tag{4.13}$$

Since $v_{j,h} \in W^{l,2}(\rho_h(\tilde{U}_h))$ it follows that for almost all $\bar{x} \in G_{\tilde{U}_h}$ the function $v_{j,h}(\bar{x}, \cdot)$ belongs to the space $W^{l,2}(\psi_{\tilde{U}_h}(\bar{x}), \eta_{\tilde{U}_h}(\bar{x}))$. Moreover, by (c) it follows that $\eta_{\tilde{U}_h}(\bar{x}) - \psi_{\tilde{U}_h}(\bar{x}) \geq R$. Thus by [33, Theorem 2] there exists $\tilde{d} = \tilde{d}(m, R)$ such that

$$\begin{aligned} \|v_{j,h}(\bar{x}, \cdot)\|^2_{L^\infty(\psi_{\tilde{U}_h}(\bar{x}), \eta_{\tilde{U}_h}(\bar{x}))} \leq \tilde{d} \Bigg(& \|v_{j,h}(\bar{x}, \cdot)\|^2_{L^2(\psi_{\tilde{U}_h}(\bar{x}), \eta_{\tilde{U}_h}(\bar{x}))} \\ & + \left\| \frac{\partial^l v_{j,h}}{\partial x_N^l}(\bar{x}, \cdot) \right\|^2_{L^2(\psi_{\tilde{U}_h}(\bar{x}), \eta_{\tilde{U}_h}(\bar{x}))} \Bigg). \end{aligned} \tag{4.14}$$

Moreover, by inequality (4.14) and property (b) we have

$$
\begin{aligned}
\sum_{j=1}^{m}\int_{G_{U_h}}\int_{\psi_{U_h}(\bar{x})}^{\eta_{U_h}(\bar{x})}|v_{j,h}(\bar{x},x_N)|^2dx_Nd\bar{x}&\\
\le \sum_{j=1}^{m}\int_{G_{U_h}}(\eta_{U_h}(\bar{x})-\psi_{U_h}(\bar{x}))\|v_{j,h}(\bar{x},\cdot)\|^2_{L^\infty(\psi_{\tilde{U}_h}(\bar{x}),\eta_{U_h}(\bar{x}))}d\bar{x}&\\
\le \tilde{d}S_{U_h}\sum_{j=1}^{m}\left(\|v_{j,h}\|^2_{L^2(\rho_h(\tilde{U}_h))}+\left\|\frac{\partial^l v_{j,h}}{\partial x_N^l}\right\|^2_{L^2(\rho_h(\tilde{U}_h))}\right)&\\
\le dS_{U_h}\left(\|u\|^2_{L^2(\Omega_1)}+\sum_{|\alpha|=l}\|D^\alpha u\|^2_{L^2(\Omega_1)}\right),&\quad (4.15)
\end{aligned}
$$

where $d=d(N,l,m,R)$ is a positive constant.

We denote by $L_n[\Omega_1]$ the linear subspace of $W^{l,2}(\Omega_1)^m$ generated by the $\varphi_{1,\mathcal{N}}[\Omega_1],\dots,\varphi_{n,\mathcal{N}}[\Omega_1]$. If $u\in L_n[\Omega_1]$ and $\|u\|_{L^2(\Omega_1)}=1$ then by (4.12), (4.13), and (4.15) we obtain

$$
\begin{aligned}
\int_{\Omega_1\setminus\Omega_2}|u|^2 &\le \sigma d\max_{h=1,\dots,\sigma}S_{U_h}\left(\frac{b}{a}+a^{-1}Q_{\Omega_1}(u)\right)\\
&\le \sigma d\max_{h=1,\dots,\sigma}S_{U_h}\left(\frac{b}{a}+a^{-1}\lambda_{n,\mathcal{N}}[\Omega_1]\right).
\end{aligned}
$$

Let $\mathcal{T}_{12}$ be the restriction operator from Ω_1 to Ω_2. Clearly, $\mathcal{T}_{12}$ maps $W^{l,2}(\Omega_1)^m$ to $W^{l,2}(\Omega_2)^m$. For all $n\in\mathbb{N}$ and for all $u\in L_n[\Omega_1]$ with $\|u\|_{L^2(\Omega_1)}=1$, we have

$$
\begin{aligned}
\|\mathcal{T}_{12}u\|^2_{L^2(\Omega_2)} &= \int_{\Omega_1}|u|^2-\int_{\Omega_1\setminus\Omega_2}|u|^2\\
&\ge 1-\sigma d\max_{h=1,\dots,\sigma}S_{U_h}\left(\frac{b}{a}+a^{-1}\lambda_{n,\mathcal{N}}[\Omega_1]\right),
\end{aligned}
$$

and, thanks to condition (1.15),

$$
Q_{\Omega_2}(\mathcal{T}_{12}u)\le Q_{\Omega_1}(u)\le \lambda_{n,\mathcal{N}}[\Omega_1].
$$

Therefore, using the terminology of [37], $\mathcal{T}_{12}$ is a transition operator from $H_{W^{l,2}(\Omega_1)^m}$ to $H_{W^{l,2}(\Omega_2)^m}$ with the measure of vicinity

$$
\delta(H_{W^{l,2}(\Omega_1)^m},H_{W^{l,2}(\Omega_2)^m})=\max_{h=1,\dots,\sigma}S_{U_h},
$$

and the parameters $a_n = \sigma d a^{-1}(b + \lambda_{n,\mathcal{N}}[\Omega_1])$, $b_n = 0$. As a consequence, by a variant of the general spectral stability theorem [37, Theorem 3.2] it follows that

$$\lambda_{n,\mathcal{N}}[\Omega_2] \leq \lambda_{n,\mathcal{N}}[\Omega_1] + 2(a_n \lambda_{n,\mathcal{N}}[\Omega_1] + b_n)\delta(H_{W^{l,2}(\Omega_1)^m}, H_{W^{l,2}(\Omega_2)^m}),$$

if $\delta(H_{W^{l,2}(\Omega_1)^m}, H_{W^{l,2}(\Omega_2)^m}) < (2a_n)^{-1}$. This concludes the proof. $\square$

Lemma 4.9. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $L, a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta} \in L^\infty(\cup_{h=1}^s V_h)$ satisfy $\|A^{ij}_{\alpha\beta}\|_{L^\infty(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C(\mathcal{A})$ with the same constants a, b.*

Then for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, a, b$, such that

$$\lambda_{n,\mathcal{N}}[\Omega_2] \leq \lambda_{n,\mathcal{N}}[\Omega_1] + c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $\Omega_2 \subset \Omega_1$ and $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon_n$.

Proof. The proof can be done adapting that of [38, Lemma 6.11] and using Lemma 4.8. $\square$

Next we consider the case $\Omega_2 = \phi_\epsilon(\Omega_1)$, where ϕ_ϵ is defined in (4.6). We recall here [38, Lemma 6.13] which will be used in the sequel.

Lemma 4.10. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Then there exist $\epsilon_0, A, R > 0$ and $\sigma \in \mathbb{N}$ depending only on $N, \mathcal{A}$, and for each open set $\Omega \in C(\mathcal{A})$ and for each $0 < \epsilon < \epsilon_0$ there exist isometries $\{\rho_h\}_{h=1}^\sigma$ and sets $\{U_h\}_{h=1}^\sigma$, $\{\tilde{U}_h\}_{h=1}^\sigma$ of ρ_h-patches U_h, $\tilde{U}_h$ satisfying conditions (a), (b), (c), (d) in Lemma 4.8 with $\Omega_1 = \Omega$ and $\Omega_2 = \phi_\epsilon(\Omega)$ and such that $\max_{h=1,\dots,\sigma} S_{U_h} < A\epsilon$.*

Theorem 4.11. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $L, a, b > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta} \in C^{0,1}(\cup_{h=1}^s V_h)$ satisfy $\|A^{ij}_{\alpha\beta}\|_{C^{0,1}(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C(\mathcal{A})$ with the same constants a, b.*

Then for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, a, b$ such that

$$|\lambda_{n,\mathcal{N}}[\Omega_1] - \lambda_{n,\mathcal{N}}[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2), \tag{4.16}$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon_n$.

Proof. For the sake of simplicity, in this proof we will use c_n, ϵ_n to denote positive constants depending only on $n, N, \mathcal{A}, l, m, L, a, b$, and their values are not necessarily the same for all the inequalities below.

Let $E > 0$ be as in Lemma 4.4. Let $0 < \epsilon < E$ and $\Omega_1, \Omega_2 \in C(\mathcal{A})$ be such that $d_{\mathcal{A}}(\Omega_1, \Omega_2) \leq \epsilon/s$. We set $\Omega_3 = \Omega_1 \cap \Omega_2$. Clearly, $\Omega_3 \in C(\mathcal{A})$ and $d_{\mathcal{A}}(\Omega_3, \Omega_1), d_{\mathcal{A}}(\Omega_3, \Omega_2) < \frac{\epsilon}{2s}$. By Lemma 4.4 applied to the couple of open sets Ω_1, Ω_3 it follows that $\phi_\epsilon(\Omega_1) \subset \Omega_3$ hence

$$\phi_\epsilon(\Omega_3) \subset \phi_\epsilon(\Omega_1) \subset \Omega_3. \tag{4.17}$$

Now we apply Lemma 4.10 to the set $\Omega = \Omega_3$. It follows that if $0 < \epsilon < \epsilon_0$ there exist rotations $\{\rho_h\}_{h=1}^{\sigma}$ and two sets $\{U_h\}_{h=1}^{\sigma}$, $\{\tilde{U}_h\}_{h=1}^{\sigma}$ of ρ_h-patches U_h, $\tilde{U}_h$ satisfying conditions (a), (b), (c), (d) in Lemma 4.8 with Ω_3, $\phi_\epsilon(\Omega_3)$ replacing Ω_1, Ω_2 respectively, and such that $\max_{h=1,\dots,\sigma} S_{U_h} < A\epsilon$. In particular,

$$\Omega_3 \setminus \phi_\epsilon(\Omega_3) \subset \cup_{h=1}^{\sigma} U_h, \tag{4.18}$$

hence by (4.17), (4.18) it follows that

$$\Omega_3 \setminus \phi_\epsilon(\Omega_1) \subset \cup_{h=1}^{\sigma} U_h.$$

Now we apply Lemma 4.8 to the couple of open sets Ω_3, $\phi_\epsilon(\Omega_1)$ by using the sets of patches defined above. Since $\max_{h=1,\dots,\sigma} S_{U_h} < A\epsilon$, by Lemma 4.8 it follows that if $A\epsilon < d_n^{-1}$ then

$$\lambda_{n,\mathcal{N}}[\phi_\epsilon(\Omega_1)] \leq \lambda_{n,\mathcal{N}}[\Omega_3](1 + d_n A\epsilon), \tag{4.19}$$

where d_n is defined by (4.11). By Lemma 1.12 and inequality (4.19) it follows that there exist $c_n, \epsilon_n > 0$ such that

$$\lambda_{n,\mathcal{N}}[\phi_\epsilon(\Omega_1)] \leq \lambda_{n,\mathcal{N}}[\Omega_3] + c_n\epsilon \tag{4.20}$$

if $0 < \epsilon < \epsilon_n$. On the other hand, by Lemma 1.12, Corollary 4.3, and by inequalities (4.7), (4.8) it follows that there exist c_n, $\epsilon_n > 0$ such that

$$|\lambda_{n,\mathcal{N}}[\phi_\epsilon[\Omega_1]] - \lambda_{n,\mathcal{N}}[\Omega_1]| \leq c_n\epsilon \tag{4.21}$$

if $0 < \epsilon < \epsilon_n$. Thus by (4.20), (4.21) it follows that there exist $c_n, \epsilon_n > 0$ such that

$$\lambda_{n,\mathcal{N}}[\Omega_1] \leq \lambda_{n,\mathcal{N}}[\Omega_3] + c_n\epsilon \tag{4.22}$$

if $0 < \epsilon < \epsilon_n$. By Lemma 4.9 applied to the couple of open sets Ω_1, Ω_3 it follows that there exist $c_n, \epsilon_n > 0$ such that

$$\lambda_{n,\mathcal{N}}[\Omega_3] \leq \lambda_{n,\mathcal{N}}[\Omega_1] + c_n\epsilon \tag{4.23}$$

if $0 < \epsilon < \epsilon_n$. Thus, by (4.22), (4.23) it follows that

$$|\lambda_{n,\mathcal{N}}[\Omega_1] - \lambda_{n,\mathcal{N}}[\Omega_3]| \leq c_n\epsilon \tag{4.24}$$

if $0 < \epsilon < \epsilon_n$. It is clear that this whole discussion holds if we replace Ω_1 with Ω_2. Therefore we obtain

$$|\lambda_{n,\mathcal{N}}[\Omega_2] - \lambda_{n,\mathcal{N}}[\Omega_3]| \leq c_n\epsilon \tag{4.25}$$

if $0 < \epsilon < \epsilon_n$. By (4.24), (4.25) we finally deduce that for each $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ such that

$$|\lambda_{n,\mathcal{N}}[\Omega_1] - \lambda_{n,\mathcal{N}}[\Omega_2]| \leq c_n\epsilon,$$

for all $0 < \epsilon < \epsilon_n$ and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon$. Finally, by arguing as in the last lines of the proof of Theorem 4.5 we deduce the validity of (4.16). □

As for Dirichlet boundary conditions we have a version of Theorem 4.11 in terms of ϵ-neighborhoods with respect to the atlas distance.

Corollary 4.12. *Under the same assumptions of Theorem 4.11, for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, a, b$, such that*

$$|\lambda_{n,\mathcal{N}}[\Omega_1] - \lambda_{n,\mathcal{N}}[\Omega_2]| \leq c_n\epsilon,$$

for all $0 < \epsilon < \epsilon_n$ and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying (1.3) or (1.4).

4.4 Estimates via the lower Hausdorff-Pompeiu deviation

The atlas distance obviously depends on the choice of the atlas $\mathcal{A}$. However, as shown in Theorem 1.4, it can be controlled by the lower Hausdorff-Pompeiu deviation, and therefore by the Hausdorff distance. We have the following

Theorem 4.13. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $l, m \in \mathbb{N}$, $L, M, a, b > 0$. Let $\omega : [0, \infty[\to [0, \infty[$ be a continuous non-decreasing function satisfying $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$. Moreover, for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A^{ij}_{\alpha\beta} \in C^{0,1}(\cup_{h=1}^s V_h)$ satisfy $\|A^{ij}_{\alpha\beta}\|_{C^{0,1}(\cup_{h=1}^s V_h)} \leq L$ and conditions (1.14)-(1.16) for any $\Omega \in C^{\omega(\cdot)}_M(\mathcal{A})$ with the same constants a, b.*

Then for any $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, M, a, b, \omega$ such that, for both Dirichlet and Neumann boundary conditions, we have

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)), \tag{4.26}$$

for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying $d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) < \epsilon_n$.

Proof. We recall that, if $\Omega_1, \Omega_2 \in C(\mathcal{A})$ then also $\Omega_1, \Omega_2 \in C(\tilde{\mathcal{A}})$ where $\tilde{\mathcal{A}} = (\rho/2, s, s', \{(V_h)_{\rho/2}\}_{h=1}^s, \{r_h\}_{h=1}^s)$. Thus by inequalities (4.9), (4.16) applied to Ω_1, Ω_2 as open sets in $C(\tilde{\mathcal{A}})$ and by inequality (1.6) we obtain inequality (4.26). □

Corollary 4.14. *Under the same assumptions of Theorem 4.13, for each $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, l, m, L, M, a, b, \omega$ such that, for both Dirichlet and Neumann boundary conditions, we have*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(\epsilon),$$

for all $0 < \epsilon < \epsilon_n$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying (1.7) or (1.8).

4.5 Estimates via the Lebesgue measure

In the previous sections we have shown continuity of Dirichlet and Neumann eigenvalues with respect to the atlas distance and the lower Hausdorff-Pompeiu deviation. In this section we prove estimates involving the Lebesgue measure, under some additional assumption on the regularity of the eigenfunctions.

We recall the following result from [39], which will be used in the proof of Theorem 4.16.

Theorem 4.15. *Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$, $l \in \mathbb{N}$, $M > 0$. Let $\Omega_1, \Omega_2 \in C_M^{l-1,1}(\mathcal{A})$. For all $l \in \mathbb{N}$, $1 \leq p < \infty$ there exist linear maps*

$$\mathcal{T}_{\mathcal{D}} : W_0^{l,p}(\Omega_1) \to W_0^{l,p}(\Omega_2) \ \text{ and } \ \mathcal{T}_{\mathcal{N}} : W^{l,p}(\Omega_1) \to W^{l,p}(\Omega_2),$$

with the following properties:

(i) there exists $C_1 > 0$ depending only on $\mathcal{A}, l, M, p$ such that

$$\|\mathcal{T}_{\mathcal{D}}\|, \|\mathcal{T}_{\mathcal{N}}\| \leq C_1.$$

(ii) there exists $C_2 > 0$ depending only on $\mathcal{A}$, and an open set $\Omega_3 \subset \Omega_1 \cap \Omega_2$ such that

$$|\Omega_1 \setminus \Omega_3|,\ |\Omega_2 \setminus \Omega_3| \le C_2 |\Omega_1 \vartriangle \Omega_2|,$$

and such that

$$\mathcal{T}_{\mathcal{D}}[u](x) = u(x), \qquad \mathcal{T}_{\mathcal{N}}[v](x) = v(x),$$

for all $u \in W^{l,p}_0(\Omega_1), v \in W^{l,p}(\Omega_2)$, $x \in \Omega_3$.

If $p = \infty$, there exist linear maps

$$\mathcal{T}_{\mathcal{D}} : \tilde{W}^{l,\infty}_0(\Omega_1) \to \tilde{W}^{l,\infty}_0(\Omega_2) \ \ \text{and} \ \ \mathcal{T}_{\mathcal{N}} : W^{l,\infty}(\Omega_1) \to W^{l,\infty}(\Omega_2),$$

satisfying properties (i)-(ii), where $\tilde{W}^{l,\infty}_0(\Omega)$ is the space of those functions in $W^{l,\infty}(\Omega)$ such that their zero-extension belongs to $W^{l,\infty}(\mathbb{R}^N)$.

Theorem 4.16. *Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$, $l, m \in \mathbb{N}$, $M, a, b > 0$. For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \le l$ and for all $i, j \in \mathbb{N}$ with $i, j \le m$, let $A^{ij}_{\alpha\beta}$ be measurable real-valued functions defined on $\cup_{h=1}^s V_h$, satisfying conditions (1.14)-(1.16) for any $\Omega \in C^{l-1,1}_M(\mathcal{A})$ with the same constants a, b.*

Let $2 < p \le \infty$, $0 < M_n < \infty$ for all $n \in \mathbb{N}$, and

$$\mathcal{X} = \left\{ \Omega \in C^{l-1,1}_M(\mathcal{A}) : \|\varphi_n[\Omega]\|_{W^{l,p}(\Omega)^m} \le M_n, \forall n \in \mathbb{N} \right\}.$$

Then for any $n \in \mathbb{N}$ there exists $c_n > 0$ depending only on $n, \mathcal{A}, l, m, M, a, b, p$, $M_1, \dots, M_n$ such that

$$\lambda_n[\Omega_2] \le \lambda_n[\Omega_1] + c_n |\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}}, \tag{4.27}$$

for all $\Omega_1 \in \mathcal{X}$, $\Omega_2 \in C^{l-1,1}_M(\mathcal{A})$ such that $|\Omega_1 \vartriangle \Omega_2| < c_n^{-1}$.

Proof. Let $\Omega_1 \in \mathcal{X}$ and $\Omega_2 \in C^{l-1,1}_M(\mathcal{A})$. To shorten our notation we set $\varphi_{n,1} = \varphi_n[\Omega_1]$, for all $n \in \mathbb{N}$. We denote by $\mathcal{L}_1$ the space of the finite linear combinations of the eigenfunctions $\varphi_{n,1}$. Moreover, we define a linear operator

$$T_{12} : \mathcal{L}_1 \to \mathrm{Dom}(H^{1/2}_{\Omega_2})$$

by setting

$$T_{12}[\varphi_{n,1}]_i = \mathcal{T}_{\mathcal{D}}(\varphi_{n,1})_i,$$

in the Dirichlet case, and

$$T_{12}[\varphi_{n,1}]_i = \mathcal{T}_{\mathcal{N}}(\varphi_{n,1})_i.$$

in the Neumann case, for all $n \in \mathbb{N}$ and for all $1 \le i \le m$. Here $\mathcal{T}_{\mathcal{D}}, \mathcal{T}_{\mathcal{N}}$ are the operators provided by Theorem 4.15. Note that T_{12} is well-defined. Indeed, by assumption $\mathcal{L}_1 \subset W^{l,p}(\Omega_1)^m$, and in the Dirichlet case $\mathcal{L}_1 \subset W_0^{l,p}(\Omega_1)^m$. Moreover, T_{12} takes values in $\mathrm{Dom}(H_{\Omega_2}^{1/2})$ because in the Dirichlet case $W_0^{l,p}(\Omega_2)^m \subset W_0^{l,2}(\Omega_2)^m = \mathrm{Dom}(H_{\Omega_2}^{1/2})$, and in the Neumann case $W^{l,p}(\Omega_2)^m \subset W^{l,2}(\Omega_2)^m = \mathrm{Dom}(H_{\Omega_2}^{1/2})$.

To prove (4.27) we apply the general spectral stability theorem [37, Theorem 3.2]. We need to prove that T_{12} is a transition operator from H_{Ω_1} to H_{Ω_2}. By Theorem 4.15, $T_{12}\varphi_n = \varphi_n$ on Ω_3 where Ω_3 is as in Theorem 4.15. Thus

$$
\begin{aligned}
(H_{\Omega_2}^{1/2}&T_{12}\varphi_{k,1}, H_{\Omega_2}^{1/2}T_{12}\varphi_{r,1})_{L^2(\Omega_2)^m} = Q_{\Omega_2}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) \\
&= Q_{\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) + Q_{\Omega_2\setminus\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) \\
&= Q_{\Omega_3}(\varphi_{k,1}, \varphi_{r,1}) + Q_{\Omega_2\setminus\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) \\
&= Q_{\Omega_1}(\varphi_{k,1}, \varphi_{r,1}) - Q_{\Omega_1\setminus\Omega_3}(\varphi_{k,1}, \varphi_{r,1}) + Q_{\Omega_2\setminus\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) \\
&= (H_{\Omega_1}^{1/2}\varphi_{k,1}, H_{\Omega_1}^{1/2}\varphi_{r,1})_{L^2(\Omega_1)^m} - Q_{\Omega_1\setminus\Omega_3}(\varphi_{k,1}, \varphi_{r,1}) \\
&\qquad + Q_{\Omega_2\setminus\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}),
\end{aligned}
\tag{4.28}
$$

for all $k, r \in \mathbb{N}$. By Hölder's inequality

$$
Q_{\Omega_1\setminus\Omega_3}(\varphi_{k,1}, \varphi_{r,1}) \le c M_k M_r |\Omega_1 \setminus \Omega_3|^{1-\frac{2}{p}},
\tag{4.29}
$$

and by Theorem 4.15 we have

$$
Q_{\Omega_2\setminus\Omega_3}(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1}) \le c M_k M_r |\Omega_2 \setminus \Omega_3|^{1-\frac{2}{p}},
\tag{4.30}
$$

and

$$
|\Omega_1 \setminus \Omega_3|,\ |\Omega_2 \setminus \Omega_3| \le c|\Omega_1 \vartriangle \Omega_2|,
\tag{4.31}
$$

where $c > 0$ depends only on $\mathcal{A}, l, m, M, a, b, p$. Thus by (4.28)-(4.31) it follows that

$$
\begin{aligned}
|(H_{\Omega_2}^{1/2}T_{12}\varphi_{k,1}, H_{\Omega_2}^{1/2}T_{12}\varphi_{r,1})_{L^2(\Omega_2)^m} - (H_{\Omega_1}^{1/2}\varphi_{k,1}, H_{\Omega_1}^{1/2}\varphi_{r,1})_{L^2(\Omega_1)^m}| \\
\le \tilde{c}_1 M_k M_r |\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}},
\end{aligned}
\tag{4.32}
$$

and similarly

$$
|(T_{12}\varphi_{k,1}, T_{12}\varphi_{r,1})_{L^2(\Omega_2)^m} - (\varphi_{k,1}, \varphi_{r,1})_{L^2(\Omega_1)^m}| \le \tilde{c}_2 M_k M_r |\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}},
\tag{4.33}
$$

for all $k, r \in \mathbb{N}$, where $\tilde{c}_1, \tilde{c}_2 > 0$ depend only on $\mathcal{A}, l, m, M, a, b, p$.

By (4.32), (4.33) it follows that T_{12} is a transition operator from H_{Ω_1} to H_{Ω_2} with parameters $a_{kr} = \tilde{c}_1 M_k M_r$, $b_{kr} = \tilde{c}_2 M_k M_r$ and measure of vicinity $\delta(H_{\Omega_1}, H_{\Omega_2}) = |\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}}$ (see [37, Definition 3.1]). Thus by [37, Theorem 3.2] it follows that

$$\lambda_n[\Omega_2] \leq \lambda_n[\Omega_1] + (2a_n\lambda_n[\Omega_1] + b_n)\delta(H_{\Omega_1}, H_{\Omega_2}), \tag{4.34}$$

if $\delta(H_{\Omega_1}, H_{\Omega_2}) \leq (2a_n)^{-1}$, where $a_n = (\sum_{k,r=1}^n a_{kr}^2)^{1/2} = \tilde{c}_1 \sum_{k=1}^n M_k^2$, $b_n = (\sum_{k,r=1}^n b_{kr}^2)^{1/2} = \tilde{c}_2 \sum_{k=1}^n M_k^2$. Furthermore, by Lemma 1.12 there exists $\Lambda_n > 0$ depending only on $n, \mathcal{A}, l, m, a, b$ such that

$$\lambda_n[\Omega] \leq \Lambda_n \tag{4.35}$$

for all $\Omega \in C_M^{l-1,1}(\mathcal{A})$. Thus, inequality (4.27) follows by combining (4.34) and (4.35). $\square$

It is well known that if $\Omega_2 \subset \Omega_1$ then $\lambda_{n,\mathcal{D}}[\Omega_1] \leq \lambda_{n,\mathcal{D}}[\Omega_2]$. Thus by Theorem 4.16 we immediately deduce the following corollary concerning Dirichlet eigenvalues.

Corollary 4.17. *Under the same assumptions of Theorem 4.16, for any $n \in \mathbb{N}$ there exists $c_n > 0$ depending only on $n, \mathcal{A}, l, m, M, a, b, p, M_1, \dots, M_n$, such that*

$$\lambda_{n,\mathcal{D}}[\Omega_1] \leq \lambda_{n,\mathcal{D}}[\Omega_2] \leq \lambda_{n,\mathcal{D}}[\Omega_1] + c_n|\Omega_1 \setminus \Omega_2|^{1-\frac{2}{p}},$$

for all Ω_2 of class $C_M^{l-1,1}(\mathcal{A})$ such that $\Omega_2 \subset \Omega_1$ and $|\Omega_1 \setminus \Omega_2| < c_n^{-1}$.

If we assume that both Ω_1 and Ω_2 belong to $\mathcal{X}$ then it is possible to swap Ω_1 and Ω_2 in (4.27). In this way we obtain a two-sided estimate for both Dirichlet and Neumann eigenvalues without assuming that $\Omega_2 \subset \Omega_1$ as in Corollary 4.17.

Corollary 4.18. *Under the same assumptions of Theorem 4.16, for any $n \in \mathbb{N}$ there exists $c_n > 0$ depending only on $n, \mathcal{A}, l, m, M, a, b, p, M_1, \dots, M_n$, such that*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n|\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}},$$

for all $\Omega_1, \Omega_2 \in \mathcal{X}$ such that $|\Omega_1 \vartriangle \Omega_2| < c_n^{-1}$.

If Ω is an open set with sufficiently smooth boundary then $\Omega \in \mathcal{X}$ with $p = \infty$.

Lemma 4.19. *Let $\mathcal{A} = (\rho, s, s', \{V_h\}_{h=1}^s, \{r_h\}_{h=1}^s)$ be an atlas in $\mathbb{R}^N$, $l, m \in \mathbb{N}$, $B, M, a, b > 0$. For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq l$ and for all $i, j \in \mathbb{N}$ with $i, j \leq m$, let $A_{\alpha\beta}^{ij} \in C^l(\overline{\cup_{h=1}^s V_h})$ satisfy $\|A_{\alpha\beta}^{ij}\|_{C^l(\overline{\cup_{h=1}^s V_h})} \leq B$, and conditions (1.14)-(1.16) for any $\Omega \in C_M^{2l}(\mathcal{A})$ with the same constants a, b.*

Then $\varphi_n[\Omega] \in W^{2l-1,\infty}(\Omega)$ and there exists $C > 0$ depending only on $\mathcal{A}, l, m, B, M, a, b$ such that

$$\|\varphi_n[\Omega]\|_{W^{k,\infty}(\Omega)^m} \leq C(1 + \lambda_n[\Omega])^{\frac{N}{4l} + \frac{k}{2l}} \tag{4.36}$$

for all $k = 0, \ldots, 2l - 1$ and $\Omega \in C_M^{2l}(\mathcal{A})$.

Proof. It is well known that under our regularity assumptions $\mathrm{Dom}(H_\Omega) \subset W^{2l,2}(\Omega)^m$ (see e.g., [9, Section 10]). Moreover, since the coefficients $A_{\alpha\beta}^{ij}$ are of class C^l and we impose either Dirichlet or Neumann boundary conditions, we can resort to the general setting of [9].

Thus, by [9, Subsection 10.3] it follows that if $u \in \mathrm{Dom}(H_\Omega)$ and $H_\Omega u \in L^p(\Omega)^m$ for some $p > 1$ then $u \in W^{2l,p}(\Omega)^m$ and

$$\|u\|_{W^{2l,p}(\Omega)^m} \leq c(\|H_\Omega u\|_{L^p(\Omega)^m} + \|u\|_{L^p(\Omega)^m}),$$

where c is a positive constant. In particular, if u is an eigenfunction corresponding to an eigenvalue λ and $u \in L^p(\Omega)^m$ then

$$\|u\|_{W^{2l,p}(\Omega)^m} \leq c(1 + \lambda)\|u\|_{L^p(\Omega)^m}. \tag{4.37}$$

By the a priori estimate (4.37) and a bootstrap argument one can finally prove estimate (4.36). See for instance [37, Theorem 5.1], where in the proof one has to replace [37, formula (5.5)] by (4.37) (note that [37, formulas (5.6) and (5.7)] remain valid in the vectorial case thanks to Theorem 1.7). □

By Corollary 4.18 and Lemma 4.19 we immediately deduce the validity of the following

Corollary 4.20. *Under the same assumptions of Lemma 4.19, for all $n \in \mathbb{N}$ there exists $c_n > 0$ depending only on $n, \mathcal{A}, l, m, B, M, a, b$ such that*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n |\Omega_1 \vartriangle \Omega_2|,$$

for all $\Omega_1, \Omega_2 \in C_M^{2l}(\mathcal{A})$ satisfying $|\Omega_1 \vartriangle \Omega_2| < c_n^{-1}$.

Chapter 5

Elliptic systems of partial differential equations

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $m \in \mathbb{N}$. In this chapter we discuss the eigenvalue problem for elliptic systems of second order partial differential equations subject to boundary conditions of Dirichlet and Neumann type. More precisely, we consider the following eigenvalue problem

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{m} a_{\alpha\beta}^{ij} \frac{\partial u_i}{\partial x_\alpha} \frac{\partial \varphi_j}{\partial x_\beta} dx = \lambda \int_\Omega u \cdot \varphi dx, \tag{5.1}$$

for any $\varphi \in V(\Omega)^m$, in the unknowns $u \in V(\Omega)^m$ (the eigenfunction), $\lambda \in \mathbb{R}$ (the eigenvalue). Here $V(\Omega)$ denotes either $H_0^1(\Omega)$ (for Dirichlet boundary conditions) or $H^1(\Omega)$ (for Neumann boundary conditions). Moreover, we shall assume that $a_{\alpha\beta}^{ij} \in \mathbb{R}$ are constant coefficients satisfying condition (1.14) and the so-called Legendre-Hadamard condition, namely

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{m} a_{\alpha\beta}^{ij} \xi_i \xi_j \eta_\alpha \eta_\beta \geq \theta |\xi|^2 |\eta|^2, \quad \forall \xi \in \mathbb{R}^m, \forall \eta \in \mathbb{R}^N, \tag{5.2}$$

for some $\theta > 0$. We note that condition (5.2) is weaker than the so-called Legendre condition, namely

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{m} a_{\alpha\beta}^{ij} \xi_i^\alpha \xi_j^\beta \geq \theta |\xi|^2, \quad \forall \xi \in \mathbb{R}^{m\times N},$$

for some $\theta > 0$. However, condition (5.2) is sufficient in order to prove coercivity of the bilinear form associated with problem (5.1), in particular condition (5.2) implies conditions (1.15), (1.16). Therefore, Theorem 1.9 applies to problem (5.1).

We remark that problem (5.1) includes some important problems in linear elasticity. For instance, the Lamé eigenvalue problem

$$\int_\Omega \left(\left(\nabla u+\nabla^t u\right):\left(\nabla\varphi+\nabla^t\varphi\right)+k\mathrm{div}u\mathrm{div}\varphi\right)dx=\lambda\int_\Omega u\cdot\varphi dx, \tag{5.3}$$

with $k\in]1-\frac{2}{N},+\infty[$ for any $\varphi\in V(\Omega)^N$, corresponds to the choice $a_{\alpha\beta}^{ij}=2(\delta_{ij}\delta_{\alpha\beta}+\delta_{i\beta}\delta_{j\alpha})+k\delta_{i\alpha}\delta_{j\beta}$, where δ_{ij} is the Kronecher delta. We refer to [61] for a detailed discussion on problem (5.3). Note that, in the case of Dirichlet boundary conditions (i.e., $V(\Omega)=H_0^1(\Omega)$), thanks to an integration by parts, problem (5.3) can also be associated with the coefficients $a_{\alpha\beta}^{ij}=\delta_{ij}\delta_{\alpha\beta}+k\delta_{i\alpha}\delta_{j\beta}$, and in this case it is easy to see that, in order to get inequality (5.2), the constant k can be chosen to be non-negative, namely $k\geq 0$ (cf. e.g., [58]). We also observe that problem (5.3) is very similar to the Reissner-Mindlin system (6.1), which arises in the study of the vibrations of a clamped plate. However, since problem (6.1) presents lower order terms, it is not included in the discussion of the present chapter (see Chapter 6).

5.1 Dirichlet boundary conditions

In the sequel of this chapter we shall use Einstein notation, hence summation symbols will be dropped.

The classical formulation of the Dirichlet problem reads

$$\begin{cases}-a_{\alpha\beta}^{ij}\frac{\partial^2 u_i}{\partial x_\alpha\partial x_\beta}=\lambda u_j,\ j=1,\dots,m, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases} \tag{5.4}$$

We consider on $H_0^1(\Omega)^m$ the bilinear form

$$<u,v>=\int_\Omega a_{\alpha\beta}^{ij}\frac{\partial u_i}{\partial x_\alpha}\frac{\partial v_j}{\partial x_\beta}dx, \tag{5.5}$$

for any $u,v\in H_0^1(\Omega)^m$. One can prove that the bilinear form (5.5) defines on $H_0^1(\Omega)^m$ a scalar product whose induced norm is equivalent to the standard one defined by (1.11). In this section we shall consider the space $H_0^1(\Omega)^m$ endowed with the scalar product (5.5).

We consider the operator S as a map from $H^1_0(\Omega)^m$ to its dual defined by

$$S[u][v]=\int_\Omega a^{ij}_{\alpha\beta}\frac{\partial u_i}{\partial x_\alpha}\frac{\partial v_j}{\partial x_\beta}dx, \tag{5.6}$$

for any $u,v\in H^1_0(\Omega)^m$. The operator S is easily seen to be a linear homeomorphism of $H^1_0(\Omega)^m$ onto its dual. We also denote by $\mathcal{J}$ the continuous embedding of $H^1_0(\Omega)^m$ into its dual, defined by

$$\mathcal{J}[u][v]:=\int_\Omega u\cdot v dx,\ \forall u,v\in H^1_0(\Omega)^m.$$

Note that problem (5.4) can be written in the following weak formulation

$$S[u][v]=\lambda\mathcal{J}[u][v],\ \forall v\in H^1_0(\Omega)^m. \tag{5.7}$$

We define the operator $T:=S^{(-1)}\circ\mathcal{J}$ from $H^1_0(\Omega)^m$ to itself. We have the following result, whose proof is similar to that of Lemma 2.1.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T is a non-negative compact selfadjoint operator in the Hilbert space $H^1_0(\Omega)^m$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $Tu=\mu u$ is satisfied for some $u\in H^1_0(\Omega)^m$, $\mu>0$ if and only if equation (5.1) is satisfied with $0\neq\lambda=\mu^{-1}$ for any $\varphi\in H^1_0(\Omega)^m$.*

5.1.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (5.4) in $\phi(\Omega)$ for any $\phi\in\mathcal{A}^1_\Omega$ and study the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ.

The main result of this section is the following

Theorem 5.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F be a finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $\mathcal{A}^1_\Omega$ and the real-valued maps which take $\phi\in\mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s=1,\dots,|F|$. Moreover, if $\tilde\phi\in\Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde\phi]$ assume the common value $\lambda_F[\tilde\phi]$ for all $j\in F$, and $\tilde\phi(\Omega)$ is of class C^2 then the Frechét differential of the map $\Lambda_{F,s}$ at the point $\tilde\phi$ is delivered by the formula*

$$d|_{\phi=\tilde\phi}(\Lambda_{F,s})[\psi]=-\lambda^s_F[\tilde\phi]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde\phi(\Omega)}a^{ij}_{\alpha\beta}\frac{\partial v^{(l)}_i}{\partial y_\alpha}\frac{\partial v^{(l)}_j}{\partial y_\beta}\zeta\cdot\nu d\sigma,$$

for all $\psi \in C_b^1(\Omega;\mathbb{R}^N)$*, where* $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ *and* $\{v^{(l)}\}_{l\in F}$ *is an orthonormal basis in* $H_0^1(\tilde{\phi}(\Omega))^m$ *(with respect to the scalar product (5.5)) of the eigenspace associated with* $\lambda_F[\tilde{\phi}]$*.*

In order to prove Theorem 5.2 we consider equation (5.7) in $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$S[v][\psi] = \lambda \mathcal{J}[v][\psi], \quad \forall\ \psi \in H_0^1(\phi(\Omega))^m, \tag{5.8}$$

in the unknowns $v \in H_0^1(\phi(\Omega))^m$, $\lambda \in]0,\infty[$. We will denote by $H^1_{0,\phi}(\Omega)^m$ the space $H_0^1(\Omega)^m$ endowed with the form

$$< u, v >_\phi = S_\phi[u][v],\ \forall u, v \in H_0^2(\Omega)^m.$$

Moreover, we recall that

$$\mathcal{J}_\phi[u][w] = \int_\Omega u \cdot w |\mathrm{det}\nabla\phi| dx,\ \forall u, w \in H_0^1(\Omega)^m.$$

Note that the map from $H^1(\Omega)^m$ to $H^1(\phi(\Omega))^m$ which maps u to $u \circ \phi^{(-1)}$ for all $u \in H^1(\Omega)^m$ is a linear homeomorphism. Hence, equation (5.7) is equivalent to

$$S_\phi[u][\varphi] = \lambda \mathcal{J}_\phi[u][\varphi], \quad \forall\ \varphi \in H^1_{0,\phi}(\Omega)^m,$$

where $u = v \circ \phi$. It turns out that the operator T defined in Lemma 5.1 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator T_ϕ defined on $H^1_{0,\phi}(\Omega)^m$ by

$$T_\phi := S_\phi^{(-1)} \circ \mathcal{J}_\phi. \tag{5.9}$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3. We recall that $\mathcal{L}(H_0^1(\Omega)^m)$ denotes the space of linear bounded operators from $H_0^1(\Omega)^m$ to itself and and $\mathcal{B}_s(H_0^1(\Omega)^m)$ denotes the space of bilinear forms on $H_0^1(\Omega)^m$.

Lemma 5.3. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1*. The operator* T_ϕ *defined in (5.9) is non-negative selfadjoint and compact on the Hilbert space* $H^1_{0,\phi}(\Omega)^m$*. The equation (5.8) is satisfied for some* $v \in H_0^1(\phi(\Omega))^m$ *if and only if the equation* $T_\phi u = \mu u$ *is satisfied with* $u = v \circ \phi$ *and* $\mu = \lambda^{-1}$*. Moreover, the map from* $\mathcal{A}^1_\Omega$ *to* $\mathcal{L}(H_0^1(\Omega)^m) \times \mathcal{B}_s(H_0^1(\Omega)^m)$ *which takes* $\phi \in \mathcal{A}^1_\Omega$ *to* $(T_\phi, < \cdot, \cdot >_\phi)$ *is real-analytic.*

Proof of Theorem 5.2. First of all, we note that by standard regularity theory (see e.g., [9, Subsection 10.3]) $v^{(l)} \in H^2(\tilde{\phi}(\Omega))^m$ for all $l \in F$. We observe that the proof is very similar to that of Theorem 2.2. It only remains to compute

$$< \mathrm{d}|_{\phi=\tilde{\phi}} T_\phi[\psi][u^{(l)}], u^{(l)} >_{\tilde{\phi}} = d|_{\phi=\tilde{\phi}} \mathcal{J}_\phi[\psi][u^{(l)}][u^{(l)}] \\ - \lambda_F^{-1}[\tilde{\phi}] d|_{\phi=\tilde{\phi}} S_\phi[\psi][u^{(l)}][u^{(l)}].$$

By formula (2.12) we have

$$d|_{\phi=\tilde{\phi}} \mathcal{J}_\phi[\psi][u^{(l)}][u^{(l)}] = \int_{\tilde{\phi}(\Omega)} |v^{(l)}|^2 \mathrm{div}\zeta dy.$$

Using Lemma 5.4 below, and the fact that $v^{(l)} = 0$ on $\partial\tilde{\phi}(\Omega)$ we obtain

$$d|_{\phi=\tilde{\phi}} S_\phi[\psi][u^{(l)}][u^{(l)}] \\ = -\int_{\partial\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \frac{\partial v_i^{(l)}}{\partial y_\alpha} \frac{\partial v_j^{(l)}}{\partial y_\beta} \zeta \cdot \nu d\sigma - \lambda_F[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \nabla(|v^{(l)}|^2) \cdot \zeta dy.$$

To conclude, just observe that

$$\int_{\tilde{\phi}(\Omega)} \nabla(|v^{(l)}|^2) \cdot \zeta dy = \int_{\partial\tilde{\phi}(\Omega)} |v^{(l)}|^2 \zeta \cdot \nu d\sigma - \int_{\tilde{\phi}(\Omega)} |v^{(l)}|^2 \mathrm{div}\zeta dy. \quad (5.10)$$

□

Lemma 5.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde{\phi} \in \mathcal{A}^1_\Omega$. Let $u^{(1)}, u^{(2)} \in H^1_0(\Omega)^m$ be such that $v^{(1)} = u^{(1)} \circ \tilde{\phi}^{-1}, v^{(2)} = u^{(2)} \circ \tilde{\phi}^{-1} \in H^2(\tilde{\phi}(\Omega))^m$. Then*

$$d|_{\phi=\tilde{\phi}} S_\phi[\psi][u^{(1)}][u^{(2)}] = -\int_{\partial\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \nu_\alpha \frac{\partial v_i^{(1)}}{\partial y_r} \frac{\partial v_j^{(2)}}{\partial y_\beta} \zeta_r d\sigma \\ - \int_{\partial\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \nu_\beta \frac{\partial v_j^{(2)}}{\partial y_r} \frac{\partial v_i^{(1)}}{\partial y_\alpha} \zeta_r d\sigma + \int_{\partial\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \frac{\partial v_i^{(1)}}{\partial y_\alpha} \frac{\partial v_j^{(2)}}{\partial y_\beta} \zeta \cdot \nu d\sigma \\ + \int_{\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \frac{\partial v_i^{(1)}}{\partial y_r} \frac{\partial^2 v_j^{(2)}}{\partial y_\alpha \partial y_\beta} \zeta_r d\sigma + \int_{\tilde{\phi}(\Omega)} a^{ij}_{\alpha\beta} \frac{\partial v_j^{(2)}}{\partial y_r} \frac{\partial^2 v_i^{(1)}}{\partial y_\alpha \partial y_\beta} \zeta_r d\sigma, \quad (5.11)$$

for all $\psi \in C^1_b(\Omega; \mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{-1}$.

Proof. We have

$$d|_{\phi=\tilde{\phi}}S_{\phi}[\psi][u^{(1)}][u^{(2)}] = -\int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_r}\frac{\partial \zeta_r}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_\beta}dy$$
$$-\int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_r}\frac{\partial \zeta_r}{\partial y_\beta}dy + \int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_\beta}\mathrm{div}\zeta dy. \quad (5.12)$$

Now note that

$$-\int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_r}\frac{\partial \zeta_r}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_\beta}dy = -\int_{\partial\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\nu_\alpha\frac{\partial v_i^{(1)}}{\partial y_r}\zeta_r\frac{\partial v_j^{(2)}}{\partial y_\beta}d\sigma$$
$$+\int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_r}\zeta_r\frac{\partial^2 v_j^{(2)}}{\partial y_\alpha\partial y_\beta}dy + \int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial^2 v_i^{(1)}}{\partial y_\alpha\partial y_r}\zeta_r\frac{\partial v_j^{(2)}}{\partial y_\beta}dy$$
$$= -\int_{\partial\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\nu_\alpha\frac{\partial v_i^{(1)}}{\partial y_r}\zeta_r\frac{\partial v_j^{(2)}}{\partial y_\beta}d\sigma + \int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_r}\zeta_r\frac{\partial^2 v_j^{(2)}}{\partial y_\alpha\partial y_\beta}dy$$
$$+\int_{\partial\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_\beta}\zeta\cdot\nu d\sigma - \int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_\alpha}\frac{\partial v_j^{(2)}}{\partial y_\beta}\mathrm{div}\zeta dy$$
$$-\int_{\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(1)}}{\partial y_\alpha}\frac{\partial^2 v_j^{(2)}}{\partial y_\beta\partial y_r}\zeta_r dy. \quad (5.13)$$

If we use in (5.12) the last equality in (5.13), and the first equality in (5.13) replacing i with j, α with β and $v^{(1)}$ with $v^{(2)}$, thanks to property (1.14) we get formula (5.11). □

5.1.2 Isovolumetric perturbations

As we have done in Chapters 2 and 3, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi],$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 5.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a non-empty finite subset of $\mathbb{N}$. Assume that $\tilde{\phi}\in\Theta_{F,\Omega}$ is such that $\tilde{\phi}(\Omega)$ is of class C^2 and that the eigenvalues $\lambda_j[\tilde{\phi}]$ have the common value $\lambda_F[\tilde{\phi}]$*

for all $j \in F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint if and only if there exists an orthonormal basis $v^{(1)}, \dots, v^{(|F|)}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (5.4) in $H^1_0(\tilde{\phi}(\Omega))^m$, and a constant $c \in \mathbb{R}$ such that

$$\sum_{l=1}^{|F|} a^{ij}_{\alpha\beta} \frac{\partial v^{(l)}_i}{\partial y_\alpha} \frac{\partial v^{(l)}_j}{\partial y_\beta} = c, \tag{5.14}$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$.

Now we introduce the following generalization of the notion of rotation invariance for scalar operators.

Definition 5.6. *The operator $\mathcal{L}$ formally defined by*

$$\mathcal{L}(u)_j = -a^{ij}_{\alpha\beta} \frac{\partial^2 u_i}{\partial x_\alpha \partial x_\beta}$$

is said to be rotation invariant if there exists a group homomorphism

$$Z : O_N(\mathbb{R}) \to O_m(\mathbb{R}),$$

(i.e., $Z(AB) = Z(A)Z(B)$ for all $A, B \in O_N(\mathbb{R})$) such that

$$\mathcal{L}\left(Z(R)^t u \circ R\right) = Z(R)^t \mathcal{L}(u) \circ R,$$

for any $R \in O_N(\mathbb{R})$, and for any $u \in H^2_{loc}(\mathbb{R}^N)^m$.

Remark 5.7. *We observe that, if $a^{ij}_{\alpha\beta} = 2(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\beta}\delta_{j\alpha}) + k\delta_{i\alpha}\delta_{j\beta}$ for any constant $k > 1 - \frac{2}{N}$ (which is the choice related to the Lamé system (5.3)), then the operator $\mathcal{L}$ is easily seen to be rotation invariant, Z being the identity map in $O_N(\mathbb{R})$. Moreover, note that Definition 5.6 applies to more general operators, such as the Reissner-Mindlin problem (6.1) (see Lemma 6.14).*

We have the following

Theorem 5.8. *Let B be a ball in $\mathbb{R}^N$ centered at zero, let S (defined in (5.6)) be a rotation invariant operator and let λ be an eigenvalue of S in B. Let F be the set of all indexes $j \in \mathbb{N}$ such that the j-th eigenvalue of S in B coincides with λ. Let $v^{(1)}, \dots, v^{(|F|)}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ in $H^1_0(B)^m$ (with respect to the scalar product (5.5)). Then there exists $c \in \mathbb{R}$ such that condition (5.14) holds.*

Proof. Since S is rotation invariant, $\{(Z(R)^t v_l)\circ R : l = 1,\dots,|F|\}$ is another orthonormal basis for the eigenspace associated with λ, where $R \in O_n(\mathbb{R})$, and $Z(R)$ is as in Definition 5.6. Since both $\{v^{(l)} : l = 1,\dots,|F|\}$ and $\{(Z(R)^t v^{(l)})\circ R : l = 1,\dots,|F|\}$ are orthonormal bases, then there exists $A[R] \in O_N(\mathbb{R})$ with matrix $(A_{kh}[R])_{k,h=1,\dots,|F|}$ such that

$$(Z(R)^t v^{(k)})\circ R = \sum_{l=1}^{|F|} A_{kl}[R] v^{(l)}. \tag{5.15}$$

Using (5.15) we get

$$\begin{aligned}\sum_{k=1}^{|F|} |v^{(k)}|^2 \circ R &= \sum_{k=1}^{|F|} |(Z(R)^t v^{(k)})\circ R|^2 \\ &= \sum_{k=1}^{|F|} \left(\sum_{l=1}^{|F|} A_{kl}[R] v^{(l)}\right)\cdot\left(\sum_{h=1}^{|F|} A_{kh}[R] v^{(h)}\right) \\ &= \sum_{k=1}^{|F|}\sum_{l,h=1}^{|F|} A_{lk}[R]A_{kh}[R](v^{(l)}\cdot v^{(h)}) = \sum_{l=1}^{|F|} |v^{(l)}|^2,\end{aligned}$$

and similarly,

$$\sum_{k=1}^{|F|}\left(a_{\alpha\beta}^{ij}\frac{\partial v_i^{(k)}}{\partial y_\alpha}\frac{\partial v_j^{(k)}}{\partial y_\beta}\right)\circ R = \sum_{l=1}^{|F|}\left(a_{\alpha\beta}^{ij}\frac{\partial v_i^{(l)}}{\partial y_\alpha}\frac{\partial v_j^{(l)}}{\partial y_\beta}\right).$$

This concludes the proof. □

Thus we get the following

Corollary 5.9. *Let Ω be a domain in $\mathbb{R}^N$ of class C^1. Let S be a rotation invariant operator. Let $\tilde{\phi} \in \mathcal{A}_\Omega^1$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (5.4) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ with volume constraint, for all $s = 1,\dots,|F|$.*

5.2 Neumann boundary conditions

The classical formulation of the Neumann problem reads

$$\begin{cases} -a_{\alpha\beta}^{ij}\frac{\partial^2 u_i}{\partial x_\alpha \partial x_\beta} = \lambda u_j, \ j = 1,\dots,m, & \text{in } \Omega, \\ a_{\alpha\beta}^{ij}\nu_\beta \frac{\partial u_i}{\partial x_\alpha} = 0, \ j = 1,\dots,m, & \text{on } \partial\Omega, \end{cases} \tag{5.16}$$

where Ω is a bounded domain in $\mathbb{R}^N$ of class C^1 and ν is the outer unit normal to $\partial\Omega$. Note that, differently from Dirichlet boundary conditions, in this case there is a nontrivial kernel. It is easy to see that the kernel is m-dimensional and given by the constants.

We set

$$H^{1,0}(\Omega)^m := \left\{ u \in H^1(\Omega)^m : \int_\Omega u dx = 0 \right\},$$

where by $\int_\Omega u dx$ we mean the vector $(\int_\Omega u_1 dx, \ldots, \int_\Omega u_m dx)$. We consider on $H^1(\Omega)^m$ the bilinear form (5.5) for any $u, v \in H^1(\Omega)^m$. One can prove that it defines on $H^{1,0}(\Omega)^m$ a scalar product whose induced norm is equivalent to the standard one defined by (1.11). We shall consider the space $H^{1,0}(\Omega)^m$ endowed with the scalar product (5.5). We denote by π the map of $H^1(\Omega)^m$ to $H^{1,0}(\Omega)^m$ defined by

$$\pi[u] = u - \frac{\int_\Omega u dx}{|\Omega|},$$

for all $u \in H^1(\Omega)^m$, where $|\Omega|$ denotes the Lebesgue measure of Ω. We denote by $\pi^\sharp$ the map of $H^1(\Omega)^m/\mathbb{R}^m$ onto $H^{1,0}(\Omega)^m$ defined by the equality $\pi = \pi^\sharp \circ p$, where p is the canonical projection of $H^1(\Omega)^m$ onto $H^1(\Omega)^m/\mathbb{R}^m$.

We consider the operator S defined by (5.6) as a map from $H^{1,0}(\Omega)^m$ to its dual. Note that, thanks to the Poincaré-Wirtinger Inequality, the norm induced from the quadratic form associated with the operator S is equivalent to the standard one of $H^{1,0}(\Omega)^m$ (as a closed subspace of $H^1(\Omega)^m$), and therefore it turns out that S is a linear homeomorphism of $H^{1,0}(\Omega)^m$ onto its dual.

We denote by $\mathcal{J}$ the continuous embedding of $H^1(\Omega)^m$ into its dual, defined by

$$\mathcal{J}[u][v] := \int_\Omega u \cdot v dx, \ \forall u, v \in H^1(\Omega)^m.$$

Note that problem (5.16) can be written in the following weak form

$$S[u][v] = \lambda \mathcal{J}[u][v], \ \forall v \in H^{1,0}(\Omega)^m. \tag{5.17}$$

We define the operator $T := (\pi^\sharp)^{(-1)} \circ S^{(-1)} \circ \mathcal{J} \circ \pi^\sharp$ from $H^1(\Omega)^m/\mathbb{R}^m$ to itself. We have the following result (see Lemma 2.1).

Lemma 5.10. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T is a non-negative compact selfadjoint operator in the Hilbert space $H^1(\Omega)^m/\mathbb{R}^m$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover,* *the equation* $Tu = \mu u$ *is satisfied for some* $u \in H^{1,0}(\Omega)^m$, $\mu > 0$ *if and only if equation (5.16) is satisfied with* $0 \neq \lambda = \mu^{-1}$ *for any* $\varphi \in H^{1,0}(\Omega)^m$.

We observe that the whole spectrum of problem (5.16) is given by the non-decreasing sequence $\{\lambda_j[\Omega]\}_{j\in\mathbb{N}}$, where $\lambda_1[\Omega] = \cdots = \lambda_m[\Omega] = 0$ and the other eigenvalues are given by Lemma 5.10.

5.2.1 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (5.16) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^1_\Omega$ and study the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ.

The main result of this section is the following

Theorem 5.11. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* C^1 *and* F *be a finite set in* $\mathbb{N}$. *The set* $\mathcal{A}_{F,\Omega}$ *is open in* $\mathcal{A}^1_\Omega$ *and the real-valued maps which take* $\phi \in \mathcal{A}_{F,\Omega}$ *to* $\Lambda_{F,s}[\phi]$ *are real-analytic on* $\mathcal{A}_{F,\Omega}$ *for all* $s = 1, \dots, |F|$. *Moreover, if* $\tilde{\phi} \in \Theta_{F,\Omega}$ *is such that the eigenvalues* $\lambda_j[\tilde{\phi}]$ *assume the common value* $\lambda_F[\tilde{\phi}]$ *for all* $j \in F$, *and* $\tilde{\phi}(\Omega)$ *is of class* C^2 *then the Frechét differential of the map* $\Lambda_{F,s}$ *at the point* $\tilde{\phi}$ *is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|} \int_{\partial\tilde{\phi}(\Omega)}\left(\lambda_F|v^{(l)}|^2 - a^{ij}_{\alpha\beta}\frac{\partial v^{(l)}_i}{\partial y_\alpha}\frac{\partial v^{(l)}_j}{\partial y_\beta}\right)\zeta\cdot\nu d\sigma, \quad (5.18)$$

for all $\psi \in C^1_b(\Omega;\mathbb{R}^N)$, *where* $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ *and* $\{v^{(l)}\}_{l\in F}$ *is an orthonormal basis in* $H^{1,0}(\tilde{\phi}(\Omega))^m$ *(with respect to the scalar product (5.5)) of the eigenspace associated with* $\lambda_F[\tilde{\phi}]$.

As we have done for Theorem 5.2, in order to prove Theorem 5.11 we consider equation (5.16) on $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$S[v][\psi] = \lambda\mathcal{J}[v][\psi], \quad \forall\, \psi \in H^{1,0}(\phi(\Omega))^m, \quad (5.19)$$

in the unknowns $v \in H^{1,0}(\phi(\Omega))^m$, $\lambda \in]0,\infty[$. We consider the operator S_ϕ as an operator acting from $H^{1,0}_\phi(\Omega)^m$ to its dual, where

$$H^{1,0}_\phi(\Omega)^m = \left\{u \in H^1(\Omega)^m : \int_\Omega u|\det\nabla\phi|dx = 0\right\}.$$

We will endow the space $H^{1,0}_{\phi}(\Omega)^m$ with the form

$$< u, v >_{\phi} = S_{\phi}[u][v],\ \forall u, v \in H^{1,0}_{\phi}(\Omega)^m.$$

Moreover, we denote by π_{ϕ} the map from $H^1(\Omega)^m$ to $H^{1,0}_{\phi}(\Omega)^m$ defined by

$$\pi_{\phi}[u] = u - \frac{\int_{\Omega} u|\mathrm{det}\nabla\phi|dx}{\int_{\Omega} |\mathrm{det}\nabla\phi|dx},$$

and by $\pi^{\sharp}_{\phi}$ the map from $H^1(\Omega)^m/\mathbb{R}^m$ onto $H^{1,0}_{\phi}(\Omega)^m$defined by the equality $\pi_{\phi} = \pi^{\sharp}_{\phi} \circ p$. Note that the map from $H^1(\Omega)^m$ to $H^1(\phi(\Omega))^m$ which maps u to $u \circ \phi^{(-1)}$ for all $u \in H^1(\Omega)^m$ is a linear homeomorphism. We also recall that

$$\mathcal{J}_{\phi}[u][w] = \int_{\Omega} u \cdot w|\mathrm{det}\nabla\phi|dx,\ \forall u, w \in H^1(\Omega)^m.$$

Hence, equation (5.19) is equivalent to

$$S_{\phi}[u][\varphi] = \lambda\mathcal{J}_{\phi}[u][\varphi],\ \ \forall\ \varphi \in H^{1,0}_{\phi}(\Omega)^m$$

where $u = v \circ \phi$. It turns out that the operator T defined in Lemma 5.10 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator T_{ϕ} defined on $H^{1,0}_{\phi}(\Omega)^m/\mathbb{R}^m$ by

$$T_{\phi} := (\pi^{\sharp}_{\phi})^{(-1)} \circ S^{(-1)}_{\phi} \circ \mathcal{J}_{\phi} \circ \pi^{\sharp}_{\phi} \tag{5.20}$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3.

Lemma 5.12. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T_{ϕ} defined in (5.20) is non-negative selfadjoint and compact on the Hilbert space $H^{1,0}_{\phi}(\Omega)^m/\mathbb{R}^m$. Equation (5.19) is satisfied for some $v \in H^{1,0}(\phi(\Omega))^m$ if and only if the equation $T_{\phi}u = \mu u$ is satisfied with $u = v \circ \phi$ and $\mu = \lambda^{-1}$. Moreover, the map from $\mathcal{A}^1_{\Omega}$ to $\mathcal{L}(H^{1,0}(\Omega)^m) \times \mathcal{B}_s(H^{1,0}(\Omega)^m)$ which takes $\phi \in \mathcal{A}^1_{\Omega}$ to $(T_{\phi}, < \cdot, \cdot >_{\phi})$ is real-analytic.*

Proof of Theorem 5.11. First of all, we note that by standard regularity theory (see e.g., [9, Subsection 10.3]) $v^{(l)} \in H^2(\tilde{\phi}(\Omega))^m$ for all $l \in F$. We observe that the proof is very similar to that of Teorem 2.2. It only remains to compute

$$\begin{aligned} < \mathrm{d}|_{\phi=\tilde{\phi}} T_{\phi}[\psi][u^{(l)}], u^{(l)} >_{\tilde{\phi}} = d|_{\phi=\tilde{\phi}} \mathcal{J}_{\phi} \circ \pi_{\phi}[\psi][u^{(l)}][\pi_{\tilde{\phi}}(u^{(l)})] \\ - \lambda_F^{-1}[\tilde{\phi}] d|_{\phi=\tilde{\phi}} S_{\phi} \circ \pi_{\phi}[\psi][u^{(l)}][\pi_{\tilde{\phi}}(u^{(l)})]. \end{aligned}$$

By formula (2.12) we have

$$d|_{\phi=\tilde{\phi}}\mathcal{J}_\phi \circ \pi_\phi[\psi][u^{(l)}][\pi_{\tilde{\phi}}(u^{(l)})] = \int_{\tilde{\phi}(\Omega)} |v^{(l)}|^2 \mathrm{div}\zeta dy.$$

Using Lemma 5.4 we obtain

$$\begin{aligned} &d|_{\phi=\tilde{\phi}}S_\phi \circ \pi_\phi[\psi]u^{(l)}][\pi_{\tilde{\phi}}(u^{(l)})] \\ &\qquad = \int_{\partial\tilde{\phi}(\Omega)} a_{\alpha\beta}^{ij}\frac{\partial v_i^{(l)}}{\partial y_\alpha}\frac{\partial v_j^{(l)}}{\partial y_\beta}\zeta\cdot\nu d\sigma - \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)} \nabla(|v^{(l)}|^2)\cdot\zeta dy. \end{aligned}$$

Using formula (5.10) we get formula (5.18). □

5.2.2 Isovolumetric perturbations

As in the previous section, we consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}} \Lambda_{F,s}[\phi],$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. We have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 5.13. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let F be a non-empty finite subset of $\mathbb{N}$. Assume that $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that $\tilde{\phi}(\Omega)$ is of class C^2 and that the eigenvalues $\lambda_j[\tilde{\phi}]$ have the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint if and only if there exists an orthonormal basis $v^{(1)}, \dots, v^{(|F|)}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (5.17) in $H^{1,0}(\tilde{\phi}(\Omega))^m$ (with respect to the scalar product (5.5)), and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|}\left(\lambda_F[\tilde{\phi}]|v^{(l)}|^2 - a_{\alpha\beta}^{ij}\frac{\partial v_i^{(l)}}{\partial y_\alpha}\frac{\partial v_j^{(l)}}{\partial y_\beta}\right) = c,$$

almost everywhere in $\partial\tilde{\phi}(\Omega)$.

Using the same arguments as in Theorem 5.8 we easily get the following

Theorem 5.14. *Let Ω be a domain in $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi} \in \mathcal{A}_\Omega^1$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (5.16) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ with volume constraint, for all $s = 1, \dots, |F|$.*

Chapter 6

The Reissner-Mindlin model for the vibrations of a clamped plate

Let Ω be a bounded open set in $\mathbb{R}^N$ with $N \geq 2$, and $t, \mu_1, \mu_2, k > 0$ be fixed parameters. We consider the following eigenvalue problem

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla\mathrm{div}\beta - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \mathrm{div}\beta) = \lambda w, & \text{in } \Omega, \\ \beta = 0, \quad w = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

in the unknowns $(\beta, w) = (\beta_1, \dots, \beta_N, w)$ (the eigenvector) and λ (the eigenvalue). According to the Reissner-Mindlin model for moderately thin plates, for $N = 2$ system (6.1) describes the free vibration modes of an elastic clamped plate $\Omega \times (-t/2, t/2)$ with midplane Ω and thickness t. In that case μ_1 and μ_2 are the Lamé constants, k is the correction factor, w the transverse displacement of the midplane, $\beta = (\beta_1, \beta_2)$ the fiber rotation and λt^2 the vibration frequency (see e.g., [21]).

The behavior of the solutions to Reissner-Mindlin systems as $t \to 0$ is well known. In particular, it is proved in [47] for $N = 2$ that $\lambda_{n,t}[\Omega] \to \lambda_{n,0}[\Omega]$ as $t \to 0$, where $\lambda_{n,0}[\Omega]$ are the eigenvalues of the problem

$$\begin{cases} \frac{2\mu_1+\mu_2}{12}\Delta^2 w = \lambda w, & \text{in } \Omega, \\ w = \nabla w = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Although $N = 2$ seems to be the case of main interest in applications, our methods allow us to treat the general case without any restriction on the space dimension. Also, we set $\mathcal{V}(\Omega) = (H_0^1(\Omega))^N \times H_0^1(\Omega)$ and we denote by (β, w) the generic element of $\mathcal{V}(\Omega)$, where $\beta = (\beta_1, \dots, \beta_N) \in (H_0^1(\Omega))^N$ and $w \in H_0^1(\Omega)$.

6.1 The Reissner-Mindlin eigenvalue problem

For the sake of completeness, in this section we provide a physical justification for the study of problem (6.1) in the case $N = 2$. Assume the plate to be of the form $\Omega \times (-t/2, t/2)$, where $\Omega \subset \mathbb{R}^2$ is the midplane of the plate and t denotes its thickness. We consider the displacement of a point of coordinates (x_1, x_2, x_3) at the time T as

$$u(x, y, z, T) = (u_1(x_1, x_2, x_3, T), u_2(x_1, x_2, x_3, T), u_3(x_1, x_2, x_3, T))\,.$$

The standard assumption in the theory of plates is that the displacement u is of the form

$$\begin{aligned} u_1(x_1, x_2, x_3, T) &= -x_3\theta_1(x_1, x_2, T), \\ u_2(x_1, x_2, x_3, T) &= -x_3\theta_2(x_1, x_2, T), \\ u_3(x_1, x_2, x_3, T) &= z(x_1, x_2, T). \end{aligned} \tag{6.3}$$

Now we consider the strain tensor $\varepsilon(u)$ defined by

$$(\varepsilon(u))_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

for all $i, j = 1, 2, 3$, and the shear tensor $\sigma(u)$ which is related to $\varepsilon(u)$ via Hooke's Law

$$\sigma(u) = 2\tilde{\mu}_1\varepsilon(u) + (\tilde{\mu}_2 \mathrm{Tr}\varepsilon(u))\mathrm{I}.$$

Here $\tilde{\mu}_1, \tilde{\mu}_2$ are the Lamé constants

$$\tilde{\mu}_1 = \frac{E}{2(1+\pi)}, \quad \tilde{\mu}_2 = \frac{\pi E}{(1+\pi)(1-2\pi)},$$

where E is the Young modulus and π is the Poisson ratio. The main hypotesis in the Reissner-Mindlin model is that the stress is planar, namely

$$\sigma_{33}(u) = 0.$$

(See also [21, §VII.3]). Note that this in principle is contraddictory, since by assumption $\varepsilon_{33}(u) = 0$. As a consequence, we assume a posteriori that

$$\varepsilon_{33}(u) = -\frac{\tilde{\mu}_2(\varepsilon_{11}(u) + \varepsilon_{22}(u))}{2\tilde{\mu}_1 + \tilde{\mu}_2}.$$

At this point we are able to write the elastic potential energy

$$\mathcal{U}(u) = -\frac{1}{2}\int_{\Omega\times(-t/2,t/2)} \varepsilon(u) : \sigma(u) dV,$$

and, assuming the mass of the plate to be uniform, the kinetic energy

$$\mathcal{K}(u) = \frac{1}{2}\int_{\Omega\times(-t/2,t/2)} \dot{u}^2 dV,$$

where $dV = dx_1 dx_2 dx_3$ is the volume element. If the external force is zero, then the mechanical energy is preserved, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{U}(u) + \mathcal{K}(u)\right) = 0.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{U}(u)) = -\int_{\Omega\times(-t/2,t/2)} \varepsilon(\dot{u}) : \sigma(u) dV,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{K}(u)) = \int_{\Omega\times(-t/2,t/2)} \dot{u}\ddot{u} dV,$$

then the equation of motion becomes

$$\int_{\Omega\times(-t/2,t/2)} \varepsilon(\dot{u}) : \sigma(u) dV = \int_{\Omega\times(-t/2,t/2)} \dot{u}\ddot{u} dV.$$

By choosing $\dot{u} = \xi = (-x_3\eta_1, -x_3\eta_2, v)$ as an arbitrary admissible velocity field, we obtain the variational equation

$$\int_{\Omega\times(-t/2,t/2)} \sigma(u) : \varepsilon(\xi) dV = \int_{\Omega\times(-t/2,t/2)} \ddot{u}\xi dV. \tag{6.4}$$

Easy computations show that

$$\sigma(u) : \varepsilon(\xi) = \frac{x_3^2 E}{1-\pi^2}\left((1-\pi)\varepsilon(\theta) : \varepsilon(\eta) + \pi \mathrm{div}\theta \mathrm{div}\eta\right) + \frac{E}{2(1+\pi)}(\nabla z - \theta)\cdot(\nabla v - \eta),$$

and therefore

$$\begin{aligned}\int_{\Omega\times(-t/2,t/2)} &\sigma(u) : \varepsilon(\xi) dV \\ &= \frac{t^3}{12}\frac{E}{1-\pi^2}\int_\Omega \left((1-\pi)\varepsilon(\theta) : \varepsilon(\eta) + \pi \mathrm{div}\theta \mathrm{div}\eta\right) dS \\ &\quad + t\frac{E}{2(1+\pi)}\int_\Omega (\nabla z - \theta)\cdot(\nabla v - \eta) dS,\end{aligned} \tag{6.5}$$

where $dS = dx_1 dx_2$ is the area element. On the other hand

$$\int_{\Omega\times(-t/2,t/2)} \ddot{u} v dV = \frac{t^3}{12}\int_\Omega \ddot{\theta}\cdot\eta dS + t\int_\Omega \ddot{z} v dS. \tag{6.6}$$

By (6.4), (6.5) and (6.6) we obtain the following variational equation of motion

$$\begin{aligned} &\frac{t^3}{12}\frac{E}{1-\pi^2}\int_\Omega ((1-\pi)\varepsilon(\theta):\varepsilon(\eta)+\pi\mathrm{div}\theta\mathrm{div}\eta)\,dS \\ &+t\frac{E}{2(1+\pi)}\int_\Omega (\nabla z-\theta)\cdot(\nabla v-\eta)dS = \frac{t^3}{12}\int_\Omega \ddot{\theta}\cdot\eta dS + t\int_\Omega \ddot{z} v dS. \end{aligned} \tag{6.7}$$

We look for solutions of equation (6.7) of the type

$$\theta(x_1,x_2,T) = e^{-i\omega T}\beta(x_1,x_2), \quad z(x_1,x_2,T) = e^{-i\omega T}w(x_1,x_2). \tag{6.8}$$

Substituting (6.8) in (6.7) we get

$$\begin{aligned} &\frac{t^3}{12}\frac{E}{1-\pi^2}\int_\Omega ((1-\pi)\varepsilon(\beta):\varepsilon(\eta)+\pi\mathrm{div}\beta\mathrm{div}\eta)\,dS \\ &+t\frac{E}{2(1+\pi)}\int_\Omega (\nabla w-\beta)\cdot(\nabla v-\eta)dS = \omega^2\left(\frac{t^3}{12}\int_\Omega \beta\cdot\eta dS + t\int_\Omega wvdS\right). \end{aligned}$$

Putting $\lambda = \omega^2 t^{-2}$ and dividing by t^3 we obtain

$$a(\beta,\eta) + \mathcal{C}t^{-2}\int_\Omega (\nabla w-\beta)\cdot(\nabla v-\eta)dx = \lambda\int_\Omega\left(wv + \frac{t^2}{12}\beta\cdot\eta\right)dx, \tag{6.9}$$

where $a(\beta,\eta) = \frac{E}{12(1-\pi^2)}\int_\Omega[(1-\pi)\epsilon(\beta):\epsilon(\eta)+\pi\mathrm{div}\beta\mathrm{div}\eta]dx$ and $\mathcal{C} = \frac{Ek}{2(1+\pi)}$. Note that here we have introduced a correction factor k (usually $k = 5/6$, cf. [20, 21, 47]). By recalling Korn's indentity

$$2\int_\Omega \epsilon(\beta):\epsilon(\eta)dx = \int_\Omega \nabla\beta:\nabla\eta dx + \int_\Omega \mathrm{div}\beta\mathrm{div}\eta dx,$$

which holds for any $\beta,\eta\in\mathcal{V}(\Omega)$, and choosing

$$\mu_1 = \frac{E}{2(1+\pi)}, \quad \text{and} \quad \mu_2 = \frac{\pi E}{1-\pi^2},$$

problem (6.9) can be easily rewritten in the form

$$\frac{\mu_1}{12}\int_\Omega \nabla\beta : \nabla\eta dx + \frac{\mu_1+\mu_2}{12}\int_\Omega \mathrm{div}\beta\mathrm{div}\eta dx + \frac{\mu_1 k}{t^2}\int_\Omega (\nabla w-\beta)\cdot(\nabla v-\eta)dx \\ = \lambda\int_\Omega \left(wv + \frac{t^2}{12}\beta\cdot\eta\right)dx. \quad (6.10)$$

The formulation in (6.10) is somewhat more general since it allows other choices of constants $\mu_1, \mu_2 > 0$ including the standard Lamé constants $\mu_1 = \tilde{\mu}_1$, $\mu_2 = \tilde{\mu}_2$ as in e.g., [52]. We refer to [17] for further details.

Remark 6.1. *The Kirchhoff-Love model assumes in addition that $\theta = \nabla z$ in (6.3). This assuption leads to problem (6.2).*

As customary in Spectral Theory we interpret problem (6.10) as an eigenvalue problem for a non-negative selfadjoint operator in Hilbert space as follows. For any fixed $t > 0$, we denote by $\mathcal{L}_t^2(\Omega)$ the space $L^2(\Omega)^N \times L^2(\Omega)$ endowed with the scalar product

$$< (\beta, w), (\eta, v) >_t = \int_\Omega \left(wv + \frac{t^2}{12}\beta\cdot\eta\right)dx,$$

for any $(\beta, w), (\eta, v) \in \mathcal{L}_t^2(\Omega)$. Clearly, for each $t > 0$ the norm induced by such scalar product is equivalent to the standard L^2-norm. We also denote by R_t the operator from $\mathcal{V}(\Omega)$ to its dual defined by the left-hand side of (6.10). Note that R_t is coercive and it defines a scalar product on $\mathcal{V}(\Omega)$ which is equivalent to the standard one (in particular, Theorem 1.9 applies to problem (6.10)). Indeed, we consider the space $\mathcal{V}(\Omega)$ endowed with the following product

$$< (\beta, w), (\eta, v) >_{\mathcal{V}(\Omega)} = R_t[(\beta, w)][(\eta, v)].$$

We will denote by $Q_{\Omega,t}(\beta, w) = R_t[(\beta, w)][(\beta, w)]$ the quadratic form associated with the operator R_t in $\mathcal{V}(\Omega)$. We define the following embedding of $\mathcal{V}(\Omega)$ into its dual

$$\mathcal{J}_t[(\beta, w)][(\eta, v)] = \int_\Omega \left(wv + \frac{t^2}{12}\beta\cdot\eta\right)dx,$$

for any $(\beta, w), (\eta, v) \in \mathcal{V}(\Omega)$. Therefore, problem (6.10) can be rewritten in the following form

$$R_t[(\beta, w)][(\eta, v)] = \lambda\mathcal{J}_t[(\beta, w)][(\eta, v)], \ \forall(\eta, v) \in \mathcal{V}(\Omega). \quad (6.11)$$

We have the following result (see also Lemma 2.1).

Lemma 6.2. *Let Ω be a bounded open set in $\mathbb{R}^N$, $t > 0$. The operator $T_t \equiv R_t^{-1} \circ \mathcal{J}_t$ is a non-negative selfadjoint compact operator in the Hilbert space $\mathcal{V}(\Omega)$. The spectrum of T_t is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $T_t(\beta, w) = \rho(\beta, w)$ is satisfied for some $(\beta, w) \in \mathcal{V}(\Omega)$, $\rho > 0$ if and only if equation (6.10) is satisfied with $\lambda = \rho^{-1}$.*

6.2 Quantitative estimates

6.2.1 Estimates via diffeomorphisms

Given an open set Ω in $\mathbb{R}^N$ with finite measure, we consider a diffeomorphism from Ω onto another open set $\phi(\Omega)$ in $\mathbb{R}^N$ and we prove a quantitative stability estimate for $|\lambda_{n,t}[\phi(\Omega)] - \lambda_{n,t}[\Omega]|$ in terms of the measure of vicinity $\delta(\phi)$ defined by

$$\delta(\phi) = \max_{1 \le |\alpha| \le 2} \sup_{x \in \Omega} |D^\alpha(\phi(x) - x)|.$$

In order to obtain an estimate independent of t, we use the special transformation C_ϕ from the space $\mathcal{V}(\Omega)$ onto $\mathcal{V}(\phi(\Omega))$ defined by

$$C_\phi(\beta, w) = (\beta \nabla \phi^{-1}, w) \circ \phi^{(-1)}, \tag{6.12}$$

for all $(w, \beta) \in \mathcal{V}(\Omega)$. Here and in the sequel we denote by A^{-1} the inverse of a matrix A, as opposed to the inverse of a function f which is denoted by $f^{(-1)}$; we shall also denote by A^t the transpose of A.

It is clear that in order to guarantee that C_ϕ is well-defined, it suffices to assume that ϕ is a diffeomorphism of class $C^{1,1}$, i.e., ϕ and its inverse have Lipschitz continuous gradients. In fact, it is easy to prove the following lemma that will be used in the sequel.

Lemma 6.3. *Let Ω be an open set in $\mathbb{R}^N$ and let $\phi : \Omega \to \phi(\Omega)$ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$. Assume that*

$$\max_{1 \le |\alpha| \le 2} \sup_{x \in \Omega} |D^\alpha \phi(x)| < \infty, \quad \inf_{x \in \Omega} |\det \nabla \phi(x)| > 0.$$

Then C_ϕ is a linear homeomorphism from $\mathcal{V}(\Omega)$ onto $\mathcal{V}(\phi(\Omega))$.

Then we can prove the following

Lemma 6.4. *Let Ω be an open set in $\mathbb{R}^N$ with finite measure and let $\phi : \Omega \to \phi(\Omega)$ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$. Assume that there exist $M_1, M_2 > 0$ such that*

$$\max_{1\le|\alpha|\le 2}\sup_{x\in\Omega}|D^\alpha\phi(x)| < M_1, \quad \inf_{x\in\Omega}|\det\nabla\phi(x)| > M_2. \tag{6.13}$$

Then there exists $c > 0$ depending only on $N, M_1, M_2, \mu_1, \mu_2$ and $|\Omega|$ such that

$$|Q_{\phi(\Omega),t}(C_\phi(\beta,w)) - Q_{\Omega,t}(\beta,w)| \le cQ_{\Omega,t}(\beta,w)\delta(\phi), \tag{6.14}$$

for all $t > 0$ and $(\beta,w) \in \mathcal{V}(\Omega)$.

Proof. Let $(\beta,w) \in \mathcal{V}(\Omega)$. To shorten our notation, we denote by $C_\phi^{(1)}(\beta)$ the first entry of $C_\phi(\beta,w)$, i.e., $C_\phi^{(1)}(\beta) = (\beta\nabla\phi^{-1})\circ\phi^{(-1)}$. We begin by estimating $\int_{\phi(\Omega)}|\nabla C_\phi^{(1)}(\beta)|^2dy - \int_\Omega|\nabla\beta|^2dx$. By means of a change of variables, we get

$$\int_{\phi(\Omega)}|\nabla C_\phi^{(1)}(\beta)|^2dy = \int_\Omega|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2|\det\nabla\phi|dx. \tag{6.15}$$

It is easy to see that in order to estimate $\int_{\phi(\Omega)}|\nabla C_\phi^{(1)}(\beta)|^2dy - \int_\Omega|\nabla\beta|^2dx$ it suffices to estimate $\int_\Omega(|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2 - |\nabla\beta|^2)|\det\nabla\phi|dx$. We clearly have that

$$\begin{aligned}\left|\int_\Omega(|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}|^2 - |\nabla\beta|^2)|\det\nabla\phi|dx\right|& \\ \le \|\det\nabla\phi\|_{L^\infty(\Omega)}\|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1} - \nabla\beta\|_{L^2(\Omega)}& \\ \cdot(\|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1}\|_{L^2(\Omega)} + \|\nabla\beta\|_{L^2(\Omega)}).&\end{aligned} \tag{6.16}$$

By the triangle inequality we get

$$\begin{aligned}\|(\nabla(\beta\nabla\phi^{-1}))\nabla\phi^{-1} - \nabla\beta\|_{L^2(\Omega)}& \\ \le \|\nabla\phi^{-1}\|_{L^\infty(\Omega)}\|\nabla(\beta\nabla\phi^{-1}) - \nabla\beta\|_{L^2(\Omega)}& \\ +\|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^2(\Omega)},&\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}\|\nabla(\beta\nabla\phi^{-1}) - \nabla\beta\|_{L^2(\Omega)} &\le \|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^2(\Omega)} \\ &+ \|\nabla(\nabla\phi^{-1})\|_{L^\infty(\Omega)}\|\beta\|_{L^2(\Omega)}.\end{aligned} \tag{6.18}$$

Moreover

$$\begin{aligned} \|\nabla(\beta\nabla\phi^{-1})\|_{L^2(\Omega)} &\leq \|\nabla\phi^{-1}\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^2(\Omega)} \\ &+ \|\nabla(\nabla\phi^{-1})\|_{L^\infty(\Omega)}\|\beta\|_{L^2(\Omega)}. \end{aligned} \tag{6.19}$$

By standard calculus it follows that there exists a constant $c > 0$ depending only on N, M_1, M_2 such that

$$\|\nabla\phi^{-1}\|_{L^\infty(\Omega)} \leq c \tag{6.20}$$

and

$$\|\nabla\phi^{-1} - I\|_{L^\infty(\Omega)},\ \|\nabla(\nabla\phi^{-1}))\|_{L^\infty(\Omega)} \leq c\delta(\phi). \tag{6.21}$$

By using the Poincaré inequality $\|\beta\|_{L^2(\Omega)} \leq c\|\nabla\beta\|_{L^2(\Omega)}$ with c depending only on N and $|\Omega|$, and combining inequalities (6.15)-(6.21) we conclude that

$$\left|\int_{\phi(\Omega)} |\nabla C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla\beta|^2 dx\right| \leq c_1\delta(\phi)\int_\Omega |\nabla\beta|^2 dx, \tag{6.22}$$

where the constant c_1 depends only on N, M_1, M_2 and $|\Omega|$.

Similarly, one can also prove the existence of a constant $c_2 > 0$ depending only on N, M_1, M_2 and $|\Omega|$ such that

$$\left|\int_{\phi(\Omega)} (\operatorname{div} C_\phi^{(1)}(\beta))^2 dy - \int_\Omega (\operatorname{div}\beta)^2 dx\right| \leq c_2\delta(\phi)\int_\Omega |\nabla\beta|^2 dx. \tag{6.23}$$

Finally, we estimate $\int_{\phi(\Omega)} |\nabla(w\circ\phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla w - \beta|^2 dx$. We note that

$$\int_{\phi(\Omega)} |\nabla(w\circ\phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy = \int_\Omega |(\nabla w - \beta)\cdot\nabla\phi^{-1}|^2 |\det\nabla\phi| dx$$

and that

$$\begin{aligned} &\int_\Omega \left||(\nabla w - \beta)\cdot\nabla\phi^{-1}|^2 - |\nabla w - \beta|^2\right| dx \\ &\qquad\qquad \leq \|\nabla\phi^{-1}(\nabla\phi^{-1})^T - I\|_{L^\infty(\Omega)}\int_\Omega |\nabla w - \beta|^2 dx. \end{aligned}$$

It follows that there exists $c_3 > 0$ depending only on N, M_1, M_2 such that

$$\begin{aligned} &\left|\int_{\phi(\Omega)} |\nabla(w\circ\phi^{(-1)}) - C_\phi^{(1)}(\beta)|^2 dy - \int_\Omega |\nabla w - \beta|^2 dx\right| \\ &\qquad\qquad \leq c_3\delta(\phi)\int_\Omega |\nabla w - \beta|^2 dx. \end{aligned} \tag{6.24}$$

By combining inequalities (6.22), (6.23), (6.24), we deduce the validity of (6.14). $\square$

As in the case of elliptic systems of partial differential equations discussed in Chapter 4, we can prove the following

Theorem 6.5. *Let Ω be an open set in $\mathbb{R}^N$ with finite measure and $M_1, M_2 > 0$. Then there exists $c > 0$ depending only on μ_1, μ_2, M_1, M_2 and $|\Omega|$ such that*

$$|\lambda_{n,t}[\phi(\Omega)] - \lambda_{n,t}[\Omega]| \leq c\lambda_{n,t}[\Omega]\delta(\phi), \tag{6.25}$$

for all $t > 0$ and for all diffeomorphisms ϕ of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$ such that inequalities (6.13) are satisfied and $\delta(\phi) < c^{-1}$.

Proof. Let ϕ be a diffeomorphism of class $C^{1,1}$ from Ω onto an open set $\phi(\Omega)$ in $\mathbb{R}^N$, satisfying inequalities (6.13). Obviously we have

$$\begin{aligned} \left| \frac{Q_{\phi(\Omega)}(C_\phi(\beta, w))}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} - \frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}} \right| &\leq \frac{\left|Q_{\phi(\Omega)}(C_\phi(\beta, w)) - Q_\Omega(\beta, w)\right|}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} \\ &+ \frac{Q_\Omega(\beta, w) \left| \|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} - \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \right|}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))} \|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}}. \end{aligned} \tag{6.26}$$

As in the proof of Lemma 6.4, one can prove the existence of a constant $c > 0$ depending only on N, M_1, M_2 such that

$$\|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} \geq c\|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \tag{6.27}$$

and

$$\left| \|C_\phi(\beta, w)\|^2_{\mathcal{L}^2_t(\phi(\Omega))} - \|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)} \right| \leq c\delta(\phi)\|(\beta, w)\|^2_{\mathcal{L}^2_t(\Omega)}, \tag{6.28}$$

see also Lemma 6.3. By combining inequalities (6.14) and (6.26)-(6.28) we deduce that

$$\begin{aligned} (1 - c\delta(\phi)) &\frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}} \\ &\leq \frac{Q_{\phi(\Omega)}(C_\phi(\beta, w))}{\|C_\phi(\beta, w))\|^2_{\mathcal{L}^2_t(\phi(\Omega))}} \leq (1 + c\delta(\phi)) \frac{Q_\Omega(\beta, w)}{\|(\beta, w))\|^2_{\mathcal{L}^2_t(\Omega)}}. \end{aligned} \tag{6.29}$$

If $1 - c\delta(\phi) > 0$, it is possible to apply the Min-Max Principle to deduce (6.25) from (6.29) combined with Lemma 6.3. $\square$

Remark 6.6. *Since the weak formulation (6.10) involves only weak derivatives of the first order, one may try to obtain stability estimates also under weaker assumptions of ϕ. For example, one may think of using bi-Lipschitz domain transformations, i.e., maps ϕ of class $C^{0,1}$ together with their inverses (cf. Chapter 4). In this case, one would replace the measure of vicinity $\delta(\phi)$ by the natural weaker measure of vicinity*

$$\tilde{\delta}(\phi) = \|\nabla\phi - I\|_{L^\infty(\Omega)}.$$

In order to prove the corresponding estimate, in the proof of Theorem 6.5 one should replace the operator C_ϕ defined in (6.12) by the operator $\tilde{C}_\phi$ defined by

$$\tilde{C}_\phi(\beta, w) = (\beta \circ \phi^{(-1)}, w \circ \phi^{(-1)}),$$

for all $(\beta, w) \in \mathcal{V}(\Omega)$. The definition of the operator $\tilde{C}_\phi$ does not involve $\nabla\phi$ and establishes a linear homeomorphism between $\mathcal{V}(\Omega)$ and $\mathcal{V}(\phi(\Omega))$. Unfortunately, the summand $\int_\Omega (\nabla w - \beta)\cdot(\nabla v - \eta)dx$ in the quadratic form (6.10) does not behave well under the transformation $\tilde{C}_\phi$ and this would lead to an estimate depending on t. Namely, one would obtain the estimate

$$|\lambda_{n,t}[\phi(\Omega)] - \lambda_{n,t}[\Omega]| \le \frac{c}{t^2}\lambda_{n,t}[\Omega]\tilde{\delta}(\phi),$$

where the presence of a better measure of vicinity $\tilde{\delta}(\phi)$ is compensated by the presence of the factor t^2 which spoils the estimate for t close to zero.

6.2.2 Estimates via atlas and Hausdorff distance

Proceeding as in Chapter 4 we can prove the following

Theorem 6.7. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Then there exists $c > 0$ depending only on $\mathcal{A}, \mu_1, \mu_2$ such that*

$$|\lambda_{n,t}[\Omega_1] - \lambda_{n,t}[\Omega_2]| \le c\max\{\lambda_{n,t}[\Omega_1], \lambda_{n,t}[\Omega_2]\}d_{\mathcal{A}}(\Omega_1, \Omega_2), \tag{6.30}$$

for all $n \in \mathbb{N}$, $t > 0$ and for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < c^{-1}$.

Proof. Let $E > 0$ be as in Lemma 4.4 and let $\Omega_1, \Omega_2 \in C(\mathcal{A})$ be such that $d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon/s$. Clearly $\Omega_1 \cap \Omega_2 \in C(\mathcal{A})$ and $d_{\mathcal{A}}(\Omega_1 \cap \Omega_2, \Omega_1), d_{\mathcal{A}}(\Omega_1 \cap \Omega_2, \Omega_2) < \epsilon/s$. Thus by Lemma 4.4 we have that $\phi_\epsilon(\Omega_1), \phi_\epsilon(\Omega_2) \subset \Omega_1 \cap \Omega_2$. By the monotonicity of the eigenvalues with respect to inclusion, we immediately get

$$\lambda_{n,t}[\Omega_i] \le \lambda_{n,t}[\Omega_1 \cap \Omega_2] \le \lambda_{n,t}[\phi_\epsilon(\Omega_i)],$$

for $i = 1, 2$. Moreover, by combining Theorem 6.5 and Lemma 4.4, we deduce that there exists c as in the statement such that

$$|\lambda_{n,t}[\Omega_i] - \lambda_{n,t}[\Omega_1 \cap \Omega_2]| \leq |\lambda_{n,t}[\phi_\epsilon(\Omega_i)] - \lambda_{n,t}[\Omega_i]| \leq c\lambda_{n,t}[\Omega_i]\epsilon, \tag{6.31}$$

for $i = 1, 2$, provided $\epsilon \leq c^{-1}$. Inequality (6.30) easily follows by choosing $\epsilon = 2sd_{\mathcal{A}}(\Omega_1, \Omega_2)$ in (6.31). □

Remark 6.8. *We note that by Theorem 1.4 and estimate (6.30), it immediately follows that if ω is a modulus of continuity as in Definition 1.1 then there exist $c > 0$ depending only on $\mathcal{A}, \omega, \mu_1, \mu_2$ such that*

$$|\lambda_{n,t}[\Omega_1] - \lambda_{n,t}[\Omega_2]| \leq c \max\{\lambda_{n,t}[\Omega_1], \lambda_{n,t}[\Omega_2]\}\omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)), \tag{6.32}$$

for all $n \in \mathbb{N}$, $t > 0$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying the condition $d_{\mathcal{HP}}(\Omega_1, \Omega_2) < c^{-1}$.

In several papers devoted to stability estimates for domain perturbation problems, the vicinity of two domains is described by means of ϵ-neighborhoods of the boundaries defined by the Euclidean distance (see e.g., [34, 45]). This can be done also in the case of the Reissner-Mindlin system. Indeed, we have the following

Corollary 6.9. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$, ω a modulus of continuity as in Definition 1.1 and $M > 0$. Then there exists $c > 0$ depending only on $\mathcal{A}, \omega, \mu_1, \mu_2 M$ such that*

$$|\lambda_{n,t}[\Omega_1] - \lambda_{n,t}[\Omega_2]| \leq c \max\{\lambda_{n,t}[\Omega_1], \lambda_{n,t}[\Omega_2]\}\omega(\epsilon), \tag{6.33}$$

for all $n \in \mathbb{N}$, $t > 0$, $\epsilon \in]0, c^{-1}[$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying (1.7) or (1.8).

Proof. Note that if Ω_1 and Ω_2 satisfy either (1.7) or (1.8), then they also satisfy (1.9), which combined with inequality (6.32) allows to deduce (6.33). □

6.3 Analyticity results

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. We shall consider problem (6.10) in $\phi(\Omega)$ for any $\phi \in \mathcal{A}^1_\Omega$ and study the dependence of $\lambda_{j,t}[\phi(\Omega)]$ on ϕ.

The main result of this section is the following

Theorem 6.10. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $t > 0$ and F a finite non-empty subset of $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $\mathcal{A}^1_\Omega$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_{j,t}[\tilde{\phi}]$ assume the common value $\lambda_{F,t}[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^2 then the Frechét differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$
\begin{aligned}
d_{|_{\phi=\tilde{\phi}}}\Lambda_{F,s}[\psi] = -\lambda_{F,t}^{s-1}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Bigg(\frac{\mu_1}{12}\left|\frac{\partial\beta^{(l)}}{\partial\nu}\right|^2 \\
+\frac{\mu_1+\mu_2}{12}\left(\frac{\partial\beta^{(l)}}{\partial\nu}\cdot\nu\right)^2+\frac{\mu_1 k}{t^2}\left(\frac{\partial w^{(l)}}{\partial\nu}\right)^2\Bigg)\zeta\cdot\nu d\sigma, \quad (6.34)
\end{aligned}
$$

for all $\psi \in C^1_b(\Omega;\mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$ and $\left\{(\beta^{(l)}, w^{(l)})\right\}_{l=1}^{|F|}$ is an orthonormal basis in $\mathcal{V}(\tilde{\phi}(\Omega))$ for the eigenspace associated with $\lambda_{F,t}[\tilde{\phi}]$.

In order to prove Theorem 6.10 we consider equation (6.11) in $\phi(\Omega)$ and pull it back to Ω. Namely, we consider the equation

$$
R_t[(\beta,w)][(\eta,v)] = \lambda\mathcal{J}_t[(\beta,w)][(\eta,v)], \quad \forall\ (\eta,v)\in\mathcal{V}(\phi(\Omega)), \qquad (6.35)
$$

in the unknowns $(\beta, w) \in \mathcal{V}(\phi(\Omega))$, $\lambda \in]0,\infty[$.

We will denote by $\mathcal{V}_\phi(\Omega)$ the space $\mathcal{V}(\Omega)$ endowed with the form

$$
< (\beta,w)(\eta,v) >_{t,\phi} = R_{t,\phi}[(\beta,w)][(\eta,v)],\ \forall(\beta,w),(\eta,v)\in\mathcal{V}(\Omega).
$$

Moreover, we recall that

$$
\mathcal{J}_{t,\phi}[(\beta,w)][(\eta,v)] = \int_\Omega\left(wv+\frac{t^2}{12}\beta\cdot\eta\right)|\mathrm{det}\nabla\phi|dx,\ \forall(\beta,w),(\eta,v)\in\mathcal{V}(\Omega).
$$

Note that the map from $\mathcal{V}(\Omega)$ to $\mathcal{V}(\phi(\Omega))$ which maps (β, w) to $(\beta\circ\phi^{(-1)}, w\circ\phi^{(-1)})$ for all $(\beta, w) \in \mathcal{V}(\Omega)$ is a linear homeomorphism. Hence, equation (6.11) is equivalent to

$$
R_{t,\phi}[(\theta,u)][(\dot{\theta},\dot{u})] = \lambda\mathcal{J}_{t,\phi}[(\theta,u)][(\dot{\theta},\dot{u})], \quad \forall\ (\dot{\theta},\dot{u})\in\mathcal{V}_\phi(\Omega),
$$

where $(\theta, u) = (\beta \circ \phi, w \circ \phi)$. It turns out that the operator T_t defined in Lemma 6.2 with Ω replaced by $\phi(\Omega)$ is unitarily equivalent to the operator $T_{t,\phi}$ defined on $\mathcal{V}_\phi(\Omega)$ by

$$
T_{t,\phi} := R_{t,\phi}^{(-1)}\circ\mathcal{J}_{t,\phi}. \qquad (6.36)
$$

Thus we have the following lemma, whose proof is analogous to that of Lemma 2.3.

Lemma 6.11. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $t > 0$. The operator $T_{t,\phi}$ defined in (6.36) is non-negative selfadjoint and compact on the Hilbert space $\mathcal{V}_\phi(\Omega)$. The equation (6.35) is satisfied for some $(\beta, w) \in \mathcal{V}(\phi(\Omega))$ if and only if the equation $T_{t,\phi}(\theta, u) = \rho(\theta, u)$ is satisfied with $(\theta, u) = (\beta \circ \phi, w \circ \phi)$ and $\rho = \lambda^{-1}$. Moreover, the map from $\mathcal{A}^1_\Omega$ to $\mathcal{L}(\mathcal{V}(\Omega)) \times \mathcal{B}_s(\mathcal{V}(\Omega))$ which takes $\phi \in \mathcal{A}^1_\Omega$ to $(T_{t,\phi}, < \cdot, \cdot >_{t,\phi})$ is real-analytic.*

Proof of Theorem 6.10. First of all, we note that by standard regularity theory (see [9, §10.3]), the eigenvectors $(\beta^{(i)}, w^{(i)}) \in H^2(\tilde{\phi}(\Omega))^{N+1}$, $i = 1, 2$. We observe that the proof is very similar to that of Theorem 2.2. It only remains to compute

$$
\begin{aligned}
< \mathrm{d}|_{\phi=\tilde{\phi}} T_{t,\phi}[\psi][(\theta^{(l)}, u^{(l)})], (\theta^{(l)}, u^{(l)}) >_{t,\tilde{\phi}}& \\
= d|_{\phi=\tilde{\phi}} \mathcal{J}_{t,\phi}[\psi][(\theta^{(l)}, u^{(l)})][(\theta^{(l)}, u^{(l)})]& \\
- \lambda^{-1}_{t,F}[\tilde{\phi}] d|_{\phi=\tilde{\phi}} R_{t,\phi}[\psi][(\theta^{(l)}, u^{(l)})][(\theta^{(l)}, u^{(l)})]&.
\end{aligned}
$$

By standard calculus in normed spaces we have

$$
d|_{\phi=\tilde{\phi}} \mathcal{J}_{t,\phi}[\psi][(\theta^{(l)}, u^{(l)})][(\theta^{(l)}, u^{(l)})] = \int_{\tilde{\phi}(\Omega)} \left(|w^{(l)}|^2 + \frac{t^2}{12} |\beta^{(l)}|^2 \right) \mathrm{div}\zeta dy,
$$

see also (2.12). Using Lemma 6.12 below, we get formula (6.34). □

Lemma 6.12. *Let Ω be a bounded open set in $\mathbb{R}^N$ and $\tilde{\phi} \in \mathcal{A}^1_\Omega$ be such that $\tilde{\phi}(\Omega)$ is of class C^2. Let $t > 0$ and $(\beta^{(i)}, w^{(i)}) \in \mathcal{V}(\tilde{\phi}(\Omega))$, $i = 1, 2$ be eigenvectors associated with an eigenvalue $\tilde{\lambda}$ of the operator R_t in $\tilde{\phi}(\Omega)$. Let $(\theta^{(i)}, u^{(i)}) = (\beta^{(i)} \circ \tilde{\phi}, w^{(i)} \circ \tilde{\phi})$, $i = 1, 2$. Then we have*

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} R_{t,\phi}[\psi][(\theta^{(1)}, u^{(1)})][(\theta^{(2)}, u^{(2)})] = -\frac{\mu_1}{12} \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial \beta^{(1)}}{\partial \nu} \cdot \frac{\partial \beta^{(2)}}{\partial \nu} \zeta \cdot \nu d\sigma& \\
- \frac{\mu_1 + \mu_2}{12} \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial \beta^{(1)}}{\partial \nu} \cdot \nu \frac{\partial \beta^{(2)}}{\partial \nu} \cdot \nu \zeta \cdot \nu d\sigma - \frac{\mu_1 k}{t^2} \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial w^{(1)}}{\partial \nu} \frac{\partial w^{(2)}}{\partial \nu} \zeta \cdot \nu d\sigma& \\
+ \tilde{\lambda} \int_{\tilde{\phi}(\Omega)} \left(w^{(1)} w^{(2)} + \frac{t^2}{12} \beta^{(1)} \cdot \beta^{(2)} \right) \mathrm{div}\zeta dy,& \quad (6.37)
\end{aligned}
$$

for all $\psi \in C^1_b(\Omega; \mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{(-1)}$.

Proof. Note that, in order to shorten our notation, in the sequel summation symbols will be omitted. By standard calculus in normed space and changing variables we get

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}}\mathcal{R}_{t,\phi}[\psi][(\theta^{(1)},u^{(1)})][(\theta^{(2)},u^{(2)})] &= \frac{\mu_1}{12}\int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_j}\frac{\partial\beta_i^{(2)}}{\partial y_j}\mathrm{div}\zeta dy \\
&- \frac{\mu_1}{12}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial\beta_i^{(1)}}{\partial y_r}\frac{\partial\beta_i^{(2)}}{\partial y_j}+\frac{\partial\beta_i^{(2)}}{\partial y_r}\frac{\partial\beta_i^{(1)}}{\partial y_j}\right)\frac{\partial\zeta_r}{\partial y_j}dy \\
&- \frac{\mu_1+\mu_2}{12}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial\beta_i^{(1)}}{\partial y_r}\mathrm{div}\beta^{(2)}+\frac{\partial\beta_i^{(2)}}{\partial y_r}\mathrm{div}\beta^{(1)}\right)\frac{\partial\zeta_r}{\partial y_i}dy \\
&+ \frac{\mu_1+\mu_2}{12}\int_{\tilde{\phi}(\Omega)}\mathrm{div}\beta^{(1)}\mathrm{div}\beta^{(2)}\mathrm{div}\zeta dy \\
&- \frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)dy \\
&- \frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial w^{(1)}}{\partial y_i}-\beta_i^{(1)}\right)\frac{\partial w^{(2)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}dy \\
&+ \frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial w^{(1)}}{\partial y_i}-\beta_i^{(1)}\right)\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)\mathrm{div}\zeta dy. \quad (6.38)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
\int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_r}\frac{\partial\beta_i^{(2)}}{\partial y_j}\frac{\partial\zeta_r}{\partial y_j}dy &= \int_{\partial\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial\nu}\frac{\partial\beta_i^{(2)}}{\partial\nu}\zeta\cdot\nu d\sigma \\
&- \int_{\tilde{\phi}(\Omega)}\Delta\beta^{(2)}\cdot(\nabla\beta^{(1)}\cdot\zeta)dy - \int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(2)}}{\partial y_j}\frac{\partial^2\beta_i^{(1)}}{\partial y_j\partial y_r}\zeta_r dy \\
&= -\int_{\tilde{\phi}(\Omega)}\Delta\beta^{(2)}\cdot(\nabla\beta^{(1)}\cdot\zeta)dy + \int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_j}\frac{\partial^2\beta_i^{(2)}}{\partial y_j\partial y_r}\zeta_r dy \\
&\qquad + \int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_j}\frac{\partial\beta_i^{(2)}}{\partial y_j}\mathrm{div}\zeta dy. \quad (6.39)
\end{aligned}
$$

Note that here and in the sequel we also use the fact that if U is a smooth open set and $f\in H^2(U)\cap H_0^1(U)$ then $\nabla f=\frac{\partial f}{\partial\nu}\nu$ on ∂U; moreover, if $g\in(H^2(U)\cap H_0^1(U))^N$ then $\mathrm{div}g=\frac{\partial g}{\partial\nu}\cdot\nu$ on ∂U.

By (6.39) the sum of the first two integrals in the right-hand side of

(6.38) equals

$$
-\frac{\mu_1}{12}\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial\beta^{(1)}}{\partial\nu}\cdot\frac{\partial\beta^{(2)}}{\partial\nu}\zeta\cdot\nu d\sigma
+\frac{\mu_1}{12}\int_{\tilde{\phi}(\Omega)}\left(\Delta\beta_i^{(1)}\nabla\beta_i^{(2)}+\Delta\beta_i^{(2)}\nabla\beta_i^{(1)}\right)\cdot\zeta dy.
$$

Now we observe that

$$
\begin{aligned}
\int_{\tilde{\phi}(\Omega)}\frac{\partial\beta_i^{(1)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\mathrm{div}\beta^{(2)}dy &= \int_{\partial\tilde{\phi}(\Omega)}\frac{\partial\beta^{(1)}}{\partial\nu}\cdot\nu\mathrm{div}\beta^{(2)}\zeta\cdot\nu d\sigma \\
&\quad -\int_{\tilde{\phi}(\Omega)}\frac{\partial\mathrm{div}\beta^{(1)}}{\partial y_r}\zeta_r\mathrm{div}\beta^{(2)}dy-\int_{\tilde{\phi}(\Omega)}\frac{\partial\mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial\beta_i^{(1)}}{\partial y_r}\zeta_r dy \\
&= -\int_{\tilde{\phi}(\Omega)}\frac{\partial\mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial\beta_i^{(1)}}{\partial y_r}\zeta_r dy+\int_{\tilde{\phi}(\Omega)}\mathrm{div}\beta^{(1)}\mathrm{div}\beta^{(2)}\mathrm{div}\zeta dy \\
&\quad +\int_{\tilde{\phi}(\Omega)}\mathrm{div}\beta^{(1)}\frac{\partial\mathrm{div}\beta^{(2)}}{\partial y_r}\zeta_r dy.
\end{aligned}
$$

Thus, the sum of third and the fourth integral in the right-hand side of (6.38) is equal to

$$
\frac{\mu_1+\mu_2}{12}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial\mathrm{div}\beta^{(1)}}{\partial y_i}\frac{\partial\beta_i^{(2)}}{\partial y_r}+\frac{\partial\mathrm{div}\beta^{(2)}}{\partial y_i}\frac{\partial\beta_i^{(1)}}{\partial y_r}\right)\zeta_r dy
-\frac{\mu_1+\mu_2}{12}\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial\beta^{(1)}}{\partial\nu}\cdot\nu\frac{\partial\beta^{(2)}}{\partial\nu}\cdot\nu\zeta\cdot\nu d\sigma.
$$

Now note that

$$
\begin{aligned}
&\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_r}\frac{\partial\zeta_r}{\partial y_i}\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)dy=\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial\nu}\frac{\partial w^{(2)}}{\partial\nu}\zeta\cdot\nu d\sigma \\
&-\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_r}\zeta_r\left(\Delta w^{(2)}-\mathrm{div}\beta^{(2)}\right)dy-\int_{\tilde{\phi}(\Omega)}\frac{\partial^2 w^{(1)}}{\partial y_i\partial y_r}\zeta_r\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)dy \\
&=-\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_r}\zeta_r\left(\Delta w^{(2)}-\mathrm{div}\beta^{(2)}\right)dy+\int_{\tilde{\phi}(\Omega)}\nabla w^{(1)}(\nabla w^{(2)}-\beta^{(2)})\mathrm{div}\zeta dy \\
&\quad +\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_i}\left(\frac{\partial^2 w^{(2)}}{\partial y_i\partial y_r}-\frac{\partial\beta_i^{(2)}}{\partial y_r}\right)\zeta_r dy. \qquad (6.40)
\end{aligned}
$$

By using the second equality in (6.40), and the first equality in (6.40) with $(\beta^{(1)}, w^{(1)})$ replaced by $(\beta^{(2)}, w^{(2)})$, we get that the sum of the last three integrals in (6.38) is equal to

$$
\begin{aligned}
&-\frac{\mu_1 k}{t^2}\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial w^{(1)}}{\partial\nu}\frac{\partial w^{(2)}}{\partial\nu}\zeta\cdot\nu d\sigma \\
&\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}(\Delta w^{(1)}-\mathrm{div}\beta^{(1)})\nabla w^{(2)}\cdot\zeta dy \\
&\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}(\Delta w^{(2)}-\mathrm{div}\beta^{(2)})\nabla w^{(1)}\cdot\zeta dy \\
&-\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\beta^{(1)}(\nabla w^{(2)}-\beta^{(2)})\mathrm{div}\zeta dy+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial y_i}\frac{\partial\beta_i^{(2)}}{\partial y_r}\zeta_r dy \\
&-\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\beta_i^{(1)}\frac{\partial^2 w^{(2)}}{\partial y_i\partial y_r}\zeta_r dy=-\frac{\mu_1 k}{t^2}\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial w^{(1)}}{\partial\nu}\frac{\partial w^{(2)}}{\partial\nu}\zeta\cdot\nu d\sigma \\
&\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}(\Delta w^{(1)}-\mathrm{div}\beta^{(1)})\nabla w^{(2)}\cdot\zeta dy \\
&\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}(\Delta w^{(2)}-\mathrm{div}\beta^{(2)})\nabla w^{(1)}\cdot\zeta dy \\
&\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial w^{(1)}}{\partial y_i}-\beta_i^{(1)}\right)\frac{\partial\beta_i^{(2)}}{\partial y_r}\zeta_r dy \\
&\qquad\qquad+\frac{\mu_1 k}{t^2}\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial w^{(2)}}{\partial y_i}-\beta_i^{(2)}\right)\frac{\partial\beta_i^{(1)}}{\partial y_r}\zeta_r dy.
\end{aligned}
$$

Using the fact that

$$
-\frac{\mu_1}{12}\Delta\beta^{(i)}-\frac{\mu_1+\mu_2}{12}\nabla\mathrm{div}\beta^{(i)}-\frac{\mu_1 k}{t^2}(\nabla w^{(i)}-\beta^{(i)})=\frac{\tilde{\lambda}t^2}{12}\beta^{(i)},
$$

and

$$
-\frac{\mu_1 k}{t^2}(\Delta w^{(i)}-\mathrm{div}\beta^{(i)})=\tilde{\lambda}w^{(i)},
$$

for $i=1,2$, we get formula (6.37). □

6.4 Isovolumetric perturbations

As in the previous chapters, we consider the following extremum problem for the symmetric functions of the eigenvalues

$$
\min_{V[\phi]=\mathrm{const}}\Lambda_{F,t}^{(s)}[\phi]\quad\text{or}\quad\max_{V[\phi]=\mathrm{const}}\Lambda_{F,t}^{(s)}[\phi],
$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. we have the following result, whose proof is analogous to that of Theorem 2.7.

Theorem 6.13. *Let Ω be a bounded open set in $\mathbb{R}^N$ and $t > 0$. Let F be a non-empty finite subset of $\mathbb{N}$ and $s \in \{1, \dots, |F|\}$. Let $\tilde{\phi} \in \Theta_\Omega[F]$ be such that $\tilde{\phi}(\Omega)$ is of class C^2. Then $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint if and only if there exists an orthonormal basis $(\beta^{(1)}, w^{(1)})$, ..., $(\beta^{(|F|)}, w^{(|F|)})$ in $\mathcal{V}(\tilde{\phi}(\Omega))$ of the eigenspace associated with the eigenvalue $\lambda_{F,t}[\tilde{\phi}]$ and there exists $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\frac{\mu_1}{12} \left| \frac{\partial \beta^{(l)}}{\partial \nu} \right|^2 + \frac{\mu_1 + \mu_2}{12} \left(\frac{\partial \beta^{(l)}}{\partial \nu} \cdot \nu \right)^2 + \frac{\mu_1 k}{t^2} \left(\frac{\partial w^{(l)}}{\partial \nu} \right)^2 \right) = c, \quad (6.41)$$

almost everywhere in $\partial \tilde{\phi}(\Omega)$.

As in the case of the Laplace operator discussed in [67] and polyharmonic operators considered in the previous chapters (see also [27, 28]), it turns out that if $\tilde{\phi}(\Omega)$ is a ball then condition (6.41) is satisfied. In order to prove it, we need the following lemma. Recall that β is thought as a row vector.

Lemma 6.14. *The operator R_t is rotation invariant (in the sense of Definition 5.6). In particular, let B be a ball in $\mathbb{R}^N$ centered at zero, $t > 0$, and let (β, w) be an eigenvector of R_t in B associated with an eigenvalue λ. Let A be an orthogonal linear transformation in $\mathbb{R}^N$ and M the corresponding matrix. Then also $((\beta \circ A)M, w \circ A)$ is an eigenvector of $R_{B,t}$ associated with λ.*

Proof. First of all, we note that the rotation invariance of the Laplace operator yields

$$\Delta((\beta \circ A)M) = ((\Delta \beta) \circ A)M, \quad \text{and} \quad \Delta(w \circ A) = (\Delta w) \circ A.$$

Moreover, by standard calculus we have

$$\mathrm{div}((\beta \circ A)M) = \mathrm{Tr}\left(M^t \nabla(\beta \circ A)\right) = \mathrm{Tr}\left(M^t((\nabla \beta) \circ A)M\right) = (\mathrm{div}\beta) \circ A,$$

where Tr denotes the trace of a matrix, and

$$\nabla \mathrm{div}((\beta \circ A)M) = \nabla((\mathrm{div}\beta) \circ A) = ((\nabla \mathrm{div}\beta) \circ A)M.$$

By using the previous identities and the fact that (β, w) is a solution to (6.1), we get

$$
\begin{aligned}
&-\frac{\mu_1}{12}\Delta((\beta\circ A)M)-\frac{\mu_1+\mu_2}{12}\nabla\mathrm{div}((\beta\circ A)M)-\frac{\mu_1 k}{t^2}(\nabla(w\circ A)-(\beta\circ A)M)\\
&=-\frac{\mu_1}{12}((\Delta\beta)\circ A)M-\frac{\mu_1+\mu_2}{12}((\nabla\mathrm{div}\beta)\circ A)M-\frac{\mu_1 k}{t^2}((\nabla w)\circ A-(\beta\circ A))M\\
&\qquad\qquad=\frac{\lambda t^2}{12}(\beta\circ A)M,
\end{aligned}
$$

and

$$
-\frac{\mu_1 k}{t^2}(\Delta(w\circ A)-(\mathrm{div}((\beta\circ A)M))=-\frac{\mu_1 k}{t^2}(\Delta w-\mathrm{div}\beta)\circ A=\lambda w\circ A,
$$

which show that $((\beta\circ A)M, w\circ A)$ is an eigenvector of R_t associated with λ. □

We now prove the following

Theorem 6.15. *Let B be a ball in $\mathbb{R}^N$ centered at zero, $t>0$, and λ be an eigenvalue of R_t. Let F be the subset of $\mathbb{N}$ of indexes j such that $\lambda_{j,t}[B]=\lambda$. Let $(\beta^{(1)}, w^{(1)}), \dots, (\beta^{(|F|)}, w^{(|F|)})$ be an orthonormal basis in $\mathcal{V}(B)$ of the eigenspace associated with λ. Then the functions*

$$
\sum_{l=1}^{|F|}|\beta^{(l)}|^2,\quad \sum_{l=1}^{|F|}\left|\frac{\partial\beta^{(l)}}{\partial\nu}\right|^2,\quad \sum_{l=1}^{|F|}\left|\frac{\partial\beta^{(l)}}{\partial\nu}\cdot\nu\right|^2,\quad \sum_{l=1}^{|F|}|w^{(l)}|^2,\quad \sum_{l=1}^{|F|}\left|\frac{\partial w^{(l)}}{\partial\nu}\right|^2, \tag{6.42}
$$

where $\nu(x)=x/|x|$ for all $x\in B\setminus\{0\}$, are radial. In particular, there exists $c\in\mathbb{R}$ such that condition (6.41) holds.

Proof. Recall that $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in $\mathbb{R}^N$, and let $A\in O_N(\mathbb{R})$ be a transformation with associated matrix M. By Lemma 6.14 it follows that $\{((\beta^{(l)}\circ A)M, w^{(l)}\circ A) : l=1,\dots,|F|\}$ is another orthonormal basis of the eigenspace associated with λ. Since both $\{(\beta^{(l)}, w^{(l)}) : l=1,\dots,|F|\}$ and $\{((\beta^{(l)}\circ A)M, w^{(l)}\circ A) : l=1,\dots,|F|\}$ are orthonormal bases, then there exists $S[A]\in O_{|F|}(\mathbb{R})$ with matrix $(S_{ij}[A])_{i,j=1,\dots,|F|}$ such that

$$
((\beta^{(j)}\circ A)M, w^{(j)}\circ A)=\sum_{l=1}^{|F|}S_{jl}[A](\beta^{(l)}, w^{(l)}) \tag{6.43}
$$

By (6.43) we deduce that

$$(\beta \circ A)M = S[A]\beta \ \text{ and } \ w \circ A = S[A]w, \tag{6.44}$$

where β denotes the $l \times N$-matrix, the rows of which are given by the row vectors $\beta^{(j)}$, and w is the column vector the entries of which are given by $w^{(j)}$.

By the first equality in (6.44) we have $(\beta\beta^t) \circ A = S[A]\beta\beta^t S[A]^t$, hence

$$\sum_{l=1}^{|F|} |\beta^{(l)} \circ A|^2 = \mathrm{Tr}\,[(\beta\beta^t) \circ A] = \mathrm{Tr}\,[S[A]\beta\beta^t S[A]^t] = \mathrm{Tr}\,[\beta\beta^t] = \sum_{l=1}^{|F|} |\beta^{(l)}|^2. \tag{6.45}$$

By the arbitrary choice of A we deduce by (6.45) that $\sum_{l=1}^{|F|} |\beta^{(l)}|^2$ is a radial function. Similarly, using the second equality in (6.44), one can prove that $\sum_{l=1}^{|F|} |w^{(l)}|^2$ is a radial function as well.

We now consider the other functions in (6.42). By differentiating in the radial direction ν the first equality in (6.44), we have that for every $j = 1, \dots, l$ and $s = 1, \dots, N$,

$$\sum_{r,h,k=1}^{N} \frac{\partial \beta_r^{(j)}}{\partial x_h} \circ A M_{hk} M_{rs} \nu_k = \sum_{l=1}^{|F|} \sum_{k=1}^{N} S_{jl}[A] \frac{\partial \beta_s^{(l)}}{\partial x_k} \nu_k. \tag{6.46}$$

Taking into account that $M\nu = \nu \circ A$ we deduce by (6.46) that

$$\left(\frac{\partial \beta}{\partial \nu} \circ A \right) M = S[A] \frac{\partial \beta}{\partial \nu}. \tag{6.47}$$

By proceeding as in (6.45) we get that $\sum_{l=1}^{|F|} \left| \frac{\partial \beta^{(l)}}{\partial \nu} \right|^2$ is a radial function.

By multiplying both sides of (6.47) by ν we also get

$$\left(\frac{\partial \beta}{\partial \nu} \cdot \nu \right) \circ A = S[A] \frac{\partial \beta}{\partial \nu} \cdot \nu,$$

which implies that $\sum_{l=1}^{|F|} \left| \frac{\partial \beta^{(l)}}{\partial \nu} \cdot \nu \right|^2$ is a radial function. Similarly, one can prove that the last function in (6.42) is radial. ☐

Combining all the results in this section we get the following

Theorem 6.16. *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $t > 0$. Let $\tilde{\phi} \in \mathcal{A}_\Omega^1$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of R_t in $\tilde{\phi}(\Omega)$ and let F be the set of indexes $j \in \mathbb{N}$ such that $\lambda_{j,t}[\tilde{\phi}(\Omega)] = \tilde{\lambda}$. Then for all $s = 1, \dots, |F|$ the elementary symmetric function $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ with volume constraint.*

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