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AN APPARENTLY UNNATURAL ESTIMATE ABOUT FORWARD-BACKWARD PARABOLIC EQUATIONS

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Abstract. In this note we come back to face a problem regarding forward-backward parabolic equations like $(r(x,t)u)_t - u_{xx} = 0$ and $r(x,t)u_t - u_{xx} = 0$ (r is both positive and negative): the continuity of $t \mapsto \int u^2(x,t)|r(x,t)|\,dx$.

1. Introduction

We here want to consider a simple, but nonetheless important, problem when dealing with forward-backward parabolic equations like

$$r(x)u_t - u_{xx} = 0, \qquad t \in [0,T],\ x \in [-a,a] \quad (a > 0) \tag{1.1}$$

where r is both positive and negative. In fact we will treat a more general result (with r depending also on t), as illustrated in the next section, but to fix ideas we now focus our attention to the simple equation (1.1).

The solution of the equation (1.1) lives in the space (or in one of its subspaces)

$$W = \{u \in L^2(0,T;H^1(-a,a)) \,|\, ru' \in L^2(0,T;H^{-1}(-a,a)).$$

In [8] and [6] it is proved that

$$\begin{aligned} &[0,T] \ni t \mapsto \int_{-a}^{a} u^2(x,t)r(x)\,dx \quad \text{is continuous and there is } c \text{ s.t.} \\ &\Big| \int_{-a}^{a} u^2(x,t)r(x)\,dx \Big|^{1/2} \leqslant c\,\|u\|_W \\ &\qquad\qquad = c\Big[\|u\|_{L^2(0,T;H^1(-a,a))} + \|ru'\|_{L^2(0,T;H^{-1}(-a,a))}\Big] \end{aligned} \tag{1.2}$$

which for $r \equiv 1$ is simply the classical continuous embedding of W in $C^0([0,T];L^2(\Omega))$ (see, e.g, [12], chap. 23).

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The problem we want to deal with is the continuity, for u belonging to W, of the function

$$[0,T] \ni t \mapsto \int_{-a}^{a} u^2(x,t)|r(x)|\,dx \tag{1.3}$$

and to understand if it is possible to get a control of this function by the norm $\|u\|_W$, i.e., if it is true that there is $c > 0$ such that

$$\begin{aligned} \Big|\int_{-a}^{a} u^2(x,t)|r(x)|\,dx\Big|^{1/2} &\leqslant c\,\|u\|_W \\ &= c\Big[\|u\|_{L^2(0,T;H^1(-a,a))} + \|ru'\|_{L^2(0,T;H^{-1}(-a,a))}\Big] \end{aligned} \tag{1.4}$$

for every $u \in W$ and every $t \in [0,T]$.

This fact might seem quite meaningless, but on the contrary it turns out to be important when treating some arguments about forward-backward parabolic equations, as for instance the regularity (local boundedness and continuity of the solution, see [10], [9] and the forthcoming paper [7]) because of the need to control the quantities

$$\int_{-a}^{a} u^2(x,t)r_+(x)\,dx \qquad \text{and} \qquad \int_{-a}^{a} u^2(x,t)r_-(x)\,dx$$

(being r_+ and r_- the positive and negative part of r) knowing only that $u \in W$ and (1.2) holds.

The problem we study in the present paper have already been treated in [3] and [1], where an equation like (1.1) is considered with $r(x) = x$, but the result stated in those papers is the following:

$$\sup_{t\in[0,T]} \int u(x,t)^2|x|\,dx \leqslant c\,\|u\|_{L^2(0,T;H^1_0(-a,a))}\|xu'\|_{L^2(0,T;H^{-1}(-a,a))}\,. \tag{1.5}$$

There is clearly an oversight in this estimate, since it is sufficient to consider u independent of t to realise that this clearly cannot be true. Anyway what is true, and this is a consequence of our main result in the present paper, is that

$$\sup_{t\in[0,T]} \int u(x,t)^2|x|\,dx \leqslant c\Big[\|u\|^2_{L^2(0,T;H^1_0(-a,a))} + \|xu'\|^2_{L^2(0,T;H^{-1}(-a,a))}\Big]. \tag{1.6}$$

We stress that for a function $u \in L^2(0,1;H_0^1(-a,a))$ such that $|x|u' \in L^2(0,1;H^{-1}(-a,a))$ it is immediate to have the estimate

$$\sup_{t\in[0,T]} \int u(x,t)^2|x|\,dx \leqslant c\Big[\|u\|^2_{L^2(0,T;H_0^1(-a,a))} + \||x|u'\|^2_{L^2(0,T;H^{-1}(-a,a))}\Big]$$

but not necessarily the previous one.

We recall that two particular cases had already been considered in [8]. Roughly speaking, we can briefly explain these two situation as follows: in the first case one needs r to be locally differentiable around the region where it changes sign (see Proposition 2.8 in [8]), and this request is satisfied by the example considered by Beals and Aarão $r(x) = x$; the second is when the regions where r is negative and r is positive are well separated by a region where $r = 0$. Starting from this second situation a characterization for (1.4) to hold (with r depending also on time) is given in [11]. For this reason in the present paper, we focus our attention on a simple, but very significant, situation:

r takes only positive or negative values almost everywhere,

i.e., $r = 0$ in a set whose measure is zero,

possibly discontinuous on the interface where it changes its sign .

In this note we want to attack directly the problem and to generalise these particular cases and estimate (1.6) not only, as in (1.4), where the spatial dimension is 1, but also when the spatial dimension is higher than 1 and the function r possibly depends also on time and show that the continuity (1.3) and inequalities like (1.6) or (1.4) are possible and in fact true (at least for some r's).

In particular, we want to prove the analogous of (1.3) and (1.4) for functions u belonging to the space which contains the solutions of (here Δ_p denotes the p-Laplacian with $p \geqslant 2$)

$$(r(x,t)u)_t - \Delta_p u = 0 \qquad \text{or} \qquad r(x,t)u_t - \Delta_p u = 0. \tag{1.7}$$

We recall that existence and uniqueness for a solution to these last equation are given in [8] and [6].

Coming to the structure of the paper: in Section 2, we present the assumptions needed and the fundamental steps, i.e., Lemma 2.4, Lemma 2.6, Lemma 2.7 and Lemma 2.9, which are summarised in Proposition (2.10).

In Section 2, we suppose that the interface where r changes its sign in one and of a particular case, i.e., satisfying (H3). The essential assumptions about r are (H1) and (H2). The first one is a request about the regularity of the set where r changes its sign. The second one is just an assumption of convenience: since equations (1.7) do have solutions even if r is discontinuous (see [8] and [6]) and if this were the case r could not admit a temporal derivative, we suppose that r admits a temporal derivative outside the interface where it changes its sign, even if it may jump on that interface.

Assumptions (H3), (H4), (H5) are just a consequence of (H1), while (H4′) and (H5′) will be explained in the last section, where some examples are made.

The third section uses all the results of the second section to prove the main result, again for one only interface, but more general: we consider a partition of unity in $\Omega \times (0,T)$, and in particular around the set where r changes its sign, and in each set of this partition we apply the results of Section 2 and get a global result. A brief section, the fourth, is devoted to a more general case, when r changes its sign in more than one interface. The last section, as already said, is devoted to some examples and to an interesting counterexample.

The proof of (1.5) given by Beals leans on an idea contained in [2] which is not, in our opinion, the most direct; on the contrary the proof given by Aarão, despite the oversight, is simple and seems more natural and direct. Our proof follows, in some sense, this idea.

Finally, we recall that mixed type equations of the kind of (1.1) have been considered first in [2], than in [5], in [4] and in the already quoted [3] and [1], and some other papers by the same authors, Pagani and Beals in particular. In all these papers the coefficient r depends only on x and is always of the type

$$\operatorname{sgn}(x)|x|^p \qquad p \text{ integer greater or equal to } 1.$$

Among these only Pagani considered a function r different from a polynomial or a positive power, in its paper quoted above he considered $r(x) = \operatorname{sgn}(x)|x|^p$ with $p \in \mathbf{R}$, $p > -1$.

Just to conclude, we mention that, but we refer to [4], [5] and the several paper by Beals for the details, equations of the type (1.1) have some interest in kinetic theory, finance and stochastic processes.

2. Assumptions and preliminary results

Consider Ω an open and bounded set of $\mathbf{R}^n$ with Lipschitz boundary, $T > 0$ we will define $\mathcal{Q} := \Omega \times (0,T)$ and given $p \geqslant 2$ we will denote by $\mathcal{V}$ and $\mathcal{V}'$ the following spaces

$$\mathcal{V} := L^p(0,T;W^{1,p}(\Omega)), \qquad \mathcal{V}' := L^{p'}(0,T;(W^{1,p}(\Omega))').$$

Given $r \in L^\infty(\mathcal{Q})$, we denote by $R(t)$ and $\mathcal{R}$ the operators

$$\begin{aligned} &R(t) : L^2(\Omega) \to L^2(\Omega), \qquad t \in [0,T], \\ &R(t)u := r(\cdot,t)u(\cdot), \\ &\mathcal{R} : L^2(\mathcal{Q}) \to L^2(\mathcal{Q}), \qquad \mathcal{R}u := ru. \end{aligned} \tag{2.1}$$

Notice that $R : [0,T] \to \mathcal{L}(L^2(\Omega))$, the linear and bounded operators from $L^2(\Omega)$ to $L^2(\Omega)$, and $\mathcal{R} \in \mathcal{L}(L^2(\mathcal{Q}))$.

Definition 2.1. *We say that R, defined via $r \in L^\infty(\mathcal{Q})$, is* regular *if*

$$[0,T] \ni t \mapsto \int_\Omega u(x)v(x)r(x,t)dx \qquad \textit{is absolutely continuous}$$

and there is a constant $\Lambda > 0$ such that

$$\Big|\frac{d}{dt}\int_\Omega u(x)v(x)r(x,t)dx\Big| \leqslant \Lambda\, \|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p}(\Omega)} \tag{2.2}$$

for every $u, v \in W^{1,p}(\Omega)$.

We refer to [8] and [6] for some examples of possible r. Here, we only want to stress that r may be discontinuous, for instance r could be

$$r(x,t) = \begin{cases} \;\;1 & \text{for } (x,t) \in \mathcal{Q}_+ \\ -1 & \text{for } (x,t) \notin \mathcal{Q} \setminus \mathcal{Q}_+ \end{cases}$$

for a suitable $\mathcal{Q}_+ \subset \mathcal{Q}$.

To have an idea of a possible $\mathcal{Q}_+$ denote by $\Omega_+(t) = \mathcal{Q}_+ \cap \big(\Omega \times \{t\}\big)$, $\Omega_-(t) = (\mathcal{Q} \setminus \mathcal{Q}_+) \cap \big(\Omega \times \{t\}\big)$ and consider $n = 1$. By (2.2) we need

$$t \mapsto \int_{\Omega_+(t)} u(x)v(x)dx - \int_{\Omega_-(t)} u(x)v(x)dx$$

differentiable for every $u, v \in W^{1,p}(\Omega)$. In dimension 1 we can suppose to have $\Omega_+(t) = (0, \gamma(t))$ and $\Omega_-(t) = [\gamma(t), L)$ for some γ which is a function of time. Then one get

$$\frac{d}{dt}\Big(\int_0^{\gamma(t)} u(x)v(x)dx - \int_{\gamma(t)}^L u(x)v(x)dx\Big) = 2\gamma'(t)u(\gamma(t))v(\gamma(t)),$$

i.e., (2.2) is satisfied if γ is differentiable and γ' is bounded. For example, in a situation like that in Figure 1.a r is regular, while a situation like that in Figure 1.b is not admitted if r is discountinuous on the graph of γ.

To this purpose see also assumptions (H4$'$), (H5$'$) and the comments made in the last section, in particular (5.1).

Remark 2.2. Sometimes, we will say that the function r is *regular* instead of saying that R is regular, even if r is not differentiable with respect to time.

If R is regular we will denote by $R'(t)$ and $\mathcal{R}'$ the operators

$$\begin{aligned}
&R'(t) : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \qquad t \in [0,T],\\
&\big\langle R'(t)u, v\big\rangle_{W^{-1,p'}(\Omega)\times W^{1,p}(\Omega)} = \frac{d}{dt}\int_\Omega u(x)v(x)r(x,t)dx\\
&\mathcal{R}' : \mathcal{V} \to \mathcal{V}'\\
&\big\langle \mathcal{R}u, v\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \int_0^T \big\langle R'(t)u(t), v(t)\big\rangle_{W^{-1,p'}(\Omega)\times W^{1,p}(\Omega)}\, dt.
\end{aligned}$$

Given $r \in L^\infty(\mathcal{Q})$ such that R is regular in the sense of Definition 2.1, we can consider the two spaces

$$\mathcal{W}_r^1 := \big\{u \in \mathcal{V} \,\big|\, (ru)' \in \mathcal{V}'\big\}, \qquad \mathcal{W}_r^2 := \big\{u \in \mathcal{V} \,\big|\, ru' \in \mathcal{V}'\big\}.$$

endowed by the norms

$$\|u\|_1 := \|u\|_{\mathcal{V}} + \|(ru)'\|_{\mathcal{V}'}, \qquad \|u\|_2 := \|u\|_{\mathcal{V}} + \|ru'\|_{\mathcal{V}'}.$$

In fact the two spaces coincide (see [6]) and the two norms are equivalent. Indeed

$$(\mathcal{R}u)' = \mathcal{R}'u + \mathcal{R}u'$$

and then

$$\|(\mathcal{R}u)'\|_{\mathcal{V}'} \leqslant \|\mathcal{R}u'\|_{\mathcal{V}'} + \Lambda\|u\|_{\mathcal{V}}, \qquad \|\mathcal{R}u'\|_{\mathcal{V}'} \leqslant \|(\mathcal{R}u)'\|_{\mathcal{V}'} + \Lambda\|u\|_{\mathcal{V}}.$$

So, we will simply denote by $\mathcal{W}_r$ the two spaces $\mathcal{W}_r = \mathcal{W}_r^1 = \mathcal{W}_r^2$, and the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ will be simply denoted by . .
We recall that $C^1([0,T]; W^{1,p}(\Omega))$ is dense in $\mathcal{W}_r$ (see [8]).

If r is regular there is constant C_o such that for every $u \in \mathcal{W}_r$ (see [8] and [6]) it holds

$$\Big|\int_\Omega u^2(x,t)r(x,t)dx\Big| \leqslant C_o\,\|u\|^2_{\mathcal{W}_r}, \tag{2.3}$$

where $C_o = C_o(r)$ depends (only) on T^{-1}, $\|r\|_{L^2(\mathcal{Q})}$ and $\|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$. We will denote by

$$\begin{aligned}
&r_+ \quad \text{the positive part of } r,\\
&r_- \quad \text{the negative part of } r
\end{aligned}$$

in such a way that $r = r_+ - r_-$ and we will denote by $R_+(t)$, $R_-(t)$, $\mathcal{R}_+$, $\mathcal{R}_-$ the operators analogous to (2.1).

For simplicity, we will suppose that the cylinder $\mathcal{Q} = \Omega \times (0, T)$ is divided in two connected regions, one where r is positive, one where is negative, so that $\mathcal{Q} = \mathcal{Q}_+ \cup \mathcal{I} \cup \mathcal{Q}_-$, $\Omega = \Omega_+(t) \cup I(t) \cup \Omega_-(t)$ with

$$\begin{aligned} \mathcal{Q}_+ &:= \big\{(x,t) \in \mathcal{Q} \,\big|\, r(x,t) > 0\big\}, & \mathcal{Q}_- &:= \big\{(x,t) \in \mathcal{Q} \,\big|\, r(x,t) < 0\big\}, \\ \mathcal{I} &:= \mathcal{Q} \setminus (\mathcal{Q}_+ \cup \mathcal{Q}_-), & I(t) &:= \Omega \setminus (\Omega_+(t) \cup \Omega_-(t)), \\ \Omega_+(t) &:= \big\{x \in \Omega \,\big|\, r(x,t) > 0\big\}, & \Omega_-(t) &:= \big\{x \in \Omega \,\big|\, r(x,t) < 0\big\}. \end{aligned}$$

About the "interface" $\mathcal{I}$ we assume that $\mathcal{Q}_+$ and $\mathcal{Q}_-$ are open sets, $\mathcal{I}$ is a subset of codimension 1 in $\mathbf{R}^{n+1}$ with $\mathcal{I} = \partial\mathcal{Q}_+ \cap \partial\mathcal{Q}_-$, $I(t)$ is a subset of codimension 1 in $\mathbf{R}^n$ for every $t \in [0, T]$ with $I(t) = \partial\Omega_+(t) \cap \partial\Omega_-(t)$ and

$$\begin{gathered} \mathcal{I} \text{ is uniformly Lipschitz continuous and} \\ I(t) \text{ is uniformly } C^{1,1} \text{ for every } t \in [0,T], \end{gathered} \tag{\textbf{H1}}$$

that is $\mathcal{I}$ is locally a graph of a Lipschitz continuous function (of n variables) and for every $t \in [0, T]$ $I(t)$ is locally a graph of a C^1 function (of $n - 1$ variables) whose gradient is uniformly bounded.

We consider $r : \Omega \times (0, T) \to \mathbf{R}$, $r = r_+ - r_-$ with $r_+, r_- \geqslant 0$, satisfying

$$\begin{gathered} r \textit{ regular} \text{ in the sense defined above} \\ r_+, D_t r_+ \in L^\infty(\mathcal{Q}_+), \quad r_-, D_t r_- \in L^\infty(\mathcal{Q}_-). \end{gathered} \tag{\textbf{H2}}$$

Remark 2.3. We observe that the function r could be less regular but here, only for the sake of simplicity, we suppose $D_t r_+ \in L^\infty(\mathcal{Q}_+)$ and $D_t r_- \in L^\infty(\mathcal{Q}_-)$. Anyway

$$r \quad \text{could not have temporal derivative in } L^\infty(\mathcal{Q})$$

and in particular it might possibly be discontinuous in the interface $\mathcal{I}$.

Notice that, under assumption (H2), not only r, but also

$$r_+ \text{ and } r_- \quad \text{satisfy (2.2) and (2.3).}$$

Now, we consider the cube $C \subset \mathbf{R}^n$, $C = [-1, 1]^n$, the cylinder

$$\mathcal{C} := C \times [0, T]$$

and define

$$\begin{aligned} C_- &:= \big\{y \in C \,\big|\, y_1 < 0\big\} = [-1, 0) \times [-1, 1]^{n-1}, \\ C_+ &:= \big\{y \in C \,\big|\, y_1 > 0\big\} = t \in (0, 1] \times [-1, 1]^{n-1}, \\ \mathcal{C}_- &:= \big\{(y, s) \in Q \,\big|\, y_1 < 0\big\} = [-1, 0) \times [-1, 1]^{n-1} \times [0, T], \\ \mathcal{C}_+ &:= \big\{(y, s) \in Q \,\big|\, y_1 > 0\big\} = (0, 1] \times [-1, 1]^{n-1} \times [0, T], \\ J &:= \big\{y \in C \,\big|\, y_1 = 0\big\} = \{0\} \times [-1, 1]^{n-1}. \end{aligned}$$

Given $\mathcal{X}$ a subset of $\mathbf{R}^{n+1}$ of the type

$$\mathcal{X} = \bigcup_{t\in[0,T]} X(t) \times \{t\}, \qquad X(t) \subset \mathbf{R}^n,$$

we will denote by $(x_1, \dots x_n, t)$ the coordinates of a point belonging to $\mathcal{X}$ and by

$$\Big\{u \in C^1(\mathcal{X}) \,\Big|\, \|u\|^p_{\mathcal{V}_\mathcal{X}} := \iint_\mathcal{X} (|u|^p + |Du|^p) dxdt < +\infty\Big\} \tag{2.4}$$

with respect to the topology induced by $\|\cdot\|_{\mathcal{V}_\mathcal{X}}$, where Du denotes the vector of the derivatives of u with respect to the first n variables. We will use this notation with $\mathcal{Q}_+, \mathcal{Q}_-, \mathcal{C}, \mathcal{C}_+, \mathcal{C}_-$ while with $\mathcal{X} = \mathcal{Q}$ we will simply write $\mathcal{V}$, the space defined at the beginning of the section.

Similarly, we introduce

$$\mathcal{H}_{\mathcal{Q}_+} := L^2(\mathcal{Q}_+; r_+).$$

Define, for $y \in C$ and $s \in [0, T]$, the reflexions

$$\begin{aligned} \mathsf{S}(y_1, y_2, \dots y_n, s) &:= (-y_1, y_2, \dots y_n, s), \\ \mathsf{s}(y_1, y_2, \dots y_n) &:= (-y_1, y_2, \dots y_n). \end{aligned} \tag{2.5}$$

We suppose in this section that the interface $\mathcal{I}$ is such that $I(t)$ can be mapped in the cube C in such a way the region where r is negative is mapped in C_- and the region where r is positive in C_+. In general this is not true: for instance, if the spatial dimension is 2, one could have that $\mathcal{I}$ is a cylinder and $I(t)$ a circle for every $t \in [0, T]$. Then suppose there is (for the moment only one, but in the next section we will consider a partition of unity around $\mathcal{I}$) one function line $\phi : \mathcal{C} \to \mathcal{Q}$ bijective and such that satisfying

$$\begin{aligned}
&\phi : \mathcal{C}_+ \to \mathcal{Q}_+ \quad \text{bijective} \\
&\phi : \mathcal{C}_- \to \mathcal{Q}_- \quad \text{bijective} \\
&(-1,1)^n \ni y \mapsto \phi(y,s) \in C^1\big((-1,1)^n\big) \\
&\qquad \text{and its inverse is } C^1(\Omega) \quad \text{for every } s \in [0,T], \\
&[0,T] \ni s \mapsto \phi(y,s), \partial_{y_1}\phi(y,s), \dots \partial_{y_n}\phi(y,s) \text{ are Lipschitz continuous} \\
&\qquad \text{and } t \mapsto \phi^{-1}(x,t), \partial_{x_1}\phi^{-1}(x,t), \dots \partial_{x_n}\phi^{-1}(x,t) \\
&\qquad \text{are Lipschitz continuous} \quad \text{for every } y \in [-1,1]^n \\
&\phi(y,s) = \big(\varphi(y,s), s\big)
\end{aligned} \tag{\textbf{H3}}$$

so that

$$\varphi(\cdot,t):C_-\to\Omega_-(t)\quad\text{and}\quad\varphi(\cdot,t):C_+\to\Omega_+(t)\qquad\text{are bijective}\ . \tag{2.6}$$

We will denote by $J\phi$ the Jacobian matrix

$$J\phi(y,s)=\begin{pmatrix}\partial_{y_1}\phi_1(y,s) & \dots & \partial_{y_n}\phi_1(y,s) & \partial_s\phi_1(y,s)\\ & \vdots & & \\ \partial_{y_1}\phi_n(y,s) & \dots & \partial_{y_n}\phi_n(y,s) & \partial_s\phi_n(y,s)\\ \partial_{y_1}\phi_{n+1}(y,s) & \dots & \partial_{y_n}\phi_{n+1}(y,s) & \partial_s\phi_{n+1}(y,s)\end{pmatrix}$$

$$=\begin{pmatrix}\partial_{y_1}\varphi_1(y,s) & \dots & \partial_{y_n}\varphi_1(y,s) & \partial_s\varphi_1(y,s)\\ & \vdots & & \\ \partial_{y_1}\varphi_n(y,s) & \dots & \partial_{y_n}\varphi_n(y,s) & \partial_s\varphi_n(y,s)\\ 0 & \dots & 0 & 1\end{pmatrix}$$

and, by the assumptions, there are positive constants $M,m,N,\tilde N,K,\tilde K$ and non-negative constants $L,\tilde L$ such that (here $|\cdot|$ denotes the modulus for a vector, the determinant for a matrix)

$$\begin{aligned} m\leqslant |J\phi(y,s)| &\leqslant M, & M^{-1}\leqslant |J\phi^{-1}(x,t)| &\leqslant m^{-1}\\ \|D\phi(y,s)\| &\leqslant N, & \|D\phi^{-1}(x,t)\| &\leqslant \tilde N, \qquad \textbf{(H4)}\\ |D_i\phi_j(y,s)| &\leqslant K, & |D_i\phi_j^{-1}(x,t)| &\leqslant \tilde K,\\ |r\circ\phi(y,s)|\,|D_s\varphi(y,s)| &\leqslant L, & |r(x,t)|\,|D_t\varphi^{-1}(x,t)| &\leqslant \tilde L, \qquad \textbf{(H4}'\textbf{)}\end{aligned}$$

for every $i,j=1,\dots n$, for every $y\in C$ and $x\in\Omega$ and almost every $s,t\in[0,T]$ and where by norm of a $m\times m$ matrix A we mean

$$\|A\|:=\max_{v\in\mathbf{R}^m,\|v\|=1}\|A\cdot v\|.$$

Notice that, by the last of (H3),

$$\big[J\phi\big]_{n+1,j}=\big[J\phi^{-1}\big]_{n+1,j}=0\qquad\text{for every } j\in\{1,\dots n\}$$

and the same holds for the matrix

$$\mathcal{M}(x,t):=J\phi\big(S(\phi^{-1}(x,t))\big)\cdot JS\big(\phi^{-1}(x,t)\big)\cdot J\phi^{-1}(x,t), \tag{2.7}$$

i.e., the matrices $J\phi, J\phi^{-1}, \mathcal{M}$ (and JS which indeed is: $(JS)_{ij} = \delta_{ij}$ for every i, j except $(JS)_{11}$, which is -1) are of the type

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1\,n+1} \\ & & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n\,n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{2.8}$$

Notice moreover that

$$|\mathcal{M}(x,t)| = 1 \qquad \text{for a.e. } (x,t) \in \mathcal{Q}. \tag{2.9}$$

For this reason one has that

$$|J\phi(y,s)| = |D\varphi(y,s)| \qquad \text{and} \qquad |J\phi^{-1}(x,t)| = |D\varphi^{-1}(x,t)| \tag{2.10}$$

where $D\varphi(y,s)$ (and similarly $D\varphi^{-1}(x,t)$) is the $n \times n$ matrix

$$\begin{aligned} D\varphi(y,s) := & \begin{pmatrix} \partial_{y_1}\phi_1(\cdot,s) & \dots & \partial_{y_n}\phi_1(\cdot,s) \\ & \vdots & \\ \partial_{y_1}\phi_n(\cdot,s) & \dots & \partial_{y_n}\phi_n(\cdot,s) \end{pmatrix} \\ = & \begin{pmatrix} \partial_{y_1}\varphi_1(\cdot,s) & \dots & \partial_{y_n}\varphi_1(\cdot,s) \\ & \vdots & \\ \partial_{y_1}\varphi_n(\cdot,s) & \dots & \partial_{y_n}\varphi_n(\cdot,s) \end{pmatrix}. \end{aligned} \tag{2.11}$$

By (H4), we get that

$$|\mathcal{M}_{ij}(x,t)| \leqslant 1 + 2K\tilde{K}. \tag{2.12}$$

At this moment, we define

$$\tilde{E}u\,(y,t) = \begin{cases} \big(u \circ \phi\big)(y,t) & \text{for } y \in C_+ \\ \big(u \circ \phi \circ S\big)(y,t) & \text{for } y \in C_-. \end{cases} \tag{2.13}$$

Notice that by this choice, we have

$$\tilde{E}u\,(y_1, y_2, \dots y_n, t) = \tilde{E}u\,(-y_1, y_2, \dots y_n, t). \tag{2.14}$$

Then we define the extension operator as

$$\begin{aligned} Eu\,(x,t) &= \tilde{E}u \circ \phi^{-1}(x,t) \\ &= \begin{cases} u(x,t) & \text{for } (x,t) \in \mathcal{Q}_+ \\ u \circ \phi \circ S \circ \phi^{-1}(x,t) & \text{for } (x,t) \in \mathcal{Q}_-. \end{cases} \end{aligned} \tag{2.15}$$

Lemma 2.4. *Fix* $q \in [1, +\infty)$. *For every* $v \in L^q(\Omega)$, *one has*

$$\int_\Omega |Ev|^q(x)\,dx = 2\int_{\Omega_+(t)} |v(x)|^q\,dx.$$

Proof. Since

$$\int_\Omega |Ev|^q(x)\,dx = \int_{\Omega_+(t)} |Ev|^q(x)\,dx + \int_{\Omega_-(t)} |Eu|^q(x)\,dx,$$

we focus our attention on the second addend of the right hand side. We have

$$\begin{aligned}\int_{\Omega_-(t)} |Ev|^q(x)\,dx &= \int_{\Omega_-(t)} \big|v(\phi\circ S\circ\phi^{-1})\big|^q(x,t)\,dx\\ &= \int_{\Omega_+(t)} |v(x)|^q |J\upphi(t)|\,dx,\end{aligned}$$

where by $\upphi(t)$, we denote the function

$$\Omega_-(t) \ni x \mapsto \phi\big(S(\phi^{-1}(x,t)\big).$$

Since ($J\varphi(t)$ and $J\varphi^{-1}(t)$ are defined in (2.11), s in (2.5))

$$J\upphi(t) = J\varphi(t)\cdot J\mathsf{s}\cdot J\varphi^{-1}(t),$$

we get that $|J\upphi(t)| = 1$ for every t and then thesis. □

Corollary 2.5. *Suppose* (H1), (H2), (H3), (H4), (H4′) *hold. For every* $u \in C^1([0,T]; W^{1,p}(\Omega))$

$$\int_{\Omega_-(t)} (Eu)^2(x,t) Er_+(x,t)\,dx = \int_{\Omega_+(t)} u^2(x,t)\,r_+(x,t)\,dx$$

and the function

$$[0,T] \ni t \mapsto \int_\Omega (Eu)^2(x,t) Er_+(x,t)\,dx \qquad \text{is absolutely continuous.}$$

Proof. The result follows as in the proof of Lemma 2.4. □

Now, consider the space

$$\mathcal{H}_+ := L^2(\mathcal{Q}; Er_+).$$

Lemma 2.6. *For every* $u \in C^1([0,T]; W^{1,p}(\Omega))$

$$\|Eu\|^2_{\mathcal{H}_+} = 2\,\|u\|^2_{\mathcal{H}_{\mathcal{Q}_+}}, \quad \|Eu\|^p_{L^p(\mathcal{Q})} = 2\,\|u\|^p_{L^p(\mathcal{Q}_+)},$$

$$\|D(Eu)\|^p_{L^p(\mathcal{Q})} \leqslant 2\,C_1\,\|Du\|^p_{L^p(\mathcal{Q}_+)},$$

where $C_1 = 2^{-1}\left(1 + \sqrt{n}\,\left(1 + 2K\tilde{K}\right)^p\right)$.

Proof. Fix $u \in \mathcal{V}$. By Corollary 2.5, one immediately gets

$$\iint_{\mathcal{Q}} |Eu|^2(x,t) Er_+(x,t)\,dxdt = 2\iint_{\mathcal{Q}_+} |u(x,t)|^2 r_+(x,t)\,dxdt$$

and by Lemma 2.4

$$\iint_{\mathcal{Q}} |Eu|^p(x,t)\,dxdt = 2\iint_{\mathcal{Q}_+} |u(x,t)|^p\,dxdt.$$

The estimate about the gradient is less immediate: now for a function $u \in C^1([0,T];W^{1,p}(\Omega))$, we denote, as usual, by Du the gradient with respect to the spatial variables, i.e.,

$$Du \in C^1([0,T];L^p(\Omega)),$$
$$\big(Du(t)\big)(x) = \big(D_1 u(x,t), \dots D_n u(x,t)\big) = \big(D_{x_1} u(x,t), \dots D_{x_n} u(x,t)\big),$$

where $D_i = D_{x_i}$ denotes the weak derivative with respect to x_i, by ∂_t the partial derivative with respect to t which coincides with u', i.e.,

$$u' \in C^0([0,T];W^{1,p}(\Omega)), \qquad \big(u'(t)\big)(x) = \partial_t u(x,t),$$

while by ∇u, with an abuse of notation, we denote

$$\nabla u(x,t) = \big(Du(x,t), \partial_t u(x,t)\big) = \big(D_{x_1} u(x,t), \dots D_{x_n} u(x,t), \partial_t u(x,t)\big).$$

Since $\mathcal{M}$ is of the type (2.8) the derivatives D_i of Eu do not depend on $\partial_t u$ since $(i = 1, \dots n)$

$$D_i\big(Eu\big)(x,t) = D_i u(x,t), \qquad (x,t) \in \mathcal{Q}_+$$
$$D_i\big(Eu\big)(x,t) = \Big(\nabla u\big(\phi \circ S \circ \phi^{-1}(x,t)\big) \cdot \mathcal{M}(x,t)\Big)_i, \qquad (x,t) \in \mathcal{Q}_-.$$

Observe that, by (2.12) and since $\mathcal{M}$ is of type (2.8), for $(x,t) \in \mathcal{Q}_-$, one has that

$$|D_i\big(Eu\big)(x,t)| \leqslant \big(2K\tilde{K} + 1\big)\, |Du\big(\phi \circ S \circ \phi^{-1}\big)(x,t)|.$$

Then for $(x,t) \in \mathcal{Q}_-$

$$|D\big(Eu\big)(x,t)| \leqslant \sqrt{n}\,\big(2K\tilde{K} + 1\big)\, |Du\big(\phi \circ S \circ \phi^{-1}\big)(x,t)|.$$

We conclude that

$$\iint_{\mathcal{Q}} |D(Eu)|^p(x,t)\,dxdt$$
$$= \iint_{\mathcal{Q}_+} |Du|^p(x,t)\,dxdt + \iint_{\mathcal{Q}_-} |D(Eu)|^p(x,t)\,dxdt$$

$$\begin{aligned}&\leqslant \iint_{\mathcal{Q}_+} |Du|^p(x,t)\,dxdt\\&\qquad + \sqrt{n}\left(2K\tilde{K}+1\right)^p \iint_{\mathcal{Q}_-} \left|Du\left(\phi\circ S\circ\phi^{-1}\right)\right|^p(x,t)\,dxdt.\end{aligned}$$

As done in the proof of Lemma 2.4, one gets

$$\iint_{\mathcal{Q}_-} \left|Du\left(\phi\circ S\circ\phi^{-1}\right)\right|^p(x,t)\,dxdt = \iint_{\mathcal{Q}_+} |Du|^p(x,t)\,dydt$$

and then finally

$$\begin{aligned}&\iint_{\mathcal{Q}} |D(Eu)|^p(x,t)\,dxdt\\&\leqslant \left(1+\sqrt{n}\left(2K\tilde{K}+1\right)^p\right)\iint_{\mathcal{Q}_+} |Du|^p(x,t)\,dxdt\,. \qquad \square\end{aligned}$$

Lemma 2.7. *For every $u \in \mathcal{V}_{\mathcal{Q}_+}$ such that $(r_+u)' \in \mathcal{V}'_+$, we have that $Eu \in \mathcal{W}_{Er_+}$. In particular*

$$\|(Er_+\,Eu)'\|_{\mathcal{V}'} \leqslant \sqrt{2}\left[C_2\,\|(r_+u)'\|_{\mathcal{V}'_{\mathcal{Q}_+}} + C_3\,\|u\|_{L^2(\mathcal{Q}_+)}\right],$$
$$\|(Er_+\,Eu)'\|_{\mathcal{V}'} \leqslant \sqrt{2}\left[C_2\,\|(ru)'\|_{\mathcal{V}'} + C_4\,\|u\|_{L^2(\mathcal{Q})} + C_5\|u\|_{L^2(\mathcal{Q}_+)}\right],$$

where

$$C_2 = \sqrt{2}N^2\tilde{N}^2\frac{M^2}{m^2}, \qquad C_3 = \sqrt{2}N^2\frac{M}{m}|\mathcal{Q}_+|^{\frac{p-2}{2p}}\,\tilde{L} + |\mathcal{Q}|^{\frac{p-2}{2p}}\,L,$$
$$C_4 = N^2\frac{M}{m}|\mathcal{Q}|^{\frac{p-2}{2p}}\,\tilde{L}, \quad C_5 = |\mathcal{Q}|^{\frac{p-2}{2p}}\,L.$$

Remark 2.8. The constant 2 in the right hand side of the two estimates of the previous lemma is due to the change of sign of the function r. Notice that if r is independent of time the constants, then $C_3 = C_4 = C_5 = 0$, since in that case $L = \tilde{L} = 0$.

Proof. For $\psi \in C^1([0,T];W^{1,p}(\Omega))$ with $\psi(0) = \psi(T) = 0$, we have

$$\begin{aligned}&-\left\langle \left(Er_+\,Eu\right)',\,\psi\right\rangle_{\mathcal{V}'\times\mathcal{V}} = \left(Er_+\,Eu,\psi'\right)_{\mathcal{H}}\\&\quad = \iint_{\mathcal{Q}} Eu\,Er_+\frac{\partial\psi}{\partial t}dxdt\\&\quad = \iint_{\mathcal{C}} Eu\big(\phi(y,t)\big)\,Er_+\big(\phi(y,t)\big)\frac{\partial\psi}{\partial t}\big(\phi(y,t)\big)|J\phi(y,t)|dydt.\end{aligned}$$

Recall that (see (H3)) we have that $\phi(y,t) = (\varphi(y,t), t)$. Denoting

$$\frac{\partial \varphi}{\partial t} := \left(\frac{\partial \varphi_1}{\partial t}, \cdots \frac{\partial \varphi_n}{\partial t} \right) \quad \text{and} \quad D\psi := (D_1 \psi, \cdots D_n \psi) \, ,$$

we have that

$$\frac{\partial}{\partial t}\big(\psi(\phi(y,t))\big) = \frac{\partial \psi}{\partial t}\big(\phi(y,t)\big) + \left(D\psi(\phi(y,t)), \frac{\partial \varphi}{\partial t}(y,t) \right)$$

and then

$$\begin{aligned} &\big\langle \big(Er_+ \, Eu\big)', \psi \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\quad = - \iint_{\mathcal{C}} Eu\big(\phi(y,t)\big) \, Er_+\big(\phi(y,t)\big) \frac{\partial}{\partial t}\big(\psi(\phi(y,t))\big) |J\phi(y,t)| dydt \qquad (2.16) \\ &\qquad + \iint_{\mathcal{C}} Eu\big(\phi(y,t)\big) \, Er_+\big(\phi(y,t)\big) \left(D\psi(\phi(y,t)), \frac{\partial \varphi}{\partial t}(y,t) \right) |J\phi(y,t)| dydt. \end{aligned}$$

Now, define $\uppsi = \psi \circ \phi$ and observe that

$$\begin{aligned} \iint_{\mathcal{Q}} \psi^2(x,t) dxdt &= \iint_{\mathcal{C}} \big(\psi(\phi(y,t))\big)^2 |J\phi(y,t)| dydt \\ &\geqslant m \iint_{\mathcal{C}} \big(\psi(\phi(y,t))\big)^2 dydt \end{aligned}$$

by which

$$\iint_{\mathcal{C}} \uppsi^2(y,t) dydt \leqslant \frac{1}{m} \iint_{\mathcal{Q}} \psi^2(x,t) dxdt. \qquad (2.17)$$

Similarly,

$$\begin{aligned} \iint_{\mathcal{C}} |D\uppsi(y,t)|^2 dydt &= \iint_{\mathcal{C}} |D\psi(\phi(y,t)) \cdot D\phi(y,t)|^2 dydt \\ &\leqslant N^2 \iint_{\mathcal{C}} |D\psi(\phi(y,t))|^2 |J\phi(y,t)| |J\phi^{-1}(y,t)| dydt \\ &\leqslant N^2 m^{-1} \iint_{\mathcal{C}} |D\psi(\phi(y,t))|^2 |J\phi(y,t)| dydt \end{aligned}$$

by which

$$\iint_{\mathcal{C}} |D\uppsi(y,t)|^2 dydt \leqslant \frac{N^2}{m} \iint_{\mathcal{Q}} |D\psi|^2(x,t) dxdt. \qquad (2.18)$$

Notice that

$$\Big| \iint_{\mathcal{C}} Eu\big(\phi(y,t)\big) \, Er_+\big(\phi(y,t)\big) \Big(D\psi(\phi(y,t)), \frac{\partial \varphi}{\partial t}(y,t) \Big) |J\phi(y,t)| dydt \Big|$$

$$\leqslant L \iint_{\mathcal{C}} \big|Eu\big(\phi(y,t)\big)\big|\big|D\psi(\phi(y,t))\big||J\phi(y,t)|dydt$$
$$\leqslant L\Big(\iint_{\mathcal{C}} \big|Eu\big(\phi(y,t)\big)\big|^2 |J\phi(y,t)|dydt \iint_{\mathcal{C}} \big|D\psi(\phi(y,t))\big|^2 \,|J\phi(y,t)|dydt\Big)^{\frac{1}{2}}$$
$$= L\Big(\iint_{\mathcal{Q}} \big|Eu(x,t)\big|^2 \,dxdt \iint_{\mathcal{Q}} \big|D\psi(x,t)\big|^2 dxdt\Big)^{1/2}$$
$$= \sqrt{2}\, L \,\Big(\iint_{\mathcal{Q}_+} u^2(x,t)\, dxdt\Big)^{1/2}\Big(\iint_{\mathcal{Q}} \big|D\psi(x,t)\big|^2 dxdt\Big)^{1/2}.$$

Then coming back to (2.16), we get

$$\begin{aligned} \big|\big\langle \big(Er_+\, Eu\big)', \psi\big\rangle_{\mathcal{V}'\times\mathcal{V}}\big| &\leqslant M\Big|\big\langle \big(\tilde{E}r_+\, \tilde{E}u\big)', \psi\big\rangle_{\mathcal{V}'_{\mathcal{C}}\times\mathcal{V}_{\mathcal{C}}}\Big| \\ &\quad + \Big|\Big(\tilde{E}r_+\, \tilde{E}u, D\psi(\phi(y,t))\cdot \frac{\partial\varphi}{\partial t}(y,t)|J\phi(y,t)|\Big)_{L^2(\mathcal{C})}\Big| \\ &\leqslant M\|\big(\tilde{E}r_+\, \tilde{E}u\big)'\|_{\mathcal{V}'_{\mathcal{C}}}\|\psi\|_{\mathcal{V}_{\mathcal{C}}} + \sqrt{2}\, L\, \|u\|_{L^2(\mathcal{Q}_+)}\|D\psi\|_{L^2(\mathcal{Q})} \\ &\leqslant \frac{MN^2}{m}\|\big(\tilde{E}r_+\, \tilde{E}u\big)'\|_{\mathcal{V}'_{\mathcal{C}}}\|\psi\|_{\mathcal{V}} + \sqrt{2}\, L\, |\mathcal{Q}|^{\frac{p-2}{2p}}\, \|u\|_{L^2(\mathcal{Q}_+)}\|D\psi\|_{L^p(\mathcal{Q})} \end{aligned}$$

and, taking the supremum over all ψ whose norm in $\mathcal{V}$ is less or equal to 1,

$$\big\|\big(Er_+\, Eu\big)'\big\|_{\mathcal{V}'} \leqslant \frac{N^2M}{m}\big\|\big(\tilde{E}r_+\, \tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}}} + \sqrt{2}\, L\, |\mathcal{Q}|^{\frac{p-2}{2p}}\, \|u\|_{L^2(\mathcal{Q}_+)}. \tag{2.19}$$

Now, we estimate the first term on the right hand side.

Consider $\eta \in C^1([0,T]; W^{1,p}(C))$ such that $\eta(0) = \eta(T) = 0$. We have

$$\big\langle \big(\tilde{E}r_+\, \tilde{E}u\big)', \eta\big\rangle_{\mathcal{V}'_{\mathcal{C}}\times\mathcal{V}_{\mathcal{C}}} = -\big(\tilde{E}r_+\, \tilde{E}u, \eta'\big)_{L^2(\mathcal{C})}.$$

Define the operators $\mathcal{S}_+$ and $\mathcal{S}_-$ (S defined in (2.5))

$$\mathcal{S}_+\eta'\,(y,t) = \begin{cases} \eta'(y,t) & \text{for } y \in C_+ \\ \big(\eta'\circ S\big)(y,t) & \text{for } y \in C_-, \end{cases}$$
$$\mathcal{S}_-\eta'\,(y,t) = \begin{cases} \big(\eta'\circ S\big)(y,t) & \text{for } y \in C_+ \\ \eta'(y,t) & \text{for } y \in C_-. \end{cases}$$

Notice that, by the simmetry of the domain $\mathcal{C}$ and of $\tilde{E}r_+\, \tilde{E}u$, one has that one of the two following chain of inequalities is always true:

$$\big(\tilde{E}r_+\, \tilde{E}u, \mathcal{S}_-\eta'\big)_{L^2(\mathcal{C})} \leqslant \big(\tilde{E}r_+\, \tilde{E}u, \eta'\big)_{L^2(\mathcal{C})} \leqslant \big(\tilde{E}r_+\, \tilde{E}u, \mathcal{S}_+\eta'\big)_{L^2(\mathcal{C})}$$

or

$$\big(\tilde{E}r_+ \tilde{E}u, \mathcal{S}_+\eta'\big)_{L^2(\mathcal{C})} \leqslant \big(\tilde{E}r_+ \tilde{E}u, \eta'\big)_{L^2(\mathcal{C})} \leqslant \big(\tilde{E}r_+ \tilde{E}u, \mathcal{S}_-\eta'\big)_{L^2(\mathcal{C})}.$$

For sake of simplicity, we denote by $\mathcal{S}\eta'$, the simmetric function (chosen among $\mathcal{S}_+\eta'$ and $\mathcal{S}_-\eta'$) which makes true

$$-\big(\tilde{E}r_+ \tilde{E}u, \eta'\big)_{L^2(\mathcal{C})} \leqslant -\big(\tilde{E}r_+ \tilde{E}u, \mathcal{S}\eta'\big)_{L^2(\mathcal{C})}.$$

Then, for every η as before, one gets

$$\begin{aligned}\big\langle\big(\tilde{E}r_+ \tilde{E}u\big)', \eta\big\rangle_{\mathcal{V}'_{\mathcal{C}}\times\mathcal{V}_{\mathcal{C}}} &= -\big(\tilde{E}r_+ \tilde{E}u, \eta'\big)_{L^2(\mathcal{C})} \leqslant -\big(\tilde{E}r_+ \tilde{E}u, \mathcal{S}\eta'\big)_{L^2(\mathcal{C})} \\ &= -\big(\tilde{E}r_+ \tilde{E}u, (\mathcal{S}\eta)'\big)_{L^2(\mathcal{C})} = -2\big(\tilde{E}r_+ \tilde{E}u, (\mathcal{S}\eta)'\big)_{L^2(\mathcal{C}_+)} \\ &= 2\,\big\langle\big(\tilde{E}r_+ \tilde{E}u\big)', \mathcal{S}\eta\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}}.\end{aligned}$$

Now, since if $\|\eta\|_{\mathcal{V}_{\mathcal{C}}} \leqslant 1$, one gets that $\|\mathcal{S}\eta\|_{\mathcal{V}_{\mathcal{C}_+}} \leqslant 1$, we derive that

$$\big\|\big(\tilde{E}r_+ \tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}}} \leqslant 2\,\big\|\big(\tilde{E}r_+ \tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}_+}}. \tag{2.20}$$

Now, consider $\eta \in C^1([0,T]; W^{1,p}(C_+))$ such that $\eta(0) = \eta(T) = 0$. We have that

$$\begin{aligned}\big\langle\big(\tilde{E}r_+ \tilde{E}u\big)', \eta\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}} &= -\iint_{\mathcal{C}_+} \tilde{E}r_+ \tilde{E}u \frac{\partial\eta}{\partial t} dydt \\ &= -\iint_{\mathcal{C}_+} r_+ \circ \phi(y,t) u \circ \phi(y,t) \frac{\partial\eta}{\partial t}(y,t) dydt \\ &= -\iint_{\mathcal{Q}_+} r_+(x,t) u(x,t) \frac{\partial\eta}{\partial t}(\phi^{-1}(x,t)) |J\phi^{-1}(x,t)| dxdt.\end{aligned}$$

Now, since

$$\frac{\partial}{\partial t}\big(\eta(\phi^{-1}(y,t))\big) = \frac{\partial\eta}{\partial t}\big(\phi^{-1}(y,t)\big) + \Big(D\eta(\phi^{-1}(x,t)), \frac{\partial\varphi^{-1}}{\partial t}(x,t)\Big),$$

we get

$$\begin{aligned}&\big\langle\big(\tilde{E}r_+ \tilde{E}u\big)', \eta\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}} \\ &\quad= -\iint_{\mathcal{Q}_+} r_+(x,t) u(x,t) \frac{\partial}{\partial t}(\eta \circ \phi^{-1}(x,t)) |J\phi^{-1}(x,t)| dxdt \\ &\quad+ \iint_{\mathcal{Q}_+} r_+(x,t) u(x,t) \left(D\eta(\phi^{-1}(x,t)), \frac{\partial\varphi^{-1}}{\partial t}(x,t)\right) |J\phi^{-1}(x,t)| dxdt.\end{aligned}$$

Defining $\eta := \eta \circ \phi^{-1}$ similarly to (2.17) and (2.18), one gets

$$\iint_{\mathcal{Q}_+} \eta^2(x,t)dxdt \leqslant M \iint_{\mathcal{C}_+} \eta^2(y,t)dydt,$$
$$\iint_{\mathcal{Q}_+} |D\eta(x,t)|^2 dxdt \leqslant \tilde{N}^2 M \iint_{\mathcal{C}_+} |D\eta|^2(y,t)dydt,$$

by which, similarly to (2.19),

$$\big\|\big(\tilde{E}r_+\tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}_+}} \leqslant \frac{\tilde{N}^2 M}{m} \|(r_+u)'\|_{\mathcal{V}'_{\mathcal{Q}_+}} + \tilde{L}\,|\mathcal{Q}_+|^{\frac{p-2}{2p}}\|u\|_{L^2(\mathcal{Q}_+)}. \tag{2.21}$$

By this last estimate, (2.19) and (2.20), we finally get

$$\begin{aligned}\big\|\big(Er_+\,Eu\big)'\big\|_{\mathcal{V}'} \leqslant{}& 2\,\frac{N^2 M}{m}\,\frac{\tilde{N}^2 M}{m}\,\|(r_+u)'\|_{\mathcal{V}'_{\mathcal{Q}_+}} \\ &+ \left[2\,\frac{N^2 M}{m}\,\tilde{L}|\mathcal{Q}_+|^{\frac{p-2}{2p}} + \sqrt{2}\,L\,|\mathcal{Q}|^{\frac{p-2}{2p}}\right]\|u\|_{L^2(\mathcal{Q}_+)}\end{aligned}$$

which concludes the proof of the first inequality.

Now, reconsider (2.20). We have that for every $w \in \mathcal{V}_{\mathcal{C}}$

$$\big\langle\big(\tilde{E}r_+\,\tilde{E}u\big)', w\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}} = \big\langle\big(\tilde{E}r_+\,\tilde{E}u\,\chi_{\mathcal{C}_+}\big)', w\big\rangle_{\mathcal{V}'_{\mathcal{C}}\times\mathcal{V}_{\mathcal{C}}},$$

where $\chi_{\mathcal{C}_+} = 1$ in $\mathcal{C}_+$ and $\chi_{\mathcal{C}_+} = 0$ in $\mathcal{C}\setminus\mathcal{C}_+$. Now, in fact, for every function

$$v \in \mathcal{V}_{\mathcal{C}} \quad \text{such that } v' \in \mathcal{V}'_{\mathcal{C}} \quad \text{and} \quad v\,\chi_{\mathcal{C}_+} \equiv \tilde{E}r_+\,\tilde{E}u\,\chi_{\mathcal{C}_+} \quad \text{in } \mathcal{C}_+ \tag{2.22}$$

one has

$$\big\langle\big(\tilde{E}r_+\,\tilde{E}u\big)', w\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}} = \big\langle(v\chi_{\mathcal{C}_+})', w\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}}.$$

Then, we have

$$\big|\big\langle\big(\tilde{E}r_+\,\tilde{E}u\big)', w\big\rangle_{\mathcal{V}'_{\mathcal{C}_+}\times\mathcal{V}_{\mathcal{C}_+}}\big| \leqslant \|(v\chi_{\mathcal{C}_+})'\|_{\mathcal{V}'_{\mathcal{C}_+}}\|w\|_{\mathcal{V}_{\mathcal{C}_+}} \leqslant \|v'\|_{\mathcal{V}'_{\mathcal{C}}}\|w\|_{\mathcal{V}_{\mathcal{C}_+}}$$

and taking the supremum over all functions $w \in \mathcal{V}_{\mathcal{C}_+}$ such that $\|w\|_{\mathcal{V}_{\mathcal{C}_+}} \leqslant 1$, one finally concludes

$$\big\|\big(\tilde{E}r_+\,\tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}_+}} \leqslant \|v'\|_{\mathcal{V}'_{\mathcal{C}}}.$$

In particular, defining $\tilde{g} \in \mathcal{V}_{\mathcal{C}}$ as $\tilde{g}(y,t) := g \circ \phi(y,t)$ for $g \in \mathcal{V}$, the function

$$v = \tilde{r}\tilde{u} \qquad \text{satisfies (2.22)}$$

and then

$$\big\|\big(\tilde{E}r_+\,\tilde{E}u\big)'\big\|_{\mathcal{V}'_{\mathcal{C}_+}} \leqslant \|(\tilde{r}\tilde{u})'\|_{\mathcal{V}'_{\mathcal{C}}}. \tag{2.23}$$

Now, proceeding as done to get (2.21), one gets that

$$\|(\tilde{r}\tilde{u})'\|_{\mathcal{V}'_{\mathcal{C}}} \leqslant \frac{\tilde{N}^2 M}{m} \|(ru)'\|_{\mathcal{V}'} + \tilde{L}\,|\mathcal{Q}|^{\frac{p-2}{2p}}\,\|u\|_{L^2(\mathcal{Q})}.$$

Then from this last estimate, (2.19), (2.20), and (2.23), we derive

$$\begin{aligned}
&\big\|\big(Er_+\, Eu\big)'\big\|_{\mathcal{V}'} \\
&\leqslant\ 2\,\frac{N^2 M}{m}\Big[\frac{\tilde{N}^2 M}{m}\|(ru)'\|_{\mathcal{V}'} + \tilde{L}\,|\mathcal{Q}|^{\frac{p-2}{2p}}\,\|u\|_{L^2(\mathcal{Q})}\Big] + \sqrt{2}\,L\,|\mathcal{Q}|^{\frac{p-2}{2p}}\,\|u\|_{L^2(\mathcal{Q}_+)} \\
&= \sqrt{2}\Big[\sqrt{2}N^2\tilde{N}^2\,\frac{M^2}{m^2}\,\|(ru)'\|_{\mathcal{V}'} + \sqrt{2}N^2\frac{M}{m}\,\tilde{L}\,|\mathcal{Q}|^{\frac{p-2}{2p}}\,\|u\|_{L^2(\mathcal{Q})} + \\
&\qquad + L\,|\mathcal{Q}|^{\frac{p-2}{2p}}\,\|u\|_{L^2(\mathcal{Q}_+)}\Big]
\end{aligned}$$

which concludes the proof. □

By the previous lemma, one can immediately derive the following statement, without adapting the previous proof.

Lemma 2.9. *For every $u \in \mathcal{V}_{\mathcal{Q}_+}$ such that $r_+ u' \in \mathcal{V}'_+$, we have that $Eu \in \mathcal{W}_{Er_+}$. In particular,*

$$\begin{aligned}
&\|Er_+(Eu)'\|_{\mathcal{V}'} \\
&\leqslant \sqrt{2}\,\big[C_2\,\|ru'\|_{\mathcal{V}'} + C_4\,\|u\|_{L^2(\mathcal{Q},|r|)} + C_5\|u\|_{L^2(\mathcal{Q}_+,r_+)} + C_6\|u\|_{\mathcal{V}}\big],
\end{aligned}$$

where $C_6 = \Lambda(C_1 + C_2)$.

Proof. Since $\mathcal{R}u' = (\mathcal{R}u)' - \mathcal{R}'u$, we get that

$$\begin{aligned}
\|Er_+(Eu)'\|_{\mathcal{V}'} &\leqslant \|(Er_+\, Eu)'\|_{\mathcal{V}'} + \|(Er_+)'Eu\|_{\mathcal{V}'} \\
&\leqslant \|(Er_+\, Eu)'\|_{\mathcal{V}'} + 2\Lambda\|Eu\|_{\mathcal{V}} \\
&\leqslant \sqrt{2}\,\big[C_2\,\|(ru)'\|_{\mathcal{V}'} + C_4\,\|u\|_{L^2(\mathcal{Q})} + C_5\|u\|_{L^2(\mathcal{Q}_+)}\big] \\
&\qquad + 2\Lambda\big(2\|u\|_{L^p(\mathcal{Q})} + C_1\|Du\|_{L^p(\mathcal{Q})}\big).
\end{aligned}$$

Since

$$\|(ru)'\|_{\mathcal{V}'} \leqslant \|ru'\|_{\mathcal{V}'} + \Lambda\|u\|_{\mathcal{V}},$$

we finally conclude that

$$\begin{aligned}
\|Er_+(Eu)'\|_{\mathcal{V}'} \leqslant \sqrt{2}\,&\big[C_2\,\|ru'\|_{\mathcal{V}'} + \Lambda C_2\,\|u\|_{\mathcal{V}} + C_4\,\|u\|_{L^2(\mathcal{Q})} \\
&+ C_5\|u\|_{L^2(\mathcal{Q}_+)}\big] + 2\Lambda\big(2\|u\|_{L^p(\mathcal{Q})} + C_1\|Du\|_{L^p(\mathcal{Q})}\big).
\end{aligned}$$

□

We can summarise the estimates of Lemma 2.6, Lemma 2.7, and Lemma 2.9 in the following proposition. Here, the positive constant $\widetilde{C}$ can be explicitly derived, but depend only on $C_1, C_2, C_3, C_4, C_5, \Lambda, |\mathcal{Q}|, p$ (and then by $n, K, \tilde{K}, N, \tilde{N}, M, m, \Lambda, C_o, |\mathcal{Q}|^{\frac{p-2}{2p}}$, where C_1, C_2, C_3, C_4, C_5 are derived in the previous lemmas).

Proposition 2.10. *Under assumptions* (H1), (H2), (H3), (H4), *and* (H4′), *one can consider the operator E defined in* (2.15) *which extends in a suitable way a function from $\mathcal{Q}_+$ to $\mathcal{Q}$. Then for every $u \in \mathcal{V}_{\mathcal{Q}_+}$*

$$\|Eu\|^2_{\mathcal{V}} \leqslant 2^{2/p}\, \widetilde{C} \|u\|^2_{\mathcal{V}} \leqslant 2\, \widetilde{C} \|u\|^2_{\mathcal{V}}$$

and for every $u \in \mathcal{V}_{\mathcal{Q}_+}$ such that $(r_+ u)' \in \mathcal{V}'_+$, we have that $Eu \in \mathcal{W}_{Er_+}$ and

$$\|(Er_+ Eu)'\|^2_{\mathcal{V}'} \leqslant 2\, \widetilde{C} \Big(\|(ru)'\|^2_{\mathcal{V}'} + \|u\|^2_{\mathcal{V}} \Big)$$
$$\|Er_+ (Eu)'\|^2_{\mathcal{V}'} \leqslant 2\, \widetilde{C} \Big(\|ru'\|^2_{\mathcal{V}'} + \|u\|^2_{\mathcal{V}} \Big).$$

In all the three estimates the factor 2 *is due to the change of sign of r.*

3. The main result

In the previous section, we presented a simple situation, i.e., when it is needed only *one* function ϕ satisfying (H3), but more than one such a function could be needed. In this section we consider this more general situation.

We will still suppose that the set where r changes its sign satisfies

$$\mathcal{I} \ \text{ is a connected subset of } \mathcal{Q},$$

but we could need more than one function ϕ satisfying (H3). A typical example, if $\Omega \subset \mathbf{R}^2$, is if $I(t)$ is a circle. We recall that $\mathcal{I}, I(t), \mathcal{Q}, C, \mathcal{C}$ are defined at the beginning the previous section.

Fix $\epsilon > 0$ and, for every $t \in [0, T]$,

$$I^\epsilon(t) = \big\{ x \in \Omega \,\big|\, \mathrm{dist}\big(x, I(t)\big) < \epsilon \big\}$$

and denote

$$I^\epsilon_-(t) = \big\{ x \in I^\epsilon(t) \,\big|\, r(x,t) < 0 \big\}, \qquad I^\epsilon_+(t) = \big\{ x \in I^\epsilon(t) \,\big|\, r(x,t) > 0 \big\}$$

Moreover, denote by

$$\mathcal{I}^\epsilon := \bigcup_{t \in [0,T]} I^\epsilon(t), \quad \mathcal{I}^\epsilon_+ = I^\epsilon \cap \mathcal{Q}_+, \quad \mathcal{I}^\epsilon_- = I^\epsilon \cap \mathcal{Q}_-.$$

Notice that

$$\mathcal{I}^\epsilon \subseteq \big\{(x,t) \in \Omega \times (0,T) \,\big|\, \operatorname{dist}\big((x,t),\mathcal{I}\big) < \epsilon\big\}$$

and in general is a proper subset. An example is shown in the figure below, where Ω is an interval, time in the vertical direction and $\mathcal{Q}$ is a square. The curve divides $\mathcal{Q}$ into $\mathcal{Q}_+$ and $\mathcal{Q}_-$, no matter which is one and the other. The set $\mathcal{I}^\epsilon$ is the region between the two dashed lines and, as one can see, it may not be uniform in t. This is why we require (H5$'$) and (H4$'$), but we will discuss briefly this issue in the last section.

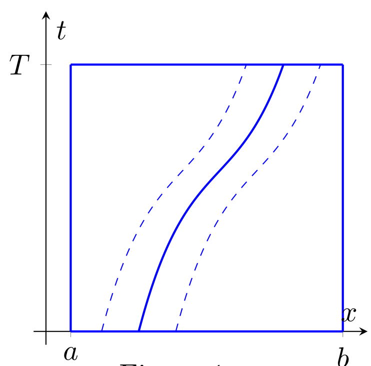

Figure 1.a

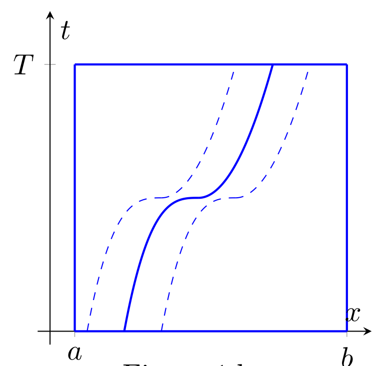

Figure 1.b

By the assumptions, we can find a finite partition of unity of $\mathcal{Q}$ made by $H+2$ functions, with bounded overlapping,

$$\eta_+ : \mathcal{U}_+ \to [0,1], \quad \eta_- : \mathcal{U}_- \to [0,1], \quad \eta_h : \mathcal{U}_h \to [0,1],\ h = 1, \dots H \qquad (3.1)$$

with

$$\eta_+^2(x,t) + \sum_{h=1}^{H} \eta_h^2(x,t) + \eta_-^2(x,t) = 1$$
$$\#\{x \in \Omega : \eta_h(x,t) \neq 0\} \leqslant A \qquad \text{for every } t \in [0,T].$$

and where

$$\mathcal{U}_+ \subset \mathcal{Q}_+ \setminus \mathcal{I}^{\epsilon/2}, \quad \mathcal{U}_- \subset \mathcal{Q}_- \setminus \mathcal{I}^{\epsilon/2}, \quad \mathcal{U}_h \subset \mathcal{I}^\epsilon,\ h = 1, \dots H,$$

and where, once denoted $U_h(t) := \mathcal{U}_h \cap \big(\Omega \times \{t\}\big)$. We suppose that

$$\bigcup_{h=1}^{H} \mathcal{U}_h = \mathcal{I}^\epsilon \qquad \text{and} \qquad \bigcup_{h=1}^{H} U_h(t) = I^\epsilon(t).$$

Finally, we define

$$\mathcal{U}_{h,+} := \mathcal{U}_h \cap \mathcal{Q}_+, \qquad \mathcal{U}_{h,-} := \mathcal{U}_h \cap \mathcal{Q}_-.$$

For every $h \in \{1, \dots H\}$, we suppose that there is ϕ_h ($\phi_h(y,s) = (\varphi_h(y,s), s)$) satisfying (H3), (H4), and (H4′) with constants independent of h. We will moreover suppose that

$$\begin{aligned} &\eta_h \in \ \mathrm{Lip}\,(\mathcal{Q}), \\ &|D\eta_h(x,t)| \leqslant \frac{2}{\epsilon}, \quad \text{a.e. in } \mathcal{Q}, \end{aligned} \tag{\textbf{H5}}$$

$$|r(x,t)\,\partial_t \eta_h(x,t)| \leqslant \hat{L} \quad \text{a.e. in } \mathcal{Q}. \tag{\textbf{H5}$'$}$$

Once defined

$$J = \big\{ y \in C : y = (0, y_2, \dots y_n),\, \tau \in [-1,1] \big\}$$

and for every $\bar{y} := (0, \bar{y}_2, \dots \bar{y}_n) \in J$

$$\Sigma_{\bar{y}} = \big\{ y \in C : y = (\tau, \bar{y}_2, \dots \bar{y}_n),\, \tau \in [-1,1] \big\},$$

thanks to assumption (H1),ST we can also suppose that

$$\begin{aligned} &\varphi_h(\cdot,t)\big(J\big) = I(t) \cap U_h(t) \\ &\varphi_h\big(\Sigma_{\bar{y}}, t\big) \ \text{ is a segment othogonal to } I(t) \text{ for every } \bar{y} \in J. \end{aligned} \tag{3.2}$$

as shown in the picture below, where in Figure 2.a, we represented the cube C, in Figure 2.b a possible $U_h(t)$.

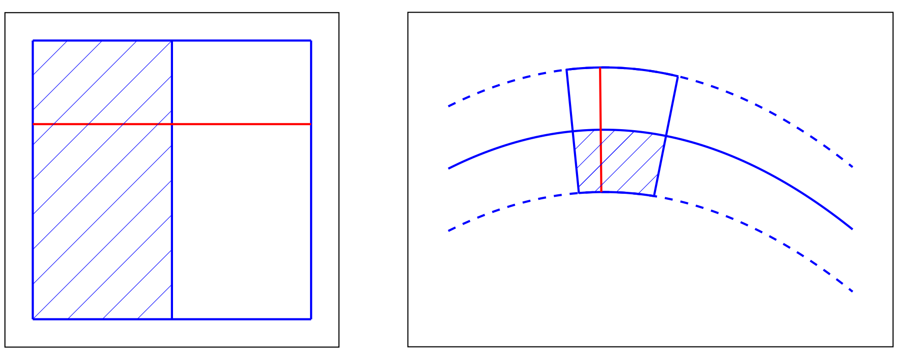

Figure 2.a Figure 2.b

Now, consider $u \in \mathcal{W}_r$. We define an operator E as follows: first, we consider

H extension operators

$$E_h v\,(x,t) = \begin{cases} v(x,t) & \text{for } (x,t) \in \mathcal{U}_{h,+} \\ v \circ \phi_h \circ S \circ \phi_h^{-1}(x,t) & \text{for } (x,t) \in \mathcal{U}_{h,-} \end{cases} \tag{3.3}$$

for a generic function $v : \mathcal{Q} \to \mathbf{R}$. Notice that

$$v \in \mathcal{W}_r \quad \Longrightarrow \eta_h v \in \mathcal{V}_h, \quad (r\eta_h v)' \in \mathcal{V}_h',$$

where

$$\mathcal{V}_h = \mathcal{V}_{\mathcal{U}_h} \qquad \text{(defined in (2.4))}.$$

By Proposition 2.10, we have

$$\big\|E_h\big(\eta_h u\big)\big\|^2_{\mathcal{V}} \leqslant 2\,\widetilde{C} H \sum_{h=1}^{H} \|\eta_h u\|^2_{\mathcal{V}} \leqslant 2\,\widetilde{C} H \Big(1 + \frac{2}{\epsilon}\Big)^2 \|u\|_{\mathcal{V}_h}.$$

Again by Proposition 2.10, we get

$$\begin{aligned} \|\big(E_h r_+ E_h(\eta_h u)\big)'\|^2_{\mathcal{V}'} &\leqslant 2\,\widetilde{C}\,\Big(\|(r\eta_h u)'\|^2_{\mathcal{V}'} + \|\eta_h u\|^2_{\mathcal{V}}\Big) \\ &\leqslant 2\,\widetilde{C}\,\Big[2\Big(1+\frac{2}{\epsilon}\Big)^2 \|(ru)'\|^2_{\mathcal{V}_h'} + 2\hat{L}^2\,\|u\|^2_{L^2(\mathcal{U}_h)} + \Big(1+\frac{2}{\epsilon}\Big)^2 \|u\|^2_{\mathcal{V}_h}\Big] \\ &\leqslant 2\,\widetilde{C}\,\Big[2\Big(1+\frac{2}{\epsilon}\Big)^2 \|(ru)'\|^2_{\mathcal{V}_h'} + \Big(2\hat{L}^2 + \Big(1+\frac{2}{\epsilon}\Big)^2\Big)\|u\|^2_{\mathcal{V}_h}\Big]. \end{aligned} \tag{3.4}$$

Similarly, again by Proposition 2.10,

$$\begin{aligned} &\|E_h r_+\big(E_h(\eta_h u)\big)'\|^2_{\mathcal{V}'} \\ &\quad\leqslant 2\,\widetilde{C}\,\left[2\left(1+\tfrac{2}{\epsilon}\right)^2 \|ru'\|^2_{\mathcal{V}_h'} + \Big(2\hat{L}^2 + \left(1+\tfrac{2}{\epsilon}\right)^2\Big)\|u\|^2_{\mathcal{V}_h}\right]. \end{aligned} \tag{3.5}$$

Theorem 3.1. *Suppose* (H1), (H2), (H3), (H4), (H4'), (H5), *and* (H5$'$) *to hold. If $u \in \mathcal{W}_r$, then the function*

$$[0,T] \ni t \mapsto \int_\Omega u^2(x,t)|r|(x,t)dx \tag{3.6}$$

is continuous and there is a constant $\widehat{C}_o$, depending only on n, K, $\tilde{K}$, N, $\tilde{N}$, M, m, $\Lambda, C_o, |\mathcal{Q}|^{\frac{p-2}{2p}}$, ϵ^{-1}, such that for every $t \in [0,T]$

$$\int_\Omega u^2(x,t)|r|(x,t)dx \leqslant 2\,\widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big], \tag{3.7}$$

$$\int_\Omega u^2(x,t)|r|(x,t)dx \leqslant 2\,\widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|ru'\|^2_{\mathcal{V}'}\Big].$$

Remark 3.2. If r is independent of t the continuity of the function in (3.6) could be stated in a simpler and more direct way as

$$\mathcal{W}_r \quad \text{continuously embeds in} \quad C^0([0,T];L^2(\Omega;|r|)),$$

where $C^0([0,T];L^2(\Omega;|r|))$ is the completion of $C^0_c(\Omega)$ with respect to the topology induced by the norm $\big(\int_\Omega u^2(x)|r|(x)\,dx\big)^{1/2}$.

Proof. We prove only the first inequality, proof for the seond one is the same. Since, by (2.3) we have that

$$\max_{t\in[0,T]}\Big|\int_\Omega u^2(x,t)r(x,t)dx\Big| \leqslant C_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big],$$

we conclude and get the thesis if we show that

$$\max_{t\in[0,T]}\int_\Omega u^2(x,t)r_+(x,t)dx \leqslant \widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big]$$

for some constant $\widehat{C}_o$. We have

$$\begin{aligned}
&\int_\Omega u^2(x,t)r_+(x,t)dx\\
&= \int_\Omega \eta_+^2 u^2(x,t)r_+(x,t)dx + \int_\Omega \Big(\sum_{h=1}^H\Big)\eta_h^2 u^2(x,t)r_+(x,t)dx\\
&= \int_{\Omega_+(t)} \eta_+^2 u^2(x,t)r_+(x,t)dx + \sum_{h=1}^H\int_\Omega \eta_h^2 u^2(x,t)r_+(x,t)dx.
\end{aligned}$$

Clearly, since η_+ is supported in $\mathcal{U}_+$ and $\eta_+ u\in\mathcal{W}_r$, we have

$$\begin{aligned}
\int_{\Omega_+(t)} \eta_+^2 u^2(x,t)r_+(x,t)dx &= \int_\Omega \eta_+^2 u^2(x,t)r(x,t)dx\\
&\leqslant C_o\Big[\|\eta_+ u\|^2_{\mathcal{V}} + \|(r\eta_+ u)'\|^2_{\mathcal{V}'}\Big].
\end{aligned}$$

Now, take into account the other terms of the sum. Notice that (2.3) holds also for $E_h r_+$ with some constant $C_{o,h}\leqslant C_o$. Then, by (3.4) and (3.5),

$$\begin{aligned}
&\int_\Omega \eta_h^2 u^2(x,t)r_+(x,t)dx \leqslant \int_\Omega \big(E_h\eta_h u\big)^2(x,t)E_h r_+(x,t)dx\\
&\leqslant C_o\Big[\|E_h(\eta_h u)\|^2_{\mathcal{V}} + \|(E_h r_+ E_h(\eta_h u))'\|^2_{\mathcal{V}'}\Big]\\
&\leqslant C_o\Big[2\,\widetilde{C}\,\|\eta_h u\|^2_{\mathcal{V}} + 2\,\widetilde{C}\,\|(r\eta_h u)'\|^2_{\mathcal{V}'}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_o\Big[2\,\widetilde{C}\Big(1+\frac{2}{\epsilon}\Big)^2\|u\|^2_{\mathcal{V}_h}\\
&\qquad + 2\,\widetilde{C}\,\Big[2\Big(1+\frac{2}{\epsilon}\Big)^2\|(ru)'\|^2_{\mathcal{V}'_h} + \Big(2\hat{L}^2 + \Big(1+\frac{2}{\epsilon}\Big)^2\Big)\|u\|^2_{\mathcal{V}_h}\Big]\Big]\\
&\leqslant 2\,\widetilde{C}_o\Big[\|u\|^2_{\mathcal{V}_h} + \|(ru)'\|^2_{\mathcal{V}'_h}\Big],
\end{aligned}
$$

where

$$\widetilde{C}_o = C_o\,\widetilde{C}\Big[2\hat{L}^2 + 2\Big(1+\frac{2}{\epsilon}\Big)^2\Big]$$

which depends on n, K, $\tilde{K}$, N, $\tilde{N}$, M, m, Λ, C_o, $|\mathcal{Q}|^{\frac{p-2}{2p}}$. Summing up, we have (remember that A is the maximum number of overlappings)

$$
\begin{aligned}
&\int_\Omega u^2(x,t)r_+(x,t)dx\\
&\leqslant (H+1)\,\Big[\Big[\|\eta_+ u\|^2_{\mathcal{V}} + \|(r\sum_{h=1}^{H} 2\,\widetilde{C}_o\Big[\|u\|^2_{\mathcal{V}_h} + \|(ru)'\|^2_{\mathcal{V}'_h}\Big]\Big]\\
&\leqslant (H+1)\Big[2\Big(1+\frac{2}{\epsilon}\Big)^2\|(ru)'\|^2_{\mathcal{V}'} + \Big(2\hat{L}^2 + \Big(1+\frac{2}{\epsilon}\Big)^2\Big)\|u\|^2_{\mathcal{V}}\\
&\qquad\qquad + 2\,A\,\widetilde{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big]\Big]\\
&= 2\,\widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big].
\end{aligned}
$$

As regards the continuity, by assumptions (H2), we get that

$$[0,T]\ni t\mapsto \int_\Omega u^2(x,t)r_+(x,t)dx \quad \text{is continuous}$$

for every $u\in C^1([0,T];W^{1,p}(\Omega)$ and then, by density, for every $u\in\mathcal{W}_r$. Since analogous arguments can be done to estimate $\int_\Omega u^2(x,t)r_-(x,t)dx$ and prove its continuity, we are done. □

4. A more general result

Suppose now to have more than one interface, i.e., $\mathcal{I}$ is not a connected set anymore, but

$$\mathcal{I} = \mathcal{I}_1\cup\ldots\cup\mathcal{I}_{\mathsf{N}}$$

for some integer $\mathsf{N}\geqslant 1$ and where each $\mathcal{I}_j$, $j=1,\ldots\mathsf{N}$, is a connected subset of $\mathcal{Q}$ and satisfies assumptions made above for $\mathcal{I}$ and $\mathcal{Q}$ is divided in $\mathsf{N}+1$ connected regions.

Suppose moreover there is $\epsilon > 0$ such that ($\mathcal{I}^\epsilon$ defined at the beginning of the previous section)

$$\mathrm{dist}(\mathcal{I}_j{}^\epsilon, \mathcal{I}_k{}^\epsilon) = \inf\left\{|x-y| : x \in \mathcal{I}_j{}^\epsilon, y \in \mathcal{I}_k{}^\epsilon\right\} > 0$$

for every j, k between 1 and N. Then the following is a direct consequence of Theorem 3.1.

The only thing to do is to prove that it is localize estimate around one interface and then sum all the N estimates. We do not know if these estimates are sharp, but in the next section, we have an example which shows that the constant in the right hand side must increase with the number of interfaces.

Theorem 4.1. *Suppose* (H1), (H2), (H3), (H4), (H4′), (H5), *and* (H5′) *hold. If* $u \in \mathcal{W}_r$, *then the function*

$$[0, T] \ni t \mapsto \int_\Omega u^2(x,t)|r|(x,t)dx$$

is continuous and there is a constant $\widehat{C}_o$, *depending only on* $n, K, \tilde{K}, N, \tilde{N}, M, m, \Lambda, C_o, |\mathcal{Q}|^{\frac{p-2}{2p}}$, ϵ^{-1}, *such that for every* $t \in [0,T]$

$$\begin{aligned} \int_\Omega u^2(x,t)|r|(x,t)dx &\leqslant 2\,\mathsf{N}\,\widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|(ru)'\|^2_{\mathcal{V}'}\Big], \\ \int_\Omega u^2(x,t)|r|(x,t)dx &\leqslant 2\,\mathsf{N}\,\widehat{C}_o\Big[\|u\|^2_{\mathcal{V}} + \|ru'\|^2_{\mathcal{V}'}\Big]. \end{aligned} \tag{4.1}$$

5. COMMENTS, EXAMPLES AND COUNTEREXAMPLES

In this section, we want to comment briefly some assumptions and to show some examples.

To simplify, we consider examples with $n = 1$. Suppose the domain is $[-a, a] \times [0, T]$ as in the figures below, suppose $r \neq 0$ almost everywhere and suppose that the two regions separated by a curve Γ in the set $[-a, a] \times [0, T]$ are the regions where r is positive and negative, no matter which is one and which is the other.

We focus our attention on the function φ defined in (H3) and in the assumption (H4′)

$$\left| r\big(\phi(y,s)\big)\frac{\partial \varphi}{\partial s}(y,s)\right| \leqslant L, \qquad \left| r(x,t)\,\frac{\partial \varphi^{-1}}{\partial t}(x,t)\right| \leqslant \tilde{L}. \tag{5.1}$$

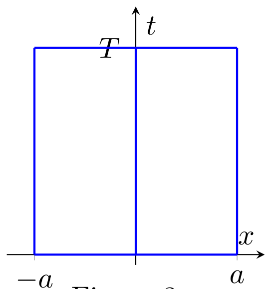

Figure 3.a

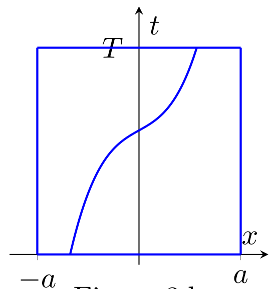

Figure 3.b

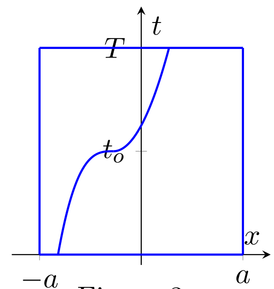

Figure 3.c

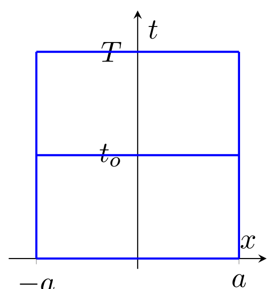

Figure 3.d

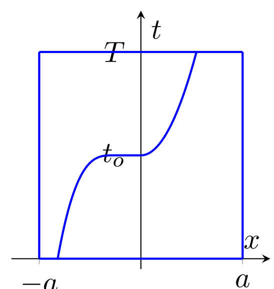

Figure 3.e

In Figure 3.a, there is the set $\mathcal{C}$, which could represent also a possible situation when r is *independent* of time and clearly in this case also φ does not depend on time and (5.1) is easily satisfied. This is the simplest situation.

The curve Γ in examples in Figure 3.b and Figure 3.c as it may be as the graph of a function $\gamma : [0, T] \to [-a, a]$. A possible choice for the function φ introduced in (H3) (suppose r is negative in the left hand side of $[-a, a] \times [0, T]$) is

$$\varphi : \mathcal{C}_- \to \mathcal{Q}_-, \qquad \varphi(y, s) = ya + (1 + y)\gamma(s)$$

by which

$$\frac{\partial \varphi}{\partial s}(y, s) = (1 + y)\gamma'(s).$$

If $\gamma \in C^1$ and γ' is bounded, (5.1) is satisfied, whatever the choice of r is. For instance, we could admit also discontinuous r like

$$r = 1 \text{ in one region} \qquad \text{and} \qquad r = -1 \text{ in the other one.} \tag{5.2}$$

If γ is differentiable everywhere except in a point t_o, as in the example in Figure 3.c, where

$$\lim_{t \to t_o} \gamma'(t) = +\infty,$$

the choice of r like in (5.2) is not possible. To have (5.1) the continuity of r through the interface where r changes its sign, at least in the point $(\gamma(t_o), t_o)$, is needed, i.e., $r(\gamma(t_o), t_o)$ has to be 0.

The example shown in Figure 3.d does not satisfy all the assumptions, since there is no a ϕ from $\mathcal{C}$ to $\mathcal{Q}$ satisfying (H3). Nevertheless, the result trivially holds anyway. Suppose for simplicity that r depends only on t. The assumption (H2) about r, in particular that (2.2) holds, implies that r is continuous (and in fact Lipschitz continuous), so that $r(t_o) = 0$. Then the estimate

$$\int_{-a}^{a} u^2(x,t)|r(t)|\, dx = |r(t)| \int_{-a}^{a} u^2(x,t)\, dx \\ \leqslant c\Big[\|u\|_{L^2(0,T;H^1(-a,a))} + \|(ru)'\|_{L^2(0,T;H^{-1}(-a,a))}\Big]$$

for $t = t_o$ becomes trivial and for $t < t_o$ and $t > t_o$ simply follows from (2.3), since in particular

$$u \in \{v \in L^2(0,T;H^1(-a,a)) \,|\, (ru)' \in L^2(0,T;H^{-1}(-a,a))\} \quad \Longrightarrow \\ u \in \{v \in L^2(0,t_o;H^1(-a,a)) \,|\, (ru)' \in L^2(0,t_o;H^{-1}(-a,a))\} \quad \text{and} \\ u \in \{v \in L^2(t_o,T;H^1(-a,a)) \,|\, (ru)' \in L^2(t_o,T;H^{-1}(-a,a))\}.$$

The example 3.e is more delicate, since we have a "flat part" in the interface, as in example 3.d, but in example 3.e, r depends both on x and t. The main problem is clearly for $t = t_o$ since $\varphi(\cdot,t)(C) = \Omega$ for every t except than for $t = t_o$. Our conjecture is that Theorem 3.1 holds also in this case, but it does not satisfy our assumptions.

The constant $\widehat{C}_o$. Suppose now to have more than one interface, as in the figure which follows, and suppose in the central strip r is positive and in two other regions is negative or vice versa.

In this case, in (3.7), we should have 4, and not 2, in front of $\widehat{C}_o$ because for every change of sign, we have a factor 2. Moreover, since $\widehat{C}_o$ depends on ϵ^{-1}, the constant may increase not only for the number of interfaces, but also depending on their proximity. For example, in the situation considered in Figure 4, one could prove the same result as in Section 3 and in particular the following estimates, as stated in Theorem 4.1:

$$\int_\Omega u^2(x,t)|r|(x,t)dx \leqslant 4\,\widehat{C}_o\big[\|u\|_{\mathcal{V}} + \|(ru)'\|_{\mathcal{V}'}\big], \\ \int_\Omega u^2(x,t)|r|(x,t)dx \leqslant 4\,\widehat{C}_o\big[\|u\|_{\mathcal{V}} + \|ru'\|_{\mathcal{V}'}\big].$$

What we want to stress is that the constant in (3.7) *does depend on* r, *but not on* $|r|$, since it depends on E, and may explode changing r even if $|r|$ does not change.

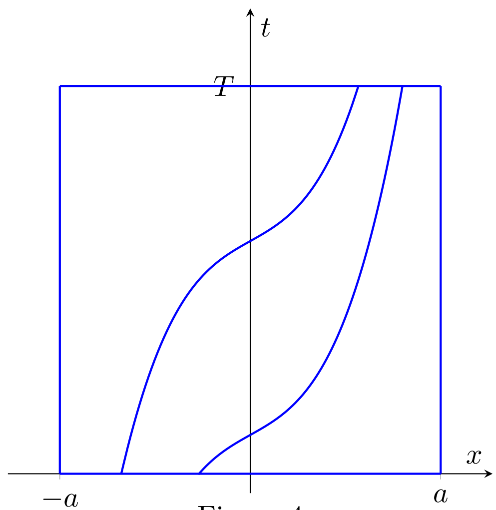

Figure 4

The following example (already shown in [11]) shows this fact: consider a function $\varphi \in H^1_0(-a,a)$ and $\psi_\alpha(t) = \sqrt{\alpha}\, e^{-\alpha t^2/2}$ consider the function $u_\alpha(x,t) = \psi_\alpha(t)\varphi(x)$. Now, consider the function $r(x)$ which is -1 in $(-a,0)$ and 1 in $(0,a)$, extend it periodically to the whole $\mathbf{R}$ and define $r_h(x) := r(hx)$ with $h \in \mathbf{N}$. Notice that $|r_h(x)| = 1$ for every $h \in \mathbf{N}$. Then u belongs to

$$\mathcal{W}_{r_h} := \{u \in L^2(-1,1;H^1_0(-a,a)) : r_h u' \in L^2(-1,1;H^{-1}(-a,a))\}$$

for every $h \in \mathbf{N}$. Moreover, notice that if we consider the function $|x|$ defined between $-a$ and a, and extend it periodically to the whole $\mathbf{R}$ and call this function s, then one has that $r(x) := s'(x)$ and, if we define $s_h(x) := s(hx)$ with $h \in \mathbf{N}$ and $x \in (-a,a)$, we also have that

$$r_h(x) = \frac{1}{h}\, s'_h(x).$$

We have that, being $r_h(u_\alpha)_t \in L^2$,

$$\| |r_h|\, u_\alpha \|^2_{C([-1,1];L^2(-a,a))} = \|u_\alpha\|^2_{C([-1,1];L^2(-a,a))} = \alpha\, \|\varphi\|^2_{L^2(-a,a)},$$

$$\|u_\alpha\|^2_{L^2(-1,1;H^1_0(-a,a))} = \alpha \int_{-1}^{1} e^{-\alpha t^2}\, dt \int_{-a}^{a} (\varphi'(x))^2\, dx \leqslant \sqrt{\alpha\, \pi}\, \|\varphi'\|^2_{L^2(-a,a)},$$

$$\|r_h(u_\alpha)_t\|^2_{L^2(-1,1;H^{-1}(-a,a))} = \frac{1}{h^2}\|s'_h\|_\infty \|\varphi\|^2_{L^2(-a,a)} \int_{-1}^{1} \left(\psi'_\alpha(t)\right)^2 dt$$
$$\leqslant \frac{1}{h^2}\,\|\varphi\|^2_{L^2(-a,a)} \frac{\alpha^{3/2}\sqrt{\pi}}{2} \to_{h\to+\infty} 0\,.$$

Then it is clear that it is not possible to have a constant C independ of h such that

$$\alpha\,\|\varphi\|^2_{L^2(-a,a)} \leqslant C\Big[\sqrt{\alpha\,\pi}\,\|\varphi'\|^2_{L^2(-a,a)} + \frac{1}{h^2}\,\|\varphi\|^2_{L^2(-a,a)} \frac{\alpha^{3/2}\sqrt{\pi}}{2}\Big].$$

Indeed letting first h go to $+\infty$ and then choosing α big enough, we can disprove the above inequality for whatever C.

Notice that, taking instead of a constant $\alpha = h^s$ for some positive power s one realizes that the right hand side is minimum for $s = 2$. Then taking $\alpha = h^2$, one gets

$$h\,\|\varphi\|^2_{L^2(-a,a)} \leqslant C\Big[\sqrt{\pi}\,\|\varphi'\|^2_{L^2(-a,a)} + \|\varphi\|^2_{L^2(-a,a)} \frac{\sqrt{\pi}}{2}\Big]$$

and this is true for every h only if C is proportional to h, i.e., to the number of points where r_h changes its sign. This is coherent with (4.1).

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