# Numerical hyperinterpolation over spherical triangles *

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**Abstract**

We present a numerical method (implemented in Matlab) for computing an orthogonal polynomial basis on spherical triangles, via a recent near-algebraic quadrature formula, and constructing the corresponding weighted orthogonal projection (hyperinterpolation) of a function sampled at the quadrature nodes.

## 1 Introduction

In this note we present some numerical algorithms for the computation of orthogonal polynomials on spherical triangles with respect to the surface measure and for the construction of hyperinterpolation polynomials.

Despite of the relevance of spherical triangles in geomathematical modelling, polynomial approximation on such regions of the sphere seems to be little explored in the numerical literature. The topic itself of algebraic quadrature, i.e. quadrature exact on polynomials up to a given degree, has been extensively studied on the whole sphere as well as on special regions such as spherical caps and spherical rectangles, whereas on spherical triangles has been considered mainly with scattered data, and only quite

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recently faced by seeking low-cardinality formulas on special nodes; cf., e.g., [1, 14, 15, 19] with the references therein.

As it is well-known, as soon as an algebraic quadrature formula is at hand, it is possible to reconstruct a continuous function from its values at the quadrature nodes by hyperinterpolation, a technique that avoids the inherent difficulties of multivariate polynomial interpolation (essentially the need to find sets of unisolvent nodes, which is already a nontrivial problem, with in addition a slowly increasing Lebesgue constant).

Introduced in a seminal paper by I.H. Sloan in 1995 [17] and initially used mainly on the whole sphere [13], hyperinterpolation is essentially an orthogonal (Fourier-like) projection on polynomial spaces, with respect to the discrete measure associated with a positive algebraic quadrature formula, or in other words a weighted least-squares polynomial approximation at the quadrature nodes. In the last twenty years the subject has been developed and extended to several 2-dimensional and 3-dimensional domains, such as cubes and balls but also less standard ones, from both the theoretical and the modelling/computational point of view; cf., e.g., [6, 11, 12, 5, 18, 21, 22] with the references therein.

In the Section 2 we shall recall for the reader's convenience some general results about hyperinterpolation, while in Section 3 we shall focus on quadrature and hyperinterpolation on spherical triangles, discussing their implementation and presenting several numerical tests of function reconstruction.

## 2 A little survey of hyperinterpolation

For the only purpose of completeness for nonexpert readers, we recall the main features of hyperinterpolation. Given an orthonormal basis of $\mathbb{P}_n^d(\Omega)$ (the subspace of $d$-variate polynomials of total-degree not exceeding $n$, restricted to a compact set or manifold $\Omega \subset \mathbb{R}^d$) with respect to a given measure $d\mu$ on $\Omega$, say $\{p_j\}$, $1 \leq j \leq N_n = dim(\mathbb{P}_n^d(\Omega))$, and a quadrature formula exact for $\mathbb{P}_{2n}^d(\Omega)$ (polynomials up to doubled degree) with nodes $X = \{x_i\} \subset \Omega$ and positive weights $\mathbf{w} = \{w_i\}$, $1 \leq i \leq M$ with $M \geq N_n$,

$$\int_\Omega q(x)\, d\mu = \sum_{i=1}^{M} w_i\, q(x_i) \ , \quad \forall q \in \mathbb{P}_{2n}^d(\Omega) \ , \tag{1}$$

one can construct the discretized orthogonal projection (hyperinterpolation) of $f \in C(\Omega)$

$$(\mathcal{L}_n f)(x) = \sum_{j=1}^{N_n} \langle f, p_j \rangle_{\ell^2_{\mathbf{w}}(X)} \, p_j(x) = \sum_{j=1}^{N_n} p_j(x) \sum_{i=1}^{M} w_i \, f(x_i) \, p_j(x_i) \qquad (2)$$

$$= \sum_{i=1}^{M} w_i \, f(x_i) \, K_n(x, x_i) = \sum_{i=1}^{M} w_i \, f(x_i) \sum_{j=1}^{N_n} p_j(x_i) \, p_j(x)$$

$$\approx (\mathcal{F}_n f)(x) = \sum_{j=1}^{N_n} \langle f, p_j \rangle_{L^2_{d\mu}(\Omega)} \, p_j(x) = \int_{\Omega} K_n(x, y) \, f(y) \, d\mu \ ,$$

$\langle f, g \rangle_{\ell^2_{\mathbf{w}}(X)}$ denoting the discrete scalar product generated by the quadrature formula and $\langle f, g \rangle_{L^2_{d\mu}(\Omega)}$ the scalar product generated by $d\mu$. The polynomial $K_n(x, y) = \sum_{j=1}^{N_n} p_j(x) \, p_j(y)$ is called the reproducing kernel of the measure and as known does not depend on the orthonormal basis, cf. [10].

We recall that necessarily $M \geq N_n$ since the quadrature nodes are $\mathbb{P}_n^d(\Omega)$-determining (polynomials $p \in \mathbb{P}_n^d(\Omega)$ vanishing there vanish everywhere on $\Omega$ by (1) with $q = p^2$), and that if $M = N_n$ (in this special case the formula is said to be minimal for $\mathbb{P}_n^d(\Omega)$) then the hyperinterpolation polynomial $\mathcal{L}_n f$ is interpolant; however, minimal formulas are known only for very few special domains and measures, cf. e.g. [23] with the references therein. We also recall that if $\Omega$ is $\mathbb{P}_n^d$-determining then $N_n = dim(\mathbb{P}_n^d) = \binom{n+d}{n} = \binom{n+d}{d}$, but the dimension can be lower when $\Omega$ lies on an algebraic variety, for example

$$N_n = dim(\mathbb{P}_n^3(\mathcal{T})) = dim(\mathbb{P}_n^3(S^2)) = (n+1)^2 \qquad (3)$$

for a spherical triangle $\mathcal{T}$ of the 2-sphere $S^2$ in $\mathbb{R}^3$ (one speaks of "spherical polynomials" of degree not exceeding $n$ in this case and is typically interested in $d\mu = d\sigma$, the standard surface measure on the sphere).

Concerning the reconstruction error estimates, one of the main results in the original paper [17] is

$$\|f - \mathcal{L}_n f\|_{L^2_{d\mu}(\Omega)} \leq 2\sqrt{\mu(\Omega)} \, E_n(f; \Omega) \ , \quad E_n(f; \Omega) = \min_{p \in \mathbb{P}_n^d(\Omega)} \|f - p\|_{L^\infty(\Omega)} \ , \qquad (4)$$

to be compared with the analogue for the continuous orthogonal projection $\|f - \mathcal{F}_n f\|_{L^2_{d\mu}(\Omega)} \leq \sqrt{\mu(\Omega)} \, E_n(f; \Omega)$ which is only half. On the other hand, using the well-known polynomial inequality involving the so-called

"Christoffel polynomial" (the diagonal of the reproducing kernel) valid for any $p \in \mathbb{P}_n^d(\Omega)$

$$\|p\|_{L^\infty(\Omega)} \le \sqrt{C_n}\, \|p\|_{L^2_{d\mu}(\Omega)} \ , \ C_n = \max_{x\in\Omega} K_n(x,x)$$

(observe that $C_n \ge N_n = \frac{1}{\mu(\Omega)} \int_\Omega K_n(x,x)\, d\mu$), one can easily estimate the uniform operator norm of the discrete orthogonal projection operator (its "Lebesgue constant" in the interpolation terminology, cf. [5])

$$\|\mathcal{L}_n f\|_{L^\infty(\Omega)} \le \sqrt{C_n}\, \|\mathcal{L}_n f\|_{L^2(\Omega)} = \sqrt{C_n}\, \|\mathcal{L}_n f\|_{\ell^2_{\mathbf{w}}(X)} \le \sqrt{C_n}\, \|f\|_{\ell^2_{\mathbf{w}}(X)}$$

$$\le \sqrt{C_n \sum_{i=1}^{N_n} w_i}\, \|f\|_{\ell^\infty(X)} = \sqrt{C_n\, \mu(\Omega)}\, \|f\|_{\ell^\infty(X)} \le \sqrt{C_n\, \mu(\Omega)}\, \|f\|_{L^\infty(\Omega)}$$

which gives immediately

$$\|\mathcal{L}_n\| = \sup_{f\ne 0} \frac{\|\mathcal{L}_n f\|_{L^\infty(\Omega)}}{\|f\|_{L^\infty(\Omega)}} \le \sqrt{C_n\, \mu(\Omega)} \ , \tag{5}$$

that is exactly what is obtained for the continuous orthogonal projection operator, $\|\mathcal{F}_n\| \le \sqrt{C_n\, \mu(\Omega)}$. From (5) one can obtain as for polynomial interpolation operators a second reconstruction error estimate, now in the sup-norm

$$\|f - \mathcal{L}_n f\|_{L^\infty(\Omega)} \le (1 + \|\mathcal{L}_n\|)\, E_n(f;\Omega) \le \left(1 + \sqrt{C_n\, \mu(\Omega)}\right)\, E_n(f;\Omega) \ , \tag{6}$$

that is again exactly what is obtained for the continuous orthogonal projection operator, $\|f - \mathcal{F}_n f\|_{L^\infty(\Omega)} \le \left(1 + \sqrt{C_n\, \mu(\Omega)}\right)\, E_n(f;\Omega)$.

The error estimates reported above show that hyperinterpolation can be considered as a reasonable alternative to continuous Fourier-like projection, with the advantage of being much easier to compute, since it requires only the availability of an algebraic quadrature formula as we shall see in the next section.

# 3 Hyperinterpolation on spherical triangles

The key tool for hyperinterpolation of degree $n$ is the availability of a positive quadrature formula $\{(X, \mathbf{w})\}$ like (1), exact up to degree $2n$. The explicit knowledge of an orthogonal polynomial basis by analytical formulas, often not available, can indeed be bypassed numerically, computing the orthogonal polynomials by linear algebra techniques or in any case solving the weighted least-squares problem

$$\|f - \mathcal{L}_n f\|_{\ell^2_{\mathbf{w}}(X)} = \min_{p \in \mathbb{P}^d_n(\Omega)} \|f - p\|_{\ell^2_{\mathbf{w}}(X)}$$

or equivalently

$$\|\sqrt{W}\,(\mathbf{f} - V\mathbf{c}^*)\|_2 = \min_{\mathbf{c} \in \mathbb{R}^{N_n}} \|\sqrt{W}\,(\mathbf{f} - V\mathbf{c})\|_2 \;, \tag{7}$$

where $W = \mathrm{diag}(w_i)$, $\mathbf{f} = \{f(x_i)\}$ is a column vector and

$$V_n = V_n(X) \in \mathbb{R}^{M \times N_n} = (v_{ij}) = (\phi_j(x_i)) \tag{8}$$

is a Vandermonde-like matrix in any given basis $(\phi_1, \dots, \phi_{N_n})$ of $\mathbb{P}^d_n(\Omega)$.

In order to orthonormalize the basis with respect to the discrete measure generated by the quadrature formula, which by polynomial exactness up to degree $2n$ is orthonormal also with respect to $d\mu$, one can compute the QR factorization with $Q \in \mathbb{R}^{M \times N_n}$, $R \in \mathbb{R}^{N_n \times N_n}$, and construct the orthonormal basis $(p_1, \dots, p_{N_n})$ as

$$\sqrt{W}\,V_n = QR \;, \quad (p_1, \dots, p_{N_n}) = (\phi_1, \dots, \phi_{N_n})\,R^{-1} \;, \tag{9}$$

$U_n = U_n(X) = V_n R^{-1} = (p_j(x_i))$ becoming the Vandermonde-like matrix in the orthonormal basis. Notice that $R$ is invertible since $V_n$ is full rank, the quadrature nodes $X$ being $\mathbb{P}^d_n(\Omega)$-determining.

At this point, the hyperinterpolation coefficients $\mathbf{c}^*$ such that

$$\mathcal{L}_n f(x) = \sum_{j=1}^{N_n} c^*_j\, p_j(x)$$

are $c^*_j = \langle f, p_j \rangle_{\ell^2_{\mathbf{w}}(X)}$, $1 \le j \le N_n$, and in view of (9) can be computed in matrix form

$$\mathbf{c}^* = U_n^t W \mathbf{f} = (\sqrt{W}\,U_n)^t \sqrt{W}\,\mathbf{f} = Q^t \sqrt{W}\,\mathbf{f} \;. \tag{10}$$

### 3.1 Near-algebraic quadrature on spherical triangles

Let us now focus on spherical triangles $\mathcal{T} = \widehat{abc}$ of the 2-sphere, that is on the case $\Omega = \mathcal{T} \subset S^2$ and $d\mu = d\sigma$ (the surface measure on the sphere). Recall that we seek a quadrature formula with positive weights and exactness degree $2n$ on $\mathcal{T}$.

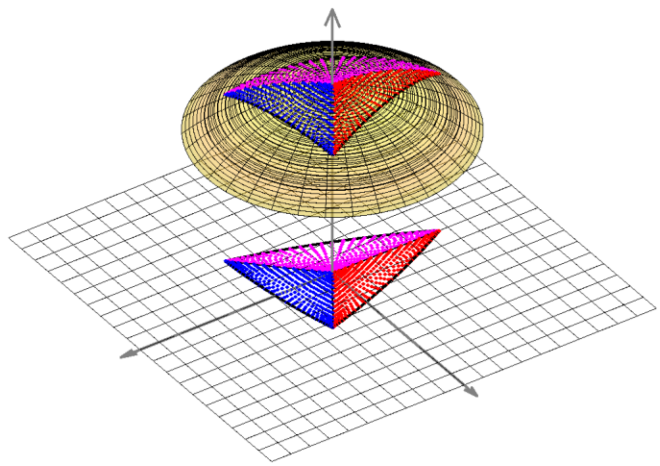

Figure 1: Quadrature nodes on a spherical triangle rotated with centroid at the north pole, lifted from the projected elliptical triangle, before compression.

In [19] we have recently constructed a near-algebraic quadrature formula on any spherical triangle that is not "too large", in the sense that it is completely contained in a hemisphere (not touching the base circle). By no loss of generality we can consider up to a rotation a spherical triangle with centroid $(a + b + c)/\|a + b + c\|_2$ at the north pole.

The method consists in projecting the spherical triangle on the equatorial plane, obtaining an "elliptical triangle" that can splitted into three elliptical sectors, say $S_1, S_2, S_3$; see Fig. 1. The surface integral of a trivariate monomial $x^\alpha\, y^\beta\, z^\gamma$, $0 \leq \alpha+\beta+\gamma \leq 2n$, becomes in cartesian coordinates that for convenience we call $(x, y, z)$

$$\int_{\mathcal{T}} x^\alpha\, y^\beta\, z^\gamma\, d\sigma = \sum_{i=1}^{3} \int_{S_i} x^\alpha\, y^\beta\, g(x,y)^{\gamma-1}\, dx\, dy\ ,\ g(x,y) = \sqrt{1 - x^2 - y^2}\ .$$

We then seek the degree $m$ of a suitable bivariate polynomial that approximates $1/g$ at machine precision on the elliptical triangle, which can be done via the univariate function $1/\sqrt{1-t}$, $0 \leq t = x^2 + y^2 \leq \rho = r^2$, where $r$ is the minimal radius of an enclosing disk. In such a way the problem is reduced to the computation of a quadrature formula of exactness degree $2n + m$ on each elliptical sector, a problem solved by subperiodic gaussian quadrature in [4]. Finally, we obtain a quadrature formula $(X_{big}, \mathbf{w}_{big})$ on the spherical triangle, by lifting the collection of nodes (see Fig. 1, taken from [19]) and scaling the weights by $1/g$. Such a formula is only nearly exact up to degree $2n$, in the sense that the error size on spherical polynomials remains close to machine-precision; see the error tables for spherical harmonics in [19].

In practice however, a further step is usually necessary, since the cardinality of the formula can be very high. Indeed, though the degree $m$ is not large unless $\rho$ approaches 1 (for example, it is still between 50 and 60 for a whole spherical octant, where $\rho \approx 0.67$), the number of nodes increases roughly like $1.5(2n+m)^2$ and becomes rapidly in the size of the thousands (see again the tables in [19]).

In order to reduce the cardinality of the quadrature formula, one can resort to what we have called "Caratheodory-Tchakaloff (CATCH) compression of discrete measures"; cf. [16] with the references therein on related approaches in the recent literature. Indeed, Tchakaloff theorem ensures the existence of an algebraic quadrature formula with positive weights for integration in any measure, with cardinality not exceeding the dimension of the exactness polynomial space (cf. e.g. [3] for a general proof).

In the case of a discrete measure (here our high-cardinality quadrature formula $(X_{big}, \mathbf{w}_{big})$ of degree $2n$ on the spherical triangle), such a result is a direct consequence of Caratheodory theorem on conic linear combinations in finite-dimensional spaces [2], applied to the columns of the transposed Vandermonde-like matrix in the underdetermined moment-matching system

$$V_{2n}^t \mathbf{u} = V_{2n}^t \mathbf{w}_{big} \ , \quad V_{2n} = V_{2n}(X_{big}) \in \mathbb{R}^{card(X_{big}) \times N_{2n}} \ . \tag{11}$$

By this theorem there exists a nonnegative solution $\mathbf{u}$ whose nonzero components are at most the number of rows of $V_{2n}^t$, that is $N_{2n} = dim(\mathbb{P}_{2n}^3(\mathcal{T})) = (2n+1)^2$. Such positive components, say $\mathbf{w} = \{u_i > 0\}$, identify a subset $X \subset X_{big}$ and are the new weights of the compressed formula $(X, \mathbf{w})$, with $card(X) \leq N_{2n}$; see Fig. 2 for an illustration. We stress that the compression ratio, at least roughly $1.5(2n+m)^2/(2n+1)^2$, is remarkable especially at low degrees $n$ and for large triangles (see [19] for a plot of $m$ as a function of $\rho$ and for some tabulated compression ratios).

The computation of such a sparse nonnegative solution can be accomplished by the Lawson-Hanson active-set method for the NonNegative Least Squares (NNLS) problem

$$\min_{\mathbf{u} \geq \mathbf{0}} \|U_{2n}^t \mathbf{u} - U_{2n}^t \mathbf{w}_{big}\|_2 \ , \quad U_{2n} = V_{2n} \, R^{-1} \ , \quad \sqrt{W_{big}} \, V_{2n} = QR \ , \tag{12}$$

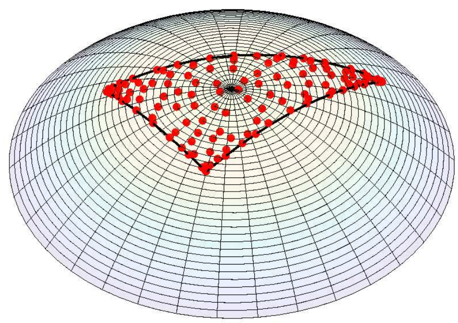

Figure 2: The 121 quadrature nodes for degree $2n = 10$ (hyperinterpolation degree $n = 5$) on the spherical triangle after compression.

with an accelerated version that has shown experimentally a speed-up by a factor from 2 to 5 with respect to standard implementations, cf. [8, 9]. In (12) the matrix $U_{2n}$ in the $\ell^2_{\mathbf{w}_{big}}(X_{big})$-orthonormal basis $(p_1, \dots, p_{N_{2n}})$ is preferred to $V_{2n}$ in order to control the conditioning.

### 3.2 Implementation and numerical results

We have implemented in Matlab the algorithms described above for quadrature and hyperinterpolation on spherical triangles, creating a devoted package available at [20]. Concerning the computation of the initial quadrature formula we resorted to our previous codes for subperiodic trigonometric gaussian quadrature and algebraic quadrature on circular/elliptical sectors discussed in [4], that have been inserted in the package.

For the implementation of the quadrature compression step in (12) and the hyperinterpolation procedure (7)-(10), we adapted to the sphere some routines of the recent general package dCATCH for discrete measure compression and polynomial fitting in $d$-variables (here $d = 3$); cf. [7, 8].

The basic routine **dCHEBVAND** computes a starting Vandermonde-like matrix in the 3-variate total-degree product Chebyshev basis of a cartesian box containing the spherical triangle, with a graded lexicographical ordering. Such a basis turns out to give a much better conditioned matrix with respect to both, the standard 3-variate monomial basis and the spherical harmonics basis.

The kernel of the package is represented by five routines:

- the routine **dORTHVANDsph** performs the computation of an orthonormal basis with respect to a given discrete measure on the spherical triangle and the corresponding Vandermonde-like matrix $U$, cf. (12) for quadrature and (9) for hyperinterpolation, automatically adaptating the polynomial space to the sphere via a preliminary QR factorization with column pivoting of the Chebyshev-Vandermonde matrix to choose the basis elements;

- the routine **dCATCHsph** performs the compression of the high-cardinality quadrature formula via NNLS, cf. (12), calling `dORTHVANDsph` and the auxiliary routine **LHDM** (an accelerated version of Lawson-Hanson algorithm by "Deviation Maximization", cf. [9]);

- the routine **dPOLYFITsph** computes the hyperinterpolation coefficients as in (10), calling `dORTHVANDsph` with the compressed formula; it is accompanied by the routine **dPOLYVAL** that computes the hyperinterpolation polynomial, given the coefficients and the transformation matrix $R$ to get the orthonormal basis in (9).

In Fig. 3 we show the reconstruction errors by hyperinterpolation of some test functions with different regularity on a large spherical triangle, namely the octant with vertices $(1, 0, 0)$,(0,1,0), $(0, 0, 1)$.

In view of (4) we have computed at degrees $n = 1, 2, \ldots, 15$ the (approximate) relative $L^2_{d\sigma}$ errors

$$\frac{\|f - \mathcal{L}_n f\|_{\ell^2_{\boldsymbol{\omega}}(\Xi)}}{\|f\|_{\ell^2_{\boldsymbol{\omega}}(\Xi)}} \approx \frac{\|f - \mathcal{L}_n f\|_{L^2_{d\sigma}(\mathcal{T})}}{\|f\|_{L^2_{d\sigma}(\mathcal{T})}}$$

(where $(\Xi, \boldsymbol{\omega})$ is the uncompressed quadrature formula of degree 40), for the following six test functions (where $P = (x, y, z)$, $P_0 = (0, 0, 1)$ is the north pole, a vertex of the octant, and $Q_0 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is the centroid)

$$f_1(x, y, z) = 1+x+y^2+x^2y+x^4+y^5+x^2y^2z^2 \ , \ \ f_2(x, y, z) = \cos(10(x+y+z)) \ ,$$

$$f_3(x, y, z) = \exp\left(-\|P - P_0\|_2^2\right) \ , \ \ f_4(x, y, z) = \exp\left(-\|P - Q_0\|_2^2\right)$$

$$f_5(x, y, z) = \|P - P_0\|_2^5 \ , \ \ f_6(x, y, z) = \|P - Q_0\|_2^5 \ .$$

In particular, $f_1$ is a polynomial of degree 6, $f_2$, $f_3$ and $f_4$ are smooth, while $f_5$ and $f_6$ are $C^4$ with a singularity of the fifth derivatives at $P_0$ and $Q_0$. We observe that, as expected, the error on $f_1$ goes down around machine precision for $n \geq 6$, and the convergence is slower for the less regular functions $f_5$ and $f_6$; see Fig. 3-top. Similar results are obtained computing the relative errors in the sup-norm, reported in Fig. 3-bottom.

It is also meaningful to compute the uniform norm of the hyperinterpolation operator and to compare it with the upper estimate in (5), since the former as known is related not only to convergence but also to stability with respect to noise in the sampled function, by the estimate

$$\|\mathcal{L}_n f - \mathcal{L}_n \tilde{f}\|_{L^\infty(\Omega)} \leq \|\mathcal{L}_n\| \, \|f - \tilde{f}\|_{\ell^\infty(X)} \leq \|\mathcal{L}_n\| \, \|f - \tilde{f}\|_{L^\infty(\Omega)} \; .$$

Indeed, by (2) it can be proved that

$$\|\mathcal{L}_n\| = \max_{x \in \Omega} \sum_{i=1}^{M} |g_i(x)| \; , \quad g_i(x) = w_i \, K_n(x_i, x) \; , \tag{13}$$

where the $\{g_i\}$ are a set of generators (in general not a basis unless $M = N_n$) of $\mathbb{P}_n^d(\Omega)$, that play a similar role to the cardinal Lagrange polynomials in interpolation. Both $\|\mathcal{L}_n\|$ and $\sqrt{C_n \, \mu(\Omega)}$ have been computed numerically by maximing on a fine discretization of the spherical triangle and reported for comparison in Fig. 4. We see that $\sqrt{C_n \, \mu(\Omega)}$ is a large overestimate of the actual norm, with a quadratic-like versus a linear-like growth.

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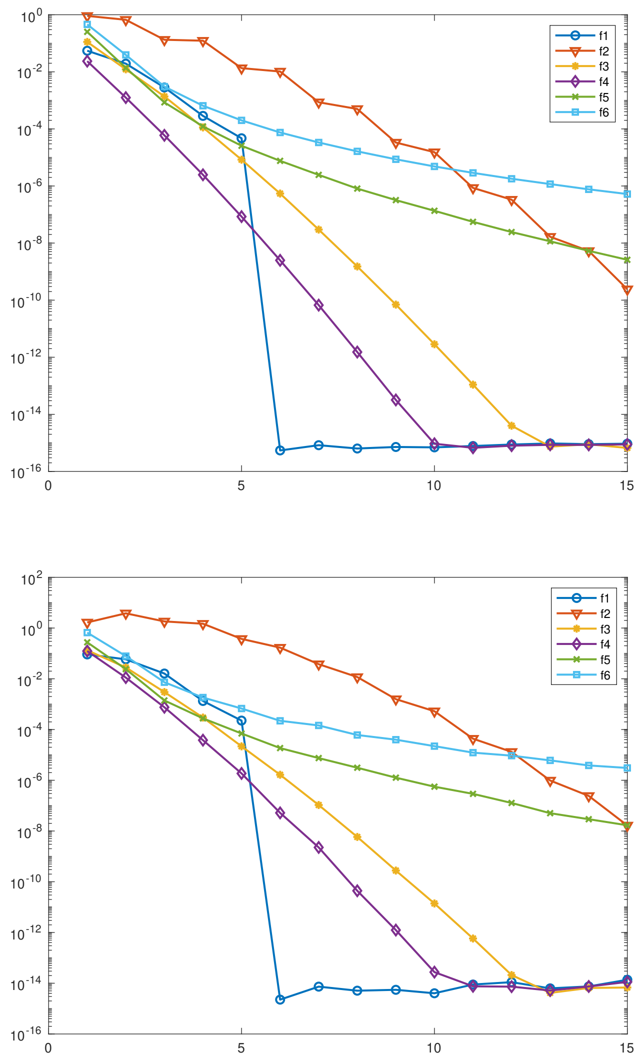


Figure 3: Relative hyperinterpolation errors on a spherical octant, in the $L^2_{d\sigma}$-norm (top) and in the sup-norm (bottom).

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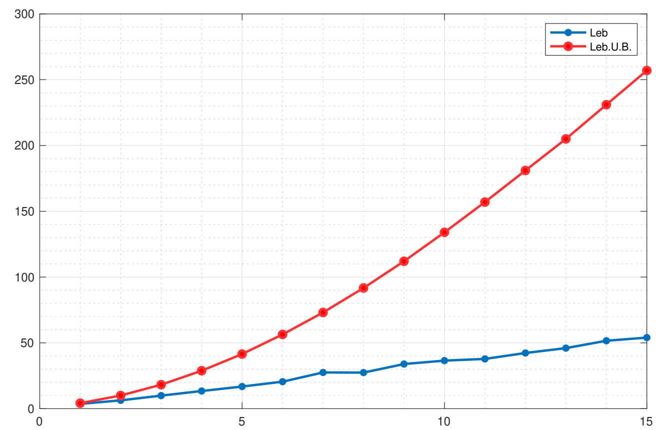


Figure 4: Uniform norm of the hyperinterpolation operator (lower curve) and the upper bound (5).


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