Università degli Studi di Padova

Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica

Scuola di dottorato di ricerca in: Scienze Matematiche.
Indirizzo: Matematica.

Ciclo: XXV

Equivalences of additive categories

Direttore della scuola: Ch.mo Prof. Paolo Dai Pra

Coordnatore d'indirizzo: Ch.mo Prof. Franco Cardin

Supervisore: Ch.ma Prof.ssa Silvana Bazzoni

Dottoranda: Alice Pavarin

Contents

Abstract
In the first part of the thesis, after an introduction of the concept of recollement and TTF triple in a triangulated category, we consider recollements of derived categories of differential graded algebras induced by self-orthogonal compact objects obtaining a generalization of Rickard's Theorem. Specializing to the case of partial tilting modules over a ring, we extend the results on triangle equivalences proved in [**B**] and [**BMT**]. After that we focus on the connection between recollements of derived categories of rings, bireflective subcategories and "generalized universal localizations". In the second part of the thesis we give some results in the setting of monoidal categories and dual qausi-bialgebras. To every dual quasi-bialgebra H and every bialgebra R in the category of Yetter-Drinfeld modules over H, one can associate a dual quasi-bialgebra, called bosonization. In this thesis, using the fundamental theorem, we characterize as bosonizations the dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode. As an application we investigate the structure of the graded coalgebra gr$A$ associated to a dual quasi-bialgebra A with the dual Chevalley property (e.g. A is pointed).

Sommario
Nella prima parte della tesi, dopo aver introdotto il concetto di incollamento e di triple TTF in una categoria triangolata, si considerano incollamenti di categorie derivate di algebre differenziali graduate indotti da oggetti compatti e auto-ortogonali, ottenendo una generalizzazione del teorema di Rickard. Considerando il caso particolare del moduli partial tilting, estendiamo i risultati sulle equivalenze tra categorie triangolate ottenute in [**B**] e [**BMT**]. Segue una parte focalizzata sulla connessione tra incollamenti di categorie derivate di anelli, sottocategorie biriflessive e localizzazioni universali generalizzate. Nella seconda parte della tesi vengono dati alcuni risultati nell'ambito di categorie monoidali e dual quasi-bialgebre. Ad ogni dual quasi-bialgebra H e ad ogni bialgebra R nella categoria dei moduli di Yetter-Drinfeld su H, è possibile associare una dual quasi-bialgebra, chiamata bosonizzazione. In questa tesi, usando il teorema fondamentale, si caratterizza come bosonizzazione ogni dual quasi-bialgebra con proiezione su una dual quasi-bialgebra con preantipode. Come applicazione si studia la struttura della coalgebra graduata gr$A$ associata ad una dual quasi-bialgebra A con la proprietà di Chevalley duale (si vedrà che A è puntata).

Introduction

This thesis is a collection of results obtained during the three years of my Ph.D. studies. In the first year I worked on monoidal categories and Hopf algebras (with the support of A. Ardizzoni). The "product" of this year is presented in the second part of this thesis. I spent the other two years doing researches on triangulated category and tilting theory (first part of this thesis, with the support of S. Bazzoni). The thesis is structured as follows: the first part of the introduction and the first three chapters are dedicated to my research on triangulated category and tilting theory, while the second part of the introduction and the other two chapters are about monoidal categories and Hopf algebras.

Tilting theory owes its origin to Bernstein, Gelfand and Ponomarev (see [**BGP**]) who invented reflection functors (reformulated, some years later, by Auslander, Platzeck and Reiten in [**APR**]). The first definition of tilting module is due to Brenner and Butler (see [**BRB**]), but the most common one, is due to Happel and Ringel (see [**HR**]). Tilting theory was born in the same philosophy as "Morita theory of equivalence", to simplify the study of the module category of an algebra A, by replacing A with another simpler algebra B. A tilting module over an algebra A is a finitely generated module of projective dimension one, such that $\mathrm{Ext}^1_A(T,T)=0$ and there exists a short exact sequence $0 \to A \to T_0 \to T_1 \to 0$ with T_0 and T_1 in Add T (the class of all direct summands of set index coproducts of T). So tilting modules can be viewed as the generalization of progenerators (finitely generated projective modules in the category of finitely generated modules A-mod). The difference between tilting theory and Morita theory is that, given a tilting module T over a finite dimensional algebra A and indicated with B its endomorphism algebra, the functors $\mathrm{Hom}_A(T,-)$ and $T\otimes_B -$ do not provide and equivalence between A-mod and B-mod, but just between two pairs of subcategories (the torsion pairs $(\mathrm{Ker}(\mathrm{Hom}_A({}_AT,-), \mathrm{Ker}(\mathrm{Ext}^1_A({}_AT,-))$ and $(\mathrm{Ker}(T\otimes_B -), \mathrm{Ker}(\mathrm{Tor}^B_1(T_B,-)))$. This result, proved by Brenner and Butler ([**BRB**]), was generalized by Miyaishita in [**Mi**]. In fact he considered tilting modules of projective dimension $n \geq 1$ (n-tilting modules) and he proved that, given the classes

$$KE_i(T) = \{M \in A\text{-Mod} \mid \mathrm{Ext}^j_A(T,M)=0 \quad \forall\ 0 \leq j \neq i\}$$

$$KT_i(T) = \{N \in B\text{-Mod} \mid \mathrm{Tor}^B_j(T,M)=0 \quad \forall\ 0 \leq j \neq i\}$$

the functors $\mathrm{Ext}^i_A(T,-)$ and $\mathrm{Tor}^B_i(T,-)$ induce equivalences between the classes $KE_i(T)$ and $KT_i(T)$. In the late 80's the study of infinitely generated tilting modules started. An infinitely generated tilting module T over a ring R is an infinitely generated module of projective dimension one, such that $\mathrm{Ext}^1_R(T,T^{(I)})=0$ for every

set I and such that there are two modules T_0 and T_1 in Add T and a short exact sequence $0 \to R \to T_0 \to T_1 \to 0$. From now on infinitely generated tilting modules will be called just tilting modules, while finitely generated tilting modules will be called classical tilting modules. The study of infinitely generated tilting modules started for different reasons. In representation theory (in particular for Tame algebras) infinitely generated modules bring a better understanding of the behavior of finitely generated modules. For example, "generic modules" permit to parametrize families of finitely generated modules ([**CB**, Introduction]). Tilting modules over a ring R are strongly linked to approximation theory (preenvelopes and precovers), while such approximations may not be available working only with finitely generated modules (e.g. [**Tr**]). Tilting modules are involved also in the finitistic dimension conjecture (as shown in [**AT**]).
In 1988 Facchini ([**F**], [**F2**]) proved that, over a commutative domain S, the divisible module ∂ introduced by Fuchs ([**Fu**]) is an infinitely generated tilting modules and it provides a pair of equivalences between two subcategories of S-Mod and $\mathrm{End}_S(\partial)$-Mod.

An important result in this direction was given by Colpi and Trlifaj ([**CT**]) who studied infinitely generated tilting modules over arbitrary rings and proved a Brenner Butler type Theorem.
In the same years works by several authors showed that a natural setting to interpret equivalences induced by classical tilting modules was that of derived categories. The first result in this direction was proved by Happel:

Theorem ([**H**]). *Let A be a finite dimensional algebra, T a finitely generated tilting module over A and set $B := \mathrm{End}_A({}_AT)$. Then there is an equivalence:*

$$\begin{tikzcd} \mathcal{D}^b(A) \arrow[r, "\mathbb{R}\mathrm{Hom}_A({}_AT,-)"] & \mathcal{D}^b(B) \arrow[l, bend right, "T\overset{\mathbb{L}}{\otimes}_B -"'] \end{tikzcd}$$

between the bounded derived categories of A and B respectively.

This result was generalized by Cline, Parshall and Scott (they removed the assumption of the finite global dimension) and then it was given in the unbounded derived categories by Rickard and Keller. In this case the equivalence arises from a generic "tilting object" as will be defined below. In order to restate Rickard-Keller Theorem we need to recall that a compact object, in a triangulated category $\mathcal{D}$ with set index coproducts, is an object M such that the functor $\mathrm{Hom}_{\mathcal{D}}(M,-)$ commutes with coproducts. Let A be a ring and $\mathcal{D}(A)$ its unbounded derived category. A complex in $\mathcal{D}(A)$ is called perfect if it is a bounded complex of finitely generated projective A-modules.

Theorem. [**Ke6**] *Let k be a commutative ring, A and B be k-algebras which are flat as modules over k. The following are equivalent:*

(1) *There is a k-linear triangle equivalence $F : \mathcal{D}(A) \to \mathcal{D}(B)$.*
(2) *There is a complex of $A^{op} - B$ modules X such that the functor*

$$X \overset{\mathbb{L}}{\otimes}_A - : \mathcal{D}(A) \to \mathcal{D}(B)$$

is an equivalence.

(3) *There is a complex T of B-modules such that the following conditions hold:*
   i) *T is perfect.*
   ii) *T generates $\mathcal{D}(B)$ as a triangulated category closed under small coproducts.*
   iii) *T is self-orthogonal and* $\mathrm{Hom}_{\mathcal{D}(B)}(T,T) = A$.

A complex T satisfying the conditions in (3) is called tilting complex. If T is a tilting module over A, with endomorphism ring B, the pair $(G,H) := (T \overset{\mathbb{L}}{\otimes}_B -, \mathbb{R}\mathrm{Hom}_A({}_AT,-))$ is no more an equivalence. Bazzoni proved that H induces an equivalence with the quotient between $\mathcal{D}(A)$ and $\mathcal{D}(B)$ modulo the kernel of G. This result was generalized to the n-dimensional case by Bazzoni, Mantese e Tonolo in [**BMT**] and later, in the more general setting of dg categories, by Yang in [**Y**]. In [**BMT**], denoted by i the inclusion functor of $\mathrm{Ker}G$ in $\mathcal{D}(B)$, the equivalence of $\mathcal{D}(A)$ with $\mathcal{D}(B)/\mathrm{Ker}G$ can be expressed by the following diagram

$$\begin{tikzcd} \mathrm{Ker}G \arrow[r, "i"] & \mathcal{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "j^*"] & \mathcal{D}(B)/\mathrm{Ker}G \simeq \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

This diagram is an example of recollement of derived categories. A recollement of a triangulated category $\mathcal{T}$ can be defined as a diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j^*"] \end{tikzcd}$$

where the six functors involved are the derived version of Grothendieck's functors. In particular, they are paired in two adjoint triples, i_* is fully faithful and $\mathcal{T}''$ is equivalent to a quotient category of $\mathcal{T}$ via j^* so that the straight arrows can be interpreted as an exact sequence of categories. The notion of recollements was introduced by Beilinson-Bernstein-Deligne [**BBD**] in a geometric context, where stratifications of varieties induce recollements of derived categories of constructible sheaves. The algebraic aspect of recollements has become more and more apparent. Equivalence classes of recollements of triangulated categories are in bijection with torsion-torsion-free triples, that is triples $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ of full triangulated subcategories of the central term $\mathcal{T}$ of a recollement, where $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{Y},\mathcal{Z})$ are torsion pairs ([**N2**, Section 9.2]). Torsion pairs in triangulated categories allow to regard a triangulated category as glued together from two other triangulated categories $(\mathcal{Y},\mathcal{Z})$ or $(\mathcal{Y},\mathcal{X})$. Particular example of recollements are the recollements of derived categories of module categories (that are generated by a single compact object). The notion of recollement in compactly generated triangulated categories is strongly linked to tilting theory. A first result in this direction was proved by König in [**K**] where necessary and sufficient conditions are given to express the bounded derived category of a ring as recollement. This result was generalized by Nicolas and Saorin in [**NS**] and by Angeleri, König and Liu in [**AKL**]. In [**NS**] is proved the following theorem:

THEOREM. [**NS**, Proposition 3.4] *The following assertions are equivalent:*

(1) $\mathcal{D}$ *is a recollement of triangulated categories generated by a single compact object.*
(2) *There are objects* P *and* Q *of* $\mathcal{D}$ *such that:*
  i) P *is compact.*
  ii) Q *is self-compact.*
  iii) $\mathrm{Hom}_{\mathcal{D}}(P[n], Q) = 0$ *for each* $n \in \mathbb{Z}$.
  iv) $\{P, Q\}$ *generates* $\mathcal{D}$.
(3) *There is a compact object* P *such that* Tria $(P)^{\perp}$ *is generated by a compact object in* Tria $(P)^{\perp}$.

A non compact version of the same result is proved in [**AKL**].
Thanks to Keller theorem, which states that every triangulated category generated by a compact object is the derived category of a differential graded algebra (a graduated algebra endowed with a differential map that satisfies the Leibniz rule), the left and right terms of the recollement in the above theorem are derived categories of suitable differential graded algebra (dg algebra). In the setting of derived categories a compact object induces a recollement of derived categories of dg algebras. Explicit instances of this situation are considered by Jørgensen in [**J**]. There, starting from results in [**DG**], [**Mi**] and [**N**], recollements of derived categories of dg algebras are characterized in terms of derived functors associated to two objects, one compact and the other self-compact. Moreover, in [**NS**] is proved that, for every dg category $\mathcal{A}$, flat over a field k, there is a bijection among TTF triples in $\mathcal{D}(\mathcal{A})$, recollements of $\mathcal{D}(\mathcal{A})$ and homological epimorphisms of dg categories $F : \mathcal{A} \to \mathcal{B}$, for a suitable dg-category $\mathcal{B}$. It is remarkable that in the connection between tilting theory and recollement there is a natural "involvement" of dg-theory, at least at the level of dg algebras, otherwise it may not be possible to express recollements induced by tilting objects as recollements of derived categories of some abelian categories. Let us note that, establishing the correspondence between tilting modules and recollements, two different approaches can be found. In [**AKL**] they start with a tilting module over a ring A and then construct a recollement of $\mathcal{D}(A)$. Another approach consists in starting from a (good) tilting module over a ring A and then construct a recollement of $\mathcal{D}(\mathrm{End}_A(T))$ (instances of this situation can be found in the work of Chen and Xi [**CX**] and Yang [**Y**]). An infinitely generated (good) n-tilting module T over a ring A with endomorphism ring B becomes a classical partial n-tilting module over B (see [**Mi**]), that is a module with a finite projective resolution consisting of finitely generated projective modules of projective dimension n, such that $\mathrm{Ext}^i_B(T, T) = 0$ for every integer $i > 0$. Hence, in particular, regarded in the derived category, it is isomorphic to a compact and self-orthogonal complex (partial tilting complex).

One of the results in this thesis can be viewed as a generalization of the Morita-type theorem proved by Rickard in [**R**] (see Theorem 2.4.6) from tilting complexes to partial tilting complexes. In fact, using a quasi-isomorphism between the endomorphism ring A of a partial tilting dg-module P_B (that is a complex of abelian groups such that the differential is compatible with the action of B) and the dg-endomorphism ring D of P, we show that the functor $P \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ induces an equivalence between the quotient of $\mathcal{D}(B)$ modulo the full triangulated subcategory $\mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ and the derived category $\mathcal{D}(A)$, that is there is the following recollement:

$$\begin{tikzcd}[column sep=large] \mathrm{Ker}(P \otimes_B -) \arrow[r, "i_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{1}$$

If P is moreover a tilting complex over a ring B with endomorphism ring A, then $\mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ is zero and we recover Rickard's Theorem. In particular we consider applications to the case of a classical partial tilting right module T over a ring B. As examples of this case we start with a possibly infinitely generated left module ${}_AT$ over a ring A, which is self-orthogonal and such that $A \in \mathrm{tria}\ T$ (that is A is in the smallest triangulated category containing T and closed under finite coproducts and direct summands). Under these assumptions, T, viewed as a right module over its endomorphism ring B, is a faithfully balanced classical partial n-tilting module and applying Theorem 2.5.6 we obtain a generalization of the result proved in [**BMT**] where the stronger assumption that ${}_AT$ was a "good n-tilting module" was assumed. Moreover, this setting provides an instance of the situation considered in [**Y**]. Finally, we analyze in more details the left end term of a recollement induced by a classical partial tilting module. It is proved, that under certain hypotheses, the kernel of the derived functor of $T \otimes_B -$ is the derived category of a dg algebra concentrated in degree zero, that is a ring, linked to B via a homological ring epimorphism. The problem is connected with the study of the following subcategory of B- modules:

$$\mathcal{E} := \{M \in B\text{-Mod} \mid Tor_i^B(T, M) = 0 \quad i \geq 0\}.$$

If T is a 1-good tilting module over A, then the left term of the recollement (1) is the derived category of the universal localization of B at the projective resolution of T_B. We generalize this situation and we prove that, for a partial n-tilting modules, if the kernel of the derived functor of the tensor product is equivalent to the derived category of a ring via a homological ring epimorphism, then this is the *generalized universal localization* of B at the projective resolution of T. Generalized universal localizations were introduced by Krause in [**Kr**] under the name of homological localization:

Definition. *Let B be a ring and Σ a set of compact objects $P \in \mathcal{D}(B)$. A ring S is a "generalized universal localization" of B at the set Σ if:*

(1) *there is a ring homomorphism $\lambda\colon B \to S$ such that $P \overset{\mathbb{L}}{\underset{B}{\otimes}} S$ is acyclic;*

(2) *for every ring homomorphism* $\mu\colon B \to R$ *such that* $P \overset{\mathbb{L}}{\underset{B}{\otimes}} R$ *is acyclic, there exists a unique ring homomorphism* $\nu\colon S \to R$ *such that* $\nu \circ \lambda = \mu$.

The first part of the thesis is structured as follows.
In Chapter 1 some results concerning recollements of triangulated categories, in particular for the compactly generated ones, are presented. We give the definition of triangulated category, the notion of Verdier localization and Bousfield localization. We recall the definition of recollements and TTF triple in a triangulated category and the bijection between them.

In chapter 2 we recall the construction of the derived category of an abelian category, and, in particular, we focus on the derived category of a dg algebra. Then we present some results on recollements from compact objects in this category. After that we specialize the situation to the case of partial tilting complexes. When P is a partial tilting right dg-module over a dg algebra B and $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is the torsion-torsion-free triple connected to the recollement induced by P, we have that $\mathcal{Y}$ coincides with $\mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$, $\mathcal{X}$ is the full triangulated subcategory of $\mathcal{D}(B)$ generated by the dual $\mathbb{R}\mathrm{Hom}_B(P, B)$ of P and $\mathcal{Z}$ is equivalent to the derived category of the endomorphism ring of P (see Theorem 2.4.6). In this setting the following generalization of Rickard theorem is proved:

Theorem. *Let B be a dg algebra and let P be a partial tilting right dg B-module. Let $A = \mathrm{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(P, P)$, $Q = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P, B)$. Then there exists a dg algebra E and a recollement:*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(E) \arrow[r, "i_*(B)\otimes^{\mathbb{L}}_E -"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(i^*(B),-)"] \arrow[r, "j^*"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

where, letting $D = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P, P)$ there is a triangle equivalence $\rho\colon \mathcal{D}(D) \to \mathcal{D}(A)$ such that:

(1) $j_! = (Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -) \circ \rho^{-1}$;
(2) $j^* = \rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$;
(3) $j_* = \mathbb{R}\mathrm{Hom}_D(P, -) \circ \rho^{-1}$ *is fully faithful;*
(4) *if* $\mathcal{Y} = \mathrm{Ker}\ (j^*)$ *and* $\mathcal{Z} = \mathrm{Im}\ j_*$, *then* $(\mathrm{Tria}\ Q, \mathcal{Y}, \mathcal{Z})$ *is a TTF TRIPLE in* $\mathcal{D}(B)$ *and* $\mathcal{Y}$ *is the essential image of* F_*;
(5) $\mathcal{D}(A)$ *is triangle equivalent to* $\mathcal{D}(B)/\mathrm{Ker}\ (j^*)$.

In particular, if P is a tilting right dg B-module, then $\mathcal{Y}$ vanishes and

$$\rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)\colon \mathcal{D}(B) \to \mathcal{D}(A)$$

is a triangle equivalence with inverse $\mathbb{R}\mathrm{Hom}_D(P, -) \circ \rho^{-1}$.

A particular example of a partial tilting complex is given by a classical partial tilting module over a ring B. In this case the following theorem is proved.

THEOREM. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. There is a dg algebra E and a recollement*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(E) \arrow[r, "i^*(B)\overset{\mathbb{L}}{\otimes}_E -"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(i^*(B),-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B -"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*=\mathbb{R}\mathrm{Hom}_A(T,-)"]
\end{tikzcd}$$

where:

(1) $j_* = \mathbb{R}\mathrm{Hom}_A(T, -)$ *is fully faithful;*

(2) $\mathcal{D}(A)$ *is triangle equivalent to* $\mathcal{D}(B)/\mathrm{Ker}\ (T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$.

The above theorem can be viewed as a generalization of the result in [**BMT**]. Moreover the concept of homological epimorphism of dg algebras is recalled and it is shown explicitly (see Proposition 2.3.9) how to exhibit a homological epimorphism of dg algebras $B \to C$ such that the left end term of the recollement induced by a compact object is the derived category $\mathcal{D}(C)$ (this result is an instance of the more general theorem proved in [**NS**]). Anyway, in Corollary 2.2.6 we prove that, without flatness conditions on B, $\mathcal{D}(B)$ is still a recollement of dg algebras, but not necessarily associated to a homological epimorphism.

In Chapter 3, given a classical partial tilting module T_B over a ring B, we look for conditions under which the class $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ is equivalent to the derived category of a ring S for which there is a homological ring epimorphism $\lambda\colon B \to S$. We show that this happens if and only if the perpendicular subcategory $\mathcal{E}$ consisting of the left B-modules N such that $\mathrm{Tor}_i^B(T, N) = 0$ for every $i \geq 0$, is bireflective and every object of $\mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$.

Results in [**GL**] and [**GP**] show that a full subcategory of a module category B-Mod is bireflective if and only if it is equivalent to a module category over a ring S linked to B via a ring epimorphism $\lambda\colon B \to S$. In the favorable case in which the left term of the recollement induced by a partial tilting module T_B is the derived category of a ring, we prove that λ is moreover a homological ring epimorphism and S is isomorphic to the endomorphism ring of the left adjoint $L(B)$. Moreover, S is the generalized universal localization of B with respect to a projective resolution of T_B.

PROPOSITION. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$, L the left adjoint of the inclusion $i\colon \mathcal{Y} \to \mathcal{D}(B)$ and $\mathcal{E}$ the subcategory of B-Mod defined above.*

Then the following conditions are equivalent:

(1) $H^i(L(B)) = 0$ *for every* $0 \neq i \in \mathbb{Z}$.

(2) *there is a ring S and a homological ring epimorphism $\lambda\colon B \to S$ inducing a recollement:*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(S) \arrow[r, "\lambda_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=S\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(S,-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*=\mathbb{R}\mathrm{Hom}_A(T,-)"]
\end{tikzcd}$$

(3) *Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$ and $\mathcal{E}$ is a bireflective subcategory of B-*Mod.

(4) *Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$ and the homologies of N belong to $\mathcal{E}$.*

Later, some properties of the "generalized universal localization" are proved in the following proposition:

Proposition. *Let P be a compact complex in $\mathcal{D}(B)$. Assume that $\lambda\colon B \to S$ is a generalized universal localization of B at $\{P\}$. Let $\mathcal{E}_P = \{N \in B$-*Mod $\mid P \underset{B}{\otimes} N$ is acyclic $\}$. *Then, the following hold:*

(1) $\lambda_*(S\text{-Mod}) \subseteq \mathcal{E}_P$.

(2) $\lambda_*(S\text{-Mod}) = \mathcal{E}_P$ *if and only if $\mathcal{E}_P$ is a bireflective subcategory of B-*Mod.

The situation illustrated above is a generalization of a recent article by Chen and Xi ([**CX**]). In fact, in [**CX**], completing the results proved in [**B**] for "good" 1-tilting modules T over a ring A with endomorphism ting B, it is shown that the derived category $\mathcal{D}(B)$ is the central term of a recollement with right term $\mathcal{D}(A)$ and left term the derived category of a ring S which is a universal localization of the differential of the projective resolution of T_B. We note that our setting is different since we fix a ring B and we obtain recollements of $\mathcal{D}(B)$ for every choice of partial tilting modules over B, while, starting with an infinitely generated good tilting module ${}_AT$ over a ring A, one obtains a recollement whose central term is the derived category of the endomorphism ring B of ${}_AT$ and B might be very large and difficult to handle. Instead, thanks to our approach, we can choose algebras of finite representation type and classical partial n-tilting modules (with $n > 1$) over them, and define a homological ring epimorphism $\lambda\colon B \to S$.

Let us set $\mathcal{Y} := \mathrm{Ker}(T \overset{\mathbb{L}}{\otimes}_B -)$. The following examples are given:

i) A class of classical partial 2-tilting modules such that the class $\mathcal{E}$ is bireflective and $\mathcal{Y}$ is equivalent to the derived category of a ring (Example 1).

ii) A classical partial 2-tilting module such that $\mathcal{E}$ is not bireflective (Example 2).

iii) A classical partial 2-tilting module such that $\mathcal{E}$ is bireflective but $\mathcal{Y}$ is not equivalent to the derived category of a ring via a homological ring epimorphism (Example 3).

iv) A classical partial n-tilting module that is also a good tilting module over its endomorphism ring, such that $\mathcal{Y}$ is not equivalent to the derived category of a ring (Example 4).

v) A classical partial n-tilting module that is also a good tilting module over its endomorphism ring, such that $\mathcal{Y}$ is equivalent to the derived category of a ring (Example 5).

The first three mentioned examples of classical partial n-tilting right modules T are not arising from good n-tilting modules with endomorphism ring B. The problem to decide when a recollement induced by a good n-tilting module over a ring A corresponds to a homological epimorphism of rings remains open. In the case of good n tilting module Chen and Xi in [**CX2**] give some equivalent conditions in terms of the functor $\mathrm{Ext}^i_A(T,-)$.

In the Appendix we regard the generator M of $\mathrm{Ker}(T \overset{\mathbb{L}}{\otimes}_B -)$ as a Milnor colimit (see [**AKL**]) and we make some computations on the homologies of the dg algebra $\mathbb{R}\mathrm{Hom}_B(M,M)$ as an attempt to prove that only a finite number of homologies are different from zero.

The second part of the thesis is a collection of results in the setting of monoidal categories and Hopf algebras. Let H be a bialgebra. Consider the functor $T := (-) \otimes H : \mathfrak{M} \to \mathfrak{M}^H_H$ from the category of vector spaces to the category of right Hopf modules. It is well-known that T determines an equivalence if and only if H has an antipode i.e. it is a Hopf algebra. The fact that T is an equivalence is the so-called fundamental (or structure) theorem for Hopf modules, which is due, in the finite-dimensional case, to Larson and Sweedler, see [**Ls**, Proposition 1, page 82]. This result is crucial in characterizing the structure of bialgebras with a projection as Radford-Majid bosonizations (see [**Ra**]). Recall that a bialgebra A has a projection onto a Hopf algebra H if there exist bialgebra maps $\sigma : H \to A$ and $\pi : A \to H$ such that $\pi \circ \sigma = \mathrm{Id}_H$. Essentially using the fundamental theorem, one proves that A is isomorphic, as a vector space, to the tensor product $R \otimes H$ where R is some bialgebra in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H. This way $R \otimes H$ inherits, from A, a bialgebra structure which is called the *Radford-Majid bosonization of R by H* and denoted by $R\#H$. It is remarkable that the graded coalgebra $\mathrm{gr}A$ associated to a pointed Hopf algebra A (here "pointed" means that all simple subcoalgebras of A are one-dimensional) always admits a projection onto its coradical. This is the main ingredient in the so-called lifting method for the classification of finite dimensional pointed Hopf algebras (see [**AS**]).

In 1989 Drinfeld introduced the concept of quasi-bialgebra in connection with the Knizhnik-Zamolodchikov system of partial differential equations. The axioms defining a quasi-bialgebra are a translation of monoidality of its representation category with respect to the diagonal tensor product. In [**Dr**], the antipode for a quasi-bialgebra (whence the concept of quasi-Hopf algebra) is introduced in order to make the category of its flat right modules rigid. If we draw our attention to the category of co-representations of H, we get the concepts of dual quasi-bialgebra and of dual quasi-Hopf algebra. These notions have been introduced in [**Maj3**] in order to prove a Tannaka-Krein type Theorem for quasi-Hopf algebras.

A fundamental theorem for dual quasi-Hopf algebras was proved by Schauenburg in [**Sch4**] but dual quasi-Hopf algebras do not exhaust the class of dual quasi-bialgebras satisfying the fundamental theorem. It is remarkable that the functor T giving the fundamental theorem in the case of ordinary Hopf algebras must be substituted, in the "quasi" case, by the functor $F := (-) \otimes H$ between the category ${}^{H}\mathfrak{M}$ of left H-comodules and the category ${}^{H}\mathfrak{M}_{H}^{H}$ of right dual quasi-Hopf H-bicomodules (essentially this is due to the fact that, unlike the classical case, a dual quasi-bialgebra H is not an algebra in the category of right H-comodules but it is still an algebra in the category of H-bicomodules). In [**AP**, Theorem 3.9], it is showed that, for a dual quasi-bialgebra H, the functor F is an equivalence if and only if there exists a suitable map $S : H \to H$ that we called a preantipode for H. Moreover for any dual quasi-bialgebra with antipode (i.e. a dual quasi-Hopf algebra) it is constructed a specific preantipode, see [**AP**, Theorem 3.10]. It is worth to notice that, by [**Sch5**, Example 4.5.1], there is a dual quasi-bialgebra H which is not a dual quasi-Hopf algebra and such that it satisfies the fundamental theorem. Then we get that H has a preantipode (cf. Theorem 4.3.7).

In this thesis we introduce and investigate the notion of bosonization in the setting of dual quasi-bialgebras. Explicitly, we associate a dual quasi-bialgebra $R\#H$ (that we call bosonization of R by H) to every dual quasi-bialgebra H and bialgebra R in ${}_{H}^{H}\mathcal{YD}$. Then, using the fundamental theorem, we characterize as bosonizations the dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode. As an application, for any dual quasi-bialgebra A with the dual Chevalley property (i.e. such that the coradical of A is a dual quasi-subbialgebra of A), under the further hypothesis that the coradical H of A has a preantipode, we prove that there is a bialgebra R in ${}_{H}^{H}\mathcal{YD}$ such that $\mathrm{gr}A$ is isomorphic to $R\#H$ as a dual quasi-bialgebra. In particular, if A is a pointed dual quasi-Hopf algebra, then $\mathrm{gr}A$ comes out to be isomorphic to $R\#\Bbbk\mathbb{G}(A)$ as dual quasi-bialgebra where R is the diagram of A and $\mathbb{G}(A)$ is the set of grouplike elements in A. We point out that the results are obtained without assuming that the dual quasi-bialgebra considered are finite-dimensional.

The second part of the thesis is structured as follows.

Chapter 4 contains preliminary results needed in the next sections. Moreover in Theorem 4.3.10, we investigate cocommutative dual quasi-bialgebras with a preantipode and we provide a Cartier-Gabriel-Kostant type theorem for dual quasi-bialgebras with a preantipode in the following corollary:

Corollary. *Let H be a dual quasi-bialgebra with a preantipode over a field $\Bbbk$ of characteristic zero. If H is cocommutative and pointed, then H is an ordinary Hopf algebra isomorphic to the biproduct $U(P(H))\#\Bbbk\mathbb{G}(H)$, where $P(H)$ denotes the Lie algebra of primitive elements in H.*

In the connected case such a result was achieved in [**Hu**, Theorem 4.3].

The central part of this chapter is devoted to the study of the category ${}_{H}^{H}\mathcal{YD}$ of Yetter-Drinfeld modules over a dual quasi-bialgebra H. Explicitly, we consider the pre-braided monoidal category $\left({}_{H}^{H}\mathcal{YD}, \otimes, \Bbbk\right)$ of Yetter-Drinfeld modules over a dual quasi-bialgebra H and we prove that the functor F, as above, induces a functor $F : {}_H^H\mathcal{YD} \to {}_H^H\mathfrak{M}_H^H$ (that is an equivalence in case H has a preantipode, see Proposition 4.4.9).

In the end this chapter we prove that the equivalence between the categories ${}_H^H\mathfrak{M}_H^H$ and ${}_H^H\mathcal{YD}$ becomes monoidal if we equip ${}_H^H\mathfrak{M}_H^H$ with the tensor product $\otimes_H$ (or $\square_H$) and unit H (see Lemma 4.5.5 and Lemma 4.5.9). As a by-product, in Lemma 4.5.12, we produce a monoidal equivalence between $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$ and $({}_H^H\mathfrak{M}_H^H, \square_H, H)$.

Chapter 5 contains the main results of this part of the thesis. In the following theorem (Theorem 5.1.1), to every dual quasi-bialgebra H and bialgebra R in ${}_H^H\mathcal{YD}$ we associate a dual quasi-bialgebra structure on the tensor product $R \otimes H$ that we call the *bosonization of R by H* and denote by $R\#H$.

THEOREM. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ be a dual quasi-bialgebra.*
Let $(R, \mu_R, \rho_R, \Delta_R, \varepsilon_R, m_R, u_R)$ be a bialgebra in ${}_H^H\mathcal{YD}$.
Let us consider on $B := F(R) = R \otimes H$ the following structures:

$$\begin{aligned}
m_B[(r \otimes h) \otimes (s \otimes k)] &= \left[\begin{array}{c} \omega_H^{-1}(r_{-2} \otimes h_1 \otimes s_{-2}k_1)\omega_H(h_2 \otimes s_{-1} \otimes k_2) \\ \omega_H^{-1}[(h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3]\omega_H(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6 k_5 \end{array} \right] \\
u_B(k) &= k1_R \otimes 1_H \\
\Delta_B(r \otimes h) &= \omega_H^{-1}(r_{-1}^1 \otimes r_{-2}^2 \otimes h_1) r_0^1 \otimes r_{-1}^2 h_2 \otimes r_0^2 \otimes h_3 \\
\varepsilon_B(r \otimes h) &= \varepsilon_R(r)\varepsilon_H(h) \\
\omega_B((r \otimes h) \otimes (s \otimes k) \otimes (t \otimes l)) &= \varepsilon_R(r)\varepsilon_R(s)\varepsilon_R(t)\omega_H(h \otimes k \otimes l).
\end{aligned}$$

Then $(B, \Delta_B, \varepsilon_B, m_B, u_B, \omega_B)$ is a dual quasi-bialgebra.

Now, let (A, H, σ, π) be a dual quasi-bialgebra with projection and assume that H has a preantipode S. In Lemma 5.2.3, we prove that such an A is an object in the category ${}_H^H\mathfrak{M}_H^H$. Therefore the fundamental theorem describes A as the tensor product $R \otimes H$ of some vector space R by H. Indeed, in Theorem 5.2.4, we prove that the dual quasi-bialgebra structure inherited by $R \otimes H$ through the claimed isomorphism is exactly the bosonization of R by H.

THEOREM. *Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ and $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ be dual quasi-bialgebras such that (A, H, σ, π) is a dual quasi-bialgebra with projection onto H. Assume that H has a preantipode S. For all $a, b \in A$, we set $a_1 \otimes a_2 := \Delta_A(a)$ and $ab = m_A(a \otimes b)$. Then, for all $a \in A$ we have*

$$\tau(a) := \omega_A[a_1 \otimes \sigma S\pi(a_3)_1 \otimes a_4] a_2 \sigma S\pi(a_3)_2$$

and $R := G(A)$ is a bialgebra $((R, \mu_R, \rho_R), m_R, u_R, \Delta_R, \varepsilon_R, \omega_R)$ in ${}_H^H\mathcal{YD}$ where, for all $r, s \in R, h \in H, k \in \Bbbk$, we have

$$h \triangleright r := \mu_R(h \otimes r) := \tau[\sigma(h) r], \qquad r_{-1} \otimes r_0 := \rho_R(r) := \pi(r_1) \otimes r_2,$$
$$m_R(r \otimes s) := rs, \qquad u_R(k) := k1_A,$$
$$r^1 \otimes r^2 := \Delta_R(r) := \tau(r_1) \otimes \tau(r_2), \qquad \varepsilon_R(r) := \varepsilon_A(r).$$

Moreover there is a dual quasi-bialgebra isomorphism $\epsilon_A : R\#H \to A$ given by

$$\epsilon_A(r \otimes h) = r\sigma(h), \qquad \epsilon_A^{-1}(a) = \tau(a_1) \otimes \pi(a_2).$$

The analogue of this result for quasi-Hopf algebras, anything but trivial, has been established by Bulacu and Nauwelaerts in [**BN**], but their proof can not be adapted to dual quasi-bialgebras with a preantipode.

In the end of the chapter we collect some applications of our results. Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Since A is an ordinary coalgebra, we can consider the associated graded coalgebra $\mathrm{gr}A$. In Proposition 5.3.2, we prove that $\mathrm{gr}A$ fits into a dual quasi-bialgebra with projection onto H. As a consequence, in Corollary 5.3.3, under the further assumption that H has a preantipode, we show that there is a bialgebra R in ${}_H^H\mathcal{YD}$ such that $\mathrm{gr}A$ is isomorphic to $R\#H$ as a dual quasi-bialgebra. When A is a pointed dual quasi-Hopf algebra it is in particular a dual quasi-bialgebra with the dual Chevalley property and its coradical has a preantipode. Using this fact, in Theorem 5.3.9 we obtain that $\mathrm{gr}A$ is of the form $R\#\Bbbk\mathbb{G}\left(A\right)$ as dual quasi-bialgebra, where R is the so-called diagram of A.

Acknowledgements. This work is the result of a useful contribution, helps and supports given me by different people I want to thank.
My supervisor, Professor Silvana Bazzoni, for the important support given me in the these three years and for the stimulating research process we have shared. Alessandro Ardizzoni for the collaboration and for the incitement in proceeding my research work. The two parts thesis referees for the useful observations and for the interesting suggestions. Professors Pedro Nicolas, Manolo Saorin, Gabriella D'Este, Chang Chang Xi, my collegue Nicola for the profitable discussions and explanations referring to some contents of the thesis. My family, for the unconditional and always inventive support. Giulia and Cecilia, for the good times spent together, inside and outside the department. All of my friends that still wonder what I have been researching in the last three years. Michele, who has accompanied me in every single moment of this "path".

Ringraziamenti. Questo lavoro è il frutto di collaborazioni, aiuti e stimoli datemi da diverse persone, che, qui di seguito, voglio ringraziare. La Prof.ssa Silvana Bazzoni per il grande sostegno datomi in questi tre anni e per gli stimoli che mi ha offerto. Alessandro Ardizzoni per la collaborazione e le spinte ad andare avanti. I referee delle due parti della tesi per le utili osservazioni e per gli spunti interessanti. I professori Pedro Nicolas, Manolo Saorin, Gabriella D'Este, Chang Chang Xi e il mio collega Nicola per le utili discussioni e spiegazioni su parte degli argomenti di questa tesi. La mia famiglia, per l'appoggio incondizionato e sempre originale. Giulia e Cecilia, per i bei momenti insieme, dentro e fuori dal dipartimento. Tutti gli amici che ancora si chiedono cos'abbia "ricercato" in questi tre anni. Michele, che mi ha accompagnato in ogni singolo momento di questo percorso.

Part 1

Recollements of triangulated categories

CHAPTER 1

Triangulated categories, recollements and TTF triples

This chapter aims to be a presentation of some tools in the context of triangulated categories, that will be useful later on. Some results on recollements and torsion-torsion free triple are proved; in particular we will focus on compactly generated triangulated categories.

1. Localizations in triangulated categories

In this section we define a triangulated category and we briefly present the two main approaches to localization in triangulated categories: the Verdier localization and the Bousfield localization. Moreover we illustrate the strict connection between them. We will follows the exposition of [**N2**] and [**Kr2**].

Definition 1.1.1. Let $\mathcal{C}$ be an additive category and $[1] : \mathcal{C} \longrightarrow \mathcal{C}$ an additive automorphism. A *candidate triangle* in $\mathcal{C}$ (with respect to [1]) is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

such that the composites $v \circ u$, $w \circ v$ and $u[1] \circ w$ are zero. A morphism $\Phi := (f, g, h)$ of candidate triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "{f[1]}"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} .$$

This defines the category of candidate triangles in $\mathcal{C}$ (with respect to [1]).

Definition 1.1.2. A *pretriangulated category* is an additive category $\mathcal{T}$ together with an additive automorphism [1] called shift functor, and a class of candidate triangles (with respect to [1]) called *distinguished triangles*, satisfying the following axioms:

[TR0] Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle. For any object X the candidate triangle

$$X \xrightarrow{Id} X \xrightarrow{0} 0 \longrightarrow X[1]$$

is distinguished.

[TR1] For any morphism $f : X \longrightarrow Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

[TR2] (*The "rotation axiom"*) Suppose we have a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

Then the following two candidate triangles are also distinguished

$$X \xrightarrow{u} Y \xrightarrow{v} X[1] \xrightarrow{-u[1]} Y[1]$$

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{-u} Y \xrightarrow{-v} Z$$

[TR3] For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & X[1] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} .$$

where the rows are distinguished triangles, there is a morphism $h : Z \longrightarrow Z'$, not necessarily unique, which makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} .$$

NOTATIONS 1.1.3. *Let $(\mathcal{T}, [1])$ be a pretriangulated category.*

- *In what follows, distinguished triangles will be called just "triangles".*
- *The opposite category $\mathcal{T}^{op}$ is a pretriangulated category with shift functor $[-1] := [1]^{-1}$.*
- *$[n]$ will indicate the composition $[1]^n$ for each $n \in \mathbb{Z}$.*

The functor [1], being an equivalence, preserves arbitrary products and coproducts (whenever they exist in $\mathcal{T}$).

DEFINITION 1.1.4. Let $(\mathcal{T}, [1])$ be a pretriangulated category and $H : \mathcal{T} \longrightarrow \mathcal{A}$ a covariant functor from $\mathcal{T}$ to some abelian category $\mathcal{A}$. H is called *homological* if, for every triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

the sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

is exact in $\mathcal{A}$. The definition of *cohomological* functor is given dually.

DEFINITION 1.1.5. Let $(\mathcal{T}, [1])$ be a pretriangulated category. Given a morphism ϕ of candidate triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

the *mapping cone* of ϕ is the candidate triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & v' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -u[1] & 0 \\ f[1] & w' \end{pmatrix}} Y[1] \oplus X[1]'$$

DEFINITION 1.1.6. A *triangulated* category is a pretriangulated category $(\mathcal{T}, [1])$ satisfying the extra axiom:
$[TR4']$ Given any commutative diagram in which the rows are triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

the morphism $h : Z \longrightarrow Z'$ making the diagram commute, given by axiom [TR3], may be chosen so that the mapping cone of the morphism $\Phi := (f, g, h)$ is a triangle.

REMARK 1.1.7. $[TR4']$ can be substituted by the Octahedral Axiom, that is, given a pretriangulated category $(\mathcal{T}, [1])$, $\mathcal{T}$ is triangulated if the following holds:
$[TR4]$ For each composable morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$,there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "1"'] \\ X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d] \\ 0 \arrow[r] \arrow[d] & Y'' \arrow[r, "1"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\ X[1] \arrow[r, "f[1]"] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2] \end{tikzcd}$$

where every row and column is a triangle.

DEFINITION 1.1.8. A *triangulated subcategory* $\mathcal{T}$ of a triangulated category $\mathcal{D}$ is a subcategory such that:

- $M[n] \in \mathcal{T}$, for all $M \in \mathcal{T}$.
- Every object C such that there is a triangle in $\mathcal{D}$, $A \to B \to C \to A[1]$ with $A, B \in \mathcal{T}$, is in $\mathcal{T}$.

In particular $\mathcal{T}$ is a triangulated category with the structures inherited by $\mathcal{D}$

DEFINITION 1.1.9. A functor $F : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between two triangulated categories is a *triangulated functor* if there is an isomorphism

$$\Phi_X : F(X[1]) \simeq F(X)[1]$$

and, for every distinguished triangle in $\mathcal{T}_1$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

$$F(X) \overset{F(u)}{\longrightarrow} F(Y) \overset{F(v)}{\longrightarrow} F(Z) \overset{\Phi_X \circ F(w)}{\longrightarrow} F(X[1])$$

is a distinguished triangle in $\mathcal{T}_2$.

DEFINITION 1.1.10. Let $F : \mathcal{T} \longrightarrow \mathcal{D}$ be a triangulated functor. The kernel of F is defined to be the full subcategory $\mathcal{C}$ of $\mathcal{T}$ whose objects are mapped to 0 by F.

REMARK 1.1.11. It turns out that, for every triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{D}$, Ker F is a triangulated subcategory of $\mathcal{T}$.

DEFINITION 1.1.12. A subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ is called *thick* if it is triangulated and closed under direct summands.

In what follows we need the concept of localizing subcategory and the connection with perpendicular subcategories and recollement. We are working in the setting of triangulated categories, therefore we will give definitions and results in this particular context, even if some of them could be given also more generally. At this step we will ignore set-theoretic issues. From now on $\mathcal{D}$ will indicate a triangulated category with the shift functor indicated by $[-]$.

1.1. Verdier localization. The notion of localization of triangulated categories was introduced by Grothendieck, and then axiomatised by Verdier in [**V**]. Here we will follows the presentation of [**Kr2**].

DEFINITION 1.1.13. [**Kr2**, Section 2.2] Let Σ be a set of morphisms of $\mathcal{D}$. Consider the *category of fractions* $\mathcal{D}[\Sigma^{-1}]$, whose objects are the same of $\mathcal{D}$ and morphisms are defined as follows (note that, at this stage, we ignore set-theoretic issues, that is, the morphisms between two objects of $\mathcal{D}[\Sigma^{-1}]$ need not to form a small set). Let us consider the quiver where the vertices are the objects of $\mathcal{D}$ and the class of arrows is made by the disjoint union $(Mor\mathcal{D}) \cup \Sigma^{-1}$. Let $\mathcal{C}$ be the set of finite sequences of composable arrows (with the composition that is the concatenation of paths indicated with $\circ_{\mathcal{C}}$). Then $Mor\mathcal{D}[\Sigma^{-1}]$ is the quotient of $\mathcal{C}$ modulo the following relations:

(1) The composition of paths in $\mathcal{C}$ coincide with the composition in $\mathcal{D}$.
(2) $id_{\mathcal{C}}X = id_{\mathcal{D}}X$ for each $X \in \mathcal{D}$.
(3) $\sigma^{-1} \circ_{\mathcal{C}} \sigma = id_{\mathcal{C}}X$ and $\sigma \circ_{\mathcal{C}} \sigma^{-1} = id_{\mathcal{C}}Y$, for each $\sigma : X \longrightarrow Y$ in Σ.

The associated *quotient functor*

$$Q_\Sigma : \mathcal{D} \longrightarrow \mathcal{D}[\Sigma^{-1}]$$

is such that:

(Q1) Q_Σ makes the morphisms in Σ invertible, that is $Q_{\Sigma^{-1}}(f)$ is invertible in $\mathcal{D}[\Sigma]$ for each $f \in \Sigma$.
(Q2) If a functor $F : \mathcal{D} \longrightarrow \mathcal{B}$ makes the morphisms in Σ invertible in $\mathcal{B}$, then there is a unique functor $\bar{F} : \mathcal{D}[\Sigma^{-1}] \longrightarrow \mathcal{B}$ such that $F = \bar{F} \circ Q_\Sigma$.

The description of the morphisms in $\mathcal{D}[\Sigma^{-1}]$ is particularly nice when Σ satisfies the conditions illustrated below. In this case it is said that Σ admits a "Calculus of fractions". The techniques arising generalize the Ore localization for non commutative rings. In the categorical setting, this concept was introduced by Grothendieck and developed also by Gabriel and Zisman in [**GZ**].

DEFINITION 1.1.14. Let Σ be a set of morphisms in $\mathcal{D}$. Then we say that Σ admits a *calculus of left fractions* if the following hold:

(LF1) Σ is closed under the composition of morphisms (whenever it exists) and, for all $M \in \mathcal{D}$, $Id_M \in \Sigma$.

(LF2) Each pair of morphisms $X' \xleftarrow{f} X \xrightarrow{g} Y$ with $f \in \Sigma$ can be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "g'"] & Y' \end{tikzcd}$$

such that $f' \in \Sigma$.

(LF3) Let $\alpha, \beta : X \longrightarrow Y$ morphisms in $\mathcal{D}$. If there is a morphism $\sigma : X' \longrightarrow X$ in Σ with $\alpha \circ \sigma = \beta \circ \sigma$, then there exists a morphism $\tau : Y \longrightarrow Y'$ in Σ with $\tau \circ \alpha = \tau \circ \beta$.

REMARK 1.1.15. Dually, we say that Σ admits a *calculus of right fractions* if it satisfies (LF1) and the dual of (LF2) and (LF3). If Σ admits a calculus of left and right fractions we say that it is a *multiplicative system*.

DEFINITION 1.1.16. A multiplicative system Σ of a triangulated category is said to be *compatible with triangulation* if

- given σ in Σ, the morphisms $\sigma[n]$ is in Σ for all $n \in \mathbb{Z}$.
- Given a morphism of triangles (f, g, h) with $f, g \in \Sigma$ then there is also a morphism between the same triangles given by (f, g, h') with $h' \in \Sigma$.

PROPOSITION 1.1.17. *Let Σ be a multiplicative system compatible with triangulation. Then $\mathcal{D}[\Sigma^{-1}]$ is a triangulated category such that the quotient functor is triangulated.*

The following theorem defines Verdier localization and gives the connection with the calculus of fractions. Verdier proved this result in [**V**] for thick subcategories of a triangulated category, and Neeman generalized it in [**N2**, Theorem 2.1.8], for every triangulated subcategory.

THEOREM 1.1.18. [**V**] *Let $\mathcal{T} \subset \mathcal{D}$ be a triangulated subcategory. Then there is a universal functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{T} \subseteq \mathrm{Ker}(F)$. In other words, there exists a triangulated category denoted by $\mathcal{D}/\mathcal{T}$ and a triangulated functor $F_{univ} : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{T}$ such that $\mathcal{T}$ is in the kernel of F_{univ} and F_{univ} is universal with respect to this property, that is, if $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a triangulated functor such that $\mathcal{T} \subset \mathrm{Ker}(F)$, then G factors uniquely as*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F_{univ}"] \arrow[d, "G"'] & \mathcal{D}/\mathcal{T} \arrow[dl] \\ \mathcal{C} & \end{tikzcd}$$

REMARK 1.1.19. [**Kr2**, 4.6.1 and 4.6.2] With the same notations as in the theorem, let us denote by $\Sigma(\mathcal{T})$ the set of morphisms $X \longrightarrow Y$ in $\mathcal{D}$ which fit into a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ with $Z \in \mathcal{T}$. Then $\Sigma(\mathcal{T})$ is a multiplicative system compatible with the triangulation. Then the quotient $\mathcal{D}/\mathcal{T}$ can be defined as $\mathcal{D}/\mathcal{T} = \mathcal{D}[\Sigma(\mathcal{T})^{-1}]$ (see [**V**, 2.2.10]).

Let us recall this key result of Gabriel and Zisman that connects pair of adjoint functors with the category of fractions.

Proposition 1.1.20. [**GZ**, Proposition I.1.3] *Let (F, G) a pair of triangulated adjoint functors, between two additive categories $\mathcal{C}$ and $\mathcal{D}$, and denote by Σ the set of morphisms in $\mathcal{C}$, such that $F(\sigma)$ is invertible in $\mathcal{D}$ for every $\sigma \in \Sigma$. Then the following are equivalent:*

(1) *The functor G is fully faithful.*
(2) *The counit of the adjunction $\epsilon : FG \to Id_{\mathcal{D}}$ is invertible.*
(3) *The functor $\overline{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ satisfying $F = \overline{F} \circ Q_{\Sigma}$ is an equivalence.*

Dually, let us set Γ the set of morphisms in $\mathcal{D}$ such that $G(\gamma)$ is invertible in $\mathcal{C}$ for every $\gamma \in \Gamma$. Then the following are equivalent:

(1') *The functor F is fully faithful.*
(2') *The unit of the adjunction $\eta : Id_{\mathcal{C}} \to GF$ is invertible.*
(3') *The functor $\overline{G} : \mathcal{D}[\Gamma^{-1}] \to \mathcal{C}$ satisfying $G = \overline{G} \circ Q_{\Gamma}$ is an equivalence.*

1.2. Bousfield localization.

Definition 1.1.21. A triangulated functor $L : \mathcal{D} \longrightarrow \mathcal{D}$ is a *localization functor* if there exists a morphism $\eta : Id_{\mathcal{D}} \longrightarrow L$ with $L\eta : L \longrightarrow L^2$ being invertible and, for each $M \in \mathcal{D}$, $L\eta_M = \eta_{L(M)}$.
Dually, a triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{D}$ is a *colocalization functor* if the opposite functor $G^{op} : \mathcal{D}^{op} \longrightarrow \mathcal{D}^{op}$ is a localization functor.

Let us note that to every pair of triangulated adjoint functor (F, G) between two triangulated categories $\mathcal{C}$ and $\mathcal{D}$ such that G is fully faithful (or F is fully faithful), it is possible to associate a localization functor (or a colocalization functor).

Proposition 1.1.22. *Let $F : \mathcal{C} \to \mathcal{D}$ a triangulated functor and G its right adjoint, then:*

(1) *if G is fully faithful then the functor $L := GF$ is a localization functor.*
(2) *If F is fully faithful then the functor $H := FG$ is a colocalization functor.*

Proof. (1) The morphism η of the definition of localization functor is given by the unit of the adjunction: $\eta : Id_{\mathcal{C}} \to GF$. Now we conclude using proposition 1.1.20.
(2) The pair (G^{op}, F^{op}) is and adjoint pair between $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$ and F^{op} is fully faithful, then by point (1) we conclude.

□

We illustrate now the connection between localization functors and orthogonal classes.

Notations 1.1.23. *Let $\mathcal{C}$ be a class of object in $\mathcal{D}$. We define respectively the right and the left orthogonal subcategories of $\mathcal{C}$ in the following way:*

$$\mathcal{C}^{\perp} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(C[n], X) = 0 \text{ for all } C \in \mathcal{C}, \quad \text{for all } n \in \mathbb{Z}\};$$

$$^{\perp}\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, C[n]) = 0 \text{for all } C \in \mathcal{C}, \quad \text{for all } n \in \mathbb{Z}\}.$$

PROPOSITION 1.1.24. [**Kr2**, Proposition 4.9.1] *Let $\mathcal{C}$ be a thick subcategory of $\mathcal{D}$. Then the following are equivalent:*

1. *There exist a localization functor $L : \mathcal{D} \longrightarrow \mathcal{D}$ with $\mathrm{Ker}L = \mathcal{C}$.*
2. *The inclusion functor $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{D}$ admits a right adjoint.*
3. *For each X in $\mathcal{D}$ there exists an exact triangle $X' \longrightarrow X \longrightarrow X'' \longrightarrow X''[1]$ with $X' \in \mathcal{C}$ and $X'' \in \mathcal{C}^{\perp}$.*
4. *The quotient functor $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ admits a right adjoint.*
5. *The composite $\mathcal{C}^{\perp} \overset{i_{\mathcal{C}^{\perp}}}{\hookrightarrow} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{C}$ is an equivalence.*
6. *The inclusion functor $\mathcal{C}^{\perp} \hookrightarrow \mathcal{D}$ admits a left adjoint and $^{\perp}(\mathcal{C}^{\perp}) = \mathcal{C}$.*

2. TTF triples and recollements

The important notion of t-structure in a triangulated category was introduced by Beilinson, Bernstein and Deligne in the celebrated paper [**BBD**]. We will recall the definition and its connection with the localization theory. Let $\mathcal{D}$ be a triangulated category with shift functor $[-]$.

DEFINITION 1.2.1. A *t-structure* on $\mathcal{D}$ is a pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathcal{D}$ that satisfies the following conditions:

1. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $\mathrm{Hom}_{\mathcal{D}}(A, B) = 0$.
2. $A[1] \in \mathcal{A}$, for all $A \in \mathcal{A}$ and $B[-1] \in \mathcal{B}$ for all $B \in \mathcal{B}$.
3. For each $M \in \mathcal{D}$ there exists a triangle

$$A_M \longrightarrow M \longrightarrow B_M \longrightarrow A_M[1] \tag{2}$$

with $A_M \in \mathcal{A}$ and $B_M \in \mathcal{B}[-1]$.

REMARK 1.2.2. From the definition it turns out that $\mathcal{A}$ and $\mathcal{B}$ are maximal with respect to property (1).

PROPOSITION 1.2.3. [**BBD**, 1.3.3] *For each $n \in \mathbb{Z}$ the inclusion functor of $\mathcal{A}[n]$ in $\mathcal{D}$ admits a right adjoint $R_{\mathcal{A}[n]}$ and the inclusion functor of $\mathcal{B}[-n]$ admits a left adjoint $L_{\mathcal{B}[n]}$.*

$$\begin{tikzcd} \mathcal{A}[n] \arrow[r, "i_{\mathcal{A}[n]}"] & \mathcal{D} \arrow[l, bend left, "R_{\mathcal{A}[n]}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}[n] \arrow[r, "i_{\mathcal{B}[n]}"'] & \mathcal{D} \arrow[l, bend right, "L_{\mathcal{B}[n]}"'] \end{tikzcd}$$

Moreover, for each $M \in \mathcal{D}$ there is a unique morphism $d \in$ $(\mathrm{Hom}_{\mathcal{D}}(L_{\mathcal{B}[-1]}(M), R_{\mathcal{A}}(M)[1])$ such that there is a triangle

$$R_{\mathcal{A}}(M) \longrightarrow M \longrightarrow L_{\mathcal{B}[-1]}(M) \xrightarrow{d} R_{\mathcal{A}}(M)[1],$$

that is isomorphic to (2).

Remark 1.2.4. So, given a t-structure $(\mathcal{A},\mathcal{B})$ in $\mathcal{D}$, for each $M \in \mathcal{D}$ the associated triangle has the form

$$R_{\mathcal{A}}(M) \longrightarrow M \longrightarrow L_{\mathcal{B}}[-1](M) \longrightarrow R_{\mathcal{A}}(M)[1]$$

Definition 1.2.5. A *torsion pair* in $\mathcal{D}$ is a pair $(\mathcal{X},\mathcal{Y})$ of full subcategories of $\mathcal{D}$, closed under isomorphisms, satisfying the following conditions:
1) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X},\mathcal{Y}) = 0$.
2) $X[1] \in \mathcal{X}$ and $Y[-1] \in \mathcal{Y}$ for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
3) for each object $M \in \mathcal{D}$, there is a triangle

$$X_M \longrightarrow C \longrightarrow Y_M \longrightarrow X_M[1]$$

in $\mathcal{D}$ with $X_M \in \mathcal{X}$ and $Y_M \in \mathcal{Y}$.

In this case $\mathcal{X}$ is called a *torsion class* and $\mathcal{Y}$ a *torsion free class.* If $\mathcal{X}$ is triangulated then $(\mathcal{X},\mathcal{Y})$ is called heditary.

Remark 1.2.6. From the definition it turns out that $\mathcal{X}$ and $\mathcal{Y}$ are maximal with respect to property 1) that is $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. Hence they are thick and we can apply Proposition 1.1.24, to conclude that

$$\mathcal{X} \simeq \mathcal{D}/\mathcal{Y} \qquad \text{and} \qquad \mathcal{Y} \simeq \mathcal{D}/\mathcal{X}$$

The following Proposition shows that in a triangulated categories t-structures and torsion pairs are in bijection.

Proposition 1.2.7 ([**BR**], [**Ke5**]). *Given a torsion pair $(\mathcal{X},\mathcal{Y})$ the pair $(\mathcal{X},\mathcal{Y}[1])$ is a t-structure. Conversely given a t-structure $(\mathcal{A},\mathcal{B})$, the pair $(\mathcal{A},\mathcal{B}[-1])$ is a torsion pair.*

Definition 1.2.8. A *torsion torsion-free triple* (TTF triple) in $\mathcal{D}$ is a triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ of full subcategories of $\mathcal{D}$, where $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{Y},\mathcal{Z})$ are torsion pairs.

Remark 1.2.9. Let us note that $(\mathcal{X},\mathcal{Y},\mathcal{Z}) = ({}^{\perp}\mathcal{Y},\mathcal{Y},\mathcal{Y}^{\perp})$. Moreover $(\mathcal{X},\mathcal{Y})$ and $(\mathcal{Y},\mathcal{Z})$ are hereditary.

In what follows we will consider triangulated categories closed under set-indexed products and coproducts, that is, using Neeman's notations, satisfying the axiom [TR5].
From 1.2.3 and 1.2.7 we have the following pairs of adjoint functors:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i_{\mathcal{X}}"] & \mathcal{D} \arrow[l, bend left, "R_{\mathcal{X}}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D} \arrow[l, bend right, "L_{\mathcal{Y}}"'] \arrow[l, bend left, "R_{\mathcal{Y}}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "i_{\mathcal{Z}}"] & \mathcal{D} \arrow[l, bend right, "L_{\mathcal{Z}}"'] \end{tikzcd}$$

DEFINITIONS 1.2.10. A *localizing* (*colocalizing*) subcategory of $\mathcal{D}$ is a triangulated subcategory of $\mathcal{D}$ closed under coproducts (or products). An *aisle* (*coaisle*) in $\mathcal{D}$ is a full triangulated subcategory of $\mathcal{D}$ such that the inclusion functor admits a right (left) adjoint.
A *strictly localizing* (*strictly colocalizing*) subcategory is an aisle (coaisle) that is also closed under coproducts (products).

Since, by definition, every localizing subcategory of $\mathcal{D}$ is closed under coproducts, then it is thick (see [**R**]).

PROPOSITION 1.2.11. ([**BR**, Corollary 2.9]) *If $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then:*

(1) *$\mathcal{X}$ and $\mathcal{Y}$ are closed under coproducts and $\mathcal{Y}$ and $\mathcal{Z}$ are closed under products. Moreover $\mathcal{X}$ is a strictly localizing subcategory, $\mathcal{Z}$ is strictly colocalizing and $\mathcal{Y}$ is strictly localizing and strictly colocalizing.*
(2) *The pairs of adjoint functors $(R_{\mathcal{X}}, i_{\mathcal{X}}), (L_{\mathcal{Y}}, i_{\mathcal{Y}}), (R_{\mathcal{Y}}, i_{\mathcal{Y}}), (L_{\mathcal{Z}}, i_{\mathcal{Z}})$ are such that:*
   i) *$R_{\mathcal{X}} i_{\mathcal{X}} = Id_{\mathcal{X}}, L_{\mathcal{Y}} i_{\mathcal{Y}} = Id_{\mathcal{Y}} = R_{\mathcal{Y}} i_{\mathcal{Y}}$ and $L_{\mathcal{Z}} i_{\mathcal{Z}} = Id_{\mathcal{Z}}$.*
   ii) *If we consider the Verdier quotient $\mathcal{D}/\mathcal{Y}$ we have the equivalences $\mathcal{X} \longrightarrow \mathcal{D}/\mathcal{Y} \longleftarrow \mathcal{Z}$ that are explicitly given by the functors*

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "L_{\mathcal{Z}} i_{\mathcal{X}}"] & \mathcal{Z} \arrow[l, bend left, "R_{\mathcal{X}} i_{\mathcal{Z}}"] \end{tikzcd}$$

   iii) *The functors $i_{\mathcal{Y}}, i_{\mathcal{X}}$ and $i_{\mathcal{Z}} L_{\mathcal{Z}} i_{\mathcal{X}}$ are fully faithful.*

PROOF. (1) Since $\mathcal{X} = \mathrm{KerHom}_{\mathcal{D}}(-, \mathcal{Y})$ and $\mathcal{Y} = \mathrm{KerHom}_{\mathcal{D}}(-, \mathcal{Z})$ then $\mathcal{X}$ and $\mathcal{Y}$ are closed under coproducts , so they are localizing. Dually $\mathcal{Z}$ and $\mathcal{Y}$ are colocalizing. Moreover $(\mathcal{X}, \mathcal{Y}[1])$ and $(\mathcal{Y}, \mathcal{Z}[1])$ are t-structures, then by Proposition 1.2.3 we conclude.

(2) i) Let us prove that, for each $X \in \mathcal{X}$, $R_{\mathcal{X}} i_{\mathcal{X}}(X) = X$. Indeed, by Proposition 1.2.3 and part (1), for each $M \in \mathcal{D}$, and in particular for X, we have a triangle

$$R_{\mathcal{X}}(X) \to X \xrightarrow{f} L_{\mathcal{Y}}(X) \to R_{\mathcal{X}}(X)[1].$$

Since $X \in \mathcal{X}$, then $f = 0$ and $R_{\mathcal{X}}(X) \simeq X \oplus L_{\mathcal{Y}}(X)[1]$. But $L_{\mathcal{Y}}(X)[1]$ is in $\mathcal{Y}$ then $R_{\mathcal{X}}(X) \simeq X$. In the same way it can be proved that $L_{\mathcal{Y}} i_{\mathcal{Y}} = Id_{\mathcal{Y}} = R_{\mathcal{Y}} i_{\mathcal{Y}}$ and $L_{\mathcal{Z}} i_{\mathcal{Z}} = Id_{\mathcal{Z}}$.

ii) See [**BR**][Corollary 2.9]

iii) $R_{\mathcal{X}} i_{\mathcal{X}} = Id_X$ implies that the counit of the adjunction $(R_{\mathcal{X}}, i_{\mathcal{X}})$ is invertible. Then $i_{\mathcal{X}}$ is fully faithful by Proposition 1.1.20. The same holds for $i_{\mathcal{Y}}$ and $i_{\mathcal{Z}}$. Now it remains to prove that $i_{\mathcal{Z}} L_{\mathcal{Z}} i_{\mathcal{X}}$ is fully faithful. But, for the previous point, $L_{\mathcal{Z}} i_{\mathcal{X}}$ is an equivalence, then we can conclude.

□

COROLLARY 1.2.12. *The functors $i_{\mathcal{X}} R_{\mathcal{X}}, i_{\mathcal{Y}} R_{\mathcal{Y}}$ are localization functors and $i_{\mathcal{Y}} L_{\mathcal{Y}}, i_{\mathcal{Z}} L_{\mathcal{Z}}$ are colocalization functors.*

Proof. See Propositions 1.2.11 and 1.1.22. ☐

Corollary 1.2.13. *There is a bijection between strictly localizing subcategories, hereditary t-structures and hereditary torsion pairs:*

$$\mathcal{X} \to (\mathcal{X}, \mathcal{X}^{\perp}) \to (\mathcal{X}, \mathcal{X}^{\perp})$$
$$(\mathcal{X}, \mathcal{Y}) \to (\mathcal{X}, \mathcal{Y}) \to \mathcal{X}.$$

Proof. If $\mathcal{X}$ is a strictly localizing subcategory then $\mathcal{X}$ and $\mathcal{X}^{\perp}$ are closed under shifts and by proposition 1.1.24 there exists a localization functor $L : \mathcal{D} \to \mathcal{D}$ such that $\mathcal{X} = \mathrm{Ker} L$. Moreover, for every $M \in \mathcal{D}$, there exists a triangle $X_M \to M \to Y_M \to X_M$ with $X_M \in \mathcal{X}$ and $\mathcal{Y}_M \in \mathcal{X}^{\perp}$. Hence $(\mathcal{X}, X^{\perp})$ is a t-structure. By Proposition 1.2.7 we conclude. In particular if $(\mathcal{X}, \mathcal{Y})$ is a hereditary torsion pair if and only it is a hereditary t-structure. ☐

From the proof of Proposition 1.2.11 we have the following.

Corollary 1.2.14.

$$L_{\mathcal{Y}} i_{\mathcal{X}}(X) = R_{\mathcal{X}} i_{\mathcal{Y}}(Y) = R_{\mathcal{Y}} i_{\mathcal{Z}}(Z) = L_{\mathcal{Z}} i_{\mathcal{Y}}(Y) = 0$$

for each $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$.

Let us recall the concept of recollement.

Definition 1.2.15. [**BBD**] Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. $\mathcal{D}$ is said to be a recollement of $\mathcal{D}'$ and $\mathcal{D}''$, expressed by the diagram

$$\begin{tikzcd}
\mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd} \tag{3}$$

if there are six triangle functors satisfying the following conditions:
i) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs;
ii) i_*, j_* and $j_!$ are fully faithful functors;
iii) $j^! i_! = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
iv) for each object $C \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow i_! i^!(C)[1], \tag{4}$$
$$j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow j_! j^!(C)[1]. \tag{5}$$

The notion of recollements was introduced by Beilinson-Bernstein-Deligne (see [**BBD**]) in a geometric context, where stratifications of varieties induce recollements of derived categories of constructible sheaves. They can be seen as "short exact sequences" of triangulated categories, in the sense that the first and the third terms are triangle equivalent respectively to a subcategory and to a quotient category of the central one. Indeed i_* is fully faithful then $\mathcal{D}'' \simeq \mathrm{Im} i_*$ and, from $j^* i_* = 0$ and i_* fully faithful, we have $\mathcal{D}' \simeq \mathcal{D}/i_*(\mathcal{D}'')$.

DEFINITION 1.2.16. Two recollements defined by the data $(\mathcal{D}, \mathcal{D}', \mathcal{D}'', i^*, i_*, i^!, j^!, j_!, j_*)$ and $(\mathcal{D}, \mathcal{T}', \mathcal{T}'', i'^*, i'_*, i'^!, j'^!, j'_!, j'_*)$ are said to be equivalent if the following equality between essential images holds :

$$(im(j_!), im(i_*), im(j_*)) = (im(j'_!), im(i'_*), im(j'_*)).$$

PROPOSITION 1.2.17. *Let $\mathcal{D}$ be a triangulated category. Then there is a bijection between TTF triples and equivalence classes of recollements.*

PROOF. Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a TTF triple in $\mathcal{D}$. Let us use notations as in Proposition 1.2.11. Then the following diagram is a recollement:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D} \arrow[l, bend right, "L_{\mathcal{Y}}"'] \arrow[l, bend left, "R_{\mathcal{Y}}"] \arrow[r, "R_{\mathcal{X}}"] & \mathcal{X} \arrow[l, bend right, "i_{\mathcal{X}}"'] \arrow[l, bend left, "i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}}"] \end{tikzcd}$$

Indeed, the third property of recollements is verified thanks to Corollary 1.2.14. Moreover, by Proposition 1.2.11, the functors $i_{\mathcal{Y}}, i_{\mathcal{X}}, i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}}$ are fully faithful. We have also that the pairs of functors $(L_{\mathcal{Y}}, i_{\mathcal{Y}})$, $(i_{\mathcal{X}}, R_{\mathcal{X}})$ and $(i_{\mathcal{Y}}, R_{\mathcal{Y}})$ are adjoint pairs. Let us prove that $(R_{\mathcal{X}}, i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}})$ is and adjoint pair too. For each $M \in \mathcal{D}$, apply the functor $R_{\mathcal{X}}$ to the triangle

$$i_{\mathcal{Y}}R_{\mathcal{Y}}(M) \to M \to i_{\mathcal{Z}}L_{\mathcal{Z}}(M) \to i_{\mathcal{Y}}R_{\mathcal{Y}}(M)[1].$$

Then we have $R_{\mathcal{X}}(M) \simeq R_{\mathcal{X}}i_{\mathcal{Z}}L_{\mathcal{Z}}(M)$, since $R_{\mathcal{X}}i_{\mathcal{Y}}R_{\mathcal{Y}}(M) = 0$. Hence $\mathrm{Hom}_{\mathcal{D}}(M, i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}}(X)) = \mathrm{Hom}_{\mathcal{Z}}(L_{\mathcal{Z}}(M), L_{\mathcal{Z}}i_{\mathcal{X}}(X))$. Recall now that the pair $(R_{\mathcal{X}}i_{\mathcal{Z}}, L_{\mathcal{Z}}i_{\mathcal{X}})$ is an equivalence between $\mathcal{Z}$ and $\mathcal{X}$. Then
$\mathrm{Hom}_{\mathcal{Z}}(L_{\mathcal{Z}}(M), L_{\mathcal{Z}}i_{\mathcal{X}}(X)) = \mathrm{Hom}_{\mathcal{X}}(R_{\mathcal{X}}i_{\mathcal{Z}}L_{\mathcal{Z}}(M), X) = \mathrm{Hom}_{\mathcal{X}}(R_{\mathcal{X}}(M), X)$, for each $X \in \mathcal{X}$.
Conversely, given a recollement as in (3) the triple

$$(j_!(\mathcal{D}'), i_*(\mathcal{D}''), j_*(\mathcal{D}'))$$

is a TTF triple. In fact, let us set $(j_!(\mathcal{D}'), i_*(\mathcal{D}''), j_*(\mathcal{D}')) = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, then $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are triangulated subcategories, because $j_!, i_*$ and j_* are triangulated functors. Moreover $\mathrm{Hom}_{\mathcal{D}}(j_!(M), i_*(N)) = \mathrm{Hom}_{\mathcal{D}'}(M, j^!i_*(N)) = 0$ (for each $M, N \in \mathcal{D}$), that is $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) = 0$. In the same way it can be proved that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{Y}, \mathcal{Z}) = 0$. Finally, using the fourth property of recollements, we have that, for each $M \in \mathcal{D}$ there are two triangles of the form

$$X_M \to M \to Y_M \qquad \text{and} \qquad Y'_M \to M \to Z_M$$

with $X_M \in \mathcal{X}$, $Y_M, Y'_M \in \mathcal{Y}$ and $Z_M \in \mathcal{Z}$. □

Remark 1.2.18. Note that, to a TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ are associated two equivalent recollements, expressed by the diagrams:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D} \arrow[l, bend right=40, "L_{\mathcal{Y}}"'] \arrow[l, bend left=40, "R_{\mathcal{Y}}"] \arrow[r, "R_{\mathcal{X}}"] & \mathcal{X} \arrow[l, bend right=40, "i_{\mathcal{X}}"'] \arrow[l, bend left=40, "i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D} \arrow[l, bend right=40, "L_{\mathcal{Y}}"'] \arrow[l, bend left=40, "R_{\mathcal{Y}}"] \arrow[r, "L_{\mathcal{Z}}"] & \mathcal{Z} \arrow[l, bend right=40, "i_{\mathcal{X}}R_{\mathcal{X}}i_{\mathcal{Z}}"'] \arrow[l, bend left=40, "i_{\mathcal{Z}}"] \end{tikzcd}$$

3. Recollements of compactly generated triangulated categories

Let $\mathcal{D}$ be a triangulated category with set indexed coproducts (i.e. satisfying axiom [TR5]).
We want to focus on recollements of compactly generated triangulated categories. In this setting there are useful results by [**NS**] and [**AKL**] that allow us to construct a correspondence between set of compact objects in the category and recollements (and then, also TTF triples).

Definition 1.3.1. $\mathcal{D}$ is said to be *generated* by a set of objects $\mathcal{P}$ of $\mathcal{D}$ if, given $X \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(P[n], X) = 0$ for each $n \in \mathbb{Z}$ and $P \in \mathcal{P}$ implies $X = 0$.

Given a class $\mathcal{C}$ of objects in $\mathcal{D}$ there are two triangulated subcategories of $\mathcal{D}$ associated to this class.

Definition 1.3.2. (1) $\mathrm{Tria}\,\mathcal{C}$ denotes the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{C}$ and closed under set indexed coproducts.
(2) $\mathrm{tria}\,\mathcal{C}$ denotes the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{C}$ and closed under finite coproducts and direct summands.

Remark 1.3.3. Note that, by [**R**], $\mathrm{Tria}\,\mathcal{C}$, being closed under coproducts, is thick.

Definition 1.3.4. $X \in \mathcal{D}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{D}}(X, -)$ commutes with set indexed coproducts. M in $\mathcal{D}$ is called *self-compact* if M is compact in $\mathrm{Tria}\, M$.

Definition 1.3.5. $\mathcal{D}$ is said *compactly generated* if it is generated by a set of compact objects. A TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathcal{D}$ is compactly generated if so is $\mathcal{X}$ as a triangulated category.

Definition 1.3.6. Let $\mathcal{C}$ be a class of objects in $\mathcal{D}$. We say that $\mathcal{D}$ satisfies the *principle of infinite devissage* with respect to $\mathcal{C}$ if $\mathcal{D} = \mathrm{Tria}\,(\mathcal{C})$.

Lemma 1.3.7. [**NS**, Lemma 2.2]

(1) *Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}'$ be a full triangulated subcategory generated by a class of objects $\mathcal{C}$. If* Tria $(\mathcal{C})$ *is an aisle in $\mathcal{D}$ contained in $\mathcal{D}'$, then $\mathcal{D}' =$* Tria $(\mathcal{C})$.

(2) *Let $\mathcal{D}$ be a triangulated category and let $(\mathcal{X}, \mathcal{Y})$ be a t-structure on $\mathcal{D}$ such that $\mathcal{X}$ is triangulated. Let $L_{\mathcal{Y}}$ be the left adjoint of the inclusion functor of $\mathcal{Y}$ in $\mathcal{D}$. Then*

   i) *If $\mathcal{C}$ is a class of generators of $\mathcal{D}$ then $L_{\mathcal{Y}}(\mathcal{C})$ is a class of generators of $\mathcal{Y}$.*

   ii) *A class $\mathcal{C}$ of objects of $\mathcal{X}$ generates $\mathcal{X}$ if an only if the objects of $\mathcal{Y}$ are precisely those which are right orthogonal to all the shifts of objects of $\mathcal{C}$.*

PROOF. (1) By Proposition 1.1.24, for each $M \in \mathcal{D}'$ there exists a triangle

$$Q_M \to M \to Q^M \to Q_M[1]$$

with $Q_M \in$ Tria $(\mathcal{C})$ and $Q^M \in$ Tria $(\mathcal{C})^{\perp}$. Moreover, $\mathcal{D}'$ is triangulated, then also Q^M is in $\mathcal{D}'$. But $\mathcal{C}$ generates $\mathcal{D}'$ so $\mathrm{Hom}_{\mathcal{D}}(C[n], Q^M) = 0$, for all $C \in \mathcal{C}$ and for every integer n, implies $Q^M = 0$ then $M \in$ Tria $(\mathcal{C})$.

(2) Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure such that $\mathcal{X}$ is triangulated.

   i) Let $\mathcal{C}$ be a class of generators of $\mathcal{D}$ and $C \in \mathcal{C}$. Let $Y \in \mathcal{Y}$. Then

$$\mathrm{Hom}_{\mathcal{D}}(L_{\mathcal{Y}}(C)[n], Y) = \mathrm{Hom}_{\mathcal{Y}}(L_{\mathcal{Y}}(C)[n], Y) = \mathrm{Hom}_{\mathcal{D}}(C[n], i_{\mathcal{Y}}(Y)),$$

   where the last equality is given by the adjunction. Then $\mathrm{Hom}_{\mathcal{Y}}(L_{\mathcal{Y}}(C)[n], Y) = 0$ if and only if $\mathrm{Hom}_{\mathcal{D}}(C, i_{\mathcal{Y}}(Y)[n]) = 0$ for every n and for every $C \in \mathcal{C}$, if and only if $Y = 0$. So $\mathrm{Hom}_{\mathcal{Y}}(L_{\mathcal{Y}}(C)[n], Y) = 0$ if and only if $Y = 0$, that is, $L_{\mathcal{Y}}(\mathcal{C})$ is a class of generators of $\mathcal{Y}$.

   ii) If $\mathcal{C}$ is a class of generators of $\mathcal{X}$ let us set
   $\mathcal{A} := \{M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(C[n], M) = 0, \forall n \in \mathbb{Z}, \text{for each C in } \mathcal{C}\}$. Then it is clear that $\mathcal{Y}$ is contained in $\mathcal{A}$. On the other hand, let $M \in \mathcal{A}$. Then we have the triangle

$$X_M \to M \to Y_M \to X_M[1]$$

   with $X_M \in \mathcal{X}$ and $Y_M \in \mathcal{Y}$. Let $C \in \mathcal{C}$. If we apply $\mathrm{Hom}(C[n], -)$ to the above triangle we obtain
   $\mathrm{Hom}_D(C[n], X_M) = 0$, for all $C \in \mathcal{C}$ and $n \in \mathbb{Z}$, then $X_M = 0$ and $M \in \mathcal{Y}$.
   Conversely, if $\mathcal{Y} = \mathcal{A}$ then, for each X in $\mathcal{X}$ such that
   $\mathrm{Hom}_{\mathcal{D}}(C, X[n]) = 0$ for each integer n and for each C in $\mathcal{C}$, we have $X \in \mathcal{Y}$, that is $X = 0$.

□

From now on $\mathcal{D}$ will denote a triangulated category compactly generated.

LEMMA 1.3.8. *A full triangulated subcategory $\mathcal{Y}$ of $\mathcal{D}$ is closed under small coproducts if and only if the inclusion functor of $\mathcal{Y}$ in $\mathcal{D}$ admits a right adjoint. That is, in $\mathcal{D}$ every localizing subcategory is strictly localizing.*

PROOF. By the dual of [**Kr2**, Corollary 10.2], we have that the inclusion functor of $\mathcal{Y}$ preserves small coproducts (that is $\mathcal{Y}$ is closed under small coproducts) if and only it has a right adjoint. □

DEFINITION 1.3.9. [**BNE**],[**Kr**] A subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *smashing* if it is the kernel of a localization functor $L : \mathcal{D} \to \mathcal{D}$ which preserves small coproducts.

REMARK 1.3.10. If $\mathcal{X}$ is a localizing subcategory then it is smashing if and only if $\mathcal{X}^\perp$ is closed under coproducts. Indeed if $\mathcal{X}$ is localizing, then, by [**Kr2**, Proposition 4.9.1] there exists a localization functor L, such that $\mathcal{X} = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathcal{X}^\perp$. It is clear that $\mathcal{X}$ is smashing if and only if $\mathrm{Im}L$ is closed under coproducts.

The following result shows that smashing subcategories can be constructed starting from sets of compact objects.

THEOREM 1.3.11. [**TLS**, 4.5] *Let $\mathcal{C}$ be a set of objects in $\mathcal{D}$. Then* Tria $\mathcal{C}$ *is a localizing subcategory of $\mathcal{D}$. If $\mathcal{C}$ is consists of compact objects then* Tria $\mathcal{C}$ *is a smashing subcategory.*

PROOF. By definition, for each set of objects $\mathcal{C}$ in $\mathcal{D}$, Tria $\mathcal{C}$ is localizing. Suppose now that $\mathcal{C}$ is a set of compact objects. Then Tria $\mathcal{C}$ is a localizing subcategory by the first part of the statement. We want to prove that $\mathcal{C}^\perp$ is closed under coproducts. Let I be a set, and $(Y_i)_{i\in I}$ a family of objects in $\mathcal{C}^\perp$. Then, by Proposition 1.3.7, $\coprod_{i\in I} Y_i \in \mathcal{C}^\perp$ if and only if $\mathrm{Hom}_\mathcal{D}(C[n], \coprod_{i\in I} Y_i) = 0$ for every $C \in \mathcal{C}$, $n \in \mathbb{Z}$. But now, for the compactness of the objects in C, we have: $\mathrm{Hom}_\mathcal{D}(C[n], \coprod_{i\in I} Y_i) = \coprod_i \mathrm{Hom}_\mathcal{D}(C[n], Y_i) = 0$. Hence $\coprod_{i\in I} Y_i \in \mathcal{C}^\perp$. □

PROPOSITION 1.3.12. [**Ni**, Proposition 4.4.3] *Let $\mathcal{X}$ be a triangulated subcategory of $\mathcal{D}$, then the following are equivalent:*

(1) *$\mathcal{X}$ is a smashing subcategory.*
(2) *$\mathcal{X}$ is closed under small coproducts and the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{X}$ admits a right adjoint that preserves small coproducts.*
(3) *$\mathcal{X}$ is the first class of the TTF triple $(\mathcal{X}, \mathcal{X}^\perp, \mathcal{X}^{\perp\perp})$.*

PROOF. 1) ⇒ 2) $\mathcal{X} = \mathrm{Ker}L$ and L preserves small coproducts, then $\mathcal{X}$ is closed under small coproducts. Let us set $\mathcal{Y} := \mathcal{D}/\mathcal{X} = \mathcal{X}^\perp$. We claim that the inclusion functor $i : \mathcal{Y} \longrightarrow \mathcal{D}$ is the right adjoint of Q. We want to prove that $\mathrm{Hom}_\mathcal{Y}(Q(M), Y) = \mathrm{Hom}_\mathcal{D}(M, i(Y))$ for each $M \in \mathcal{D}$ and $Y \in \mathcal{Y}$. By Remark 1.2.13 we have that $(\mathcal{X}, \mathcal{Y}[1]) = (\mathcal{X}, \mathcal{Y})$ is a torsion pair. Then there is a triangle in $\mathcal{D}$:

$$X_M \to M \to Y_M \to X_M[1]$$

with X_M in $\mathcal{X}$ and Y_M in $\mathcal{Y}$. Applying the functor Q to the triangle, we get $Q(M) \simeq Y_M$. Apply now the functor $\mathrm{Hom}_\mathcal{D}(-, Y)$ to the triangle

$$X_M \to M \to Q(M) \to X[1],$$

recalling that $\mathcal{Y} = \mathcal{X}^\perp$ we obtain $\mathrm{Hom}_\mathcal{Y}(Q(M), Y) = \mathrm{Hom}_\mathcal{D}(M, Y)$. So Q is the left adjoint of the inclusion functor of $\mathcal{Y}$ and, by [**Kr2**, Proposition 3.5.1] it preserves coproducts. Moreover, thank to [**NS**, Lemma 2.3] Q preserves compact objects. Now we use [**Kr2**, Lemma 11.2] to conclude that $i_\mathcal{Y}$ preserves coproducts.

2) $\Rightarrow$ 3) Set $\mathcal{Y} := \mathcal{D}/\mathcal{X}$. From the proof of 1) $\Rightarrow$ 2) we can argue that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair. Moreover $\mathcal{Y}$ is closed under small coproducts, then by [**Be**, Proposition 5.14], we can conclude that there exists a triangulated subcategory $\mathcal{Z}$ such that $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple.
3) $\Rightarrow$ 1) Given the TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, one has $\mathcal{X} = \mathrm{Ker}(i_{\mathcal{Y}} L_{\mathcal{Y}})$ and it is easy to see that $i_{\mathcal{Y}} L_{\mathcal{Y}}$ is a localization functor. Moreover $L_{\mathcal{Y}}$ has a right adjoint $(i_{\mathcal{Y}})$ then it preserves coproducts. So $\mathcal{X}$ is the kernel of a localization functor that preserves coproducts (see Proposition 1.3.8).

□

The following result illustrates a connection between compact objects in $\mathcal{D}$ and compact and self-compact objects in the classes of a TTF triple.

PROPOSITION 1.3.13. [**NS**] *Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a TTF triple in $\mathcal{D}$. Then:*

(1) *If M is compact in $\mathcal{D}$ then $L_{\mathcal{Y}}(M)$ is compact in $\mathcal{Y}$. In particular, if $\mathcal{Y} =$* Tria $(L_{\mathcal{Y}}(M))$ *then $L_{\mathcal{Y}}(M)$ is self-compact.*
(2) *If N is compact in $\mathcal{X}$ then M is compact in $\mathcal{D}$.*

PROOF. (1) [**NS**, Lemma 2.4]
(2) Let $(M_i)_{i\in I}$ a family of objects in $\mathcal{D}$ and, for each i let

$$X_i \to M_i \to Y_i$$

be the triangle for the M_i associated to the torsion pair $(\mathcal{X}, \mathcal{Y})$. Then the coproduct of these triangles $\coprod_i X_i \to \coprod_i M_i \to \coprod_i Y_i$ is the triangle for $\coprod_i M_i$ with respect to the torsion pair $(\mathcal{X}, \mathcal{Y})$. If we apply the homological functor $\mathrm{Hom}_{\mathcal{D}}(N, -)$ to both triangles, from $\mathcal{Y} = \mathcal{X}^{\perp}$, $\mathcal{Y}$ closed under coproducts, and N compact in $\mathcal{X}$, we obtain $\mathrm{Hom}_{\mathcal{D}}(N, \coprod_i M_i) \simeq \coprod_i \mathrm{Hom}_{\mathcal{D}}(N, M_i)$.

□

In what follows we will focus on recollements of derived categories of dg algebras (or rings), which are particular cases of recollements of triangulated categories generated by a single objects. So we recall some well known results on singly generated triangulated categories.

DEFINITION 1.3.14. An object $X \in \mathcal{D}$ is called *self-orthogonal* if

$$\mathrm{Hom}_{\mathcal{D}}(X, X[n]) = 0, \text{ for every } 0 \neq \mathrm{n} \in \mathbb{Z}.$$

THEOREM 1.3.15. [**NS**, Proposition 3.4] *The following assertions are equivalent:*

(1) *$\mathcal{D}$ is a recollement of triangulated categories generated by a single compact object.*
(2) *There are objects P and Q of $\mathcal{D}$ such that:*
    i) *P is compact.*
    ii) *Q is self-compact.*
    iii) $\mathrm{Hom}_{\mathcal{D}}(P[n], Q) = 0$ *for each $n \in \mathbb{Z}$.*
    iv) *$\{P, Q\}$ generates $\mathcal{D}$.*
(3) *There is a compact object P such that* Tria $(P)^{\perp}$ *is generated by a compact object in* Tria $(P)^{\perp}$.

Proof. 1) $\Rightarrow$ 2) Consider a recollement of $\mathcal{D}$ as (3), then we have a TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (j_!(\mathcal{D}), i_*(\mathcal{D}''), j_*(\mathcal{D}'))$ and set P for the compact generators of $\mathcal{X}$ and Q for the compact generator of $\mathcal{Y}$. Then, Q is self-compact and by Proposition 1.3.13, part (2), P is compact in $\mathcal{D}$. Then we have points i) and ii). Point iii) is obvious since $P \in \mathcal{X}$ and $Q \in \mathcal{Y}$. Let us prove point iv). Let $M \in \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(P[n], M) = 0 = \mathrm{Hom}_{\mathcal{D}}(Q[n], M)$ for every integer n. Thus, since $\mathcal{X} = \mathrm{Tria}\ P$ and $\mathcal{Y} = \mathrm{Tria}\ Q$, by Proposition, 1.3.7 $M \in \mathcal{Y} \cap \mathcal{Z}$, so $M = 0$.
2) $\Rightarrow$ 3) Let us set $(\mathcal{X}, \mathcal{Y}) := (\mathrm{Tria}\ P, \mathrm{Tria}\ P^{\perp})$. Then, by Proposition 1.3.7 $\{L_{\mathcal{Y}}(P), L_{\mathcal{Y}}(Q)\}$ is a class of generators of $\mathcal{Y}$. Now, $L_{\mathcal{Y}}(P) = 0$ and, using again Proposition 1.3.7, Q is in $\mathrm{Tria}\ P^{\perp}$, then $L_{\mathcal{Y}}(Q) = Q$. Moreover $\mathrm{Tria}\ Q$ is an aisle in $\mathcal{Y}$, then by Proposition 1.3.7, $\mathcal{Y} = \mathrm{Tria}\ Q$.
3) $\Rightarrow$ 1) We have that $\mathrm{Tria}\ (P)$ is a smashing subcategory by Proposition 1.3.11, and the TTF triple associated is $(\mathrm{Tria}\ P, \mathrm{Tria}\ P^{\perp}, \mathrm{Tria}\ P^{\perp\perp})$ where $\mathrm{Tria}\ P^{\perp}$ is generated by a single object that is compact in $\mathrm{Tria}\ P^{\perp}$. Then by Proposition 1.2.17 there is a recollement where the first and the third terms are triangulated categories generated by a single, compact object. $\square$

Remark 1.3.16. In Theorem 1.3.15, if we assume, in point 1), 2) or 3) that the compact objects are self-orthogonal, then they will be self-orthogonal also in the other points.

In a similar way, using the characterization of smashing subcategories in a compactly generated triangulated category, an alternative version of the above theorem can be proved in a “non compact” version.

Theorem 1.3.17. [**AKL**, Theorem 1.6] *The following assertions are equivalent:*

(1) *$\mathcal{D}$ is a recollement of triangulated categories generated by a single compact object.*
(2) *There is an object P such that* $\mathrm{Tria}\ (P)$ *is a smashing subcategory of $\mathcal{D}$.*
(3) *There is an object P in $\mathcal{D}$ such that* $\mathrm{KerHom}_{\mathcal{D}}(\mathrm{Tria}\ (P), -)$ *is closed under coproducts.*
(4) *There are objects P and Q of D such that:*
   i) $\mathrm{KerHom}_{\mathcal{D}}(\mathrm{Tria}\ (P), -)$ *is closed under coproducts.*
   ii) *Q is self-compact.*
   iii) $\mathrm{Hom}_{\mathcal{D}}(P[n], Q) = 0$ *for each $n \in \mathbb{Z}$.*
   iv) *$\{P, Q\}$ generates $\mathcal{D}$.*

CHAPTER 2

Recollements of derived categories of dg algebras

In this chapter we want to introduce the well known concept of derived category of an abelian category, in particular the derived category of a ring and of a dg algebra.

1. Derived categories of dg algebras

DEFINITION 2.1.1. Let A be an abelian category. Let us denote by $\mathcal{C}(A)$ the category of cochains of objects of A

$$... \to X^n \overset{d_n}{\to} X^{n+1} \overset{d_{n+1}}{\to} X^{n+2} \to ...$$

where the morphisms are the chain maps.
The *homotopy category* $\mathcal{H}(A)$ is the category whose objects are complexes in A and whose morphisms are homotopy equivalence classes of morphisms of complexes.

THEOREM 2.1.2. [**W**, Section 10.9] *In the additive category $\mathcal{H}(A)$ it can be defined a class of distinguished triangles such that $(\mathcal{H}(A), [1])$ is a triangulated category where $[1]$ is the shift of complexes.*

REMARK 2.1.3. For any n in $\mathbb{Z}$ the additive cohomology functor

$$H^n : \mathcal{C}(A) \longrightarrow \mathcal{A}$$

induces an additive functor $\mathcal{H}(A) \longrightarrow \mathcal{A}$ which we also denote H^n. If we set $H = H^0$ then it is easy to check that $H^n = H \circ [n]$ for any $n \in \mathbb{Z}$.

DEFINITION 2.1.4. A morphism of complexes $f : X \longrightarrow Y$ is a *quasi-isomorphism* if the morphism $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ is an isomorphism in $\mathcal{A}$ for every $n \in \mathbb{Z}$. Since this property is stable under homotopy equivalence, it makes sense to say that a morphism in $\mathcal{H}(A)$ is a quasi-isomorphism.

PROPOSITION 2.1.5. *A morphism of complexes $f : X \longrightarrow Y$ is a quasi-isomorphism if and only its mapping cone is exact.*

REMARK 2.1.6. The exact complexes form a thick triangulated subcategory $\mathcal{Z}$ of $\mathcal{H}(A)$ where the morphisms are the quasi-isomorphisms in $\mathcal{H}(A)$. In particular, if we denote by Σ the class of quasi-isomorphisms in $\mathcal{H}(A)$, then every $\sigma \in \Sigma$ has its mapping cone in $\mathcal{Z}$, so, by definition, we have $\mathcal{H}(A)/\mathcal{Z} = \mathcal{H}(A)[\Sigma^{-1}]$.

DEFINITION 2.1.7. Let $\mathcal{A}$ be an abelian category. The *derived category* of $\mathcal{A}$ is the Verdier quotient $\mathcal{D}(A) := \mathcal{H}(A)/\mathcal{Z} = \mathcal{H}(A)[\Sigma^{-1}]$ with the canonical triangulated functor $\mathcal{H}(A) \longrightarrow \mathcal{H}(A)/\mathcal{Z} = \mathcal{D}(A)$

A classical example of derived category is given by the derived category of a module category. We start from a ring R, then we consider the abelian category of its left (or right) modules R-Mod, then the category of R-complexes and we quotient the homotopic category $\mathcal{H}(R)$ by the class of quasi-isomorphisms.

Now we briefly introduce the concept of dg categories, dg functors and we focus on derived categories of dg algebras.
Let k be a commutative ring.

DEFINITIONS 2.1.8. A *graded category* $\mathcal{A}$ is a k-linear category whose morphisms spaces are $\mathbb{Z}$-graded modules. A *graded functor* is a functor of graded categories $F : \mathcal{A} \to \mathcal{B}$ such that the map

$$\Phi_{(M,N)} : \mathcal{A}(M, N) \to \mathcal{B}(F(M), F(N)), f \mapsto F(f)$$

is homogeneous of degree 0.
A *differential graded category* (dg category) is a graded category $\mathcal{A}$ whose morphisms spaces are complexes of k-modules, that is they are endowed with a differential such that, for each $M, N, P \in \mathcal{A}, f : N \to P, g : M \to N$ we have:

$$d(fg) = (df)g + (-1)^p f(dg)$$

where f is homogeneous of degree p.

The simplest dg category is the dg category $\mathcal{A}$ with one object $*$ and space of endomorphism $B := \mathrm{Hom}_{\mathcal{A}}(*, *)$. Then B is not just a complex of k-modules but it has also a "multiplicative structure". It is an example of differential graded algebra over k.

DEFINITION 2.1.9. A *differential graded algebra* over k (dg algebra) is a $\mathbb{Z}$-graded k-algebra $B = \bigoplus_{p\in\mathbb{Z}} B^p$ endowed with a differential d of degree one, satisfying the Leibniz rule:

$$d(ab) = d(a)b + (-1)^p ad(b)$$

for all $a \in B^p, b \in B$.

In particular, a ring is a dg $\mathbb{Z}$-algebra concentrated in degree 0.

DEFINITION 2.1.10. Let B be a dg algebra over k with differential d_B. A *differential graded (left) B-module* (dg B-module) is a $\mathbb{Z}$-graded (left) B-module $M = \bigoplus_{p\in\mathbb{Z}} M^p$ endowed with a differential d_M of degree 1 such that

$$d_M(bm) = bd_M(m) + (-1)^p d_B(b)m$$

for all $m \in M^p$, $b \in B$.

In the sequel we will simply talk about a dg algebra without mentioning the ground ring k.

NOTATIONS 2.1.11. *We denote by B^{op} the opposite dg algebra of B. Thus, dg right B-modules will be identified with left dg B^{op}-modules. Also $\mathcal{D}(B^{\mathrm{op}})$ will denote the derived category of right dg B-modules.*

M is a B-A dg-bimodule if it is a left dg B-module and a left dg A^{op}-module, with compatible B and A^{op} module structure. In this case we also write ${}_BM_A$.

DEFINITION 2.1.12. A *morphism between dg B-modules* is a morphism of the underlying graded B-modules, homogeneous of degree zero and commuting with the differentials.
A morphism $f\colon M \to N$ of dg B-modules is said to be *null-homotopic* if there exists a morphism of graded modules $s\colon M \to N$ of degree -1 such that $f = sd_M + d_N s$.

The category of left dg B-modules is abelian and it will be denoted by $\mathcal{C}(B)$. If B is concentrated in degree zero, then $\mathcal{C}(B)$ is the usual category of complexes over the algebra B.

REMARK 2.1.13. (1) The homotopy category $\mathcal{H}(B)$ is the category with the same objects as $\mathcal{C}(B)$ and with morphisms the equivalence classes of morphisms in $\mathcal{C}(B)$ modulo the null-homotopic ones. The derived category $\mathcal{D}(B)$ is the localization of $\mathcal{H}(B)$ with respect to quasi-isomorphisms, that is morphisms in $\mathcal{C}(B)$ inducing isomorphisms in homology.
(2) $\mathcal{H}(B)$ and $\mathcal{D}(B)$, as in the case of rings, are triangulated categories with shift functor [1] (the usual shift of complexes).
(3) $\mathcal{D}(B)$ is a compactly generated triangulated category generated by the single object B.

DEFINITION 2.1.14. $\mathcal{C}_{dg}(B)$ denotes the category of dg B-modules where the morphism space $\mathrm{Hom}_{\mathcal{C}_{dg}(B)}(M,N)$ between dg B-modules M,N is the complex $\mathcal{H}om_B(M,N)$ with $[\mathcal{H}om_B(M,N)]^n = \mathrm{Hom}_B(M,N[n])$ (here $\mathrm{Hom}_B(M,N)$ denotes the group of morphisms of graded B-modules, homogeneous of degree zero) and differential defined, for each $f \in \mathcal{H}om_B(M,N)$,

$$d(f) = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

Observe that, if X is a dg B-module, then $\mathcal{H}om_B(X,X)$ is a dg algebra called the dg-endomorphism ring of X.

DEFINITION 2.1.15. A dg B-module is *acyclic* if it has zero homology.

REMARK 2.1.16. Let $Z^0\mathcal{C}_{dg}(B)$ and $H^0(\mathcal{C}_{dg}(B))$ the categories having exactly the same objects as $\mathcal{C}(B)$ and morphisms, for each $f: M \longrightarrow N$ in $\mathcal{C}_{dg}(B)$, respectively $Z^0(f)$ and $H^0(f)$. Then the following equalities hold:

$$Z^0\mathcal{C}_{dg}(B) = \mathcal{C}(B) \text{ and } H^0(\mathcal{C}_{dg}(B)) = \mathcal{H}(B).$$

DEFINITION 2.1.17. $\mathcal{H}_p(B)$ indicates the category of the $\mathcal{H}$-projective modules, that is a full subcategory of $\mathcal{H}(B)$ consisting of the dg modules M such that $\mathrm{Hom}_{\mathcal{H}(B)}(M,N) = 0$ for each acyclic module N. Dually we define the category $\mathcal{H}_i(A)$ of the $\mathcal{H}$-injective modules as the full subcategory of $\mathcal{H}(B)$ of all modules I such that $\mathrm{Hom}_{\mathcal{H}(B)}(N,I) = 0$ for each $I \in \mathcal{H}_i(B)$ and for each acyclic module N.

PROPOSITION 2.1.18. [**Ke2**, Theorem 3.1] *Let us denote by inc_p and inc_i respectively the inclusion of $\mathcal{H}_p(B)$ and $\mathcal{H}_i(B)$ in $\mathcal{H}(B)$, then there are two pairs of adjoint functors*

$$\mathcal{H}_p(B) \underset{inc_p}{\overset{p}{\leftrightarrows}} \mathcal{H}(B) \qquad \textit{and} \qquad \mathcal{H}_i(B) \underset{inc_i}{\overset{i}{\leftrightarrows}} \mathcal{H}(B)$$

such that $p \circ inc_p = Id_{\mathcal{H}_p(B)}$ *and* $i \circ inc_i = Id_{\mathcal{H}_i(B)}$. *Moreover there are two triangle equivalences*

$$\mathcal{H}_p(B) \xrightarrow{p} \mathcal{H}(B) \longrightarrow \mathcal{D}(B) \qquad \text{and} \qquad \mathcal{H}_i(B) \xrightarrow{i} \mathcal{H}(B) \longrightarrow \mathcal{D}(B).$$

REMARK 2.1.19. For each M in $\mathcal{H}(B)$, $\underline{p}M$ and $\underline{i}M$ will indicate the $\mathcal{H}$*-projective* and $\mathcal{H}$*-injective* resolution of M. Proposition 2.1.18 tells us that for each M in $\mathcal{D}(B)$ projective and injective resolutions always exist and we have the quasi-isomorphisms:

$$\underline{p}M \simeq M \simeq \underline{i}M.$$

DEFINITION 2.1.20. Let A and B two dg algebras over k and

$$F : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$$

an additive functor. Then F induces a functor on the homotopy category which we still denote by F. It also induces a functor between $\mathcal{C}_{dg}(A) \longrightarrow \mathcal{C}_{dg}(B)$. With abuse of notations we denote by F the functor induced at the level of homotopic categories. We recall the definition of the *total right derived functor* of F, $\mathbb{R}F : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ as $\mathbb{R}F(X) = F(\underline{i}X)$ and of the *total left derived functor* of F, $\mathbb{L}F : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ as $\mathbb{L}F(X) = F(\underline{p}X)$, for every X in $\mathcal{D}(A)$.

NOTATIONS 2.1.21.
(1) *Let M be an A-B^{op} bimodule, then the total right derived functor of* $\mathrm{Hom}_A(M, -)$ *is denoted by* $\mathbb{R}\mathrm{Hom}_A(M, -)$ *and it is defined by:*

$$\mathbb{R}\mathrm{Hom}_A(M, N) = \mathcal{H}om_A(M, \underline{i}N) = \mathcal{H}om_A(\underline{p}M, N).$$

(2) *the following equalities hold, for each M,N in* $\mathcal{D}(A)$*:*

$$H^n(\mathbb{R}\mathrm{Hom}_A(M, N)) = H^n(\mathcal{H}om_A(M, \underline{i}N)) =$$
$$= H^0(\mathcal{H}om_A(M, \underline{i}N[n])) = \mathrm{Hom}_{\mathcal{H}(A)}(M, \underline{i}N[n]) = \mathrm{Hom}_{\mathcal{D}(A)}(M, N[n]).$$

(3) *Let T be a left B dg-modules with dg-endomorphism ring A. The total left derived functor of* $T \otimes_B -$ *is denoted by* $T \overset{\mathbb{L}}{\otimes}_B -$ *and, for every N in* $\mathcal{D}(B)$

$$T \overset{\mathbb{L}}{\underset{B}{\otimes}} N = T \underset{B}{\otimes} \underline{p}N \text{ in } \mathcal{D}(A).$$

DEFINITION 2.1.22. (see [**Ke1**, Sec 2.6]) Let A be a dg algebra. A dg A-module X is called *perfect* if it is $\mathcal{H}$-projective and compact in $\mathcal{D}(A)$. The full subcategory of $\mathcal{H}(A)$ consisting of perfect dg A-modules is denoted by per A; it coincides with the subcategory tria A of $\mathcal{H}(A)$.

By Ravenel-Neeman's result, an object of $\mathcal{D}(A)$ is compact if and only if it is quasi-isomorphic to a perfect dg B-module.

If A is an ordinary algebra, then the perfect complexes are the bounded complexes with finitely generated projective terms, that is $\mathcal{H}_p^b(A)$.

Let us recall that if we take a compact object Q in the derived category $\mathcal{D}(A)$ of a dg algebra A, then the triangulated subcategory Tria Q is a smashing subcategory by Proposition 1.3.11. We want to recall a fundamental result proved by Keller (in the more general setting of dg-categories in [**Ke2**, Theorem 4.3]) that establishes a triangle equivalence between Tria Q and the derived category of a suitable dg algebra.

THEOREM 2.1.23. [**Ke1**, Theorem 3.3] *Let A be a dg algebra and Q be a self-compact and $\mathcal{H}$-projective object in $\mathcal{D}(A)$. Set $B := \mathbb{R}\mathrm{Hom}_A(Q,Q)$, then there is a triangle equivalence*

$$F : \mathcal{D}(B) \longrightarrow \mathrm{Tria}\ Q$$

with $H^n(B) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(Q, Q[n])$.

PROOF. Let us set $\mathcal{Y} := \mathrm{Tria}\ Q$. Then $\mathcal{Y}$ is a triangulated category. Consider the functor

$$F_1 : \mathcal{Y} \to \mathcal{C}(B) : M \mapsto \mathcal{H}om_A(Q, M).$$

Set F for the composition of F_1 with the quotient functor $\mathcal{C}(B) \to \mathcal{D}(B)$. We want to prove that F is an equivalence. First of all we can see that F commutes with coproducts, indeed, for every set of objects $(M_j)_{j\in J}$ of $\mathcal{Y}$ we have:

$$\mathcal{H}om_A(Q, \coprod_j M_j) = \coprod_j \mathcal{H}om_A(Q, M_j)$$

since Q is compact in $\mathcal{Y}$ and $\mathcal{H}$-projective in $\mathcal{H}(A)$. Moreover $F(Q) = B$ and

$$\mathrm{Hom}_{\mathcal{D}(B)}(F(Q), F(Q[n])) = \mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) = H^n(B) =$$

$$= H^n F(Q) = \mathrm{Hom}_{\mathcal{D}(A)}(Q, Q[n]) = \mathrm{Hom}_{\mathcal{Y}}(Q, Q[n]).$$

Then F commutes with coproducts, sends a generator of $\mathcal{Y}$ in a generator of $\mathcal{D}(B)$ and it is fully faithful on the compact objects of $\mathcal{Y}$. Thanks to [**Ke2**, Lemma 4.2], we can conclude that F is an equivalence. $\square$

REMARK 2.1.24. Let us note that, if Q is perfect in $\mathcal{D}(A)$ then it is in particular $\mathcal{H}$-projective, then $\mathbb{R}\mathrm{Hom}_A(Q,Q) = \mathcal{H}om_A(Q,Q)$.

2. Recollements from compact objects

Now we have all the necessary notions to explain some results on the construction of recollements of derived categories of dg algebras arising from compact objects. Our approach follows the exposition in [**J**] which generalizes to dg algebras the situation considered in [**DG**] for derived categories of rings.

Let B be a dg algebra and let Q be a perfect left dg B-module. Consider the dg-endomorphism ring D of Q, that is $D = \mathcal{H}om_B(Q,Q)$; then Q becomes a B-D dg-bimodule and, by Theorem 2.1.23, Tria $Q \simeq \mathcal{D}(D)$. Let $P = Q^* = \mathbb{R}\mathrm{Hom}_B(Q, B)$, then P is a D-B dg-bimodule.

REMARK 2.2.1. Let Q be a perfect left dg B-module with dg-endomorphism ring D and $P = Q^* = \mathbb{R}\mathrm{Hom}_B(Q,B)$. Then P is a perfect right dg B-module and the following hold

(1) ([**DG**, Sec 2.5] or [**J**, Sec 2.1]) The functors

$$H = \mathbb{R}\mathrm{Hom}_B(Q, -),\ G = P \overset{\mathbb{L}}{\underset{B}{\otimes}} - : \mathcal{D}(B) \to \mathcal{D}(D).$$

are isomorphic.

(2) The functor $\mathcal{H}om_B(-,B)$ induces an equivalence

$$\mathcal{H}om_B(-,B)\colon \text{per } B \to \text{per } B^{\text{op}}$$

with inverse $\mathcal{H}om_{B^{\text{op}}}(-,B)$. Thus, $P^* = \mathbb{R}\text{Hom}_B(P,B)$ is isomorphic to Q. Moreover, it follows that the functor $\mathcal{H}om_B(-,B)\colon \mathcal{C}(B) \to \mathcal{C}(B^{\text{op}})$ induces a quasi-isomorphism between the dg algebras $\mathcal{H}om_B(Q,Q)$ and $\mathcal{H}om_{B^{\text{op}}}(P,P)$. So we can identify $\mathcal{H}om_{B^{\text{op}}}(P,P)$ with D.

Remark 2.2.2. By Proposition 1.3.11, given a compact object Q in $\mathcal{D}(B)$, Tria Q is a smashing subcategory of $\mathcal{D}(B)$ and so (Tria $Q, Q^{\perp}, Q^{\perp\perp}$) is a TTF triple. Hence there is the recollement:

$$\begin{tikzcd}[column sep=huge]
\mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D}(B) \arrow[l, bend right, "L_{\mathcal{Y}}"'] \arrow[l, bend left, "R_{\mathcal{Y}}"] \arrow[r, "R_{\mathcal{X}}"] & \mathcal{X} \arrow[l, bend right, "i_{\mathcal{X}}"'] \arrow[l, bend left, "i_{\mathcal{Z}}L_{\mathcal{Z}}i_{\mathcal{X}}"]
\end{tikzcd}$$

Thank's to Keller theorem and to some observations on derived functors we can rewrite the above recollement in a different way. The following result appears in different forms in papers by Dwyer and Greenless [**DG**, Sec. 2], Miyachi [**Mi**, Proposition 2.7] and Jørgensen [**J**, Proposition 3.2]. We restate it and give an alternative proof following the arguments used by Yang in the proof of [**Y**, Theorem 1].

Proposition 2.2.3. *Let B be a dg algebra and let Q be a perfect left dg B-module. Let $D = \mathcal{H}om_B(Q,Q)$ and $P = \mathbb{R}\text{Hom}_B(Q,B)$. Then the following diagram is a recollement:*

$$\begin{tikzcd}[column sep=huge]
Q^{\perp} \arrow[r, "i_*=inc"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "\mathbb{R}\text{Hom}_B(Q,-)\cong P\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(D) \arrow[l, bend right, "Q\overset{\mathbb{L}}{\underset{D}{\otimes}}-"'] \arrow[l, bend left, "\mathbb{R}\text{Hom}_D(P,-)"]
\end{tikzcd}$$

Proof. We first show that the functor $j_! = Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -$ is fully faithful.

By construction we have that $Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -$ induces an equivalence between tria $D \to$ tria Q. In other words the pair $(D,\ {}_BQ_D)$ is a standard lift (see [**Ke2**, Sec.7]). The functor $j_!$ commutes with set index coproducts, its restriction to tria D is fully faithful and $j_!(D) = Q$ is a compact object. Thus by [**Ke2**, Lemma 4.2 b] we conclude that $j_!$ is fully faithful, since D is a generator of $\mathcal{D}(D)$.

So the functor $\mathbb{R}\text{Hom}_B(Q,-) \cong (P \overset{\mathbb{L}}{\underset{D}{\otimes}} -)$ has a fully faithful left adjoint and a right adjoint $\mathbb{R}\text{Hom}_D(P,-)$. By [**Mi**, Proposition 2.7], the functor $\mathbb{R}\text{Hom}_D(P,-)$ is fully faithful, so the right part of the diagram in the statement can be completed to a recollement with left term the kernel of the functor $\mathbb{R}\text{Hom}_B(Q,-)$, which coincides with the category ${}_BQ^{\perp}$, since Q is a compact object. Moreover we have that this recollement is equivalent to the one in Remark 2.2.2 so $Q^{\perp} = \mathcal{Y}$ and $i^* = L_{\mathcal{Y}}$ and $i^! = R_{\mathcal{Y}}$. $\square$

Corollary 2.2.4. *In the same notations as in Proposition 2.2.3 the following hold:*

(1) $Q^{\perp} = \mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$;

(2) *The functor* $P \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ *induces an equivalence*

$$\mathcal{D}(B)/\mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \to \mathcal{D}(D);$$

(3) $\big(\mathrm{Tria}\, Q,\, Q^{\perp},\, \mathrm{Im}\, \mathbb{R}\mathrm{Hom}_D(P,-)\big)$ *is the same TTF triple in* $\mathcal{D}(B)$ *shown in Remark 2.2.2.*

Proof. (1) It follows by well known results about recollements [**BBD**, Proposition 1.4.5]). Let $M \in Q^{\perp}$, that is $\mathrm{Hom}_{\mathcal{D}(B)}(Q, M[n]) = 0$ for all integer n. Now, $P \overset{\mathbb{L}}{\underset{B}{\otimes}} M \simeq \mathbb{R}\mathrm{Hom}_B(Q, M)$ and $H^i(\mathbb{R}\mathrm{Hom}_B(Q, M)) = \mathrm{Hom}_{\mathcal{D}(B)}(Q, M[i]) = 0$ for all integer i, that is $P \overset{\mathbb{L}}{\underset{B}{\otimes}} M \simeq 0$ in $\mathcal{D}(D)$.

On the other side, if $P \overset{\mathbb{L}}{\underset{B}{\otimes}} M \simeq \mathbb{R}\mathrm{Hom}_B(Q, M) \simeq 0$, then $H^i(\mathbb{R}\mathrm{Hom}_B(Q, M)) = \mathrm{Hom}_{\mathcal{D}(B)}(Q, M[i]) = 0$ for all integer i, i.e. $M \in Q^{\perp}$.

(2) and (3) As recalled in Proposition 1.2.17, if

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

gives $\mathcal{D}$ as a recollement of $\mathcal{D}'$and $\mathcal{D}''$, then the data $(j_!(\mathcal{D}''), i_*(\mathcal{D}), j_*(\mathcal{D}''))$ is a TTF triple on $\mathcal{D}$.

Thus to conclude the proof of (2) and (3) it remains to show that Tria Q is the essential image of the functor $j_! = Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -$. This follows from the facts that the fully faithful functor $Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -$ is a triangle functor which commutes with coproducts and sends the generator D of the category $\mathcal{D}(D)$ to the object Q of $\mathcal{D}(B)$, hence its image is Tria Q. Then, from 1.2.11, we have that the Verdier quotient $\mathcal{D}(B)/\mathrm{Ker}(P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ is equivalent to Tria Q, that is triangle equivalent, by 2.1.23, to $\mathcal{D}(D)$. We conclude that the recollements in Remark 2.2.2 and in Proposition 2.2.3 are equivalent then the TTF triples associated are the same. $\square$

Remark 2.2.5. From the proof of Theorem 1.3.15 we have that the central class $\mathcal{Y}$ is generated by the self-compact and $\mathcal{H}$-projective object $L_{\mathcal{Y}}(B)$. Let us write simply L for $L_{\mathcal{Y}}$, then, if we set $E := \mathbb{R}\mathrm{Hom}_B(L(B), L(B))$, by Keller's Theorem there is a derived equivalence $\mathcal{D}(E) \simeq \mathcal{Y}$; moreover, from [**J**, Theorem 1.6], there is an adjoint pair of functors:

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(E) \arrow[r, bend left, "L(B)\overset{\mathbb{L}}{\otimes}_E -"] & \mathcal{D}(B) \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(L(B),-)"]
\end{tikzcd}$$

whose restriction to $\mathcal{Y}$ gives the equivalence $\mathcal{D}(E) \simeq \mathcal{Y}$

Keeping the same notation for the restriction and the corestriction to $\mathcal{Y}$ of the adjoint pair and composing with the functors in the recollement of Proposition 2.2.3, we deduce the following corollary.

Corollary 2.2.6. *In the same setting as in Corollary 2.2.4 and Remark 2.2.5 there is a recollement*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(E) \arrow[r, "i_{\mathcal{Y}}\circ(L(B)\overset{\mathbb{L}}{\otimes}_E -)"] & \mathcal{D}(B) \arrow[l, bend right, "\mathbb{R}\mathrm{Hom}_B(L(B),-)\circ L"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(L(B),-)\circ R"] \arrow[r, "\mathbb{R}\mathrm{Hom}_B(Q,-)\cong P\overset{\mathbb{L}}{\otimes}_B -"] & \mathcal{D}(D) \arrow[l, bend right, "Q\overset{\mathbb{L}}{\underset{D}{\otimes}} -"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_D(P,-)"]
\end{tikzcd}$$

3. Homological epimorphisms

Let B be a dg algebra and $\mathcal{D}(B)$ its derived category. From the previous section we know that, given a compact object in $\mathcal{D}(B)$ we have a TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ where the central class $\mathcal{Y}$ is equivalent to the derived category of a dg algebra. In this section we prove that if B is a k-flat dg algebra every TTF triple gives rise to a homological epimorphism $F : B \to C$ for a suitable dg algebra C. The main theorem of this section can be viewed as a generalization of [**J**, Theorem 3.3] (in the case of k-flat dg algebras), since it characterizes the left term of the recollement as the derived category of a dg algebra obtained by a homological epimorphism.
We recall the notions of homological epimorphisms of rings and dg algebras.

Definition 2.3.1. A morphism $f : R \longrightarrow S$ between two rings is called a *ring epimorphism* if, for every morphisms $g, h : S \longrightarrow T$ of rings such that $gf = hf$, one has $g = h$. Equivalently ([**GL**, Theorem 4.4]), f is a ring epimorphism if and only if the multiplication map $S \otimes_R S \longrightarrow S$ is an isomorphism of S right, S left bimodules.

Definition 2.3.2. Two ring epimorphisms $f : R \longrightarrow S$ and $g : R \longrightarrow S'$ are said to be equivalent if there exists an isomorphism of rings $h : S \longrightarrow S'$ such that $hf = g$.

Definition 2.3.3. [**GL**, Section 4] Let $f : R \longrightarrow S$ be a ring epimorphism, then f is said to be *homological* if one of the following equivalent conditions holds:

1) $S \overset{\mathbb{L}}{\otimes}_R S = S$ in $\mathcal{D}(R)$;

2) for all S^{op}-modules N and all S-modules M, the canonical map $N \overset{\mathbb{L}}{\otimes}_R M \longrightarrow N \overset{\mathbb{L}}{\otimes}_S M$ is an isomorphism;

3) for all S-modules M, M', the canonical map $\mathbb{R}Hom_S({}_SM, {}_SM') \longrightarrow \mathbb{R}Hom_R({}_RM, {}_RM')$ is an isomorphism;

4) the induced functor $f_* : \mathcal{D}(S) \longrightarrow \mathcal{D}(R)$ is a full embedding of derived categories.

Definition 2.3.4. Two homological epimorphisms of rings $f : R \longrightarrow S$ and $g : R \longrightarrow S'$ are said to be equivalent if they are equivalent as ring epimorphisms.

The concept of homological epimorphism of rings can be "naturally" generalized to the setting of dg algebras ([**P**]) and to the more general setting of dg categories ([**NS**]). Here we give the definition of homological epimorphism of dg algebras and its characterization at the level of derived categories.

Theorem 2.3.5. [**P**, Theorem 3.9] *Let C and D be dg k-algebras and $F : C \longrightarrow D$ a morphism of dg algebras. Then the following are equivalent:*

1) *there is an isomorphism ${}_DD \overset{\mathbb{L}}{\otimes}_C D \longrightarrow {}_DD_D$ given by the canonical map;*

2) *for all dg D-C module M, the canonical map ${}_DD \overset{\mathbb{L}}{\otimes}_C M \longrightarrow {}_DM$ is an isomorphism;*

3) *for all dg right D-modules N and all left dg D-modules M, the canonical map $N_C \overset{\mathbb{L}}{\otimes}_C {}_CM \longrightarrow N \overset{\mathbb{L}}{\otimes}_D M$ is an isomorphism;*

4) *for all dg D-C modules M, the canonical map ${}_DM \longrightarrow \mathbb{R}Hom_C({}_CD_D, {}_CM)$ is an isomorphism;*

5) *for all dg D-modules M, M', the canonical map $\mathbb{R}Hom_D({}_DM, {}_DM') \longrightarrow \mathbb{R}Hom_C({}_CM, {}_CM')$ is an isomorphism;*

6) *the induced functor $F^* : \mathcal{D}(D) \longrightarrow \mathcal{D}(C)$ is a full embedding of derived categories.*

Proof. 1) $\Rightarrow$ 2)

$$ {}_DM = {}_DD \overset{\mathbb{L}}{\otimes}_D M \simeq ({}_DD \overset{\mathbb{L}}{\otimes}_C D_D) \overset{\mathbb{L}}{\otimes}_D M \simeq_D D \overset{\mathbb{L}}{\otimes}_C (D_D \overset{\mathbb{L}}{\otimes}_D M) \simeq {}_DD \overset{\mathbb{L}}{\otimes}_C M $$

2) $\Rightarrow$ 3)

$$ N_D \overset{\mathbb{L}}{\otimes}_D M \simeq N \overset{\mathbb{L}}{\otimes}_D ({}_DD \overset{\mathbb{L}}{\otimes}_C M) \simeq (N \overset{\mathbb{L}}{\otimes}_D D_C) \overset{\mathbb{L}}{\otimes}_C M \simeq N_C \overset{\mathbb{L}}{\otimes}_C M $$

3) $\Rightarrow$ 1)

$$ D_C \overset{\mathbb{L}}{\otimes}_C D \simeq D \overset{\mathbb{L}}{\otimes}_D D \simeq {}_DD_D $$

1) $\Rightarrow$ 4) Let us note that there is an adjoint pair $({}_DD \overset{\mathbb{L}}{\otimes}_C -, \mathbb{R}\mathrm{Hom}_D(D,-))$:

$$ \begin{tikzcd} \mathcal{D}(D) \arrow[r, "\mathbb{R}\mathrm{Hom}_D(D,-)"] & \mathcal{D}(C) \arrow[l, bend right, "{}_DD\overset{\mathbb{L}}{\otimes}_C-"'] \end{tikzcd} $$

Then:

$$ {}_DM \simeq \mathbb{R}\mathrm{Hom}_D({}_DD_D, {}_CM) \simeq \mathbb{R}\mathrm{Hom}_D({}_DD \overset{\mathbb{L}}{\otimes}_C D_D, {}_DM) \simeq $$

$$\simeq \mathbb{R}\mathrm{Hom}_C({}_CD_D, \mathbb{R}\mathrm{Hom}_D({}_DD_C, {}_DM)) \simeq \mathbb{R}Hom_C({}_CD_D, {}_CM).$$

4) $\Rightarrow$ 5) Let us note that there is an adjoint pair $({}_CD \overset{\mathbb{L}}{\otimes}_D -, \mathbb{R}\mathrm{Hom}_C({}_CD, -))$:

$$\begin{tikzcd} \mathcal{D}(C) \arrow[r, "\mathbb{R}\mathrm{Hom}_C(C,-)"] & \mathcal{D}(D) \arrow[l, bend right, "{}_CD\overset{\mathbb{L}}{\otimes}_D-"'] \end{tikzcd}$$

Then:

$$\mathbb{R}\mathrm{Hom}_D({}_DM, {}_DM') \simeq \mathbb{R}\mathrm{Hom}_D({}_DM, \mathbb{R}Hom_C({}_CD_D, {}_CM')) \simeq$$

$$\simeq \mathbb{R}\mathrm{Hom}_C({}_CD \overset{\mathbb{L}}{\otimes}_D M, {}_CM') \simeq \mathbb{R}\mathrm{Hom}_C({}_CM_D, {}_C M').$$

5) $\Rightarrow$ 6) the assertion in 5) is a reformulation of the fact that $F_* = D \overset{\mathbb{L}}{\otimes}_D -$ is a full embedding.

6) $\Rightarrow$ 4) If F_* is fully faithful, then, for each D-C module M:

$$\mathbb{R}Hom_C({}_CD_D, {}_CM) \simeq \mathbb{R}Hom_D(F_*({}_CD_D), F_*({}_DM)) \simeq$$

$$\simeq \mathbb{R}Hom_D({}_DD_D, {}_DM) \simeq {}_DM.$$

5) $\Rightarrow$ 1) Recall that $({}_CD \overset{\mathbb{L}}{\otimes}_D -, \mathbb{R}\mathrm{Hom}_C({}_CD, -))$ is an adjoint pair, hence we have, for each C-D module M:

$$\mathbb{R}\mathrm{Hom}_D({}_DD, {}_DM) \simeq \mathbb{R}\mathrm{Hom}_C({}_CD, {}_CM) \simeq$$

$$\mathbb{R}\mathrm{Hom}_C({}_CD, \mathbb{R}\mathrm{Hom}_D({}_CD_D, {}_DM)) \simeq \mathbb{R}\mathrm{Hom}_D({}_DD \overset{\mathbb{L}}{\otimes}_C {}_CD_D, {}_D M).$$

So, for each D-module M, $\mathbb{R}\mathrm{Hom}_D(D, M) \simeq \mathbb{R}\mathrm{Hom}_D(D \overset{\mathbb{L}}{\otimes}_C D, M)$ then ${}_DD_D \simeq D \overset{\mathbb{L}}{\otimes} D$. □

Definition 2.3.6. A morphism of dg algebras $F : C \to D$ is a said to be a *homological epimorphism* if it satisfies one of the equivalent conditions of Theorem 2.3.5.

Definition 2.3.7. Two homological epimorphisms of dg k-algebras $F : C \longrightarrow D$ and $G : C \longrightarrow D'$ are said to be equivalent if there exists an isomorphism of dg k-algebras $H : D \longrightarrow D'$ such that $HF = G$.

Remark 2.3.8. From the definition it is clear that a homological epimorphism of rings is exactly a homological epimorphism of dg algebras over $\mathbb{Z}$ concentrated in degree 0.

In [**NS**, Theorem 5] it is proved that for a flat small dg-category $\mathcal{B}$ there are bijections between equivalence classes of recollements of $\mathcal{D}(\mathcal{B})$, TTF triples on $\mathcal{D}(\mathcal{B})$ and equivalence classes of homological epimorphisms of dg-categories $F \colon \mathcal{B} \to \mathcal{C}$.

Moreover, in [**NS**, Lemma 5] it is observed that every derived category of a small dg-category is triangle equivalent to the derived category of a small flat dg-category. To achieve this last result ones uses the construction of a model structure on the category of all small dg-categories defined by Tabuada (see [**T**].)

We now state [**NS**, Theorem 4] for the case of a flat dg k-algebra and we give a proof, since in this case the construction of the homological epimorphism becomes more explicit and it will also be used later on in Theorem 3.1.13.

PROPOSITION 2.3.9. *Let B be a dg algebra flat as k-module and $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a TTF triple in $\mathcal{D}(B)$. Then there is a dg algebra C and a homological epimorphism $F\colon B \to C$ such that $\mathcal{Y}$ is the essential image of the restriction of scalars functor $F_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.*

PROOF. Since $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}(B)$ there exists a triangle

$$X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1], \quad \text{with } X \in \mathcal{X} \text{ and } \ Y \in \mathcal{Y}. \tag{6}$$

where φ_B is the unit morphism of the adjunction. Without loss of generality, we may assume that Y is an $\mathcal{H}$-injective left dg B-module and that φ_B is a morphism in $\mathcal{C}(B)$.

Let $E = \mathbb{R}\mathrm{Hom}_B(Y, Y) = \mathcal{H}om_B(Y, Y)$, then ${}_BY_E$ is a dg B-E-bimodule. Applying the functor $\mathbb{R}\mathrm{Hom}_B(-, Y)$ to the triangle (6) we obtain a triangle in the derived category $\mathcal{D}(E^{\mathrm{op}})$:

$$\mathbb{R}\mathrm{Hom}_B(X[1], Y) \longrightarrow \mathbb{R}\mathrm{Hom}_B(Y, Y) \xrightarrow{\beta} \mathbb{R}\mathrm{Hom}_B(B, Y) \longrightarrow \mathbb{R}\mathrm{Hom}_B(X, Y).$$

where $\beta = \mathbb{R}\mathrm{Hom}_B(\varphi_B, Y) = \varphi_*$. Since $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and Y is $\mathcal{H}$-injective, we have, for each $i, n \in \mathbb{Z}$:

$$\mathrm{H}^n\mathbb{R}\mathrm{Hom}_B(X[i], Y) \cong \mathrm{Hom}_{\mathcal{D}(B)}(X[i], Y[n]) = 0$$

Therefore we deduce that β is a quasi-isomorphism, so we have

$$E = \mathbb{R}\mathrm{Hom}_B(Y, Y) \overset{\beta}{\simeq} \mathbb{R}\mathrm{Hom}_B(B, Y) \overset{\gamma}{\simeq} Y \text{ in } \mathcal{D}(E^{\mathrm{op}}). \tag{7}$$

Let $\xi\colon Y \to Y'$ be a quasi isomorphism of dg B-E-bimodules such that Y' is an $\mathcal{H}$-injective resolution of Y as a dg B-E-bimodule. Since B is assumed to be k-flat, we have that the restriction functor from dg B-E-bimodules to dg E-modules preserves $\mathcal{H}$-injectivity. In fact, its left adjoint $B \otimes_k -$ preserves acyclicity. Then, Y'_E is an $\mathcal{H}$-injective right dg E-module. Consider the dg algebra $C = \mathcal{H}om_{E^{\mathrm{op}}}({}_BY'_E, {}_BY'_E) = \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}({}_BY'_E, {}_BY'_E)$ and a morphism of dg algebras defined by:

$$F\colon B \longrightarrow C$$

$$b \longmapsto F(b)\colon y'^{|b||y'|}by',$$

where $|\cdot|$ denotes the degree.

Since Y'_E is $\mathcal{H}$-injective we have quasi-isomorphisms:

$$C = \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}({}_BY'_E, {}_BY'_E) \xrightarrow{\xi_*} \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}({}_BY_E, {}_BY'_E) \xrightarrow{\beta_*} \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}(E, {}_BY'_E) \cong Y'.$$

We regard C as a dg B-B-bimodule with the action induced by F, so F is also a morphism of dg B-B-bimodules and the morphism $\beta_* \circ \xi_*\colon C \to Y'$ is a quasi-isomorphism of left dg B-modules; moreover, $\xi \circ \varphi_B = \beta_* \circ \xi_* \circ F$. Now define the morphism $\varepsilon := \xi^{-1} \circ \beta_* \circ \xi_*\colon C \to Y$ in $\mathcal{D}(B)$. Then ε is a quasi isomorphism of left dg B-modules such that $\varepsilon \circ F = \varphi$ and we get an isomorphism of triangles:

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & B \arrow[r, "F"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\varepsilon"'] & X[1] \arrow[d, equal] \\ X \arrow[r] & B \arrow[r, "\varphi_B"] & Y \arrow[r] & X[1] \end{tikzcd}$$

Consider the restriction of scalars functor $F_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$. F_* is a triangulated functor admitting a right adjoint, hence it commutes with small coproducts. Moreover, $F_*(C) = {}_BC \cong Y' \cong Y \in \mathcal{Y}$, hence $F_*(\mathrm{Tria}\, C) = F_*(\mathcal{D}(C))$ is a subcategory of $\mathcal{Y}$, closed under coproducts and containing the generator ${}_BY$. Now we notice that, F being a morphism of dg B-B-bimodules, one has a triangle of B-B bimodules:

$$X \longrightarrow B \xrightarrow{F} C \longrightarrow X[1]. \tag{8}$$

Consider the adjunction:

$$\begin{tikzcd} \mathcal{D}(C) \arrow[r, "F^*"'] & \mathcal{D}(B) \arrow[l, bend right, "C \overset{\mathbb{L}}{\underset{B}{\otimes}} -"'] \end{tikzcd}$$

and let $M \in \mathcal{X}$ and N in $\mathcal{D}(C)$, then

$$\mathrm{Hom}_{\mathcal{D}(C)}(C \overset{\mathbb{L}}{\underset{B}{\otimes}} M, N) \cong \mathrm{Hom}_{\mathcal{D}(B)}(M, F^*(N)) = 0$$

since $F^*(N) \in \mathcal{Y}$. Then $C \overset{\mathbb{L}}{\underset{B}{\otimes}} M = 0$ for each $M \in \mathcal{X}$. Hence, applying the functor $C \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ to the triangle (8), we obtain

$$C \overset{\mathbb{L}}{\underset{B}{\otimes}} B \cong C \overset{\mathbb{L}}{\underset{B}{\otimes}} C,$$

which shows that F is a homological epimorphism of dg algebras. In particular, $\mathrm{Im}\, F_*$ is a triangulated subcategory of $\mathcal{Y}$, hence $\mathrm{Im}\, F_* = \mathcal{Y}$, by the principle of infinite dévissage. □

THEOREM 2.3.10. [**NS**, 5.4.4] *Let B be a k-flat dg algebra then there exists a bijection between:*

(1) *Smashing subcategories $\mathcal{X}$ of $\mathcal{D}(B)$.*
(2) *TTF triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ on $\mathcal{D}(B)$.*
(3) *Equivalence classes of recollements of $\mathcal{D}(B)$.*
(4) *Equivalence classes of homological epimorphisms of dg algebras of the form $F : B \longrightarrow C$.*

PROOF. The bijection between recollements, TTF triples and smashing subcategories in $\mathcal{D}(B)$ is given by Theorem 1.2.17 and Proposition 1.3.12. Moreover, given a homological epimorphism $F : B \to C$, we have that $F_*(\mathcal{D}(C))$ is a localizing and a colocalizing subcategory of $\mathcal{D}(B)$ closed under products and coproducts. Then it is the central class of a TTF triple. On the other hand, given a recollement of $\mathcal{D}(B)$:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r] & \mathcal{X} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

we have that $M := i^*(B)$ is a self-compact generator of $\mathcal{Y}$ and, by Proposition 2.3.9, since $\varphi_B(B) = i_*i^*(B) \simeq i^*(B)$, there exists a dg algebra C and a homological epimorphism $F : B \to C$, such that $\mathcal{Y} \simeq \mathcal{D}(C)$.

□

REMARK 2.3.11. Is it possible to prove the same result as in the above theorem without assuming that B is flat over k? In the particular case in which B is a ring the answer might be yes, as proved in the following proposition by Angeleri, König, Liu.

PROPOSITION 2.3.12. [**AKL**, Proposition 1.7] *Let B be a ring. Then there is a bijection between the equivalence classes of homological epimorphisms starting in R and the equivalence classes of recollements such that $i^*(B)$ is a self-orthogonal object of $\mathcal{Y}$.*

PROOF. If $F : B \longrightarrow C$ is a homological epimorphism of rings then we have the recollement:

$$\begin{tikzcd} \mathcal{D}(C) \arrow[r, "F^*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=C\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(C{,}-)"] \arrow[r] & \mathcal{X} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Moreover $i^*(B) \simeq C$ that is exceptional in $\mathcal{D}(C)$.
Conversely, let us take the recollement

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r] & \mathcal{X} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

such that $Y := i^*(B)$ is self-orthogonal in $\mathcal{Y}$. Thus, by Keller's Theorem 2.1.23, we have $\mathcal{Y} \simeq \mathcal{D}(E)$, where $E := \mathbb{R}\mathrm{Hom}_B(Y, Y)$ has homology concentrated in degree zero and $H^0(E) \cong \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y)$.

Consider a triangle

$$X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1], \quad \text{with } X \in {}^{\perp}\mathcal{Y}. \tag{9}$$

where φ_B is the unit of the adjunction morphism and set $C = \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y)$. Let us define a ring homomorphism $\lambda\colon B \to C$ by $\lambda(b) = L(\dot{b})$, where $\dot{b}$ denotes the right multiplication by b on B. We have ${}_BC = \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y) \cong \mathrm{Hom}_{\mathcal{D}(B)}(B, Y) \cong H^0(Y) \cong Y$. So we have a quasi-isomorphism

$\varepsilon\colon {}_BC \to {}_BY$ and from the definition one sees that $\varepsilon \circ \lambda = \varphi_B$. Thus we have an isomorphism of triangles:

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & B \arrow[r, "\lambda"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\varepsilon"'] & X[1] \arrow[d, equal] \\ X \arrow[r] & B \arrow[r, "\varphi_B"] & Y \arrow[r] & X[1] \end{tikzcd} .$$

Now we can continue arguing as in the last part of the proof of Proposition 2.3.9 to conclude that λ is a homological epimorphism and that $\mathcal{Y}$ is the essential image of λ_*.

□

Combining the previous results we can state the main theorem of this section.

THEOREM 2.3.13. *Let B be a k-flat dg algebra and let Q be a perfect left dg B-module. Let $D = \mathcal{H}om_B(Q,Q)$ and $P = \mathbb{R}\mathrm{Hom}_B(Q,B)$. Then there is a homological epimorphism of dg algebras $F\colon B \to C$ and a recollement:*

$$\begin{tikzcd}[column sep=huge] \mathcal{D}(C) \arrow[r, "F_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=C\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(C,-)"] \arrow[r, "\mathbb{R}\mathrm{Hom}_B(Q,-)\cong P\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(D) \arrow[l, bend right, "Q\overset{\mathbb{L}}{\underset{D}{\otimes}}-"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_D(P,-)"] \end{tikzcd}$$

Moreover, the following hold:

(1) *The triple* $(\mathrm{Tria}\ Q, \mathcal{Y}, \mathcal{Z})$ *with* $\mathcal{Y} = \mathrm{Ker}\ (P\overset{\mathbb{L}}{\underset{B}{\otimes}}-)$ *and* $\mathcal{Z} = \mathrm{Im}\ (\mathbb{R}\mathrm{Hom}_D(P,-))$ *is a TTF triple in $\mathcal{D}(B)$;*
(2) *the essential image of F_* is $\mathcal{Y}$;*
(3) *the functor* $\mathbb{R}\mathrm{Hom}_D(P,-)$ *is fully faithful;*
(4) *$\mathcal{D}(D)$ is triangle equivalent to* $\mathcal{D}(B)/\mathrm{Ker}\ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$.

In particular, if $B \in \mathrm{tria}\ Q$, then $\mathcal{Y}$ vanishes and the functor $\mathbb{R}\mathrm{Hom}_D(P,-)$ *induces an equivalence between $\mathcal{D}(D)$ and $\mathcal{D}(B)$ with inverse* $P \overset{\mathbb{L}}{\underset{B}{\otimes}} -$.

PROOF. (1) See Corollary 2.2.4.
(2) See the last part of the proof of Proposition 2.3.10.
(3) and (4) follow by definition of recollement. □

4. Partial tilting complexes

In this section we specialize the situation illustrated by Theorem 2.3.13 to the case of self-orthogonal compact objects.

Our next result, Theorem 2.4.6, can be viewed as a generalization of the Morita-type theorem proved by Rickard in [**R**] in the sense that we consider partial tilting complexes instead of tilting complexes and dg algebras instead of algebras.

REMARK 2.4.1. Note that some generalizations were obtained also by König in [**K**] in the case of right bounded derived categories of rings. One of the tools used by König is the following: for bounded derived category, there is the following characterization of perfect objects ([**R**, Proof of Proposition 8.1]): let A be a ring and $D^-(A)$ its right bounded derived category. Then X in $\mathcal{D}^-(A)$ is a perfect object if and only if, for each Y in $\mathcal{D}^-(A)$, there exists a natural number N such that $\mathrm{Hom}_{\mathcal{D}^-(A)}(X, Y[n]) = 0$ for each $n \geq N$. Unfortunately, in the case of unbounded derived category this criterion is no more valid.

By Remark 2.2.1 we have that if ${}_BQ$ is a partial tilting left dg B-module, then $P = \mathbb{R}\mathrm{Hom}_B(Q, B)$ is a partial tilting right dg B-module and $P^* = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P, B)$ is isomorphic to Q. Moreover, $D = \mathbb{R}\mathrm{Hom}_B({}_BQ, {}_BQ) \cong \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P_B, P_B)$.

Let us recall the important result proved by Rickard and then by Keller and try to "generalize" it using the considerations above.

THEOREM 2.4.2. [**Ke6**] *Let k be a commutative ring, A and B be k-algebras which are flat as modules over k. The following are equivalent:*

(1) *There is a k-linear triangle equivalence $F : \mathcal{D}(A) \to \mathcal{D}(B)$.*
(2) *There a complex of $A^{op} - B$ modules X such that the functor*

$$X \overset{\mathbb{L}}{\otimes}_A - : \mathcal{D}(A) \to \mathcal{D}(B)$$

is an equivalence.
(3) *There is a complex T of B-modules such that the following conditions hold:*
   i) *T is perfect.*
   ii) *T generates $\mathcal{D}(B)$ as a triangulated category closed under small coproducts.*
   iii) *T is self-orthogonal and* $\mathrm{Hom}_{\mathcal{D}(B)}(T, T) = A$.

The hypotheses of flatness is essential to prove that the complex ${}_BT$ is isomorphic in $\mathcal{D}(B)$ to a complex of A^{op}-B bimodule. In fact the action of A is global on the complex, not on the terms. Thus we need the flatness condition.

REMARK 2.4.3. Rickard's Theorem states that if B is a flat k-algebra over a commutative ring k and P_B is a tilting complex of right B-modules with endomorphism ring A, then there is a complex ${}_AX_B$, with terms that are A-B bimodules, isomorphic to P_B in $\mathcal{D}(B)$, and such that $X \underset{B}{\overset{\mathbb{L}}{\otimes}} -\colon \mathcal{D}(B) \to \mathcal{D}(A)$ is an equivalence with inverse the functor $\mathbb{R}\mathrm{Hom}_A(X, -)$. Equivalences of this form are called *standard equivalences* (see [**Ke1**, Sec. 1.4]). It is still an open problem to decide if all triangle equivalences between derived categories of rings (or dg algebras) are isomorphic to standard equivalence (see [**Ke4**, Sec.6.1]).

In the same assumptions as in Rickard's Theorem, but without any flatness condition on B, our next Theorem 2.4.6 provides an equivalence between $\mathcal{D}(B)$ and $\mathcal{D}(A)$. An analysis of the way in which this equivalence is constructed, shows that it is induced by the composite derived functor $A \underset{D_-}{\overset{\mathbb{L}}{\otimes}} ({}_{D_-}P \underset{B}{\overset{\mathbb{L}}{\otimes}} -)$ where $D_- = \tau_{\leq 0}(D)$ and P is viewed as a dg D_--B-bimodule. Let us note that, without the flatness assumption (or, in particular, without assuming that k is a field, as in Yang's paper, see [**Y**]), we cannot conclude that there is an A-B dg-bimodule ${}_AX_B$ such that $A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} ({}_{D_-}P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) = {}_AX_B \overset{\mathbb{L}}{\underset{B}{\otimes}} -$. In fact, in general, the notation $A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} {}_{D_-}P_B$ may not be regarded as an A-B dg-bimodule, because to have it we would need an $\mathcal{H}$-projective resolution of A (or of P) as A-B bimodule that is also an $\mathcal{H}$-projective resolution just as B-module. But without flatness condition this $\mathcal{H}$-projective resolution may not exist.

DEFINITION 2.4.4. Let B be a dg algebra. A right (left) dg B-module P is called *partial tilting* if it is perfect and self-orthogonal. A right (left) dg B-module P is called *tilting* if it is partial tilting and $B^{\mathrm{op}} \in \operatorname{tria} P$ ($B \in \operatorname{tria} P$).

NOTATIONS 2.4.5. *Let P be a partial tilting right dg B module. Let $D = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P_B, P_B)$ and $A = \mathrm{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(P, P)$. Then, $H^n(D) \cong \mathrm{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(P, P[n]) = 0$, for every $0 \neq n \in \mathbb{Z}$, hence the dg algebra D has homology concentrated in degree zero and $H^0(D) \cong A$. Thus, by* [**Ke4**, Sec. 8.4] *there is a triangle equivalence $\rho\colon \mathcal{D}(D) \to \mathcal{D}(A)$. For later purposes we give explicitly the functors defining this equivalence and its inverse.*

Stalk algebras

Let $\tau_{\leq 0}$ be the truncation functor and consider the subalgebra $D_- = \tau_{\leq 0}(D)$. Then the inclusion $f\colon D_- \to D$ and $\pi\colon D_- \to H^0(D) = A$ are quasi-isomorphisms of dg algebras, inducing equivalences f_* and π_* between the corresponding derived categories. Thus we have the following diagrams:

$$\begin{tikzcd} \mathcal{D}(D) \arrow[r, "f_*"] & \mathcal{D}(D_-) \arrow[l, bend right, "D \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_{D_-}(D,-)"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}(D_-) \arrow[r, bend left, "A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -"] \arrow[r, bend right, "\mathbb{R}\mathrm{Hom}_{D_-}(A,-)"'] & \mathcal{D}(A) \arrow[l, "\pi_*"] \end{tikzcd}$$

So $\rho = (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ f_*$ (with its inverse $\rho^{-1} = (D \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ \pi_*$) is an equivalence between $\mathcal{D}(D)$ and $\mathcal{D}(A)$.

Note that $f_* \cong {}_{D_-}D \overset{\mathbb{L}}{\underset{D}{\otimes}} -$ and $\pi_* \cong {}_{D_-}A \overset{\mathbb{L}}{\underset{A}{\otimes}} -$

THEOREM 2.4.6. *Let B be a dg algebra and let P be a partial tilting right dg B-module. Let $A = \mathrm{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(P, P)$, $Q = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P, B)$. Then there exists a dg algebra E and a recollement:*

$$\begin{tikzcd} \mathcal{D}(E) \arrow[r, "L(B)\otimes^{\mathbb{L}}_E -"] & \mathcal{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_B(L(B),-)"] \arrow[r, "j^*"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where, letting $D = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P,P)$ *there is a triangle equivalence* $\rho\colon \mathcal{D}(D) \to \mathcal{D}(A)$ *such that:*

(1) $j_! = (Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -) \circ \rho^{-1}$;
(2) $j^* = \rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$;
(3) $j_* = \mathbb{R}\mathrm{Hom}_D(P,-) \circ \rho^{-1}$ *is fully faithful;*
(4) *if* $\mathcal{Y} = \mathrm{Ker}\ (j^*)$ *and* $\mathcal{Z} = \mathrm{Im}\ j_*$, *then* $(\mathrm{Tria}\ Q, \mathcal{Y}, \mathcal{Z})$ *is a TTF triple in* $\mathcal{D}(B)$ *and* $\mathcal{Y}$ *is the essential image of* F_*;
(5) $\mathcal{D}(A)$ *is triangle equivalent to* $\mathcal{D}(B)/\mathrm{Ker}\ (j^*)$.
(6) *Moreover, if* B *is* k*-flat there exists a homological epimorphism of dg algebras* $F : B \longrightarrow C$ *such that the above recollement becomes:*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(C) \arrow[r, "F_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=C\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(C,-)"] \arrow[r, "j^*"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

(7) *In particular, if* P *is a tilting right dg* B*-module, then* $\mathcal{Y}$ *vanishes and*

$$\rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)\colon \mathcal{D}(B) \to \mathcal{D}(A)$$

is a triangle equivalence with inverse $\mathbb{R}\mathrm{Hom}_D(P,-) \circ \rho^{-1}$.

PROOF. By Remark 2.2.1 we can identify P with $\mathbb{R}\mathrm{Hom}_B(Q,B)$ and $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P,P)$ with $\mathbb{R}\mathrm{Hom}_B(Q,Q)$.

By Stalk algebras 2.4.5 there is an equivalence $\rho\colon \mathcal{D}(D) \to \mathcal{D}(A)$ given by $\rho = (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ f_*$, with inverse $\rho^{-1} = (D \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ \pi_*$. Hence, if we compose the functor in the right side of the recollement (2.3.13 we get the functors $j_! = (Q \overset{\mathbb{L}}{\underset{D}{\otimes}} -) \circ \rho^{-1}$, $j^* = \rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ and $j_* = \mathbb{R}\mathrm{Hom}_D(P,-) \circ \rho^{-1}$ (that is fully faithful since it is the composition of two fully faithful functors). So points (1), (2), (3) are proved. Now, an application of Corollary 2.2.6 proves points (4) while point (5) derives from the properties of recollements. Finally, if B is k-flat we use Theorem 2.3.13 to prove point (6).
Finally, if P is tilting, then $\mathcal{Y}$ vanishes. In fact, assume that there exists $M \in \mathcal{D}(B)$, such that $M \in \mathcal{Y}$. Then, since $B \in \mathrm{tria}P$, $B \overset{\mathbb{L}}{\otimes}_B M = 0$, that is $M = 0$ in $\mathcal{D}(B)$. $\square$

5. Tilting and partial tilting modules

In this section we concentrate on the connection between recollements and tilting (or partial tilting) modules. Indeed, using the results of the previous sections we characterize recollements induced by partial n-tilting modules. Moreover we recall results in [**AKL**] that show the correspondence between tilting objects and recollements.

We recall the definition of (partial) tilting modules over a ring R by using the canonical embedding of the category of R-modules into the derived category $\mathcal{D}(R)$ and we restate the various definitions using the terminology of derived categories theory.

Definition 2.5.1. Let R be a ring and T an R-module. Consider the following conditions on T viewed as an object of $\mathcal{D}(R)$ under the canonical embedding:

(T1) T is isomorphic to a bounded complex with projective terms;
(T1') T is a compact object of $\mathcal{D}(R)$;
(T2) T is orthogonal to coproducts of copies of T, that is $\mathrm{Hom}_{\mathcal{D}(R)}(T, T^{(\alpha)}[n]) = 0$ for every $0 \neq n \in \mathbb{Z}$ and every set α.
(T2') T is self-orthogonal, that is $\mathrm{Hom}_{\mathcal{D}(R)}(T, T[n]) = 0$ for every $0 \neq n \in \mathbb{Z}$.
(T3) $R \in \mathrm{Tria}\ T$.
(T3') $R \in \mathrm{tria}\ T$.

If the projective dimension of T is at most n, then T is called a *classical n-tilting* module if it satisfies (T1'), (T2') and (T3'), and a *classical partial n-tilting* module if it satisfies (T1') and (T2'). T is called an *n-tilting* module (possibly infinitely generated), if it satisfies (T1), (T2) and (T3) and it is called a *good n-tilting* module if it satisfies (T1), (T2) and (T3').

In [**H**] and [**CPS**] it was shown that a classical n-tilting module over an artin algebra A with endomorphism algebra B induces a triangle equivalence between $\mathcal{D}(A)$ and $\mathcal{D}(B)$.

The following theorem shows the construction of a recollement in the derived category of the ring A, starting from an infinitely generated 1-tilting module on A.

Remark 2.5.2. Let T be a 1-tilting module over a ring A. Then there is a short exact sequence: $0 \to A \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{Add}\ T$. It is well known, by results in [**BH**], that to every 1-tilting module ${}_AT$ in A-Mod is associated a class $\mathcal{C}$ of finitely presented modules of projective dimension one (in particular $\mathcal{C}$ consists of perfect objects) such that $\mathcal{C}^{\perp} = \mathrm{Gen}T$. Then in [**AA**] it is proved that $\mathcal{C}^{\perp} = \mathrm{Gen}T = \mathrm{KerExt}^1_A(T_1, -)$. So $\mathrm{Tria}\ \mathcal{C}$ is a smashing subcategory of $\mathcal{D}(A)$ and $\mathcal{C}^{\perp} = \mathrm{KerExt}^1_A(T_1, -)$.

Theorem 2.5.3. [**AKL**, Theorem 4.8] *Let A be a ring and T a 1-tilting A-module. Then there is a class $\mathcal{C}$ of finitely presented modules of projective dimension one and a module $T_1 \in \mathrm{Add}\ (T)$ such that, there is a recollement:*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D}(A) \arrow[l, bend right=50, "L_{\mathcal{Y}}"'] \arrow[l, bend left=50] \arrow[r] & \mathcal{X} \arrow[l, bend right=50, "i_{\mathcal{X}}"'] \arrow[l, bend left=50]
\end{tikzcd}$$

where $\mathcal{Y} = \mathrm{KerExt}^1_A(T_1, -)$ and $\mathcal{X} = \mathrm{Tria}\ \mathcal{C}$.

Proof. See Remark 2.5.2 and Proposition 1.2.17. $\square$

Infinitely generated tilting modules do not provide equivalences between derived categories of rings, but Bazzoni proved in [**B**] that, if T is a good 1-tilting module over a ring A with endomorphism ring B, then the total left derived functor $T \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ induces an equivalence between $\mathcal{D}(B)/\mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ and $\mathcal{D}(A)$. This result has been generalized in [**BMT**] to the case of good n-tilting modules and in [**Y**] in the more general setting of dg categories. Let us note that, in [**AKL**] the recollement is given in $\mathcal{D}(A)$, with T tilting over A. In [**BMT**] they start with a good n-tilting module T over A, and they exhibit a TTF triple in $\mathcal{D}(\mathrm{End}_A(T))$.

Theorem 2.5.4. [**BMT**] *Let ${}_AT$ be a good n-tilting module and $B := End({}_AT)$. Then the following hold:*

(1) *The counit of the adjunction morphism*

$$(T_B \overset{\mathbb{L}}{\otimes} -) \circ \mathbb{R}\mathrm{Hom}_A({}_AT, -) \longrightarrow Id_{\mathcal{D}(A)}$$

is invertible.

(2) *Let us set $\mathcal{Y} := \mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes} -)$. Then there is a triangle equivalence*

$$\mathcal{D}(B)/\mathcal{Y} \longrightarrow \mathcal{D}(A).$$

Proof. (1) We prove that $T_B \overset{\mathbb{L}}{\otimes} \mathbb{R}\mathrm{Hom}_A({}_AT, M) \simeq {}_AM$ for all $M \in \mathcal{D}(A)$. Let $\underline{i}M$ a $\mathcal{H}$-injective resolution of M in $\mathcal{D}(A)$, then:

$$T_B \overset{\mathbb{L}}{\otimes} \mathbb{R}\mathrm{Hom}_A({}_AT, M) = T_B \otimes_B \mathrm{Hom}_A({}_AT, \underline{i}M).$$

Now, by [**Mi**, Lemma 1.8], for each integer n there is a natural isomorphism: $T_B \otimes_B \mathrm{Hom}_A({}_AT, (\underline{i}M)^n) \simeq (\underline{i}M)^n$. Hence $T_B \otimes_B \mathrm{Hom}_A({}_AT, \underline{i}M) \simeq \underline{i}M$.

(2) By point (1) and [**GZ**, Proposition 1.3].

□

Corollary 2.5.5. *Let ${}_AT$ be a good n-tilting module and $B := End({}_AT)$. Let us set $\mathcal{Y} = \mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes} -)$. Then there is a recollement of the form:*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] & \mathcal{D}(B) \arrow[l, bend right, "i_{\mathcal{Y}}"'] \arrow[l, bend left] \arrow[r, "j^*"] & \mathcal{D}(B)/\mathcal{Y} \simeq \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Proof. We have that $\mathcal{Y}$ is the central class of a TTF triple $({}^{\perp}\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^{\perp})$, where ${}^{\perp}\mathcal{Y} \simeq \mathcal{D}(B)/\mathcal{Y}$, then, by Proposition 1.2.17, we can conclude. □

We will prove now the same result as in the previous theorem, but with weaker hypotheses, namely without asking that it is a good n-tilting A-module, but only that it satisfies conditions (T2') and (T3'). In our approach, indeed, we can fix a ring B and obtain recollements of $\mathcal{D}(B)$ for every choice of classical partial tilting modules.

Moreover we want to point out that the disadvantage of starting with an infinitely generated n-tilting module ${}_AT$ over a ring A, is that a good n-tilting module T' equivalent to ${}_AT$ is obtained as a summand of a possibly infinite direct sum of copies of T and this procedure produces a very large endomorphism ring B of T'. So the recollement induced by T' concerns the derived category of a ring which is hardly under control. More precisely an instance of Theorem 2.4.6 yields the following generalization of Theorem 2.5.4.

THEOREM 2.5.6. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. There is a dg algebra E and a recollement*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(E) \arrow[r, "L(B)\overset{\mathbb{L}}{\otimes}_E -"] & \mathcal{D}(B) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "\mathbb{R}\mathrm{Hom}_B(L(B){,}-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B -"] & \mathcal{D}(A) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*=\mathbb{R}\mathrm{Hom}_A(T{,}-)"]
\end{tikzcd}$$

where:

(1) $j_* = \mathbb{R}\mathrm{Hom}_A(T, -)$ *is fully faithful;*

(2) $\mathcal{D}(A)$ *is triangle equivalent to* $\mathcal{D}(B)/\mathrm{Ker}\ (T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$.

Moreover, if B is k-flat, there is a homological epimorphism of dg algebras $F\colon B \to C$ and the recollement above becomes

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(C) \arrow[r, "F_*"] & \mathcal{D}(B) \arrow[l, bend right=40, "i^*=C\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left=40, "i^!=\mathbb{R}\mathrm{Hom}_B(C{,}-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B -"] & \mathcal{D}(A) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*=\mathbb{R}\mathrm{Hom}_A(T{,}-)"]
\end{tikzcd}$$

PROOF. Let P be a projective resolution of the module T in Mod-B. Then P is a partial tilting complex of $\mathcal{D}(B^{op})$ so that we may apply Theorem 2.4.6 which states that there is a triangle equivalence $\rho\colon \mathcal{D}(D) \to \mathcal{D}(A)$ where $D = \mathbb{R}\mathrm{Hom}_{B^{op}}(P, P)$.

As shown in Stalk algebras 2.4.5 we have:

$$\rho = (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ f_*.$$

where $f_*\colon \mathcal{D}(D) \to \mathcal{D}(A)$ is the restriction of scalar functors induced by the quasi-isomorphism of dg algebras $f\colon D_- \to D$

To conclude the proof we must show that

$$(a) \quad \rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \cong T \overset{\mathbb{L}}{\underset{B}{\otimes}} -,$$

$$(b) \quad \mathbb{R}\mathrm{Hom}_D(P, -) \circ \rho^{-1} \cong \mathbb{R}\mathrm{Hom}_A(T, -).$$

We first prove (a).

Let $\sigma\colon P_B \to T_B$ be a morphism of complexes inducing a quasi-isomorphsm in $D(B)$. From the dg algebra morphisms $f\colon D_- \to D$ and $\pi\colon D_- \to A$ we have that P

and T are left dg D_--modules. Checking the action of the dg algebra D_- on P and T we see that σ is a morphism of dg D_--modules. Thus, σ is a quasi isomorphism between P and T as dg D_--B-bimodules.

This implies that the functors $P \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ and $T \overset{\mathbb{L}}{\underset{B}{\otimes}} -$ from $\mathcal{D}(B)$ to $\mathcal{D}(D_-)$ are isomorphic (see [**Ke2**, Lemma 6.1 b]). Consequently, , in the notations of Stalk algebras 2.4.5, we have:

$$j^* = \rho \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) = (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ f_* \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \cong (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ ({}_{D_-}P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \cong$$

$$\cong ((A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ ({}_{D_-}T \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \cong (A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ \pi_* \circ ({}_AT \overset{\mathbb{L}}{\underset{B}{\otimes}} -).$$

Since $(A \overset{\mathbb{L}}{\underset{D_-}{\otimes}} -) \circ \pi_*$ is isomorphic to the identity of $\mathcal{D}(A)$, we conclude that $j^* \cong_A T \overset{\mathbb{L}}{\underset{B}{\otimes}} -$.

Next, from the uniqueness of right adjoint up to isomorphisms, we also get

$$\mathbb{R}\mathrm{Hom}_D(P, -) \circ \rho_*^{-1} \cong \mathbb{R}\mathrm{Hom}_A(T, -).$$

□

NOTE 2.5.7. In the assumption of Theorem 2.5.6 if we let $Q = \mathbb{R}\mathrm{Hom}_{B^{op}}(P, B)$, then, by Remark 2.2.1 (1) we have $\mathbb{R}\mathrm{Hom}_B(Q, -) \cong P \overset{\mathbb{L}}{\underset{B}{\otimes}} -$, hence also

$$f_* \circ \mathbb{R}\mathrm{Hom}_B(Q, -) \cong f_* \circ (P \overset{\mathbb{L}}{\underset{B}{\otimes}} -) \cong {}_AT \overset{\mathbb{L}}{\underset{B}{\otimes}} -.$$

We translate now the recollement of Theorem 2.5.6 in terms of its associated TTF triple.

COROLLARY 2.5.8. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let $Q = \mathbb{R}\mathrm{Hom}_B(T, B)$. The triple:*

$$(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \left(\mathrm{Tria}\ Q,\ Q^\perp = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -),\ \mathrm{Im}\ \mathbb{R}\mathrm{Hom}_A(T, -)\right)$$

is a TTF triple in $\mathcal{D}(B)$ and the left adjoint of the inclusion functor of $\mathcal{Z}$ in $\mathcal{D}(B)$ is given by $L_{\mathcal{Z}} = HG$ where $H = \mathbb{R}\mathrm{Hom}_A(T, -)$ and $G = T \overset{\mathbb{L}}{\underset{B}{\otimes}} -$.

PROOF. Follows by Corollary 2.2.4, Theorem 2.5.6 and the properties of recollements.

□

Let us conclude this section with the result by Angeleri, König and Liu on the possibility to construct tilting objects, starting from recollements of derived categories. These results cover and generalize the concept of Bongartz complement ([**Bo**]).

Recall that an object T in $\mathcal{D}(B)$ is called *tilting* if it is perfect, self-orthogonal and $B \in \mathrm{Tria}\ T$.

REMARK 2.5.9. A tilting object concentrated in degree zero is a finitely generated tilting module over a ring B.

THEOREM 2.5.10. [**AKL**, Theorem 2.4] *Let $\mathcal{D}$ be a triangulated category with small coproducts, admitting a TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and so a recollement of the form:*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D} \arrow[l, bend right=50, "L_{\mathcal{Y}}"'] \arrow[l, bend left=50, "R_{\mathcal{Y}}"] \arrow[r, "R_{\mathcal{X}}"] & \mathcal{X} \arrow[l, bend right=50, "i_{\mathcal{X}}"'] \arrow[l, bend left=50, "i_{\mathcal{Z}} L_{\mathcal{Z}} i_{\mathcal{X}}"]
\end{tikzcd}$$

Suppose, moreover, that there exists a self-orthogonal generator Q of $\mathcal{X}$ and a tilting object P in $\mathcal{Y}$ such that $\mathrm{Hom}_{\mathcal{D}}(P, Q[n]) = 0$ *for every integer $n \neq 0, 1$. Set $I :=$* $\mathrm{Hom}_{\mathcal{D}}(P, Q[1])$*. Consider the map $f : P^{(I)} \to Q[1]$ and let*

$$Q \to T \to P^{(I)} \xrightarrow{f} Q[1]$$

be the triangle determined by f in $\mathcal{D}$. Then $T \oplus P$ is a self-orthogonal generator of $\mathcal{D}$.

REMARK 2.5.11. The condition $\mathrm{Hom}_{\mathcal{D}}(P, Q[n]) = 0$, for every integer $n \neq 0, 1$, is a generalization of the projective dimension one of a tilting module.

CHAPTER 3

Recollements of derived categories of rings

In this chapter we are interested in characterizing the case in which the subcategory $\mathcal{Y} := Q^{\perp}$ in the TTF triple of Lemma 2.5.8 is the derived category of a ring. The problem is related to the notion of bireflective and perpendicular categories. In particular we will prove that, if $\mathcal{Y} \simeq \mathcal{D}(S)$ for a ring S via a homological ring epimorphism, then S is the "generalized universal localization" of B at P, where P is the projective resolution of T_B.

1. Bireflective subcategories and ring epimorphisms

In this section we will recall the notion and the characterization of bireflective subcategories and the well known bijection existing between equivalence classes of ring epimorphisms and bireflective subcategories of module categories.

Definition 3.1.1. Let $\mathcal{E}$ be a full subcategory of R-Mod. A morphism $f : M \longrightarrow E$, with E in $\mathcal{E}$, is called an *$\mathcal{E}$-reflection* if for every map $g : M \longrightarrow E'$, with E' in $\mathcal{E}$, there is a unique map $h : E \longrightarrow E'$ such that $hf = g$. A subcategory $\mathcal{E}$ of R-Mod is said to be *reflective* if every R-module X admits an $\mathcal{E}$-reflection. The definition of *coreflective* subcategory is given dually. A subcategory that is both reflective and coreflective is called *bireflective*.

Remark 3.1.2. It is clear that a full subcategory $\mathcal{E}$ of R-Mod is reflective if and only if the inclusion functor $i : \mathcal{E} \longrightarrow R$-Mod admits a left adjoint

$$\begin{tikzcd} \mathcal{E} \arrow[r, "i"] & R\text{-Mod} \arrow[l, bend right, "l"'] \end{tikzcd}$$

Moreover, in this case, $l(R)$ is a generator of $\mathcal{E}$ (i.e. every object M in $\mathcal{E}$ can be seen as the image of a homomorphism $l(R)^{(I)} \to M$). Indeed, for every $Y \in \mathcal{E}$:

$$\mathrm{Hom}_{\mathcal{E}}(l(R), Y) = \mathrm{Hom}_R(R, i(Y)) = 0$$

if and only if $Y = 0$. Dually, a subcategory $\mathcal{X}$ is coreflecting if and only if the inclusion functor $j : \mathcal{X} \longrightarrow R$-Mod admits a right adjoint:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "j"] & R\text{-Mod} \arrow[l, bend left, "r"] \end{tikzcd}$$

Proposition 3.1.3. *([**GL**] and [**GP**]) Let $\mathcal{E}$ be a full subcategory of B-Mod. The following assertions are equivalent:*

1) *$\mathcal{E}$ is a bireflective subcategory of B-Mod;*

2) *there is a ring epimorphism* $f : B \longrightarrow S$ *such that* $\mathcal{E}$ *is the essential image of the restriction of scalars functor* $f_* : S\text{-Mod} \to B\text{-Mod}$.

3) $\mathcal{E}$ *is closed under isomorphic images, direct sums, direct products, kernels and cokernels.*

In particular there is a bijection between the bireflective subcategory of B-Mod and the equivalence classes of ring epimorphisms starting from B. *Moreover the map* $f : B \longrightarrow S$ *as in* 2) *is an* $\mathcal{E}$*-reflection.*

Proof. 1) $\Rightarrow$ 2) Set $l : B\text{-Mod} \to \mathcal{E}$ the left adjoint of the inclusion functor of $\mathcal{E}$ in B-Mod. Then $\mathrm{Hom}_B(B, M) = \mathrm{Hom}_B(l(B), M) = \mathrm{Hom}_{\mathcal{E}}(l(B), M)$ for every $M \in \mathcal{E}$. Thus $l(B)$ is a progenerator of $\mathcal{E}$, that is a finitely generated projective module in $\mathcal{E}$ and every object $N \in \mathcal{E}$ can be seen as the image of an homomorphism $f : l(B)^{(I)} \to N$ for some set I. Set $S := \mathrm{End}_B(l(B))$. Then there is an equivalence between $\mathcal{E}$ and S-Mod:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathrm{Hom}_B(l(B),-)"] & S\text{-Mod} \arrow[l, bend right=60, "l(B)\otimes_B -"'] \end{tikzcd}$$

Let us regard B as $\mathrm{End}_B(B)$ and denote by $\dot{b}$ the multiplication by the element $b \in B$. We want to prove that the ring homomorphism
$f : B \to S : \dot{b} \mapsto l(\dot{b})$ is a ring epimorphism. In particular we have to prove that the restriction of scalars functor is fully faithful $f_* = l(B) \otimes_S -$, that is: $\mathrm{Hom}_S(M, N) = \mathrm{Hom}_B(f_*(M), f_*(N))$ for every $M, N \in S$-Mod. We have:

$$\mathrm{Hom}_B(l(B) \otimes_S M, l(B) \otimes_S N) = \mathrm{Hom}_S(M, \mathrm{Hom}_B(l(B), l(B) \otimes_S N)) = \mathrm{Hom}_S(M, N)$$

where we have used that $(l(B) \otimes_S -, \mathrm{Hom}_B(l(B), -))$ is an equivalence.

2) $\Rightarrow$ 3) Since f_* is fully faithful, $f_*(S\text{-Mod})$ is an abelian subcategory of B-Mod.
3) $\Rightarrow$ 1) See [**GP**].

□

Theorem 3.1.4. [**Sc**] *Let* Σ *be a set of morphisms between finitely generated projective left* B*-modules. Then there are a ring* B_Σ *and a morphism of rings* $f : B \to B_\Sigma$ *such that:*

(1) f *is* Σ *inverting, that is if* $g : M \to N$ *is in* Σ, *then*

$$g \otimes_B Id_{B_\Sigma} : B_\Sigma \otimes_B M \to B_\Sigma \otimes_B N$$

is an isomorphism of left B_Σ*-modules.*

(2) f *is universal with respect to this property, that is if* S *is a ring such that there exists a* Σ*-inverting morphism* $\phi : B \to S$, *then there exists a unique morphism of rings* $\psi : B_\Sigma \to S$ *such that* $\psi f = \phi$.

Remark 3.1.5. The ring B_Σ is called *universal localization* of B at Σ and the morphism f is a ring epimorphism. If Σ is a set of maps between finitely generated projective B-modules, let us set $\mathcal{C}$ the set of these objects. Then B_Σ will also be denoted by $B_{\mathcal{C}}$ and we call it the *universal localization* of B with respect to $\mathcal{C}$. If $\mathcal{E}$ is a class of finitely presented B-module of projective dimension one, then, for every M in $\mathcal{E}$, we have a projective resolution $0 \to P_{0M} \overset{d_M}{\to} P_{1M} \to M \to 0$ with

P_{0M} and P_{1M} finitely generated projective modules. Then we can regard $\mathcal{E}$ as the class of maps d_M between the projective modules P_{0M}, P_{1M}, with M in $\mathcal{E}$, and thus it makes sense to consider the universal localization $B_{\mathcal{E}}$.

Notations 3.1.6. *Let $\mathcal{C}$ be a class of left B-modules. We denote with $\mathcal{C}^{\perp}$ the subcategory of B-*Mod

$$\mathcal{C}^{\perp} = \{M \in B\text{-Mod} \mid \mathrm{Hom}_B(C, M) = \mathrm{Ext}^i_B(C, M) = 0,\ \forall\ C \in \mathcal{C}\}.$$

Proposition 3.1.7. [**AA**, Proposition 1.7] *Let $\mathcal{C}$ be a set of finitely presented left B-modules of projective dimension at most* 1*. Then $\mathcal{C}^{\perp}$ coincides with the essential image of the restriction functor $B_{\mathcal{C}}$-*Mod $\to B$-Mod *induced by the universal localization at $\mathcal{C}$. In particular $\mathcal{C}^{\perp}$ is bireflective.*

Corollary 3.1.8. *If the subcategory $\mathcal{E}$ of Lemma 3.1.3 is the perpendicular subcategory of a class $\mathcal{C}$ of finitely presented B-modules of projective dimension one, then $\mathcal{E}$ coincides with the essential image of the restriction functor $B_{\mathcal{C}}$-*Mod $\to$ B-Mod .

Let us recall the following result that connects recollements of triangulated categories with universal localization.

Proposition 3.1.9. ([**CX**, Proposition 3.5]) *With the same notations as in Corollary 3.1.8, let us denote by $f : B \to B_{\mathcal{C}}$ the ring epimorphism defining the universal localization and set*

$$\mathcal{Y} = \{M \in \mathcal{D} \mid H^n(M) \in \mathcal{C}^{\perp},\ \forall\ n \in \mathbb{Z}\}.$$

Then there is a recollement:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "i_{\mathcal{Y}}"] & \mathcal{D}(B) \arrow[l, bend right=40, "L_{\mathcal{Y}}"'] \arrow[l, bend left=40] \arrow[r] & \mathrm{Tria}\ \mathcal{C} \arrow[l, bend right=40] \arrow[l, bend left=40]
\end{tikzcd}$$

such that $L_{\mathcal{Y}}$ is the left adjoint of the inclusion functor $i_{\mathcal{Y}}$. Moreover the following are equivalent:

(1) *$f : B \to B_{\mathcal{C}}$ is homological.*
(2) *f determines an equivalence $f_* : \mathcal{D}(B_{\mathcal{C}}) \to \mathcal{Y}$.*
(3) *$L(B) \simeq B_{\mathcal{C}}$ in $\mathcal{D}(B)$.*
(4) *$L(B)$ is quasi-isomorphic in $\mathcal{D}(B)$ to a complex with terms in $\mathcal{E}$.*

Remark 3.1.10. Under the same hypotheses as in Proposition 3.1.9, the recollement 3.1.9 of $\mathcal{D}(B)$ is equivalent to:

$$\begin{tikzcd}
\mathcal{D}(B_{\mathcal{C}}) \arrow[r, "f^*"] & \mathcal{D}(B) \arrow[l, bend right=40, "B_{\mathcal{C}}\overset{\mathbb{L}}{\otimes}_B -"'] \arrow[l, bend left=40, "\mathbb{R}\mathrm{Hom}_B(B_{\mathcal{C}}{,}-)"] \arrow[r] & \mathrm{Tria}\ \mathcal{C} \arrow[l, bend right=40] \arrow[l, bend left=40]
\end{tikzcd}$$

Let us take now a classical partial n-tilting module T_B and set $A := \mathrm{End}_B(T_B)$. How can we use these results on bireflective subcategories and homological epimorphisms to study the recollement induced by T_B? Associated to T_B we can define the following full subcategory of B-Mod (consider the canonical embedding of B-Mod in $\mathcal{D}(B)$):

$$\mathcal{E} = \{N \in B\text{-Mod} \mid N \in \mathcal{Y}\} = \{N \in B\text{-Mod} \mid T \overset{\mathbb{L}}{\underset{B}{\otimes}} N = 0\}$$

Then, $\mathcal{E} = \{N \in B\text{-Mod} \mid \mathrm{Tor}_i^B(T, N) = 0 \ \text{ for all } i \geq 0\}$.

Remark 3.1.11. Note that $\mathcal{E}$ is closed under extensions, direct sums and direct products (since T is a perfect object in $\mathcal{D}(B)$). So $\mathcal{E}$ is bireflective if and only if it is closed under kernel and/or cokernels. Moreover, if $n = 1$, then $\mathcal{E}$ is always bireflective. Indeed, for every morphism $f : M \to N$ in $\mathcal{E}$, if we apply the functor $(T_B \otimes_B -)$ to the short exact sequences

$$0 \to \mathrm{Ker} f \to M \to \mathrm{Im} f \to 0$$

$$0 \to \mathrm{Im} f \to N \to \mathrm{Coker} f \to 0$$

we have that $\mathrm{Ker} f$ and $\mathrm{Coker} f$ are in $\mathcal{E}$.

In the case of $n = 1$, if ${}_AT$ is a good 1-tilting module, we have the following important result proved in [**CX**] that characterizes $\mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -)$ in terms of $\mathcal{E}$ and prove that the left term of the recollement induced by T_B is the derived category of a ring.

Proposition 3.1.12. [**CX**, Proposition 4.6, Theorem 1.1] *Let ${}_AT$ be a good 1-tilting module over a ring A, with endomorphism ring B. Then:*

$$\mathcal{Y} := \mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -) = \{Y' \in \mathcal{D}(B) \mid Y' \simeq Y \quad \textit{such that} \quad Y^n \in \mathcal{E} \quad \forall n \in \mathbb{Z}\}.$$

Moreover there is a ring $S := \mathrm{End}_B(L(B), L(B))$ and a homological ring epimorphism

$$F : S \to B$$

that gives the recollement

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(S) \arrow[r, "F^*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=S\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(S,-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*=\mathbb{R}\mathrm{Hom}_A(T,-)"]
\end{tikzcd}$$

Moreover S is the universal localization of B at the projective resolution P of T_B.

We partially generalize these results to a classical partial n-tilting module T_B (with n possibly grater than one). When $n > 1$ the problem is that $\mathcal{E}$ may not be bireflective (indeed for the non vanishing of $\mathrm{Tor}_2^B(T, -)$ the techniques used in

Remark 3.1.11 are now not sufficient). In the last section we will present some examples of this situation.

THEOREM 3.1.13. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$, L the left adjoint of the inclusion $i\colon \mathcal{Y} \to \mathcal{D}(B)$ and $\mathcal{E}$ the subcategory of B-*Mod *defined above.*

Then the following conditions are equivalent:

(1) *$H^i(L(B)) = 0$ for every $0 \neq i \in \mathbb{Z}$.*
(2) *there is a ring S and a homological ring epimorphism $\lambda\colon B \to S$ inducing a recollement:*

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(S) \arrow[r, "\lambda_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=S\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(S{,}-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*=\mathbb{R}\mathrm{Hom}_A(T{,}-)"]
\end{tikzcd}$$

(3) *Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$ and $\mathcal{E}$ is a bireflective subcategory of B-*Mod.
(4) *Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$ and the homologies of N belong to $\mathcal{E}$.*

PROOF. Note that the equivalence between (1) and (2) was somehow known to topologists, as shown for instance in [**D**].
(1) $\Rightarrow$ (2) Let $Y = L(B)$. First note that, by adjunction, we have $\mathrm{Hom}_{\mathcal{D}(B)}(Y, Y[i]) \cong \mathrm{Hom}_{\mathcal{D}(B)}(B, Y[i]) \cong H^i(Y)$. Thus, by [**Ke4**, Theorem 8.7], condition (1) implies that the dg algebra $E = \mathbb{R}\mathrm{Hom}_B(Y, Y)$ has homology concentrated in degree zero and $H^0(E) \cong \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y)$.

Consider a triangle

$$X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1], \quad \text{with } X \in {}^{\perp}\mathcal{Y}. \tag{10}$$

where φ_B is the unit of the adjunction morphism and set $S = \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y)$. As in [**AKL**, Proposition 1.7], define a ring homomorphism $\lambda\colon B \to S$ by $\lambda(b) = L(\dot{b})$, where $\dot{b}$ denotes the right multiplication by b on B. We have ${}_BS = \mathrm{Hom}_{\mathcal{D}(B)}(Y, Y) \cong \mathrm{Hom}_{\mathcal{D}(B)}(B, Y) \cong H^0(Y) \cong Y$. So we have a quasi-isomorphism $\varepsilon\colon {}_BS \to_B Y$ and from the definition one sees that $\varepsilon \circ \lambda = \varphi_B$. Thus we have an isomorphism of triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & B \arrow[r, "\lambda"] \arrow[d, equal] & S \arrow[r] \arrow[d, "\varepsilon"'] & X[1] \arrow[d, equal] \\
X \arrow[r] & B \arrow[r, "\varphi_B"] & Y \arrow[r] & X[1]
\end{tikzcd}$$

Now we can continue arguing as in the last part of the proof of Proposition 2.3.9 to conclude that λ is a homological epimorphism and that $\mathcal{Y}$ is the essential image of λ_*. So condition (2) follows.

(2) $\Rightarrow$ (3) $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$ is the essential image of the functor λ_*, hence the image of S-Mod under λ_* is the category $\mathcal{E}$. Every object in $\mathcal{Y}$ is quasi-isomorphic to a complex with S-modules terms, hence in $\mathcal{E}$. Moreover, since λ is an epimorphism of rings, the differentials are S-module morphisms. Hence, Lemma 3.1.3 tells us that $\mathcal{E}$ is bireflective.

(3) $\Rightarrow$ (4) Clear from the fact that $\mathcal{E}$ is closed under kernel and cokernels.

(4) $\Rightarrow$ (1) We first show that condition (4) implies that $\mathcal{E}$ is bireflective. Indeed, let $E_0 \xrightarrow{f} E_1$ be a morphism in $\mathcal{E}$. Then, the complex $E' = \dots 0 \to E_0 \xrightarrow{f} E_1 \to 0 \to \dots$ has $(T \underset{B}{\otimes} -)$-acyclic terms so $T \overset{\mathbb{L}}{\underset{B}{\otimes}} E' = T \underset{B}{\otimes} E' = 0$. By (4) the kernel and the cokernel of f belong to $\mathcal{E}$. Thus $\mathcal{E}$ is bireflective by Remark 3.1.11. By Lemma 3.1.3 there is a ring S and a ring epimorphism $\lambda \colon R \to S$ such that $\mathcal{E} = \lambda_*(S\text{-Mod})$ where $\lambda_* \colon S\text{-Mod} \to B\text{-Mod}$ is the restriction functor.

We show now that $L(B) \cong \lambda_*(S)$.

To this aim we follows the arguments used in [**CX**, Proposition 3.6]. Let Y_0 be a complex in $\mathcal{Y}$ with terms in $\mathcal{E}$ and quasi-isomorphic to $L(B)$. Let $B \xrightarrow{\varphi} Y_0$ be the unit adjunction morphism associated to the adjoint pair (L, i). Since S viewed as a left B-module belongs to $\mathcal{Y}$ we have that $\mathrm{Hom}_{\mathcal{Y}}(Y_0, S) \cong \mathrm{Hom}_{\mathcal{D}(B)}(B, S)$, so there is a unique morphism $f \colon Y_0 \to S$ such that $\lambda = f \circ \varphi$.

We have $\mathrm{Hom}_{\mathcal{H}(B)}(S, Y_0) \cong H^0(\mathrm{Hom}_B(S, Y_0))$ and, since $\lambda \colon B \to S$ is a ring epimorphism, $\mathrm{Hom}_B(S, Y_0) = \mathrm{Hom}_S(S, Y_0)$, and the terms of Y_0 are S-modules. Thus, $\mathrm{Hom}_{\mathcal{H}(B)}(S, Y_0) \cong H^0(Y_0) \cong \mathrm{Hom}_{\mathcal{H}(B)}(B, Y_0)$. Now, every morphism in $\mathrm{Hom}_{\mathcal{D}(B)}(S, Y_0)$ is the image under the canonical quotient functor of a morphism in $\mathrm{Hom}_{\mathcal{H}(B)}(S, Y_0)$, hence going through the construction of the above isomorphisms, we conclude that there is $g \in \mathrm{Hom}_{\mathcal{D}(B)}(S, Y_0)$ such that $g \circ \lambda = \varphi$. Consequently, $g \circ f \circ \varphi = \varphi$ and $\lambda = f \circ g \circ \lambda$. Since λ is an $\mathcal{E}$-reflection of B and φ is the unit morphism of the adjunction, we conclude that $f \circ g = id_S$ and $g \circ f = id_{Y_0}$. So $S \cong Y_0 \cong L(B)$, hence (1) follows.

$\square$

Remark 3.1.14. Note that if condition (2) of Proposition 3.1.13 holds, then there is a homological ring epimorphism $\lambda \colon B \to S$ even without the assumption of flatness on B. The key point is the existence of a quasi-isomorphism between the ring S and $L(B)$ (compare with Remark 2.3.11).

We add another property related to the situation considered above.

Proposition 3.1.15. *In the notations of Theorem 3.1.13 consider the following condition:*

- (*a*) *a complex of $\mathcal{D}(B)$ belongs to $\mathcal{Y}$ if and only if all its homologies belong to $\mathcal{E}$.*
- (*b*) *$\mathcal{E}$ is bireflective.*
- (*c*) *There is a ring R and a ring epimorphism $\mu \colon B \to R$ such that ${}_BR \in \mathcal{E}$ and $\mathcal{Y}$ is contained in the essential image of the restriction functor $\mu_* \colon \mathcal{D}(R) \to \mathcal{D}(B)$.*

Then (a) implies (b) and (a) together with (c) is equivalent to any one of the conditions in Theorem 3.1.13.

In particular, if ${}_AT$ is a good n-tilting module with endomorphism ring B, then (a) is equivalent to any one of the conditions in Theorem 3.1.13 .

PROOF. Assume that condition (a) holds. Arguing as in the first part of the proof of (4) $\Rightarrow$ (1) in Theorem 3.1.13, we see that $\mathcal{E}$ is bireflective.

Condition (c) imply $\mu_*(R\text{-Mod}) \subseteq \mathcal{E}$, hence every complex in $\mathcal{Y}$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$. Thus, assuming both (a) and (c), we have that condition (4) in Theorem 3.1.13 is satisfied.

Conversely, if condition (2) of Theorem 3.1.13 is satisfied, then by [**AKL**, Lemma 4.6] (a) holds; moreover, (c) is satisfied by choosing the ring epimorphism $\lambda\colon B \to S$.

To prove the last statement it is enough to show that, if ${}_AT$ is a good n-tilting module then condition (c) holds. This follows as in the proof of [**CX**, Proposition 4.6], which is stated for the case of 1-good tilting module, but the argument used there works also in case of higher projective dimension.

$\square$

2. Generalized universal localization

As recalled in the previous section, Chen and Xi in [**CX**] consider the case of a good 1-tilting module ${}_AT$ with endomorphism ring B. In particular T_B becomes a classical partial 1-tilting module over B. They show that $\mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -)$ is equivalent to the derived category of the universal localization of B at the projective resolution P of T_B (see Theorem 3.1.12).

If S is a universal localization for a morphism $P_1 \xrightarrow{f} P_0$ between finitely generated projective right modules, then the morphism $f \otimes_B S$ is an isomorphism, hence the complex

$$\ldots \to 0 \to P_1 \underset{B}{\otimes} S \overset{f\otimes_B S}{\to} P_0 \underset{B}{\otimes} S \to 0 \to \ldots$$

is acyclic.

Inspired by the above interpretation of universal localization, there is a natural way to generalize this notion as follows. We define the concept, which was first introduced by Krause under the name "homological localization". This notion was given in connection with the *Chain map lifting problem*, presented by Neeman and Ranicki in [**NR**].

DEFINITION 3.2.1. (See [**Kr**, Section 15]) Let B be a ring and Σ be a set of perfect complexes $P \in \mathcal{H}(B)$. A ring S is a *generalized universal localization* of B at the set Σ if:

(1) there is a ring homomorphism $\lambda\colon B \to S$ such that $P \underset{B}{\otimes} S$ is acyclic;
(2) for every ring homomorphism $\mu\colon B \to R$ such that $P \underset{B}{\otimes} R$ is acyclic, there exists a unique ring homomorphism $\nu\colon S \to R$ such that $\nu \circ \lambda = \mu$.

LEMMA 3.2.2. *If $\lambda\colon B \to S$ is a "generalized universal localization" of B at a set Σ of perfect objects P of $\mathcal{D}(B)$, then λ is a ring epimorphism.*

PROOF. Let $\delta\colon S \to R$ be a ring homomorphism. Then, for every $P \in \Sigma$ we have:

$$P \underset{B}{\otimes} R = (P \underset{B}{\otimes} S) \underset{S}{\otimes} R.$$

Now $P \underset{B}{\otimes} S$ is split acyclic (that is null-homotopic, see [**W**, pag. 17]) and since its terms are finitely generated projective right S-modules, the complex $(P \underset{B}{\otimes} S) \underset{S}{\otimes} R$ is still split acyclic. By the universal property satisfied by S we conclude that δ is the only possible ring homomorphism extending $\mu = \delta \circ \lambda$. □

Now we can relate the result stated in Theorem 3.1.13 with the notion of "generalized universal localization".

PROPOSITION 3.2.3. *Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let P be a projective resolution of T_B in $\mathcal{D}(B)$.*

If condition (2) in Theorem 3.1.13 is satisfied, then $\lambda\colon B \to S$ is a "generalized universal localization" of B at the set $\{P\}$.

PROOF. As usual let $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\underset{B}{\otimes}} -)$. By assumptions $\lambda_*(S) \in \mathcal{Y}$, thus $T \overset{\mathbb{L}}{\underset{B}{\otimes}} S = 0$, so $P \underset{B}{\otimes} S$ is acyclic. Moreover, $\mathcal{Y} \cap B\text{-Mod} = \mathcal{E}$ is bireflective and, by [**GL**, Proposition 3.8], we have that $\lambda_*(S) = l(B)$, where $l\colon B\text{-Mod} \to \mathcal{E}$ is the left adjoin of the inclusion of $i\colon \mathcal{E} \to B\text{-Mod}$. Let $\mu\colon B \to S'$ be a ring homomorphism such that $P \underset{B}{\otimes} S'$ is acyclic, then also $T \overset{\mathbb{L}}{\underset{B}{\otimes}} S' = 0$, hence $S' \in \mathcal{E}$. Thus, $\mathrm{Hom}_B(l(B), S') \cong \mathrm{Hom}_B(B, S')$, hence there is a unique morphism $\rho\colon l(B) \to S'$ of right B-modules such that $\rho \circ \eta_B = \mu$, where $\eta_B\colon B \to l(B)$ is the unit morphism of the adjunction. Using the fact that $S = \mathrm{End}_B(l(B))$ and the naturality of the maps induced by the adjunction (l, j), it is not hard to see that ρ induces a unique ring homomorphism $\nu\colon S \to S'$ such that $\nu \circ \lambda = \mu$. □

REMARK 3.2.4. Note that the converse of the above statement does not hold in general. In fact, as shown in [**AKL**, Example 5.4] even in the case of a classical 1-tilting module over an algebra, the universal localization does not give rise to a homological epimorphism.

We now illustrate another property of the "generalized universal localization" .

PROPOSITION 3.2.5. *Let P be a perfect complex in $\mathcal{D}(B)$. Assume that $\lambda\colon B \to S$ is a "generalized universal localization" of B at $\{P\}$. Let $\mathcal{E}_P = \{N \in B\text{-Mod} \mid P \underset{B}{\otimes} N$ is acyclic $\}$. Then, the following hold:*

(1) $\lambda_*(S\text{-Mod}) \subseteq \mathcal{E}_P$.
(2) $\lambda_*(S\text{-Mod}) = \mathcal{E}_P$ *if and only if $\mathcal{E}_P$ is a bireflective subcategory of* B-Mod.

PROOF. (1) Let ${}_BM \in \lambda_*(S\text{-Mod})$. We have

$$P \underset{B}{\otimes} M \cong P \underset{B}{\otimes} (S \underset{S}{\otimes} M) \cong (P \underset{B}{\otimes} S) \underset{S}{\otimes} M$$

and $(P \underset{B}{\otimes} S)$ is a complex in $\mathcal{D}(S)$ whose terms are finitely generated projective right S-modules and by assumption it is acyclic. Thus, $P \underset{B}{\otimes} M$ is acyclic too, so $M \in \mathcal{E}_P$.

(2) By Lemma 3.2.2, λ is a ring epimorphism, hence, if $\lambda_*(S\text{-Mod}) = \mathcal{E}_P$, then $\mathcal{E}_P$ is bireflective, by Lemma 3.1.3.

Conversely, assume that $\mathcal{E}_P$ is bireflective. By Lemma 3.1.3, there is a ring R and a ring epimorphism $\mu\colon B \to R$ such that $\mu_*(R\text{-Mod}) = \mathcal{E}_P$. In particular, $\mu_*(R) \in \mathcal{E}_P$, hence $P \underset{B}{\otimes} R$ is an acyclic complex. Thus, by the universal property satisfied by S, there is a unique ring homomorphism $\nu\colon S \to R$ such that $\nu \circ \lambda = \mu$. By Lemma 3.1.3, $\mu\colon B \to R$ is an $\mathcal{E}_P$-reflection of B and $S \in \mathcal{E}_P$ by part (1). We infer that there is a unique morphism $\rho\colon R \to S$ such that $\rho \circ \mu = \lambda$. By the unicity of the rings homomorphisms ν and ρ it follows that they are inverse to each other. □

Let A be a ring and ${}_AT$ be a good n-tilting module with $n \geq 2$ and denote by B its endomorphism ring. Then Xi and Chen give the following characterization of the fact that $\mathcal{Y} := \operatorname{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -)$ is equivalent to the derived category of a ring.

Theorem 3.2.6. [**CX2**, Theorem 1.1] *Let A be a ring and ${}_AT$ be a good n-tilting module with $n \geq 2$ and denote by B its endomorphism ring. Set*

$$\mathcal{E} := \{Y \in \mathcal{D}(B) \mid \operatorname{Tor}_i^B(T, Y) = 0 \quad \textit{for all} \quad i \geq 0\}$$

and

$$M := 0 \to M_n \to \ldots \to M_1 \xrightarrow{\sigma} M_0 \xrightarrow{\pi} T \to 0$$

the projective resolution of ${}_AT$. Then the following are equivalent:

(1) *there exists a ring S such that $\mathcal{Y} := \operatorname{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -) \simeq \mathcal{D}(S)$.*
(2) *$\mathcal{E}$ is bireflective.*
(3) *$H^m(\operatorname{Hom}_A(M, A) \otimes_A T_B) = 0$ for all $m \geq 2$.*
(4) *Let us regard ${}_AT_B$ as $\operatorname{Hom}_A(A, T)$ and set, for $i = 0, 1$:*

$$\varphi_i : \operatorname{Hom}_A(M_i, A) \otimes_A \operatorname{Hom}_A(A, T) \to \operatorname{Hom}_A(M_i, A) : f \otimes t \mapsto t \circ f.$$

Consider the map

$$\psi : \operatorname{Coker}(\varphi_0) \longrightarrow \operatorname{Coker}(\varphi_1)$$

induced by $\sigma : M_1 \to M_0$. Then the kernel K of ψ satisfies

$$\operatorname{Ext}^i_{B^{op}}(T, K) = 0$$

for $i \geq 0$.

In particular, if $n = 2$ then (1) holds if an only if $\operatorname{Ext}^2_A(T, A) \otimes_A T = 0$.

Corollary 3.2.7. [**CX2**, Corollary 1.2] *With the hypotheses of the previous theorem, the following statement are true:*

(1) *if ${}_AT$ decomposes into $M \oplus N$ such that the projective dimension of ${}_AM$ is at most 1 and the first syzygy of ${}_AN$ is finitely generated, then the category $\mathcal{Y}$ is equivalent to the derived category of a ring.*

(2) *If A is commutative and* $\mathrm{Hom}_A(T_{i+1}, T_i) = 0$ *for all* T_i *as in* $(T3')$ *with* $1 \leq i \leq n-1$*, then* $\mathcal{Y}$ *is equivalent to the derived category of a ring if and only if the projective dimension of* ${}_AT$ *is at most* 1*, that is* ${}_AT$ *is a* 1*-tilting module.*

3. Examples

Using the notations of the previous section, we give some examples of different behavior of n-partial tilting modules with respect to the class $\mathcal{E}$. In what follows k will indicate an algebraically closed field.

EXAMPLE 1. We exhibit a class of examples of classical partial tilting modules T of projective dimension two over an artin algebra B such that there exists a "generalized universal localization" S of B at the projective resolution of T_B and the class $\mathcal{Y}$ is triangle equivalent to $\mathcal{D}(S)$.

Consider a representation-finite type algebra $\Lambda := kQ/I$ of an acyclic connected quiver Q (with $n > 1$ vertices) with a unique sink j and the category of its finite dimensional right modules mod-Λ. Let $T_\Lambda = \tau^{-1}(S(j)) \oplus (\bigoplus_{i \neq j} P(i))$ be an APR tilting module over Λ (see [**APR**]). Then $pdT_\Lambda = 1$ and its projective resolution is given by

$$0 \longrightarrow S(j) \longrightarrow (\bigoplus_{i \neq j} P(i)) \oplus E \longrightarrow T_\Lambda \longrightarrow 0$$

where

$$0 \longrightarrow S(j) \longrightarrow E \longrightarrow \tau^{-1}(S(j)) \longrightarrow 0$$

is an almost split exact sequence with E a projective Λ-module. Let $S(j)^d := \mathrm{Hom}_k(S(j), k)$ and consider $B := \begin{pmatrix} k & 0 \\ S(j)^d & \Lambda \end{pmatrix}$ the one point coextension of Λ by the non injective simple $S(j)_\Lambda$ (see [**ASS**]). In particular $B \simeq kQ'/J$ *where* Q' is exactly Q with the adjoint of a sink $*$ and of an arrow $j \longrightarrow *$. Let $I(*)$ and $S(*)$ be respectively the indecomposable injective B-module and the simple B-module at the vertex $*$, then $I(*) = \begin{smallmatrix} j \\ * \end{smallmatrix}$ and letting $P(*) = I(*)^d = \mathrm{Hom}_k(I(*), k)$ be the indecomposable projective at the vertex $*$ (regarded as right module on B^{op}), then $P(*) = \begin{smallmatrix} * \\ j \end{smallmatrix}$.

Every Λ-module can be regarded as a B-module via the natural embedding $\varphi : \text{mod-}\Lambda \hookrightarrow \text{mod-}B$.

PROPOSITION 3.3.1. *The following hold:*

(1) T_B *has projective dimension* 2.
(2) T_B *is self-orthogonal.*
(3) $\mathcal{E}_\Lambda = \{M \in \Lambda\text{-Mod} \mid \mathrm{Tor}_i^\Lambda(T, M) = 0,\ \forall i \geq 0\} = 0$ *and*

$$\mathcal{E}_B = \{M \in B\text{-Mod} \mid \mathrm{Tor}_i^B(T, M) = 0\ \forall i \geq 0\} = \mathrm{Add}\ I(*)^d = \mathrm{Add}\ P(*)$$

where for every module M, Add M *denotes the class of all direct summands of arbitrary direct sums of copies of* M.

PROOF. (1) We have that $S(j)$ regarded as B-module is non projective and its projective cover is given by

$$I(*) \longrightarrow S(j) \longrightarrow 0.$$

Hence a projective resolution of T_B is

$$0 \longrightarrow S(*) \longrightarrow I(*) \longrightarrow (\bigoplus_{i\neq j} P(i)) \oplus E \longrightarrow \tau^{-1}(S(j)) \oplus (\bigoplus_{i\neq j} P(i)) \longrightarrow 0.$$

(2) To prove the self-orthogonality of T_B we can observe that mod-Λ is equivalent to the class mod-$B \bigcap \mathrm{KerHom}_B(-, I(*))$, then, in particular, it is closed under extensions in mod-B. Then it is clear that

$$\mathrm{Ext}^1_B(T_B, T_B) \simeq \mathrm{Ext}^1_\Lambda(T_\Lambda, T_\Lambda) = 0.$$

Moreover

$$\mathrm{Ext}^2_B(T_B, T_B) = \mathrm{Ext}^1_B(S(j)_B, T_B) = \mathrm{Ext}^1_\Lambda(S(j), T_\Lambda) = 0.$$

(3) $\mathcal{E}_\Lambda = 0$ because T_Λ is a tilting module. Now, ind-$B\backslash$ind-$\Lambda = \{I(*), S(*)\}$ and $I(*) = \begin{smallmatrix} j \\ * \end{smallmatrix}$. We want to compute the class

$$\begin{aligned} \mathcal{E}_B &= \{M \in B\text{-Mod such that } \mathrm{Tor}^B_i(T, M) = 0 \text{ for each } i \geq 0\} \\ &= \{M \in \text{Mod-}B^{op} \text{ such that } \mathrm{Ext}^i_{B^{op}}(M, T^d) = 0 \text{ for each } i \geq 0\} \end{aligned}$$

where $T^d_B := \mathrm{Hom}_k(T_B, k)$. We can regard B-Mod as Mod-B^{op} and B^{op} is the one point extension of Λ^{op} by the simple $S(j)^d = S(j)$. Let $P(*)$ be the indecomposable projective at the vertex $*$ (the dual of $I(*)$, regarded as right module on B^{op}), then we claim that $\mathcal{E}_B = \mathrm{Add}\ (P(*))$. Note that, as in the previous case, ind-$B^{op}\backslash$ind-$\Lambda^{op} = \{P(*), S(*)\}$ and $P(*) = \begin{smallmatrix} * \\ j \end{smallmatrix}$. From the fact that $\mathcal{E}_\Lambda = 0$ and that every Λ-module can be regarded as a B-module, only Add $\{P(*), S(*)\}$ could be contained in $\mathcal{E}_B$.

We prove that $S(*) \notin \mathrm{KerExt}^2_B(-, T^d_B)$. Since $S(j)$ is the first cosyzygy of the injective resolution of $T^d_\Lambda = \tau^{-1}(S(j))^d \oplus (\bigoplus_{i\neq j} I(i))$, we show that $S(*) \notin \mathrm{KerExt}^1_B(-, S(j))$. Indeed there is the non split short exact sequence

$$0 \longrightarrow S(j) \longrightarrow \begin{smallmatrix} * \\ j \end{smallmatrix} \longrightarrow S(*) \longrightarrow 0.$$

Hence $S(*) \notin \mathcal{E}_B$. To show that $P(*) \in \mathcal{E}_B$ we only have to check that $\mathrm{Hom}_{B^{op}}(P(*), T^d_B) = 0$, since $P(*)$ is projective. It is true from the fact that $\mathrm{top}P(*) = S(*)$ does not belongs to any composition series of T^d_B. Then $\mathcal{E}_B = \mathrm{Add}\ (P(*))$.

□

Set now $A := \mathrm{End}_B(T_B) = \mathrm{End}_\Lambda(T_\Lambda)$, *then* $\Lambda = \mathrm{End}_A({}_AT)$ *because* T_Λ *is tilting (hence balanced) over* Λ. *So* ${}_AT$ *is* 1*-tilting but* $End_A({}_AT) \neq B$. *Let* $\mathcal{E} = \mathcal{E}_B$.

Lemma 3.3.2. For each projective left B-module P the unit morphism of the adjunction $(T \otimes_B -, \mathrm{Hom}_A({}_AT, -))$

$$\overline{\eta}_P : P \longrightarrow \mathrm{Hom}_A(T, T \otimes_B P)$$

is surjective and $\mathrm{Ker}\overline{\eta}_P \in \mathcal{E} = \mathrm{Add}\ (P(*))$.

Proof. Let us note that we can regard

$$\overline{\eta}_B : B \longrightarrow \mathrm{Hom}_A({}_AT, T_B \otimes_B B) \simeq \Lambda$$

as the projection $\pi : B \longrightarrow \Lambda \simeq B/Be_*B$, hence it is surjective and the kernel is the annihilator of T_B as right B-module, that is $\mathrm{Ker}\overline{\eta}_P$ is the projective B-module $P(*)$. Now, since $\mathcal{E}$ is closed under direct summand, we can prove the statement just for free modules. Let α be a cardinal, then the map

$$\overline{\eta}_{B^{(\alpha)}} : B^{(\alpha)} \longrightarrow \mathrm{Hom}_A(T, T \otimes_B B^{(\alpha)}) = \Lambda^{(\alpha)}$$

is exactly $\pi^{(\alpha)}$ and the kernel of $\overline{\eta}_{B^{(\alpha)}}$ is $P(*)^{(\alpha)}$. □

Proposition 3.3.3. *There is a homological ring epimorphism*

$$\lambda : B \longrightarrow S$$

with $n = \dim_k P(*)$ *and* $S = \mathrm{End}(P(*)^{\oplus n})$.

Proof. $\mathcal{E}$ being bireflective, there exists an object $M \in \mathcal{E}$ such that $S := \mathrm{End}_B(M) \simeq M$ as B-modules and $\mathcal{E} \simeq S$-Mod. If $n = \dim_k P(*)$, then $M = P(*)^{\oplus n}$ and $S \simeq M_n(k)$ and there exists a ring epimorphism $\lambda : B \longrightarrow S$. We now prove that λ is homological. In view of Theorem 3.1.13 we have just to prove that every object in $\mathcal{Y} = \mathrm{Ker}(T \overset{\mathbb{L}}{\otimes}_B -)$ is quasi-isomorphic to a complex with terms in $\mathcal{E}$. Set $H = \mathbb{R}\mathrm{Hom}_A({}_AT_B, -)$ and $G = {}_AT_B \overset{\mathbb{L}}{\otimes}_B -$and consider the triangle

$$B \xrightarrow{\eta_B} HG(B) \longrightarrow Y \longrightarrow B[1].$$

We have

$$HG(B) = \mathbb{R}\mathrm{Hom}_A({}_AT_B, {}_AT_B \overset{\mathbb{L}}{\otimes}_B B) = \mathbb{R}\mathrm{Hom}_A({}_AT_B, {}_AT_B) = \mathrm{Hom}_A({}_AT_B, {}_AT_B) = \Lambda$$

(because $T_B \simeq T_\Lambda$ that is self-orthogonal in A-Mod, hence $\mathrm{Hom}_A({}_AT, -)$-acyclic). Then $\eta_B = \overline{\eta}_B$ and, considering the long exact sequence of the homologies, we can conclude that Y is quasi-isomorphic to the stalk complex $\mathrm{Ker}\eta_B[1]$, that is $P(*)[1]$. We now follow [**CX**, Prop. 4.6]. Let M be an object in $\mathcal{D}(B)$, then there is the triangle

$$M \xrightarrow{\eta_M} HG(M) \longrightarrow Y_M \longrightarrow M[1] \tag{11}$$

where

$$HG(M) = \mathbb{R}\mathrm{Hom}_A({}_AT_B, {}_AT_B \overset{\mathbb{L}}{\otimes}_B M) = \mathrm{Hom}_A({}_AT_B, {}_AT_B \otimes_B W)$$

with W a $\mathcal{H}$-projective resolution of the complex M. Therefore $\mathrm{Hom}_A({}_AT_B, {}_AT_B \otimes_B W)$ has terms of the form $\mathrm{Hom}_A(T, T_i)$ with $T_i \in \mathrm{Add}\ ({}_AT)$. ${}_AT$ being finitely generated, we have that the module $\mathrm{Hom}_A(T, T_i)$ is in $\mathrm{Add}\ (\Lambda_\Lambda)$. Regard the triangle in (11) as the triangle

$$W \xrightarrow{\eta_M} \mathrm{Hom}(T, T^\bullet) \longrightarrow Y_M \longrightarrow W[1] \tag{12}$$

where $T^\bullet$ is the complex $({}_AT_B \otimes_B W)$. Therefore the morphism n_M can be regarded in $\mathcal{C}(B)$ as the family $(\overline{\eta}^i)_{i\in\mathbb{Z}}$ with $\eta_i : W^i \longrightarrow \mathrm{Hom}_A(T, T_i)$. Then for Lemma 3.3.2, noting that $(\mathrm{Ker}\eta_M)^i = \mathrm{Ker}\overline{\eta}^i \in \mathcal{E}$, we can conclude that $Y_M \simeq \mathrm{Ker}\eta_M[1]$ has terms in $\mathcal{E}$. Now, for every Y in $\mathcal{Y}$, there is the triangle

$$Y \xrightarrow{\eta_Y} 0 \longrightarrow Y \longrightarrow Y[1]$$

then Y is exactly $\mathrm{Ker}\eta_Y$ that has terms in $\mathcal{E}$. $\square$

Let us show a particular instance of the situation just described. Using the same notations, set Λ equal to the path algebra of the quiver $\underset{1}{\circ} \xrightarrow{a} \underset{2}{\circ}$. Then B is the path algebra of the quiver: $\underset{1}{\circ} \xrightarrow{a} \underset{2}{\circ} \xrightarrow{b} \underset{3}{\circ}$ with the relation $ab = 0$. So

$$T_\Lambda = \begin{matrix} 1 \\ 2 \end{matrix}$$

and

$$T_B = \begin{matrix} 1 \\ 2 \end{matrix} \oplus 1$$

Moreover

$$\mathcal{E} = \mathrm{Add} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \subseteq B\text{-Mod},$$

$S = M_2(k)$ and we can express B as the matrix

$$\begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}.$$

Then there exists a homological ring epimorphism $\lambda : B \to S$ defined by, for all $a, b, c, d, e, f \in k$,

$$\lambda \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} = \begin{pmatrix} c & 0 \\ e & f \end{pmatrix}.$$

Moreover, since T_Λ is classical tilting over Λ we have that the functor $\mathbb{R}\mathrm{Hom}_A({}_AT, -) : \mathcal{D}(A) \to \mathcal{D}(\Lambda)$ is an equivalence. Let us express Λ as the triangular matrix

$$\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

and consider the natural projection of B over Λ,

$$\pi : \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix} \to \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}.$$

π is a ring epimorphism and the restriction of scalars functor $\pi_* : \mathcal{D}(\Lambda) \to \mathcal{D}(B)$ is fully faithful. Then the composition $\pi_* \circ \mathbb{R}\mathrm{Hom}_A({}_AT, -) : \mathcal{D}(A) \to \mathcal{D}(B)$ is fully faithful and we have the following recollement:

$$\begin{tikzcd}
\mathcal{D}(S) \arrow[r, "\lambda_*"] & \mathcal{D}(B) \arrow[l, bend right, "i^*=S\overset{\mathbb{L}}{\underset{B}{\otimes}}-"'] \arrow[l, bend left, "i^!=\mathbb{R}\mathrm{Hom}_B(S,-)"] \arrow[r, "j^*=T\overset{\mathbb{L}}{\otimes}_B-"] & \mathcal{D}(A) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "\pi_*\circ\mathbb{R}\mathrm{Hom}_A(T,-)"]
\end{tikzcd}$$

Remark 3.3.4. The previous example can be generalized considering a situation similar to [**Mi**, Corollary 5.5]. Let us point out the key steps used in the previous Example 1. Assume that I is a non-zero projective, idempotent two-sided ideal of an ordinary k-algebra B. Then the projective dimension of Λ, viewed as a right B-module, is one. By [**NS**, Example in Section 4] the canonical projection $\pi : B \to \Lambda := B/I$ is a homological ring epimorphism and Λ_B is self-orthogonal. Let now T_Λ be a classical n-tilting module over Λ and view T as a right B-module via π. Then I is the annihilator of T_B (and of Λ_B) and T_B is a classical $n+1$-partial tilting module, since $\text{proj.dim}(T_\Lambda) \leq \text{proj.dim}(T_B) \leq \text{proj.dim}(T_\Lambda)+1$. Set $A := \mathrm{End}_\Lambda(T_\Lambda) = \mathrm{End}_B(T_B)$ (where the last equality holds since π is a ring epimorphism). The functor ${}_AT \overset{\mathbb{L}}{\otimes}_\Lambda -\colon \mathcal{D}(\Lambda) \to \mathcal{D}(A)$ is a triangle equivalence, since T_Λ is a classical n-tilting module. Moreover the functor ${}_AT \overset{\mathbb{L}}{\otimes}_B -\colon \mathcal{D}(B) \to \mathcal{D}(\Lambda)$ is given by the composition of functors $({}_AT \overset{\mathbb{L}}{\otimes}_\Lambda -) \circ ({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -)$, so the kernel of ${}_AT \overset{\mathbb{L}}{\otimes}_B -$ is exactly the kernel of $({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -)$. Thus, $\mathrm{Ker}({}_AT \overset{\mathbb{L}}{\otimes}_B -)$ is equivalent to the derived category of a ring via a homological ring epimorphism if and only so is $\mathrm{Ker}({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -)$. But, Λ_B is a classical 1-partial tilting module with $\mathrm{End}_B(\Lambda) = \Lambda$, so the class $\mathcal{E} = \mathrm{Ker}({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -) \cap B\text{-Mod}$ is bireflective. Now, similarly to the proof of Proposition 2, we let $G = ({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -)$ and $H = \mathbb{R}\mathrm{Hom}_\Lambda({}_\Lambda\Lambda_B, -)$. Then, a complex $Y \in \mathrm{Ker}({}_\Lambda\Lambda \overset{\mathbb{L}}{\otimes}_B -)$ if and only if Y is quasi isomorphic to $HG(Y)$. Computing $HG(Y)$ by means of an $\mathcal{H}$-projective resolution of Y in $\mathcal{D}(B)$ we obtain that $HG(Y)$ is a direct summand of complex with terms of the form $\Lambda^{(I)}$ for some set I, viewed as left B-modules, hence in the class $\mathcal{E}$. By Theorem 3.1.13, we conclude that the kernel of the functor ${}_AT \overset{\mathbb{L}}{\otimes}_B -$ is triangle equivalent to the derived category of a ring via the homological epimorphism π.

Example 2. Now we give a simple example of a finitely generated partial tilting module T over a finite dimensional algebra B, such that the class $\mathcal{E} = \bigcap_{i\geq 0} \mathrm{KerTor}_i^B(T,-)$ is not bireflective (in particular there are no homological ring epimorphisms $B \to S$ such that $\mathrm{Ker}(T \overset{\mathbb{L}}{\otimes}_B -) \simeq \mathcal{D}(S)$).

Let us take the quiver $\underset{1}{\circ} \overset{a}{\longrightarrow} \underset{2}{\circ} \overset{b}{\longrightarrow} \underset{3}{\circ}$ with relation $ab = 0$ and the right modules over its path algebra B. Consider the simple injective right module S_1. The projective dimension of S_1 is two and its projective resolution is given by:

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0$$

It is easy to see that S_1 is partial tilting over B. A calculation shows that the class $\mathcal{E} = \mathrm{Add}\left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right\}$ is not bireflective. In fact there is a morphism

$$f : \begin{matrix} 2 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 3 \\ 2 \end{matrix}$$

such that the kernel is not in $\mathcal{E}$.

Example 3. Let us consider the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, "a"] & \underset{2}{\circ} \arrow[l, bend right, "b"'] \end{tikzcd}$$

with relation $ab = 0$ and consider the partial tilting module $\begin{matrix} 1 \\ 2 \end{matrix}$ of projective dimension 2. Here, as shown in [**B2**, Example 1], $\mathcal{E} = 0$ then it is bireflective but, obviously, the complexes in $\mathcal{Y}$ don't have terms in $\mathcal{E}$.

Example 4. [**CX2**, Section 7.1] The following is an example of a good n-tilting module ${}_AT$ with $B = \mathrm{End}_A({}_AT)$, such that $\mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -)$ is not triangle equivalent to the derived category of a ring via a homological ring epimorphism.

Let A be a commutative n-Gorestein ring and consider a minimal injective resolution of the regular module ${}_AA$ of the form:

$$0 \to A \to \bigoplus_{p \in \mathcal{P}_0} E(A/p) \to \ldots \to \bigoplus_{p \in \mathcal{P}_n} E(A/p) \to 0$$

where $\mathcal{P}_i$ is the set of all prime ideals of A of height i (see [**Bas**, Theorem 1, Theorem 6.2]). Then, the module

$${}_AT := \bigoplus_{0 \leq i \leq n} \bigoplus_{p \in \mathcal{P}_i} E(A/p)$$

is an n-tilting module by [**GT**, Example 5.16] and it is moreover good. Set, for all $0 \leq i \leq n$, $T_i := \bigoplus_{p \in \mathcal{P}_i} E(A/p)$, then we have $\mathrm{Hom}_A(T_j, T_i) = 0$ for all $0 \leq i \leq j \leq n$. Assume that $n \geq 2$ and that the injective dimension of A is exactly n; then T has projective dimension n (see [**B2**, Proposition 3.5]). Note that $T_i \neq 0$ for every $2 \leq i \leq n$ so T satisfies the hypotheses of [**CX2**, Corollary 1.2], hence $\mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -)$ cannot be realized as the derived category $\mathcal{D}(S)$ of a ring S linked to B via a homological ring epimorphism $B \to S$.

Example 5. [**CX2**, Section 5] An easy application of Corollary 3.2.7 leads to the construction of a class of good n-tilting modules such that the Kernel of the tensor functor is equivalent to the derived category of a ring. Let us take T a classical n-tilting module over a ring A with endomorphism ring B, such that $T = M \oplus N$ with M a nonzero A-module of projective dimension at most one. Let J be an infinite set and take $T_1 := M^{(I)} \oplus N$. Then T_1 is a good n-tilting module and, by Corollary 3.2.7, there exists a ring R such that $\mathrm{Ker}(T_1 \overset{\mathbb{L}}{\otimes}_B -) \simeq \mathcal{D}(R)$.

Appendix A

Recall that, if T_B is an n-partial tilting module, of projective resolution P and with endomorphisms ring A, then $Q := \mathbb{R}\mathrm{Hom}_{B^{op}}(P, B^{op})$ is a perfect and self-orthogonal object of $\mathcal{D}(B)$ (see Theorem 2.4.6) and moreover, by Proposition 2.5.8, there is a TTF triple

$$(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (\mathrm{Tria}\ Q, \mathrm{Ker}(T_B \overset{\mathbb{L}}{\otimes}_B -), \mathrm{Im}\mathbb{R}\mathrm{Hom}_A({}_AT, -)).$$

Let us denote by L the left adjoint of the inclusion functor $i : \mathcal{Y} \to \mathcal{D}(B)$. Then, by Theorem 2.1.23, the central class $\mathcal{Y}$ is triangle equivalent to $\mathcal{D}(E)$ where $E = \mathbb{R}\mathrm{Hom}_B(L(B), L(B))$ and $H^i(E) = H^i(L(B))$. If T_B is a classical partial 1-tilting module then $E \simeq H^0(L(B))$, that is $L(B)$ is concentrated in degree zero and it can be expressed directly as a universal localization as Chen Xi shows in [**CX**, proof of Proposition 3.9]. If the projective dimension of T_B is greater than one, in [**AKL**, Appendix] is proved that $L(B)$ is a Milnor colimit of a sequence of morphisms in $\mathcal{D}(B)$. In what follows we adapt the results in [**AKL**] to our case and we present some computations on the homologies of $L(B)$ in order to understand when it is quasi-isomorphic to a bounded complex.

Definition 3.3.5. [**N2**, Definition 1.6.4] Let $\mathcal{D}$ be a triangulated category and let

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$$

be a sequence of morphisms of $\mathcal{D}$ such that the coproduct $\coprod_{i\in\mathbb{N}} M_i$ exists in $\mathcal{D}$. The *Milnor colimit* (or *homotopy colimit*) of this sequence, denoted by $\mathrm{Mcolim}M_n$, is given, up to non-unique isomorphism, by the triangle

$$\coprod_{i\in\mathbb{N}} M_i \xrightarrow{1-\sigma} \coprod_{i\in\mathbb{N}} M_i \xrightarrow{\pi} \mathrm{Mcolim}M_n \longrightarrow \coprod_{i\in\mathbb{N}} M_i[1]$$

where the morphism σ has components

$$M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{can} \coprod_{i\in\mathbb{N}} M_i$$

We specialize the situation of Lemma A.2 in [**AKL**] to the case of the perfect self-orthogonal object Q.

Proposition 3.3.6. [**AKL**, Lemma A.2] *Given a partial tilting module T_B, there exists a sequence of maps in $\mathcal{D}(B)$*

$$B_0 := B \overset{\sigma_0}{\to} B_1 \overset{\sigma_1}{\to} \dots \to B_n \overset{\sigma_n}{\to} B_{n+1} \to \dots \tag{13}$$

such that:

(1) $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B_n) = 0$ *for every* $i \neq n$.

(2) *$L(B)$ is isomorphic to the homotopy colimit B_∞ of the sequence $(B_n)_{n\in\mathbb{N}}$*

$$\bigoplus_{n\in\mathbb{N}} B_n \xrightarrow{\mathbb{I}-\sigma} \bigoplus_{n\in\mathbb{N}} B_n \xrightarrow{\pi} B_\infty \tag{14}$$

$$\textit{where } (\mathbb{I}-\sigma)(b_0,\dots,b_k) \longmapsto (b_0, b_1-\sigma_0 b_0, b_2-\sigma_1 b_1, \dots, -\sigma_k b_k).$$

Proof. Thank's to Lemma A.2 in [**AKL**] the only thing that remained to prove in this particular case is that $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B_n) = 0$ for every $i \neq n$. Let us prove it by induction. Let $n = 0$ and set $I_0 = \mathrm{Hom}_{\mathcal{D}(B)}(Q, B)$. The sequence (13) is constructed recursively, starting from the triangle

$$Q^{(I_0)} \overset{\alpha_0}{\to} B \overset{\sigma_0}{\to} B_1 \to Q^{(I_0)}[1]$$

Apply $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], -)$ to get the long exact sequence

$$.. \to \mathrm{Hom}_{\mathcal{D}(B)}(Q[i], Q^{(I_0)}) \to \mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B) \to \mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B_1) \to \mathrm{Hom}_{\mathcal{D}(B)}(Q[i], Q^{(I_0)}[1]) \to \dots$$

Now we have: $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B) = 0$ for every $i \neq 0$, $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], Q^{(I_0)}[j]) = 0$ for every $i \neq j$ and moreover the map $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], Q^{(I_0)}) \overset{\alpha_0}{\to} \mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B)$ is an epimorphism. Then $\mathrm{Hom}_{\mathcal{D}(B)}(Q, B_1) = 0$ for every $i \neq 1$.
$n \Rightarrow n+1$
Set $I_n := \mathrm{Hom}_{\mathcal{D}(B)}(Q[n], B_n)$ and apply the functor $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], -)$ to the triangle

$$Q^{(I_n)} \to B_n \overset{\sigma_n}{\to} B_{n+1} \to Q^{(I_n)}.$$

Analogously to the zero step of the induction we can conclude that $\mathrm{Hom}_{\mathcal{D}(B)}(Q[i], B_n) = 0$ for every $i \neq n$. □

Lemma 3.3.7. *With notation as in the previous proposition, we have:*

$$H^i(B_\infty) = \varinjlim_{n\in\mathbb{N}} H^i(B_n), \ \textit{for each } n \in \mathbb{Z}.$$

Proof. For each $i \in \mathbb{Z}$ there is the following long exact sequence

$$\dots \longrightarrow \bigoplus_{n\in\mathbb{N}} H^i(B_n) \xrightarrow{\mathbb{I}-\overline{\sigma_n}} \bigoplus_{n\in\mathbb{N}} H^i(B_{n+1}) \longrightarrow \varinjlim_{n\in\mathbb{N}} H^i(B_n) \to \bigoplus_{n\in\mathbb{N}} H^{i+1}(B_n) \longrightarrow \dots$$

Now, for each fixed $i \in \mathbb{Z}$, the map

$$\bigoplus_{n\in\mathbb{N}} H^i(B_n) \xrightarrow{\mathbb{I}-\overline{\sigma_n}} \bigoplus_{n\in\mathbb{N}} H^i(B_{n+1})$$

is a monomorphism since $\{H^i(B_n), H^i(\sigma_n)\}$ is a countable direct system, thus, from the long exact sequence in homology from the triangle (14), we have the following short exact sequence:

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} H^i(B_n) \xrightarrow{\mathbb{I}-\overline{\sigma_n}} \bigoplus_{n\in\mathbb{N}} H^i(B_{n+1}) \longrightarrow H^i(B_\infty) \longrightarrow 0.$$

Therefore:

$$H^i(B_\infty) \simeq \varinjlim_{n\in\mathbb{N}} H^i(B_n), \ \text{for each } i \in \mathbb{Z}.$$

□

Let us suppose now that T, seen as left module over A, is a good n-tilting module. Then there exists the exact sequence:

$$0 \to T_0 \to T_1 \to \dots \to T_n \to 0$$

with $T_i \in \operatorname{add}\, {}_AT$. In this case the projective resolution of T_B is given by the complex obtained applying the contravariant functor $\mathrm{Hom}_A(-, T)$ to the sequence above. Then, indicated with P the projective resolution of T_B, one has: $P_{-i} = \mathrm{Hom}_A(T_i, T)$ for each $0 \leq i \leq n$. Then, for each $0 \leq j \leq n$ we have $Q_n = \mathbb{R}\mathrm{Hom}_{B^{op}}(P, B) = \mathrm{Hom}_A({}_AT, T_i)$ and $\mathbb{R}\mathrm{Hom}_A({}_AT, A) = Q$.

Notations 3.3.8. *Set* $k_0 := \mathrm{Hom}_A(A, T)$. *Regarding* T *as e left* A*-module we have the following short exact sequences:*

$$\begin{aligned}
0 &\longrightarrow k_1 \longrightarrow A^{(T)} \longrightarrow T \longrightarrow 0;\\
0 &\longrightarrow k_2 \longrightarrow A^{(Hom_A(A,k_1))} \longrightarrow k_1 \longrightarrow 0;\\
&\dots\\
0 &\longrightarrow k_m \longrightarrow A^{(Hom_A(A,k_{m-1}))} \longrightarrow k_{m-1} \longrightarrow 0.
\end{aligned}$$

When $m \geq n$ *we have that* k_m *is projective as* A*-module.*

Proposition 3.3.9. *There is the following isomorphism of abelian groups:*

$$Hom_{\mathcal{D}(B)}(Q[n], B_n) = k_n \text{ for } n \geq 0$$

Proof. Let us prove it by induction. Since $P_{-i} = \mathrm{Hom}_A(T_i, T)$, then $\mathrm{Hom}_{\mathcal{D}(B)}(Q, B) = H^0(P) =$ Set $\mathrm{Hom}_{\mathcal{H}(B)}(Q[n], B_n) = k_n$. Let us apply the functor $\mathbb{R}\mathrm{Hom}_B(Q, -)$ to the triangle

$$Q[n]^{(k_n)} \xrightarrow{\alpha_n} B_n \longrightarrow B_{n+1} \longrightarrow Q[n+1]^{(k_n)}. \tag{15}$$

Looking at the long exact sequence of the homologies we obtain

$$Hom_{\mathcal{H}(B)}(Q[n], Q[n])^{(k_n)} \longrightarrow k_n \longrightarrow Hom_{\mathcal{H}(B)}(Q[n], B_{n+1}) \longrightarrow Hom_{\mathcal{H}(B)}(Q[n], Q[n+1])^{(k_n)}$$

We know that $Hom_{\mathcal{H}(B)}(Q[n], B_{n+1}) = 0$, hence

$$0 \longrightarrow Hom_{\mathcal{H}(B)}(Q[n], B_{n+1}[-1]) \longrightarrow Hom_{\mathcal{H}(B)}(Q[n], Q[n])^{(k_n)} \longrightarrow k_n \longrightarrow 0. \tag{16}$$

Moreover $\mathrm{Hom}_{\mathcal{H}(B)}(Q, Q) = A$, that is (16) becomes

$$0 \longrightarrow Hom_{\mathcal{H}(B)}(Q[n], B_{n+1}[-1]) \longrightarrow A^{(k_n)} \longrightarrow k_n \longrightarrow 0.$$

□

Lemma 3.3.10. *Denote by* G *the functor* $T \overset{\mathbb{L}}{\otimes}_B -$. *For every* n*, the map* $\overline{G(\alpha_n)}$ *induced by* $G(\alpha_n)$ *in homology is surjective.*

Proof. Apply G to the triangle (15) and consider the long exact sequence in homology. Using the isomorphisms of the functors $\mathbb{R}\mathrm{Hom}_B(Q, -)$ and $G = T \overset{\mathbb{L}}{\otimes}_B -$ (see Remark 2.2.1) we have:

$$\begin{aligned}
&0 \longrightarrow \mathrm{Hom}_{\mathcal{H}(B)}(Q[n], B_{n+1}[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{H}(B)}(Q[n], Q^{(k_n)}[n])\\
&\xrightarrow{\overline{\mathbb{R}\mathrm{Hom}_B(Q,\alpha_n)}} \mathrm{Hom}_{\mathcal{H}(B)}(Q[n], B_n) \longrightarrow \mathrm{Hom}_{\mathcal{H}(B)}(Q[n], B_{n+1}) \longrightarrow 0.
\end{aligned} \tag{17}$$

But we have proved that $\mathrm{Hom}_{\mathcal{D}(B)}(Q[n], B_{n+1}) = 0$ then we can conclude that $\overline{\mathbb{R}\mathrm{Hom}_B(Q\alpha_n)} = \overline{G(\alpha_n)}$ is surjective. □

Proposition 3.3.11. *we have*

$$G(B_n) \simeq k_n[n] \text{ for each } n \geq 0.$$

Proof. For $n = 0$ it is obvious.

Suppose $G(B_n) = k_n[n]$. Since H is fully faithful, the counit of the adjunction (G, H) an isomorphism (see Theorem 2.5.6), then $G(Q[n]^{(k_n)}) = G(Q[n])^{(k_n)} \simeq A[n]^{(k_n)}$. Consider the long exact sequence

$$0 \longrightarrow H^{-n}G(B_{n+1})[-1] \longrightarrow H^{-n}A[n]^{(k_n)} \stackrel{\overline{G(\alpha_n)}}{\longrightarrow} k_n[n] \longrightarrow H^{-n}G(B_{n+1}) \longrightarrow 0.$$

From the previous Lemma we have that $\overline{G(\alpha_n)}$ is surjective, than $H^{-n}G(B_{n+1}) = 0$ and $H^{-n}G(B_{n+1})[-1] = k_{n+1}[n+1]$. □

Corollary 3.3.12. *From the triangle above we obtain the short exact sequence*

$$0 \longrightarrow k_{n+1} \longrightarrow A^{(k_n)} \stackrel{\overline{G(\alpha_n)}}{\longrightarrow} k_n \longrightarrow 0.$$

We make now some computations in order to try to understand when it may happen that $L(B)$ has bounded homologies. We suppose that ${}_AT$ is a good 2-tilting module. It is possible to consider also the general case $n \geq 2$ but in order to simplify the index notation we show the calculations just in the case $n = 2$.

Remark 3.3.13. By construction we have, for each $i \geq -1$:
i) $B_\infty^{-i} = B_{2-i}^{-i}$.
ii) $H^{-i}(B_\infty) = H^{-i}(B_{3-i})$.

Notations 3.3.14. *Let $\delta_\infty^{-i}, \delta_n^{-i}$ be the differentials of the complex B_∞, B_n respectively. For every $n \geq 2$ consider the exact sequence of complexes*

$$(*) \quad 0 \to \tau_{\leq 2-n}(B_n) \to B_n \to \overline{B_n} \to 0.$$

By Remark 3.3.13 we have $\delta_\infty^{2-n} = \delta_n^{2-n}$ so

$$\overline{B_n} = \frac{B_\infty}{\tau_{\leq 2-n}(B_\infty)}.$$

Thus we also have a short exact sequence of complexes:

$$(a) \quad 0 \to \tau_{\leq 2-n}(B_\infty) \to B_\infty \to \overline{B_n} \to 0.$$

In general

a *For every $n \geq 2$, $\overline{B_n}$ is quasi-isomorphic to the complex*

$$0 \to \mathrm{Coker}\delta_n^{2-n} \to B_n^{4-n} \to \ldots \to B_n^0 \to B_n^1 \to 0$$

in degrees $3-n, 4-n, \ldots, -1, 0, 1$..

b *Let $X_n = 0 \to B_n^{4-n} \to \ldots \to B_n^0 \to B_n^1 \to 0$*
in degrees $4-n, \ldots, -1, 0, 1$.
X_n has projective terms so $G(X_n) \cong T \otimes_B X_n$.

c *$H^{-i}(G(X_n)) = 0$, for every $-i \leq 3-n$ (i.e. $i \geq n-3$).*

d *We have a triangle*

$$(b) \quad \mathrm{Coker}\delta_n^{2-n}[n-4] \to X_n \to \overline{B_n} \to$$

Lemma 3.3.15. *In the previous notations the following hold true for every $n \geq 2$.*

(1) $H^{-i}(G(\overline{B_n})) = 0$ *for every* $i \neq n-1$.
(2) $H^{1-n}(G(\overline{B_n})) \cong \mathrm{Tor}_2^B(T, \mathrm{Coker}\delta_n^{2-n})$.
(3) $H^{2-n}(G(\overline{B_n})) \cong \mathrm{Tor}_1^B(T, \mathrm{Coker}\delta_n^{2-n}) = 0$

PROOF. (1) We have that $\tau_{\leq 2-n}(B_\infty)$ has terms in degrees $\leq 2-n$, so $H^j(G(\tau_{\leq 2-n}(B_\infty))) = 0$ for every $j > 2-n$; and $H^j(G(B_\infty)) = 0$, for every j. From sequence (a) we obtain the exact sequence

$$0 \to H^j(G(\overline{B_n})) \to H^{j+1}(G(\tau_{\leq 2-n}(B_n))) \to 0$$

so we conclude that $H^j(G(\overline{B_n})) = 0$, for every $j \geq 2-n$.
Secondly, we show that $H^{-i}(G(\overline{B_n})) = 0$ for every $i \geq n$.
From the triangle (b) we obtain the exact sequence

$$H^{-i}(G(X_n)) \to H^{-i}(G(\overline{B_n})) \to H^{-i+1}(G(\mathrm{Coker}\delta_n^{2-n})[n-4])$$

and $H^{-i+1}(G(\mathrm{Coker}\delta_n^{2-n})[n-4]) = H^{-i+n-3}(G(\mathrm{Coker}\delta_n^{2-n})) = 0$, for every $i \geq n$ since $\mathrm{Tor}_j(T, -) = 0$, for every $j \geq 3$. Moreover, we already noticed in **3** that $H^{-i}(G(X_n)) = 0$ for every $i \geq n-3$.
In conclusion, $H^{-i}(G(\overline{B_n})) = 0$ for every $i \geq n$ and $i \leq n-2$, so $G(\overline{B_n})$ has cohomology at most in degree $1-n$.
(2) From triangle (b) we have

$$H^{1-n}(G(X_n)) \to H^{1-n}(G(\overline{B_n})) \to H^{1-n+n-3}(G(\mathrm{Coker}\delta_n^{2-n})) \to H^{2-n}(G(X_n))$$

where $H^{1-n}(G(X_n)) = 0 = H^{2-n}(G(X_n))$ and

$$H^{-2}(G(\mathrm{Coker}\delta_n^{2-n})) \cong \mathrm{Tor}_2(T, \mathrm{Coker}\delta_n^{2-n}).$$

(3) From triangle (b) we also have

$$0 = H^{2-n}(G(X_n) \to H^{2-n}(G(\overline{B_n})) \to H^{-1}(G(\mathrm{Coker}\delta_n^{2-n})) \to H^{3-n}(G(X_n)) = 0$$

so $H^{2-n}(G(\overline{B_n})) \cong H^{-1}(G(\mathrm{Coker}\delta_n^{2-n}))$ and by part (2) they are zero.

□

LEMMA 3.3.16. *In the previous notations the following hold true.*
(1) $\mathrm{Tor}_2(T, H^{3-n}(B_n)) \cong \mathrm{Tor}_2(T, \mathrm{Coker}\delta_n^{2-n}) \cong \mathrm{Tor}_1(T, \mathrm{Im}\ \delta_n^{2-n})$.
(2) $\mathrm{Tor}_1(T, H^{3-n}(B_n)) \cong \mathrm{Tor}_1(T, \mathrm{Coker}\delta_n^{2-n}) = 0$.

PROOF. (1) Note that $\mathrm{Tor}_2^B(T, \mathrm{Im}\ \delta_n^i) = 0$, for every $i \leq 0$, since $\mathrm{Im}\ \delta_n^i$ is a submodule of a projective module and $\mathrm{Tor}_3(T, -) = 0$. Thus, from the exact sequences:

$$0 \to H^{3-n}(B_n) \to \mathrm{Coker}\delta_n^{2-n} \to \mathrm{Im}\ \delta_n^{3-n} \to 0,$$

$$0 \to \mathrm{Im}\ \delta_n^{2-n} \to B_n^{3-n} \to \mathrm{Coker}\delta_n^{2-n} \to 0,$$

we obtain

$$0 \to \mathrm{Tor}_2(T, H^{3-n}(B_n)) \cong \mathrm{Tor}_2(T, \mathrm{Coker}\delta_n^{2-n}) \to 0,$$

$$0 \to \mathrm{Tor}_2(T, \mathrm{Coker}\delta_n^{2-n}) \cong \mathrm{Tor}_1(T, \mathrm{Im}\ \delta_n^{2-n}) \to 0.$$

(2) From the above exact sequences we obtain also

$$0 \to \mathrm{Tor}_1(T, H^{3-n}(B_n)) \to \mathrm{Tor}_1(T, \mathrm{Coker}\delta_n^{2-n}),$$

and $\mathrm{Tor}_1(T, \mathrm{Coker}\delta_n^{2-n}) = 0$ by Lemma 3.3.16 (2).

□

Lemma 3.3.17. *If* ${}_BN \in B$-Mod, *then*

$$\mathrm{Tor}_2^B(T,N) \cong Hom_B(H^2(Q),N).$$

Proof. We have $\mathrm{Tor}_2^B(T,N) \cong H^{-2}(G(N)) \cong H^{-2}($
$Hom_B(Q,N) = \mathrm{Ker}($
$Hom_B(d_Q^1,N) \cong$
$Hom_B(H^2(Q),N)$. □

Remark 3.3.18. If B_∞ has bounded cohomology, let's say $H^{-i}(B_\infty) = 0$ for every $-i \leq 2-n$, then B_∞ is quasi-isomorphic to $\dfrac{B_\infty}{\tau_{\leq 2-n}(B_\infty)} \cong \overline{B}_n$.

So necessarily $\mathrm{Tor}_2^B(T, \mathrm{Coker}\delta_n^{2-n}) = 0$ for some n.

Part 2

Equivalences of monoidal categories and bosonization for dual quasi-bialgebras

CHAPTER 4

Dual quasi-bialgebras and monoidal categories

In this chapter we recall the definitions and results that will be needed later on.

1. Monoidal categories

NOTATIONS 4.1.1. *Let $\Bbbk$ be a field. All vector spaces will be defined over $\Bbbk$. The unadorned tensor product $\otimes$ will denote the tensor product over $\Bbbk$ if not stated otherwise.*

DEFINITION 4.1.2. Recall that (see [**Ka**, Chap. XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom,* that is the equality

$$(U \otimes a_{V,W,X}) \circ a_{U,V\otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom,* that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories.

DEFINITION 4.1.3. Let $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. An (associative) algebra in $\mathcal{M}$ is a tern (A, m, u) where A is an object in the category, and
$m : A \otimes A \to A$ (multiplication)
$u : \mathbf{1} \to A$ (unit)
are morphisms in $\mathcal{M}$ obeying the *associativity* and *unity* axioms:

$$\begin{tikzcd}
 & (A\otimes A)\otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[dl, "m\otimes A"'] & A\otimes(A\otimes A) \arrow[dr, "A\otimes m"] & \\
A\otimes A \arrow[drr, "m"'] & & & A\otimes A \arrow[dl, "m"] \\
 & & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}\otimes A \arrow[r, "l_A"] \arrow[dr, "u\otimes A"'] & A & A\otimes\mathbf{1} \arrow[l, "r_A"'] \arrow[dl, "A\otimes u"] \\
 & A\otimes A \arrow[u, "m"] &
\end{tikzcd}$$

REMARK 4.1.4. In a dual way is define the concept of *coalgebra* in a monoidal category.

DEFINITION 4.1.5. Given an algebra A in $\mathcal{M}$ one can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of *left*, *right* and *two-sided modules* over A respectively. Given an object $V \in {}_A\mathcal{M}$, the associativity of the left action ρ of A over V is expressed by the diagram:

$$\begin{tikzcd} & (A \otimes A) \otimes V \arrow[r, "\alpha_{A,A,A}"] \arrow[dl, "m\otimes V"'] & A \otimes (A \otimes V) \arrow[dr, "A\otimes\rho"] & \\ A \otimes V \arrow[drr, "\rho"] & & & A \otimes V \arrow[dl, "\rho"'] \\ & & V & \end{tikzcd}$$

Similarly, given a coalgebra C in $\mathcal{M}$, one can define the categories of C-*comodules* ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$. For more details, the reader is refereed to [**Ka**].

REMARK 4.1.6. Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ is abelian and both the functors $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and right exact, for any $X \in \mathcal{M}$. Given an algebra A in $\mathcal{M}$, there exists a suitable functor $\otimes_A : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A$ and constraints that make the category $({}_A\mathcal{M}_A, \otimes_A, A)$ monoidal, see [**AMS1**, 1.11]. The tensor product over A in $\mathcal{M}$ of a right A-module (V, μ_V^r) and a left A-module (W, μ_W^l) is defined to be the coequalizer:

$$\begin{tikzcd} (V \otimes A) \otimes W \arrow[rr, "\mu_V^r \otimes W"] \arrow[dr, "a_{V,A,W}"'] & & V \otimes W \arrow[r, "{}_A\chi_{V,W}"] & V \otimes_A W \arrow[r] & 0 \\ & V \otimes (A \otimes W) \arrow[ur, "V\otimes\mu_W^l"'] & & & \end{tikzcd}$$

Note that, since $\otimes$ preserves coequalizers, then $V \otimes_A W$ is also an A-bimodule, whenever V and W are A-bimodules.

Dually, given a coalgebra (C, Δ, ε) in a monoidal category $\mathcal{M}$, abelian and with additive and left exact tensor functors, there exist a suitable functor $\Box_C : {}^C\mathcal{M}^C \times {}^C\mathcal{M}^C \to {}^C\mathcal{M}^C$ and constraints that make the category $({}^C\mathcal{M}^C, \Box_C, C)$ monoidal. The cotensor product over C in $\mathcal{M}$ of a right C-comodule (V, ρ_V^r) and a left C-comodule (W, ρ_W^l) is defined to be the equalizer:

$$\begin{tikzcd} 0 \arrow[r] & V\Box_C W \arrow[r, "{}_C\varsigma_{V,W}"] & V \otimes W \arrow[rr, "V\otimes\rho_W^l"] \arrow[dr, "\rho_V^r\otimes W"'] & & V \otimes (C \otimes W) \\ & & & (V \otimes C) \otimes W \arrow[ur, "a_{V,C,W}"'] & \end{tikzcd}$$

Note that, since $\otimes$ preserves equalizers, then $V\Box_C W$ is also a C-bicomodule, whenever V and W are C-bicomodules.

DEFINITION 4.1.7. A *dual quasi-bialgebra* is a datum $(H, m, u, \Delta, \varepsilon, \omega)$ where

- (H, Δ, ε) is a coassociative coalgebra;
- $m : H \otimes H \to H$ and $u : \Bbbk \to H$ are coalgebra maps called multiplication and unit respectively; we set $1_H := u(1_\Bbbk)$;

- $\omega : H \otimes H \otimes H \to \Bbbk$ is a unital 3-cocycle i.e. it is convolution invertible and satisfies

$$\omega\left(H \otimes H \otimes m\right) * \omega\left(m \otimes H \otimes H\right) = m_{\Bbbk}\left(\varepsilon \otimes \omega\right) * \omega\left(H \otimes m \otimes H\right) * m_{\Bbbk}\left(\omega \otimes \varepsilon\right) \tag{18}$$

$$\text{and} \quad \omega\left(h \otimes k \otimes l\right) = \varepsilon\left(h\right) \varepsilon\left(k\right) \varepsilon\left(l\right) \quad \text{whenever} \quad 1_H \in \{h, k, l\}; \tag{19}$$

- m is quasi-associative and unitary i.e. it satisfies

$$m\left(H \otimes m\right) * \omega = \omega * m\left(m \otimes H\right), \tag{20}$$

$$m\left(1_H \otimes h\right) = h, \text{ for all } h \in H, \tag{21}$$

$$m\left(h \otimes 1_H\right) = h, \text{ for all } h \in H. \tag{22}$$

ω is called *the reassociator* of the dual quasi-bialgebra.

A morphism of dual quasi-bialgebras (see e.g. [**Sch1**, Section 2])

$$\alpha : (H, m, u, \Delta, \varepsilon, \omega) \to (H', m', u', \Delta', \varepsilon', \omega')$$

is a coalgebra homomorphism $\alpha : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ such that

$$m'(\alpha \otimes \alpha) = \alpha m, \qquad \alpha u = u', \qquad \omega'\left(\alpha \otimes \alpha \otimes \alpha\right) = \omega.$$

It is an isomorphism of quasi-bialgebras if, in addition, it is invertible.

A dual quasi-subbialgebra of a dual quasi-bialgebra $(H', m', u', \Delta', \varepsilon', \omega')$ is a quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ such that H is a vector subspace of H' and the canonical inclusion $\alpha : H \to H'$ yields a morphism of dual quasi-bialgebras.

We shall see examples of dual quasi-bialgebras that are not ordinary bialgebras. In order to do it let us introduce the following concepts.

Definition 4.1.8. For any coalgebra C and algebra A we set $\mathrm{Reg}(C, A) := U(\mathrm{Hom}_k(C, A))$, i.e. the group of units in the monoid $(\mathrm{Hom}_k(C, A), *, u\varepsilon)$. Here, * denotes the convolution product which is defined as follows:

$$(f * g)(c) = \sum f(c_1) \cdot_A g(c_2).$$

A *Gauge transformation* γ on H is an element of $\mathrm{Reg}(H \otimes H, k)$ that is unital.

Theorem 4.1.9. *Let $(H, m, u, \Delta, \varepsilon)$ be a dual quasi bialgebra and γ a Gauge transformation on H. Let us define the maps $m_\gamma : (H \otimes H) \otimes H \to H$ and $\omega_\gamma : (H \otimes H) \otimes H \to \Bbbk$ as:*

$$m_\gamma(z) := \gamma(z_1) m(z_2) \gamma^{-1}(z_3), \text{ for all } z \in H^{\otimes 3};$$

$$\omega_\gamma := m_k(\varepsilon \otimes \gamma) * \gamma(H \otimes m) * \omega * \gamma^{-1}(m \otimes H) * m_k(\gamma^{-1} \otimes \varepsilon).$$

Then $(H, m_\gamma, u, \Delta, \varepsilon, \omega_\gamma)$ is a dual quasi-bialgebra, that is said to be equivalent *to H.*

Theorem 4.1.10. *Let $(H, m, u, \Delta, \varepsilon)$ be a dual quasi-bialgebra. The following assertions are equivalent.*

1. *$(H, M, u, \Delta, \varepsilon, \omega)$ is equivalent to a dual quasi bialgebra with trivial reassociator (that is an ordinary bialgebra).*
2. *$\omega = \gamma^{-1}(H \otimes m) * m_k(\varepsilon \otimes \gamma^{-1}) * m_k(\gamma \otimes \varepsilon) * \gamma(m \otimes H)$ for some Gauge transformation γ on H.*

REMARK 4.1.11. It can be proved that the situation of Theorem 4.1.10 does not always occur, i.e. there are non trivial dual quasi-bialgebras (for more details consider the dual situation in [**Ka**, XV, Sections 1,2,3]).

Let us introduce now the category of bicomodules for a dual quasi-bialgebras.

Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. It is well-known that the category $\mathfrak{M}^H$ of right H-comodules becomes a monoidal category as follows. Given a right H-comodule V, we denote by $\rho = \rho_V^r : V \to V \otimes H, \rho(v) = v_0 \otimes v_1$, its right H-coaction. The tensor product of two right H-comodules V and W is a comodule via diagonal coaction i.e. $\rho(v \otimes w) = v_0 \otimes w_0 \otimes v_1 w_1$. The unit is $\Bbbk$, which is regarded as a right H-comodule via the trivial coaction i.e. $\rho(k) = k \otimes 1_H$. The associativity and unit constraints are defined, for all $U, V, W \in \mathfrak{M}^H$ and $u \in U, v \in V, w \in W, k \in \Bbbk$, by

$$a^H_{U,V,W}((u \otimes v) \otimes w) := u_0 \otimes (v_0 \otimes w_0)\omega(u_1 \otimes v_1 \otimes w_1),$$
$$l_U(k \otimes u) := ku \qquad \text{and} \qquad r_U(u \otimes k) := uk.$$

The monoidal category we have just described will be denoted by $(\mathfrak{M}^H, \otimes, \Bbbk, a^H, l, r)$.

Similarly, the monoidal categories $({}^H\mathfrak{M}, \otimes, \Bbbk, {}^Ha, l, r)$ and $({}^H\mathfrak{M}^H, \otimes, \Bbbk, {}^Ha^H, l, r)$ are introduced. We just point out that

$${}^H a_{U,V,W}((u \otimes v) \otimes w) := \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes (v_0 \otimes w_0),$$
$${}^H a^H{}_{U,V,W}((u \otimes v) \otimes w) := \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes (v_0 \otimes w_0)\omega(u_1 \otimes v_1 \otimes w_1).$$

REMARK 4.1.12. We know that, if $(H, m, u, \Delta, \varepsilon, \omega)$ is a dual quasi-bialgebra, we cannot construct the category $\mathfrak{M}_H$, because H is not an algebra in this category. Moreover H is not an algebra in $\mathfrak{M}^H$ or in ${}^H\mathfrak{M}$. On the other hand $((H, \rho^l_H, \rho^r_H), m, u)$ is an algebra in the monoidal category $({}^H\mathfrak{M}^H, \otimes, \Bbbk, {}^Ha^H, l, r)$ with $\rho^l_H = \rho^r_H = \Delta$. Thus, the only way to construct the category ${}^H\mathfrak{M}^H_H$ is to consider the right H-modules in ${}^H\mathfrak{M}^H$. Hence, we can set

$${}^H\mathfrak{M}^H_H := ({}^H\mathfrak{M}^H)_H.$$

The category ${}^H\mathfrak{M}^H_H$ is the so-called category of *right dual quasi-Hopf H-bicomodules* [**BC**, Remark 2.3].

REMARK 4.1.13. Let (A, m, u) be an algebra in a given monoidal category $(\mathcal{M}, \otimes, 1, a, l, r)$. Then the assignments $M \longmapsto (M \otimes A, (M \otimes m) \circ a_{A,A,A})$ and $f \longmapsto f \otimes A$ define a functor $T : \mathcal{M} \to \mathcal{M}_A$. Moreover the forgetful functor $U : \mathcal{M}_A \to \mathcal{M}$ is a right adjoint of T.

2. An adjunction between ${}^H\mathfrak{M}^H_H$ and ${}^H\mathfrak{M}$

We are going to construct an adjunction between ${}^H\mathfrak{M}^H_H$ and ${}^H\mathfrak{M}$ that will be crucial afterwards.

REMARK 4.2.1. Consider the functor $L : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H$ defined on objects by $L({}^\bullet V) := {}^\bullet V^\circ$ where the upper empty dot denotes the trivial right coaction while the upper full dot denotes the given left H-coaction of V. The functor L has a right adjoint $R : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}$ defined on objects by $R({}^\bullet M^\bullet) := {}^\bullet M^{coH}$, where $M^{coH} := \{m \in M \mid m_0 \otimes m_1 = m \otimes 1_H\}$ is the space of right H-coinvariant elements

in M. Indeed, since $L(V)$ has the trivial right coaction, each $f : L(V) \to W \in {}^H\mathfrak{M}^H$ can be regarded as a morphism from V to $G(W)$ in ${}^H\mathfrak{M}$. On the other side, every morphism $g : V \to G(W) \in {}^H\mathfrak{M}$ is also right H-colinear with respect to the trivial coaction, that is g can be regarded as a morphism from $L(V)$ to W in ${}^H\mathfrak{M}^H$.
Moreover, by Remark 4.1.13, the forgetful functor $U : {}^H\mathfrak{M}^H_H \to {}^H\mathfrak{M}^H, U({}^\bullet M^\bullet_\bullet) := {}^\bullet M^\bullet$ has a right adjoint, namely the functor $T : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}^H_H, T({}^\bullet M^\bullet) := {}^\bullet M^\bullet \otimes {}^\bullet H^\bullet_\bullet$. Here the upper dots indicate on which tensor factors we have a codiagonal coaction and the lower dot indicates where the action takes place. Explicitly, the structure of $T({}^\bullet M^\bullet)$ is given as follows:

$$\begin{aligned}
\rho^l_{M\otimes H}(m \otimes h) \quad &: \quad = m_{-1}h_1 \otimes (m_0 \otimes h_2), \\
\rho^r_{M\otimes H}(m \otimes h) \quad &: \quad = (m_0 \otimes h_1) \otimes m_1 h_2, \\
\mu^r_{M\otimes H}\left[(m \otimes h) \otimes l\right] \quad &= \quad (m \otimes h)l := \omega^{-1}(m_{-1} \otimes h_1 \otimes l_1)m_0 \otimes h_2 l_2 \omega(m_1 \otimes h_3 \otimes l_3).
\end{aligned}$$

Define the functors $F := TL : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$ and $G := RU : {}^H\mathfrak{M}^H_H \to {}^H\mathfrak{M}$. Explicitly $G({}^\bullet M^\bullet_\bullet) = {}^\bullet M^{coH}$ and $F({}^\bullet V) := {}^\bullet V^\circ \otimes {}^\bullet H^\bullet_\bullet$ so that, for every $v \in V, h, l \in H$,

$$\begin{aligned}
\rho^l_{V\otimes H}(v \otimes h) &= v_{-1}h_1 \otimes (v_0 \otimes h_2), \\
\rho^r_{V\otimes H}(v \otimes h) &= (v \otimes h_1) \otimes h_2, \\
\mu^r_{V\otimes H}\left[(v \otimes h) \otimes l\right] &= (v \otimes h)l = \omega^{-1}(v_{-1} \otimes h_1 \otimes l_1)v_0 \otimes h_2 l_2.
\end{aligned}$$

Remark 4.2.2. By the right-hand version of [**Sch4**, Lemma 2.1], the functor $F : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$ is a left adjoint of the functor G, where the counit and the unit of the adjunction are given respectively by $\epsilon_M : FG(M) \to M, \epsilon_M(x \otimes h) := xh$ and by $\eta_N : N \to GF(N), \eta_N(n) := n \otimes 1_H$, for every $M \in {}^H\mathfrak{M}^H_H, N \in {}^H\mathfrak{M}$. Moreover η_N is an isomorphism for any $N \in {}^H\mathfrak{M}$. In particular the functor F is fully faithful.

3. The notion of preantipode

In what follows we will show that, for a dual quasi-bialgebra H, the functor F is an equivalence if and only if there exists a suitable map $S : H \to H$ that we called a preantipode for H. Moreover for any dual quasi-bialgebra with antipode (i.e. a dual quasi-Hopf algebra) we constructed a specific preantipode, see [**AP**, Theorem 3.10].

Remark 4.3.1. It is worth to notice that, by [**Sch5**, Example 4.5.1], there is a dual quasi-bialgebra H which is not a dual quasi-Hopf algebra and such that the category ${}^H\mathfrak{M}_f$ of finite-dimensional left H-comodules is left and right rigid so that, by the right-handed version of [**Sch4**, Theorem 3.1], we get that H has a preantipode. Nevertheless, for a finite-dimensional dual quasi-bialgebra, the existence of an antipode is equivalent to the existence of a preantipode. This follows by duality in view of [**Sch4**, Theorem 3.1]. Next result characterizes when the adjunction (F, G) is an equivalence of categories in term of the existence of a suitable map τ.

Proposition 4.3.2. [**AP**, Proposition 3.3] *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The following assertions are equivalent.*

(*i*) *The adjunction (F, G) is an equivalence.*

(*ii*) *For each* $M \in {}^{H}\mathfrak{M}_{H}^{H}$, *there exists a* $\Bbbk$*-linear map* $\tau : M \to M^{coH}$ *such that:*

$$\tau(mh) = \omega^{-1}[\tau(m_0)_{-1} \otimes m_1 \otimes h]\tau(m_0)_0, \text{ for all } h \in H, m \in M, \tag{23}$$

$$m_{-1} \otimes \tau(m_0) = \tau(m_0)_{-1}m_1 \otimes \tau(m_0)_0, \text{ for all } m \in M, \tag{24}$$

$$\tau(m_0)m_1 = m \ \forall m \in M. \tag{25}$$

(*iii*) *For each* $M \in {}^{H}\mathfrak{M}_{H}^{H}$, *there exists a* $\Bbbk$*-linear map* $\tau : M \to M^{coH}$ *such that (25) holds and*

$$\tau(mh) = m\varepsilon(h), \text{ for all } h \in H, m \in M^{coH}. \tag{26}$$

REMARK 4.3.3. Let $\tau : M \to M^{coH}$ be a $\Bbbk$-linear map such that (25) holds. By [**AP**, Remark 3.4], the map τ fulfills (26) if and only if it fulfills (23) and (24).

DEFINITION 4.3.4. Following [**AP**, Definition 3.6] we will say that a preantipode for a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ is a $\Bbbk$-linear map $S : H \to H$ such that, for all $h \in H$,

$$S(h_1)_1h_2 \otimes S(h_1)_2 = 1_H \otimes S(h), \tag{27}$$

$$S(h_2)_1 \otimes h_1S(h_2)_2 = S(h) \otimes 1_H, \tag{28}$$

$$\omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h). \tag{29}$$

REMARK 4.3.5. [**AP**, Remark 3.7] Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. Then the following equalities hold

$$h_1S(h_2) = \varepsilon S(h)1_H = S(h_1)h_2 \text{ for all } h \in H. \tag{30}$$

LEMMA 4.3.6. [**AP**, Lemma 3.8] *Let* $(H, m, u, \Delta, \varepsilon, \omega, S)$ *be a dual quasi-bialgebra with a preantipode. For any* $M \in {}^{H}\mathfrak{M}_{H}^{H}$ *and* $m \in M$, *set*

$$\tau(m) := \omega[m_{-1} \otimes S(m_1)_1 \otimes m_2]m_0S(m_1)_2. \tag{31}$$

Then (31) defines a map $\tau : M \to M^{coH}$ *which fulfills (23), (24) and (25).*

THEOREM 4.3.7. [**AP**, Theorem 3.9] *For a dual quasi-bialgebra* $(H, m, u, \Delta, \varepsilon, \omega)$ *the following are equivalent.*

(*i*) *The adjunction* (F, G) *of Remark 4.2.2 is an equivalence of categories.*
(*ii*) *There exists a preantipode.*

We include here some new results that will be needed later on.

LEMMA 4.3.8. *Let* $(H, m, u, \Delta, \varepsilon, \omega, S)$ *be a dual quasi-bialgebra with a preantipode. Then*

$$\omega^{-1}\left[S\left(h_1\right) \otimes h_2 \otimes S\left(h_3\right)\right] = \varepsilon S\left(h\right), \text{ for all } h \in H. \tag{32}$$

PROOF. Set $\alpha := \omega\left(H \otimes H \otimes m\right) * \omega\left(m \otimes H \otimes H\right) * m_{\Bbbk}\left(\omega^{-1} \otimes \varepsilon\right)$ and $\beta = m_{\Bbbk}\left(\varepsilon \otimes \omega\right) * \omega\left(H \otimes m \otimes H\right)$. Fix $h \in H$. We have

$$\begin{aligned}
&\alpha\left(S\left(h_1\right) \otimes h_2 \otimes S\left(h_3\right) \otimes h_4\right) \\
= {}& \omega\left[S\left(h_1\right)_1 \otimes h_2 \otimes S\left(h_5\right)_{(1)} h_6\right] \omega\left[S\left(h_1\right)_2 h_3 \otimes S\left(h_5\right)_{(2)} \otimes h_7\right] \omega^{-1}\left[S\left(h_1\right)_3 \otimes h_4 \otimes S\left(h_5\right)_{(3)}\right] \\
\overset{(27)}{=} {}& \omega\left[S\left(h_1\right)_1 \otimes h_2 \otimes 1_H\right] \omega\left[S\left(h_1\right)_2 h_3 \otimes S\left(h_5\right)_{(1)} \otimes h_6\right] \omega^{-1}\left[S\left(h_1\right)_3 \otimes h_4 \otimes S\left(h_5\right)_{(2)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \omega\left[S\left(h_1\right)_1 h_2 \otimes S\left(h_4\right)_{(1)} \otimes h_5\right] \omega^{-1}\left[S\left(h_1\right)_2 \otimes h_3 \otimes S\left(h_4\right)_{(2)}\right] \\
&\overset{(27)}{=} \omega\left[1_H \otimes S\left(h_4\right)_{(1)} \otimes h_5\right] \omega^{-1}\left[S\left(h_1\right) \otimes h_3 \otimes S\left(h_4\right)_{(2)}\right] \\
&= \omega^{-1}\left[S\left(h_1\right) \otimes h_2 \otimes S\left(h_3\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& \beta\left(S\left(h_1\right) \otimes h_2 \otimes S\left(h_3\right) \otimes h_4\right) \\
= {} & \omega\left[h_2 \otimes S\left(h_4\right)_{(1)} \otimes h_5\right] \omega\left[S\left(h_1\right) \otimes h_3 S\left(h_4\right)_{(2)} \otimes h_6\right] \\
\overset{(28)}{=} {} & \omega\left[h_2 \otimes S\left(h_3\right) \otimes h_4\right] \omega\left[S\left(h_1\right) \otimes 1_H \otimes h_5\right] \\
= {} & \omega\left[h_2 \otimes S\left(h_3\right) \otimes h_4\right] \varepsilon S\left(h_1\right) \\
\overset{(29)}{=} {} & \varepsilon S(h).
\end{aligned}
$$

By the cocycle condition we have $\alpha = \beta$. □

Definition 4.3.9. [**Maj1**, page 66] A dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra anti-homomorphism

$$s : H \to H$$

and two maps α, β in H^*, such that, for all $h \in H$:

$$
\begin{aligned}
&(33) \qquad & h_1 \beta(h_2) s(h_3) &= \beta(h) 1_H, \\
&(34) \qquad & s(h_1) \alpha(h_2) h_3 &= \alpha(h) 1_H, \\
&(35) \qquad & \omega(h_1 \otimes \beta(h_2) s(h_3) \alpha(h_4) \otimes h_5) &= \varepsilon(h) = \omega^{-1}(s(h_1) \otimes \alpha(h_2) h_3 \beta(h_4) \otimes s(h_5)).
\end{aligned}
$$

In [**AP**, Theorem 3.10], we proved that any dual quasi-Hopf algebra has a preantipode, but the converse, as pointed out in Remark 4.3.1, in general is not true. The following result proves that the converse holds true whenever H is also cocommutative.

Theorem 4.3.10. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. If H is cocommutative, then $(H, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra, where, for all $h \in H$,*

$$s(h) := S\left(h_3\right)_1 \omega\left[h_1 \otimes S\left(h_3\right)_2 \otimes h_2\right].$$

Furthermore $(H, m, u, \Delta, \varepsilon, \omega, \alpha, \beta, s)$ is a dual quasi-Hopf algebra, where $\alpha := \varepsilon$ and $\beta := \varepsilon S$. Moreover one has $S = \beta * s$.

Proof. By (20), cocommutativity and convolution invertibility of ω, we get that $(hk)l = h(kl)$ for all $h, k, l \in H$. Therefore m is associative and hence $(H, m, u, \Delta, \varepsilon)$ is an ordinary bialgebra. Let us check that s is an antipode for H. Using cocommutativity, (27) and (29) one proves that $s\left(h_1\right) h_2 = 1_H \varepsilon\left(h\right)$ for all $h \in H$. Similarly one gets $h_1 s(h_2) = 1_H \varepsilon\left(h\right)$ for all $h \in H$. Hence $(H, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra. Note that, for all $h \in H$,

$$(36) \qquad S(h) = S(h_1)\left[h_2 s(h_3)\right] = \left[S(h_1) h_2\right] s(h_3) \overset{(30)}{=} \varepsilon S(h_1) s(h_2) = \beta\left(h_1\right) s(h_2).$$

Let us check that $(H, m, u, \Delta, \varepsilon, \omega, \alpha, \beta, s)$ is a dual quasi-Hopf algebra. For all $h \in H$,

$$h_1\beta(h_2)s(h_3) \overset{(36)}{=} h_1 S(h_2) \overset{(30)}{=} 1_H\varepsilon S(h),$$
$$s(h_1)\alpha(h_2)h_3 = s(h_1)h_2 = 1_H\varepsilon(h) = 1_H\alpha(h),$$
$$\omega[h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5] \overset{(36)}{=} \omega[h_1 \otimes S(h_2) \otimes h_3] \overset{(29)}{=} 1_H\varepsilon(h).$$

Now, since $(H, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra, we have that s is an anti-coalgebra map. Thus

$$\begin{aligned} S(h)_1 \otimes S(h)_2 &\overset{(36)}{=} \beta(h_1)s(h_2)_1 \otimes s(h_2)_2 = \beta(h_1)s(h_3) \otimes s(h_2) \\ &\overset{\text{cocom.}}{=} \beta(h_1)s(h_2) \otimes s(h_3) \overset{(36)}{=} S(h_1) \otimes s(h_2) \end{aligned}$$

so that

$$\begin{aligned} & \omega^{-1}[s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)] \\ \overset{(36)}{=} \; & \omega^{-1}[s(h_1) \otimes h_2 \otimes S(h_3)] \\ = \; & \omega^{-1}[S(h_3)_1 \otimes h_4 \otimes S(h_5)]\,\omega[h_1 \otimes S(h_3)_2 \otimes h_2] \\ = \; & \omega^{-1}[S(h_3) \otimes h_5 \otimes S(h_6)]\,\omega[h_1 \otimes s(h_4) \otimes h_2] \\ \overset{\text{cocom.}}{=} \; & \omega^{-1}[S(h_2) \otimes h_3 \otimes S(h_4)]\,\omega[h_1 \otimes s(h_5) \otimes h_6] \\ \overset{(32)}{=} \; & \varepsilon S(h_2)\,\omega[h_1 \otimes s(h_3) \otimes h_4] \\ \overset{(36)}{=} \; & \omega[h_1 \otimes S(h_2) \otimes h_3] \overset{(29)}{=} 1_H\varepsilon(h). \end{aligned}$$

□

Definition 4.3.11. A dual quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \omega)$ is called pointed if the underlying coalgebra is pointed, i.e. all its simple subcoalgebras are one dimensional.

Definition 4.3.12. Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The set

$$\mathbb{G}(A) = \{a \in A \mid \Delta(a) = a \otimes a \text{ and } \varepsilon(a) = 1\}$$

is called the set of the grouplike elements of A.

Remark 4.3.13. Let A be a pointed dual quasi-bialgebra. We know that the 1-dimensional subcoalgebras of A are exactly those of the form $\Bbbk g$ for $g \in G$ ([**Sw**, page 57]). Thus the coradical of A is $A_0 = \sum_{g\in G} \Bbbk g = \Bbbk\mathbb{G}(A)$.

The following results extends the so-called "Cartier-Gabriel-Kostant" to dual quasi-bialgebras with a preantipode. In the connected case such a result was achieved in [**Hu**, Theorem 4.3].

Corollary 4.3.14. *Let H be a dual quasi-bialgebra with a preantipode over a field $\Bbbk$ of characteristic zero. If H is cocommutative and pointed, then H is an ordinary Hopf algebra isomorphic to the biproduct $U(P(H))\#\Bbbk\mathbb{G}(H)$, where $P(H)$ denotes the Lie algebra of primitive elements in H.*

Proof. By Theorem 4.3.10, H is an ordinary Hopf algebra. By [**Sw**, Section 13.1, page 279], we conclude (see also [**Mo**, page 79]). □

4. Yetter-Drinfeld modules over a dual quasi-bialgebra

The main aim of this section is to restrict the equivalence between ${}^{H}\mathfrak{M}_{H}^{H}$ and ${}^{H}\mathfrak{M}$ of Theorem 4.3.7, to an equivalence between ${}_{H}^{H}\mathfrak{M}_{H}^{H}$ and ${}_{H}^{H}\mathcal{YD}$ (the category of Yetter-Drinfeld modules over H) for any dual quasi-bialgebra H with a preantipode.

4.1. Yetter-Drinfeld modules.

DEFINITION 4.4.1. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The category ${}_{H}^{H}\mathcal{YD}$ of Yetter-Drinfeld modules over H, is defined as follows. An object in ${}_{H}^{H}\mathcal{YD}$ is a tern $(V, \rho_V, \triangleright)$, where

- (V, ρ) is an object in ${}^{H}\mathfrak{M}$
- $\triangleright$: $H \otimes V \to V$ is a $\Bbbk$-linear map such that, for all $h, l \in L$ and $v \in V$

$$(hl) \triangleright v = \left[\begin{array}{c} \omega^{-1}(h_1 \otimes l_1 \otimes v_{-1})\, \omega\left(h_2 \otimes (l_2 \triangleright v_0)_{-1} \otimes l_3\right) \\ \omega^{-1}\left((h_3 \triangleright (l_2 \triangleright v_0)_0)_{-1} \otimes h_4 \otimes l_4\right)(h_3 \triangleright (l_2 \triangleright v_0)_0)_0 \end{array} \right], \tag{37}$$

$$1_H \triangleright v = v \qquad \text{and} \tag{38}$$

$$(h_1 \triangleright v)_{-1}\, h_2 \otimes (h_1 \triangleright v)_0 = h_1 v_{-1} \otimes (h_2 \triangleright v_0) \tag{39}$$

A morphism $f : (V, \rho, \triangleright) \to (V', \rho', \triangleright')$ in ${}_{H}^{H}\mathcal{YD}$ is a morphism $f : (V, \rho) \to (V', \rho')$ in ${}^{H}\mathfrak{M}$ such that $f(h \triangleright v) = h \triangleright' f(v)$.

DEFINITION 4.4.2. Let us recall that a *prebraided* monoidal category $\mathcal{A}$, is a monoidal category, such that, for all $X, Y \in \mathcal{A}$, there is a natural morphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

such that the following equalities hold true for all $X, Y, Z, \in \mathcal{A}$:

$$(c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) = c_{X \otimes Y, Z};$$

$$(Y \otimes c_{X,Z})(c_{X,Y} \otimes Z) = c_{X, Y \otimes Z}.$$

$\mathcal{A}$ is said *braided* if $c_{X,Y}$ is invertible for all $X, Y \in \mathcal{A}$.

REMARK 4.4.3. The category ${}_{H}^{H}\mathcal{YD}$ is isomorphic to the weak right center of ${}^{H}\mathfrak{M}$ (regarded as a monoidal category as at the beginning of this Chapter 4, see Theorem 5.3.11). As a consequence ${}_{H}^{H}\mathcal{YD}$ has a pre-braided monoidal structure given as follows. The unit is $\Bbbk$ regarded as an object in ${}_{H}^{H}\mathcal{YD}$ via trivial structures i.e. $\rho_{\Bbbk}(k) = 1_H \otimes k$ and $h \triangleright k = \varepsilon(h)\, k$. The tensor product is defined by

$$(V, \rho_V, \triangleright) \otimes (W, \rho_W, \triangleright) = (V \otimes W, \rho_{V \otimes W}, \triangleright)$$

where $\rho_{V \otimes W}(v \otimes w) = v_{-1} w_{-1} \otimes v_0 \otimes w_0$ and

$$h \triangleright (v \otimes w) = \left[\begin{array}{c} \omega(h_1 \otimes v_{-1} \otimes w_{-2})\, \omega^{-1}\left((h_2 \triangleright v_0)_{-2} \otimes h_3 \otimes w_{-1}\right) \\ \omega\left((h_2 \triangleright v_0)_{-1} \otimes (h_4 \triangleright w_0)_{-1} \otimes h_5\right)(h_2 \triangleright v_0)_0 \otimes (h_4 \triangleright w_0)_0 \end{array} \right]. \tag{40}$$

The constraints are the same of ${}^{H}\mathfrak{M}$ viewed as morphisms in ${}_{H}^{H}\mathcal{YD}$. The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by

$$c_{V,W}(v \otimes w) = (v_{-1} \triangleright w) \otimes v_0. \tag{41}$$

REMARK 4.4.4. It is easily checked that condition (37) holds for all $h,l\in L$ and $v\in V$ if and only if

$$c_{H\otimes H,V}={}^{H}a_{V,H,H}\circ(c_{H,V}\otimes H)\circ{}^{H}a^{-1}_{H,V,H}\circ(H\otimes c_{H,V})\circ{}^{H}a_{H,H,V},$$

where ${}^{H}a$ is the associativity constraint in ${}^{H}\mathfrak{M}$. Now, the displayed equality above, can be written as

$${}^{H}a^{-1}_{V,H,H}\circ c_{H\otimes H,V}\circ{}^{H}a^{-1}_{H,H,V}=(c_{H,V}\otimes H)\circ{}^{H}a^{-1}_{H,V,H}\circ(H\otimes c_{H,V})\,.$$

One easily checks that this is equivalent to ask that

$$\begin{aligned}&\omega\left(h_{1}\otimes l_{1}\otimes v_{-1}\right)\omega\left(\left((h_{2}l_{2})\rhd v_{0}\right)_{-1}\otimes h_{3}\otimes l_{3}\right)\left((h_{2}l_{2})\rhd v_{0}\right)_{0}\\ =\;&\omega(h_{1}\otimes(l_{1}\rhd v)_{-1}\otimes l_{2})h_{3}\rhd(l_{1}\rhd v)_{0}\end{aligned}$$

holds for all $h,l\in L$ and $v\in V$. This equation is the left-handed version of [**Ba**, (3.3)]. In conclusion, the axioms defining the category ${}_{H}^{H}\mathcal{YD}$ are the left-handed version of the ones appearing in [**Ba**, Definition 3.1].

4.2. The restriction of the equivalence (F,G). Let H be a dual quasi-bialgebra. From Theorem 4.3.7, we know that the adjunction (F,G) of Remark 4.2.2 is an equivalence of categories when H has a preantipode.
Next aim is to prove that (F,G) restricts to an equivalence between the categories ${}_{H}^{H}\mathcal{YD}$ and ${}_{H}^{H}\mathfrak{M}_{H}^{H}$, where ${}_{H}^{H}\mathfrak{M}_{H}^{H}={}_{H}({}^{H}\mathfrak{M}^{H})_{H}$ is the subcategory of ${}^{H}\mathfrak{M}^{H}$ given by the objects that are also bimodules over H.

Inspired by [**Sch4**, page 541] we get the following result.

LEMMA 4.4.5. *Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. For all $U\in{}^{H}\mathfrak{M}$ and $M\in{}_{H}^{H}\mathfrak{M}_{H}^{H}$, we have a map*

$$\begin{aligned}\xi_{U,M}&:&F(U)\otimes_{H}M\to U\otimes M,\\ \xi_{U,M}\left((u\otimes h)\otimes_{H}m\right)&=&\omega^{-1}\left(u_{-1}\otimes h_{1}\otimes m_{-1}\right)u_{0}\otimes h_{2}m_{0}\end{aligned}$$

which is a $\Bbbk$-linear natural isomorphism with inverse given by $\xi_{U,M}^{-1}\left(u\otimes m\right)=(u\otimes 1_{H})\otimes_{H}m$. Moreover:

1) *the map $\xi_{U,M}$ is a natural isomorphism in ${}^{H}\mathfrak{M}_{H}^{H}$ where $U\otimes M$ has the following structures:*

$$\begin{aligned}\rho^{l}_{U\otimes M}\left(u\otimes m\right)&=&u_{-1}m_{-1}\otimes(u_{0}\otimes m_{0}),\\ \rho^{r}_{U\otimes M}\left(u\otimes m\right)&=&(u\otimes m_{0})\otimes m_{1},\\ \mu^{r}_{U\otimes M}((u\otimes m)\otimes h)&=&\omega^{-1}(u_{-1}\otimes m_{-1}\otimes h_{1})u_{0}\otimes m_{0}h_{2};\end{aligned}$$

2) *if $U\in{}_{H}^{H}\mathcal{YD}$, the map $\xi_{U,M}$ is a natural isomorphism in ${}_{H}^{H}\mathfrak{M}_{H}^{H}$ where $U\otimes M$ has the structures above along with the following left module structure:*

$${\mu^{l}}_{U\otimes M}(h\otimes(u\otimes m))=\omega\left(h_{1}\otimes u_{-1}\otimes m_{-2}\right)\omega^{-1}\left((h_{2}\rhd u_{0})_{-1}\otimes h_{3}\otimes m_{-1}\right)(h_{2}\rhd u_{0})_{0}\otimes h_{4}m_{0}.$$

PROOF. Clearly $U\otimes M\in{}^{H}\mathfrak{M}^{H}$ via $\rho^{l}_{U\otimes M}$ and $\rho^{r}_{U\otimes M}$. Let $\xi'_{U,M}:F\left(U\right)\otimes M\to U\otimes M$ be defined by $\xi'_{U,M}\left((u\otimes h)\otimes m\right)=\omega^{-1}\left(u_{-1}\otimes h_{1}\otimes m_{-1}\right)u_{0}\otimes h_{2}m_{0}$.

Using the quasi-associativity condition (20), one easily checks that $\xi'_{U,M}$ is in ${}^{H}\mathfrak{M}^{H}$.

Let us check that $\xi'_{U,M}$ is balanced in ${}^{H}\mathfrak{M}^{H}$ i.e. that it equalizes the maps

$$\left(F\left(U\right)\otimes H\right)\otimes M \underset{\left(F(U)\otimes\mu^{l}_{M}\right)\circ {}^{H}a^{H}_{F(U),H,M}}{\overset{\mu^{r}_{F(U)}\otimes M}{\rightrightarrows}} F\left(U\right)\otimes M$$

We have

$$\begin{array}{cl} & \xi'_{U,M}\left(\mu^{r}_{F(U)}\otimes M\right)\left(\left(\left(u\otimes h\right)\otimes l\right)\otimes m\right)\\ = & \omega^{-1}\left(u_{-1}\otimes h_{1}\otimes l_{1}\right)\xi'_{U,M}\left(\left(u_{0}\otimes h_{2}l_{2}\right)\otimes m\right)\\ = & \omega^{-1}\left(u_{-2}\otimes h_{1}\otimes l_{1}\right)\omega^{-1}\left(u_{-1}\otimes h_{2}l_{2}\otimes m_{-1}\right)u_{0}\otimes\left(h_{3}l_{3}\right)m_{0}\\ = & \left[\begin{array}{c}\omega^{-1}\left(u_{-2}\otimes h_{1}\otimes l_{1}\right)\omega^{-1}\left(u_{-1}\otimes h_{2}l_{2}\otimes m_{-2}\right)\omega^{-1}\left(h_{3}\otimes l_{3}\otimes m_{-1}\right)\\ u_{0}\otimes h_{4}\left(l_{4}m_{0}\right)\omega\left(h_{5}\otimes l_{5}\otimes m_{1}\right)\end{array}\right]\\ \overset{(18)}{=} & \omega^{-1}\left(u_{-2}h_{1}\otimes l_{1}\otimes m_{-2}\right)\omega^{-1}\left(u_{-1}\otimes h_{2}\otimes l_{2}m_{-1}\right)u_{0}\otimes h_{3}\left(l_{3}m_{0}\right)\omega\left(h_{4}\otimes l_{4}\otimes m_{1}\right)\\ = & \omega^{-1}\left(u_{-1}h_{1}\otimes l_{1}\otimes m_{-2}\right)\xi'_{U,M}\left(F\left(U\right)\otimes\mu^{l}_{M}\right)\left(\left(u_{0}\otimes h_{2}\right)\otimes\left(l_{2}\otimes m_{0}\right)\right)\omega\left(h_{3}\otimes l_{3}\otimes m_{1}\right)\\ = & \xi'_{U,M}\left(F\left(U\right)\otimes\mu^{l}_{M}\right){}^{H}a^{H}_{F(U),H,M}\left(\left(\left(u\otimes h\right)\otimes l\right)\otimes m\right).\end{array}$$

Hence there exists a unique morphism $\xi_{U,M}: F\left(U\right)\otimes_{H} M\rightarrow U\otimes M$ in ${}^{H}\mathfrak{M}^{H}$ such that $\xi_{U,M}\left(\left(u\otimes h\right)\otimes_{H} m\right)=\xi'_{U,M}\left(\left(u\otimes h\right)\otimes m\right)$. This proves that $\xi_{U,M}$ is well-defined.

We now check that $\xi_{U,M}$ is invertible. Define

$$\overline{\xi}_{U,M}: U\otimes M\rightarrow F\left(U\right)\otimes_{H} M,\qquad \overline{\xi}_{U,M}\left(u\otimes m\right)=\left(u\otimes 1_{H}\right)\otimes_{H} m.$$

We have $\xi_{U,M}\circ\overline{\xi}_{U,M}=\mathrm{Id}_{U\otimes M}$ and

$$\begin{array}{cl} & \overline{\xi}_{U,M}\xi_{U,M}\left(\left(u\otimes h\right)\otimes_{H} m\right)\\ = & \omega^{-1}\left(u_{-1}\otimes h_{1}\otimes m_{-1}\right)\left(u_{0}\otimes 1_{H}\right)\otimes_{H} h_{2}m_{0}\\ \overset{\mathrm{def}\otimes_{H}}{=} & \left[\begin{array}{c}\omega^{-1}\left(u_{-1}\otimes h_{1}\otimes m_{-2}\right)\omega\left(\left(u_{0}\otimes 1_{H}\right)_{-1}\otimes h_{2}\otimes m_{-1}\right)\\ \left(u_{0}\otimes 1_{H}\right)_{0}h_{3}\otimes_{H} m_{0}\omega^{-1}\left(\left(u_{0}\otimes 1_{H}\right)_{1}\otimes h_{2}\otimes m_{1}\right)\end{array}\right]\\ = & \left[\begin{array}{c}\omega^{-1}\left(u_{-2}\otimes h_{1}\otimes m_{-2}\right)\omega\left(u_{-1}\otimes h_{2}\otimes m_{-1}\right)\\ \left(u_{0}\otimes 1_{H}\right)h_{3}\otimes_{H} m_{0}\omega^{-1}\left(1_{H}\otimes h_{4}\otimes m_{1}\right)\end{array}\right]\\ = & \left(u\otimes 1_{H}\right)h\otimes_{H} m=\left(u\otimes h\right)\otimes_{H} m.\end{array}$$

The proof that $\xi^{-1}_{U,M}:=\overline{\xi}_{U,M}$ is natural in U and M is straightforward.

1) In order to have that $\xi_{U,M}$ is in ${}^{H}\mathfrak{M}^{H}_{H}$, it suffices to prove that $\xi'_{U,M}$ is in ${}^{H}\mathfrak{M}^{H}_{H}$. Clearly, $\xi'_{U,M}$ is in ${}^{H}\mathfrak{M}^{H}$ being an inverse of $\xi_{U,M}$.

The map $\xi'_{U,M}$ is right H-linear in ${}^{H}\mathfrak{M}^{H}$:

$$\begin{array}{cl} & \xi'_{U,M}\left[\left(\left(u\otimes h\right)\otimes m\right)l\right]\\ = & \omega^{-1}\left(\left(u\otimes h\right)_{-1}\otimes m_{-1}\otimes l_{1}\right)\xi'_{U,M}\left[\left(u\otimes h\right)_{0}\otimes m_{0}l_{2}\right]\omega\left(\left(u\otimes h\right)_{1}\otimes m_{1}\otimes l_{3}\right)\\ = & \omega^{-1}\left(u_{-1}h_{1}\otimes m_{-1}\otimes l_{1}\right)\xi'_{U,M}\left[\left(u_{0}\otimes h_{2}\right)\otimes m_{0}l_{2}\right]\omega\left(h_{3}\otimes m_{1}\otimes l_{3}\right)\\ = & \left[\begin{array}{c}\omega^{-1}\left(u_{-2}h_{1}\otimes m_{-2}\otimes l_{1}\right)\omega^{-1}\left(u_{-1}\otimes h_{2}\otimes m_{-1}l_{2}\right)\\ u_{0}\otimes h_{3}\left(m_{0}l_{3}\right)\omega\left(h_{4}\otimes m_{1}\otimes l_{4}\right)\end{array}\right]\end{array}$$

$$
\begin{aligned}
&\overset{(20)}{=} \left[\begin{array}{c} \omega^{-1}(u_{-2}h_1 \otimes m_{-3} \otimes l_1)\,\omega^{-1}(u_{-1} \otimes h_2 \otimes m_{-2}l_2)\,\omega(h_3 \otimes m_{-1} \otimes l_3) \\ u_0 \otimes (h_4 m_0) l_4 \end{array} \right] \\
&\overset{(18)}{=} \omega^{-1}(u_{-2} \otimes h_1 \otimes m_{-2})\,\omega^{-1}(u_{-1} \otimes h_2 m_{-1} \otimes l_1)\, u_0 \otimes (h_3 m_0) l_2 \\
&= \omega^{-1}(u_{-1} \otimes h_1 \otimes m_{-1})\,(u_0 \otimes h_2 m_0)\, l \\
&= \xi'_{U,M}((u \otimes h) \otimes m)\, l
\end{aligned}
$$

2) $\xi'_{U,M}$ is left H-linear in ${}^H\mathfrak{M}^H$:

$$
\begin{aligned}
& \xi'_{U,M}[l((u \otimes h) \otimes m)] \\
= \;& \omega\left(l_1 \otimes (u \otimes h)_{-1} \otimes m_{-1}\right) \xi'_{U,M}[l_2 (u \otimes h)_0 \otimes m_0]\, \omega^{-1}(l_3 \otimes (u \otimes h)_1 \otimes m_1) \\
= \;& \omega(l_1 \otimes u_{-1}h_1 \otimes m_{-1})\, \xi'_{U,M}[l_2(u_0 \otimes h_2) \otimes m_0]\, \omega^{-1}(l_3 \otimes h_3 \otimes m_1) \\
= \;& \left[\begin{array}{c} \omega(l_1 \otimes u_{-2}h_1 \otimes m_{-1})\,\omega(l_2 \otimes u_{-1} \otimes h_2)\omega^{-1}((l_3 \rhd u_0)_{-1} \otimes l_4 \otimes h_3) \\ \xi'_{U,M}[\{(l_3 \rhd u_0)_0 \otimes l_5 h_4\} \otimes m_0]\, \omega^{-1}(l_6 \otimes h_5 \otimes m_1) \end{array} \right] \\
= \;& \left[\begin{array}{c} \omega(l_1 \otimes u_{-2}h_1 \otimes m_{-2})\,\omega(l_2 \otimes u_{-1} \otimes h_2)\omega^{-1}((l_3 \rhd u_0)_{-2} \otimes l_4 \otimes h_3) \\ \omega^{-1}((l_3 \rhd u_0)_{-1} \otimes l_5 h_4 \otimes m_{-1})\,(l_3 \rhd u_0)_0 \otimes (l_6 h_5)\, m_0 \omega^{-1}(l_7 \otimes h_6 \otimes m_1) \end{array} \right] \\
\overset{(20)}{=} \;& \left[\begin{array}{c} \omega(l_1 \otimes u_{-2}h_1 \otimes m_{-3})\,\omega(l_2 \otimes u_{-1} \otimes h_2)\omega^{-1}((l_3 \rhd u_0)_{-2} \otimes l_4 \otimes h_3) \\ \omega^{-1}((l_3 \rhd u_0)_{-1} \otimes l_5 h_4 \otimes m_{-2})\,\omega^{-1}(l_6 \otimes h_5 \otimes m_{-1})\,(l_3 \rhd u_0)_0 \otimes l_7(h_6 m_0) \end{array} \right] \\
\overset{(18)}{=} \;& \left[\begin{array}{c} \omega(l_1 \otimes u_{-2}h_1 \otimes m_{-3})\,\omega(l_2 \otimes u_{-1} \otimes h_2)\omega^{-1}((l_3 \rhd u_0)_{-2} l_4 \otimes h_3 \otimes m_{-2}) \\ \omega^{-1}((l_3 \rhd u_0)_{-1} \otimes l_5 \otimes h_4 m_{-1})\,(l_3 \rhd u_0)_0 \otimes l_6(h_5 m_0) \end{array} \right] \\
\overset{(39)}{=} \;& \left[\begin{array}{c} \omega(l_1 \otimes u_{-3}h_1 \otimes m_{-3})\,\omega(l_2 \otimes u_{-2} \otimes h_2)\omega^{-1}(l_3 u_{-1} \otimes h_3 \otimes m_{-2}) \\ \omega^{-1}\left((l_4 \rhd u_0)_{-1} \otimes l_5 \otimes h_4 m_{-1}\right)(l_4 \rhd u_0)_0 \otimes l_6(h_5 m_0) \end{array} \right] \\
\overset{(18)}{=} \;& \left[\begin{array}{c} \omega^{-1}(u_{-2} \otimes h_1 \otimes m_{-3})\,\omega(l_1 \otimes u_{-1} \otimes h_2 m_{-2}) \\ \omega^{-1}\left((l_2 \rhd u_0)_{-1} \otimes l_3 \otimes h_3 m_{-1}\right)(l_2 \rhd u_0)_0 \otimes l_4(h_4 m_0) \end{array} \right] \\
= \;& \omega^{-1}(u_{-1} \otimes h_1 \otimes m_{-1})\, l\, [u_0 \otimes h_2 m_0] = l \xi'_{U,M}((u \otimes h) \otimes m).
\end{aligned}
$$

$\square$

LEMMA 4.4.6. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. For all $U, V \in {}^H\mathfrak{M}$, consider the map*

$$
\begin{aligned}
\alpha_{U,V} \;&:\; U \otimes (V \otimes H) \to (U \otimes V) \otimes H \\
\alpha_{U,V}(u \otimes (v \otimes k)) \;&=\; \omega(u_{-1} \otimes v_{-1} \otimes k_1)(u_0 \otimes v_0) \otimes k_2.
\end{aligned}
$$

1) *The map $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ is a natural isomorphism in ${}^H\mathfrak{M}^H_H$, where $U \otimes F(V)$ has the structure described in Lemma 4.4.5 for $M = F(V)$.*
2) *If $U, V \in {}^H_H\mathcal{YD}$, then $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$, where $U \otimes F(V)$ has the structure described in Lemma 4.4.5 for $M = F(V)$.*

PROOF. Note that $\alpha_{U,V} = \left({}^H a_{U,V,H}\right)^{-1}$ so that $\alpha_{U,V} \in {}^H\mathfrak{M}$ and it is invertible.

1) Let us check that $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ is a morphism in ${}^H\mathfrak{M}^H_H$, where $U \otimes F(V)$ has the structure described in Lemma 4.4.5 for $M = F(V)$.

Let us check that $\alpha_{U,V}$ is right H-colinear.

$$\alpha_{U,V}[(u \otimes (v \otimes k))_0] \otimes (u \otimes (v \otimes k))_1$$

$$
\begin{aligned}
&= \alpha_{U,V}\left[u\otimes(v\otimes k)_0\right]\otimes(v\otimes k)_1\\
&= \alpha_{U,V}\left[u\otimes(v\otimes k_1)\right]\otimes k_2\\
&= \omega\left(u_{-1}\otimes v_{-1}\otimes k_1\right)\left(u_0\otimes v_0\right)\otimes k_2\otimes k_3\\
&= \rho^r_{F(U\otimes V)}\left[\omega\left(u_{-1}\otimes v_{-1}\otimes k_1\right)\left(u_0\otimes v_0\right)\otimes k_2\right]\\
&= \rho^r_{F(U\otimes V)}\alpha_{U,V}\left(u\otimes(v\otimes k)\right).
\end{aligned}
$$

Moreover the 3-cocycle condition (18) yields that $\alpha_{U,V}$ is right H-linear in ${}^H\mathfrak{M}^H$, i.e. that $\alpha_{U,V}$ is a morphism in ${}^H\mathfrak{M}^H_H$.

2) Let us check that $\alpha_{U,V}$ is left H-linear in ${}^H\mathfrak{M}^H$. On the one hand we have

$$
\begin{aligned}
\alpha_{U,V}\left[h\left(u\otimes(v\otimes k)\right)\right] &= \begin{bmatrix} \omega\left(h_1\otimes u_{-1}\otimes(v\otimes k)_{-2}\right)\\ \omega^{-1}\left((h_2\rhd u_0)_{-1}\otimes h_3\otimes(v\otimes k)_{-1}\right)\alpha_{U,V}\left[(h_2\rhd u_0)_0\otimes h_4(v\otimes k)_0\right]\end{bmatrix}\\
&= \begin{bmatrix} \omega\left(h_1\otimes u_{-1}\otimes v_{-2}k_1\right)\\ \omega^{-1}\left((h_2\rhd u_0)_{-1}\otimes h_3\otimes v_{-1}k_2\right)\alpha_{U,V}\left[(h_2\rhd u_0)_0\otimes h_4(v_0\otimes k_3)\right]\end{bmatrix}\\
&= \begin{bmatrix} \omega\left(h_1\otimes u_{-1}\otimes v_{-3}k_1\right)\\ \omega^{-1}\left((h_2\rhd u_0)_{-1}\otimes h_3\otimes v_{-2}k_2\right)\omega(h_4\otimes v_{-1}\otimes k_3)\\ \omega^{-1}((h_5\rhd v_0)_{-1}\otimes h_6\otimes k_4)\alpha_{U,V}\left[(h_2\rhd u_0)_0\otimes\left[(h_5\rhd v_0)_0\otimes h_7k_5\right]\right]\end{bmatrix}\\
&= \begin{bmatrix} \omega\left(h_1\otimes u_{-1}\otimes v_{-3}k_1\right)\\ \omega^{-1}\left((h_2\rhd u_0)_{-2}\otimes h_3\otimes v_{-2}k_2\right)\omega(h_4\otimes v_{-1}\otimes k_3)\\ \omega^{-1}((h_5\rhd v_0)_{-2}\otimes h_6\otimes k_4)\omega\left((h_2\rhd u_0)_{-1}\otimes(h_5\rhd v_0)_{-1}\otimes h_7k_5\right)\\ \left[(h_2\rhd u_0)_0\otimes(h_5\rhd v_0)_0\right]\otimes h_8k_6\end{bmatrix}
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&h\alpha_{U,V}\left(u\otimes(v\otimes k)\right)=\omega\left(u_{-1}\otimes v_{-1}\otimes k_1\right)h\left[(u_0\otimes v_0)\otimes k_2\right]\\
&= \begin{bmatrix} \omega\left(u_{-1}\otimes v_{-1}\otimes k_1\right)\omega(h_1\otimes(u_0\otimes v_0)_{-1}\otimes k_2)\\ \omega^{-1}((h_2\rhd(u_0\otimes v_0)_0)_{-1}\otimes h_3\otimes k_3)(h_2\rhd(u_0\otimes v_0)_0)_0\otimes h_4k_4\end{bmatrix}\\
&= \begin{bmatrix} \omega\left(u_{-2}\otimes v_{-2}\otimes k_1\right)\omega(h_1\otimes u_{-1}v_{-1}\otimes k_2)\\ \omega^{-1}((h_2\rhd(u_0\otimes v_0))_{-1}\otimes h_3\otimes k_3)(h_2\rhd(u_0\otimes v_0))_0\otimes h_4k_4\end{bmatrix}\\
&\overset{(40)}{=} \begin{bmatrix} \omega\left(u_{-2}\otimes v_{-2}\otimes k_1\right)\omega(h_1\otimes u_{-1}v_{-1}\otimes k_2)\\ \omega\left((h_2)_1\otimes(u_0)_{-1}\otimes(v_0)_{-2}\right)\omega^{-1}\left(((h_2)_2\rhd(u_0)_0)_{-2}\otimes(h_2)_3\otimes(v_0)_{-1}\right)\\ \omega\left(((h_2)_2\rhd(u_0)_0)_{-1}\otimes((h_2)_4\rhd(v_0)_0)_{-1}\otimes(h_2)_5\right)\\ \omega^{-1}(\left[((h_2)_2\rhd(u_0)_0)_0\otimes((h_2)_4\rhd(v_0)_0)_0\right]_{-1}\otimes h_3\otimes k_3)\\ \left[((h_2)_2\rhd(u_0)_0)_0\otimes((h_2)_4\rhd(v_0)_0)_0\right]_0\otimes h_4k_4\end{bmatrix}\\
&= \begin{bmatrix} \omega\left(u_{-3}\otimes v_{-4}\otimes k_1\right)\omega(h_1\otimes u_{-2}v_{-3}\otimes k_2)\omega\left(h_2\otimes u_{-1}\otimes v_{-2}\right)\\ \omega^{-1}\left((h_3\rhd u_0)_{-2}\otimes h_4\otimes v_{-1}\right)\omega\left((h_3\rhd u_0)_{-1}\otimes(h_5\rhd v_0)_{-1}\otimes h_6\right)\\ \omega^{-1}(((h_3\rhd u_0)_0\otimes(h_5\rhd v_0)_0)_{-1}\otimes h_7\otimes k_3)\\ ((h_3\rhd u_0)_0\otimes(h_5\rhd v_0)_0)_0\otimes h_8k_4\end{bmatrix}\\
&\overset{(18)}{=} \begin{bmatrix} \omega(h_1\otimes u_{-2}\otimes v_{-3}k_1)\omega(h_2u_{-1}\otimes v_{-2}\otimes k_2)\omega^{-1}\left((h_3\rhd u_0)_{-2}\otimes h_4\otimes v_{-1}\right)\\ \omega\left((h_3\rhd u_0)_{-1}\otimes(h_5\rhd v_0)_{-1}\otimes h_6\right)\omega^{-1}(((h_3\rhd u_0)_0\otimes(h_5\rhd v_0)_0)_{-1}\otimes h_7\otimes k_3)\\ ((h_3\rhd u_0)_0\otimes(h_5\rhd v_0)_0)_0\otimes h_8k_4\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\begin{array}{c}
\omega(h_1 \otimes u_{-2} \otimes v_{-3}k_1)\omega(h_2u_{-1} \otimes v_{-2} \otimes k_2)\omega^{-1}\left((h_3 \rhd u_0)_{-3} \otimes h_4 \otimes v_{-1}\right) \\
\omega\left((h_3 \rhd u_0)_{-2} \otimes (h_5 \rhd v_0)_{-2} \otimes h_6\right)\omega^{-1}((h_3 \rhd u_0)_{-1}(h_5 \rhd v_0)_{-1} \otimes h_7 \otimes k_3) \\
((h_3 \rhd u_0)_0 \otimes (h_5 \rhd v_0)_0) \otimes h_8k_4
\end{array}\right] \\
&\overset{(18)}{=} \left[\begin{array}{c}
\omega(h_1 \otimes u_{-2} \otimes v_{-3}k_1)\omega(h_2u_{-1} \otimes v_{-2} \otimes k_2)\omega^{-1}\left((h_3 \rhd u_0)_{-3} \otimes h_4 \otimes v_{-1}\right) \\
\omega^{-1}((h_3 \rhd u_0)_{-2} \otimes (h_5 \rhd v_0)_{-3}h_6 \otimes k_3)\omega^{-1}((h_5 \rhd v_0)_{-2} \otimes h_7 \otimes k_4) \\
\omega((h_3 \rhd u_0)_{-1} \otimes (h_5 \rhd v_0)_{-1} \otimes h_8k_5)((h_3 \rhd u_0)_0 \otimes (h_5 \rhd v_0)_0) \otimes h_9k_6
\end{array}\right] \\
&\overset{(39)}{=} \left[\begin{array}{c}
\omega(h_1 \otimes u_{-2} \otimes v_{-4}k_1)\omega(h_2u_{-1} \otimes v_{-3} \otimes k_2)\omega^{-1}\left((h_3 \rhd u_0)_{-3} \otimes h_4 \otimes v_{-2}\right) \\
\omega^{-1}((h_3 \rhd u_0)_{-2} \otimes h_5v_{-1} \otimes k_3)\omega^{-1}((h_6 \rhd v_0)_{-2} \otimes h_7 \otimes k_4) \\
\omega((h_3 \rhd u_0)_{-1} \otimes (h_6 \rhd v_0)_{-1} \otimes h_8k_5)((h_3 \rhd u_0)_0 \otimes (h_6 \rhd v_0)_0) \otimes h_9k_6
\end{array}\right] \\
&\overset{(18)}{=} \left[\begin{array}{c}
\omega(h_1 \otimes u_{-2} \otimes v_{-5}k_1)\omega(h_2u_{-1} \otimes v_{-4} \otimes k_2)\omega^{-1}((h_3 \rhd u_0)_{-3}h_4 \otimes v_{-3} \otimes k_3) \\
\omega^{-1}((h_3 \rhd u_0)_{-2} \otimes h_5 \otimes v_{-2}k_4)\omega(h_6 \otimes v_{-1} \otimes k_5)\omega^{-1}((h_7 \rhd v_0)_{-2} \otimes h_8 \otimes k_6) \\
\omega((h_3 \rhd u_0)_{-1} \otimes (h_7 \rhd v_0)_{-1} \otimes h_9k_7)((h_3 \rhd u_0)_0 \otimes (h_7 \rhd v_0)_0) \otimes h_{10}k_8
\end{array}\right] \\
&\overset{(39)}{=} \left[\begin{array}{c}
\omega(h_1 \otimes u_{-3} \otimes v_{-5}k_1)\omega(h_2u_{-2} \otimes v_{-4} \otimes k_2)\omega^{-1}(h_3u_{-1} \otimes v_{-3} \otimes k_3) \\
\omega^{-1}((h_4 \rhd u_0)_{-2} \otimes h_5 \otimes v_{-2}k_4)\omega(h_6 \otimes v_{-1} \otimes k_5)\omega^{-1}((h_7 \rhd v_0)_{-2} \otimes h_8 \otimes k_6) \\
\omega((h_4 \rhd u_0)_{-1} \otimes (h_7 \rhd v_0)_{-1} \otimes h_9k_7)((h_4 \rhd u_0)_0 \otimes (h_7 \rhd v_0)_0) \otimes h_{10}k_8
\end{array}\right] \\
&= \left[\begin{array}{c}
\omega(h_1 \otimes u_{-1} \otimes v_{-3}k_1)\omega^{-1}((h_2 \rhd u_0)_{-2} \otimes h_3 \otimes v_{-2}k_2)\omega(h_4 \otimes v_{-1} \otimes k_3) \\
\omega^{-1}((h_5 \rhd v_0)_{-2} \otimes h_6 \otimes k_4)\omega((h_2 \rhd u_0)_{-1} \otimes (h_5 \rhd v_0)_{-1} \otimes h_7k_5) \\
((h_2 \rhd u_0)_0 \otimes (h_5 \rhd v_0)_0) \otimes h_8k_6
\end{array}\right].
\end{aligned}
$$

Summing up, we have proved that $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ is an isomorphism in ${}^H_H\mathfrak{M}^H_H$. Now, since $\alpha_{U,V} = \left({}^H a_{U,V,H}\right)^{-1}$, we have that $\alpha_{U,V}$ is natural in U, V for all morphisms in ${}^H\mathfrak{M}$ (in particular in ${}^H_H\mathcal{YD}$). $\square$

LEMMA 4.4.7. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The functor $F : (-) \otimes H : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$ of 4.2.1. induces a functor $F : {}^H_H\mathcal{YD} \to {}^H_H\mathfrak{M}^H_H$. Explicitly $F(M) \in {}^H_H\mathfrak{M}^H_H$ with the following structures, for all $m \in M, h, l \in H$,*

$$
\begin{aligned}
&\mu^l_{M\otimes H}[l \otimes (m \otimes h)] := l \cdot (m \otimes h) \qquad (42)\\
&l \cdot (m \otimes h) := \omega(l_1 \otimes m_{-1} \otimes h_1)(l_2 \rhd m_0 \otimes l_3) \cdot h_2 \\
&= \omega(l_1 \otimes m_{-1} \otimes h_1)\omega^{-1}((l_2 \rhd m_0)_{-1} \otimes l_3 \otimes h_2)(l_2 \rhd m_0)_0 \otimes l_4h_3 \\
&\mu^r_{M\otimes H}[(m \otimes h) \otimes l] := (m \otimes h) \cdot l \qquad (43)\\
&(m \otimes h) \cdot l := \omega^{-1}(m_{-1} \otimes h_1 \otimes l_1)m_0 \otimes h_2l_2, \\
&\rho^l_{M\otimes H}(m \otimes h) := m_{-1}h_1 \otimes (m_0 \otimes h_2), \\
&\rho^r_{M\otimes H}(m \otimes h) := (m \otimes h_1) \otimes h_2,
\end{aligned}
$$

PROOF. Let $M \in {}^H_H\mathcal{YD}$. Consider $H \otimes M$ as an object in ${}^H\mathfrak{M}^H$ via

$$
\begin{aligned}
\rho^r_{H\otimes M}(h \otimes m) &:= (h_1 \otimes m) \otimes h_2, \\
\rho^l_{H\otimes M}(h \otimes m) &:= h_1m_{-1} \otimes (h_2 \otimes m_0).
\end{aligned}
$$

Since $(H\otimes M, \rho^l_{H\otimes M}) \in {}^H\mathfrak{M}$, by Lemma 4.4.6, the map $\alpha_{H,M} : H \otimes F(M) \to F(H \otimes M)$ is a natural isomorphism in ${}^H\mathfrak{M}^H_H$, where $H \otimes F(M)$ has the structure described in Lemma 4.4.5 for "M" $= F(M)$, i.e. for all $h \in H, x \in M \otimes H$

$$
\rho^l_{H\otimes F(M)}(h \otimes x) = h_1x_{-1} \otimes (h_2 \otimes x_0),
$$

$$\begin{aligned}\rho^r_{H\otimes F(M)}(h\otimes x) &= (h\otimes x_0)\otimes x_1,\\ \mu^r_{H\otimes F(M)}((h\otimes x)\otimes k) &= \omega^{-1}(h_1\otimes x_{-1}\otimes k_1)h_2\otimes x_0k_2.\end{aligned}$$

In particular, we have

$$\rho^l_{T(H\otimes M)}\alpha_{H,M} = \rho^l_{F(H\otimes M)}\alpha_{H,M} = (H\otimes\alpha_{H,M})\,\rho^l_{H\otimes F(M)}$$

where $T : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}^H_H, T({}^\bullet M^\bullet) := {}^\bullet M^\bullet \otimes {}^\bullet H^\bullet_\bullet$ is the functor of 4.2.1. Now, consider on $H\otimes F(M)$ the following new structures

$$\begin{aligned}\widetilde{\rho}^l_{H\otimes F(M)}(h\otimes x) &= h_1x_{-1}\otimes(h_2\otimes x_0),\\ \widetilde{\rho}^r_{H\otimes F(M)}(h\otimes x) &= (h_1\otimes x_0)\otimes h_2x_1,\\ \widetilde{\mu}^r_{H\otimes F(M)}((h\otimes x)\otimes k) &= \omega^{-1}(h_1\otimes x_{-1}\otimes k_1)h_2\otimes x_0k_2\omega(h_3\otimes x_1\otimes k_3),\end{aligned}$$

note that $\widetilde{\mu}^r_{H\otimes F(M)} = \left(H\otimes\mu^r_{F(M)}\right)\circ {}^Ha^H_{H,F(M),H}$. Moreover one gets

$$\rho^r_{T(H\otimes M)}\alpha_{H,M} = (\alpha_{H,M}\otimes H)\,\widetilde{\rho}^r_{H\otimes F(M)}$$

and

$$\begin{aligned}&\mu^r_{T(H\otimes M)}(\alpha_{H,M}\otimes H)([h\otimes(m\otimes k)]\otimes l)\\ =\;&\omega(h_1\otimes m_{-2}\otimes k_1)\,\omega^{-1}(h_2m_{-1}\otimes k_2\otimes l_1)(h_3\otimes m_0)\otimes k_3l_2\omega(h_4\otimes k_4\otimes l_3)\\ =\;&\mu^r_{F(H\otimes M)}(\alpha_{H,M}\otimes H)\left[(h_1\otimes(m\otimes k_1)_0)\otimes l_1\right]\omega(h_2\otimes(m\otimes k_1)_1\otimes l_2)\\ =\;&\alpha_{H,M}\mu^r_{H\otimes F(M)}\left[(h_1\otimes(m\otimes k_1)_0)\otimes l_1\right]\omega(h_2\otimes(m\otimes k_1)_1\otimes l_2)\\ =\;&\omega^{-1}(h_1\otimes(m\otimes k)_{-1}\otimes l_1)\alpha_{H,M}\left[h_2\otimes(m\otimes k_1)_0\cdot l_2\right]\omega(h_3\otimes(m\otimes k_1)_1\otimes l_3)\\ =\;&\alpha_{H,M}\widetilde{\mu}^r_{H\otimes F(M)}([h\otimes(m\otimes k)]\otimes l)\end{aligned}$$

We have so proved that $\alpha_{H,M}$ can be regarded as a morphism in ${}^H\mathfrak{M}^H_H$ from $H\otimes F(M)$ to $T(H\otimes M)$, where $H\otimes F(M)$ has structures $\widetilde{\rho}^l_{H\otimes F(M)}, \widetilde{\rho}^r_{H\otimes F(M)}$ and $\widetilde{\mu}^r_{H\otimes F(M)}$.

Consider the map $c_{H,M} : H\otimes M\to M\otimes H$, as in (41) i.e. $c_{H,M}(h\otimes m) = (h_1\triangleright m)\otimes h_2$.

Using (39) one can prove that $c_{H,M} : H\otimes M\to F(M)$ is a morphism in ${}^H\mathfrak{M}^H$ (where $H\otimes M$ is regarded as an object in ${}^H\mathfrak{M}^H$ as at the beginning of this proof) whence $T(c_{H,M})$ is in ${}^H\mathfrak{M}^H_H$ (note that we do not know that H is in ${}^H_H\mathcal{YD}$ so that we cannot say that $c_{H,M}$ is in ${}^H_H\mathcal{YD}$ directly).

Now, consider the morphism $\mu^r_{F(M)} : F(M)\otimes H\to F(M)$. Clearly $\mu^r_{F(M)}$ can be regarded as a morphism in ${}^H\mathfrak{M}^H_H$ from $TF(M)$ to $F(M)$. Summing up we can consider in ${}^H\mathfrak{M}^H_H$ the composition

$$\mu^l_{M\otimes H} := \left(H\otimes F(M)\xrightarrow{\alpha_{H,M}} T(H\otimes M)\xrightarrow{T\left(c_{H,M}\right)} TF(M)\xrightarrow{\mu^r_{F(M)}} F(M)\right)$$

where $H\otimes F(M)$ has structures $\widetilde{\rho}^l_{H\otimes F(M)}, \widetilde{\rho}^r_{H\otimes F(M)}$ and $\widetilde{\mu}^r_{H\otimes F(M)}$. Thus $\mu^l_{M\otimes H}$ is a morphism in ${}^H\mathfrak{M}^H$ such that

$$\mu^r_{M\otimes H}\circ\left(\mu^l_{M\otimes H}\otimes H\right) = \mu^l_{M\otimes H}\circ\left(H\otimes\mu^r_{M\otimes H}\right)\circ {}^Ha^H_{H,M\otimes H,H}. \tag{44}$$

It remains to prove that $\left(M\otimes H,\mu^l_{M\otimes H}\right)$ is a left H-module in ${}^H\mathfrak{M}^H$. Let us prove that

$$\mu^l_{M\otimes H}\circ\left(H\otimes\mu^l_{M\otimes H}\right)\circ{}^Ha^H_{H,H,M\otimes H}=\mu^l_{M\otimes H}\circ\left[m\otimes(M\otimes H)\right]. \tag{45}$$

First note that, using (44) and (18) one checks that

$$\begin{aligned}&\mu^l_{M\otimes H}\left(H\otimes\mu^l_{M\otimes H}\right){}^Ha^H_{H,H,M\otimes H}\left[(h\otimes k)\otimes(m\otimes l)\right]\\ =\ &\omega(h_1k_1\otimes m_{-1}\otimes l_1)\left[\mu^l_{M\otimes H}\left(H\otimes\mu^l_{M\otimes H}\right){}^Ha^H_{H,H,M\otimes H}\left[(h_2\otimes k_2)\otimes(m_0\otimes 1_H)\right]\right]l_2\end{aligned}$$

and

$$\begin{aligned}&\mu^l_{M\otimes H}\left[m\otimes(M\otimes H)\right]\left[(h\otimes k)\otimes(m\otimes l)\right]\\ \overset{(44)}{=}\ &\omega(h_1k_1\otimes m_{-1}\otimes l_1)\left[(h_2k_2)\left(m_0\otimes 1_H\right)\right]l_2\\ =\ &\omega(h_1k_1\otimes m_{-1}\otimes l_1)\mu^l_{M\otimes H}\left[m\otimes(M\otimes H)\right]\left[(h_2\otimes k_2)\otimes(m_0\otimes 1_H)\right]l_2\end{aligned}$$

Thus we have to prove that (45) holds on elements of the form $(h\otimes k)\otimes(m\otimes 1_H)$.

We have

$$\begin{aligned}&\mu^l_{M\otimes H}\left(H\otimes\mu^l_{M\otimes H}\right){}^Ha^H_{H,H,M\otimes H}\left[(h\otimes k)\otimes(m\otimes 1_H)\right]\\ =\ &\begin{bmatrix}\omega^{-1}(h_1\otimes k_1\otimes m_{-1})\omega(h_2\otimes(k_2\rhd m_0)_{-1}\otimes k_3)\\ \left[h_3\rhd(k_2\rhd m_0)_0\otimes h_4\right]k_4\end{bmatrix}\\ =\ &\begin{bmatrix}\omega^{-1}(h_1\otimes k_1\otimes m_{-1})\omega(h_2\otimes(k_2\rhd m_0)_{-1}\otimes k_3)\\ \omega^{-1}((h_3\rhd(k_2\rhd m_0)_0)_{-1}\otimes h_4\otimes k_4)(h_3\rhd(k_2\rhd m_0)_0)_0\otimes h_5k_5\end{bmatrix}\\ \overset{(37)}{=}\ &h_1k_1\rhd m\otimes h_2k_2=(hk)\left(m\otimes 1_H\right)=\mu^l_{M\otimes H}\left[m\otimes(M\otimes H)\right]\left[(h\otimes k)\otimes(m\otimes 1_H)\right].\end{aligned}$$

Finally one checks that, for each morphism $f:M\to N$ in ${}^H_H\mathcal{YD}$, we have $F(f):=f\otimes H\in{}^H_H\mathfrak{M}^H_H$.

□

LEMMA 4.4.8. *Let $(H,m,u,\Delta,\varepsilon,\omega,S)$ be a dual quasi-bialgebra with a preantipode. The functor $G:(-)^{coH}:{}^H\mathfrak{M}^H_H\to{}^H\mathfrak{M}$ of 4.2.1 induces a functor $G:{}^H_H\mathfrak{M}^H_H\to{}^H_H\mathcal{YD}$. Explicitly $G(M)\in{}^H_H\mathcal{YD}$ with the following structures, for all $m\in M^{coH},h\in H$,*

$$\begin{aligned}\rho^l_{M^{coH}}(m)\ &:=\rho^l_M(m),\\ \mu^l_{M^{coH}}(h\otimes m)\ &:=h\rhd m:=\tau(hm)=\omega[h_1m_{-1}\otimes S(h_3)_1\otimes h_4]\left(h_2m_0\right)S(h_3)_2.\end{aligned}$$

PROOF. Let $M\in{}^H_H\mathfrak{M}^H_H$. We already know that $G(M)\in{}^H\mathfrak{M}$. In order to prove that $G(M)$ is in ${}^H_H\mathcal{YD}$, we consider the canonical isomorphism $\epsilon_M:FG(M)\to M$ of Remark 4.2.2. A priori, this is a morphism in ${}^H\mathfrak{M}^H_H$. Since M is in ${}^H_H\mathfrak{M}^H_H$, we can endow $FG(M)$ with a left H-module structure as follows

$$\begin{aligned}l\cdot(m\otimes h)\ &:=\ \epsilon_M^{-1}\left(l\epsilon_M(m\otimes h)\right)=\epsilon_M^{-1}\left(l(mh)\right)=\tau\left[l_1(m_0h_1)\right]\otimes l_2(m_1h_2)\\ &=\ \tau\left[l_1(mh_1)\right]\otimes l_2h_2=l_1\rhd(mh_1)\otimes l_2h_2\end{aligned}$$

so that

$$l\cdot(m\otimes h)=l_1\rhd(mh_1)\otimes l_2h_2,\ \text{for all } m\in M^{\mathrm{co}H},h\in H. \tag{46}$$

By associativity we have

$$(lk)\cdot(m\otimes h)=\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}h_1\right)l_2\left(k_2\left(m_0\otimes h_2\right)\right)\omega\left(l_3\otimes k_3\otimes h_3\right)$$

i.e., for $h=1_H$,

$$(lk)\cdot(m\otimes 1_H)=\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)l_2\left(k_2\left(m_0\otimes 1_H\right)\right).$$

The first term is

$$(lk)\cdot(m\otimes 1_H)\overset{(46)}{=}(l_1k_1)\rhd m\otimes l_2k_2.$$

The second term is

$$\begin{aligned}
&\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)l_2\left(k_2\left(m_0\otimes 1_H\right)\right)\overset{(46)}{=}\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)l_2\left(k_2\rhd m_0\otimes k_3\right)\\
\overset{(46)}{=}\;&\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)l_2\rhd\left(\left(k_2\rhd m_0\right)k_3\right)\otimes l_3k_4\\
=\;&\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)\tau\left[l_2((k_2\rhd m_0)\,k_3)\right]\otimes l_3k_4\\
=\;&\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)\omega\left(l_2\otimes\left(k_2\rhd m_0\right)_{-1}\otimes k_3\right)\tau\left[\left(l_3\left(k_2\rhd m_0\right)_0\right)k_4\right]\otimes l_4k_5\\
=\;&\left[\begin{array}{c}\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)\omega\left(l_2\otimes\left(k_2\rhd m_0\right)_{-1}\otimes k_3\right)\\ \omega^{-1}\left(\left(l_3\rhd\left(k_2\rhd m_0\right)_0\right)_{-1}\otimes l_4\otimes k_4\right)\left(l_3\rhd\left(k_2\rhd m_0\right)_0\right)_0\otimes l_5k_5\end{array}\right]
\end{aligned}$$

Hence, we obtain

$$(l_1k_1)\rhd m\otimes l_2k_2=\left[\begin{array}{c}\omega^{-1}\left(l_1\otimes k_1\otimes m_{-1}\right)\omega\left(l_2\otimes\left(k_2\rhd m_0\right)_{-1}\otimes k_3\right)\\ \omega^{-1}\left(\left(l_3\rhd\left(k_2\rhd m_0\right)_0\right)_{-1}\otimes l_4\otimes k_4\right)\left(l_3\rhd\left(k_2\rhd m_0\right)_0\right)_0\otimes l_5k_5\end{array}\right].$$

By applying $M\otimes\varepsilon_H$ on both sides, we arrive at (37). Moreover, by (26), we have $1_H\rhd m=\tau(m)=m$ and

$$\begin{aligned}
&\left(h_1\rhd m\right)_{-1}h_2\otimes\left(h_1\rhd m\right)_0=\tau\left(h_1m\right)_{-1}h_2\otimes\tau\left(h_1m\right)_0=\tau\left((hm)_0\right)_{-1}(hm)_1\otimes\tau\left((hm)_0\right)_0\\
\overset{(24)}{=}\;&(hm)_{-1}\otimes\tau\left((hm)_0\right)=h_1m_{-1}\otimes\tau\left(h_2m_0\right)=h_1m_{-1}\otimes\left(h_2\rhd m_0\right).
\end{aligned}$$

We have so proved that $G(M)\in{}_H^H\mathcal{YD}$. Now it is easy to verify that for every $g:M\to N\in{}_H^H\mathfrak{M}_H^H$, we have that $G(g):M^{coH}\to N^{coH}\in{}_H^H\mathcal{YD}$. $\square$

Proposition 4.4.9. *Let $(H,m,u,\Delta,\varepsilon,\omega,S)$ be a dual quasi-bialgebra with a preantipode. (F,G) is an equivalence between ${}_H^H\mathfrak{M}_H^H$ and ${}_H^H\mathcal{YD}$, i.e. the morphisms ε_M and η_N of Remark 4.2.2 are in ${}_H^H\mathfrak{M}_H^H$ and in ${}_H^H\mathcal{YD}$ respectively, for each $M\in{}_H^H\mathfrak{M}_H^H$, $N\in{}_H^H\mathcal{YD}$.*

Proof. We already know that $\varepsilon_M\in{}^H\mathfrak{M}_H^H$. Let us check that ε_M is left H-linear.

$$\begin{aligned}
&\varepsilon_M\mu_{M^{coH}\otimes H}(h\otimes m\otimes k)=\varepsilon_M(h\cdot(m\otimes k))\overset{(4.4.7)}{=}\varepsilon_M[\omega(h_1\otimes m_{-1}\otimes k_1)(h_2\rhd m_0\otimes h_3)k_2]\\
\overset{\varepsilon_M\text{ right lin}}{=}\;&\omega(h_1\otimes m_{-1}\otimes k_1)\varepsilon_M[(h_2\rhd m_0\otimes h_3)]k_2=\omega(h_1\otimes m_{-1}\otimes k_1)[(h_2\rhd m_0)h_3]k_2\\
=\;&\omega(h_1\otimes m_{-1}\otimes k_1)[\tau(h_2m_0)h_3]k_2=\omega(h_1\otimes m_{-1}\otimes k_1)[\tau(h_2m_0)(h_3m_1)]k_2\\
\overset{(25)}{=}\;&\omega(h_1\otimes m_{-1}\otimes k_1)(h_2m_0)k_2\overset{(20)}{=}h_1(m_0k_1)\omega(h_2\otimes m_1\otimes k_2)=h(mk)=\mu_M(H\otimes\varepsilon_M)(h\otimes m\otimes k).
\end{aligned}$$

Now let us check the compatibility of η with $\rhd$. For $N\in{}_H^H\mathcal{YD}$ and $n\in N$,

$$\left[\mu^l_{(N\otimes H)^{coH}}\circ{}^Ha_{H,N,H}\circ(H\otimes\eta_N)\right](h\otimes n)=\left[\mu^l_{(N\otimes H)^{coH}}\circ{}^Ha_{H,N,H}\right](h\otimes(n\otimes 1_H))$$

$= \omega^{-1}(h_1 \otimes n_{-1} \otimes 1_H)\mu^l_{(N\otimes H)^{coH}}(h_2 \otimes (n_0 \otimes 1_H)) = \mu^l_{(N\otimes H)^{coH}}(h \otimes (n \otimes 1_H)) = \tau(h\,(n \otimes 1_H))$
$\overset{(4.4.7)}{=} \tau(h_1 \rhd n \otimes h_2) \overset{(43)}{=} \tau((h_1 \rhd n \otimes 1_H)\,h_2) = \tau(\eta_N(h_1 \rhd n)h_2) \overset{(26)}{=} \eta_N(h_1 \rhd n)\varepsilon_H\,(h_2) = \eta_N(h \rhd n)$

So $\eta_N \in {}^H_H\mathcal{YD}$, for each $N \in {}^H_H\mathcal{YD}$. □

5. Monoidal equivalences

In this section we prove that the equivalence between the categories ${}^H_H\mathfrak{M}^H_H$ and ${}^H_H\mathcal{YD}$ becomes monoidal if we equip ${}^H_H\mathfrak{M}^H_H$ with the tensor product $\otimes_H$ (or $\square_H$) and unit H. As a by-product we produce a monoidal equivalence between $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$ and $({}^H_H\mathfrak{M}^H_H, \square_H, H)$.

Lemma 4.5.1. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The category $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$ is monoidal with respect to the following constraints:*

$$\begin{aligned} a_{U,V,W}((u \otimes_H v) \otimes_H w) &= \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes_H (v_0 \otimes_H w_0)\omega(u_1 \otimes v_1 \otimes w_1) \\ l_U(h \otimes_H u) &= hu \\ r_U(u \otimes_H h) &= uh \end{aligned}$$

Proof. See e.g. [**AMS1**, Theorem 1.12]. □

Lemma 4.5.2. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Let $U \in {}^H\mathfrak{M}^H$, $V \in {}^H\mathfrak{M}^H_H$. Then $(U \otimes V, \rho^l, \rho^r, \mu) \in {}^H\mathfrak{M}^H_H$ with the following structures:*

$$\begin{aligned} \rho^l_{U\otimes V}(u \otimes v) &= u_{-1}v_{-1} \otimes (u_0 \otimes v_0) \\ \rho^r_{U\otimes V}(u \otimes v) &= (u_0 \otimes v_0) \otimes u_1 v_1 \\ \mu^r_{U\otimes V}((u \otimes v) \otimes h) &= \omega^{-1}(u_{-1} \otimes v_{-1} \otimes h_1)u_0 \otimes v_0 h_2 \omega(u_1 \otimes v_1 \otimes h_3). \end{aligned}$$

Proof. It is left to the reader. □

Definition 4.5.3. We recall that a *lax monoidal functor*

$$(F, \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{M}', \otimes', \mathbf{1}', a', l', r')$$

between two monoidal categories consists of

- a functor $F : \mathcal{M} \to \mathcal{M}'$,
- a natural transformation $\phi_2(U, V) : F(U) \otimes' F(V) \to F(U \otimes V)$, with $U, V \in \mathcal{M}$, and
- a natural transformation $\phi_0 : \mathbf{1}' \to F(\mathbf{1})$ such that the diagram

(47)
$$\begin{tikzcd} (F(U) \otimes' F(V)) \otimes' F(W) \arrow[r, "\phi_2(U,V)\otimes' F(W)"] \arrow[d, "a'_{F(U),F(V),F(W)}"'] & F(U \otimes V) \otimes' F(W) \arrow[r, "\phi_2(U\otimes V,W)"] & F((U \otimes V) \otimes W) \arrow[d, "F(a_{U,V,W})"] \\ F(U) \otimes' (F(V) \otimes' F(W)) \arrow[r, "F(U)\otimes'\phi_2(V,W)"] & F(U) \otimes' F(V \otimes W) \arrow[r, "\phi_2(U,V\otimes W)"] & F(U \otimes (V \otimes W)) \end{tikzcd}$$

commutes and the following conditions are satisfied:

$$F(l_U) \circ \phi_2(\mathbf{1}, U) \circ (\phi_0 \otimes F(U)) = l'_{F(U)}, \tag{48}$$

$$F(r_U) \circ \phi_2(U, \mathbf{1}) \circ (F(U) \otimes \phi_0) = r'_{F(U)}. \tag{49}$$

The morphisms $\phi_2(U,V)$ and ϕ_0 are called *structure morphisms.*

Colax monoidal functors are defined similarly but with the directions of the structure morphisms reversed. A strong monoidal functor or simply a *monoidal functor* is a lax monoidal functor with invertible structure morphisms.

Examples of lax and colax functors that are not monoidal are given in the nex Lemma.

LEMMA 4.5.4. *Let $(\mathcal{M},\otimes,\mathbf{1})$ be a monoidal category which is abelian.*

(1) *Let A be an algebra in $\mathcal{M}$. Assume that the tensor functors are additive and right exact (see* [**AMS1**, Theorem 1.12]*). Then the forgetful functor*
$$D:({}_A\mathcal{M}_A\,,\otimes_A,A)\longrightarrow(\mathcal{M},\otimes,\mathbf{1})$$
is a lax monoidal functor with structure morphisms
$$\zeta_2(M,N):D(M)\otimes D(N)\to D(M\otimes_A N)\qquad\text{and}\qquad\zeta_0:\mathbf{1}\to D(A),$$
where ζ_2 is the canonical epimorphism and ζ_0 is the unity of A.

(2) *Let C be a coalgebra in $\mathcal{M}$. Assume that the tensor functors are additive and left exact. Then the forgetful functor*
$$D:({}^C\mathcal{M}^C,\square_C,C)\longrightarrow(\mathcal{M},\otimes,\mathbf{1})$$
is a colax monoidal functor with structure morphisms
$$\zeta_2(M,N):D(M\square_C N)\to D(M)\otimes D(N)\qquad\text{and}\qquad\zeta_0:D(C)\to\mathbf{1},$$
where ζ_2 is the canonical monomorphism and ζ_0 is the counit of C.

PROOF. 1) From [**AMS1**, 1.11], for all $M,N,S\in{}_A\mathcal{M}_A$, we deduce
$$\begin{aligned}&D\left({}^Aa^A_{M,N,S}\right)\circ\zeta_2(M\otimes_A N,S)\circ[\zeta_2(M,N)\otimes D\left(S\right)]\\=\;&\zeta_2(M,N\otimes_A S)\circ[D\left(M\right)\otimes\zeta_2(N,S)]\circ a_{M,N,S}.\end{aligned}$$
Moreover, for all $M\in{}_A\mathcal{M}_A$, we have
$$\begin{aligned}D\left({}^Al^A_M\right)\circ\zeta_2(A,M)\circ[\zeta_0\otimes D\left(M\right)]&={}^Al^A_M\circ\zeta_2(A,M)\circ(\zeta_0\otimes M)\\&=\mu^l_M\circ(u_A\otimes M)=l_M.\end{aligned}$$
Similarly $D\left({}^Ar^A_M\right)\circ\zeta_2(M,A)\circ[D\left(M\right)\otimes\zeta_0]=r_M$. We have so proved that D is a lax monoidal functor.

2) It follows by dual arguments. $\square$

LEMMA 4.5.5. *Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. The functor F : ${}^H_H\mathcal{YD}\to{}^H_H\mathfrak{M}^H_H$ defines a monoidal functor $F:({}^H_H\mathcal{YD},\otimes,\Bbbk)\to({}^H_H\mathfrak{M}^H_H,\otimes_H,H)$. For $U,V\in{}^H_H\mathcal{YD}$, the structure morphisms are*
$$\varphi_2(U,V):F(U)\otimes_H F(V)\to F(U\otimes V)\qquad\text{and}\qquad\varphi_0:H\to F(\Bbbk)$$
which are defined, for every $u\in U,v\in V,h,k\in H,$ by
$$\varphi_2(U,V)[(u\otimes h)\otimes_H(v\otimes k)]:=\left[\begin{array}{c}\omega^{-1}\left(u_{-2}\otimes h_1\otimes v_{-2}k_1\right)\omega(h_2\otimes v_{-1}\otimes k_2)\\ \omega^{-1}((h_3\rhd v_0)_{-2}\otimes h_4\otimes k_3)\omega\left(u_{-1}\otimes(h_3\rhd v_0)_{-1}\otimes h_5k_4)\right)\\ (u_0\otimes(h_3\rhd v_0)_0)\otimes h_6k_5\end{array}\right]$$

and

$$\varphi_0(h) := 1_{\Bbbk} \otimes h.$$

Moreover

$$\varphi_2(U,V)^{-1}((u\otimes v)\otimes k) = \omega^{-1}(u_{-1}\otimes v_{-1}\otimes k_1)(u_0\otimes 1_H)\otimes_H (v_0\otimes k_2).$$

PROOF. Let us check that φ_0 is a morphism in ${}^H_H\mathfrak{M}^H_H$. Since $\varphi_0 = l_H^{-1} : H \to \Bbbk\otimes H$, i.e. the inverse of the left unit constraint in ${}^H\mathfrak{M}^H$, then φ_0 is in ${}^H\mathfrak{M}^H$ and it is invertible. It is easy to check it is H-bilinear in ${}^H\mathfrak{M}^H$.

Let us consider now $\varphi_2(U,V)$.

By Lemma 4.4.5, for all $U,V \in {}^H_H\mathcal{YD}$, the map $\xi_{U,F(V)} : F(U)\otimes_H F(V) \to U\otimes F(V)$, is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$. By Lemma 4.4.6, $\alpha_{U,V} : U\otimes F(V) \to F(U\otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$, where $U\otimes F(V)$ has the structure described in Lemma 4.4.5 for $M = F(V)$.

Thus $\alpha_{U,V}\xi_{U,F(V)} : F(U)\otimes_H F(V) \to F(U\otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$. A direct computation shows that $\varphi_2(U,V) = \alpha_{U,V}\xi_{U,F(V)}$ and hence $\varphi_2(U,V)$ is a well-defined isomorphism in ${}^H_H\mathfrak{M}^H_H$. Moreover $\varphi_2(U,V)^{-1} = \xi^{-1}_{U,F(V)}\alpha^{-1}_{U,V}$ fulfills (4.5.5).

In order to check the commutativity of the diagram (47) it suffices to prove the following equality:

$$[\varphi_2^{-1}(U,V)\otimes_H F(W)]\varphi_2^{-1}(U\otimes V,W)F(a^{-1}_{U,V,W}) = a^{-1}_{F(U),F(V),F(W)}[F(U)\otimes_H\varphi_2^{-1}(V,W)]\varphi_2^{-1}(U,V\otimes W)$$

Since these maps are right H-linear, it suffices to check this equality on elements of the form $(u\otimes(v\otimes w))\otimes 1_H$, where $u\in U, v\in V, w\in W$. This computation and the ones of (48) and (49) are straightforward. $\square$

We now compute explicitly the braiding induced on ${}^H_H\mathfrak{M}^H_H$ through the functor F in Lemma 4.5.5 in case F comes out to be an equivalence i.e. when H has a preantipode.

LEMMA 4.5.6. *Let $(H,m,u,\Delta,\varepsilon,\omega,S)$ be a dual quasi-bialgebra with a preantipode. Through the monoidal equivalence (F,G) we have that $({}^H_H\mathfrak{M}^H_H,\otimes_H,H)$ becomes a pre-braided monoidal category, with braiding defined as follows:*

$$c_{M,N}(m\otimes_H n) = \omega(m_{-2}\otimes\tau(n_0)_{-1}\otimes n_1)(m_{-1}\triangleright\tau(n_0)_0\otimes_H m_0)\cdot n_2,$$

where $M,N\in{}^H_H\mathfrak{M}^H_H$ and $m\in M, n\in N$.

PROOF. First of all, for any $U,V\in{}^H_H\mathcal{YD}$, let us consider the following composition:

$$\lambda_{U,V} := \left(F(U)\otimes_H F(V) \overset{\varphi_2(U,V)}{\longrightarrow} F(U\otimes V) \overset{F(c_{U,V})}{\longrightarrow} F(V\otimes U) \overset{\varphi_2^{-1}(V,U)}{\longrightarrow} F(V)\otimes_H F(U)\right).$$

This map is right H-linear, so, if we compute

$$\begin{aligned} &\lambda_{U,V}[(u\otimes h)\otimes_H(v\otimes 1_H)] \\ =\ &\left[\begin{array}{c} \omega^{-1}(u_{-4}\otimes h_1\otimes v_{-1})\,\omega(u_{-3}\otimes(h_2\triangleright v_0)_{-1}\otimes h_3)) \\ \omega^{-1}\left((u_{-2}\triangleright(h_2\triangleright v_0)_0)_{-1}\otimes u_{-1}\otimes h_4\right)\left((u_{-2}\triangleright(h_2\triangleright v_0)_0)_0\otimes 1_H\right)\otimes_H u_0\otimes h_5 \end{array}\right] \\ \overset{(37)}{=}\ &((u_{-1}h_1)\triangleright v\otimes 1_H)\otimes_H(u_0\otimes h_2), \end{aligned}$$

we obtain

$$\begin{aligned}
&\lambda_{U,V}[(u\otimes h)\otimes_H (v\otimes k)]\\
=\;&\lambda_{U,V}[(u\otimes h)\otimes_H (v\otimes 1_H)\cdot k]\\
=\;&\omega(u_{-1}h_1\otimes v_{-1}\otimes k_1)\lambda_{U,V}[[(u_0\otimes h_2)\otimes_H (v_0\otimes 1_H)]\cdot k_2]\omega^{-1}(h_3\otimes 1_H\otimes k_3)\\
=\;&\omega(u_{-1}h_1\otimes v_{-1}\otimes k_1)\lambda_{U,V}[[(u_0\otimes h_2)\otimes_H (v_0\otimes 1_H)]\cdot k_2]\\
=\;&\omega(u_{-1}h_1\otimes v_{-1}\otimes k_1)\lambda_{U,V}[(u_0\otimes h_2)\otimes_H (v_0\otimes 1_H)]\cdot k_2\\
=\;&\omega(u_{-2}h_1\otimes v_{-1}\otimes k_1)[(u_{-1}h_2)\triangleright v_0\otimes 1_H)\otimes_H (u_0\otimes h_3)]\cdot k_2.
\end{aligned}$$

Now, using the map $\lambda_{U,V}$, we construct the braiding of ${}^H_H\mathfrak{M}^H_H$ in this way:

$$M\otimes_H N \stackrel{\epsilon_M^{-1}\otimes_H\epsilon_N^{-1}}{\longrightarrow} FG(M)\otimes_H FG(N) \stackrel{\lambda_{G(M),G(N)}}{\longrightarrow} FG(N)\otimes_H FG(M) \stackrel{\epsilon_N\otimes_H\epsilon_M}{\longrightarrow} N\otimes_H M.$$

Therefore

$$\begin{aligned}
&(\epsilon_N\otimes_H\epsilon_M)\lambda_{G(M),G(N)}(\epsilon_M^{-1}\otimes_H\epsilon_N^{-1})(m\otimes_H n)\\
=\;&(\epsilon_N\otimes_H\epsilon_M)\lambda_{G(M),G(N)}\left\{[\tau(m_0)\otimes m_1]\otimes_H[\tau(n_0)\otimes n_1]\right\}\\
=\;&\left[\begin{array}{c}\omega(\tau(m_0)_{-2}m_1\otimes\tau(n_0)_{-1}\otimes n_1)\\ (\epsilon_N\otimes_H\epsilon_M)\left\{[(\tau(m_0)_{-1}m_2)\triangleright\tau(n_0)_0\otimes 1_H)\otimes_H(\tau(m_0)_0\otimes m_3)]\cdot n_2\right\}\end{array}\right]\\
\stackrel{(24)}{=}\;&\left[\begin{array}{c}\omega(m_{-2}\otimes\tau(n_0)_{-1}\otimes n_1)\\ (\epsilon_N\otimes_H\epsilon_M)[(m_{-1}\triangleright\tau(n_0)_0\otimes 1_H)\otimes_H(\tau(m_0)\otimes m_1)]\cdot n_2\end{array}\right]\\
=\;&\omega(m_{-2}\otimes\tau(n_0)_{-1}\otimes n_1)[(m_{-1}\triangleright\tau(n_0)_0\otimes_H\tau(m_0)m_1]\cdot n_2\\
\stackrel{(25)}{=}\;&\omega(m_{-2}\otimes\tau(n_0)_{-1}\otimes n_1)[(m_{-1}\triangleright\tau(n_0)_0\otimes_H m_0]\cdot n_2.
\end{aligned}$$

□

Next aim is to prove that the equivalence between the categories ${}^H_H\mathfrak{M}^H_H$ and ${}^H_H\mathcal{YD}$ becomes monoidal if we equip ${}^H_H\mathfrak{M}^H_H$ with the tensor product $\square_H$ (see Remark 4.1.6) and unit H.

Remark 4.5.7. Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. Note that, since H is an ordinary coalgebra, we have that $\left({}^H\mathfrak{M}^H,\square_H,H,b,r,l\right)$ is a monoidal category with constraints defined, for all $L,M,N\in{}^H\mathfrak{M}^H$, by

$$b_{L,M,N}:(L\square_H M)\square_H N\to L\square_H(M\square_H N):(l\square_H m)\square_H n\mapsto l\square_H(m\square_H n),$$

$$\begin{aligned}
r_M &: M\square_H H\longrightarrow M: m\square_H h\mapsto m\varepsilon_H(h),\\
l_M &: H\square_H M\longrightarrow M: h\square_H m\mapsto\varepsilon_H(h)m.
\end{aligned}$$

where, for sake of brevity we just wrote $m\square_H n$ in place of the more precise $\sum_i m^i\square_H n^i$.

We want to endow ${}^H_H\mathfrak{M}^H_H$ with a monoidal structure, following the dual version of [**HN**] (see also [**Sch3**, Definition 3.2]). The definition of the structure is given in such a way that the forgetful functor ${}^H_H\mathfrak{M}^H_H\to{}^H\mathfrak{M}^H$ is a strict monoidal functor. Hence the constraints are induced by the ones of ${}^H\mathfrak{M}^H$ (i.e. $b_{L,M,N}$,l_M and r_M), and the tensor product is given by $M\square_H N$ with structures

$$\begin{aligned}
\rho^l_{M\square_H N}(m\square_H n) &= m_{-1}\otimes(m_0\square_H n),\\
\rho^r_{M\square_H N}(m\square_H n) &= (m\square_H n_0)\otimes n_1,
\end{aligned}$$

$$\begin{aligned}\mu^l_{M\square_H N}\left[h\otimes(m\square_H n)\right] &= h\cdot(m\square_H n)=h_1m\square_H h_2n,\\ \mu^r_{M\square_H N}\left[(m\square_H n)\otimes h\right] &= (m\square_H n)\cdot h=mh_1\square_H nh_2.\end{aligned}$$

The unit of the category is H endowed with the following structures:

$$\begin{aligned}\rho^l_H(h) &= h_1\otimes h_2, \qquad \rho^r_H(h)=h_1\otimes h_2,\\ h\cdot l &= hl, \qquad l\cdot h=lh.\end{aligned}$$

The following result is similar to 2) in Lemma 4.4.5.

LEMMA 4.5.8. *Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. For all $V\in{}^H_H\mathcal{YD}$ and $M\in{}^H_H\mathfrak{M}^H_H$, the map*

$$\beta_{V,M}:F(V)\square_H M\longrightarrow V\otimes M:(v\otimes h)\square_H m\mapsto v\varepsilon(h)\otimes m$$

is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$ where $V\otimes M$ has the structures as in Lemma 4.4.5. The inverse of $\beta_{V,M}$ is given by

$$\beta^{-1}_{V,M}:V\otimes M\longrightarrow(V\otimes H)\square_H M:v\otimes m\mapsto(v\otimes m_{-1})\,\square_H m_0.$$

PROOF. The proof is straightforward and is based on the fact that $(v\otimes h)\,\square_H m\in(V\otimes H)\square_H M$ implies

$$(v\otimes h)\otimes m=(v\varepsilon\,(h)\otimes m_{-1})\otimes m_0. \tag{50}$$

∎

LEMMA 4.5.9. *(cf. [**Sch3**, Proposition 3.6]) Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. The functor $F:{}^H_H\mathcal{YD}\to{}^H_H\mathfrak{M}^H_H$ defines a monoidal functor $F:({}^H_H\mathcal{YD},\otimes,\Bbbk)\to({}^H_H\mathfrak{M}^H_H,\square_H,H)$. For $U,V\in{}^H_H\mathcal{YD}$, the structure morphisms are*

$$\psi_2(U,V):F(U)\square_H F(V)\to F(U\otimes V)\qquad\text{and}\qquad\psi_0:H\to F(\Bbbk)$$

which are defined, for every $u\in U,v\in V,h,k\in H$, by

$$\psi_2(U,V)[(u\otimes h)\otimes(v\otimes k)]:=\omega(u_{-1}\otimes v_{-1}\otimes k_1)u_0\varepsilon(h)\otimes v_0\otimes k_2 \tag{51}$$

and

$$\psi_0(h):=1_\Bbbk\otimes h.$$

Moreover

$$\psi_2(U,V)^{-1}\left((u\otimes v)\otimes h\right)=\omega^{-1}(u_{-1}\otimes v_{-2}\otimes h_1)(u_0\otimes v_{-1}h_2)\otimes(v_0\otimes h_3). \tag{52}$$

PROOF. Since $\psi_0=\varphi_0$ as in Lemma 4.5.5, we already know that ψ_0 is an isomorphism in ${}^H_H\mathfrak{M}^H_H$. Let us deal with $\psi_2(U,V)$. By Lemma 4.4.6, the map $\alpha_{U,V}:U\otimes F\,(V)\to F\,(U\otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$, where $U\otimes F\,(V)$ has the structure described in Lemma 4.4.5 for $M=F\,(V)$. By Lemma 4.5.8, $\beta_{U,F(V)}=\beta:F(U)\square_H F(V)\longrightarrow U\otimes F(V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$, where $U\otimes F\,(V)$ has the structure described in Lemma 4.4.5 for $M=F\,(V)$. Hence it makes sense to consider the composition $\psi_2(U,V):=\alpha_{U,V}\circ\beta_{U,V\otimes H}$. Then $\psi_2(U,V)$ fulfills (51). It is clear that $\psi_2(U,V):F(U)\square_H F(V)\to F\,(U\otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$ with inverse given by $\psi_2(U,V)^{-1}:=\beta^{-1}_{U,V\otimes H}\circ\alpha^{-1}_{U,V}$. Moreover $\psi_2(U,V)^{-1}$ satisfies (52).

In order to check the commutativity of the diagram (47) it suffices to prove the following equality:

$$\begin{aligned} & (\psi_2(U,V)^{-1}\otimes F(W))\psi_2(U\otimes V,W)^{-1}F(a_{U,V,W}^{-1})[(u\otimes(v\otimes w))\otimes h] \\ = \quad & b_{F(U),F(V).F(W)}^{-1}[F(U)\otimes\psi_2(V,W)^{-1}]\psi_2(U,V\otimes W)^{-1}[(u\otimes(v\otimes w))\otimes h]. \end{aligned}$$

By right H-linearity, it suffices to check the displayed equality for $h=1_H$. The proof of this fact and of (48) and (49) is straightforward. $\square$

If H has a preantipode, the functor F of Lemma 4.5.9 is an equivalence. As a consequence, its adjoint G is monoidal too. For future reference we include here its explicit monoidal structure.

LEMMA 4.5.10. *Let $(H,m,u,\Delta,\varepsilon,\omega,S)$ be a dual quasi-bialgebra with a preantipode. The right adjoint $G:{}_H^H\mathfrak{M}_H^H\to{}_H^H\mathcal{YD}$ of the functor F, defines a monoidal functor $G:({}_H^H\mathfrak{M}_H^H,\square_H,H)\to({}_H^H\mathcal{YD},\otimes,\Bbbk)$. For $M,N\in{}_H^H\mathfrak{M}_H^H$, the structure morphisms are*

$$\psi_2^G(M,N):G(M)\otimes G(N)\to G(M\square_H N)\qquad \textit{and}\qquad \psi_0^G:\Bbbk\to G(H)$$

which are defined, for every $m\in M,n\in N,k\in H$, by

$$\psi_2^G(M,N)\,(m\otimes n)=mn_{-1}\square_H n_0\qquad \textit{and}\qquad \psi_0^G(k):=k1_H.$$

Moreover, for all $m\in M,n\in N$,

$$\psi_2^G(M,N)^{-1}\,(m\square_H n)=\tau\,(m)\otimes\tau\,(n)\,.$$

PROOF. Apply [**Sch6**, Section 2] and [**SR**, Proposition 4.4.2] to the functor F. Then G is monoidal with structure morphisms

$$\begin{aligned} \psi_2^G(M,N) \quad &: = G\left(\epsilon_M\square_H\epsilon_M\right)\circ G\left(\psi_2(GM,GN)^{-1}\right)\circ\eta_{GM\otimes GN}, \\ \psi_0^G \quad &: = G\left(\psi_0^{-1}\right)\circ\eta_{\Bbbk} \end{aligned}$$

A direct computation shows that they are the desired maps.

The inverse of $\psi_2^G(M,N)$ can be computed by

$$\psi_2^G(M,N)^{-1}:=\eta_{GM\otimes GN}^{-1}\circ G\left(\psi_2(GM,GN)\right)\circ G\left(\epsilon_M^{-1}\square_H\epsilon_M^{-1}\right)$$

$\square$

REMARK 4.5.11. Consider the composition

$$\kappa=\kappa(U,V):=\psi_2(U,V)^{-1}\circ\varphi_2(U,V):(U\otimes H)\otimes_H(V\otimes H)\longrightarrow(U\otimes H)\square_H(V\otimes H).$$

We have

$$\begin{aligned} & \kappa(U,V)\,[(u\otimes h)\otimes_H(v\otimes k)] \\ = \quad & \psi_2(U,V)^{-1}\varphi_2(U,V)\,[(u\otimes h)\otimes_H(v\otimes k)] \\ = \quad & \left[\begin{array}{c} \omega^{-1}\left(u_{-2}\otimes h_1\otimes v_{-2}k_1\right)\omega(h_2\otimes v_{-1}\otimes k_2) \\ \omega^{-1}((h_3\rhd v_0)_{-2}\otimes h_4\otimes k_3)\omega\left(u_{-1}\otimes(h_3\rhd v_0)_{-1}\otimes h_5k_4)\right) \\ \psi_2(U,V)^{-1}[(u_0\otimes(h_3\rhd v_0)_0)\otimes(h_6k_5)] \end{array}\right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\begin{array}{c}
\omega^{-1}\left(u_{-2} \otimes h_{1} \otimes v_{-2} k_{1}\right) \omega\left(h_{2} \otimes v_{-1} \otimes k_{2}\right) \\
\omega^{-1}((h_{3} \rhd v_{0})_{-2} \otimes h_{4} \otimes k_{3}) \omega\left(u_{-1} \otimes (h_{3} \rhd v_{0})_{-1} \otimes h_{5} k_{4}\right)) \\
\omega^{-1}(u_{0-1} \otimes (h_{3} \rhd v_{0})_{0-2} \otimes (h_{6} k_{5})_{1}) \\
(u_{00} \otimes (h_{3} \rhd v_{0})_{0-1}(h_{6} k_{5})_{2}) \square_{H}((h_{3} \rhd v_{0})_{00} \otimes (h_{6} k_{5})_{3})
\end{array}\right] \\
&= \left[\begin{array}{c}
\omega^{-1}\left(u_{-3} \otimes h_{1} \otimes v_{-2} k_{1}\right) \omega\left(h_{2} \otimes v_{-1} \otimes k_{2}\right) \\
\omega^{-1}((h_{3} \rhd v_{0})_{-4} \otimes h_{4} \otimes k_{3}) \omega(u_{-2} \otimes (h_{3} \rhd v_{0})_{-3} \otimes h_{5} k_{4}) \\
\omega^{-1}(u_{-1} \otimes (h_{3} \rhd v_{0})_{-2} \otimes h_{6} k_{5})(u_{0} \otimes (h_{3} \rhd v_{0})_{-1}(h_{7} k_{6})) \square_{H}((h_{3} \rhd v_{0})_{0} \otimes h_{8} k_{7})
\end{array}\right] \\
&= \left[\begin{array}{c}
\omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-2} k_{1}\right) \omega\left(h_{2} \otimes v_{-1} \otimes k_{2}\right) \\
\omega^{-1}((h_{3} \rhd v_{0})_{-2} \otimes h_{4} \otimes k_{3}) \\
(u_{0} \otimes (h_{3} \rhd v_{0})_{-1}(h_{5} k_{4})) \square_{H}((h_{3} \rhd v_{0})_{0} \otimes h_{6} k_{5})
\end{array}\right] \\
&\overset{(20)}{=} \left[\begin{array}{c}
\omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-2} k_{1}\right) \omega\left(h_{2} \otimes v_{-1} \otimes k_{2}\right) \\
(u_{0} \otimes ((h_{3} \rhd v_{0})_{-2} h_{4}) k_{3})) \\
\omega^{-1}((h_{3} \rhd v_{0})_{-1} \otimes h_{5} \otimes k_{4}) \square_{H}((h_{3} \rhd v_{0})_{0} \otimes h_{6} k_{5})
\end{array}\right] \\
&= \left[\begin{array}{c}
\omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-2} k_{1}\right) \omega\left(h_{2} \otimes v_{-1} \otimes k_{2}\right) \\
(u_{0} \otimes ((h_{3} \rhd v_{0})_{-1} h_{4}) k_{3})) \square_{H}((h_{3} \rhd v_{0})_{0} \otimes h_{5}) \cdot k_{4})
\end{array}\right] \\
&\overset{(39)}{=} \left[\begin{array}{c}
\omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-3} k_{1}\right) \omega\left(h_{2} \otimes v_{-2} \otimes k_{2}\right) \\
(u_{0} \otimes ((h_{3} v_{-1}) k_{3})) \square_{H}((h_{4} \rhd v_{0}) \otimes h_{5}) \cdot k_{4})
\end{array}\right] \\
&\overset{(20)}{=} \left[\begin{array}{c}
\omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-3} k_{1}\right) \\
(u_{0} \otimes (h_{2}(v_{-2} k_{2})) \\
\omega(h_{3} \otimes v_{-1} \otimes k_{3}) \square_{H}((h_{4} \rhd v_{0}) \otimes h_{5}) \cdot k_{4})
\end{array}\right] \\
&\overset{(4.4.7)}{=} \omega^{-1}\left(u_{-1} \otimes h_{1} \otimes v_{-2} k_{1}\right)\left(u_{0} \otimes (h_{2}(v_{-1} k_{2})) \square_{H}(h_{3} \cdot (v_{0} \otimes k_{3}))\right. \\
&= (u_{0} \otimes h_{1}) \cdot (v_{-1} k_{1}) \square_{H} h_{3} \cdot (v_{0} \otimes k_{3}) \\
&= (u \otimes h)_{0} \cdot (v \otimes k)_{-1} \square_{H} (u \otimes h)_{1} \cdot (v \otimes k)_{0}.
\end{aligned}
$$

so that

$$\kappa(U,V)\left[(u \otimes h) \otimes_{H} (v \otimes k)\right] = (u \otimes h)_{0} \cdot (v \otimes k)_{-1} \square_{H} (u \otimes h)_{1} \cdot (v \otimes k)_{0}.$$

Thus, for $M, N \in {}_{H}^{H}\mathfrak{M}_{H}^{H}$,, using that the counit ϵ is in ${}_{H}^{H}\mathfrak{M}_{H}^{H}$, one gets

$$\left[(\epsilon_{M} \square_{H} \epsilon_{N}) \circ \kappa(M^{coH}, N^{coH}) \circ \left(\epsilon_{M}^{-1} \otimes_{H} \epsilon_{N}^{-1}\right)\right](m \otimes_{H} n) = m_{0} n_{-1} \square_{H} m_{1} n_{0}.$$

We can also compute $\kappa(U,V)^{-1} := \varphi_{2}(U,V)^{-1} \circ \psi_{2}(U,V)$. We have:

$$\kappa(U,V)^{-1}((u \otimes h) \square_{H} (v \otimes k)) = (u\varepsilon(h) \otimes 1_{H}) \otimes_{H} (v \otimes k).$$

We are now able to provide a monoidal equivalence between $({}_{H}^{H}\mathfrak{M}_{H}^{H}, \otimes_{H}, H)$ and $({}_{H}^{H}\mathfrak{M}_{H}^{H}, \square_{H}, H)$. This result is similar to [**Sch2**, Corollary 6.1].

LEMMA 4.5.12. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. The identity functor on ${}_{H}^{H}\mathfrak{M}_{H}^{H}$ defines a monoidal functor $E : ({}_{H}^{H}\mathfrak{M}_{H}^{H}, \otimes_{H}, H) \to ({}_{H}^{H}\mathfrak{M}_{H}^{H}, \square_{H}, H)$. For $M, N \in {}_{H}^{H}\mathfrak{M}_{H}^{H}$, the structure morphisms are*

$$\vartheta_{2}(M,N) : E(M) \square_{H} E(V) \to E(M \otimes_{H} N) \qquad \textit{and} \qquad \vartheta_{0} : H \to E(H) = H$$

which are defined, for every $m \in M, n \in N, h \in H$, by

$$\vartheta_{2}(M,N)(m \square_{H} n) := \tau(m) \otimes_{H} n \qquad \textit{and} \qquad \vartheta_{0}(h) := h.$$

Moreover

$$\vartheta_2(M,N)^{-1}(m\otimes_H n) = m_0 n_{-1}\square_H m_1 n_0, \tag{53}$$

$$\vartheta_2(FU,FV) = \varphi_2(U,V)^{-1}\circ\psi_2(U,V). \tag{54}$$

PROOF. Using the map κ of Remark 4.5.11, for each $M,\ N\in {}_H^H\mathfrak{M}_H^H$, we set

$$\vartheta_2(M,N) := (\epsilon_M\otimes_H\epsilon_N)\circ\kappa(M^{coH},N^{coH})^{-1}\circ\left(\epsilon_M^{-1}\square_H\epsilon_N^{-1}\right).$$

Clearly, by Remark 4.5.11, $\vartheta_2(M,N)^{-1}$ fulfills (53). Moreover, using (25), one gets

$$\vartheta_2(M,N)(m\square_H n) = \tau(m)\otimes_H n.$$

It is straightforward to check that ϑ_2^{-1} makes commutative the diagram (47) and that (48) and (49) hold. Let us check that (54) holds:

$$\begin{aligned}
&\vartheta_2(FU,FV) = (\epsilon_{FU}\otimes_H\epsilon_{FV})\circ\kappa(GFU,GFV)^{-1}\circ\left(\epsilon_{FU}^{-1}\square_H\epsilon_{FV}^{-1}\right)\\
=\ &(\epsilon_{FU}\otimes_H\epsilon_{FV})\circ\varphi_2(GFU,GFV)^{-1}\circ\psi_2(GFU,GFV)\circ\left(\epsilon_{FU}^{-1}\square_H\epsilon_{FV}^{-1}\right)\\
=\ &\left[\begin{array}{l}(\epsilon_{FU}\otimes_H\epsilon_{FV})\circ\varphi_2(GFU,GFV)^{-1}\circ F(\eta_U\otimes\eta_V)\\ F\left(\eta_U^{-1}\otimes\eta_V^{-1}\right)\circ\psi_2(GFU,GFV)\circ\left(\epsilon_{FU}^{-1}\square_H\epsilon_{FV}^{-1}\right)\end{array}\right]\\
=\ &\left[\begin{array}{l}(\epsilon_{FU}\otimes_H\epsilon_{FV})\circ(F\eta_U\otimes F\eta_V)\circ\varphi_2(U,V)^{-1}\\ \psi_2(U,V)\circ\left(F\eta_U^{-1}\otimes F\eta_V^{-1}\right)\circ\left(\epsilon_{FU}^{-1}\square_H\epsilon_{FV}^{-1}\right)\end{array}\right] = \varphi_2(U,V)^{-1}\circ\psi_2(U,V).
\end{aligned}$$

□

The following result is similar to [**Sch3**, Proposition 3.11].

COROLLARY 4.5.13. *Let $(H,m,u,\Delta,\varepsilon,\omega)$ be a dual quasi-bialgebra. The identity functor on ${}_H^H\mathfrak{M}_H^H$ defines a monoidal functor $\Xi : ({}_H^H\mathfrak{M}_H^H,\square_H,H)\to({}_H^H\mathfrak{M}_H^H,\otimes_H,H)$. For $M,N\in{}_H^H\mathfrak{M}_H^H$, the structure morphisms are*

$$\gamma_2(M,N):\Xi(M)\otimes_H\Xi(V)\to\Xi(M\square_H N)\qquad\text{and}\qquad\gamma_0 : H\to\Xi(H)$$

which are defined by $\gamma_2(M,N) := \vartheta_2^{-1}(M,N)$ and $\gamma_0 := \vartheta_0^{-1}$ using Lemma 4.5.12.

PROOF. It follows by [**Sch6**, Section 2] and [**SR**, Proposition 4.4.2]. □

CHAPTER 5

The main results and some applications

Let H be a Hopf algebra, let A be a bialgebra and let $\sigma : H \to A$ and $\pi : A \to H$ be morphisms of bialgebras such that $\pi\sigma = \mathbb{I}_H$. In this case A is called a bialgebra with projection onto H and $A \in {}^H_H\mathfrak{M}^H_H$ through

$$\rho^r(a) = a_1 \otimes \pi(a_2), \qquad \rho^l(a) = \pi(a_1) \otimes a_2,$$
$$\mu^r(a \otimes h) = a\sigma(h), \qquad \mu^l(h \otimes a) = \sigma(h)a.$$

Define now the map $\tau : A \to A : a \longmapsto a_1\sigma S(a_2)$. It can be proved that $\mathrm{Im}\tau = A^{coH} =: R$ and, when H is the coradical of A, that R is connected. Indeed it is well-known that R becomes a connected bialgebra in the pre-braided monoidal category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H (cf. [**Ra**]).

Now, from the fact that (F,G) is an equivalence we know that $\epsilon_A : R \otimes H \to A$ is an isomorphism. Conversely, it can be proved that, given a Hopf algebra H and a braided bialgebra R in ${}^H_H\mathcal{YD}$, we can endow $R \otimes H$ with a bialgebra structure and define two bialgebras morphisms σ and π such that $\pi\sigma = \mathrm{Id}_H$, see ([**Ra**]). This bialgebra is called *Radford-Majid Bosonization* (or Radford biproduct) and permits to classify different kinds of bialgebras as "compositions" (crossed product) of different objects in the same category.

The main aim of this chapter is to extend the results above to the setting of dual quasi-bialgebras and to give some applications of it.

1. The bosonization of R by H

THEOREM 5.1.1. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ be a dual quasi-bialgebra.*

Let $(R, \mu_R, \rho_R, \Delta_R, \varepsilon_R, m_R, u_R)$ be a bialgebra in ${}^H_H\mathcal{YD}$ and use the following notations

$$\begin{aligned} h \triangleright r \ &: = \mu_R(h \otimes r), \qquad r_{-1} \otimes r_0 := \rho_R(r), \\ r \cdot_R s \ &: = m_R(r \otimes s), \qquad 1_R := u_R(1_{\Bbbk}), \\ r^1 \otimes r^2 \ &: = \Delta_R(r). \end{aligned}$$

Let us consider on $B := F(R) = R \otimes H$ the following structures:

$$\begin{aligned} m_B[(r \otimes h) \otimes (s \otimes k)] &= \begin{bmatrix} \omega_H^{-1}(r_{-2} \otimes h_1 \otimes s_{-2}k_1)\omega_H(h_2 \otimes s_{-1} \otimes k_2) \\ \omega_H^{-1}[(h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3]\omega_H(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6k_5 \end{bmatrix} \\ u_B(k) &= k1_R \otimes 1_H \\ \Delta_B(r \otimes h) &= \omega_H^{-1}(r^1_{-1} \otimes r^2_{-2} \otimes h_1)r^1_0 \otimes r^2_{-1}h_2 \otimes r^2_0 \otimes h_3 \end{aligned}$$

$$\begin{aligned}\varepsilon_B(r\otimes h) &= \varepsilon_R(r)\varepsilon_H(h)\\ \omega_B((r\otimes h)\otimes(s\otimes k)\otimes(t\otimes l)) &= \varepsilon_R(r)\varepsilon_R(s)\varepsilon_R(t)\omega_H(h\otimes k\otimes l).\end{aligned}$$

Then $(B,\Delta_B,\varepsilon_B,m_B,u_B,\omega_B)$ *is a dual quasi-bialgebra.*

Proof. The following proof is the dual version of [**BN**, Lemma 3.1]. Recall that, by Lemma 4.5.5, the functor $F:{}_H^H\mathcal{YD}\to{}_H^H\mathfrak{M}_H^H$ defines a monoidal functor $F:({}_H^H\mathcal{YD},\otimes,\Bbbk)\to({}_H^H\mathfrak{M}_H^H,\otimes_H,H)$ where, for $U,V\in{}_H^H\mathcal{YD}$, the structure morphisms are given by $\varphi_2(U,V),\varphi_0$. Directly by the definition we have that (B,m_B',u_B') is an algebra in $({}_H^H\mathfrak{M}_H^H,\otimes_H,H)$ where

$$m_B':=F(m_R)\circ\varphi_2(R,R),\qquad u_B':=F(u_R)\circ\varphi_0.$$

Explicitly we have

$$\begin{aligned}m_B'((r\otimes h)\otimes_H(s\otimes k)) &= \begin{bmatrix}\omega_H^{-1}(r_{-2}\otimes h_1\otimes s_{-2}k_1)\,\omega_H(h_2\otimes s_{-1}\otimes k_2)\\ \omega_H^{-1}((h_3\rhd s_0)_{-2}\otimes h_4\otimes k_3)\omega_H(r_{-1}\otimes(h_3\rhd s_0)_{-1}\otimes h_5k_4))\\ r_0\cdot_R(h_3\rhd s_0)_0\otimes h_6k_5\end{bmatrix}\\ &= m_B[(r\otimes h)\otimes(s\otimes k)],\end{aligned}$$

$$u_B'(h)=u_R(1_\Bbbk)\otimes h=1_R\otimes h.$$

Since m_B' is associative in $({}_H^H\mathfrak{M}_H^H,\otimes_H,H)$, we have that

$$m_B'\circ(m_B'\otimes_H B)=m_B'\circ(B\otimes_H m_B')\circ a_{B,B,B}$$

where $a_{B,B,B}$ is the one defined in Lemma 4.5.1. Let $\pi:B\to H$ be defined by $\pi(r\otimes h):=\varepsilon_R(r)h$. Then

$$\omega_H(\pi\otimes\pi\otimes\pi)=\omega_B. \tag{55}$$

One easily gets that

$$\pi(x_1)\otimes x_2\otimes\pi(x_3)=x_{-1}\otimes x_0\otimes x_1,\ \text{for all } x\in B. \tag{56}$$

Let $x,y,z\in B$, then

$$m_B'(m_B'\otimes_H B)((x\otimes_H y)\otimes_H z) \;=\; m_B(m_B\otimes B)((x\otimes y)\otimes z)$$

and

$$\begin{aligned}&m_B'(B\otimes_H m_B')\,a_{B,B,B}((x\otimes_H y)\otimes_H z)\\ =\;&\omega_H^{-1}(x_{-1}\otimes y_{-1}\otimes z_{-1})\,m_B'(B\otimes_H m_B')(x_0\otimes_H(y_0\otimes_H z_0))\,\omega_H(x_1\otimes y_1\otimes z_1)\\ =\;&\omega_H^{-1}(x_{-1}\otimes y_{-1}\otimes z_{-1})\,m_B(B\otimes m_B)(x_0\otimes(y_0\otimes z_0))\,\omega_H(x_1\otimes y_1\otimes z_1)\\ \overset{(56)}{=}\;&\omega_H^{-1}(\pi(x_1)\otimes\pi(y_1)\otimes\pi(z_1))\,m_B(B\otimes m_B)(x_2\otimes(y_2\otimes z_2))\,\omega_H(\pi(x_3)\otimes\pi(y_3)\otimes\pi(z_3))\\ \overset{(55)}{=}\;&\omega_B^{-1}(x_1\otimes y_1\otimes z_1)\,m_B(B\otimes m_B)(x_2\otimes(y_2\otimes z_2))\,\omega_B(x_3\otimes y_3\otimes z_3)\\ =\;&\left[\omega_B^{-1}*[m_B(B\otimes m_B)]*\omega_B\right]((x\otimes y)\otimes z)\end{aligned}$$

so that $m_B(m_B\otimes B)=\omega_B^{-1}*[m_B(B\otimes m_B)]*\omega_B$.

Since m_B' is unitary in $({}_H^H\mathfrak{M}_H^H,\otimes_H,H)$, we have that $m_B'(u_B'\otimes_H B)=l_B$. From this equality, we get $m_B(u_B\otimes B)=l_B$. Similarly $m_B(B\otimes u_B)=r_B$. Let us recall that, by Lemma 4.5.9, the functor $F:{}_H^H\mathcal{YD}\to{}_H^H\mathfrak{M}_H^H$ defines a monoidal functor

$F:({}^H_H\mathcal{YD},\otimes,\Bbbk)\to({}^H_H\mathfrak{M}^H_H,\square_H,H)$, with structure morphisms $\psi_2(U,V),\psi_0$, with $U,V\in{}^H_H\mathcal{YD}$. We have that $\left(B,\overline{\Delta}_B,\overline{\varepsilon}_B\right)$ is a coalgebra in $({}^H_H\mathfrak{M}^H_H,\square_H,H)$ where

$$\overline{\Delta}_B:=\psi_2(R,R)^{-1}\circ F\left(\Delta_R\right),\qquad \overline{\varepsilon}_B:=\psi_0^{-1}\circ F\left(\varepsilon_R\right).$$

Explicitly we have

$$\begin{aligned}\overline{\Delta}_B\left(r\otimes h\right) &= \psi_2(R,R)^{-1}\left(\left(r^1\otimes r^2\right)\otimes h\right)\\ &= \omega^{-1}(r^1_{-1}\otimes r^2_{-2}\otimes h_1)(r^1_0\otimes r^2_{-1}h_2)\square_H(r^2_0\otimes h_3)\\ &= \Delta_B(r\otimes h),\end{aligned}$$

and

$$\overline{\varepsilon}_B\left(r\otimes h\right)=\psi_0^{-1}\left(\varepsilon_R\left(r\right)\otimes h\right)=\varepsilon_R\left(r\right)h.$$

From the fact that $\left(B,\overline{\Delta}_B,\overline{\varepsilon}_B\right)$ is a coalgebra in $({}^H_H\mathfrak{M}^H_H,\square_H,H)$ one easily gets that $(B,\Delta_B,\varepsilon_B)$ is an ordinary coalgebra.

It is straightforward to prove that π is multiplicative, comultiplicative, counitary and unitary i.e.

$$\pi m_B=m_H\left(\pi\otimes\pi\right),\quad \left(\pi\otimes\pi\right)\Delta_B=\Delta_H\pi,\quad \varepsilon_B=\varepsilon_H\pi,\quad \pi u_B=u_H. \tag{57}$$

Using these equalities plus (55), one easily gets that the cocycle and unitary conditions for ω_B follow from the ones of ω_H.

Now we want to prove that m_B is a morphism of coalgebras. It is counitary as

$$\varepsilon_B m_B\overset{(57)}{=}\varepsilon_H\pi m_B\overset{(57)}{=}\varepsilon_H m_H\left(\pi\otimes\pi\right)=m_\Bbbk\left(\varepsilon_H\otimes\varepsilon_H\right)\left(\pi\otimes\pi\right)\overset{(57)}{=}m_\Bbbk\left(\varepsilon_B\otimes\varepsilon_B\right).$$

Hence we just have to prove that

$$\Delta_B[(r\otimes h)\cdot_B(s\otimes k)]=(r\otimes h)_1\cdot_B(s\otimes k)_1\otimes(r\otimes h)_2\cdot_B(s\otimes k)_2,$$

where $x\cdot_B y:=m_B\left(x\otimes y\right)$ and $x_1\otimes x_2:=\Delta_B\left(x\right)$, for all $x,y\in B$. Equivalently we will prove that

$$\Delta_B m_B=\left(m_B\otimes m_B\right)\Delta_{B\otimes B}.$$

Since ${}^H_H\mathcal{YD}$ is a pre-braided monoidal category and $(R,\Delta_R,\varepsilon_R)$ is a coalgebra in this category, then we can define two morphisms $\Delta_{R\otimes R}$ and $\varepsilon_{R\otimes R}$ in ${}^H_H\mathcal{YD}$ such that $(R\otimes R,\Delta_{R\otimes R},\varepsilon_{R\otimes R})$ is a coalgebra in ${}^H_H\mathcal{YD}$ too. We have:

$$\begin{aligned}\Delta_{R\otimes R} &:= a^{-1}_{R,R,R\otimes R}\circ(R\otimes a_{R,R,R})\circ(R\otimes(c_{R,R}\otimes R))\circ(R\otimes a^{-1}_{R,R,R})\circ a_{R,R,R\otimes R}\circ(\Delta_R\otimes\Delta_R),\\ \varepsilon_{R\otimes R} &:= \varepsilon_R\otimes\varepsilon_R.\end{aligned}$$

Explicitly we obtain

$$\begin{aligned}\Delta_{R\otimes R}(r\otimes s) &= \left[\begin{array}{c}\omega^{-1}(r^1_{-2}\otimes r^2_{-5}\otimes s^1_{-2}s^2_{-4})\omega(r^2_{-4}\otimes s^1_{-1}\otimes s^2_{-3})\\ \omega^{-1}[(r^2_{-3}\rhd s^1_0)_{-2}\otimes r^2_{-2}\otimes s^2_{-2})\\ \omega(r^1_{-1}\otimes(r^2_{-3}\rhd s^1_0)_{-1}\otimes r^2_{-1}s^2_{-1})\\ \left[r^1_0\otimes(r^2_{-3}\rhd s^1_0)_0\right]\otimes(r^2_0\otimes s^2_0)\end{array}\right], \\ \varepsilon_{R\otimes R}(r\otimes s) &:= \varepsilon_R\left(r\right)\varepsilon_R\left(s\right).\end{aligned} \tag{58}$$

Consider the canonical maps

$$j_{M,N}:M\square_H N\to M\otimes N\qquad\text{and}\qquad \chi_{M,N}:M\otimes N\to M\otimes_H N,$$

for all $M,N\in {}_H^H\mathfrak{M}_H^H$. Set

$$\begin{aligned}
\widehat{\Delta_R m_R} &: = j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F(\Delta_R m_R)\circ\varphi_2(R,R)\circ\chi_{F(R),F(R)},\\
\widehat{\Delta_R} &: = j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F(\Delta_R),\\
\widehat{m_R} &: = F(m_R)\circ\varphi_2(R,R)\circ\chi_{F(R),F(R)},\\
(m_R\widehat{\otimes m_R)}\Delta_{R\otimes R} &: = j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F((m_R\otimes m_R)\Delta_{R\otimes R})\circ\varphi_2(R,R)\circ\chi_{F(R),F(R)}.
\end{aligned}$$

We have

$$\begin{aligned}
\widehat{\Delta_R m_R} &= j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F(\Delta_R)\circ F(m_R)\circ\varphi_2(R,R)\circ\chi_{F(R),F(R)}\\
&= \widehat{\Delta_R}\circ\widehat{m_R}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\widehat{\Delta_R} &= j_{F(R),F(R)}\circ\overline{\Delta}_B=\Delta_B,\\
\widehat{m_R} &= m'_B\circ\chi_{F(R),F(R)}=m_B,
\end{aligned}$$

so that, since $(m_R\otimes m_R)\Delta_{R\otimes R}=\Delta_R m_R$, we obtain

$$(m_R\widehat{\otimes m_R)}\Delta_{R\otimes R}=\widehat{\Delta_R m_R}=\Delta_B m_B. \tag{59}$$

It remains to prove that

$$(m_R\widehat{\otimes m_R)}\Delta_{R\otimes R}=(m_B\otimes m_B)\Delta_{B\otimes B}. \tag{60}$$

First, one checks that $(m_B\otimes m_B)\Delta_{B\otimes B}$ is H-balanced. Hence there is a unique map $\zeta:B\otimes_H B\to B\otimes B$ such that

$$\zeta\circ\chi_{F(R),F(R)}=(m_B\otimes m_B)\Delta_{B\otimes B}.$$

Our aim is to prove that (60) holds i.e. that

$$j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F((m_R\otimes m_R)\Delta_{R\otimes R})\circ\varphi_2(R,R)\circ\chi_{F(R),F(R)}=\zeta\circ\chi_{F(R),F(R)}.$$

Since $\chi_{F(R),F(R)}$ is an epimorphism, the latter displayed equality is equivalent to

$$j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F((m_R\otimes m_R)\Delta_{R\otimes R})=\zeta\circ\varphi_2(R,R)^{-1}. \tag{61}$$

Now

$$\begin{aligned}
\zeta(x\otimes_H y) &= \zeta\circ\chi_{F(R),F(R)}(x\otimes y)=(m_B\otimes m_B)\Delta_{B\otimes B}(x\otimes y)\\
&= x_1\cdot_B y_1\otimes x_2\cdot_B y_2.
\end{aligned}$$

One proves that $\zeta(x\otimes_H y)\in B\square_H B$. Then there is a unique map $\zeta':B\otimes_H B\to B\square_H B$ such that $j_{F(R),F(R)}\circ\zeta'=\zeta$. Hence (61) is equivalent to

$$j_{F(R),F(R)}\circ\psi_2(R,R)^{-1}\circ F((m_R\otimes m_R)\Delta_{R\otimes R})=j_{F(R),F(R)}\circ\zeta'\circ\varphi_2(R,R)^{-1}$$

i.e. to

$$F((m_R\otimes m_R)\Delta_{R\otimes R})=\psi_2(R,R)\circ\zeta'\circ\varphi_2(R,R)^{-1}. \tag{62}$$

By construction

$$\zeta'(x\otimes_H y)=x_1\cdot_B y_1\square_H x_2\cdot_B y_2.$$

It is straightforward to prove that ζ' is right H-linear. Thus it suffices to check that (62) holds on elements of the form $(r\otimes s)\otimes 1_H$. Thus, for $r,s\in R, h\in H$

$$
\begin{array}{cl}
 & \left[\psi_2(R,R)\circ\zeta'\circ\varphi_2(R,R)^{-1}\right]((r\otimes s)\otimes 1_H) \\
= & \psi_2(R,R)\left[(r\otimes 1_H)_1\cdot_B(s\otimes 1_H)_1\square_H(r\otimes 1_H)_2\cdot_B(s\otimes 1_H)_2\right] \\
= & \psi_2(R,R)\left[\left(r^1\otimes r^2_{-1}\right)\cdot_B\left(s^1\otimes s^2_{-1}\right)\otimes\left(r^2_0\otimes 1_H\right)\cdot_B\left(s^2_0\otimes 1_H\right)\right] \\
= & \psi_2(R,R)\left[\left(r^1\otimes r^2_{-1}\right)\cdot_B\left(s^1\otimes s^2_{-1}\right)\otimes\left(r^2_0\cdot_R s^2_0\otimes 1_H\right)\right] \\
= & \left[\begin{array}{c}
\omega_H^{-1}((r^1)_{-2}\otimes\left(r^2_{-1}\right)_1\otimes(s^1)_{-2}\left(s^2_{-1}\right)_1)\omega_H(\left(r^2_{-1}\right)_2\otimes(s^1)_{-1}\otimes\left(s^2_{-1}\right)_2) \\
\omega_H^{-1}[(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_{-2}\otimes\left(r^2_{-1}\right)_4\otimes\left(s^2_{-1}\right)_3] \\
\omega_H((r^1)_{-1}\otimes(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_{-1}\otimes\left(r^2_{-1}\right)_5\left(s^2_{-1}\right)_4) \\
\psi_2(R,R)\left[(r^1)_0\cdot_R(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_0\otimes\left(r^2_{-1}\right)_6\left(s^2_{-1}\right)_5\otimes(r^2_0\cdot_R s^2_0\otimes 1_H)\right]
\end{array}\right] \\
= & \left[\begin{array}{c}
\omega_H^{-1}((r^1)_{-2}\otimes\left(r^2_{-1}\right)_1\otimes(s^1)_{-2}\left(s^2_{-1}\right)_1)\omega_H(\left(r^2_{-1}\right)_2\otimes(s^1)_{-1}\otimes\left(s^2_{-1}\right)_2) \\
\omega_H^{-1}[(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_{-2}\otimes\left(r^2_{-1}\right)_4\otimes\left(s^2_{-1}\right)_3] \\
\omega_H((r^1)_{-1}\otimes(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_{-1}\otimes\left(r^2_{-1}\right)_5\left(s^2_{-1}\right)_4) \\
\left[(r^1)_0\cdot_R(\left(r^2_{-1}\right)_3\rhd(s^1)_0)_0\otimes(r^2_0\cdot_R s^2_0\otimes 1_H)\right]
\end{array}\right] \\
= & \left[\begin{array}{c}
\omega_H^{-1}(r^1_{-2}\otimes r^2_{-5}\otimes s^1_{-2}s^2_{-4})\omega_H(r^2_{-4}\otimes s^1_{-1}\otimes s^2_{-3}) \\
\omega_H^{-1}[(r^2_{-3}\rhd s^1_0)_{-2}\otimes r^2_{-2}\otimes s^2_{-2}] \\
\omega_H(r^1_{-1}\otimes(r^2_{-3}\rhd s^1_0)_{-1}\otimes r^2_{-1}s^2_{-1}) \\
\left[r^1_0\cdot_R(r^2_{-3}\rhd s^1_0)_0\otimes(r^2_0\cdot_R s^2_0\otimes 1_H)\right]
\end{array}\right] \\
= & \left[(m_R\otimes m_R)\Delta_{R\otimes R}(r\otimes s)\right]\otimes 1_H \\
= & F\left((m_R\otimes m_R)\Delta_{R\otimes R}\right)((r\otimes s)\otimes 1_H).
\end{array}
$$

Hence we have proved that (62) holds and hence (60) is fulfilled. Thus, from (59), we can conclude that m_B is a coalgebra morphism. Finally, it is easy to prove that u_B is a coalgebra map. $\square$

Remark 5.1.2. Let us point out that the coalgebra structure of $F(R)$ in the previous result is a smash coproduct one, see [**BN1**, Definition 3.4].

Definition 5.1.3. With hypotheses and notations as in Theorem 5.1.1, the bialgebra B will be called the *bosonization of* R *by* H and denoted by $R\#H$.

2. Dual quasi-bialgebras with a projection

Definition 5.2.1. Let $(H,m,u,\Delta,\varepsilon,\omega)$ and $(A,m_A,u_A,\Delta_A,\varepsilon_A,\omega_A)$ be dual quasi-bialgebras, and suppose there exist morphisms of dual quasi-bialgebras

$$\sigma: H\to A \qquad \text{and} \qquad \pi: A\to H$$

such that $\pi\sigma=\mathrm{Id}_H$. Then (A,H,σ,π) is called a *dual quasi-bialgebra with a projection onto* H.

Proposition 5.2.2. *Keep the hypotheses and notations of Theorem 5.1.1. Then* $(R\#H,H,\sigma,\pi)$ *is a dual quasi-bialgebra with projection onto* H *where*

$$\sigma: H\to R\#H, \sigma(h):=1_R\#h, \qquad \pi: R\#H\to H, \pi(r\#h):=\varepsilon_R(r)\,h.$$

PROOF. The proof that σ is a morphism of dual quasi-bialgebras is straightforward.

The map π is a morphism of dual quasi-bialgebras in view of (55) and (57). Finally, we have $\pi\sigma(h) = \pi(1_R\#h) = \varepsilon_R(1_R)h = h$. $\square$

Next aim is to characterize dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode as bosonizations.

LEMMA 5.2.3. *Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ and $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ be dual quasi-bialgebras such that (A, H, σ, π) is a dual quasi-bialgebra with a projection onto H. Then A is an object in ${}^H_H\mathfrak{M}^H_H$ through*

$$\rho^r_A(a) = a_1 \otimes \pi(a_2), \qquad \rho^l_A(a) = \pi(a_1) \otimes a_2,$$
$$\mu^r_A(a \otimes h) = a\sigma(h), \qquad \mu^l_A(h \otimes a) = \sigma(h)\,a.$$

PROOF. It is straightforward. $\square$

THEOREM 5.2.4. *Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ and $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ be dual quasi-bialgebras such that (A, H, σ, π) is a dual quasi-bialgebra with projection onto H. Assume that H has a preantipode S. For all $a, b \in A$, we set $a_1 \otimes a_2 := \Delta_A(a)$ and $ab = m_A(a \otimes b)$. Then, for all $a \in A$ we have*

$$\tau(a) := \omega_A[a_1 \otimes \sigma S\pi(a_3)_1 \otimes a_4]a_2\sigma S\pi(a_3)_2$$

and $R := G(A)$ is a bialgebra $((R, \mu_R, \rho_R), m_R, u_R, \Delta_R, \varepsilon_R, \omega_R)$ in ${}^H_H\mathcal{YD}$ where, for all $r, s \in R, h \in H, k \in \Bbbk$, we have

$$h \rhd r := \mu_R(h \otimes r) := \tau[\sigma(h)r], \qquad r_{-1} \otimes r_0 := \rho_R(r) := \pi(r_1) \otimes r_2,$$
$$m_R(r \otimes s) := rs, \qquad u_R(k) := k1_A,$$
$$r^1 \otimes r^2 := \Delta_R(r) := \tau(r_1) \otimes \tau(r_2), \qquad \varepsilon_R(r) := \varepsilon_A(r).$$

Moreover there is a dual quasi-bialgebra isomorphism $\epsilon_A : R\#H \to A$ given by

$$\epsilon_A(r \otimes h) = r\sigma(h), \qquad \epsilon_A^{-1}(a) = \tau(a_1) \otimes \pi(a_2).$$

PROOF. We have

$$\rho^r_A(a_1) \otimes a_2 = a_1 \otimes \pi(a_2) \otimes a_3 = a_1 \otimes \rho^l_A(a_2)$$

so that $\Delta_A(a) \in A\square_H A$ for all $a \in A$. Let $\overline{\Delta}_A : A \to A\square_H A$ be the corestriction of Δ_A to $A\square_H A$. Using that $\omega_H = \omega_A(\pi \otimes \pi \otimes \pi)$, we obtain

$$m_A \circ \left(A \otimes \mu^l_A\right) \circ {}^H a^H_{A,H,A} = m_A \circ (\mu^r_A \otimes A).$$

Denote by $\chi_{X,Y} : X \otimes Y \to X \otimes_H Y$ the canonical projection, for all X, Y objects in ${}^H_H\mathfrak{M}^H_H$.

Since $(A \otimes_H A, \chi_{A,A})$ is the coequalizer of $\left(\left(A \otimes \mu^l_A\right) {}^H a^H_{A,H,A}, (\mu^r_A \otimes A)\right)$, we get that m_A factors through to a map $m'_A : A \otimes_H A \to A$ such that $m'_A \circ \chi_{A,A} = m_A$. Consider the canonical map $\vartheta_2(M, N) : M\square_H N \to M \otimes_H N$ of Lemma 4.5.12 defined by $\vartheta_2(M, N)(m\square_H n) := \tau(m) \otimes_H n$ and let $\overline{m}_A := m'_A \circ \vartheta_2(A, A)$. Then

$$\overline{m}_A(a\square_H b) = m'_A(\tau(a) \otimes_H b) = \tau(a)b.$$

Note that, by Lemma 4.3.6, the map $\tau : A \to A^{coH}$ is defined, for all $a \in A$, by

$$\tau(a) = \omega_H[a_{-1} \otimes S(a_1)_1 \otimes a_2]a_0 S(a_1)_2$$

$$
\begin{aligned}
&= \omega_H[\pi(a_1) \otimes S\pi(a_3)_1 \otimes \pi(a_4)]a_2\sigma[S\pi(a_3)_2] \\
&= \omega_H[\pi(a_1) \otimes \pi\sigma[S\pi(a_3)_1] \otimes \pi(a_4)]a_2\sigma[S\pi(a_3)_2] \\
&= \omega_A[a_1 \otimes \sigma[S\pi(a_3)_1] \otimes a_4]a_2\sigma[S\pi(a_3)_2] \\
&= \omega_A[a_1 \otimes \sigma S\pi(a_3)_1 \otimes a_4]a_2\sigma S\pi(a_3)_2.
\end{aligned}
$$

It is straightforward to prove that $\left(A, \overline{\Delta}_A, \overline{\varepsilon}_A := \pi\right)$ is a coalgebra in $({}^H_H\mathfrak{M}^H_H, \square_H, H)$.

One checks that (A, m'_A, σ) is an algebra in $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$.

Now, given the monoidal functor $E : ({}^H_H\mathfrak{M}^H_H, \otimes_H, H) \to ({}^H_H\mathfrak{M}^H_H, \square_H, H)$ of Lemma 4.5.12 we have that $\left(E(A), m_{E(A)}, u_{E(A)}\right)$ is an algebra in $({}^H_H\mathfrak{M}^H_H, \square_H, H)$ where

$$m_{E(A)} = E(m'_A) \circ \vartheta_2(A, A) \qquad \text{and} \qquad u_{E(A)} = E(\sigma) \circ \vartheta_0.$$

It is clear that $\left(E(A), m_{E(A)}, u_{E(A)}\right) = (A, \overline{m}_A, \overline{u}_A = \sigma)$. Thus $(A, \overline{m}_A, \overline{u}_A)$ is an algebra in $({}^H_H\mathfrak{M}^H_H, \square_H, H)$.

Now, we apply [**AMS2**, Proposition 1.5] to the functor $G : {}^H_H\mathfrak{M}^H_H \to {}^H_H\mathcal{YD}$ of Lemma 4.5.10. Set $R := G(A) = A^{coH}$. Then R is both an algebra and a coalgebra in ${}^H_H\mathcal{YD}$ through

$$
\begin{aligned}
m_R &:= G(\overline{m}_A) \circ \psi_2^G(A, A), \qquad u_R := G(\overline{u}_A) \circ \psi_0^G, \\
\Delta_R &:= \psi_2^G(A, A)^{-1} \circ G\left(\overline{\Delta}_A\right), \qquad \varepsilon_R := \left(\psi_0^G\right)^{-1} \circ G(\overline{\varepsilon}_A).
\end{aligned}
$$

Explicitly, for all $r, s \in R, k \in \Bbbk$

$$m_R(r \otimes s) = \tau(rs_{-1})s_0 \overset{(26)}{=} r\varepsilon_H(s_{-1})s_0 = rs,$$

$$u_R(k) = G(\overline{u}_A)\psi_0^G(k) = \overline{u}_A(k1_H) = k\sigma(1_H) = k1_A,$$

$$\Delta_R(r) = \tau(r_1) \otimes \tau(r_2),$$

$$\varepsilon_R(r) = \left(\psi_0^G\right)^{-1} G(\overline{\varepsilon}_A)(r) = \left(\psi_0^G\right)^{-1} \pi(r) = \pi(r) = \varepsilon_A(r_1)\pi(r_2) = \varepsilon_A(r_0)r_1 = \varepsilon_A(r)1_H.$$

We will use the following notations for all $r, s \in R$,

$$r \cdot_R s := m_R(r \otimes s), \qquad 1_R := u_R(1_\Bbbk).$$

Now, by [**AMS2**, Corollary 1.7], we have that $\epsilon_A : FG(A) \to A$ is an algebra and a coalgebra isomorphism in $({}^H_H\mathfrak{M}^H_H, \square_H, H)$. Let us write the algebra and coalgebra structure of $FG(A) = R \otimes H$. By construction, we have

$$
\begin{aligned}
\overline{m}_{F(R)} &:= F(m_R) \circ \psi_2(R, R) : F(R) \square_H F(R) \to F(R), \\
\overline{u}_{F(R)} &:= F(u_R) \circ \psi_0 : H \to F(R), \\
\overline{\Delta}_{F(R)} &:= \psi_2(R, R)^{-1} \circ F(\Delta_R) : F(R) \to F(R) \square_H F(R), \\
\overline{\varepsilon}_{F(R)} &:= \psi_0^{-1} \circ F(\varepsilon_R) : F(R) \to H.
\end{aligned}
$$

Explicitly we have

$$\overline{m}_{F(R)}((r \otimes h) \square_H (s \otimes k)) = \omega(r_{-1} \otimes s_{-1} \otimes k_1)r_0\varepsilon(h) \cdot_R s_0 \otimes k_2,$$

$$\overline{u}_{F(R)}(h) = F(u_R)\psi_0(h) = 1_R \otimes h,$$

$$\overline{\Delta}_{F(R)}(r \otimes h) = \omega^{-1}(r^1_{-1} \otimes r^2_{-2} \otimes h_1)(r^1_0 \otimes r^2_{-1}h_2)\square_H(r^2_0 \otimes h_3),$$

$$\overline{\varepsilon}_{F(R)}(r \otimes h) = \psi_0^{-1}F(\varepsilon_R)(r \otimes h) = \psi_0^{-1}(\varepsilon_R(r) \otimes h) = \varepsilon_R(r)h.$$

In view of 4.5.7, the forgetful functor $({}_H^H\mathfrak{M}_H^H, \square_H, H) \to ({}^H\mathfrak{M}^H, \square_H, H)$ is a strict monoidal functor. $\epsilon_A : (F(R), \overline{\Delta}_{F(R)}, \overline{\varepsilon}_{F(R)}) \to (A, \overline{\Delta}_A, \overline{\varepsilon}_A = \pi)$ being a coalgebra morphism in $({}_H^H\mathfrak{M}_H^H, \square_H, H)$, we have that $\epsilon_A : (F(R), \overline{\Delta}_{F(R)}, \overline{\varepsilon}_{F(R)}) \to (A, \overline{\Delta}_A, \overline{\varepsilon}_A = \pi)$ is a coalgebra morphism in $({}^H\mathfrak{M}^H, \square_H, H)$. Apply Lemma 4.5.4 to the case $(\mathcal{M}, \otimes, \mathbf{1}) = (\mathfrak{M}, \otimes, \Bbbk)$ and $C = H$. Let $j_{X,Y} : X\square_H Y \to X \otimes Y$ be the canonical map. Then $\epsilon_A : (F(R), j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)}, \varepsilon_H \circ \overline{\varepsilon}_{F(R)}) \to (A, j_{A,A} \circ \overline{\Delta}_A, \varepsilon_H \circ \overline{\varepsilon}_A)$ is a coalgebra morphism in $(\mathfrak{M}, \otimes, \Bbbk)$. In other words it is an ordinary coalgebra morphism. Note that $(A, j_{A,A} \circ \overline{\Delta}_A, \varepsilon_H \circ \overline{\varepsilon}_A) = (A, \Delta_A, \varepsilon_A)$. Set $(\Delta_{F(R)}, \varepsilon_{F(R)}) := (j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)}, \varepsilon_H \circ \overline{\varepsilon}_{F(R)})$. Let us compute explicitly these maps. We have

$$\begin{aligned}
\Delta_{F(R)}(r \otimes h) &= \left(j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)}\right)(r \otimes h) = \omega^{-1}(r^1_{-1} \otimes r^2_{-2} \otimes h_1)(r^1_0 \otimes r^2_{-1} h_2) \otimes (r^2_0 \otimes h_3), \\
\varepsilon_{F(R)}(r \otimes h) &= \left(\varepsilon_H \circ \overline{\varepsilon}_{F(R)}\right)(r \otimes h) = \varepsilon_R(r)\, \varepsilon_H(h).
\end{aligned}$$

Thus $\epsilon_A : (F(R), \Delta_{F(R)}, \varepsilon_{F(R)}) \to (A, \Delta_A, \varepsilon_A)$ is an ordinary coalgebra morphism.
$\epsilon_A : (F(R), \overline{m}_{F(R)}, \overline{u}_{F(R)}) \to (A, \overline{m}_A, \overline{u}_A = \sigma)$ being an algebra morphism in $({}_H^H\mathfrak{M}_H^H, \square_H, H)$, then, in view of Lemma 4.5.12,

$$\epsilon_A : (F(R), \Xi\left(\overline{m}_{F(R)}\right)\circ\gamma_2(F(R), F(R)), \Xi\left(\overline{u}_{F(R)}\right)\circ\gamma_0) \to (A, \Xi(\overline{m}_A)\circ\gamma_2(A, A), \Xi(\overline{u}_A)\circ\gamma_0)$$

is an algebra morphism in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$. Note that

$$\begin{aligned}
\Xi(\overline{m}_A) \circ \gamma_2(A, A) &= \overline{m}_A \circ \vartheta_2^{-1}(A, A) = m'_A, \\
\Xi(\overline{u}_A) \circ \gamma_0 &= \overline{u}_A = \sigma
\end{aligned}$$

so that

$$(A, \Xi(\overline{m}_A) \circ \gamma_2(A, A), \Xi(\overline{u}_A) \circ \gamma_0) = (A, m'_A, \sigma).$$

Set $\left(m'_{F(R)}, u'_{F(R)}\right) := \left(\Xi\left(\overline{m}_{F(R)}\right) \circ \gamma_2(F(R), F(R)), \Xi\left(\overline{u}_{F(R)}\right) \circ \gamma_0\right)$. We have

$$\begin{aligned}
& m'_{F(R)}((r \otimes h) \otimes_H (s \otimes k)) \\
= & \left[\Xi\left(\overline{m}_{F(R)}\right) \circ \gamma_2(F(R), F(R))\right]((r \otimes h) \otimes_H (s \otimes k)) \\
= & \overline{m}_{F(R)}\left[(r \otimes h)_0 (s \otimes k)_{-1} \otimes_H (r \otimes h)_1 (s \otimes k)_0\right] \\
= & \overline{m}_{F(R)}\left[(r \otimes h_1)(s_{-1}k_1) \otimes_H h_2 (s_0 \otimes k_2)\right] \\
= & \omega^{-1}\left[r_{-1} \otimes h_1 \otimes s_{-2}k_1\right] \overline{m}_{F(R)}\left[r_0 \otimes \left[h_2(s_{-1}k_2)\right] \otimes_H h_3(s_0 \otimes k_3)\right] \\
= & \left[\begin{array}{c} \omega^{-1}\left[r_{-1} \otimes h_1 \otimes s_{-3}k_1\right] \omega(h_3 \otimes s_{-1} \otimes k_3)\omega^{-1}((h_4 \rhd s_0)_{-1} \otimes h_5 \otimes k_4) \\ \overline{m}_{F(R)}\left[r_0 \otimes \left[h_2(s_{-2}k_2)\right] \otimes_H \left[(h_4 \rhd s_0)_0 \otimes h_6 k_5\right]\right. \end{array}\right] \\
= & \left[\begin{array}{c} \omega^{-1}\left[r_{-2} \otimes h_1 \otimes s_{-3}k_1\right] \omega(h_3 \otimes s_{-1} \otimes k_3)\omega^{-1}((h_4 \rhd s_0)_{-2} \otimes h_5 \otimes k_4) \\ \omega(r_{-1} \otimes (h_4 \rhd s_0)_{-1} \otimes h_6 k_5) r_0 \varepsilon_H\left[h_2(s_{-2}k_2)\right] \cdot_R (h_4 \rhd s_0)_0 \otimes h_7 k_6 \end{array}\right] \\
= & \left[\begin{array}{c} \omega^{-1}\left[r_{-2} \otimes h_1 \otimes s_{-2}k_1\right] \omega(h_2 \otimes s_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \rhd s_0)_{-2} \otimes h_4 \otimes k_3)\omega(r_{-1} \otimes (h_3 \rhd s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \rhd s_0)_0 \otimes h_6 k_5 \end{array}\right]
\end{aligned}$$

so that

$$m'_{F(R)}((r \otimes h) \otimes_H (s \otimes k))$$

$$= \left[\begin{array}{c} \omega^{-1}\left[r_{-2}\otimes h_1\otimes s_{-2}k_1\right]\omega(h_2\otimes s_{-1}\otimes k_2) \\ \omega^{-1}((h_3\triangleright s_0)_{-2}\otimes h_4\otimes k_3)\omega(r_{-1}\otimes (h_3\triangleright s_0)_{-1}\otimes h_5k_4) \\ r_0\cdot_R (h_3\triangleright s_0)_0\otimes h_6k_5 \end{array} \right].$$

Moreover

$$u'_{F(R)}(h) = \left[\Xi\left(\overline{u}_{F(R)}\right)\circ\gamma_0\right](h) = \overline{u}_{F(R)}(h) = 1_R\otimes h.$$

Apply Lemma 4.5.4 to the case $(\mathcal{M},\otimes,\mathbf{1}) = ({}^H\mathfrak{M}^H,\otimes,\Bbbk)$ and $A = H$. Then

$$\epsilon_A : (F(R), m'_{F(R)}\circ\chi_{F(R),F(R)}, u'_{F(R)}\circ u_H) \to (A, m'_A\circ\chi_{A,A},\sigma\circ u_H)$$

is an algebra homomorphism in $({}^H\mathfrak{M}^H,\otimes,\Bbbk)$. Note that $(A, m'_A\circ\chi_{A,A},\sigma\circ u_H) = (A, m_A, u_A)$. Moreover, if we set $(m_{F(R)}, u_{F(R)}) := (m'_{F(R)}\circ\chi_{F(R),F(R)}, u'_{F(R)}\circ u_H)$, we get

$$m_{F(R)}\left((r\otimes h)\otimes(s\otimes k)\right)$$
$$= \left[\begin{array}{c} \omega^{-1}\left[r_{-2}\otimes h_1\otimes s_{-2}k_1\right]\omega(h_2\otimes s_{-1}\otimes k_2) \\ \omega^{-1}((h_3\triangleright s_0)_{-2}\otimes h_4\otimes k_3)\omega(r_{-1}\otimes (h_3\triangleright s_0)_{-1}\otimes h_5k_4) \\ r_0\cdot_R (h_3\triangleright s_0)_0\otimes h_6k_5 \end{array} \right].$$

Moreover

$$u_{F(R)}(k) = 1_R\otimes k.$$

Thus $\epsilon_A : (F(R), m_{F(R)}, u_{F(R)}) \to (A, m_A, u_A)$ is an algebra isomorphism in $({}^H\mathfrak{M}^H,\otimes,\Bbbk)$ and $\epsilon_A : (F(R),\Delta_{F(R)},\varepsilon_{F(R)}) \to (A,\Delta_A,\varepsilon_A)$ is an ordinary coalgebra isomorphism. Thus

$$\begin{aligned} m_A\circ(\epsilon_A\otimes\epsilon_A) &= \epsilon_A\circ m_{F(R)}, \qquad \epsilon_A\circ u_{F(R)} = u_A, \\ (\epsilon_A\otimes\epsilon_A)\circ\Delta_{F(R)} &= \Delta_A\circ\epsilon_A, \qquad \varepsilon_A\circ\epsilon_A = \varepsilon_{F(R)}, \end{aligned}$$

so that $m_{F(R)}, u_{F(R)}, \Delta_{F(R)}, \varepsilon_{F(R)}$ are exactly the morphisms induced by $m_A, u_A, \Delta_A, \varepsilon_A$ via the vector space isomorphism $\epsilon_A : F(R)\to A$. Let $\omega_{F(R)}$ be the map induced by ω_A via the vector space isomorphism ϵ_A i.e.

$$\omega_{F(R)} := \omega_A\circ(\epsilon_A\otimes\epsilon_A\otimes\epsilon_A) : F(R)\otimes F(R)\otimes F(R)\to\Bbbk.$$

Then $\epsilon_A : \left(F(R),\Delta_{F(R)},\varepsilon_{F(R)}, m_{F(R)}, u_{F(R)},\omega_{F(R)}\right) \to (A, m_A, u_A,\Delta_A,\varepsilon_A,\omega_A)$ is clearly an isomorphism of dual quasi-bialgebras. Since, for all $r\in R$, we have $\pi(r) = \varepsilon_A(r_1)\pi(r_2) = \varepsilon_A(r)1_H$, then, for $r,s,t\in R, h,k,l\in H$, we get

$$\begin{aligned} &\omega_{F(R)}\left[(r\otimes h)\otimes(s\otimes k)\otimes(t\otimes l)\right] = \omega_A\left(r\sigma(h)\otimes s\sigma(k)\otimes t\sigma(l)\right) \\ = \;&\omega_H\left[\pi(r\sigma(h))\otimes\pi(s\sigma(k))\otimes\pi(t\sigma(l))\right] = \omega_H\left[\pi(r)h\otimes\pi(s)k\otimes\pi(t)l\right] \\ = \;&\omega_H\left[\varepsilon_A(r)h\otimes\varepsilon_A(s)k\otimes\varepsilon_A(t)l\right] = \varepsilon_A(r)\varepsilon_A(t)\varepsilon_A(s)\omega_H(h\otimes k\otimes l) \end{aligned}$$

so that

$$\omega_{F(R)}\left[(r\otimes h)\otimes(s\otimes k)\otimes(t\otimes l)\right] = \varepsilon_A(r)\varepsilon_A(t)\varepsilon_A(s)\omega_H(h\otimes k\otimes l).$$

Note that $\left(F(R),\Delta_{F(R)},\varepsilon_{F(R)}, m_{F(R)}, u_{F(R)},\omega_{F(R)}\right) = R\#H$ once proved that $(R, m_R, u_R,\Delta_R,\varepsilon_R)$ is a bialgebra in the monoidal category $\left({}_H^H\mathcal{YD},\otimes,\Bbbk\right)$. It remains to prove that m_R and u_R are coalgebra maps. Since ${}_H^H\mathcal{YD}$ is a pre-braided monoidal category and $(R,\Delta_R,\varepsilon_R)$ is a coalgebra in this category, then we can define two morphisms $\Delta_{R\otimes R}$

and $\varepsilon_{R\otimes R}$ in ${}_H^H\mathcal{YD}$ such that $(R\otimes R,\Delta_{R\otimes R},\varepsilon_{R\otimes R})$ is a coalgebra in ${}_H^H\mathcal{YD}$ too. We have

$$\begin{aligned}\Delta_{R\otimes R} &: = a^{-1}_{R,R,R\otimes R}\circ(R\otimes a_{R,R,R})\circ(R\otimes c_{R,R}\otimes R)\circ(R\otimes a^{-1}_{R,R,R})\circ a_{R,R,R\otimes R}\circ(\Delta_R\otimes\Delta_R),\\ \varepsilon_{R\otimes R} &: = \varepsilon_R\otimes\varepsilon_R.\end{aligned}$$

Explicitly $\Delta_{R\otimes R}$ satisfies (58). In order to prove that m_R is a morphism of coalgebras in ${}_H^H\mathcal{YD}$, we have to check the following equality

$$(m_R\otimes m_R)\Delta_{R\otimes R}=\Delta_R m_R.$$

Since we already obtained that $B:=F(R)$ is a dual quasi-bialgebra, we know that

$$\Delta_B[(r\otimes 1_H)\cdot_B(s\otimes 1_H)]=(r\otimes 1_H)_1\cdot_B(s\otimes 1_H)_1\otimes(r\otimes 1_H)_2\cdot_B(s\otimes 1_H)_2.$$

By applying $R\otimes\varepsilon_H\otimes R\otimes\varepsilon_H$ on both sides we get:

$$\begin{aligned}
&(r\cdot_R s)^1\otimes(r\cdot_R s)^2\\
&=\left[\begin{array}{c}
\omega^{-1}((r^1)_{-2}\otimes(r^2_{-1})_1\otimes(s^1)_{-2}\left(s^2_{-1}\right)_1)\\
\omega(\left(r^2_{-1}\right)_2\otimes(s^1)_{-1}\otimes\left(s^2_{-1}\right)_2)\\
\omega^{-1}[(\left(r^2_{-1}\right)_3\triangleright(s^1)_0)_{-2}\otimes\left(r^2_{-1}\right)_4\otimes\left(s^2_{-1}\right)_3]\\
\omega((r^1)_{-1}\otimes(\left(r^2_{-1}\right)_3\triangleright(s^1)_0)_{-1}\otimes\left(r^2_{-1}\right)_5\left(s^2_{-1}\right)_4)\\
(r^1)_0\cdot_R(\left(r^2_{-1}\right)_3\triangleright(s^1)_0)_0
\end{array}\right]\otimes(r^2_0\cdot_R s^2_0)\\
&=\left[\begin{array}{c}
\omega^{-1}(r^1_{-2}\otimes r^2_{-5}\otimes s^1_{-2}s^2_{-4})\omega(r^2_{-4}\otimes s^1_{-1}\otimes s^2_{-3})\\
\omega^{-1}[(r^2_{-3}\triangleright s^1_0)_{-2}\otimes r^2_{-2}\otimes s^2_{-2}]\\
\omega(r^1_{-1}\otimes(r^2_{-3}\triangleright s^1_0)_{-1}\otimes r^2_{-1}s^2_{-1})\\
r^1_0\cdot_R(r^2_{-3}\triangleright s^1_0)_0\otimes(r^2_0\cdot_R s^2_0)
\end{array}\right]\\
&=(m_R\otimes m_R)\Delta_{R\otimes R}(r\otimes s).
\end{aligned}$$

The compatibility of m_R with ε_R and the fact that u_R is a coalgebra morphism can be easily proved. □

3. Applications

Here we collect some applications of the results of the results just exposed.

3.1. The associated graded coalgebra.

EXAMPLE 6. *Let $(A,m_A,u_A,\Delta_A,\varepsilon_A,\omega_A)$ be a dual quasi-bialgebra with the dual Chevalley property i.e. such that the coradical H of A is a dual quasi-subbialgebra of A. Since A is an ordinary coalgebra, we can consider the associated graded coalgebra*

$$\mathrm{gr}A:=\bigoplus_{n\in\mathbb{N}}\mathrm{gr}^n A\qquad \text{where } \mathrm{gr}^n A:=\frac{A_n}{A_{n-1}}.$$

Here $A_{-1}:=\{0\}$ and, for all $n\geq 0$, A_n is the nth term of the coradical filtration of A. The coalgebra structure of $\mathrm{gr}A$ is given as follows. The nth graded component of the counit is the map $\varepsilon^n_{\mathrm{gr}A}:A_n/A_{n-1}\to\Bbbk$ defined by setting

$$\varepsilon^n_{\mathrm{gr}A}(x+A_{n-1})=\delta_{n,0}\varepsilon_A(x).$$

The nth graded component of comultiplication is the map

$$\Delta^n_{\mathrm{gr}A} : \mathrm{gr}^{a+b}A \to \bigoplus_{a+b=n,a,b\geq 0} \mathrm{gr}^a A \otimes \mathrm{gr}^b A$$

defined as the diagonal map of the family $(\Delta^{a,b}_{\mathrm{gr}A})_{a+b=n,a,b\geq 0}$ *where*

$$\Delta^{a,b}_{\mathrm{gr}A} : \mathrm{gr}^{a+b}A \to \mathrm{gr}^a A \otimes \mathrm{gr}^b A, \Delta^{a,b}_{\mathrm{gr}A}(x + A_{a+b-1}) = (x_1 + A_{a-1}) \otimes (x_2 + A_{b-1}).$$

PROPOSITION 5.3.1. *Let A be a dual quasi-bialgebra with the dual Chevalley property. Then*

$$(\mathrm{gr}A, m_{\mathrm{gr}A}, u_{\mathrm{gr}A}, \Delta_{\mathrm{gr}A}, \varepsilon_{\mathrm{gr}A}, \omega_{\mathrm{gr}A})$$

is a dual quasi-bialgebra where the graded components of the structure maps are given by the maps

$$m^{a,b}_{\mathrm{gr}A} : \mathrm{gr}^a A \otimes \mathrm{gr}^b A \to \mathrm{gr}^{a+b}A, \qquad u^n_{\mathrm{gr}A} : \Bbbk \to \mathrm{gr}^n A,$$
$$\Delta^{a,b}_{\mathrm{gr}A} : \mathrm{gr}^{a+b}A \to \mathrm{gr}^a A \otimes \mathrm{gr}^b A, \qquad \varepsilon^n_{\mathrm{gr}A} : \mathrm{gr}^n A \to \Bbbk,$$
$$\omega^{a,b,c}_{\mathrm{gr}A} : \mathrm{gr}^a A \otimes \mathrm{gr}^b A \otimes \mathrm{gr}^c A \to \Bbbk,$$

defined by

$$m^{a,b}_{\mathrm{gr}A}\left[(x + A_{a-1}) \otimes (y + A_{b-1})\right] := xy + A_{a+b-1}, \qquad u^n_{\mathrm{gr}A}(k) := \delta_{n,0}1_A + A_{-1} = \delta_{n,0}1_A,$$
$$\Delta^{a,b}_{\mathrm{gr}A}(x + A_{a+b-1}) := (x_1 + A_{a-1}) \otimes (x_2 + A_{b-1}), \qquad \varepsilon^n_{\mathrm{gr}A}(x + A_{n-1}) := \delta_{n,0}\varepsilon_A(x),$$
$$\omega^{a,b,c}_{\mathrm{gr}A}[(x + A_{a-1}) \otimes (y + A_{b-1}) \otimes (z + A_{c-1})] := \delta_{a,0}\delta_{b,0}\delta_{c,0}\omega_A(x \otimes y \otimes z).$$

Here $\delta_{i,j}$ denotes the Kronecker delta.

PROOF. The proof of the facts that $m_{\mathrm{gr}A}$ and $u_{\mathrm{gr}A}$ are well-defined, are coalgebra maps and that $m_{\mathrm{gr}A}$ is unitary is analogous to the classical case, and depend on the fact that the coradical filtration is an algebra filtration. This can be proved mimicking [**Mo**, Lemma 5.2.8]. The cocycle condition and the quasi-associativity of $m_{\mathrm{gr}A}$ are straightforward. □

PROPOSITION 5.3.2. *Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Then* $(\mathrm{gr}A, H, \sigma, \pi)$ *is a dual quasi-bialgebra with projection onto H, where*

$$\sigma : H \longrightarrow \mathrm{gr}A : h \longmapsto h + A_{-1},$$
$$\pi : \mathrm{gr}A \longrightarrow H : a + A_{n-1} \longmapsto \delta_{n,0}a, \text{ for all } a \in A_n.$$

PROOF. It is straightforward. □

COROLLARY 5.3.3. *Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Assume that H has a preantipode. Then there is a bialgebra R in ${}^H_H\mathcal{YD}$ such that* $\mathrm{gr}A$ *is isomorphic to $R\#H$ a dual quasi-bialgebra.*

PROOF. It follows by Proposition 5.3.2 and Theorem 5.2.4. □

DEFINITION 5.3.4. Following [**AS**, Definition, page 659], the bialgebra R in ${}^H_H\mathcal{YD}$ of Corollary 5.3.3, is called the *diagram* of A.

3.2. On pointed dual quasi-bialgebras. We conclude this chapter considering the pointed case.

LEMMA 5.3.5. *Let G be a monoid and consider the monoid algebra $H := \Bbbk G$. Suppose there is a map $\omega \in (H \otimes H \otimes H)^*$ such that (H, ω) is a dual quasi-bialgebra. Then (H, ω) has a preantipode S if and only if G is a group. In this case*

$$S(g) = [\omega(g \otimes g^{-1} \otimes g)]^{-1} g^{-1}.$$

PROOF. Suppose that S is a preantipode for (H, ω). Since H is a cocommutative ordinary bialgebra, by Theorem 4.3.10, we have that $\Bbbk G$ is an ordinary Hopf algebra, where the antipode is defined, for all $g \in G$, by

$$s(g) := S(g)_1 \, \omega \left[g \otimes S(g)_2 \otimes g\right].$$

Moreover one has $S = \varepsilon S * s$. Now, since $\Bbbk G$ is a Hopf algebra, one has that the set of grouplike elements in $\Bbbk G$, namely G itself, form a group, where $g^{-1} := s(g)$, for all $g \in G$.

Now, since s is an anti-coalgebra map, we have

$$S(g)_1 \otimes S(g)_2 = \varepsilon S(g) s(g)_1 \otimes s(g)_2 = \varepsilon S(g) s(g) \otimes s(g) = S(g) \otimes g^{-1}$$

so that $s(g) = S(g)_1 \, \omega \left[g \otimes S(g)_2 \otimes g\right] = S(g) \omega \left(g \otimes g^{-1} \otimes g\right)$. Hence $S(g) = [\omega(g \otimes g^{-1} \otimes g)]^{-1} g^{-1}$.

The other implication is trivial (see [**AP**, Example 3.14]). □

The motivation for the previous result is Corollary 5.3.8 below.

PROPOSITION 5.3.6. *Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Then the set of grouplike elements $\mathbb{G}(A)$ of A is a monoid and the monoid algebra $\Bbbk \mathbb{G}(A)$ is a dual quasi-subbialgebra of A.*

PROOF. It is straightforward. □

COROLLARY 5.3.7. *Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a pointed dual quasi-bialgebra. Then $A_0 = \Bbbk \mathbb{G}(A)$ is a dual quasi-subbialgebra of A.*

PROOF. By Remark 4.3.13, $A_0 = \Bbbk \mathbb{G}(A)$. In view of Proposition 5.3.6, we conclude. □

COROLLARY 5.3.8. *Let $(A, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a pointed dual quasi-Hopf algebra. Then $\mathbb{G}(A)$ is a group and $A_0 = \Bbbk \mathbb{G}(A)$ is a dual quasi-Hopf algebra with respect to the induced structures.*

PROOF. Set $G := \mathbb{G}(A)$. By Corollary 5.3.7, $A_0 = \Bbbk G$ is a dual quasi-subbialgebra of A. It remains to check that the antipode on A induces an antipode on A_0. We have

$$\begin{aligned} \Delta s(g) &= s(g_2) \otimes s(g_1) = s(g) \otimes s(g), \\ \varepsilon s(g) &= \varepsilon(g) = 1, \end{aligned}$$

i.e. $s(g) \in G$, for any $g \in G$. Let $s_0, \alpha_0, \beta_0, \omega_0, m_0, u_0, \Delta_0, \varepsilon_0$ be the induced maps from $s, \alpha, \beta, \omega, m, u, \Delta, \varepsilon$, respectively. It is then clear from the definition that A_0, with respect to these structures, is a dual quasi-Hopf algebra. Since any dual quasi-Hopf algebra has a preantipode, by Lemma 5.3.5, G is a group. □

Pointed dual quasi-Hopf algebras have been investigated also in [**Hu**, page 2] under the name of pointed Majid algebras. In view of Corollary 5.3.8, which seems to be implicitly assumed in [**Hu**, page 2], we can apply Corollary 5.3.3 to obtain the following result.

THEOREM 5.3.9. *Let A be a pointed dual quasi-Hopf algebra. Then* $\mathrm{gr}A$ *is isomorphic to* $R\#\Bbbk\mathbb{G}(A)$ *as dual quasi-bialgebra where R is the diagram of A.*

Appendix B

Definition 5.3.10. [**BCP**, Section 1.5] Let $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. The weak right center $\mathcal{W}_r(\mathcal{M})$ of $\mathcal{M}$ is a category defined as follows. The objects in $\mathcal{W}_r(\mathcal{M})$ are all the objects V of $\mathcal{M}$ such that there exists an associated class of morphisms $c_{-,V}$ ($c_{X,V} : X \otimes V \to V \otimes X$, for any object X in $\mathcal{M}$), which are natural in the first entry and satisfying, for all $X, Y \in \mathcal{M}$:

$$a_{V,X,Y}^{-1} \circ c_{X\otimes Y,V} \circ a_{X,Y,V}^{-1} = (c_{X,V} \otimes Y) \circ a_{X,V,Y}^{-1} \circ (X \otimes c_{Y,V}) \tag{63}$$

and such that $r_V \circ c_{\mathbf{1},V} = l_V$. A morphism $f : (V, c_{-,V}) \to (W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{M}$ such that, for each $X \in \mathcal{M}$ we have

$$(f \otimes X) \circ c_{X,V} = c_{X,W} \circ (X \otimes f).$$

$\mathcal{W}_r(\mathcal{M})$ becomes a monoidal category with unit $(\mathbf{1}, l^{-1} \circ r)$ and tensor product

$$(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V\otimes W})$$

where, for all $X \in \mathcal{M}$, the morphism $c_{X,V\otimes W} : X \otimes (V \otimes W) \to (V \otimes W) \otimes X$ is defined by

$$c_{X,V\otimes W} := a_{V,W,X}^{-1} \circ (V \otimes c_{X,W}) \circ a_{V,X,W} \circ (c_{X,V} \otimes W) \circ a_{X,V,W}^{-1}.$$

The constraints are the same of $\mathcal{M}$ viewed as morphisms in $\mathcal{W}_r(\mathcal{M})$. Moreover the monoidal category $\mathcal{W}_r(\mathcal{M})$ is pre-braided, with braiding

$$c_{(V,c_{-,V}),(W,c_{-,W})} : (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V})$$

given by $c_{V,W}$.

Theorem 5.3.11. *Let H be a dual quasi-bialgebra. The categories $\mathcal{W}_r\left({}^H\mathfrak{M}\right)$ and ${}_H^H\mathcal{YD}$ are isomorphic, where ${}^H\mathfrak{M}$ is regarded as a monoidal category as in Chapter 4.*

Proof. The proof is analogue to [**Ba**, Theorem 3.5].

□

Remark 5.3.12. Let us point out that, by [**Ba**, Theorem 3.3], given a dual quasi Hopf algebra H with a bijective antipode, the weak right center $\mathcal{W}_r({}^H\mathfrak{M})$ coincides with the common center of ${}^H\mathfrak{M}$.

Example : the group algebra

We now investigate the category of Yetter-Drinfeld modules over a particular dual quasi-Hopf algebra.

Let G be a group. Let $\theta : G\times G\times G\rightarrow \Bbbk^{\ast}:=\Bbbk\backslash\left\{ 0\right\}$ be a normalized 3-cocycle on the group G in the sense of [**Maj1**, Example 2.3.2, page 54] i.e. a map such that, for all $g,h,k,l\in H$

$$\begin{aligned} \theta\left(g,1_{G},h\right) &= 1 \\ \theta\left(h,k,l\right)\theta\left(g,hk,l\right)\theta\left(g,h,k\right) &= \theta\left(g,h,kl\right)\theta\left(gh,k,l\right). \end{aligned}$$

Then θ can be extended by linearity to a reassociator $\omega : \Bbbk G\otimes\Bbbk G\otimes\Bbbk G\rightarrow\Bbbk$ making $\Bbbk G$ a dual quasi-bialgebra with usual underlying algebra and coalgebra structures. This dual quasi-bialgebra is denoted by $\Bbbk^{\theta}G$. Note that in particular $\Bbbk^{\theta}G$ is an ordinary bialgebra but with nontrivial reassociator. In particular it is associative as an algebra. Let us investigate the category ${}_{\Bbbk^{\theta}G}^{\Bbbk^{\theta}G}\mathcal{YD}$ of Yetter-Drinfeld module over $\Bbbk^{\theta}G$.

Definition 5.3.13. Let $\theta : G\times G\times G\rightarrow\Bbbk^{\ast}$ be a normalized 3-cocycle on a group G. The category of cocycle crossed left G-modules (G,θ)-Mod is defined as follows. An object in (G,θ)-Mod is a pair $\left(V,\blacktriangleright\right)$, where $V=\oplus_{g\in G}V_{g}$ is a G-graded vector space endowed with a map $\blacktriangleright : G\times V\rightarrow V$ such that, for all $g,h,l\in H$ and $v\in V$, we have

$$h\blacktriangleright V_{g}\in V_{hgh^{-1}}, \tag{64}$$

$$h\blacktriangleright\left(l\blacktriangleright v\right)=\frac{\theta\left(hlgl^{-1}h^{-1},h,l\right)\theta\left(h,l,g\right)}{\theta\left(h,lgl^{-1},l\right)}\left(hl\right)\blacktriangleright v, \tag{65}$$

$$1_{H}\blacktriangleright v=v. \tag{66}$$

A morphism $f:(V,\blacktriangleright)\rightarrow(V',\blacktriangleright')$ in (G,θ)-Mod is a morphism $f:V\rightarrow V'$ of G-graded vector spaces such that, for all $h\in H,v\in V$, we have $f(h\blacktriangleright v)=h\blacktriangleright' f(v)$.

The following result is inspired by [**Maj2**, Proposition 3.2].

Proposition 5.3.14. *Let $\theta : G\times G\times G\rightarrow\Bbbk^{\ast}$ be a normalized 3-cocycle on a group G. Then the category ${}_{\Bbbk^{\theta}G}^{\Bbbk^{\theta}G}\mathcal{YD}$ is isomorphic to* (G,θ)-Mod.

Proof. Set $H:=\Bbbk^{\theta}G$ and let $\left(V,\rho_{V},\rhd\right)\in{}_{H}^{H}\mathcal{YD}$. Then $\left(V,\rho_{V}\right)$ is an object in ${}^{\Bbbk G}\mathfrak{M}$. Hence, see e.g. [**Mo**, Example 1.6.7], we have that $V=\oplus_{g\in G}V_{g}$ where $V_{g}=\left\{v\in V\mid\rho_{V}\left(v\right)=g\otimes v\right\}$. Define the map $\blacktriangleright : G\times V\rightarrow V$, by setting $g\blacktriangleright v:=g\rhd v$. It is easy to prove that the assignments

$$\left(V,\rho_{V},\rhd\right)\mapsto\left(V=\oplus_{g\in G}V_{g},\blacktriangleright\right)\qquad f\mapsto f$$

define a functor $L:{}_{H}^{H}\mathcal{YD}\rightarrow(G,\theta)$-Mod. Conversely, let $\left(V=\oplus_{g\in G}V_{g},\blacktriangleright\right)$ be an object in (G,θ)-Mod. Then $\blacktriangleright$ can be extended by linearity to a map $\rhd:\Bbbk G\otimes V\rightarrow V$. Define $\rho_{V}:V\rightarrow\Bbbk G\otimes V$, by setting $\rho_{V}\left(v\right)=g\otimes v$ for all $v\in V_{g}$. Therefore, the assignments

$$\left(V=\oplus_{g\in G}V_{g},\blacktriangleright\right)\mapsto\left(V,\rho_{V},\rhd\right)\qquad f\mapsto f$$

define a functor $R:\left(G,\theta\right)$-Mod $\rightarrow{}_{H}^{H}\mathcal{YD}$. It is clear that $LR=\mathrm{Id}$ and $RL=\mathrm{Id}$. $\square$

Remark 5.3.15. As a consequence of the previous result, the pre-braided monoidal structure on ${}^{\Bbbk^{\theta}G}_{\Bbbk^{\theta}G}\mathcal{YD}$ induces a pre-braided monoidal structure on (G,θ)-Mod as follows. The unit is $\Bbbk$ regarded as a G-graded vector space whose homogeneous components are all zero excepted the one corresponding to 1_G. Moreover $h \blacktriangleright k = \varepsilon_H(h)\,k$ for all $h \in H, k \in \Bbbk$. The tensor product is defined by

$$(V, \blacktriangleright) \otimes (W, \blacktriangleright) = (V \otimes W, \blacktriangleright)$$

where

$$(V \otimes W)_g = \oplus_{h \in H}(V_h \otimes W_{h^{-1}g})$$

and, for all $v \in V_g, w \in W_l$, we have

$$h \blacktriangleright (v \otimes w) = \frac{\theta\left(hgh^{-1}, hlh^{-1}, h\right)\theta\left(h, g, l\right)}{\theta\left(hgh^{-1}, h, l\right)}\left(h \blacktriangleright v\right) \otimes \left(h \blacktriangleright w\right).$$

The constraints are the same of ${}^{H}\mathfrak{M}$ viewed as morphisms in ${}^{H}_{H}\mathcal{YD}$.

The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given, for all $v \in V_g, w \in W_l$, by

$$c_{V,W}\left(v \otimes w\right) = \left(g \blacktriangleright w\right) \otimes v.$$

Bibliography

[AA] L. Angeleri Hügel, M. Archetti Tilting modules and universal localization, arXiv:0908.1985v1, 2009

[AC] L. Angeleri Hügel and F. U. Coelho. Infinitely generated tilting modules of finite projective dimension. *Forum Math.* 13, no. 2, 239250, 2001.

[AKL] L. Angeleri Hügel, S. König and Q. Liu. Recollements and tilting objects. *J. Pure Appl. Algebra*, 215(2): 420–438, 2011.

[AMS1] A. Ardizzoni, C. Menini and D. Stefan,Hochschild Cohomology And 'Smoothness' In Monoidal Categories, J. *Pure Appl. Algebra*, Vol. 208 (2007), 297-330.

[AMS2] A. Ardizzoni, C. Menini and D. Stefan, A Monoidal Approach to Splitting Morphisms of Bialgebras, *Trans. Amer. Math. Soc.*, 359 (2007), 991-1044.

[AP] A. Ardizzoni, A. Pavarin, Preantipodes for Dual Quasi-Bialgebras, *Israel J. Math.*, to appear. (arXiv:submit/0160766)

[APR] M. Auslander, M.I. Platzeck, I. Reiten. Coxeter functors without diagrams. *Trans. Amer. Math. Soc*, 250, 146, 1979.

[AS] N. Andruskiewitsch, H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra* 209 (1998), no. 2, 658–691.

[ASS] I. Assem, D. Simson and A. Skowroński. Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory. *London Mathematical Society Student Texts* 65. Cambridge University Press, Cambridge, 2006.

[AT] L. Angeleri Hüge, J. Trlifaj Tilting theory and the finitistic dimension conjecture. *Trans. Amer. Math. Soc.* 354, 4345-4358, 2002

[B] S. Bazzoni. Equivalences induced by infinitely generated tilting modules. *Proc. Amer. Math. Soc.*, 138(2): 533-544, 2009.

[B2] S. Bazzoni. A characterization of n-cotilting and n-tilting modules. *J. Algebra*, 273, no. 1, 359-372, 2004

[Ba] A. Balan, Yetter-Drinfeld modules and Galois extensions over coquasi-Hopf algebras. (English, Romanian summary) *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* 71 (2009), no. 3, 43–60.

[Bas] H. Bass On the ubiquity of Gorenstein rings. *Math. Z.*, 82 1963 828.

[BB] S. Brenner and M. C. R. Butler. Generalizations of the Bernšteĭn-Gel'fand-Ponomarev reflection functors. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Math.*, pages 103–169. Springer, Berlin, 1980.

[BBD] A.A. Beilinson, J. Bernšteĭn and P. Deligne. Faisceaux pervers. *Astérisque*, 100, 1982.

[BC] D. Bulacu, S.Caenepeel,Integrals for (dual) quasi-Hopf algebras. Applications.*J. Algebra*, 266 (2003), no. 2, 552–583.

[BCP] D. Bulacu, S. Caenepeel, F. Panaite, Yetter-Drinfeld categories for quasi-Hopf algebras.*Comm. Algebra* 34. 2006, no. 1, 1–35.

[Be] A. Beligiannis. Auslander-Reiten triangles, Ziegler spectra and Gorenstein rings. *K-Theory* 32(2004)1-82.

[BGP] I.N. Bernšteĭn, I. M. Gel'fand and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[BH] S. Bazzoni, D. Herbera. One dimensional tilting modules are of finite type. *Algebr. Represent. Theory* 11, no. 1, 4361, 2008

[BMT] S. Bazzoni, F. Mantese and A. Tonolo. Derived equivalence induced by infinitely generated n-tilting modules. *Proc. Amer. Math. Soc.*, 139(12): 4225-4234, 2011.

[BN] D. Bulacu, E.Nauwelaerts, Radford's biproduct for quasi-Hopf algebras and bosonization, *J. Pure Appl. Algebra* 174, no. 1, 1–42, 2002

[BN1] D. Bulacu and E. Nauwelaerts, Dual quasi-Hopf algebra coactions, smash coproducts and relative Hopf modules, *Rev. Roum. Math. Pures Appl.* 47, No. 4, 415-443, 2002.

[BNE] M. Böstedt, A. Neeman. Homotopy limits in triangulated categories. *Compositio Mathematica*, tome 86, n. 2, pag. 209-234, 1993.

[Bo] K. Bongartz Tilted algebras *Representations of algebras* (Puebla, 1980), pp. 2638, *Lecture Notes in Math.*, 903, Springer, Berlin-New York, 1981.

[BR] A. Beligiannis, I. Reiten Homological and homotopical aspects of torsion theories. *Mem. Amer. Math. Soc.* 188, no. 883, viii+207 pp, 2007

[BRB] S. Brenner, M. C. R. Butler The equivalence of certain functors occurring in the representation theory of algebras and species. *J. London Math. Soc.*, 14(1976) 18387.

[CB] W. Crawley-Boevey Infinite-dimensional modules in the representation theory of finite-dimensional algebras. *Canadian Math. Soc. Conf. Proc*, 23 , 29-54, 1998

[CF] R.R. Colby, K.R. Fuller. Tilting, cotilting and serially tilted rings. *Comm. Algebra*, 18(5), 15851615, 1990.

[CPS] E. Cline, B. Parshall and L. Scott. Derived categories and Morita theory. *J. Algebra*, 104(2):397–409, 1986.

[CT] R. Colpi and J. Trlifaj. Tilting modules and tilting torsion theories. *J. Algebra*, 178(2):614–634, 1995.

[CX] H.X. Chen and C.C. Xi. Good tilting modules and recollements of derived module categoires. *Proc. Lond. Math. Soc.*, 104:959–996, 2012.

[CX2] H.X. Chen and C.C. Xi. Ringel modules and homological subcategories, arXiv:1206.0522

[D] W.G. Dwyer. Noncommutative localization in homotopy theory.
Noncommutative localization in algebra and topology, 24–39 London Math. Soc. Lecture Note, 330, 2006.

[DG] W.G. Dwyer and J.P.C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124: 199-220, 2002.

[Dr] V. G. Drinfeld, Quasi-Hopf algebras. (Russian) *Algebra i Analiz 1* (1989), no. 6, 114–148; translation in Leningrad Math. J., 1, no. 6, 1419–1457, 1990

[F] A. Facchini. A tilting module over commutative integral domains. *Comm. Algebra*,15(11):22352250, 1987

[F2] A. Facchini. Divisible modules over integral domains. *Ark. Mat.*, 26(1):6785, 1988.

[Fu] L. Fuchs. On divisible modules over domains. In Abelian groups and modules (Udine, 1984), volume 287 of CISM *Courses and Lectures*, pages 341356. Springer, Vienna, 1984

[GL] W. Geigle and H. Lenzing. Perpendicular categories with applications to representations and sheaves. *J. Algebra*, 144(2): 273–343, 1991.

[GP] P. Gabriel and A.J. de la Peña. Quotients of representation-finite algebras. *Comm. Algebra*, 15: 2235-2250, 1987.

[GT] R. Göbel J and J. Trlifaj, Approximations and endomorphism algebras of modules. *De Grutyter Expositions in Mathematics*, de Gruyter, Berlin, 14, 2006.

[GZ] P. Gabriel, M. Zisman Calculus of fractions and homotopy theory. *M. Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 35 Springer-Verlag New York, Inc., New York x+168 pp., 1967.

[H] D. Happel. On the derived category of a finite-dimensional algebra. *Comment. Math. Helv.*, 62(3): 339–389, 1987.

[HN] F. Hausser and F. Nill, Integral theory for quasi Hopf algebras, preprint (arXiv:math/9904164v2).

[HR] D. Happel and C. M. Ringel. Tilted algebras. *Trans. Amer. Math. Soc.*, 274(2): 399–443, 1982.

[Hu] H.-L. Huang, Quiver approaches to quasi-Hopf algebras, *J. Math. Phys.*, 50, no. 4, 2009.

[J] P. Jørgensen. Recollements for differential graded algebras. *J. Algebra*, 299: 589–601, 1991.

[K] S. König. Tilting complexes, perpendicular categories and recollements of derived categories of rings. *J. Pure Appl. Algebra*, 73: 211–232, 1991.

[Ka] C. Kassel, *Quantum groups*,Graduate Text in Mathematics **155**, Springer, 1995.

[Ke1] B. Keller. On the construction of triangle equivalences. Lectures Workshop Pappenheim 1994, 1994.

[Ke2] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup.*, 27(4): 63–102, 1994.

[Ke3] B. Keller. On differential graded categories. *International Congress of Mathematicians*, vol. II, Eur. Math. Soc., Zurich, pp. 151-190, 2006.

[Ke4] B. Keller. Derived categories and tilting. *Handbook of tilting theory, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge*, 332: 49–104, 2007

[Ke5] B. Keller, D. Vossieck Aisles in derived categories. *Deuxième Contact Franco-Belge en Algèbre* ,(Faulx-les-Tombes, 1987). *Bull. Soc. Math. Belg. Sr. A 40*, no. 2, 239253, 1988.

[Ke6] B. Keller Introduction to abelian and derived categories. *Representations of reductive groups*, 4161, Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1998.

[Kr] H. Krause. Cohomological quotients and smashing localizations. Amer. J. Math. 127(6) (2005) 1191-1246.

[Kr2] H. Krause Localization theory for triagulated categories *London Math. Soc. Lecture Note Ser.*, 375, Cambridge Univ. Press, Cambridge, 2010.

[Ls] R. G. Larson, M. E. Sweedler An associative orthogonal bilinear form for Hopf algebras. *Amer. J. Math.* 91, 75–94, 1969.

[Maj1] S. Majid, Foundations of quantum group theory, *Cambridge University Press*, 1995.

[Maj2] S. Majid, Quantum double for quasi-Hopf algebras. *Lett. Math. Phys.*, 45, no. 1, 1–9, 1998.

[Maj3] S. Majid, Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results. *Deformation theory and quantum groups with applications to mathematical physics* (Amherst, MA, 1990), 219–232, Contemp. Math. 134, Amer. Math. Soc., Providence, RI, 1992.

[Mi] J. Miyachi. Localization of triangulated categories and derived categories. *J. Algebra*, 141(2): 463–483, 1991

[Miy] Y. Miyashita. Tilting modules of finite projective dimension. *Math. Z.*, 193(1):113–146, 1986.

[Mo] S. Montgomery, Hopf algebras and their actions on rings. *CBMS Regional Conference Series in Mathematics* 82. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[N] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Am. Math. Soc.*, 9: 205–236, 1996.

[N2] A. Neeman. Triangulated Categories. *Ann. of Math. Stud.*, vol. 148, Princeton University Press, 2001.

[N3] A. Neeman. The connection between the K-theory localization theorem of Thomason and Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. *Ann. Sci. cole Norm. Sup. (4)*25, no. 5, 547566, 1992

[Ni] P. Nicolás On torsion torsion-free triple, Ph. D. thesis, arXiv:0801.0507

[NR] A. Neeman, A. Ranicki. Noncommutative localization and chain complexes I. Algebraic Kand L-theory. Preprint (2001)

[NS] P. Nicolás and M. Saorín. Parametrizing recollement data from triangulated categories. *J. Algebra*, 322:1220–1250, 2009.

[P] D. Pauksztello. Homological epimorphisms of differential graded algebras. *Comm. Algebra*, in press.

[R] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3):436–456, 1989.

[Ra] D. E. Radford,*The structure of Hopf algebras with a projection.* J. Algebra **92** (1985), no. 2, 322–347.

[Sc] A.H Schofield Representations of rings over skew fields. *LMS Lecture Note Series* 92, Cambridge, 1985

[Sch1] P. Schauenburg, A generalization of Hopf crossed products. *Comm. Algebra*, 27, no. 10, 4779–4801, 1999.

[Sch2] P. Schauenburg, Hopf modules and Yetter-Drinfeld modules. *J. Algebra*, 169, no. 3, 874–890, 1993.

[Sch3] P. Schauenburg, Hopf modules and the double of a quasi-Hopf algebra. *Trans. Amer. Math. Soc.* 354, no. 8, 3349–3378, 2002.

[Sch4] P. Schauenburg, Two characterizations of finite quasi-Hopf algebras. *J. Algebra*, 273, no. 2, 538–550, 2004.

[Sch5] P. Schauenburg, Hopf algebra extensions and monoidal categories. *New directions in Hopf algebras*, 321–381, *Math. Sci. Res. Inst. Publ.*, 43, Cambridge Univ. Press, Cambridge, 2002.

[Sch6] P. Schauenburg, Turning monoidal categories into strict ones, *New York J. Math.*, 7, 257265, 2001.

[SR] N. Saavedra Rivano, Categories Tannakiennes, *Lecture Notes in Mathematics*, Vol. 265. Springer-Verlag, Berlin-New York, 1972

[Sw] M. E. Sweedler, Hopf algebras. *Mathematics Lecture Note Series W. A. Benjamin, Inc.*, New York, 1969.

[T] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris* , 340(1):15–19, 2005

[TLS] A. Tarro, J. Lpez, S. Salorio. Localization in categories of complexes and unbounded resolutions *Canad. J. Math.*,52 no. 2, 225247, 2000

[TP] J. Trlifaj, D. Pospil Tilting and cotilting classes over Gorenstein rings. *Rings, modules and representations*, 319334, Contemp. Math., 480, Amer. Math. Soc., Providence, RI, 2009

[Tr] J. Trlifaj Infinite dimensional tilting modules and cotorsion pairs. *Handbook of tilting theory*, pp. 279-322

[V] J.L Verdier Des catgories drives des catgories abliennes. *Astrisque* No. 239, xii+253 pp., 1996

[W] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Y] D. Yang. Recollements from generalized tilting. *Proc. Amer. Math. Soc.*, 140(1):83–91, 2012.