N-QUASI-ABELIAN CATEGORIES
VS
N-TILTING TORSION PAIRS

WITH AN APPLICATION TO FLOPS OF HIGHER RELATIVE DIMENSION

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ABSTRACT. It is a well established fact that the notions of quasi-abelian categories and tilting torsion pairs are equivalent. This equivalence fits in a wider picture including tilting pairs of t-structures.

Firstly, we extend this picture into a hierarchy of n-quasi-abelian categories and n-tilting torsion classes. We prove that any n-quasi-abelian category $\mathcal{E}$ admits a "derived" category $D(\mathcal{E})$ endowed with a n-tilting pair of t-structures such that the respective hearts are derived equivalent.

Secondly, we describe the hearts of these t-structures as quotient categories of coherent functors, generalizing Auslander's Formula.

Thirdly, we apply our results to Bridgeland's theory of perverse coherent sheaves for flop contractions. In Bridgeland's work, the relative dimension 1 assumption guaranteed that f_*-acyclic coherent sheaves form a 1-tilting torsion class, whose associated heart is derived equivalent to $D(Y)$. We generalize this theorem to relative dimension 2.

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INTRODUCTION

In [7, 3.3.1] Beilinson, Bernstein and Deligne introduced the notion of a t-structure obtained by tilting the natural one on $D(\mathcal{A})$ (derived category of an abelian category $\mathcal{A}$) with respect to a torsion pair $(\mathcal{X}, \mathcal{Y})$. In [24] Happel Reiten and Smalø developed this procedure, they proved the Tilting Theorem: whenever $\mathcal{X}$ is a *tilting torsion class* (1.2) on $\mathcal{A}$ there is a triangulated equivalence $D(\mathcal{H}) \simeq D(\mathcal{A})$ where $\mathcal{H}$ is the heart of the tilted t-structure.

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In [49] J.-P. Schneiders associated to any quasi-abelian category $\mathcal{E}$ (1.11) a triangulated category $D(\mathcal{E})$ endowed with a 1-tilting pair of t-structures $(\mathcal{R}, \mathcal{L})$ (1.9) such that $\mathcal{E} = \mathcal{H}_{\mathcal{R}} \cap \mathcal{H}_{\mathcal{L}}$.

Rump in [45], followed by Bondal and Van den Bergh in [9, App. B], established an equivalence between the previous notions: given an additive category $\mathcal{E}$, the following properties are equivalent: 1) $\mathcal{E}$ is a 1-quasi-abelian category, 2) $\mathcal{E}$ is a 1-tilting torsion class, 3) $\mathcal{E}$ is a 1-cotilting torsion-free class, 4) $\mathcal{E}$ is the intersection of the hearts of a 1-tilting pair of t-structures $(\mathcal{R}, \mathcal{L})$ on $D(\mathcal{E})$.

This paper contains three main results.

0.1. We propose an higher analog of the previous equivalence: given an additive category $\mathcal{E}$, the following properties are equivalent: 1) $\mathcal{E}$ is a n-quasi-abelian (6.6), 2) $\mathcal{E}$ is a n-tilting torsion class (6.7), 3) $\mathcal{E}$ is a 1-cotilting torsion-free class, 4) $\mathcal{E}$ is the intersection of the hearts of a n-tilting pair of t-structures $(\mathcal{R}, \mathcal{L})$ on $D(\mathcal{E})$.

In particular, we prove that the derived category of an n-quasi-abelian category $\mathcal{E}$ has two canonical t-structures (the left and the right one). We can view the hearts of these t-structures as canonical abelian envelopes for $\mathcal{E}$.

0.2. We establish a new description for the hearts of these t-structures as Gabriel quotients of the category of coherent functors with respect to a suitable Serre subcategory of effaceable functors (6.11). For an abelian (0-quasi-abelian) category $\mathcal{A}$ this result reduces to the Auslander's Formula introduced by H. Krause in [37].

0.3. Our main application is the generalization of Bridgeland's theory of perverse coherent sheaves. Let consider $Y \xrightarrow{f} X$ a flop contraction with X and Y varieties over $\mathbb{C}$, Y smooth and $Y^+ \xrightarrow{f^+} X$ its flop. The Bondal-Orlov conjecture predicts that the derived categories $D(Y)$ and $D(Y^+)$ (of coherent $\mathcal{O}$-modules) are equivalent. Bridgeland proved the Bondal-Orlov conjecture for threefolds ([12]) and Van den Bergh proposed a different proof relaxing some hypotheses, but always assuming that f has relative dimension 1 ([57]).

Bridgeland considered the t-structures on $D(Y)$ obtained by tilting the natural t-structure with respect to the 1-tilting torsion classes

$$\mathcal{T}_0 := \{T \in \mathrm{coh}(Y) \mid \mathbf{R}f_*T \simeq f^*T\} \quad \text{and} \quad \mathcal{T}_{-1} := \{T \in \mathrm{coh}(Y) \mid \eta_T : f^*f_*T \twoheadrightarrow T\}.$$

He denoted by ${}^0\mathrm{Per}(Y/X)$ and ${}^{-1}\mathrm{Per}(Y/X)$ their respective hearts. These categories form the first main ingredient in Bridgeland's and Van den Bergh's proofs of the Bondal-Orlov conjecture in the relative dimension 1 case. In these proofs the use of 1-tilting torsion classes is dictated by the geometry of the problem since the fibers of the flops have dimension ≤ 1. In the case of relative dimension n

$$\mathcal{T}_0 := \{T \in \mathrm{coh}(Y) \mid \mathbf{R}f_*T \simeq f^*T\} \quad \text{and} \quad \mathcal{T}_{-1} := \{T \in \mathcal{T}_0 \mid \eta_T : f^*f_*T \twoheadrightarrow T\}$$

are n analogues of the previous classes. We prove that for $n = 2$, these are 2-tilting torsion classes. We denote by ${}^i\mathrm{Per}(Y/X))$ the respective hearts and we prove that

$$D(Y) \simeq D({}^0\mathrm{Per}(Y/X)) \simeq D({}^{-1}\mathrm{Per}(Y/X)).$$

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1. 1-tilting torsion classes

In what follows any full subcategory $\mathcal{C}'$ of an additive category $\mathcal{C}$ will be strictly full (i.e., closed under isomorphisms) and additive. We will use the notation $\mathcal{C}' \subseteq \mathcal{C}$ to indicate such a subcategory. Any functor between additive categories will be an additive functor.

1.1. Torsion pairs in abelian categories ([18]). A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{A}$ such that: $\mathcal{A}(X, Y) = 0$, for every $X \in \mathcal{X}$ (*torsion class*) and $Y \in \mathcal{Y}$ (*torsion-free class*), and $\forall C \in \mathcal{A}$ there exists a short exact sequence $0 \to X \to C \to Y \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Hence the "inclusion" functor $i_{\mathcal{X}} : \mathcal{X} \to \mathcal{A}$ has a right adjoint τ, while $i_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{A}$ has a left adjoint ϕ. The class $\mathcal{X}$ (resp. $\mathcal{Y}$) is closed under extensions, quotients (resp. subobjects) representable direct sums (resp. direct products). As observed in [9, 5.4] both $\mathcal{X}$ and $\mathcal{Y}$ admit kernels and cokernels such that: $\mathrm{Ker}_{\mathcal{X}} = \tau \circ \mathrm{Ker}_{\mathcal{A}}$, $\mathrm{Coker}_{\mathcal{X}} = \mathrm{Coker}_{\mathcal{A}}$; $\mathrm{Ker}_{\mathcal{Y}} = \mathrm{Ker}_{\mathcal{A}}$ and $\mathrm{Coker}_{\mathcal{Y}} = \phi \circ \mathrm{Coker}_{\mathcal{A}}$.

Definition 1.2. ([24]) A torsion pair $(\mathcal{X}, \mathcal{Y})$ is called *tilting* if $\mathcal{X}$ *cogenerates* $\mathcal{A}$ (i.e., every object in $\mathcal{A}$ is a subobject of an object in $\mathcal{X}$) and $\mathcal{X}$ is called a 1-*tilting torsion class* (in $\mathcal{A}$). Dually $(\mathcal{X}, \mathcal{Y})$ is *cotilting* if $\mathcal{Y}$ *generates* $\mathcal{A}$ (i.e., every object in $\mathcal{A}$ is a quotient of an object in $\mathcal{Y}$) and $\mathcal{Y}$ is called a 1-*cotilting torsion-free class.*

Lemma 1.3. *Let $\mathcal{A}$ be an abelian category. The full subcategory $\mathcal{E} \overset{i_{\mathcal{E}}}{\hookrightarrow} \mathcal{A}$ is a 1-tilting torsion class if and only if*

(1) *$\mathcal{E}$ cogenerates $\mathcal{A}$;*
(2) *$\mathcal{E}$ is closed under extensions in $\mathcal{A}$;*
(3) *$\mathcal{E}$ is closed under representable direct sums in $\mathcal{A}$;*
(4) *for any exact sequence $0 \to A \to X \to B \to 0$ in $\mathcal{A}$ with $X \in \mathcal{E}$ and $A, B \in \mathcal{A}$ we have $B \in \mathcal{E}$;*
(5) *$\mathcal{E}$ has kernels.*

Proof. Any tilting torsion class satisfies these conditions. On the other side let $\mathcal{E} \overset{i_{\mathcal{E}}}{\hookrightarrow} \mathcal{A}$ be a full subcategory satisfying the previous conditions. Hence, by the first property, we can co-present any $A \in \mathcal{A}$ as $A = \mathrm{Ker}_{\mathcal{A}} f$ with $X_1 \overset{f}{\to} X_2$ and $X_i \in \mathcal{E}$ for $i = 1, 2$ and so, since the functor $\text{Mod-}\mathcal{E} \ni \mathcal{A}(i_{\mathcal{E}}(\_), A) \cong \mathcal{E}(\_, \mathrm{Ker}_{\mathcal{E}} f)$, we can define $\tau(A) := \mathrm{Ker}_{\mathcal{E}} f$ (using the last property) which gives a right adjoint for the functor $i_{\mathcal{E}}$. We remark that the functoriality of this construction is guaranteed by the fact that if we change the co-presentation of A as $A = \mathrm{Ker}_{\mathcal{A}}\, g$ with $Y_1 \overset{g}{\to} Y_2$ and $Y_i \in \mathcal{E}$ for $i = 1, 2$ there exists a unique isomorphism $\mathrm{Ker}_{\mathcal{E}} f \overset{\phi}{\simeq} \mathrm{Ker}_{\mathcal{E}}\, g$ such that the following triangle commutes:

$$\begin{tikzcd} \mathrm{Ker}_{\mathcal{E}} f \arrow[rr, "\phi"] \arrow[dr] & & \mathrm{Ker}_{\mathcal{E}}\, g \arrow[dl] \\ & A & \end{tikzcd}$$

The fourth property implies that for any $A \in \mathcal{A}$ the co-unit of the adjunction $\varepsilon_A : i_{\mathcal{E}}\tau(A) \to A$ is a monomorphism. So for any $A \in \mathcal{A}$ we have a short exact sequence $0 \to i_{\mathcal{E}}\tau(A) \overset{\varepsilon_A}{\to} A \to \mathrm{Coker}(\varepsilon_A) \to 0$. Moreover $\mathrm{Coker}_{\mathcal{A}}(\varepsilon_A) \in \mathcal{E}^{\perp}$ (see C.1 for the notion of orthogonal class) since given any morphism $f : E \to \mathrm{Coker}_{\mathcal{A}}(\varepsilon_A)$ with $E \in \mathcal{E}$ its $\mathcal{A}$ pull-back $A \times_{\mathrm{Coker}_{\mathcal{A}}(\varepsilon_A)} E$ belongs to $\mathcal{E}$ (by the second property since it is an extension of E by $i_{\mathcal{E}}\tau(A)$), hence the pull-back morphism $f' : A \times_{\mathrm{Coker}_{\mathcal{A}}(\varepsilon_A)} E \to A$ factors (by adjunction) through $i_{\mathcal{E}}\tau(A)$ which implies that $f = 0$. $\square$

Corollary 1.4. *Let $\mathcal{A}$ be a well powered abelian category with arbitrary direct sums. The full subcategory $\mathcal{E} \overset{i_{\mathcal{E}}}{\hookrightarrow} \mathcal{A}$ is a 1-tilting torsion class if and only if*

(1) *$\mathcal{E}$ cogenerates $\mathcal{A}$;*
(2) *$\mathcal{E}$ is closed under extensions in $\mathcal{A}$;*
(3) *$\mathcal{E}$ is closed under direct sums in $\mathcal{A}$;*
(4) *for any exact sequence $0 \to A \to X \to B \to 0$ in $\mathcal{A}$ with $X \in \mathcal{E}$ and $A, B \in \mathcal{A}$ we have $B \in \mathcal{E}$.*

We note that the torsion pair $(\mathcal{A}, 0)$ in an abelian category $\mathcal{A}$ is tilting while $(0, \mathcal{A})$ is cotilting. So the identity $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ represents $\mathcal{A}$ as a 1-tilting torsion class and also as a 1-cotilting torsion-free class.

We will refer to Appendix C for some generalities on t-structures. In particular in order to assure that any category introduced in this work has Hom sets we will suppose in the whole paper the following:

1.5. Hypothesis HS. Given $\mathcal{E}$ a *projectively complete*[1] category (i.e., additive category such that any idempotent splits) its derived category $D(\mathcal{E}) := D(\mathcal{E}, \mathcal{E}x_{\max})$ (endowed with its maximal Quillen exact structure see Appendix A) has Hom sets.

In the following we will always suppose that $\mathcal{E}$ is a projectively complete category.

1.6. Happel-Reiten-Smalø tilted t-structure. [24, Prop. I.2.1, Prop. I.3.2] [13, Prop. 2.5]. Let $\mathcal{H}_{\mathcal{D}}$ be the heart of a non degenerate t-structure $\mathcal{D} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{C}$ and let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair on $\mathcal{H}_{\mathcal{D}}$. Then the pair $\mathcal{T} := (\mathcal{T}^{\leq 0}_{(\mathcal{X},\mathcal{Y})}, \mathcal{T}^{\geq 0}_{(\mathcal{X},\mathcal{Y})})$ of full subcategories of $\mathcal{C}$

$$\begin{aligned} \mathcal{T}^{\leq 0}_{(\mathcal{X},\mathcal{Y})} = &\quad \{C \in \mathcal{C} \mid H^0_{\mathcal{D}}(C) \in \mathcal{X},\ H^i_{\mathcal{D}}(C) = 0\ \forall i > 0\} \\ \mathcal{T}^{\geq 0}_{(\mathcal{X},\mathcal{Y})} = &\quad \{C \in \mathcal{C} \mid H^{-1}_{\mathcal{D}}(C) \in \mathcal{Y},\ H^i_{\mathcal{D}}(C) = 0\ \forall i < -1\} \end{aligned}$$

is a t-structure on $\mathcal{C}$. Following [13] we say that $\mathcal{T}$ is obtained *by right tilting* $\mathcal{D}$ *with respect to the torsion pair* $(\mathcal{X}, \mathcal{Y})$ while the t-structure $\overline{\mathcal{T}} := \mathcal{T}[-1]$ is called the t-structure obtained *by left tilting* $\mathcal{T}$ *with respect to the torsion pair* $(\mathcal{X}, \mathcal{Y})$. The right tilted heart is:

$$\mathcal{H}_{\mathcal{T}} = \{C \in \mathcal{C} \mid H^0_{\mathcal{D}}(C) \in \mathcal{X},\ H^{-1}_{\mathcal{D}}(C) \in \mathcal{Y},\ H^i_{\mathcal{D}}(C) = 0\ \forall i \notin \{-1, 0\}\}.$$

In this paper we simply call tilting the right one. In [43, Lem. 1.1.2] Polishchuk proved that given any pair of t-structures $(\mathcal{D}, \mathcal{T})$ on a triangulated category $\mathcal{C}$ such that $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$, the t-structure $\mathcal{T}$ is obtained by right tilting $\mathcal{D}$ with respect to the torsion pair $(\mathcal{X} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}, \mathcal{H}_{\mathcal{T}}[-1] \cap \mathcal{H}_{\mathcal{D}} =: \mathcal{Y})$ while $\mathcal{D}$ is obtained by left tilting $\mathcal{T}$ with respect to the tilted torsion pair $(\mathcal{Y}[1] = \mathcal{H}_{\mathcal{D}}[1] \cap \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}} =: \mathcal{X})$.

1.7. Notation. In this paper whenever we have a pair of t-structures $(\mathcal{D}, \mathcal{T})$ on a triangulated category $\mathcal{C}$ we will denote by $\delta^{\leq 0}$ the truncation functor with respect to $\mathcal{D}$ and by $\tau^{\leq 0}$ the one with respect to $\mathcal{T}$.

Theorem 1.8. 1-Tilting Theorem. *(*[24, Th. I.3.3]*,* [15]*). Given a tilting torsion pair $(\mathcal{E}, \mathcal{Y})$ in $\mathcal{A}$ there exists a triangle equivalence $D(\mathcal{H}_{\mathcal{T}}) \overset{\simeq}{\to} D(\mathcal{A})$ (where $\mathcal{H}_{\mathcal{T}}$ is the heart of the t-structure obtained by right tilting the natural t-structure with respect to the torsion pair $(\mathcal{E}, \mathcal{Y})$) which is compatible with the natural inclusion $\mathcal{H}_{\mathcal{T}} \subseteq D(\mathcal{A})$. Moreover $(\mathcal{Y}[1], \mathcal{E})$ is a cotilting torsion pair in $\mathcal{H}_{\mathcal{T}}$.*

Definition 1.9. A pair of t-structures $(\mathcal{D}, \mathcal{T})$ on a triangulated category $\mathcal{C}$ is called 1-*tilting* if the following two conditions hold:

(1) $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$;
(2) denoting by $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$, the following equivalent conditions are satisfied:
  - **(i):** $\mathcal{C} \simeq K(\mathcal{E})/\mathcal{N}$ and $D(\mathcal{H}_{\mathcal{D}}) \overset{\simeq}{\hookleftarrow} K(\mathcal{E})/\mathcal{N} \overset{\simeq}{\hookrightarrow} D(\mathcal{H}_{\mathcal{T}})$ where $\mathcal{N}$ is the null system of complexes in $K(\mathcal{E})$ acyclic in $\mathcal{H}_{\mathcal{D}}$ or equivalently in $\mathcal{H}_{\mathcal{T}}$;
  - **(ii):** $\mathcal{C} \simeq D(\mathcal{H}_{\mathcal{D}})$ and $\mathcal{E}$ cogenerates $\mathcal{H}_{\mathcal{D}}$;
  - **(iii):** $\mathcal{C} \simeq D(\mathcal{H}_{\mathcal{T}})$ and $\mathcal{E}$ generates $\mathcal{H}_{\mathcal{T}}$.

Proposition 1.10. *The pair $(\mathcal{D}, \mathcal{T})$ is a* 1*-tilting pair of t-structures if and only if $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ is a* 1*-tilting torsion class (resp.* 1*-tilting torsion-free class) in $\mathcal{H}_{\mathcal{D}}$ (resp. in $\mathcal{H}_{\mathcal{T}}$).*

[1] A skeletally small projectively complete category is called a variety of annuli in [4].

Proof. One implication is a consequence of the 1-Tilting Theorem 1.8: if $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ is a 1-tilting torsion class (resp. 1-tilting torsion-free class) in $\mathcal{H}_{\mathcal{D}}$ (resp. in $\mathcal{H}_{\mathcal{T}}$) we obtain that $(\mathcal{D}, \mathcal{T})$ is a 1-tilting pair of t-structures. On the other side if $(\mathcal{D}, \mathcal{T})$ is a 1-tilting pair of t-structures by [43, Lem. 1.1.2] $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ is a torsion class in $\mathcal{H}_{\mathcal{D}}$ so we have only to prove that $\mathcal{E}$ cogenerates $\mathcal{H}_{\mathcal{D}}$. By hypothesis $K(\mathcal{E})/\mathcal{N} \overset{\simeq}{\hookrightarrow} D(\mathcal{H}_{\mathcal{D}})$ so any $A \in \mathcal{H}_{\mathcal{D}}$ can be represented by a complex $E^\bullet \in K(\mathcal{E})$, hence $A \hookrightarrow \operatorname{Coker}_{\mathcal{H}_{\mathcal{D}}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ (and $\operatorname{Coker}_{\mathcal{H}_{\mathcal{D}}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ since it is a quotient of a torsion object in $\mathcal{H}_{\mathcal{D}}$). Dually if $K(\mathcal{E})/\mathcal{N} \overset{\simeq}{\hookrightarrow} D(\mathcal{H}_{\mathcal{T}})$ we have that $\mathcal{E}$ generates $\mathcal{H}_{\mathcal{T}}$ and it is a torsion-free class in $\mathcal{H}_{\mathcal{T}}$. $\square$

Definition 1.11. ([49]). An additive category $\mathcal{E}$ is called 1-*quasi-abelian* if it admits kernels and cokernels, and any push-out of a kernel is a kernel, and any pullback of a cokernel is a cokernel. A zero sequence $0 \to E \overset{u}{\to} F \overset{v}{\to} G \to 0$ is called *exact* if and only if (E, u) is the kernel of v and (G, v) is the cokernel of u. A complex $X^\bullet$ with entries in $\mathcal{E}$ is called *acyclic* if each differential $d^n : X^n \to X^{n+1}$ decomposes in $\mathcal{E}$ as $d^n = m_n \circ e_n :\ X^n \overset{e_n}{\twoheadrightarrow} D^n \overset{m_n}{\rightarrowtail} X^{n+1}$ where m_n is the kernel of e_{n+1}, and e_{n+1} is the cokernel of m_n for any $n \in \mathbb{Z}$.

Remark 1.12. The class of kernel-cokernel exact sequences provides the maximal Quillen exact structure on $\mathcal{E}$ if and only if $\mathcal{E}$ is 1-quasi-abelian (see A.1 for the notion of maximal Quillen exact structure).

1.13. Left and Right t-structures on the derived category of a quasi-abelian category ([49, §1.2]). Let $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ (resp. $\mathcal{RK}^{\geq 0}_{\mathcal{E}}$) denote the full subcategory of $K(\mathcal{E})$ formed by complexes which are isomorphic in $K(\mathcal{E})$ to complexes whose entries in each strictly positive (resp. strictly negative) degree are zero. Let now suppose that $\mathcal{E}$ admits kernels and cokernels, hence the pairs $\mathcal{LK}_{\mathcal{E}} := (\mathcal{LK}^{\leq 0}_{\mathcal{E}}, (\mathcal{LK}^{\leq -1}_{\mathcal{E}})^{\perp})$ and $\mathcal{RK}_{\mathcal{E}} := ({}^{\perp}(\mathcal{RK}^{\geq 1}_{\mathcal{E}}), \mathcal{RK}^{\geq 0}_{\mathcal{E}})$ define two t-structures on $K(\mathcal{E})$ whose truncation functors are resp.:

$$\begin{array}{rcccccccc}
\tau^{\leq 0}_{\mathcal{L}} E^\bullet := \cdots \longrightarrow & E^{-2} & \longrightarrow & E^{-1} & \longrightarrow & \overset{\bullet}{\operatorname{Ker}_{\mathcal{E}} d^0} & \longrightarrow & 0 \longrightarrow \cdots \\
\tau^{\geq 1}_{\mathcal{L}} E^\bullet := \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{Ker}_{\mathcal{E}} d^0 & \longrightarrow & \overset{\bullet}{E^0} & \longrightarrow & E^1 \longrightarrow \cdots \\
\tau^{\leq -1}_{\mathcal{R}} E^\bullet := \cdots \longrightarrow & E^{-1} & \longrightarrow & \overset{\bullet}{E^0} & \longrightarrow & \operatorname{Coker}_{\mathcal{E}} d^{-1} & \longrightarrow & 0 \longrightarrow \cdots \\
\tau^{\geq 0}_{\mathcal{R}} E^\bullet := \cdots \longrightarrow & 0 & \longrightarrow & \overset{\bullet}{\operatorname{Coker}_{\mathcal{E}} d^{-1}} & \longrightarrow & E^1 & \longrightarrow & E^2 \longrightarrow \cdots
\end{array}$$

(as in C.2 we use a point to indicate the object placed in degree 0). The left t-structure $\mathcal{LK}_{\mathcal{E}}$ is the one considered by Schneiders in [49, Prop. 1.2.4]. We will denote by $\mathcal{LK}(\mathcal{E})$ (resp. $\mathcal{RK}(\mathcal{E})$) the heart associated to the t-structure $\mathcal{LK}_{\mathcal{E}}$ (resp. $\mathcal{RK}_{\mathcal{E}}$). We have $\mathcal{E} \simeq \mathcal{LK}^{\leq 0}_{\mathcal{E}} \cap \mathcal{RK}^{\geq 0}_{\mathcal{E}} = \mathcal{LK}(\mathcal{E}) \cap \mathcal{RK}(\mathcal{E})$ in $K(\mathcal{E})$ and moreover $\mathcal{RK}^{\leq -2}_{\mathcal{E}} \subseteq \mathcal{LK}^{\leq 0}_{\mathcal{E}} \subseteq \mathcal{RK}^{\leq 0}_{\mathcal{E}}$ (since for any $E^\bullet \in K(\mathcal{E})$ its $\tau^{\leq -2}_{\mathcal{R}}(E^\bullet) \in \mathcal{LK}^{\leq 0}_{\mathcal{E}}$). In $K(\mathcal{E})$ we have that $\mathcal{RK}^{\leq -1}_{\mathcal{E}}$ is contained in $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ if and only if any cokernel map is a split epimorphism or equivalently any kernel map is a split monomorphism. If this is not the case in order to reduce the "gap" ([21, Def. 2.1]) between the left and the right t-structures (without changing the intersection $\mathcal{E}$) we can try to localize by a null system formed by acyclic complexes with respect to a Quillen exact structure. In this case, if the previous t-structures satisfy the conditions of Lemma C.4, they will induce a pair of t-structures $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ on the localized category $D(\mathcal{E}, \mathcal{E}x)$. In order to obtain $\mathcal{RD}^{\leq -1}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{RD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ we need to prove that for any $E^\bullet \in D(\mathcal{E}, \mathcal{E}x)$ the canonical morphism of complexes

$\alpha_{E^\bullet} : \tau_{\mathcal{L}}^{\leq 0}(\tau_{\mathcal{R}}^{\leq -1}E^\bullet) \to \tau_{\mathcal{R}}^{\leq -1}E^\bullet$ is an isomorphism in $D(\mathcal{E}, \mathcal{E}x)$:

$$\begin{tikzcd}
\tau_{\mathcal{L}}^{\leq 0}(\tau_{\mathcal{R}}^{\leq -1}E^\bullet) := \cdots \arrow[r] \arrow[d, "\alpha_{E^\bullet}"'] & E^{-1} \arrow[r] \arrow[d] & \overset{\bullet}{\mathrm{Im}_{\mathcal{E}}}(d^{-1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\mathcal{R}}^{\leq -1}E^\bullet := \cdots \arrow[r] & E^{-1} \arrow[r, "d^{-1}"] & \overset{\bullet}{E^0} \arrow[r] & \mathrm{Coker}_{\mathcal{E}}(d^{-1}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which is equivalent to require the acyclicity of the mapping cone $M(\alpha_{E^\bullet})$ (which is homotopically isomorphic to $\mathrm{Ex}(d^0)$):

$$\begin{tikzcd}
M(\alpha_{E^\bullet}) := \cdots \arrow[r] \arrow[d, "\cong"] & E^{-2} \oplus E^{-1} \arrow[r] \arrow[d] & E^{-1} \oplus \mathrm{Im}_{\mathcal{E}}(d^{-1}) \arrow[r] \arrow[d] & \overset{\bullet}{E^0} \arrow[r] \arrow[d] & \mathrm{Coker}_{\mathcal{E}}(d^{-1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\mathrm{Ex}(d^0) := \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Im}_{\mathcal{E}}(d^{-1}) \arrow[r] & \overset{\bullet}{E^0} \arrow[r] & \mathrm{Coker}_{\mathcal{E}}(d^{-1}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Hence we would like to use a null system containing the complexes $\mathrm{Ex}(d^0)$ for any $d^0 : E^0 \to E^1$ which is possible if and only if these short exact sequences satisfy the axioms of a Quillen exact structure. Therefore if $\mathcal{E}$ is a 1-quasi-abelian category the previous truncation functors induce, by [49, Lem. 1.2.17; 1.18] (see Lemma C.4 and Lemma 3.11), the t-structure $\mathcal{LD}_{\mathcal{E}}$ (resp. $\mathcal{RD}_{\mathcal{E}}$) in the derived category $D(\mathcal{E}) = K(\mathcal{E})/\mathcal{N}$. Moreover, since $0 \to \mathrm{Im}_{\mathcal{E}}(d^{-1}) \to E^0 \to \mathrm{Coker}_{\mathcal{E}}(d^{-1}) \to 0$ is a kernel-cokernel exact sequence, it is exact for the maximal Quillen exact structure on $\mathcal{E}$, hence $\mathcal{RD}_{\mathcal{E}}^{\leq -1} \subseteq \mathcal{LD}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RD}_{\mathcal{E}}^{\leq 0}$ and $\mathcal{E} = \mathcal{LD}_{\mathcal{E}}^{\leq 0} \cap \mathcal{RD}_{\mathcal{E}}^{\geq 0}$. The t-structure $\mathcal{LD}_{\mathcal{E}}$ (resp. $\mathcal{RD}_{\mathcal{E}}$) is called the *left t-structure* (resp. the *right t-structure*), whose aisle $\mathcal{LD}_{\mathcal{E}}^{\leq 0}$ (resp. co-aisle $\mathcal{RD}_{\mathcal{E}}^{\geq 0}$) is the class of complexes isomorphic in $D(\mathcal{E})$ to complexes whose entries in each strictly positive (resp. negative) degree are zero. The heart of $\mathcal{LD}_{\mathcal{E}}$ (resp. $\mathcal{RD}_{\mathcal{E}}$) is denoted by $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) and we denote by $I_{\mathcal{L}}$ (resp. $I_{\mathcal{R}}$) the canonical embedding into $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$)

$$I_{\mathcal{L}} : \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{LH}(\mathcal{E}) \\ E & \longmapsto & 0 \to \overset{\bullet}{E} \end{array} \qquad I_{\mathcal{R}} : \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{RH}(\mathcal{E}) \\ E & \longmapsto & \overset{\bullet}{E} \to 0 \end{array}$$

which preserves and reflects exact sequences. Moreover $\mathcal{E}$ is stable under extensions in $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$).

Proposition 1.14. *Let $\mathcal{E}$ be a 1-quasi-abelian category. The t-structures $\mathcal{LD}_{\mathcal{E}} = \mathcal{RD}_{\mathcal{E}}$ coincide if and only if $\mathcal{E}$ is an abelian category.*

Proof. Let $\mathcal{E}$ be a 1-quasi-abelian category, $\mathcal{LD}_{\mathcal{E}} = \mathcal{RD}_{\mathcal{E}}$ if and only if for any complex $E^\bullet \in D(\mathcal{E})$ the canonical map $\beta_{E^\bullet} : \tau_{\mathcal{L}}^{\leq 0}E^\bullet \to \tau_{\mathcal{R}}^{\leq 0}E^\bullet \cong \tau_{\mathcal{L}}^{\leq 1}\tau_{\mathcal{R}}^{\leq 0}E^\bullet$

$$\begin{tikzcd}
\tau_{\mathcal{L}}^{\leq 0}E^\bullet := \cdots \arrow[r] \arrow[d, "\beta_{E^\bullet}"'] & E^{-1} \arrow[r] \arrow[d] & \overset{\bullet}{\mathrm{Ker}_{\mathcal{E}}}\, d^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\mathcal{L}}^{\leq 1}\tau_{\mathcal{R}}^{\leq 0}E^\bullet = \cdots \arrow[r] & E^{-1} \arrow[r, "d^{-1}"] & \overset{\bullet}{E^0} \arrow[r] & \mathrm{Im}_{\mathcal{E}}(d^0) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is an isomorphism in $D(\mathcal{E})$ which holds true if and only if the short sequence $0 \to \mathrm{Ker}_{\mathcal{E}}\, d^0 \to E^0 \to \mathrm{Im}_{\mathcal{E}}(d^0) \to 0$ is exact on $\mathcal{E}$ i.e.; $\mathcal{E}$ is an abelian category. $\square$

Theorem 1.15. [45], [9, Prop. B.3]. *Let $\mathcal{E}$ be an additive category. The following properties are equivalent:*

1. *$\mathcal{E}$ is a 1-cotilting torsion-free class in an abelian category $\mathcal{A}$;*
2. *$\mathcal{E}$ is a 1-tilting torsion class in an abelian category $\mathcal{A}'$;*
3. *$\mathcal{E}$ is a 1-quasi-abelian category;*
4. *$\mathcal{E}$ is the intersection of the hearts $\mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ of a 1-tilting pair of t-structures.*

Moreover $\mathcal{A} \simeq \mathcal{LH}(\mathcal{E})$, $\mathcal{A}' \simeq \mathcal{RH}(\mathcal{E})$ and $(\mathcal{D}, \mathcal{T}) = (\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}})$.

Proof. The equivalence between (1), (2) and (4) is a consequence of Theorem 1.8 and Proposition 1.10. Given $\mathcal{E}$ a 1-quasi-abelian category as recovered in 1.13 Schneiders proved that $(\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}})$ is a 1-tilting pair of t-structures with $\mathcal{LH}(\mathcal{E}) \cap \mathcal{RH}(\mathcal{E}) \simeq \mathcal{E}$, so (3) implies (4). On the other direction given any 1-tilting pair of t-structures $(\mathcal{D}, \mathcal{T})$ by Proposition 1.10 the class $\mathcal{E} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}$ is a tilting torsion class in $\mathcal{H}_{\mathcal{D}}$, hence a 1-quasi-abelian category and thus (4) implies (3). □

We have seen in 1.1 that given any torsion pair $(\mathcal{X}, \mathcal{Y})$ in an abelian category $\mathcal{A}$ both $\mathcal{X}$ and $\mathcal{Y}$ are 1-quasi-abelian categories, so in particular $\mathcal{X}$ is a 1-tilting torsion class after a suitable replacement of the abelian category:

Proposition 1.16. *Let $(\mathcal{X}, \mathcal{Y})$ be any torsion pair in an abelian category $\mathcal{A}$. Let consider $\mathcal{A}_{\mathcal{X}}$ to be the full subcategory of $\mathcal{A}$ whose objects are cogenerated by $\mathcal{X}$. Then $\mathcal{A}_{\mathcal{X}}$ is abelian, the canonical embedding functor $\mathcal{A}_{\mathcal{X}} \hookrightarrow \mathcal{A}$ is exact and the pair $(\mathcal{X}, \mathcal{Y} \cap \mathcal{A}_{\mathcal{X}})$ is a* 1*-tilting torsion pair in $\mathcal{A}_{\mathcal{X}}$ therefore $\mathcal{A}_{\mathcal{X}} \simeq \mathcal{RH}(\mathcal{X})$.*
Dually let consider $\mathcal{A}_{\mathcal{Y}}$ to be the full subcategory of $\mathcal{A}$ whose objects are generated by $\mathcal{Y}$. Then $\mathcal{A}_{\mathcal{Y}}$ is abelian, the functor $\mathcal{A}_{\mathcal{Y}} \hookrightarrow \mathcal{A}$ is exact and the pair $(\mathcal{X} \cap \mathcal{A}_{\mathcal{Y}}, \mathcal{Y})$ is a 1*-cotilting torsion pair in $\mathcal{A}_{\mathcal{Y}}$ therefore $\mathcal{A}_{\mathcal{Y}} \simeq \mathcal{LH}(\mathcal{Y})$.*

Proof. Let us prove that for any $X \xrightarrow{f} Y$ morphism in $\mathcal{A}_{\mathcal{X}}$, its kernel and cokernel in $\mathcal{A}$ belong to $\mathcal{A}_{\mathcal{X}}$. By definition of $\mathcal{A}_{\mathcal{X}}$ there exist $X \overset{\alpha_X}{\hookrightarrow} T_X$ and $Y \overset{\alpha_Y}{\hookrightarrow} T_Y$ with T_X, T_Y in $\mathcal{X}$. Hence $\mathrm{Ker}_{\mathcal{A}}(f) \hookrightarrow X \overset{\alpha_X}{\hookrightarrow} T_X$ implies $\mathrm{Ker}_{\mathcal{A}}(f) \in \mathcal{A}_{\mathcal{X}}$ while $\mathrm{Coker}_{\mathcal{A}}(f) \hookrightarrow \mathrm{Coker}_{\mathcal{A}}(\alpha_Y f) \in \mathcal{X}$, since $\mathcal{X}$ is closed under quotients and $T_Y \in \mathcal{X}$. Let $X \in \mathcal{A}_{\mathcal{X}}$ and let consider its short exact sequence $0 \to T(X) \to X \to F(X) \to 0$ where $T(X)$ (resp. $F(X)$) is its torsion (resp. torsion-free) part with respect to the torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}$. Then $T(X) \in \mathcal{X} \subseteq \mathcal{A}_{\mathcal{X}}$, hence $F(X) \in \mathcal{A}_{\mathcal{X}}$ (since it is a cokernel of a morphism in $\mathcal{A}_{\mathcal{X}}$) which proves that $(\mathcal{X}, \mathcal{Y} \cap \mathcal{A}_{\mathcal{X}})$ is a torsion pair in $\mathcal{A}_{\mathcal{X}}$. The second statement follows dually. □

2. n-Tilting Theorem

2.1. Let $\mathcal{C}$ be a triangulated category endowed with a pair of t-structures $(\mathcal{D}, \mathcal{T})$: $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{E} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}$. The following statements hold true:

(1) any complex $\cdots \to 0 \to E^{-s} \to \cdots \to E^{-1} \to \overset{\bullet}{E^0} \to 0 \to \cdots$ with $s \geq 0$ belongs to $\mathcal{T}^{[-s,0]} \cap \mathcal{D}^{[-s,0]}$ ([20, Lem. 1.1]);

(2) if $n \geq 1$, given an exact sequence in $\mathcal{H}_{\mathcal{D}}$ (resp. $\mathcal{H}_{\mathcal{T}}$)

$$0 \longrightarrow M \xrightarrow{g} E_{-n+1} \xrightarrow{d_E^{-n+1}} \cdots \xrightarrow{d_E^{-1}} E_0 \xrightarrow{f} N \longrightarrow 0 \qquad E_{-i} \in \mathcal{E} \quad \forall i = 0, \ldots, n-1$$

implies $N = \mathrm{Coker}_{\mathcal{H}_{\mathcal{D}}} d_E^{-1} \in \mathcal{E}$ (resp. $M = \mathrm{Ker}_{\mathcal{H}_{\mathcal{T}}} d_E^{-n+1} \in \mathcal{E}$). The argument of [20, Lem. 1.2] gives a distinguished triangle $M[n-1] \to [E_{-n+1} \to \cdots \to \overset{\bullet}{E_0}] \to N[0] \xrightarrow{+}$ hence $M[n-1] \in \mathcal{H}_{\mathcal{D}}[n-1] \subseteq \mathcal{T}^{\leq 1}$ and $[E_{-n+1} \to \cdots \to \overset{\bullet}{E_0}] \in \mathcal{T}^{\leq 0}$ so $N[0] \in \mathcal{H}_{\mathcal{D}} \cap \mathcal{T}^{\leq 0} = \mathcal{E}$;

(3) a complex $E^\bullet \in K(\mathcal{E})$ is acyclic in $\mathcal{H}_{\mathcal{D}}$ if and only if it is acyclic in $\mathcal{H}_{\mathcal{T}}$ and in this case for any i we have $\mathrm{Ker}_{\mathcal{H}_{\mathcal{D}}} d^i_{E^\bullet} \cong \mathrm{Ker}_{\mathcal{H}_{\mathcal{T}}} d^i_{E^\bullet} \in \mathcal{E}$ ([20, Prop. 1.3]);

(4) $\mathcal{E}$ is projectively complete (any idempotent in $\mathcal{E}$ splits in $\mathcal{H}_{\mathcal{D}}$ and it belongs to $\mathcal{H}_{\mathcal{T}}$ too); $\mathcal{E}$ is closed under extensions both in $\mathcal{H}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{T}}$, hence the class of short exact sequences $0 \to E_1 \to E \to E_2 \to 0$ (in $\mathcal{H}_{\mathcal{D}}$ or equivalently $\mathcal{H}_{\mathcal{T}}$) provides a Quillen exact structure $(\mathcal{E}, \mathcal{E}x)$ on $\mathcal{E}$.

Remark 2.2. Let consider $\mathcal{C} = D(\mathcal{H}_{\mathcal{D}})$ and $\mathcal{E}$ a cogenerating class in $\mathcal{H}_{\mathcal{D}}$. By [20, Lem. 1.4] $\mathcal{E}$ is generating in $\mathcal{H}_{\mathcal{T}}$ and by point (2) of 2.1 any $A \in \mathcal{H}_{\mathcal{D}}$ admits a copresentation of length at most n. Dually any $B \in \mathcal{H}_{\mathcal{T}}$ has a presentation of length at most n.

All the previous results combine into the following n version of Theorem 1.8

Theorem 2.3. n-Tilting Theorem. *(*[20, Th. 1.5] *) Let $\mathcal{A}$ be abelian category such that its derived category $D(\mathcal{A})$ has Hom sets, let $\mathcal{D}$ be the natural t-structure in $D(\mathcal{A})$ and $\mathcal{T}$ a t-structure such that $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$. Let us suppose that $\mathcal{E} := \mathcal{A} \cap \mathcal{H}_{\mathcal{T}}$ cogenerates $\mathcal{A}$, hence there exists a triangle equivalence E:*

$$\begin{tikzcd} & \frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} \arrow[dl, hook', "\cong"'] \arrow[dr, hook, "\cong"] & \\ D(\mathcal{H}_{\mathcal{T}}) \arrow[rr, dotted, "\cong", "E"'] & & D(\mathcal{A}) \end{tikzcd}$$

(where $\mathcal{N}_{\mathcal{E}x}$ is the null system of complexes in $K(\mathcal{E})$ acyclic in $\mathcal{A}$ or equivalently in $\mathcal{H}_{\mathcal{T}}$) such that the restriction of E to $\mathcal{H}_{\mathcal{T}}$ is naturally isomorphic to the inclusion $\mathcal{H}_{\mathcal{T}} \subseteq D(\mathcal{A})$. Moreover $\mathcal{E}$ is generating in $\mathcal{H}_{\mathcal{T}}$.

Definition 2.4. A *pair of t-structures* $(\mathcal{D}, \mathcal{T})$ on a triangulated category $\mathcal{C}$ is called *n-tilting* if the following statements hold:

(1) $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$;
(2) the following equivalent conditions are satisfied:
  - **(i):** given $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ we get $\mathcal{C} \simeq K(\mathcal{E})/\mathcal{N}$ and $D(\mathcal{H}_{\mathcal{D}}) \overset{\simeq}{\hookleftarrow} K(\mathcal{E})/\mathcal{N} \overset{\simeq}{\hookrightarrow} D(\mathcal{H}_{\mathcal{T}})$ where $\mathcal{N}$ is the null system of complexes in $K(\mathcal{E})$ acyclic in $\mathcal{H}_{\mathcal{D}}$ or equivalently in $\mathcal{H}_{\mathcal{T}}$;
  - **(ii):** $\mathcal{C} \simeq D(\mathcal{H}_{\mathcal{D}})$ and $\mathcal{E}$ cogenerates $\mathcal{H}_{\mathcal{D}}$;
  - **(iii):** $\mathcal{C} \simeq D(\mathcal{H}_{\mathcal{T}})$ and $\mathcal{E}$ generates $\mathcal{H}_{\mathcal{T}}$.

If $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ by Theorem 2.3 we have that (ii) implies (i) and (iii), dually (iii) implies (i) and (ii) (by the cotilting version of Theorem 2.3) so (ii) is equivalent to (iii). If (i) holds $\mathcal{C} \simeq D(\mathcal{H}_{\mathcal{D}})$ and $\mathcal{E}$ cogenerates $\mathcal{H}_{\mathcal{D}}$ since any $A \in \mathcal{H}_{\mathcal{D}}$ can be represented by a complex $E^\bullet \in K(\mathcal{E})$ and so $A \hookrightarrow \mathrm{Coker}_{\mathcal{H}_{\mathcal{D}}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ $(\mathrm{Coker}_{\mathcal{H}_{\mathcal{D}}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ by (2) of 2.1) which proves that (i) implies (ii).

We note that any n-tilting pair of t-structures is also m-tilting for any $m \geq n$.

Proposition 2.5. *Let $(\mathcal{D}, \mathcal{T})$ be a n-tilting pair of t-structures in a triangulated category $\mathcal{C}$. Hence the equivalence $F : \mathcal{C} \overset{\simeq}{\to} K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} = D(\mathcal{E}, \mathcal{E}x)$ (where the Quillen exact structure on $\mathcal{E}$ is the one of 2.1 (4)) gives:*

$$F(\mathcal{T}^{\leq 0}) = \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\leq 0} \textit{ in } K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \textit{ with } E^\bullet_{\leq 0} \in \mathcal{L}\mathcal{K}^{\leq 0}_{\mathcal{E}}\} =: \mathcal{L}\mathcal{D}^{\leq 0}_{(\mathcal{E}, \mathcal{E}x)}$$

$$F(\mathcal{D}^{\geq 1}) = \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\geq 1} \textit{ in } K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \textit{ with } E^\bullet_{\geq 1} \in \mathcal{R}\mathcal{K}^{\geq 1}\} =: \mathcal{R}\mathcal{D}^{\geq 1}_{(\mathcal{E}, \mathcal{E}x)}.$$

Proof. By definition $D(\mathcal{E}, \mathcal{E}x) = \frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}}$ ([41] see A.2). Since $(\mathcal{D}, \mathcal{T})$ is n-tilting , we have that under the n-tilting equivalence $D^{\leq 0}(\mathcal{H}_{\mathcal{T}})$ corresponds to $\mathcal{T}^{\leq 0}$, while $\mathcal{D}^{\geq 1}$ corresponds to $D^{\geq 1}(\mathcal{H}_{\mathcal{D}})$. Moreover the class $\mathcal{E}$ generates $\mathcal{H}_{\mathcal{T}}$ and so any object in $D^{\leq 0}(\mathcal{H}_{\mathcal{T}})$ can be represented in $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$ by a complex in $K^{\leq 0}(\mathcal{E})$. On the other side since $\mathcal{E}$ cogenerates $\mathcal{H}_{\mathcal{D}}$ any object in $D^{\geq 1}(\mathcal{H}_{\mathcal{D}})$ can be represented in $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$ by a complex in $K^{\geq 1}(\mathcal{E})$. □

Remark 2.6. The proof of Theorem 2.3 produces the desired equivalence on the derived categories of the hearts passing trough an equivalence with the triangulated category $\frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} = D(\mathcal{E}, \mathcal{E}x)$ where $\mathcal{E}$ is the intersection of the hearts. We remark that the role of the Quillen exact structure is important in order to define $D(\mathcal{E}, \mathcal{E}x)$. The previous proposition proves that the category $\mathcal{E}$ encodes the data of the t-structures since $(\mathcal{D}, \mathcal{T}) \simeq (\mathcal{R}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{L}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)})$.

3. 2-tilting torsion classes

As we will see soon the case $n = 2$ is neatly easier than $n > 2$ and so we will first analyze this case in detail.

Lemma 3.1. *Let $(\mathcal{D}, \mathcal{T})$ be a 2-tilting pair of t-structures in $\mathcal{C} \simeq D(\mathcal{H}_\mathcal{D}) \simeq D(\mathcal{H}_\mathcal{T})$. Hence $\mathcal{E} := \mathcal{H}_\mathcal{D} \cap \mathcal{H}_\mathcal{T}$ is closed under extensions (both in $\mathcal{H}_\mathcal{D}$ and $\mathcal{H}_\mathcal{T}$); it admits kernels and cokernels and given $d : E \to F$ in $\mathcal{E}$ we have $\mathrm{Ker}_\mathcal{E}(d) = \mathrm{Ker}_{\mathcal{H}_\mathcal{T}}(d) \in \mathcal{E}$ while $\mathrm{Coker}_\mathcal{E}(d) = \mathrm{Coker}_{\mathcal{H}_\mathcal{D}}(d) \in \mathcal{E}$. Moreover the inclusion functor $i : \mathcal{E} \hookrightarrow \mathcal{H}_\mathcal{D}$ admits a right adjoint $t : \mathcal{H}_\mathcal{D} \to \mathcal{E}$ while the inclusion functor $j : \mathcal{E} \hookrightarrow \mathcal{H}_\mathcal{T}$ admits a left adjoint $f : \mathcal{H}_\mathcal{T} \to \mathcal{E}$.*

Proof. Let $d : E \to F$ be a morphism in $\mathcal{E}$, by point (2) of 2.1 we have: $\mathrm{Ker}_{\mathcal{H}_\mathcal{T}} d \in \mathcal{E}$ while $\mathrm{Coker}_{\mathcal{H}_\mathcal{D}} d \in \mathcal{E}$ and so they provide the kernel, resp. the cokernel, of d in $\mathcal{E}$. By hypothesis $\mathcal{D}^{\leq -2} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ so, by orthogonality $\mathcal{T}^{\geq 1} \subseteq \mathcal{D}^{\geq -1}$. Let $A \in \mathcal{H}_\mathcal{D}$, the distinguished triangle $\tau^{\leq 0}(A) \to A \to \tau^{\geq 1}(A) \xrightarrow{+}$ proves that $\tau^{\leq 0}(A) \in \mathcal{D}^{\geq 0}$ since $A \in \mathcal{H}_\mathcal{D}$ and $\tau^{\geq 1}(A) \in \mathcal{T}^{\geq 1} \subseteq \mathcal{D}^{\geq -1}$ so $t(A) := \tau^{\leq 0}(A) \in \mathcal{T}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{E}$ (recall notation 1.7). Hence $\mathcal{H}_\mathcal{D}(i(E), A) = \mathcal{C}(E, A) \simeq \mathcal{C}(E, \tau^{\leq 0}A) = \mathcal{E}(E, t(A))$ for any $E \in \mathcal{E}$, which proves that t is a right adjoint of i. Dually the functor $\delta^{\geq 0}$ restricted to $\mathcal{H}_\mathcal{T}$ takes image in $\mathcal{E}$ and provides the left adjoint f of j. □

Following Lemma 1.3 we define the a 2-tilting torsion class in the following way.

Definition 3.2. Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{E} \hookrightarrow \mathcal{A}$ is a 2-*tilting torsion class* if

1. $\mathcal{E}$ cogenerates $\mathcal{A}$;
2. $\mathcal{E}$ is closed under extensions in $\mathcal{A}$;
3. $\mathcal{E}$ has kernels;
4. for any exact sequence $0 \to A \to X_1 \to X_2 \to B \to 0$ in $\mathcal{A}$ with $X_i \in \mathcal{E}$ for $i \in \{1, 2\}$ and $A, B \in \mathcal{A}$ we have $B \in \mathcal{E}$.

Moreover $\mathcal{E}$ is endowed with a canonical Quillen exact structure whose short exact sequences are exact sequences in $\mathcal{A}$ with terms in $\mathcal{E}$. Any 1-tilting torsion class as in Definition 1.2 is also a 2-tilting torsion class.

Dually a 2-*cotilting torsion-free class* in $\mathcal{A}$ is a full generating extension closed subcategory $\mathcal{E}$ of $\mathcal{A}$ admitting cokernels and closed under kernels in $\mathcal{A}$.

Proposition 3.3. *Given $(\mathcal{D}, \mathcal{T})$ a 2-tilting pair of t-structures the category $\mathcal{E} := \mathcal{H}_\mathcal{D} \cap \mathcal{H}_\mathcal{T}$ is a 2-tilting torsion class (resp. a 2-tilting torsion-free class) in $\mathcal{H}_\mathcal{D}$ (resp. in $\mathcal{H}_\mathcal{T}$).*

Proof. By Definition 2.4 $\mathcal{E}$ cogenerates $\mathcal{H}_\mathcal{D}$ and generates $\mathcal{H}_\mathcal{T}$. By point (4) of 2.1 $\mathcal{E}$ is closed under extensions both in $\mathcal{H}_\mathcal{D}$ and $\mathcal{H}_\mathcal{T}$. Given a morphism $d : X_1 \to X_2$ in $\mathcal{E}$ by point (2) of 2.1 we deduce that $\mathrm{Ker}_\mathcal{E} d \cong \mathrm{Ker}_{\mathcal{H}_\mathcal{T}} d \in \mathcal{E}$ and $\mathrm{Coker}_\mathcal{E} d \cong \mathrm{Coker}_{\mathcal{H}_\mathcal{D}} d \in \mathcal{E}$ which concludes the proof. □

Theorem 3.4. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{D}$ be the natural t-structure on the triangulated category $D(\mathcal{A})$. Let $i : \mathcal{E} \hookrightarrow \mathcal{A}$ be a 2-tilting torsion class on $\mathcal{A}$. Hence $\mathcal{T}^{\leq 0} := \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$ (see C.1) is an aisle in $D(\mathcal{A})$ such that $\mathcal{E} = \mathcal{A} \cap \mathcal{H}_\mathcal{T}$ and the pair $(\mathcal{D}, \mathcal{T})$ is a 2-tilting pair of t-structures. We will say that the t-structure $\mathcal{T}$ is obtained by* tilting *$\mathcal{D}$ with respect to the 2-tilting torsion class $\mathcal{E}$.*

Proof. The class $\mathcal{T}^{\leq 0}$ is extension closed by Lemma 3.5; $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ since the suspension of any factor is contained in a factor. By definition $\mathcal{D}^{\leq -2} \subseteq \mathcal{T}^{\leq 0}$ and, since any factor is contained in $\mathcal{D}^{\leq 0}$ (which is extension closed), $\mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$.

Let us prove that the functor $i_{\mathcal{T}^{\leq 0}} : \mathcal{T}^{\leq 0} \to D(\mathcal{A})$ has a right adjoint $\tau^{\leq 0} : D(\mathcal{A}) \to \mathcal{T}^{\leq 0}$. Let us notice that the functor $i : \mathcal{E} \to \mathcal{A}$ has a right adjoint t defined as in Lemma 1.3: for any $A \in \mathcal{A}$ let consider a copresentation $0 \to A \to X_1 \xrightarrow{f} X_2$ and let us pose $t(A) = \mathrm{Ker}_{\mathcal{E}}(f)$. For any $L \in \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$ we have $H^0_{\mathcal{D}}(L) \in \mathcal{E}$ (since it is a cokernel in $\mathcal{A}$ between two objects in $\mathcal{E}$). Let $A \in \mathcal{A}$, we have $D(\mathcal{A})(M, A) \cong \mathcal{A}(H^0_{\mathcal{D}}(M), A) \cong \mathcal{A}(H^0_{\mathcal{D}}(M), t(A)) \cong D(\mathcal{A})(M, t(A)), \forall M \in \mathcal{T}^{\leq 0}$. So our truncation functor $\tau^{\leq 0}$ restricted to $\mathcal{A}$ coincides with t: $\tau^{\leq 0}_{|\mathcal{A}} = t$ (hence the mapping cone $[t(A) \to \overset{\bullet}{A}]$ belongs to $\mathcal{T}^{\geq 1}$ by [35, Prop. 1.1]). Even if we have to choose a morphism in order to define this functor, the functoriality of the construction is guaranteed by the fact that, for another choice, there exists a unique isomorphism compatible with this construction ([7, Prop. 1.3.3]).

Let us now compute the restriction of $\tau^{\leq 0}$ to $\mathcal{D}^{[-1,0]}$. Any object $D \in \mathcal{D}^{[-1,0]}$ can be represented as $[A \xrightarrow{f} \overset{\bullet}{B}]$ (see C.2). Since $\mathcal{E}$ is cogenerating in $\mathcal{A}$ there exists an immersion $h : A \hookrightarrow E$ with $E \in \mathcal{E}$ and so D is isomorphic in $D(\mathcal{A})$ to $[E \xrightarrow{\overline{f}} E \overset{\bullet}{\oplus}_A B]$. Let define $\tau^{\leq 0}(D)$ to be $[E \xrightarrow{t(\overline{f})} t(E \overset{\bullet}{\oplus}_A B)]$. Let consider the following commutative diagram whose rows and columns are distinguished triangles (obtained by applying the octahedron axiom to the composition $E \to t(E \oplus_A B) \to E \oplus_A B$ [7, Prop. 1.1.11])

$$\begin{tikzcd}
E \arrow[r, "t(\overline{f})"] \arrow[d] & t(E \oplus_A B) \arrow[r] \arrow[d] & \tau^{\leq 0}(D) \arrow[r, "+"] \arrow[d] & {} \\
E \arrow[r, "\overline{f}"] \arrow[d] & E \oplus_A B \arrow[r] \arrow[d] & D \arrow[r, "+"] \arrow[d] & {} \\
0 \arrow[r] \arrow[d, "+"] & \tau^{\geq 1}(E \oplus_A B) \arrow[r] \arrow[d, "+"] & \tau^{\geq 1}(E \oplus_A B) \arrow[r, "+"] \arrow[d, "+"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

By the previous case $\tau^{\geq 1}(E \oplus_A B) \in \mathcal{T}^{\geq 1}$. Since $\tau^{\geq 1}(E \oplus_A B)$ and $\tau^{\geq 1}(E \oplus_A B)[-1]$ belong to $\mathcal{T}^{\geq 1}$; $D(\mathcal{A})(M, \tau^{\leq 0}(D)) \cong D(\mathcal{A})(M, D)$ for any $M \in \mathcal{T}^{\leq 0}$.

For any $X \in D(\mathcal{A})$, $\tau^{\leq 0}(X) \cong \tau^{\leq 0}(\delta^{\leq 0}(X))$ since $\mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ (one can see by the octahedron axiom that the mapping cone of the composition $\tau^{\leq 0}(\delta^{\leq 0}(X)) \to \delta^{\leq 0}(X) \to X$ lyes in $\mathcal{T}^{\geq 1}$). Given $C \in \mathcal{D}^{\leq 0}$; the following commutative diagram (whose rows and columns are distinguished triangles)

$$\begin{tikzcd}
\delta^{\leq -2}(C) \arrow[r] \arrow[d] & \tau^{\leq 0}(C) \arrow[r] \arrow[d] & \tau^{\leq 0}(\delta^{[-1,0]}(C)) \arrow[r, "+"] \arrow[d] & {} \\
\delta^{\leq -2}(C) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \delta^{[-1,0]}(C) \arrow[r, "+"] \arrow[d] & {} \\
0 \arrow[r] \arrow[d, "+"] & \tau^{\geq 1}(\delta^{[-1,0]}(C)) \arrow[r] \arrow[d, "+"] & \tau^{\geq 1}(\delta^{[-1,0]}(C)) \arrow[r, "+"] \arrow[d, "+"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

permits us to compute $\tau^{\leq 0}(C)$ for any $C \in \mathcal{D}^{\leq 0}$. We recall that, whenever two rows and any column of such a digram are distinguished, the third row is distinguished too [7]. The functoriality of this construction is guaranteed by the orthogonality of the classes $\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}$.

Let us prove that $\mathcal{E} = \mathcal{A} \cap \mathcal{H}_{\mathcal{T}}$; we recall that $\mathcal{A} \simeq \mathcal{H}_{\mathcal{D}}$. Let consider $A^\bullet \in \mathcal{A} \cap \mathcal{H}_{\mathcal{T}}$, hence $A^\bullet \in \mathcal{T}^{\leq 0} = \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$ and so it fits into a distinguished triangle $B^\bullet \to A^\bullet \to E^{[-1,0]} \xrightarrow{+}$ for suitable $B^\bullet \in \mathcal{D}^{\leq -2}$ and $E^{[-1,0]} \in \mathcal{E} \star \mathcal{E}[1]$; but since $A^\bullet \in \mathcal{D}^{\geq 0}$ we deduce that $B^\bullet \in \mathcal{D}^{\leq -2} \cap \mathcal{D}^{\geq 0} = 0$ so $A^\bullet \in \mathcal{E} \star \mathcal{E}[1]$. Therefore $A^\bullet = [E^{-1} \xrightarrow{d} \overset{\bullet}{E^0}]$ and $A^\bullet \cong H^0_{\mathcal{D}}(A^\bullet)$ since $A^\bullet \in \mathcal{A} \simeq \mathcal{H}_{\mathcal{D}}$, so by point (4) of Definition 3.2 we obtain $A^\bullet \in \mathcal{E}$ which proves that $\mathcal{T}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{E}$. We can apply the Tilting Theorem 2.3 ($\mathcal{E}$ cogenerates $\mathcal{A}$) thus obtaining that $(\mathcal{D}, \mathcal{T})$ is a 2-tilting pair of t-structures. □

Lemma 3.5. *Let $\mathcal{E}$ be a 2-tilting torsion class in an abelian category $\mathcal{A}$. The full subcategory $\mathcal{T}^{\leq 0} := \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$ of $D(\mathcal{A})$ is closed under extensions.*

Proof. Let us denote by $\mathcal{D}$ the natural t-structure on $D(\mathcal{A})$.

Step 1. Let us prove that $\mathcal{E}[1]\star\mathcal{E} \subseteq \mathcal{E}\star\mathcal{E}[1]$. Any $X^\bullet \in \mathcal{E}[1]\star\mathcal{E}$ can be represented by a complex $E^\bullet \in K^{\geq -1}(\mathcal{E})$ (since $\mathcal{E}$ cogenerates $\mathcal{A}$). The distinguished triangle $H^{-1}_{\mathcal{D}}(E^\bullet)[1] \to E^\bullet \to H^0_{\mathcal{D}}(E^\bullet) \overset{+}{\to}$ is the unique realizing $E^\bullet \in \mathcal{A}[1] \star \mathcal{A}$ hence $H^{-1}_{\mathcal{D}}(E^\bullet) = \operatorname{Ker}_{\mathcal{A}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ and $H^0_{\mathcal{D}}(E^\bullet) = \frac{\operatorname{Ker}_{\mathcal{A}}(d^0_{E^\bullet})}{\operatorname{Im}_{\mathcal{A}}(d^{-1}_{E^\bullet})} \in \mathcal{E}$. The short exact sequence $0 \to \operatorname{Ker}_{\mathcal{A}}(d^{-1}_{E^\bullet}) \to E^{-1} \to \operatorname{Im}_{\mathcal{A}}(d^{-1}_{E^\bullet}) \to 0$ implies that $\operatorname{Im}_{\mathcal{A}}(d^{-1}_{E^\bullet}) \in \mathcal{E}$ (by property (4) of Definition 3.2) and so $\operatorname{Ker}_{\mathcal{A}}(d^0_{E^\bullet}) \in \mathcal{E}$ (since it is an extension of objects in $\mathcal{E}$). This proves that $X^\bullet \cong [E^{-1} \to \overset{\bullet}{\operatorname{Ker}_{\mathcal{A}}}(d^0_{E^\bullet})] \in \mathcal{E} \star \mathcal{E}[1]$.

Step 2. It remains to prove that $(\mathcal{E} \star \mathcal{E}[1]) \star \mathcal{D}^{\leq -2} \subseteq \mathcal{T}^{\leq 0}$ which is equivalent to require that $\delta^{\geq -1}Z^\bullet \in \mathcal{E}\star\mathcal{E}[1]$ for any $Z^\bullet \in (\mathcal{E}\star\mathcal{E}[1])\star\mathcal{D}^{\leq -2}$. The complex $Z^\bullet$ can be represented as $[\cdots Y^{-3} \overset{d^{-3}}{\to} Y^{-2} \overset{d^{-2}}{\to} E^{-1} \overset{d^{-1}}{\to} \overset{\bullet}{E^0}] \in K^{\leq 0}(\mathcal{A})$. We can choose an inclusion $\operatorname{Coker} d^{-2}_{Z^\bullet} \overset{i}{\hookrightarrow} F^{-1}$ with $F^{-1} \in \mathcal{E}$ (since $\mathcal{E}$ cogenerates $\mathcal{A}$), hence $\delta^{\geq -1}Z^\bullet = [\operatorname{Coker} d^{-2}_{Z^\bullet} \to \overset{\bullet}{E^0}] \cong [F^{-1} \to F^{-1} \oplus_{E^{-1}} \overset{\bullet}{E^0}] \in \mathcal{E} \star \mathcal{E}[1]$ since $F^{-1} \oplus_{E^{-1}} E^0 \in \mathcal{E}$. $\square$

Remark 3.6. Theorem 3.4 admits a dual version: given $\mathcal{T}$ a t-structure on $D(\mathcal{H}_{\mathcal{T}})$ and $j : \mathcal{E} \to \mathcal{H}_{\mathcal{T}}$ a 2-cotilting torsion-free class on $\mathcal{H}_{\mathcal{T}}$, the class $\mathcal{D}^{\geq 0} := \mathcal{E}[-1] \star \mathcal{E} \star \mathcal{T}^{\geq 2}$ is a co-aisle in $D(\mathcal{H}_{\mathcal{T}})$ such that $\mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}} = \mathcal{E}$. Moreover the pair $(\mathcal{D}, \mathcal{T})$ is a 2-tilting pair of t-structures.

Definition 3.7. A 2-*quasi-abelian category* is the data $(\mathcal{E}, \mathcal{E}x)$ of an additive category $\mathcal{E}$ with a Quillen exact structure $\mathcal{E}x$ such that $\mathcal{E}$ admits kernels and cokernels.

Remark 3.8. Clearly any 1-quasi-abelian category is also 2-quasi-abelian. Any 2-tilting torsion class $\mathcal{E}$ is a 2-quasi-abelian category since by Definition 3.2 (3) it admits kernels and by (4) it admits cokernels.

Let us start by studying the case of a 2-quasi-abelian category $(\mathcal{E}, \mathcal{E}x_{\text{split}})$ whose Quillen exact structure is the minimal one (i.e., any conflation splits).

Proposition 3.9. *Let $\mathcal{E}$ be an additive category admitting kernels and cokernels. The category $K(\mathcal{E})$ admits a canonical 2-tilting pair of t-structures $(\mathcal{RK}_{\mathcal{E}}, \mathcal{LK}_{\mathcal{E}})$ such that $\mathcal{E} = \mathcal{RK}(\mathcal{E}) \cap \mathcal{LK}(\mathcal{E})$ and so $\mathcal{E} \hookrightarrow \mathcal{RK}(\mathcal{E})$ is a 2-tilting torsion class while $\mathcal{E} \hookrightarrow \mathcal{LK}(\mathcal{E})$ is a 2-cotilting torsion-free class.*

Proof. We can endow $\mathcal{E}$ with its minimal Quillen exact structure $\mathcal{E}x_{\text{split}}$ (split short exact sequences A.1). So $(\mathcal{E}, \mathcal{E}x_{\text{split}})$ is a 2-quasi-abelian category whose derived category $D(\mathcal{E}, \mathcal{E}x_{\text{split}}) = K(\mathcal{E})$. In 1.13 we provided the construction of the left and right t-structures on $K(\mathcal{E})$ for $\mathcal{E}$ a 1-quasi-abelian category. This construction is based on the existence of kernels and cokernels, so it works unchanged in this case and it provides the t-structures $\mathcal{LK}_{\mathcal{E}}$ and $\mathcal{RK}_{\mathcal{E}}$ on $K(\mathcal{E})$ whose associated truncated functors are those described in 1.13. Moreover $\mathcal{RK}^{\leq -2}_{\mathcal{E}} \subseteq \mathcal{LK}^{\leq 0}_{\mathcal{E}} \subseteq \mathcal{RK}^{\leq 0}_{\mathcal{E}}$. The heart of $\mathcal{LK}_{\mathcal{E}}$ (resp. $\mathcal{RK}_{\mathcal{E}}$) is denoted by $\mathcal{LK}(\mathcal{E})$ (resp. $\mathcal{RK}(\mathcal{E})$). Any short exact sequence $0 \to K \overset{\alpha}{\to} L \overset{\pi}{\to} E \to 0$ in $\mathcal{LK}(\mathcal{E})$ with $E \in \mathcal{E}$ is a distinguished triangle in $K(\mathcal{E})$. It induces the exact sequence $\mathcal{K}(\mathcal{E})(E, L) \to \mathcal{K}(\mathcal{E})(E, E) \to \mathcal{K}(\mathcal{E})(E, K[1]) = 0$ (since $K[1]$ is a complex in $\mathcal{LK}^{\leq -1}_{\mathcal{E}}$ with 0 entries in degrees greater or equal to 0), hence π is a split epimorphism. Thus $\mathcal{E}$ coincides with the class of projective objects in $\mathcal{LK}(\mathcal{E})$. Any object $L \in \mathcal{LK}(\mathcal{E})$ can be represented as a complex $L \cong C(d) := [\operatorname{Ker}_{\mathcal{E}}(d) \overset{\alpha}{\to} X \overset{d}{\to} \overset{\bullet}{Y}] \in K(\mathcal{E})$ (since $L \cong \tau^{\geq 0}_{\mathcal{L}} \tau^{\leq 0}_{\mathcal{L}} L$ see 1.13) which permits to prove that L has a projective resolution of at most length 2: the distinguished triangles (where $C(\alpha) := [\mathrm{Ker}_{\mathcal{E}}(d) \xrightarrow{\alpha} \overset{\bullet}{X}]$)

$$\text{(1)} \qquad \mathrm{Ker}_{\mathcal{E}}(d)[0] \longrightarrow X[0] \longrightarrow C(\alpha) \overset{+}{\rightarrow} \qquad C(\alpha) \longrightarrow Y[0] \longrightarrow C(d) \overset{+}{\rightarrow} \quad \text{in } \mathcal{K}(\mathcal{E})$$

give the short exact sequences

$$0 \to \mathrm{Ker}_{\mathcal{E}}(d) \to X \to C(\alpha) \to 0 \quad 0 \to C(\alpha) \to Y \to C(d) \to 0\ ; \quad \text{in } \mathcal{LK}(\mathcal{E})$$

from which we obtain the projective resolution $0 \to \mathrm{Ker}_{\mathcal{E}}(d) \to X \to Y \to C(d) \to 0$ of $C(d)$ in $\mathcal{LK}(\mathcal{E})$. We have $K(\mathcal{E}) \simeq D(\mathcal{LK}(\mathcal{E}))$ which proves that $(\mathcal{RK}_{\mathcal{E}}, \mathcal{LK}_{\mathcal{E}})$ is a 2-tilting pair of t-structures, hence by Proposition 3.3 $\mathcal{E}$ is a 2-tilting torsion (resp. 2-cotilting torsion-free) class in $\mathcal{RK}(\mathcal{E})$ (resp. $\mathcal{LK}(\mathcal{E})$). In particular $\mathcal{E}$ coincides with the class of injective objects in $\mathcal{RK}(\mathcal{E})$ (resp. projective objects in $\mathcal{LK}(\mathcal{E})$). □

Corollary 3.10. *Given $(\mathcal{E}, \mathcal{E}x_{\mathrm{split}})$ a 2-quasi-abelian category, $\mathcal{LK}(\mathcal{E}) \simeq \mathrm{coh}\text{-}\mathcal{E}$ and $\mathcal{RK}(\mathcal{E}) \simeq (\mathcal{E}\text{-}\mathrm{coh})^{\circ}$.*

Proof. The category $\mathcal{E}$ has kernels and cokernels, hence it is a coherent category (see Definition B.7 and Proposition B.10). Both coh-$\mathcal{E}$ and $\mathcal{LK}(\mathcal{E})$ are abelian categories whose projective objects coincide with $\mathcal{E}$, and such that any object has a projective resolution of at most length 2. The functor $I_{\mathcal{L}} : \mathcal{E} \to \mathcal{LK}(\mathcal{E})$ extends uniquely to a functor $I_{\mathcal{L}}^c : \mathrm{coh}\text{-}\mathcal{E} \to \mathcal{LK}(\mathcal{E})$ cokernel preserving (see B.6) which is an equivalence of categories (any object in $L \in \mathcal{LK}(\mathcal{E})$ has a projective resolution therefore $I_{\mathcal{L}}^c$ is essentially surjective and fully faithful since any object in $\mathcal{E}$ is projective in $\mathcal{LK}(\mathcal{E})$). Thus the left heart is equivalent to the category of right coherent functors. The right statement follows dually. □

Let us now turn to the case of a general 2-quasi-abelian category $(\mathcal{E}, \mathcal{E}x)$:

Lemma 3.11. *Given any 2-quasi-abelian category $(\mathcal{E}, \mathcal{E}x)$ the left and right t-structures on $\mathcal{K}(\mathcal{E})$ induce a 2-tilting pair of t-structures $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ on the derived category $D(\mathcal{E}, \mathcal{E}x)$ such that $\mathcal{E} = \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$.*

Proof. Let us denote by $\mathcal{N}_{\mathcal{E}x}$ the null system of acyclic complexes with respect to $(\mathcal{E}, \mathcal{E}x)$ (see A.2). Let us prove that the t-structure $\mathcal{LK}_{\mathcal{E}}$ on $K(\mathcal{E})$ satisfies the hypothesis of Lemma C.4 thus inducing (passing trough the quotient) the t-structure $\mathcal{LD}_{(\mathcal{E},\mathcal{E}x)}$ on $D(\mathcal{E}, \mathcal{E}x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$. We have to prove that, given any distinguished triangle $Y^\bullet \to X^\bullet \to N^\bullet \overset{+}{\to}$ in $K(\mathcal{E})$ such that $Y^\bullet \in \mathcal{LK}_{\mathcal{E}}^{\geq 1}$, $X^\bullet \in \mathcal{LK}_{\mathcal{E}}^{\leq 0}$ and $N^\bullet \in \mathcal{N}_{\mathcal{E}x}$ we have $Y^\bullet, X^\bullet \in \mathcal{N}_{\mathcal{E}x}$. We can suppose $Y^\bullet = \tau_{\mathcal{L}}^{\geq 1} Y^\bullet$ and $X^\bullet \in K^{\leq 0}(\mathcal{E})$. Let consider the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & 0 \arrow[rr] \arrow[d] & & \mathrm{Ker}(d_Y^0) \arrow[rr] \arrow[d] & & Y^0 \arrow[rr, "d_Y^0"] \arrow[d] & & Y^1 \arrow[d] \\
{} \arrow[r] & X^{-2} \arrow[rr, "d_X^{-2}"] \arrow[d] & & X^{-1} \arrow[rr, "d_X^{-1}"] \arrow[d] & & X^0 \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
{} \arrow[r] & \mathrm{Ker}(d_Y^0) \oplus X^{-2} \arrow[rr] \arrow[dr, two heads] & & Y^0 \oplus X^{-1} \arrow[rr] \arrow[dr, two heads] & & Y^1 \oplus X^0 \arrow[rr] \arrow[dr, two heads] & & Y^2 \\
\mathrm{Im}(d_X^{-3}) \arrow[ur, tail] & & \mathrm{Ker}(d_Y^0) \oplus \mathrm{Coker}(d_X^{-3}) \arrow[ur, tail] & & \mathrm{Ker}(d_Y^1) \oplus X^0 \arrow[ur, tail] & & \mathrm{Ker}(d_Y^2) \arrow[ur, tail] &
\end{tikzcd}$$

one has to start looking the last row, for $i \leq -3$ we have $N^i = X^i$, while for $j \geq 1$ we have $N^j = Y^{j+1}$; so we can write $\mathrm{Im}(d_X^{-3})$ on the left and $\mathrm{Ker}(d_Y^2)$ on the right. We complete taking resp. the cokernel and the kernel and we are able to decompose $Y^\bullet$ and $X^\bullet$ via conflations. The following pullback diagram

$$\begin{tikzcd}
\mathrm{Ker}(d_Y^0) \oplus \mathrm{Coker}(d_X^{-3}) \arrow[r, tail] & Y^0 \oplus X^{-1} \arrow[r, two heads] & \mathrm{Ker}(d_Y^1) \oplus X^0 \\
\mathrm{Ker}(d_Y^0) \oplus \mathrm{Coker}(d_X^{-3}) \arrow[r, tail] \arrow[u] & \mathrm{Ker}(d_Y^0) \oplus X^{-1} \arrow[r, two heads] \arrow[u] & X^0 \arrow[u]
\end{tikzcd}$$

proves that $\mathrm{Coker}(d_X^{-3}) \rightarrowtail X^{-1} \twoheadrightarrow X^0$ is a conflation, thus $X^\bullet, Y^\bullet \in \mathcal{N}_{\mathcal{E}x}$.

Therefore we obtain a pair of t-structures $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ on $D(\mathcal{E},\mathcal{E}x)$ such that $\mathcal{RD}^{\leq -2}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{RD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$. Clearly $\mathcal{E} \subseteq \mathcal{RH}(\mathcal{E},\mathcal{E}x) \cap \mathcal{LH}(\mathcal{E},\mathcal{E}x)$. If $E^\bullet \in \mathcal{RH}(\mathcal{E},\mathcal{E}x) \cap \mathcal{LH}(\mathcal{E},\mathcal{E}x)$, we can suppose $E^\bullet \in K^{\leq 0}(\mathcal{E})$ and $E^\bullet \simeq \tau^{\geq 0}_{\mathcal{R}} E^\bullet = \operatorname{Coker}_{\mathcal{E}} d^{-1}_{E^\bullet} \in \mathcal{E}$ so $E^\bullet \in \mathcal{E}$.

It remains to prove that the derived category of the heart is equivalent to $D(\mathcal{LH}(\mathcal{E},\mathcal{E}x)) \simeq D(\mathcal{E},\mathcal{E}x) =: K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$. Now $\mathcal{E}$ is a full subcategory of $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ and a sequence $S : 0 \to E_1 \to E \to E_2 \to 0$ belongs to $\mathcal{E}x$ if and only if the triangle $E_1[0] \to E[0] \to E_2[0] \xrightarrow{+}$ is distinguished in $D(\mathcal{E},\mathcal{E}x)$, hence (since any term is in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$) if and only if S is exact in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$. We note that given any morphism $f : E \to F$ in $\mathcal{E}$ we have $\operatorname{Ker}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}(f) = H^0_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}([\overset{\bullet}{E} \to F]) \in \mathcal{E}$ (due to the inclusion $\mathcal{RD}^{\leq -2}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$) and so $\operatorname{Ker}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}(f) = \operatorname{Ker}_{\mathcal{E}}(f)$. Hence any complex in $K(\mathcal{E})$ which is acyclic in $D(\mathcal{LH}(\mathcal{E},\mathcal{E}x))$ can be decomposed into short exact sequences in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ whose terms belong to $\mathcal{E}$ and so we deduce that $\mathcal{N}_{\mathcal{E}x} = \mathcal{N}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)} \cap K(\mathcal{E})$. Moreover any object $X^\bullet \in \mathcal{LH}(\mathcal{E},\mathcal{E}x)$ can be represented as a complex $X^\bullet \in K^{\leq 0}(\mathcal{E})$ such that $\tau^{\geq 0}_{\mathcal{L}} X^\bullet \cong X^\bullet$ and so (as in the proof of Proposition 3.9) it can be represented by a complex $C(d) := [\operatorname{Ker}_{\mathcal{E}}(d) \xrightarrow{\alpha} X \xrightarrow{d} \overset{\bullet}{Y}] \in \mathcal{LH}(\mathcal{E},\mathcal{E}x)$ whose terms belong to $\mathcal{E}$. This suggests that $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ is a Gabriel quotient of the heart $\mathcal{LK}(\mathcal{E})$ as we will see in Theorem 6.11. The same argument of Proposition 3.9 (1) proves that the exact sequence $0 \to \operatorname{Ker}_{\mathcal{E}}(d) \to X \to Y \to C(d) \to 0$ is exact in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$,thus any object in the left heart admits a $\mathcal{E}$-resolution of length at most 2. Therefore the subcategory $\mathcal{E}$ in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ satisfies the hypotheses of [32, Prop. 13.2.6] (see Proposition C.3), hence $\frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} \simeq D(\mathcal{LH}(\mathcal{E},\mathcal{E}x))$. □

Now we have a definition for any property appearing in Theorem 1.15 whose generalization is the following theorem:

Theorem 3.12. *Let $(\mathcal{E},\mathcal{E}x)$ be an additive category endowed with a Quillen exact structure $\mathcal{E}x$. The following properties are equivalent:*

(1) *$\mathcal{E}$ is a 2-cotilting torsion-free class in an abelian category $\mathcal{C}$ (and sequences in $\mathcal{E}x$ are short exact sequences in $\mathcal{C}$ whose terms belong to $\mathcal{E}$);*
(2) *$\mathcal{E}$ is a 2-tilting torsion class in an abelian category $\mathcal{C}'$ (and sequences in $\mathcal{E}x$ are short exact sequences in $\mathcal{C}'$ whose terms belong to $\mathcal{E}$);*
(3) *$(\mathcal{E},\mathcal{E}x)$ is a 2-quasi-abelian category;*
(4) *$\mathcal{E}$ is the intersection of the hearts $\mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ of a 2-tilting pair of t-structures on $D(\mathcal{E},\mathcal{E}x)$.*

Moreover $\mathcal{C} \simeq \mathcal{LH}(\mathcal{E})$, $\mathcal{C}' \simeq \mathcal{RH}(\mathcal{E})$ and $(\mathcal{D},\mathcal{T}) = (\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}})$.

Proof. By Proposition 3.3 given any 2-tilting pair of t-structures $(\mathcal{D},\mathcal{T})$ we obtain that $\mathcal{E}$ is a 2-tilting torsion (resp. 2-cotilting torsion-free) class in $\mathcal{H}_{\mathcal{D}}$ (resp. $\mathcal{H}_{\mathcal{T}}$) and by Remark 3.8 $\mathcal{E}$ is 2-quasi-abelian. So (4) implies (1), (2) and (3). By Theorem 3.4 given $\mathcal{E}$ a 2-tilting torsion class in $\mathcal{H}_{\mathcal{D}}$, the pair $(\mathcal{D},\mathcal{T})$ (on $D(\mathcal{H}_{\mathcal{D}})$) is a 2-tilting pair of t-structures (where $\mathcal{T}$ is the t-structure obtained by tilting $\mathcal{D}$ with respect to $\mathcal{E}$). By Proposition 2.5 the pair $(\mathcal{D},\mathcal{T})$ coincides with $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$. So (2) implies (4) and by the dual of Theorem 3.4 (1) implies (4). Given $(\mathcal{E},\mathcal{E}x)$ a 2-quasi-abelian category endowed with a Quillen exact structure by Lemma 3.11 one can associate the 2-tilting pair of t-structures $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ on $D(\mathcal{E},\mathcal{E}x)$ such that $\mathcal{E} = \mathcal{RH}(\mathcal{E},\mathcal{E}x) \cap \mathcal{LH}(\mathcal{E},\mathcal{E}x)$. So (3) implies (4). □

Remark 3.13. We have proved that for any n-quasi-abelian category $(\mathcal{E},\mathcal{E}x)$ with $n \in \{1,2\}$ we have a derived equivalence $D(\mathcal{LD}(\mathcal{E},\mathcal{E}x)) \simeq D(\mathcal{RD}(\mathcal{E},\mathcal{E}x))$ even if the category $\mathcal{E}$ does not contain a tilting object.

4. Effaceable functors

We prove that the left $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ is a Gabriel quotient of the heart $\mathcal{LK}(\mathcal{E}) \simeq$ coh-$\mathcal{E}$ (as suggested in Lemma 3.11). This section is devoted to the tool of effaceable functors which we will use in Section 6 to define a Serre subcategory of coh-$\mathcal{E}$.

Proposition 4.1. *Let $\mathcal{E}$ be a projectively complete category and let* fp-$\mathcal{E}$ *be the Freyd category of (contravariant) finitely presented functors. The maximal Quillen exact structure on* fp-$\mathcal{E}$ *is the one whose conflations are $0 \to F_1 \rightarrowtail F \twoheadrightarrow F_2 \to 0$ such that for any $E \in \mathcal{E}$ the sequence of abelian groups $0 \to F_1(E) \to F(E) \to F_2(E) \to 0$ is exact.*

Proof. Let us recall that fp-$\mathcal{E}$ admits cokernels which are calculated pointwise and if a morphism admits a kernel it is also computed pointwise; moreover any functor which is (pointwise or in Mod-$\mathcal{E}$) extension of finitely presented functors is finitely presented too. Hence the push-out of any inflation is an inflation, resp. the pull-back of any deflation is a deflation and they are stable by compositions so these conflations define a Quillen exact structure on fp-$\mathcal{E}$. For any other Quillen exact structure on fp-$\mathcal{E}$ a conflation $0 \to G_1 \rightarrowtail G \twoheadrightarrow G_2 \to 0$ is a kernel-cokernel sequence and so for any $E \in \mathcal{E}$ we get a short exact sequence $0 \to G_1(E) \to G(E) \to G_2(E) \to 0$ of abelian groups. □

Let us recall the definition of *right filtering* subcategory of an exact category and some related results due to Schlichting ([48]). Let us recall that for any inflation $A \rightarrowtail B$ the object A is called an admissible subobject of B.

Definition 4.2. [48, Def. 1.3.] Let $\mathcal{U}$ be an exact category (i.e., an additive category with a Quillen exact structure) and $\mathcal{A} \subset \mathcal{U}$. Then the inclusion $\mathcal{A} \subset \mathcal{U}$ is called *right filtering* and $\mathcal{A}$ is called *right filtering in $\mathcal{U}$* if:

(1) $\mathcal{A}$ is an extension closed full subcategory of $\mathcal{U}$;
(2) $\mathcal{A}$ is closed under taking admissible subobjects and admissible quotients;
(3) every map $f : U \to A$ with $U \in \mathcal{U}$ and $A \in \mathcal{A}$ admits a factorisation $f = g\pi$ $U \xrightarrow{\pi} \!\!\!\!\!\rightarrow B \xrightarrow{g} A$ with $B \in \mathcal{A}$ and π a deflation.

Definition 4.3. [48, Def. 1.12.] Let $\mathcal{U}$ be an exact category and $\mathcal{A} \subset \mathcal{U}$ be an extension closed full subcategory. A $\mathcal{U}$-morphism is called a *weak isomorphism* if it is a finite composition of inflations with cokernel in $\mathcal{A}$ and deflations with kernel in $\mathcal{A}$. We write $\Sigma_{\mathcal{A} \subset \mathcal{U}}$ for the class of weak isomorphisms.

Lemma 4.4. [48, Lem. 1.13.] *If $\mathcal{A}$ is right filtering in $\mathcal{U}$ then $\Sigma_{\mathcal{A} \subset \mathcal{U}}$ admits a calculus of right fractions.*

By passing to the opposite category one obtains the dual results in the left filtering case.

In the following, we will define a right filtering subcategory eff-$_{\mathcal{E}x}\mathcal{E}$ of fp-$\mathcal{E}$ whose objects are the quotients in fp-$\mathcal{E}$ of deflations in $\mathcal{E}x$, they are called *effaceable functors* ([54, p.14], [58, p.28] and [37, p.4]). When $\mathcal{A}$ is an abelian category, the right orthogonal class of eff-$\mathcal{A}$ coincides with the full subcategory of coherent functors which respects monomorphisms, hence the quotient category $\frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}}$ is the category of coherent left exact functors. Following Krause's denomination the equivalence $\mathcal{A} \simeq \frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}}$ is called Auslander's formula ([37, Th. 2.2]).

This procedure is analog to the procedure one needs to do in order to define the category of sheaves in abelian groups associated to a topological space. One first defines the localizing Serre subcategory of pre-sheaves which have stalk 0 at any point, hence its right orthogonal class is formed by separated pre-sheaves, while the quotient category provides the category of sheaves in abelian groups.

It turns out that the approach via Quillen exact structures is equivalent to the one via Grothendieck topologies as recently explained by Kaledin and Lowen in their paper [30, 2.2, 2.5]. The deflations (resp. the inflations) of a Quillen exact structure provide a Grothendieck pre-topology in $\mathcal{E}$ (resp. in $\mathcal{E}^\circ$). In this equivalence the notion of pre-sheaf with stalk 0 at any point would give rise to the notion of weak effaceable functor which is equivalent to the notion of effaceable functor in the finitely presented case (see Proposition 4.5).

Following the analogy with abelian sheaves on a topological space X, a pre-sheaf $\mathcal{F}$ has stalk 0 in any point $x \in X$ if and only if for any U open subset of X and $\eta \in \mathcal{F}(U)$ there exists an open covering $p : \coprod_{i\in I} U_i \twoheadrightarrow U$ such that the restriction $\mathcal{F}(p)(\eta) = \prod_{i\in I} \eta_{|U_i} = 0$. In the additive context we have the following counterpart: let $\mathcal{E}$ be a projectively complete category endowed with a Quillen exact structure $(\mathcal{E}, \mathcal{E}x)$ and fp-$\mathcal{E}$ its Freyd category. We denote by

$$\text{eff-}_{\mathcal{E}x}\mathcal{E} := \{\mathrm{Coker}_{\text{fp-}\mathcal{E}}(q) \mid q \text{ is a deflation in } \mathcal{E}x\}$$

the full subcategory of fp-$\mathcal{E}$ whose objects are cokernels of morphisms induced by deflations of $\mathcal{E}x$. We call the elements of eff-$_{\mathcal{E}x}\mathcal{E}$ effaceable functors.

Proposition 4.5. *Let $F \in$ fp-$\mathcal{E}$; the following are equivalent:*

(1) *F is effaceable;*
(2) *for any $U \in \mathcal{E}$ and $\eta \in F(U)$, there exists a deflation $p : Y \twoheadrightarrow U$ such that $F(p)(\eta) = 0$ (weak effaceable).*

Proof. Let us prove that (1) $\Rightarrow$ (2). We have to prove that for any $\eta \in F(U) \cong \mathrm{Hom}_{\text{fp-}\mathcal{E}}(\mathcal{E}_U, F)$ there exists a deflation $p : Y \twoheadrightarrow U$ such that $F(p)(\eta) = 0$. Let consider $\mathcal{E}_{E_1} \xrightarrow{q} \mathcal{E}_{E_2} \xrightarrow{\gamma} F \to 0$ with $q : E_1 \twoheadrightarrow E_2$ a deflation in $\mathcal{E}$, then there exists $\mathcal{E}_U \xrightarrow{h} \mathcal{E}_{E_2}$ (since $\mathcal{E}_U$ is projective in fp-$\mathcal{E}$) such that $\gamma h = \eta$. Let consider the following commutative diagram where $Y := E_1 \times_{E_2} U$ and p is a deflation since it is the pull-back of a deflation (the axiomatic of Quillen exact structure guarantees the existence of the fiber product Y):

$$\begin{tikzcd}
\mathcal{E}_Y \arrow[r, "p"] \arrow[d] & \mathcal{E}_U \arrow[d, "h"] \arrow[dr, "\eta"] & & \\
\mathcal{E}_{E_1} \arrow[r, "q"] \arrow[rr, bend right, "0"'] & \mathcal{E}_{E_2} \arrow[r, "\gamma"] & F \arrow[r] & 0.
\end{tikzcd}$$

hence $F(p)(\eta) = \eta p = 0$.

Let us prove that (2) $\Rightarrow$ (1). Since $F \in$ fp-$\mathcal{E}$ is finitely presented there exists $f \in \mathcal{E}(E_1, E_2)$ such that $\mathcal{E}_{E_1} \xrightarrow{f} \mathcal{E}_{E_2} \xrightarrow{\eta} F \to 0$ and by hypothesis (2) there exists a deflation $p : Y \twoheadrightarrow E_2$ such that $\eta p = 0$ hence (since $\mathcal{E}_{E_1} \twoheadrightarrow \mathrm{Ker}_{\text{fg-}\mathcal{E}}(\eta)$ and $\mathcal{E}_Y$ is projective in fg-$\mathcal{E}$) there exists $g : Y \to E_1$ such that $p = fg$ which proves that f is a deflation. $\square$

Remark 4.6. Following (2) implies (1) in the previous Proposition 4.5 we have also proved that, given any presentation $\mathcal{E}_{E_1} \xrightarrow{f} \mathcal{E}_{E_2} \xrightarrow{\eta} F \to 0$ of an effaceable functor, the map f is a deflation.

Proposition 4.7. *Let consider* fp-$\mathcal{E}$ *endowed with its maximal Quillen exact structure. The full subcategory* eff-$_{\mathcal{E}x}\mathcal{E} \subset$ fp-$\mathcal{E}$ *is right filtering; if $\mathcal{E}$ is right coherent, hence* eff-$_{\mathcal{E}x}\mathcal{E}$ *is a Serre subcategory of the abelian category* fp-$\mathcal{E}$ = coh-$\mathcal{E}$*. Dually* $\mathcal{E}$-eff$_{\mathcal{E}x} \subset \mathcal{E}$-fp *is left filtering in* $\mathcal{E}$-fp *and if $\mathcal{E}$ is left coherent, hence* $\mathcal{E}$-eff$_{\mathcal{E}x}$ *is a Serre subcategory of the abelian category* $\mathcal{E}$-fp $= \mathcal{E}$-coh.

Proof. Let us prove that eff-$_{\mathcal{E}x}\mathcal{E} \subset$ fp-$\mathcal{E}$ is right filtering; by Definition 4.2 we have to verify:

(1) eff-$_{\mathcal{E}x}\mathcal{E}$ is an extension closed full subcategory of fp-$\mathcal{E}$;
(2) eff-$_{\mathcal{E}x}\mathcal{E}$ is closed under taking admissible subobjects and admissible quotients in fp-$\mathcal{E}$;
(3) every map $f : U \to A$ with $U \in$ fp-$\mathcal{E}$ and $A \in$ eff-$_{\mathcal{E}x}\mathcal{E}$ admits a factorisation $f = g\pi$ with $U \overset{\pi}{\twoheadrightarrow} B \overset{g}{\to} A$, π a deflation and $B \in$ eff-$_{\mathcal{E}x}\mathcal{E}$.

Let us verify that eff-$_{\mathcal{E}x}\mathcal{E}$ is closed under extension in fp-$\mathcal{E}$. Let consider a conflation $0 \to T_1 \rightarrowtail T \twoheadrightarrow T_2 \to 0$ such that both T_1, T_2 are effaceable functors and let us prove that T satisfies condition (2) of Proposition 4.5. Given $\eta \in T(U) \cong$ fp-$\mathcal{E}(\mathcal{E}_U, T)$ with $U \in \mathcal{E}$, let us consider the following commutative diagram (explained below):

$$\begin{tikzcd}
\mathcal{E}_W \arrow[r, "q"] \arrow[ddr, "0"'] & \mathcal{E}_Y \arrow[dd, "\xi"] \arrow[dr, "p"] \arrow[ddrr, bend left, "0"] & & & \\
& & \mathcal{E}_U \arrow[d, "\eta"] \arrow[dr, "\beta\eta"] & & \\
0 \arrow[r] & T_1 \arrow[r, "\alpha"] & T \arrow[r, "\beta"] & T_2 \arrow[r] & 0
\end{tikzcd}$$

Because T_2 is effaceable, there exists a deflation $p : Y \twoheadrightarrow U$ such that $\beta\eta p = 0$, and so $\eta p = T(p)(\eta)$ factors through α via $\xi \in$ fp-$\mathcal{E}(\mathcal{E}_Y, T_1)$. Now, since T_1 is effaceable, there exists $q : W \twoheadrightarrow Y$ such that $\xi q = T_1(q)(\xi) = 0$, hence $0 = \alpha\xi q = \eta p q = T(pq)(\eta)$. We remark that pq is a deflation since it is a composition of two deflations, therefore the previous construction proves that T is effaceable.

Let us prove that eff-$_{\mathcal{E}x}\mathcal{E}$ is closed under admissible subobjects and admissible quotients. Let $0 \to T_1 \rightarrowtail T \twoheadrightarrow T_2 \to 0$ be a conflation in fp-$\mathcal{E}$ with $T \in$ eff-$_{\mathcal{E}x}\mathcal{E}$. Given $U \in \mathcal{E}$ and $\eta \in T_1(U)$, there exists a deflation $p : Y \twoheadrightarrow U$ such that $\alpha(Y)(T_1(p)(\eta)) = T(p)(\alpha(U)(\eta)) = 0$, which proves that $T_1(p)(\eta) = 0$ (since $\alpha(Y)$ is a monomorphism of abelian groups by Proposition 4.1). Given an object V of $\mathcal{E}$ and $\xi \in T_2(V) \cong$ fp-$\mathcal{E}(\mathcal{E}_V, T_2)$, there exists $\sigma : \mathcal{E}_V \to T$ such that $\xi = \beta\sigma$ (because β is a deflation). Since T is effaceable, there exists $q : W \twoheadrightarrow V$ such that $\sigma q = 0$, which implies $\xi q = T_2(q)(\xi) = 0$ and so T_2 is effaceable.

Let consider: $f : U \to A$, $\mathcal{E}_{U_2} \overset{h}{\to} \mathcal{E}_{U_1} \to U \to 0$ a presentation of $U \in$ fp-$\mathcal{E}$ and $\mathcal{E}_{A_2} \overset{p}{\to} \mathcal{E}_{A_1} \to A \to 0$ a presentation of $A \in$ eff-$_{\mathcal{E}x}\mathcal{E}$ with p a deflation. Since representable functors are projective in fp-$\mathcal{E}$ there exist $f_i : \mathcal{E}_{U_i} \to \mathcal{E}_{A_i}$ with $i \in \{1, 2\}$ such that $pf_2 = f_1 h$. Hence the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}_{U_2} \arrow[r, "r"] \arrow[d, "h"'] \arrow[rr, dotted, bend right, "f_2"'] & \mathcal{E}_{U_1 \times_{A_1} A_2} \arrow[r] \arrow[d, "q"] & \mathcal{E}_{A_2} \arrow[d, "p"] \\
\mathcal{E}_{U_1} \arrow[r, equal] \arrow[d] & \mathcal{E}_{U_1} \arrow[r, "f_1"] \arrow[d] & \mathcal{E}_{A_1} \arrow[d, two heads] \\
U \arrow[r, "\pi"] \arrow[rr, dotted, bend right, "f"'] & \mathrm{Coker}_{\text{fp-}\mathcal{E}}(q) \arrow[r, "g"] & A
\end{tikzcd}$$

Thus $\ker\pi$ belongs to fp-$\mathcal{E}$, because $\mathcal{E}_{U_2} \overset{r}{\to} \mathcal{E}_{U_1 \times_{A_1} A_2} \to \ker\pi \to 0$ is exact, and the sequence $0 \to \ker\pi \to U \overset{\pi}{\to} \mathrm{Coker}_{\text{fp-}\mathcal{E}}(q) \to 0$ is a conflation since it is pointwise exact. This proves that $\mathrm{Coker}_{\text{fp-}\mathcal{E}}(q)$ belongs to eff-$_{\mathcal{E}x}\mathcal{E}$ (q is a deflation since it is the pullback of p). When $\mathcal{A}$ is an abelian category conditions (1) and (2) prove that eff-$_{\mathcal{E}x}\mathcal{A}$ is a Serre subcategory of coh-$\mathcal{A}$. The left statement follows by duality. $\square$

5. n-coherent categories

We have seen that the main difference between 1-quasi-abelian categories and 2-quasi-abelian ones is the need of Quillen exact structures. The passage from $n = 2$ to $n \geq 3$ requires a new technicality due to the possible absence of kernels and cokernels. Let $(\mathcal{E}, \mathcal{E}x)$ be a projectively complete category endowed with a Quillen exact structure. We are looking for a definition of n-quasi-abelian category which permits us to associate to $(\mathcal{E}, \mathcal{E}x)$ a n-tilting pair of t-structures on $D(\mathcal{E}, \mathcal{E}x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$. By Proposition 2.5 we know that ,if these t-structures exist, they are the left and right t-structures:

$$\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} := \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\leq 0} \text{ in } D(\mathcal{E}, \mathcal{E}x) \text{ with } E^\bullet_{\leq 0} \in K^{\leq 0}(\mathcal{E})\}$$

$$\mathcal{RD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)} := \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\geq 1} \text{ in } D(\mathcal{E}, \mathcal{E}x) \text{ with } E^\bullet_{\geq 1} \in K^{\geq 1}(\mathcal{E})\}.$$

In the following we will use the notions of coherent functor, coherent category (Definition B.7) weak kernels and cokernels; we refer to Appendix B for more details. First of all we study the case of $(\mathcal{E}, \mathcal{E}x_{\mathrm{split}})$ which gives $D(\mathcal{E}, \mathcal{E}x_{\mathrm{split}}) = K(\mathcal{E})$.

Proposition 5.1. *The followings hold:*

(1) *the class $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ is an aisle in $K(\mathcal{E})$ if and only if $\mathcal{E}$ is right coherent;*
(2) *the class $\mathcal{RK}^{\geq 1}_{\mathcal{E}}$ is a co-aisle in $K(\mathcal{E})$ if and only if $\mathcal{E}$ is left coherent.*

If $\mathcal{E}$ is a right coherent category we have $\mathcal{LK}(\mathcal{E}) \simeq$ coh-$\mathcal{E}$; dually if $\mathcal{E}$ is left-coherent $\mathcal{RK}(\mathcal{E}) \simeq (\mathcal{E}\text{-coh})^\circ$. Moreover given $\mathcal{E}$ a coherent category $\mathcal{RK}^{\leq -n}_{\mathcal{E}} \subseteq \mathcal{LK}^{\leq 0}_{\mathcal{E}} \subseteq \mathcal{RK}^{\leq 0}_{\mathcal{E}}$ (with n minimal) if and only if coh-$\mathcal{E}$ *(or equivalently* $\mathcal{E}$-coh*) has projective dimension n.*

Proof. Statement (2) is dual to (1). Let us recall that by Proposition B.10 $\mathcal{E}$ is right coherent if and only if it admits weak kernels.

Let $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ be an aisle (we denote by $\tau^{\leq 0}_{\mathcal{L}}$ its truncation functor) and let us prove that $\mathcal{E}$ is right coherent. Let $d : E_0 \to E_1$ be a morphism in $\mathcal{E}$ and let us regard it as a complex $E^\bullet := [\overset{\bullet}{E_0} \xrightarrow{d} E_1]$. The universal property of the truncation $[\cdots \to K^{-1} \to \overset{\bullet}{K^0}] = \tau^{\leq 0}_{\mathcal{L}}(E^\bullet) \xrightarrow{\alpha^\bullet} E^\bullet$ implies that (K^0, α^0) is a weak kernel for d.

On the other side let us suppose that $\mathcal{E}$ is right coherent and let us prove that $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ is an aisle in $K(\mathcal{E})$. Notice that $\mathcal{LK}^{\leq 0}_{\mathcal{E}}$ is extension closed in $K(\mathcal{E})$ and $\mathcal{LK}^{\leq 0}_{\mathcal{E}}[1] \subseteq \mathcal{LK}^{\leq 0}_{\mathcal{E}}$. Since $\mathcal{E}$ is right coherent, the Freyd category of (contravariant) finitely presented functor is abelian fp-$\mathcal{E}$ = coh-$\mathcal{E}$ (Proposition B.10) and $\mathcal{E}$ coincides with the class of projective objects in coh-$\mathcal{E}$; thus $D^-(\text{coh-}\mathcal{E}) \simeq K^-(\mathcal{E})$, $D^{\leq 0}(\text{coh-}\mathcal{E}) \simeq \mathcal{LK}^{\leq 0}_{\mathcal{E}}$ and their hearts are equivalent: $\mathcal{LK}(\mathcal{E}) \simeq$ coh-$\mathcal{E}$. The category coh-$\mathcal{E}$ has finite projective dimension n if and only if given any $E^\bullet \in \mathcal{RK}^{\geq 0}_{\mathcal{E}}$ the kernel $\mathrm{Ker}_{\text{coh-}\mathcal{E}}(d^0_E)$ admits a resolution of length at most $n-2$ (since $0 \to \mathrm{Ker}_{\text{coh-}\mathcal{E}}(d^0_E) \to E^0 \to E^1 \to \mathrm{Coker}_{\text{coh-}\mathcal{E}}(d^0_E) \to 0$ is exact and any projective resolution of $\mathrm{Coker}_{\text{coh-}\mathcal{E}}(d^0_E)$ has at most length n). This is equivalent to require that $\tau^{\geq 1}_{\mathcal{L}} X^\bullet \cong \tau^{\geq 1}_{\mathcal{L}} X^{\geq 0} \subseteq \mathcal{RK}^{\leq -n+2}_{\mathcal{E}}$ for any $X^\bullet \in K(\mathcal{E})$ (see 1.13) which is equivalent to $\mathcal{RK}^{\leq -n}_{\mathcal{E}} \subseteq \mathcal{LK}^{\leq 0}_{\mathcal{E}} \subseteq \mathcal{RK}^{\leq 0}_{\mathcal{E}}$. In this case n is called the *global dimension* of $\mathcal{E}$. $\square$

Definition 5.2. A coherent category of global dimension at most n will be said *n-coherent*. For example the category proj-R of projective (right) modules of finite type on a coherent ring R with global dimension n is n-coherent.

6. n-TILTING TORSION CLASSES FOR $n > 2$

Definition 6.1. Let $(\mathcal{E}, \mathcal{E}x)$ be a projectively complete category endowed with a Quillen exact structure and let $f : A \to B$ be a morphism in $\mathcal{E}$. A *$(\mathcal{E}, \mathcal{E}x)$-pre-kernel* of f is a map $i : K \to A$ in $\mathcal{E}$ such that $f \circ i = 0$ and for any $j : X \to A$ such that $f \circ j = 0$ there exist (possibly many) a deflation $\pi : N \twoheadrightarrow X$ and a map $k : N \to K$ such that $j\pi = ik$:

$$\begin{tikzcd}
N \arrow[r, dotted, two heads, "\pi"] \arrow[d, dotted, "k"'] & X \arrow[d, "j"'] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B.
\end{tikzcd}$$

The category $(\mathcal{E}, \mathcal{E}x)$ has $(\mathcal{E}, \mathcal{E}x)$*-pre-pull-back squares* if, given any pair $f_i : X_i \to Y$ with $i = 1, 2$, there exist an object Z and $g_i : Z \to X_i$ such that $f_1 g_1 = f_2 g_2$ and, for any pair of arrows $\alpha_i : W \to X_i$ with $i \in \{1, 2\}$ such that $\alpha_1 f_1 = \alpha_2 f_2$, there exist (not necessary unique) a deflation $\pi : N \twoheadrightarrow W$ and a map $k : N \to Z$ such that the diagram below commutes:

$$\begin{tikzcd}
W \arrow[rrr, bend left, "\alpha_1"] \arrow[drr, "\alpha_2"'] & N \arrow[l, two heads, dotted, "\pi"'] \arrow[r, dotted, "k"] & Z \arrow[r, dashed, "g_1"] \arrow[d, dashed, "g_2"'] & X_1 \arrow[d, "f_1"] \\
 & & X_2 \arrow[r, "f_2"] & Y.
\end{tikzcd} \tag{2}$$

Passing throughout the opposite category one obtains the dual notion of $(\mathcal{E}, \mathcal{E}x)$-*pre-cokernel* and $(\mathcal{E}, \mathcal{E}x)$*-pre-push-out square.*

Remark 6.2. The notion of $(\mathcal{E}, \mathcal{E}x)$-pre-kernel (resp. $(\mathcal{E}, \mathcal{E}x)$-pre-cokernel) is not functorial due to the lack of unicity of the arrows involved in in its definition. Nevertheless its existence is equivalent to require the existence of kernels in the quotient category $\frac{\text{fp-}\mathcal{E}}{\text{eff-}_{\mathcal{E}x}\mathcal{E}}$ (see Remark 6.2) which is a necessary and sufficient condition to prove that this quotient category is an abelian category (see Theorem 6.11). When the Quillen exact structure coincides with the minimal one, we have $D(\mathcal{E}, \mathcal{E}x_{\text{split}}) = K(\mathcal{E})$ and the previous definitions reduce to the notions *weak kernel* and *weak pull-back square* (see Definition B.9).

If $\mathcal{E}$ admits weak kernels it admits $(\mathcal{E}, \mathcal{E}x)$-pre-kernels for any Quillen exact structure on $\mathcal{E}$, since any weak kernel is also a $(\mathcal{E}, \mathcal{E}x)$-pre-kernel. More generally if $\mathcal{E}$ admits $(\mathcal{E}, \mathcal{E}x)$-pre-kernels, given any other Quillen exact structure $\overline{\mathcal{E}x}$ containing the conflations of $\mathcal{E}x$, we have that $\mathcal{E}$ admits $(\mathcal{E}, \overline{\mathcal{E}x})$-pre-kernels.

Lemma 6.3. *Let $(\mathcal{E}, \mathcal{E}x)$ be a projectively complete category endowed with a Quillen exact structure and $D(\mathcal{E}, \mathcal{E}x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$ its derived category. The classes*

$$\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} := \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\leq 0} \textit{ in } D(\mathcal{E}, \mathcal{E}x) \textit{ with } E^\bullet_{\leq 0} \in C^{\leq 0}(\mathcal{E})\}$$

$$\mathcal{RD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)} := \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E^\bullet_{\geq 1} \textit{ in } D(\mathcal{E}, \mathcal{E}x) \textit{ with } E^\bullet_{\geq 1} \in C^{\geq 1}(\mathcal{E})\}$$

are extension closed in $D(\mathcal{E}, \mathcal{E}x)$.

Proof. We have to prove that for any morphism $Y^\bullet[-1] \xrightarrow{\delta} X^\bullet$ in $D(\mathcal{E}, \mathcal{E}x)$ with $X^\bullet, Y^\bullet \in C^{\leq 0}(\mathcal{E})$ the mapping cone $M(\delta) \in \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$. We can represent δ as $Y^\bullet[-1] \xrightarrow{\delta'} F^\bullet \underset{\cong}{\xleftarrow{\alpha}} X^\bullet$ with δ' and α morphisms in $K(\mathcal{E})$. Since any $N^\bullet \in \mathcal{N}_{\mathcal{E}x}$ fits in the distinguished triangle in $K(\mathcal{E})$: $N^\bullet_0 \to N^\bullet \to N^\bullet_1 \xrightarrow{+}$, with $N^\bullet_0 = [\cdots \to N^{-1} \to \overset{\bullet}{\mathrm{Ker}}\, d^0_{N^\bullet}] \in C^{\leq 0}(\mathcal{E}) \cap \mathcal{N}_{\mathcal{E}x}$ and $N^\bullet_1 = [\overset{\bullet}{\mathrm{Ker}}\, d^1_{N^\bullet} \to N^1 \to \cdots] \in \mathcal{N}_{\mathcal{E}x}$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{\alpha_0} & F^\bullet_0 & \longrightarrow & N^\bullet_0 & \xrightarrow{+} & \\
\downarrow & & \downarrow\varphi & & \downarrow & & \\
X^\bullet & \xrightarrow{\alpha} & F^\bullet & \longrightarrow & N^\bullet & \xrightarrow{+} & \\
\downarrow & & \downarrow\psi & & \downarrow & & \\
0 & \longrightarrow & N^\bullet_1 & \longrightarrow & N^\bullet_1 & \xrightarrow{+} & \\
\downarrow+ & & \downarrow+ & & \downarrow+ & &
\end{array}$$

Notice that α_0 and φ are isomorphisms in $D(\mathcal{E}, \mathcal{E}x)$ with $F^\bullet_0 \in C^{\leq 0}(\mathcal{E})$ and, since the composition $\psi \circ \delta' = 0$ (as a morphism in $K(\mathcal{E})$), we obtain $\delta' = \varphi \circ \overline{\delta}$ (in $K(\mathcal{E})$). Thus $M(\delta') \in C^{\leq 0}(\mathcal{E})$ and $M(\delta) \cong M(\delta')$ in $D(\mathcal{E}, \mathcal{E}x)$ which concludes the proof.

The analog result for $\mathcal{RD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)}$ follows dually. $\square$

Lemma 6.4. *Under the previous hypotheses:*

(1) *The subcategory $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ is an aisle in $D(\mathcal{E},\mathcal{E}x)$ if and only if $\mathcal{E}$ has $(\mathcal{E},\mathcal{E}x)$-pre-kernels.*

(2) *The subcategory $\mathcal{RD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)}$ is a co-aisle in $D(\mathcal{E},\mathcal{E}x)$ if and only if $\mathcal{E}$ has $(\mathcal{E},\mathcal{E}x)$-pre-cokernels.*

If the previous equivalent conditions hold $\mathcal{E} = \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \cap \mathcal{RD}^{\geq 0}_{(\mathcal{E},\mathcal{E}x)}$; any object in the heart $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ can be represented as a complex $K^\bullet \in C^{\leq 0}(\mathcal{E})$ such that $K^i = (\mathcal{E},\mathcal{E}x)$-pre-kernel of d^{i+1}_K for any $i \leq -2$. Dually objects in $\mathcal{RH}(\mathcal{E},\mathcal{E}x)$ are complexes $C^\bullet \in C^{\geq 0}(\mathcal{E})$ such that $C^i = (\mathcal{E},\mathcal{E}x)$-pre-cokernel of d^{i-2}_C for any $i \geq 2$.

Proof. (1). Let $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ be an aisle in $D(\mathcal{E},\mathcal{E}x)$. Any morphism $f : A \to B$ in $\mathcal{E}$ can be regarded as a complex $M^\bullet := [\overset{\bullet}{A} \overset{f}{\to} B] \in C^{\geq 0}(\mathcal{E})$. Let denote by $\alpha : K^\bullet \to M^\bullet$ with $K^\bullet = [\cdots \to K^{-1} \to K^0] \in \mathcal{C}^{\leq 0}(\mathcal{E})$ its truncation with respect to $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ (notice that $\alpha \in K(\mathcal{E})$ by A.2). Thus $K^0 \overset{\alpha^0}{\to} A$ is a $(\mathcal{E},\mathcal{E}x)$-pre-kernel for f: any morphism j in $\mathcal{E}$, such that $fj = 0$, induces a morphism $[\overset{\bullet}{X}] \overset{j}{\to} M^\bullet$ which factorizes trought $K^\bullet$ in $D(\mathcal{E},\mathcal{E}x)$, i.e.; $\alpha\beta = j$ with $[\overset{\bullet}{X}] \underset{\cong}{\overset{\varphi}{\leftarrow}} N^\bullet \overset{\beta}{\to} K^\bullet$, providing a deflation $N^0 \overset{\varphi^0}{\twoheadrightarrow} X$ and a morphism $N^0 \overset{\beta^0}{\to} K^0$ such that $j\varphi^0 = \alpha^0\beta^0$.

On the other side let us suppose that $\mathcal{E}$ has $(\mathcal{E},\mathcal{E}x)$-pre-kernels. The full subcategory $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ of $D(\mathcal{E},\mathcal{E}x)$ is closed by [1] and extensions (Lemma 6.3). Let us construct the truncation functor $\tau^{\leq 0}_{\mathcal{L}} : D(\mathcal{E},\mathcal{E}x) \to \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ in two steps.

Step 1. Given $L^\bullet := [\overset{\bullet}{L^0} \overset{d^0_L}{\to} L^1 \to \cdots] \in C^{\geq 0}(\mathcal{E})$, let $K^0 \overset{i}{\to} L^0$ be a $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d^0_L, $K^{-1} \overset{d^{-1}_K}{\to} K^0$ a $(\mathcal{E},\mathcal{E}x)$-pre-kernel of i and, recursively, $K^{-i} \overset{d^{-i}_K}{\to} K^{-i+1}$ be a $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d^{-i+1}_K with $i \geq 2$. Hence i induces a morphism of complexes $K^\bullet \overset{i}{\to} L^\bullet$ with $K^\bullet := [\cdots \overset{d^{-2}_K}{\to} K^{-1} \overset{d^{-1}_K}{\to} K^0] \in \mathcal{C}^{\leq 0}(\mathcal{E})$. It remains to prove that the mapping cone of i, $M^\bullet := M(i)$, belongs to $\mathcal{LD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)} := (\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)})^\perp$.

Notice that $d^{-i}_{M^\bullet}$ is the $(\mathcal{E},\mathcal{E}x)$-pre-kernel of $d^{-i+1}_{M^\bullet}$ for any $i \geq 1$ and, whenever $\psi : M^\bullet \to Y^\bullet$ is a qis in $D(\mathcal{E},\mathcal{E}x)$, the same property holds true for $Y^\bullet$ (since the mapping cone of ψ is in $\mathcal{N}_{\mathcal{E}x}$). Let $X^\bullet \in C^{\leq 0}(\mathcal{E})$ and $X^\bullet \overset{\gamma}{\to} Y^\bullet \underset{\cong}{\overset{\psi}{\leftarrow}} M^\bullet$ be a morphism in $D(\mathcal{E},\mathcal{E}x)$ (γ and ψ are morphisms of complexes). Hence $d^0_Y\gamma_0 = 0$ and, since d^{-1}_Y is the $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d^0_Y, there exists $p^0 : W^0 \twoheadrightarrow X^0$ and $s^0 : W^0 \to Y^{-1}$ such that $\gamma^0 p^0 = d^{-1}_Y s^0$. Let consider the cartesian square

$$\begin{tikzcd}
\widetilde{X}^{-1} \arrow[r, "\tilde{d}^{-1}"] \arrow[d, twoheadrightarrow, "p^{-1}"'] & W^0 \arrow[d, twoheadrightarrow, "p^0"] \\
X^{-1} \arrow[r, "d^{-1}_X"'] & X^0
\end{tikzcd}$$

and let us denote by $\phi_{-1} := \gamma^{-1}p^{-1} - s^0\tilde{d}^{-1}$. We have $d^{-1}_Y\phi_{-1} = 0$ and, since d^{-2}_Y is the $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d^{-1}_Y, there exists $q^{-1} : W^{-1} \twoheadrightarrow \widetilde{X}^{-1}$ and $s^{-1} : W^{-1} \to Y^{-2}$ such that $\phi_{-1}q^{-1} = d^{-2}_Y s^{-1}$ which gives $\gamma^{-1}p^{-1}q^{-1} = d^{-2}_Y s^{-1} + s^0\tilde{d}^{-1}q^{-1}$. Let define $p^{-1} := \tilde{p}^0 q^{-1}$ and $d^{-1}_W := \tilde{d}^{-1}q^{-1}$. Iterating the argument we construct a qis $W^\bullet \overset{p^\bullet}{\to} X^\bullet$ such that $\gamma p^\bullet$ is null up to homotopy (via the s^i) which proves that $\gamma = 0$ in $D(\mathcal{E},\mathcal{E}x)$.

Step 2. Given $E^\bullet \in C(\mathcal{E})$, its truncation $\tau^{\leq 0}_{\mathcal{L}}(E^\bullet)$ is the mapping cone of the morphism $\tau^{\leq 0}_{\mathcal{L}}(d^{-1}_{E^\bullet})$, described in the following commutative diagram, since by the previous step $\tau_{\mathcal{L}}^{\geq 1}(E^\bullet) \in \mathcal{LD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)}$:

$$\begin{tikzcd}
E^{\leq -1}[-1] \arrow[r, "id"] \arrow[d, "\tau^{\leq 0}(d^{-1}_{E^\bullet})"] & E^{\leq -1}[-1] \arrow[r] \arrow[d, "d^{-1}_{E^\bullet}"] & 0 \arrow[r, "+"] \arrow[d] & {} \\
\tau_{\mathcal{L}}^{\leq 0}(E^{\geq 0}) \arrow[r] \arrow[d] & E^{\geq 0} \arrow[r] \arrow[d] & \tau_{\mathcal{L}}^{\geq 1}(E^{\geq 0}) \arrow[r, "+"] \arrow[d, "\cong"] & {} \\
\tau_{\mathcal{L}}^{\leq 0}(E^\bullet) \arrow[r] \arrow[d, "+"] & E^\bullet \arrow[r] \arrow[d, "+"] & \tau_{\mathcal{L}}^{\geq 1}(E^\bullet) \arrow[r, "+"] \arrow[d, "+"] & {} \\
{} & {} & {} &
\end{tikzcd} \tag{3}$$

An object in the heart $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ can be represented as a complex $K^\bullet \in K^{\leq 0}(\mathcal{E})$ such that $\tau_{\mathcal{L}}^{\leq -1}(K^\bullet) \in \mathcal{N}_{\mathcal{E}x}$ and so $K^i = D(\mathcal{E},\mathcal{E}x)$-kernel of d_K^{i+1} for any $i \leq -2$. Statement (2) is dual to (1).

If $\mathcal{E}$ admits $(\mathcal{E},\mathcal{E}x)$-pre-kernels and $(\mathcal{E},\mathcal{E}x)$-pre-cokernels, let us denote by $\tau_{\mathcal{L}}^{\leq 0}$ (resp. $\delta_{\mathcal{R}}^{\geq 1}$) the truncation functor with respect to $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ (resp. $\mathcal{RD}^{\geq 1}_{(\mathcal{E},\mathcal{E}x)}$). Hence $E^\bullet \in \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \cap \mathcal{RD}^{\geq 0}_{(\mathcal{E},\mathcal{E}x)}$ if and only if the composition

$$\gamma : K^\bullet := \tau_{\mathcal{L}}^{\leq 0} E^\bullet \to E^\bullet \to \delta_{\mathcal{R}}^{\geq 0}(E^\bullet) =: C^\bullet$$

is an isomorphism in $D(\mathcal{E},\mathcal{E}x)$ i.e., if and only if the mapping cone $M(\gamma) \in \mathcal{N}_{\mathcal{E}x}$:

$$\begin{tikzcd}
K^\bullet : \arrow[d, "\gamma"] & \cdots \arrow[r] \arrow[d] & K^{-1} \arrow[r, "d_K^{-1}"] \arrow[d] & K^0 \arrow[r] \arrow[d, "\gamma^0"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & {} \\
C^\bullet : \arrow[d] & \cdots \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C^0 \arrow[r, "d_C^0"] \arrow[d] & C^1 \arrow[r, "d_C^1"] \arrow[d] & \cdots \arrow[r] \arrow[d] & {} \\
M(\gamma) : & K^{-1} \arrow[rr, "d_K^{-1}"] \arrow[dr, two heads] & & K^0 \arrow[r, "\gamma^0"] \arrow[dr, two heads] & C^0 \arrow[r, "d_C^0"] \arrow[dr, two heads] & C^1 \arrow[r, "d_C^1"] \arrow[dr, two heads] & \cdots \arrow[r] & {} \\
& & W^{-1} \arrow[ur, tail] & & W^0 \arrow[ul, phantom] & W^1 & W^2 \quad \cdots
\end{tikzcd}$$

this proves that $K^\bullet \cong W^0[0] \in \mathcal{E}$. ☐

Lemma 6.5. *Let us suppose that $(\mathcal{E},\mathcal{E}x)$ admits $(\mathcal{E},\mathcal{E}x)$-pre-kernels and $(\mathcal{E},\mathcal{E}x)$-pre-cokernels. Then $\mathcal{RD}^{\leq -n}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{RD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ (with $n \geq 2$) if and only if one of the following equivalent conditions hold:*

1. *given any complex $K^{[-n+1,0]} := K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} K^0$ with $K^i = (\mathcal{E},\mathcal{E}x)$-kernel of d_K^{i+1} for any $i \leq -2$, the morphism d_K^{-n+1} has a kernel in $\mathcal{E}$;*
2. *given any complex $C^{[-n+1,0]} := C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \cdots \xrightarrow{d_C^{-1}} C^0$ with $C^i = (\mathcal{E},\mathcal{E}x)$-cokernel of d_C^{i-2} for any $i \geq -n-1$, the morphism d_C^{-1} has a cokernel in $\mathcal{E}$.*

In this case the sequence in (1) is exact in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ while the one in (2) is exact in $\mathcal{RH}(\mathcal{E},\mathcal{E}x)$ and the pair $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ is a n-tilting pair of t-structures on $D(\mathcal{E},\mathcal{E}x)$.

Proof. Let us suppose that $n \geq 2$ and $\mathcal{RD}^{\leq -n}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{RD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$. Given a complex $K^{[-n+1,0]}$ with $K^i = (\mathcal{E},\mathcal{E}x)$-kernel of d_K^{i+1} for any $i \leq -2$, by the proof of Lemma 6.4 the complex $\tau_{\mathcal{L}}^{\leq -n+1}(K^{[-n+1,0]})$ is constructed taking in degree $-n+1$ the $(\mathcal{E},\mathcal{E}x)$-kernel of d_K^{-n+1} and taking in degrees $i < -n+1$ the $(\mathcal{E},\mathcal{E}x)$-kernel of the differential $i+1$. Thus $\tau_{\mathcal{L}}^{\geq -n+2}(K^{[-n+1,0]}) \cong \tau_{\mathcal{L}}^{\geq 0}(K^{[-n+1,0]}) \in \mathcal{LH}(\mathcal{E},\mathcal{E}x)$ since any term of this complex is a $(\mathcal{E},\mathcal{E}x)$-kernel of its successive differential and the complex $K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} K^0$ is exact in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$.

By hypothesis $\mathcal{LH}(\mathcal{E},\mathcal{E}x) \subseteq \mathcal{RD}^{\geq -n}_{(\mathcal{E},\mathcal{E}x)}$ and since $K^{[-n+1,0]} \in \mathcal{RD}^{[-n+1,0]}_{(\mathcal{E},\mathcal{E}x)}$ we get $\tau^{\leq -n+1}_{\mathcal{L}}(K^{[-n+1,0]}) \in \mathcal{RD}^{\geq -n+1}_{(\mathcal{E},\mathcal{E}x)} \cap \mathcal{LD}^{\leq -n+1}_{(\mathcal{E},\mathcal{E}x)} = \mathcal{E}[n-1]$.

The dual argument proves that (2) holds true and the sequence in (2) is exact in $\mathcal{RH}(\mathcal{E},\mathcal{E}x)$.

On the other side if (1) holds true, given $X^\bullet \in D(\mathcal{E},\mathcal{E}x)$, we have $\tau^{\geq 1}_{\mathcal{L}} X^\bullet \cong \tau^{\geq 1}_{\mathcal{L}} X^{\geq 0} \subseteq \mathcal{RD}^{\geq -n+1}_{(\mathcal{E},\mathcal{E}x)}$ (since $\tau^{\leq 0}_{\mathcal{L}} X^{\geq 0} \in \mathcal{RD}^{\geq -n}_{(\mathcal{E},\mathcal{E}x)}$), hence $\mathcal{RD}^{\leq -n}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{RD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$. Therefore any object $K^\bullet \in \mathcal{LH}(\mathcal{E},\mathcal{E}x)$ can be represented as a complex $K^\bullet \in K^{\leq 0}(\mathcal{E})$ such that $\tau^{\leq -1}_{\mathcal{L}}(K^\bullet) \in \mathcal{N}_{\mathcal{E}x}$ and so it can be represented by a complex

$$C(d_K^{-n},\dots,d_K^{-1}) := [\ \mathrm{Ker}(d_K^{-n+1}) \xrightarrow{d_K^{-n}} K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} \overset{\bullet}{K^0}\ ]$$

such that K^i is a $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d_K^{i+1} for any $i \leq -2$. The following distinguished triangle in $D(\mathcal{E},\mathcal{E}x)$ provides a short exact sequence in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$

$$0 \longrightarrow C(0,d_K^{-n},\dots,d_K^{-2}) \longrightarrow K^0[0] \longrightarrow C(d_K^{-n},\dots,d_K^{-1}) \longrightarrow 0$$

which proves that $\mathcal{E}$ generates $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$. Hence $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \simeq D(\mathcal{LH}(\mathcal{E},\mathcal{E}x))$ since the full subcategory $\mathcal{E}$ in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ satisfies the hypotheses of Proposition C.3. Dually $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \simeq D(\mathcal{RH}(\mathcal{E},\mathcal{E}x))$. □

Definition 6.6. A projectively complete category $(\mathcal{E},\mathcal{E}x)$ endowed with a Quillen exact structure is called *n-quasi-abelian* (for $n \geq 2$) if it admits $(\mathcal{E},\mathcal{E}x)$-pre-kernels and $(\mathcal{E},\mathcal{E}x)$-pre-cokernels and one of the following equivalent conditions holds:

(1) For any complex $K^{[-n+1,0]} := K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} K^0$ such that K^i is $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d_K^{i+1} for any $i \leq -2$ the morphism d_K^{-n+1} has a kernel in $\mathcal{E}$.

(2) For any complex $C^{[-n+1,0]} := C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \cdots \xrightarrow{d_C^{-1}} C^0$ such that C^i is $(\mathcal{E},\mathcal{E}x)$-pre-cokernel of d_C^{i-2} for any $i \geq -n-1$ the morphism d_C^{-1} has a cokernel in $\mathcal{E}$.

Whenever the exact structure is not specified, we will consider $\mathcal{E}$ endowed with its maximal Quillen exact structure $\mathcal{E}x_{\max}$.

Theorem (see 2.1) and Definition 3.2 suggest the following n-level generalization of the notion of 1-tilting torsion class in an abelian category:

Definition 6.7. Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{E} \hookrightarrow \mathcal{A}$ is a *n-tilting torsion class* if

(1) $\mathcal{E}$ cogenerates $\mathcal{A}$;
(2) $\mathcal{E}$ is extension closed in $\mathcal{A}$, hence it is endowed with the Quillen exact structure $\mathcal{E}x$ whose conflations are sequences in $\mathcal{E}$ which are exact in $\mathcal{A}$;
(3) $\mathcal{E}$ has $(\mathcal{E},\mathcal{E}x)$-pre-kernels;
(4) for any exact sequence in $\mathcal{A}$: $0 \to A \to X_1 \overset{d_X^1}{\to} \cdots \overset{d_X^{n-1}}{\to} X_n \to B \to 0$ with $X_i \in \mathcal{E}$ for any $1 \leq i \leq n$ and $A, B \in \mathcal{A}$, we have $B \in \mathcal{E}$.

Dually a *n-cotilting torsion-free class* in $\mathcal{A}$ is a full generating extension closed subcategory $\mathcal{E}$ of $\mathcal{A}$ admitting $(\mathcal{E},\mathcal{E}x)$-cokernels and such that for any exact sequence in $\mathcal{A}$: $0 \to A \to Y_1 \overset{d_Y^1}{\to} \cdots \overset{d_Y^{n-1}}{\to} Y_n \to B \to 0$ with $Y_i \in \mathcal{E}$ we have $A \in \mathcal{E}$.

Remark 6.8. Given $(\mathcal{E},\mathcal{E}x)$ a n-quasi-abelian category by Lemma 6.5 $\mathcal{E}$ is a n-tilting torsion class in $\mathcal{RH}(\mathcal{E},\mathcal{E}x)$ and $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ is a n-tilting pair of t-structures on $D(\mathcal{E},\mathcal{E}x)$.

On the other hand, given a n-tilting pair of t-structures $(\mathcal{D},\mathcal{T})$ on $\mathcal{C}$ by Proposition 2.5 and Lemma 6.4, the category $\mathcal{E}=\mathcal{T}^{\leq 0}\cap\mathcal{D}^{\geq 0}$ (with the Quillen exact structure induced by $D(\mathcal{E},\mathcal{E}x)$) admits $(\mathcal{E},\mathcal{E}x)$-pre-kernels and $(\mathcal{E},\mathcal{E}x)$-pre-cokernels, hence it is n-quasi-abelian.

Theorem 6.9. *Any n-tilting torsion class $\mathcal{E}$ in $\mathcal{A}$, endowed with the Quillen exact structure induced by $\mathcal{A}$, is n-quasi-abelian; $\mathcal{A}\simeq\mathcal{RH}(\mathcal{E},\mathcal{E}x)$ and $\frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}}\simeq D(\mathcal{A})$.*

Proof. Conditions (1) and (4) of Definition 6.7 imply that $\mathcal{E}$ satisfies the hypotheses of Proposition C.3 and so $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}\simeq D(\mathcal{A})$. Since $D^{\geq 0}(\mathcal{A})\simeq\mathcal{RD}^{\geq 0}_{(\mathcal{E},\mathcal{E}x)}$ we obtain that $\mathcal{RH}(\mathcal{E},\mathcal{E}x)\simeq\mathcal{A}$.

By Lemma 6.4 $\mathcal{E}$ has $(\mathcal{E},\mathcal{E}x)$-pre-cokernels and by point (3) of Definition 6.7 $\mathcal{E}$ admits $(\mathcal{E},\mathcal{E}x)$-pre-kernels. Moreover by Lemma 6.5 the complex $C^{[-n+1,0]}:=$ $C^{-n+1}\xrightarrow{d_C^{-n+1}}C^{-n+2}\longrightarrow\cdots\xrightarrow{d_C^{-1}}C^0$ is exact in $\mathcal{A}$ if and only if C^i is a $(\mathcal{E},\mathcal{E}x)$-pre-kernel of d_C^{i-2} for any $i\geq -n-1$, hence by Definition 6.7, the morphism d_C^{-1} has a cokernel in $\mathcal{E}$. $\square$

Corollary 6.10. *Let $\mathcal{D}$ be the natural t-structure on the triangulated category $D(\mathcal{H}_\mathcal{D})$ and $i:\mathcal{E}\to\mathcal{H}_\mathcal{D}$ a n-tilting torsion class on $\mathcal{H}_\mathcal{D}$. Hence $\mathcal{T}^{\leq 0}:=\mathcal{D}^{\leq -n}\star\mathcal{E}\star\mathcal{E}[1]\star\cdots\star\mathcal{E}[n-1]$ is an aisle in $D(\mathcal{H}_\mathcal{D})$ such that $\mathcal{E}=\mathcal{H}_\mathcal{D}\cap\mathcal{H}_\mathcal{T}$ and the pair $(\mathcal{D},\mathcal{T})$ is a n-tilting pair of t-structures. We will say that the t-structure $\mathcal{T}$ is obtained by tilting $\mathcal{D}$ with respect to the n-tilting torsion class $\mathcal{E}$.*

Proof. By Theorem 6.9, the n-tilting torsion class $\mathcal{E}$ is a n-quasi-abelian category and $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)},\mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ is a n-tilting pair of t-structures on $D(\mathcal{E},\mathcal{E}x)\simeq D(\mathcal{H}_\mathcal{D})$. The right t-structure coincides with the natural one on $D(\mathcal{H}_\mathcal{D})$ (i.e., $\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}=\mathcal{D}$) while the left t-structure satisfies $\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}\subseteq\mathcal{T}^{\leq 0}$. On the other hand, since $\mathcal{D}^{\leq -n}\simeq\mathcal{RD}^{\leq -n}_{(\mathcal{E},\mathcal{E}x)}\subseteq\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ and $\mathcal{E}[i]\subseteq\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$, for any $1\leq i\leq n-1$, we deduce that $\mathcal{T}^{\leq 0}\subseteq\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$. This proves that $\mathcal{T}^{\leq 0}\simeq\mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ is an aisle, $\mathcal{E}=\mathcal{H}_\mathcal{D}\cap\mathcal{H}_\mathcal{T}$ and $(\mathcal{D},\mathcal{T})=(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)},\mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ is a n-tilting pair of t-structures on $D(\mathcal{H}_\mathcal{D})$. $\square$

Theorem 6.11. *Let $(\mathcal{E},\mathcal{E}x)$ be a n-quasi-abelian category. One has the following equivalences of categories*

$$\mathcal{LH}(\mathcal{E},\mathcal{E}x)\simeq\frac{\text{fp-}\mathcal{E}}{\text{eff-}_{\mathcal{E}x}\mathcal{E}};\qquad\mathcal{RH}(\mathcal{E},\mathcal{E}x)\simeq\left(\frac{\mathcal{E}\text{-fp}}{\mathcal{E}\text{-eff}_{\mathcal{E}x}}\right)^\circ.$$

In the special case of an abelian category endowed with its maximal Quillen exact structure $(\mathcal{A},\mathcal{A}x_{\max})$, these equivalences give the Auslander's formulas:

$$\mathcal{A}\simeq\frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}};\qquad\mathcal{A}\simeq\left(\frac{\mathcal{A}\text{-coh}}{\mathcal{A}\text{-eff}}\right)^\circ.$$

Proof. The second statement is dual to the first one. By the universal property of the Freyd category fp-$\mathcal{E}$, there exists a unique functor L cokernel preserving such that the diagram below commutes:

$$\begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[dr] & \\ \text{fp-}\mathcal{E} \arrow[rr, "Q_L"] & & \mathcal{LH}(\mathcal{E},\mathcal{E}x). \end{tikzcd}$$

If $F=\operatorname{Coker}_{\text{fp-}\mathcal{E}}(f)$, we have $Q_L(F)=\operatorname{Coker}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}(f)$. The functor Q_L is essentially surjective since any object $L\in\mathcal{LH}(\mathcal{E},\mathcal{E}x)$ admits a resolution $0\to K^{-n}\overset{d_K^{-n}}{\to}\cdots\overset{d_K^{-1}}{\to}K^0\to L\to 0$ (due to the fact that $\mathcal{E}$ is a n-cotilting torsion-free class in $\mathcal{LH}(\mathcal{E},\mathcal{E}x)$), thus $L=\operatorname{Coker}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}(d_k^{-1})$ and $L\cong[K^{-n}\overset{d_K^{-n}}{\to}\cdots\overset{d_K^{-1}}{\to}\overset{\bullet}{K^0}]=:C(d_K^{-n},\ldots,d_K^{-1})$ in $D(\mathcal{E},\mathcal{E}x)$.

We notice that $K = \mathrm{Coker}_{\text{fp-}\mathcal{E}}(g)$ satisfies $Q_L(K) = 0$ if and only if g is a deflation in $\mathcal{E}$, hence $K \in \text{eff-}_{\mathcal{E}x}\mathcal{E}$. This proves that the functor Q_L induces a canonical faith and essentially surjective functor $\overline{Q_L}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[dr] & \\ \text{fp-}\mathcal{E} \arrow[r, "\pi"] \arrow[rr, bend right, "Q_L"'] & \frac{\text{fp-}\mathcal{E}}{\text{eff-}_{\mathcal{E}x}\mathcal{E}} \arrow[r, "\overline{Q_L}"] & \mathcal{LH}(\mathcal{E}, \mathcal{E}x). \end{tikzcd}$$

It remains to prove that $\overline{Q_L}$ is full. Given K and L in fp-$\mathcal{E}$ with presentations $K^{-1} \overset{d_K^{-1}}{\to} K^0 \to K \to 0$ and $L^{-1} \overset{d_L^{-1}}{\to} L^0 \to L \to 0$, their images under Q_L are $C(d_K^{-n}, \dots, d_K^{-1})$ and $C(d_L^{-n}, \dots, d_L^{-1})$. A morphism $Q_L(K) \overset{h}{\to} Q_L(L)$ is a morphism in $D(\mathcal{E}, \mathcal{E}x)$; hence there exists a complex $C^\bullet \in K^{\leq 0}(\mathcal{E})$ (up to truncation) and morphisms $Q_L(K) \xleftarrow[\simeq]{\varphi} C^\bullet \xrightarrow{\tilde{h}} Q_L(L)$ such that the mapping cone $M(\varphi) \in \mathcal{N}_{\mathcal{E}x}$. The zigzag

$$\pi(K) \xleftarrow{\mathrm{Coker}_{\text{fp-}\mathcal{E}}(\varphi^0)} \mathrm{Coker}_{\text{fp-}\mathcal{E}}(d_{C^\bullet}^{-1}) \xrightarrow{\mathrm{Coker}_{\text{fp-}\mathcal{E}}(\tilde{h})} \pi(L) \tag{4}$$

viewed as a a morphism in $\frac{\text{fp-}\mathcal{E}}{\text{eff-}_{\mathcal{E}x}\mathcal{E}}$, is sent to h by $\overline{Q_L}$.

Since $M(\varphi) \in \mathcal{N}_{\mathcal{E}x} \cap K^{\leq 0}(\mathcal{E})$, its -1 differential $K^{-1} \oplus C^0 \xrightarrow{(d_K^{-1}, \varphi^0)} K^0$ has to be a deflation and the sequence $K^{-2} \oplus C^{-1} \to K^{-1} \oplus C^0 \to K^0$ is exact. Therefore the sequence

$$0 \to \mathrm{Coker}_{\text{fp-}\mathcal{E}}(d_{C^\bullet}^{-1}) \rightarrowtail \mathrm{Coker}_{\text{fp-}\mathcal{E}}(d_K^{-1}) \twoheadrightarrow \mathrm{Coker}_{\text{fp-}\mathcal{E}}(d_K^{-1}, \varphi^0) \to 0$$

is a conflation and $\mathrm{Coker}_{\text{fp-}\mathcal{E}}(d_K^{-1}, \varphi^0) \in \text{eff-}_{\mathcal{E}x}\mathcal{E}$ which proves that (4) is a morphism in $\frac{\text{fp-}\mathcal{E}}{\text{eff-}_{\mathcal{E}x}\mathcal{E}}$ and by construction it maps to h by $\overline{Q_L}$. $\square$

Corollary 6.12. *Let $(\mathcal{E}, \mathcal{E}x)$ be a n-quasi-abelian category and $\overline{\mathcal{E}x}$ a Quillen exact structure on $\mathcal{E}$ finer than $\mathcal{E}x$. Hence the class*

$$\overline{\text{eff-}_{\mathcal{E}x}\mathcal{E}} := \{\mathrm{Coker}_{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}(g) \mid g \text{ is a deflation in } \overline{\mathcal{E}x}\}$$

is a Serre subcategory of $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ and $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x}) \simeq \frac{\mathcal{LH}(\mathcal{E},\mathcal{E}x)}{\overline{\text{eff-}_{\mathcal{E}x}\mathcal{E}}}$.

Corollary 6.13. *Let $\mathcal{E}$ be a 1-quasi-abelian category. Hence*

$$\mathcal{LH}(\mathcal{E}) \simeq \frac{\text{coh-}\mathcal{E}}{\text{eff-}\mathcal{E}} \qquad \mathcal{RH}(\mathcal{E}) \simeq \left(\frac{\mathcal{E}\text{-coh}}{\mathcal{E}\text{-eff}}\right)^\circ.$$

In this case the Serre subcategories of effaceable functors are:

$$\text{eff-}\mathcal{E} := \{\mathrm{Coker}_{\text{coh-}\mathcal{E}}(q) \mid q \text{ is a cokernel map in } \mathcal{E}\}$$
$$\mathcal{E}\text{-eff} := \{\mathrm{Coker}_{\mathcal{E}\text{-coh}}(i) \mid i \text{ is a kernel map in } \mathcal{E}\}$$

since any cokernel map is a deflation (resp. any kernel map is an inflation) if and only if $\mathcal{E}$ is a 1-quasi-abelian category.

Remark 6.14. Let consider $(\mathcal{E}, \mathcal{E}x)$ a n-quasi-abelian category, with $n \geq 3$, which is not a $n-1$-quasi-abelian category (i.e., such that $\mathcal{RD}^{\leq -n}_{(\mathcal{E},\mathcal{E}x)} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$ but $\mathcal{RD}^{\leq -n+1}_{(\mathcal{E},\mathcal{E}x)} \nsubseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\mathcal{E}x)}$). Hence for any Quillen exact structure $\overline{\mathcal{E}x}$ on $\mathcal{E}$ finer than $\mathcal{E}x$ (i.e., which contains the conflations of $\mathcal{E}x$) we have that $(\mathcal{E}, \overline{\mathcal{E}x})$ is a n-quasi-abelian category which is not a $n-1$-quasi-abelian category. Otherwise if $\mathcal{RD}^{\leq -n+1}_{(\mathcal{E},\overline{\mathcal{E}x})} \subseteq \mathcal{LD}^{\leq 0}_{(\mathcal{E},\overline{\mathcal{E}x})}$, any object $L \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ which has a presentation $0 \to K^{-n} \overset{d_K^{-n}}{\to} \cdots \overset{d_K^{-1}}{\to}$

$K^0 \to L \to 0$ would short in $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x})$ i.e., d_K^{-n+2} would have a kernel (computed in $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x})$) which belongs to $\mathcal{E}$ but (since $\mathcal{E}$ is fully faithful in $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ this would be a kernel for d_K^{-n+2} also in $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ which contradicts the hypothesis.

So for $n \geq 3$ the index n of quasi-abelianity for $\mathcal{E}$ is independent from the Quillen exact structure on $\mathcal{E}$, hence it can be computed using the maximal Quillen exact structure.

We are now able to prove the n version of Theorem 1.15.

Theorem 6.15. *Let $(\mathcal{E}, \mathcal{E}x)$ be an additive category endowed with a Quillen exact structure. The following properties are equivalent:*

1. *$\mathcal{E}$ is a n-cotilting torsion-free class in an abelian category $\mathcal{A}$;*
2. *$\mathcal{E}$ is a n-tilting torsion class in an abelian category $\mathcal{A}'$;*
3. *$(\mathcal{E}, \mathcal{E}x)$ is a n-quasi-abelian category;*
4. *$\mathcal{E}$ is the intersection of the hearts $\mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ of a n-tilting pair of t-structures in $D(\mathcal{E}, \mathcal{E}x)$.*

Moreover $\mathcal{A} \simeq \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$, $\mathcal{A}' \simeq \mathcal{RH}(\mathcal{E}, \mathcal{E}x)$ and $(\mathcal{D}, \mathcal{T}) = (\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}})$.

Proof. We can visualize the links between properties (1) to (4) by the following diagram:

$$\begin{tikzcd}
\{n\text{-tilting torsion classes}\} \arrow[rrr, leftrightarrow] \arrow[dd, leftrightarrow] & & & \{n\text{-cotilting torsion-free classes}\} \\
& \mathcal{E} \text{ in } \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \arrow[r, leftrightarrow] & \mathcal{E} \text{ in } \mathcal{LH}(\mathcal{E}, \mathcal{E}x) \arrow[ur] \arrow[dr] & \\
\{n\text{-quasi-abelian categories}\} \arrow[rrr, leftrightarrow] \arrow[ur] \arrow[d] & & & \{n\text{-tilting pairs of } t\text{-structures}\} \arrow[ul] \arrow[d] \\
\mathcal{E} = \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x) \arrow[rrr, leftrightarrow] & & & (\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)}) \text{ on } \mathcal{C} = D(\mathcal{E}, \mathcal{E}x).
\end{tikzcd}$$

If $(\mathcal{D}, \mathcal{T})$ is a n-tilting pair of t-structures in $\mathcal{C}$, by Remark 6.8 $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ is a n-quasi-abelian category and by Proposition 2.5 $(\mathcal{D}, \mathcal{T}) = (\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$.

Let $(\mathcal{E}, \mathcal{E}x)$ be a n-quasi-abelian category. By Lemma 6.5 and Remark 6.8 the pair of t-structures $(\mathcal{RD}_{(\mathcal{E},\mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E},\mathcal{E}x)})$ is n-tilting and $\mathcal{E}$ is a n-tilting torsion class in $\mathcal{RH}(\mathcal{E}, \mathcal{E}x)$ (resp. $\mathcal{E}$ is a n-cotilting torsion-free class in $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$).

By Theorem 6.9, if $\mathcal{E}$ is a n-tilting torsion class in $\mathcal{A}'$ (resp. n-cotilting torsion-free class in $\mathcal{A}$), $\mathcal{A}' \simeq \mathcal{RH}(\mathcal{E}, \mathcal{E}x)$ (resp. $\mathcal{A} \simeq \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$) and $\mathcal{E} \simeq \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ which concludes the proof. □

6.1. **Examples.**

Example 6.16. Given R a commutative ring, the following categories are 1-quasi-abelian:

- The category of filtered modules over any ring ([1, Exam. 1.2.13]).
- The category of torsion-free coherent sheaves over a reduced irreducible analytic space or algebraic variety X. For X a normal curve, the previous category is that of vector bundles (of finite rank) ([1, Exam. 1.2.13]).
- In the contest of $\mathcal{D}$-modules the category of strict relative coherent $\mathcal{D}_{X\times S/S}$-modules with $X \times S$ a complex manifold and $\dim S = 1$ ([19] and [17]).
- Let R be a (left and right) coherent ring with global dimension $\mathrm{gl.dim}(R) = 1$ and $\mathcal{E} := \mathrm{add}(R)$ (see Appendix B.3). The maximal Quillen exact structure on $\mathcal{E}$ coincides with the minimal one and $\mathcal{E}$ is a 1-quasi-abelian category; its left heart is $\mathcal{LK}(\mathcal{E}) \simeq \text{coh-}R$ (and so $\mathcal{E} = \text{proj-}\mathcal{E}$ is 1-cotilting torsion-free class with its minimal Quillen exact structure) while $\mathcal{RK}(\mathcal{E}) \simeq (\mathcal{E}\text{-coh})^{\circ}$.

The following list contains more examples of 1 and 2-quasi-abelian categories:

- Let R be a (left and right) coherent ring with global dimension $\mathrm{gl.dim}(R) = 2$ and $\mathcal{E} := \mathrm{add}(R)$. Hence $\mathcal{E}$ admits kernels and cokernels: given a morphism $f : P_1 \to P_2$ in $\mathcal{E}$, its kernel $\mathrm{Ker}_{\mathcal{E}}(f) = \mathrm{Ker}_{\text{coh-}R}(f) \in \mathcal{E}$ while $\mathrm{Coker}_{\mathcal{E}}(f) = (\mathrm{Ker}_{R\text{-coh}}(f^*))^*$ where $(\_)^* := \mathrm{Hom}_R(\_, R)$. Therefore for any Quillen exact structure $\mathcal{E}$ is 2-quasi-abelian. In [46] Rump constructed a tilted algebra A, of type $\mathbb{E}_6$, such that its category of projective modules of finite type has kernels and cokernels (since A has global dimension 2), but it is not 1-quasi-abelian.
- Let us consider the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(R)$ with $R = k[x, y]$ and k a field; hence R has projective dimension 2 and it is Noetherian therefore coherent; this assures that $\mathcal{E} := \mathrm{add}(R)$ has kernels and cokernels. In this case $\mathcal{E}$ coincides with the category of free R-modules of finite type (this result was proved by Seshadri in [50], while the general statement, known as Serre problem, was proved by Quillen and Suslin [44], [53]). Its left heart as a 2-quasi-abelian category endowed with its minimal Quillen exact structure, is the category $\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})$ of coherent sheaves on the affine plane $\mathbb{A}^2_k$. A sequence $0 \to \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \to 0$ is exact in $\mathcal{E}$ for its maximal Quillen exact structure if and only if $\mathcal{E}_3 \cong (\mathrm{Ker}_R(\beta^*))^*$ and so the cokernel of β in $\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})$ is a torsion sheaf whose support has dimension 0 (finite union of closed points). On the other side any coherent sheaf supported on a finite union of closed points can be represented as a cokernel of such a β. Let us denote by $\mathcal{T}_0$ the class of torsion sheaves supported on points; this is a Serre subcategory of $\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})$ and the functor $I_{\mathcal{L}} : \mathcal{E} \to \mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})/\mathcal{T}_0$ is fully faithful and $\mathcal{E}$ is a 1-cotilting torsion-free class in $\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})/\mathcal{T}_0$ and so $\mathcal{E}$ is 1-quasi-abelian category (an hence the left heart of $\mathcal{E}$ as a 1-quasi-abelian category is the quotient abelian category $\mathrm{Coh}(\mathcal{O}_{\mathbb{A}^2_k})/\mathcal{T}_0$).

The following list contains examples of n-quasi-abelian categories for $n > 2$:

- Let R be a (left and right) coherent ring with global dimension $\mathrm{gl.dim}(R) = n$, the full subcategory $\mathcal{E} := \mathrm{add}(R)$ of Mod-R is n-quasi-abelian.
- Let X be a smooth algebraic variety of $\dim X = n$. The full subcategory $\mathcal{E}$ in $\mathrm{Coh}(\mathcal{O}_X)$ formed by locally-free sheaves of finite rank is n-quasi-abelian (it is n-cotilting in $\mathrm{Coh}(\mathcal{O}_X)$). The full-subcategory $\mathcal{F}$ of $\mathrm{Qcoh}(\mathcal{O}_X)$ (quasi-coherent sheaves) formed by flat quasi-coherent modules is n-quasi-abelian (it is n-cotilting in $\mathrm{Qcoh}(\mathcal{O}_X)$) and the dual t-structure can be described in terms of support conditions (see [31]).

Example 6.17. Let $\mathcal{E}$ be the category of free abelian groups of finite type. It is a 1-quasi-abelian category and its maximal Quillen exact structure coincides with the minimal one (split short exact sequences). Its left heart $\mathcal{LK}(\mathcal{E})$ is the whole category of finitely generated abelian groups while $\mathcal{RK}(\mathcal{E}) = (\mathcal{E}\text{-coh})^\circ$ is equivalent to the opposite category of the category of abelian groups of finite type. The derived equivalence $D(\mathcal{A}b) \simeq D(\mathcal{A}b^\circ)$ is given by $\mathbf{R}\,\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$ and the intersection of the hearts is given by the finitely generated abelian groups F such that $\mathbf{R}\,\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z})$ which are the free abelian groups of finite type. One can also interpret the right heart as the tilt of the abelian category of finitely generated abelian groups with respect to the cotilting torsion-free class of free abelian groups of finite type: i.e., objects are complexes $d : F_0 \to F_1$ (in degree 0 e 1) of free abelian groups such that $\mathrm{Coker}(d)$ is a torsion group.

Example 6.18. [6, Exam. 3.6.(5), Exer. 3.7.(12)]. Let X be a smooth projective curve, $\mu \in \mathbb{R}$ a real number and let $A_{\geq \mu}$ be the full subcategory of $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$

generated by torsion sheaves and vector bundles whose HN-filtration quotients have slope $\geq \mu$. Hence $A_{\geq\mu}$ is a tilting torsion class in $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$. In particular let $X = \mathbb{P}^1_k$ the projective line over a field k. Let us recall that any coherent sheaf $\mathcal{F} \in \mathcal{C}\mathrm{oh}(\mathcal{O}_{\mathbb{P}^1_k})$ decomposes as $\mathcal{F} \cong \mathcal{F}_{\mathrm{tor}} \oplus \mathcal{F}_{\mathrm{free}}$ and, by the Birkhoff-Grothendieck theorem, the torsion-free part is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1_k}(d_i)$. So $\mathcal{E} := A_{\geq 0}$ is a a 1-tilting torsion class in $\mathcal{C}\mathrm{oh}(\mathcal{O}_{\mathbb{P}^1_k})$ (hence it is a 1-quasi-abelian category). In this case the maximal Quillen exact structure on $\mathcal{E}$ does not coincide with the minimal one since the sequence $0 \to \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}(1)^2 \to \mathcal{O}_{\mathbb{P}^1_k}(2) \to 0$ does not split (i.e., $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1_k}}(\mathcal{O}_{\mathbb{P}^1_k}(2), \mathcal{O}_{\mathbb{P}^1_k}) \neq 0$). So we have a right heart (as a 2-quasi-abelian category with $\mathcal{E}$ endowed with the split exact structure) in $K(\mathcal{E})$ which is the category $(\mathcal{E}\text{-coh})^\circ$ while its right heart in $D(\mathcal{E})$ as 1-quasi-abelian category is the category of coherent sheaves $\mathcal{C}\mathrm{oh}(\mathcal{O}_{\mathbb{P}^1_k})$ (since $\mathcal{E}$ is a 1-tilting torsion class in it). Concerning the left heart $\mathcal{LD}(\mathcal{E})$ its objects are complexes $X = [\mathcal{E}^{-1} \stackrel{d}{\to} \mathcal{E}^0]$ with $\mathcal{E}^i \in \mathcal{E}$ and d a monomorphism in $\mathcal{E}$. Since any object in $\mathcal{E}$ admits a finite resolution whose terms are direct factors of finite direct sums of $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$ (and so in $\mathrm{add}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1))$ see Appendix B.3) we can represent X as a bounded complex $X = [X^{-m} \to \cdots \to X^0] \in K^{\leq 0}(\mathrm{add}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)))$. Thus for any $X \in \mathcal{LD}(\mathcal{E})$ and for any $i > 0$ we have $\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1), X) \cong \mathcal{D}(\mathcal{E})(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1), X[i]) = 0$ and (via the associated distinguished triangle) we get a short exact sequence in the left heart $0 \to X^{[-m,-1]}[-1] \to X^0 \to X \to 0$ which proves that $T = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$ is a projective generator of the left heart $\mathcal{LD}(\mathcal{E})$. Hence $\mathcal{LD}(\mathcal{E})$ is equivalent to the category of left modules of finite type on the ring $R := \mathrm{End}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1))$ which is the path algebra of the Kronecker quiver Q

$$\bullet \rightrightarrows \bullet$$

The derived equivalence $D^b(\mathcal{C}\mathrm{oh}((\mathcal{O}_{\mathbb{P}^1_k})) \simeq D^b(\mathrm{Rep}_k(Q))$ (which holds true also in the unbounded derived categories by Theorem 1.8) is due to A. Beilinson and $T = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$ is an example of *tilting sheaf.*

Example 6.19. Given $\mathcal{A}$ a Grothendieck category and T a classical n-tilting object in $\mathcal{A}$ one can associate to T the t-structure:

$$\mathcal{T}^{\leq 0} := \{X^\bullet \in D(\mathcal{A}) \mid \mathrm{Hom}_{D(\mathcal{A})}(T, X^\bullet) = 0 \text{ for all } i > 0\}$$

$$\mathcal{T}^{\leq 0} := \{X^\bullet \in D(\mathcal{A}) \mid \mathrm{Hom}_{D(\mathcal{A})}(T, X^\bullet) = 0 \text{ for all } i > 0\}.$$

The pair $(\mathcal{D}, \mathcal{T})$ is a n-tilting pair of t-structures. The intersection $\mathcal{E}$ of their hearts is the full subcategory of $\mathcal{A}$ whose objects are n-presented by T. It is a n-tilting torsion class in $\mathcal{A}$ (see [21, Prop. 6.2] for more details).

7. Perverse coherent sheaves

This section provides a generalisation of Bridgeland categories of perverse coherent sheaves by the use of n-tilting torsion classes.

This problem has been studied in [56] where the authors proposed a category of perverse coherent sheaves via the used of iterated 1-tilting classes (see also [21] for a general treatment of this iterated Happel Reiten Smalø procedure). The construction in [56] requires the use of a tilting complex, which is proved to exist in the case of relative dimension 2 under a technical assumption. In our approach we will follow Bridgeland paper and we will define (for $n = 2$) a category of perverse coherent sheaves without the use of a tilting complex.

Let X be a Noetherian scheme over $\mathbb{C}$, we denote by $\mathrm{Qcoh}(X)$ (resp. $\mathrm{coh}(X)$) the category of quasi-coherent (resp. coherent) sheaves on X and by $D(\mathrm{Qcoh}(X))$ its derived category. We denote by $D(X)$ the derived category of $\mathrm{coh}(X)$ and we recall that it is equivalent to the derived category of quasi-coherent sheaves with coherent cohomologies $D_{\mathrm{coh}}(\mathrm{Qcoh}(X))$.

7.1. Assumptions. For the rest of this section we will assume that $f : Y \to X$ is a projective birational morphism of Noetherian locally $\mathbb{Q}$-factorial semiseparated schemes over $\mathbb{C}$ such that $\mathbf{R}f_*(\mathcal{O}_Y) = \mathcal{O}_X$ with relative dimension n. The condition of being Noetherian locally $\mathbb{Q}$-factorial semiseparated assures that the schemes involved have the resolution property i.e.; every coherent sheaf is a quotient of some vector bundle. Moreover we get:

- for any coherent $\mathcal{O}_Y$-module $\mathcal{G}$ we have $\mathbf{R}f_*(\mathcal{G}) \in D^{[0,n]}(X)$;
- $\mathrm{id}_{D(X)} \simeq \mathbf{R}f_*\mathbf{L}f^*$, hence the functor $\mathbf{L}f^*$ is fully faithful;
- $\mathbf{R}f_*f^! \simeq \mathrm{id}_{D(X)}$, therefore the functor $f^!$ is fully faithful;
- $f^!(D^{\geq 0}(X)) \subseteq D^{\geq -n}(Y)$ (this is the n-version of [57, Lem. 3.1.4] whose proof coincides with that one with -1 replaced by $-n$ and -2 replaced by $-n-1$ at the beginning of the proof).

In the case of $n = 1$ Van den Bergh proved in [57, Lem. 3.1.2, Lem. 3.1.3, Lem. 3.1.5] (following [12, Prop. 5.1]) that the following classes

$$\begin{array}{lcl} \mathcal{T}_0 = \{T \in \mathrm{coh}(Y) \mid \mathbf{R}^1 f_*T = 0\} & ; & \mathcal{F}_0 = \{F \in \mathrm{coh}(Y) \mid F \overset{\phi_F}{\hookrightarrow} H^{-1}f^!\mathbf{R}^1 f_*F\} \\ \mathcal{T}_{-1} = \{T \in \mathrm{coh}(Y) \mid \eta_T : f^*f_*T \twoheadrightarrow T\} & ; & \mathcal{F}_{-1} = \{F \in \mathrm{coh}(Y) \mid f_*F = 0\} \end{array}$$

define torsion pairs in $\mathrm{coh}(Y)$ (which we will prove to be tilting in Lemma 7.4). We recall that $\eta_T : f^*f_*T \to T$ is the co-unit of the adjunction (f^*, f_*) while the map $\phi_F : F \to H^{-1}f^!\mathbf{R}^1 f_*F$ is the morphism obtained by taking the zero cohomology of the composition $F \to f^!\mathbf{R}f_*F \to f^!\mathbf{R}^1 f_*F[-1]$ (where the first map is the unit of the adjunction $(\mathbf{R}f_*, f^!)$). Notice that $\mathcal{T}_{-1} = \mathcal{T}_0 \cap \mathcal{X}$ where $\mathcal{X} := \{\mathcal{F} \in \mathrm{coh}(Y) \mid \mathrm{Hom}(\mathcal{F}, C) = 0 \ \forall\ C \in \mathrm{coh}(Y) : \mathbf{R}f_*C = 0\}$. The heart of the t-structure obtained by tilting the natural t-structure with respect to the tilting torsion pair $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$ (resp. $(\mathcal{T}_0, \mathcal{F}_0)$) is called ${}^{-1}\mathrm{Perv}(Y/X)$ (resp. ${}^{0}\mathrm{Perv}(Y/X)$). Hence $D(Y) \simeq D({}^{-1}\mathrm{Perv}(Y/X)) \simeq D({}^{0}\mathrm{Perv}(Y/X))$.

7.2. Higher analog of $\mathcal{T}_{-1}$ and $\mathcal{T}_0$. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. In the case of relative dimension $n > 1$ we propose the following generalization of the previous tilting torsion classes:

$$\mathcal{T}_0 = \{T \in \mathrm{coh}(Y) \mid \mathbf{R}f_*T \cong f_*T\} \qquad \mathcal{T}_{-1} = \{T \in \mathcal{T}_0 \mid \eta_T : f^*f_*T \twoheadrightarrow T\}.$$

Conjecture 7.3. We conjecture that under the previous assumptions the classes $\mathcal{T}_0$ and $\mathcal{T}_{-1}$ are n-tilting in $\mathrm{coh}(Y)$.

We will prove that for any n these classes satisfy conditions (1), (2) and (4) of Definition 6.7. For $n = 1$ they are tilting torsion classes by Lemma 7.4. We will prove in Theorem 7.7 that for $n = 2$ they are 2-tilting in $\mathrm{coh}(Y)$.

Let us prove that under the assumptions of 7.1 the class $\mathcal{T}_{-1}$ cogenerates $\mathrm{coh}(Y)$; this statement is the relative version of McMurray Price's Lemma [39, Lem. 5.2] which we prove with the same argument in the following Lemma.

Lemma 7.4. *Let $f : Y \to X$ be a projective morphism as in 7.1 and let $\mathcal{L}$ be an f-ample vector bundle. For any $\mathcal{F} \in \mathrm{coh}(Y)$ there exists a monomorphism $\alpha : \mathcal{F} \hookrightarrow T$ with $T \in \mathcal{T}_{-1}$.*

Proof. The relative Serre vanishing Theorem ([25, Ch. III.5]), guarantees that given $\mathcal{F} \in \mathrm{coh}(Y)$ for $m \gg 0$ we have: $\mathbf{R}^i f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) = 0$ for any $i > 0$ and the counit $f^*f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m$ of the adjunction (f^*, f_*) is an epimorphism; which is equivalent to require that $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m \in \mathcal{T}_{-1}$. Let $\mathcal{F} \in \mathrm{coh}(Y)$ and let consider m

big enough such that both $\mathcal{L}^m$ and $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m$ belong to $\mathcal{T}_{-1}$. Let $\mathcal{E} \twoheadrightarrow f_*(\mathcal{L}^m)$ be an epimorphism in $\mathrm{coh}(X)$ with $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank (it exists since X has the resolution property). Hence the composition

$$\eta : f^*(\mathcal{E}) \twoheadrightarrow f^* f_*(\mathcal{L}^m) \twoheadrightarrow \mathcal{L}^m$$

is a locally splitting epimorphism since $\mathcal{L}^m$ is a locally free sheaf, hence its dual $\eta^\vee : \mathcal{L}^{-m} \to \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{O}_Y)$ is a locally splitting monomorphism and so it is pure (i.e., universally injective) which implies that the morphism $\delta := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m \otimes_{\mathcal{O}_Y} \eta^\vee$ is injective

$$\delta : \mathcal{F} \hookrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{O}_Y) \cong \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m).$$

Moreover

$$\begin{aligned} \mathbf{R}f_*\mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \cong{} & \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{L}f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \cong \\ \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathbf{R}f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)) \cong{} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)) \cong \\ \cong{} & f_*\mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m). \end{aligned}$$

The first isomorphism holds true since both $\mathcal{E}$ and $f^*(\mathcal{E})$ are locally free coherent sheaves, hence $f^*(\mathcal{E})$ is $\mathcal{H}om_{\mathcal{O}_Y}(\_, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)$-acyclic, while $\mathcal{E}$ is f^*-acyclic. The second isomorphism is induced by the adjunction $(\mathbf{L}f^*, \mathbf{R}f_*)$. Since $\mathcal{E}$ is locally free $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m))$, hence the third isomorphism is deduced by the fact that we choose m such that $\mathbf{R}f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)$. The last isomorphism is induced by the adjunction (f^*, f_*). It remains to prove that the counit of the adjunction $f^* f_* \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \to \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m)$ is an epimorphism. By the last isomorphism of the previous list we have

$$\begin{aligned} f^* f_* \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \cong{} & f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, f_*(\mathcal{L}^m \otimes_{\mathcal{O}_Y} \mathcal{F})) \cong \\ \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), f^* f_*(\mathcal{L}^m \otimes_{\mathcal{O}_Y} \mathcal{F})) & \end{aligned}$$

and by hypothesis the counit $f^* f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m) \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^m$ is an epimorphism which implies that $\mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), f^* f_*(\mathcal{L}^m \otimes_{\mathcal{O}_Y} \mathcal{F})) \twoheadrightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*(\mathcal{E}), \mathcal{L}^m \otimes_{\mathcal{O}_Y} \mathcal{F})$ (because $f^*(\mathcal{E})$ is locally free). □

Lemma 7.5. *The full subcategories $\mathcal{T}_i$, with $i \in \{0, -1\}$, are closed under extensions in* $\mathrm{coh}(Y)$.

Proof. Let us prove that $\mathcal{T}_0$ is closed under extensions in $\mathrm{coh}(Y)$. Given any short exact sequence

$$0 \to T_1 \to \mathcal{F} \to T_2 \to 0 \qquad \text{with } T_1, T_2 \in \mathcal{T}_0; \text{ and } \mathcal{F} \in \mathrm{coh}(Y) \tag{5}$$

we get a distinguished triangle $\mathbf{R}f_*T_1 \to \mathbf{R}f_*\mathcal{F} \to \mathbf{R}f_*T_2 \xrightarrow{+}$ with $\mathbf{R}f_*T_1, \mathbf{R}f_*T_2$ coherent $\mathcal{O}_X$-modules (thus complexes concentrated in degree 0) which proves that $\mathbf{R}f_*\mathcal{F} \cong f_*\mathcal{F}$ is a complex concentrated in degree 0.

Let us prove that $\mathcal{T}_{-1}$ is closed under extensions in $\mathrm{coh}(Y)$. Let us start with the short exact sequence (5) by supposing that $T_1, T_2 \in \mathcal{T}_{-1}$. Since $\mathcal{T}_{-1} \subseteq \mathcal{T}_0$, we deduce by the previous argument that $\mathcal{F} \in \mathcal{T}_0$. Thus the sequence $0 \to f_*T_1 \to f_*\mathcal{F} \to f_*T_2 \to 0$ is exact. Hence the following diagram commutes

$$\begin{tikzcd} & f^*f_*(T_1) \arrow[r] \arrow[d, two heads] & f^*f_*(\mathcal{F}) \arrow[r] \arrow[d] & f^*f_*(T_2) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & T_1 \arrow[r] & \mathcal{F} \arrow[r] & T_2 \arrow[r] & 0 \end{tikzcd} \tag{6}$$

therefore the canonical map $f^*f_*\mathcal{F} \twoheadrightarrow \mathcal{F}$ is an epimorphism. □

Lemma 7.6. *(Under the assumptions 7.1), the full subcategories $\mathcal{T}_i$, $i \in \{0, -1\}$, satisfy condition (4) of Definition 6.7, namely:*
for any exact sequence in $\mathrm{coh}(Y)$

$$0 \longrightarrow A \longrightarrow X_1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X_n \longrightarrow B \longrightarrow 0 \tag{7}$$

with $X_j \in \mathcal{T}_i$ for any $1 \leq j \leq n$ and $A, B \in \mathrm{coh}(Y)$ we have $B \in \mathcal{T}_i$.

Proof. Consider $X^\bullet := [\cdots \to 0 \to X_1 \to \cdots \to \overset{\bullet}{X}_n \to 0 \to \cdots]$ where X_n is placed in degree 0. The sequence (7) produces the distinguished triangle $A[n-1] \to X^\bullet \to B[0] \overset{+}{\to}$ which induces the distinguished triangle $\mathbf{R}f_*(A)[n-1] \to \mathbf{R}f_*(X^\bullet) \to \mathbf{R}f_*(B) \overset{+}{\to}$. Since f has relative dimension n, $\mathbf{R}f_*(A)[n-1] \in D^{\leq 1}(X)$. Hence $\mathbf{R}f_*(B) \in D^{\leq 0}(X)$ (since $\mathbf{R}f_*(X^\bullet) \in D^{\leq 0}(X)$), therefore B belongs to $\mathcal{T}_0$.

If $X_j \in \mathcal{T}_{-1}$ for any $1 \leq j \leq n$, by the previous argument we deduce that B belongs to $\mathcal{T}_0$ and, since it is a quotient of X_n, $f^*f_*(B) \twoheadrightarrow B$. □

Theorem 7.7. *For $n = 2$ the classes*

$$\mathcal{T}_0 = \{T \in \mathrm{coh}(Y) \mid f_*T = \mathbf{R}f_*T\} \qquad \mathcal{T}_{-1} = \{T \in \mathcal{T}_0 \mid f^*f_*T \twoheadrightarrow T\}$$

are 2-tilting torsion classes in $\mathrm{coh}(Y)$.

Proof. Points (1), (2) and (4) of Definition 3.2 have been proved in Lemma 7.4, Lemma 7.5 and Lemma 7.6 respectively. We have to prove that $\mathcal{T}_{-1}$ and $\mathcal{T}_0$ have kernels.

The full subcategory $\mathcal{X}_0 = \{T \in \mathrm{Qcoh}(Y) \mid \mathbf{R}^2 f_*T = 0\}$ is a 1-tilting torsion class in $\mathrm{Qcoh}(Y)$ i.e.; it is closed under direct sums, extension, quotients and it cogenerates $\mathrm{Qcoh}(Y)$ (since it contains any injective sheaf)

Given $\mathcal{F} \in \mathrm{Qcoh}(Y)$ we will denote by $t_{\mathcal{X}_0}(\mathcal{F})$ its torsion part (i.e.; the biggest subsheaf of $\mathcal{F}$ lying in $\mathcal{X}_0$). Notice that if $\mathcal{F} \in \mathrm{coh}(Y)$ even $t_{\mathcal{X}_0}(\mathcal{F}) \in \mathrm{coh}(Y)$.

Step 1. Let us prove that $\mathcal{T}_{-1}$ admits kernels.

Give any locally free sheaf $\mathcal{E} \in \mathrm{coh}(X)$, the sheaf $f^*\mathcal{E} \in \mathcal{T}_{-1}$ since, by 7.1, we have $\mathcal{E} = \mathbf{R}f_*\mathbf{L}f^*\mathcal{E} \cong \mathbf{R}f_*f^*\mathcal{E} \cong f_*f^*\mathcal{E}$. Hence, given any coherent sheaf $\mathcal{M} \in \mathrm{coh}(X)$, the sheaf $f^*\mathcal{M}$ belongs to $\mathcal{T}_{-1}$ (since it is the cokernel of a map in $\mathcal{T}_{-1}$).

Let $\mathcal{E}_1 \overset{\alpha}{\to} \mathcal{E}_2$ be a morphism in $\mathcal{T}_{-1}$ whose kernel in $\mathrm{coh}(Y)$ is $\mathcal{K} := \operatorname{Ker}\alpha$. Let us denote by $\eta_{\mathcal{K}} : f^*f_*\mathcal{K} \to \mathcal{K}$ the counit of the adjunction (f^*, f_*). The short exact sequence $0 \to t_{\mathcal{X}_0}(\operatorname{Ker}\eta_{\mathcal{K}}) \overset{j}{\to} f^*f_*\mathcal{K} \to \overline{\mathcal{K}} \to 0$, ($\overline{\mathcal{K}} := \operatorname{Coker} j$), induces the distinguished triangle $\mathbf{R}f_*(t_{\mathcal{X}_0}(\operatorname{Ker}\eta_{\mathcal{K}})) \to f_*f^*f_*\mathcal{K} \to \mathbf{R}f_*(\overline{\mathcal{K}}) \overset{+}{\to}$ which proves that $\mathbf{R}f_*(\overline{\mathcal{K}}) \in D^{\leq 0}(X)$, therefore $\overline{\mathcal{K}} \in \mathcal{T}_{-1}$. Let us verify that $\overline{\mathcal{K}} = \ker_{\mathcal{T}_{-1}}(\alpha)$. Let $\mathcal{L} \overset{\phi}{\to} \mathcal{E}_1$ be a morphism in $\mathcal{T}_{-1}$ such that $\alpha\phi = 0$ and consider the following functorial commutative diagram obtained by the universal property of the kernel and by the adjunction (f^*, f_*):

$$\begin{tikzcd}
 & & \operatorname{Ker}\eta_{\mathcal{L}} \arrow[r, hook] \arrow[dll] \arrow[ddl, "\gamma"'] & f^*f_*\mathcal{L} \arrow[r, two heads, "\eta_{\mathcal{L}}"] \arrow[ddl, "f^*f_*\beta"'] & \mathcal{L} \arrow[dd, "\beta"', "\exists!"] \arrow[dddl, "\overline{\beta}"'] \arrow[ddr, "\phi"] \arrow[ddrr, "0"] & & \\
t_{\mathcal{X}_0}(\operatorname{Ker}\eta_{\mathcal{K}}) \arrow[dr] \arrow[drr] & & & & & & \\
 & \operatorname{Ker}\eta_{\mathcal{K}} \arrow[r, hook] & f^*f_*\mathcal{K} \arrow[rr, "\eta_{\mathcal{K}}"] \arrow[dr, two heads, "\pi"'] & & \mathcal{K} \arrow[r] & \mathcal{E}_1 \arrow[r, "\alpha"] & \mathcal{E}_2 \\
 & & & \overline{\mathcal{K}} \arrow[ur] & & &
\end{tikzcd}$$

we note that $\ker\eta_{\mathcal{L}} \in \mathcal{X}_0$ since $\mathcal{L} \in \mathcal{T}_{-1}$, hence γ factors through $t_{\mathcal{X}_0}(\operatorname{Ker}\eta_{\mathcal{K}})$. Therefore there exists a unique $\overline{\beta} : \mathcal{L} \to \overline{\mathcal{K}}$ such that the diagram commutes.

Step 2. Let us prove that $\mathcal{T}_0$ admits kernels.

Let $\mathcal{F}_1 \overset{\alpha}{\to} \mathcal{F}_2$ be a morphism in $\mathcal{T}_0$ whose kernel in $\mathrm{coh}(Y)$ is $\mathcal{K} := \operatorname{Ker}\alpha$. Let $\mathcal{M}[1]$ be the mapping cone of $\chi : \mathcal{K} \to f^!\mathbf{R}f_*\mathcal{K} \to f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K}$. By 7.1 we have $f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K} \in \mathcal{D}^{\geq -1}(Y)$, hence $\mathcal{M} \in \mathcal{D}^{\geq 0}(Y)$.

Let us prove that $\overline{\mathcal{K}} := t_{\mathcal{X}_0}(H^0(\mathcal{M}))$ belongs to $\mathcal{T}_0$ and $\overline{\mathcal{K}} = \ker_{\mathcal{T}_0}(\alpha)$. Let consider the following commutative diagram with distinguished rows and columns:

$$\begin{tikzcd}
H^0(\mathcal{M}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & \delta^{\geq 1}(\mathcal{M}) \arrow[r, "+"] \arrow[d] & {} \\
\mathcal{K} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d, "\chi"] & 0 \arrow[r, "+"] \arrow[d] & {} \\
\mathcal{N} \arrow[r] \arrow[d, "+"] & f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K} \arrow[r] \arrow[d, "+"] & \delta^{\geq 1}(\mathcal{M})[1] \arrow[r, "+"] \arrow[d, "+"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

By applying to it the functor $\mathbf{R}f_*$ (using $\mathbf{R}f_*f^! = \mathrm{id}_{\mathcal{D}(X)}$) we deduce the following facts: $\mathbf{R}f_*\mathcal{N} \in \mathcal{D}^{\geq 1}(X)$, $f_*H^0(\mathcal{M}) \cong f_*\mathcal{K}$, the map $\mathbf{R}^1f_*\mathcal{K} \to \mathbf{R}^1f_*\mathcal{N} \to \mathbf{R}^1f_*\mathcal{K}$ (induced by the sud-ovest square) is the identity, hence $\mathbf{R}^1f_*(H^0(\mathcal{M})) = 0$. Thus $\overline{\mathcal{K}} \in \mathcal{T}_0$ (since $f_*\left(\frac{H^0(\mathcal{M})}{\overline{\mathcal{K}}}\right) = 0$). Any $\mathcal{L} \overset{\phi}{\to} \mathcal{F}_1$ in $\mathcal{T}_0$ such that $\alpha\phi = 0$ factors uniquely through $\mathcal{L} \overset{\phi'}{\to} \mathcal{K}$. In the exact sequence

$$\mathrm{Hom}^{-1}_{D^b(Y)}(\mathcal{L}, f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K}) \to \mathrm{Hom}_{D^b(Y)}(\mathcal{L}, \mathcal{M}) \to \mathrm{Hom}_{D^b(Y)}(\mathcal{L}, \mathcal{K}) \to \mathrm{Hom}_{D^b(Y)}(\mathcal{L}, f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K})$$

the first and the last terms are zero since

$$\mathrm{Hom}^i_{D^b(Y)}(\mathcal{L}, f^!\delta^{\geq 1}\mathbf{R}f_*\mathcal{K}) \cong \mathrm{Hom}^i_{D^b(X)}(f_*\mathcal{L}, \delta^{\geq 1}\mathbf{R}f_*\mathcal{K}) = 0 \quad \forall i \in \{-1, 0\}.$$

This proves that $\mathrm{Hom}_Y(\mathcal{L}, H^0(\mathcal{M})) \cong \mathrm{Hom}_{D^b(Y)}(\mathcal{L}, \mathcal{M}) \cong \mathrm{Hom}_Y(\mathcal{L}, \mathcal{K})$ (remember that $\mathcal{M} \in D^{\geq 0}(Y)$). Thus we obtain that ϕ' factors uniquely through $\mathcal{L} \overset{\phi''}{\to} H^0(\mathcal{M})$. Therefore the morphism ϕ'' factors uniquely through $\mathcal{L} \overset{\overline{\phi''}}{\to} \overline{\mathcal{K}}$ (since $\mathcal{L} \in \mathcal{X}_0$). $\square$

Definition 7.8. By Theorem 3.4 (for $n = 2$ or supposing that Conjecture 7.3 holds true $n > 2$), we define $({}^i\mathcal{D}^{\leq 0}, {}^i\mathcal{D}^{\geq 0})$ (with $i \in \{0, -1\}$) to be the t-structures obtained by tilting $\mathcal{D}$ with respect to the n-tilting torsion classes $\mathcal{T}_i$. Their hearts are denoted by ${}^i\mathrm{Per}(Y/X)$ for $i \in \{-1, 0\}$ and objects in ${}^i\mathrm{Per}(Y/X)$ are called *perverse coherent sheaves*.

Theorem 7.9. *(Theorem 6.15). For $n = 2$ or assuming Conjecture 7.3*

$$D(Y) \simeq D({}^0\mathrm{Per}(Y/X)) \simeq D({}^{-1}\mathrm{Per}(Y/X)).$$

Remark 7.10. In higher dimension, Toda remarked in [55] that Bridgeland proof, of the derived equivalence between $\mathcal{D}^b(Y)$ and $\mathcal{D}^b(Y^+)$ via the intersection theorem, produces also the smoothness of the flop. Nevertheless there are examples of 4 dimensional flops which do not preserve the smoothness. We think that the use of the previous n-tilted torsion classes (which produce equivalences $\mathcal{D}(Y) \simeq \mathcal{D}({}^i\mathrm{Per}(Y/X))$) could permit to attack the problem of the equivalence $\mathcal{D}({}^{-1}\mathrm{Per}(Y/X)) \simeq \mathcal{D}({}^0\mathrm{Per}(Y^+/X))$ as in [57].

8. Comparison between n-abelian and $n+1$-quasi-abelian categories

Recently Jasso in [28] introduced the notion of *n-abelian category* whose basic example is an n-cluster-tilting subcategory of an abelian category. Let us briefly recall this definition and the principal results of [28].

Given $\mathcal{C}$ an additive category and $d^0 : X^0 \to X^1$ a morphism in $\mathcal{C}$ an *n-cokernel of d^0* ([28, Def. 2.2]) is a sequence

$$(d^1, \ldots, d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^n} X^{n+1}$$

such that for all $Y \in \mathcal{C}$ the sequence of abelian groups
(8)

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{d^n \circ -} \mathcal{C}(X^n, Y) \xrightarrow{d^{n-1} \circ -} \cdots \longrightarrow \mathcal{C}(X^1, Y) \xrightarrow{d^0 \circ -} \mathcal{C}(X^0, Y)$$

is exact. In terms of coherent functors in $\mathcal{E}$-coh the previous sequence (8) proves that (following the notation of Appendix B.2) the kernel of the morphism ${}_{X^1}\mathcal{C} \xrightarrow{d^0 \circ -} {}_{X^0}\mathcal{C}$ is a coherent functor which admits a projective presentation

$$0 \longrightarrow {}_{X^{n+1}}\mathcal{C} \xrightarrow{d^n \circ -} {}_{X^n}\mathcal{C} \xrightarrow{d^{n-1} \circ -} \cdots \longrightarrow \mathrm{Ker}({}_{X^1}\mathcal{C} \xrightarrow{d^0 \circ -} {}_{X^0}\mathcal{C}) \longrightarrow 0$$

of length n in $\mathcal{E}$-coh. The dual concept of n-kernel implies that the kernel of the morphism $\mathcal{C}_{X^0} \xrightarrow{-\circ d^0} \mathcal{C}_{X^1}$ is a coherent functor admitting a projective presentation of length n in coh-$\mathcal{E}$.

An *n-exact sequence in $\mathcal{C}$* ([28, Def. 2.4]) is a complex $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ such that $(d^0, \dots, d^{n-1})$ is a n-kernel of d^n and $(d^1, \dots, d^n)$ is an n-cokernel of d^0.

Definition 8.1. ([28, Def. 3.1]). Let n be a positive integer. An *n-abelian category* is an additive category $\mathcal{M}$ satisfying the following axioms:

- $(A0)$**:** the category $\mathcal{M}$ is projectively complete;
- $(A1)$**:** every morphism in $\mathcal{M}$ has an n-kernel and an n-cokernel;
- $(A2)$**:** for every monomorphism $f^0 : X^0 \to X^1$ in $\mathcal{M}$ and for every n-cokernel $(f^1, \dots, f^n)$ of f^0 the following sequence is n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}$$

- $(A2^{op})$**:** for every epimorphism $g^n : X^n \to X^{n+1}$ in $\mathcal{M}$ and for every n-kernel $(g^0, \dots, g^{n-1})$ of g^n the following sequence is n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}$$

Proposition 8.2. *Any n-abelian category $\mathcal{M}$ is an $n+1$-coherent category, hence $\mathcal{M}$ is an $n+1$-quasi-abelian category for any Quillen exact structure on $\mathcal{M}$.*

Proof. Axioms $(A0)$ and $(A1)$ prove that the category $\mathcal{M}$ is coherent (see Definition D1.3) since any kernel of a morphism between representable functors is finitely presented. Thus coh-$\mathcal{E}$ and $\mathcal{E}$-coh are abelian categories. Moreover any coherent functor $F \in$ coh-$\mathcal{E}$ admits a presentation $\mathcal{C}_{X^n} \longrightarrow \mathcal{C}_{X^{n+1}} \longrightarrow\!\!\!\!\rightarrow F \longrightarrow 0$, hence by axiom $(A1)$ it admits a projective resolution of length ad most $n+1$ which proves that $\mathcal{M}$ is $n+1$-coherent (Definition 5.2). Therefore, by Definition 6.6, $\mathcal{M}$ endowed with its minimal Quillen exact structure is an $n+1$-quasi-abelian category. □

There are $n+1$-coherent categories which are not n-abelian. For example 1-abelian categories are precisely abelian categories while 2-quasi-abelian categories are projective complete categories admitting kernels and cokernels. For example 1-quasi-abelian categories which are not abelian categories are never n-abelian ones.

Appendix A. Maximal Quillen exact structure

A.1. Minimal and maximal Quillen exact structures. See [33], [14] for the notion of Quillen exact structure. We denote by $\mathcal{E}x$ an *exact structure* on $\mathcal{E}$ (i.e., elements in $\mathcal{E}x$ are conflations). We recall that an additive category $\mathcal{E}$ can admit different exact structures, since any split short exact sequence is a conflation for any exact structure, they form the *minimal exact structure* $\mathcal{E}x_{\mathrm{split}}$ on $\mathcal{E}$.

Any additive category admits a *maximal exact structure* ([47]). By [16, Th. 3.5] (which generalizes [51]), any *weakly idempotent complete* additive category $\mathcal{E}$[2] the class of all kernel-cokernel pairs stable by push-outs and pull-backs is the maximal Quillen exact structure on $\mathcal{E}$. Whenever the exact structure on $\mathcal{E}$ is not specified, we will endow $\mathcal{E}$ with its maximal Quillen exact structure $\mathcal{E}x_{\max}$.

A.2. Let $(\mathcal{E}, \mathcal{E}x)$ be an exact category. A complex $X^\bullet \in C(\mathcal{E})$ is called *acyclic* if each differential $d^n = m_n \circ e_n$ where m_n is an inflation, e_n is a deflation and the sequence $e_{n+1} \circ m_n$ belongs to $\mathcal{E}x$ for any $n \in \mathbb{Z}$. Following Neeman [41], the "derived" category of a projectively complete exact category $(\mathcal{E}, \mathcal{E}x)$ is the quotient of $K(\mathcal{E})$ by $\mathcal{N}_{\mathcal{E}x}$ (the full subcategory of $K(\mathcal{E})$ whose objects are acyclic complexes). Moreover $D(\mathcal{E}, \mathcal{E}x)(X^\bullet, Y^\bullet) = K(\mathcal{E})(X^\bullet, Y^\bullet), \forall X^\bullet \in C^{\leq 0}(\mathcal{E}); Y^\bullet \in C^{\geq 0}(\mathcal{E})$.

Appendix B. Freyd categories and coherent functors

We will consider $\mathcal{C}$ a category in the classical terminology (for which any homomorphism class $\mathcal{C}(X, Y)$, with X, Y objects in $\mathcal{C}$, is a set)[3]

Definition B.1. Let us recall that a category is called:

(1) $\mathcal{C}$ is *pre-additive* if hom-sets are groups with bilinear composition;
(2) $\mathcal{B}$ is *additive* if it is pre-additive with zero object and biproducts;
(3) *idempotent complete* [4] if any idempotent splits;
(4) $\mathcal{P}$ is *projectively complete* [5] when it is additive and idempotent complete.

B.2. We denote by **Mod-**$\mathcal{C}$ the enriched category of additive contravariant functors (i.e., $F : \mathcal{C}^\circ \to \mathcal{A}b$) from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups, and by $\mathcal{C}$**-Mod** the one of covariant functors (see [37] [2], [52], [40]). The following functors are the enriched version of the Yoneda ones and they admit an additive analogue of the Yoneda Lemma:

$$\begin{array}{rcl} Y_{\mathcal{C}} : \mathcal{C} & \longrightarrow & \text{Mod-}\mathcal{C} \\ X & \longmapsto & \mathcal{C}_X := \mathcal{C}(\_, X) \end{array} \qquad \begin{array}{rcl} {}_{\mathcal{C}}Y : \mathcal{C} & \longrightarrow & (\mathcal{C}\text{-Mod})^\circ \\ X & \longmapsto & {}_X\mathcal{C} := \mathcal{C}(X, \_). \end{array}$$

Remark B.3. Let $\mathcal{C}$ be a pre-additive category, one can perform the *projective completion* $\text{add}(\mathcal{C})$ of $\mathcal{C}$ formally adding the zero object and finitely direct sums of objects in $\mathcal{C}$, hence taking its idempotent completion ([5]). Let proj-$\mathcal{C}$ (resp. $\mathcal{C}$-proj) be the full subcategory of Mod-$\mathcal{C}$ (resp. of $\mathcal{C}$-Mod) whose objects are direct summands of finite direct sums of representable functors. Hence $\text{add}(\mathcal{C})$, proj-$\mathcal{C}$ and $\mathcal{C}$-proj are equivalent (and if $\mathcal{C}$ is projectively complete they are equivalent $\mathcal{C}$). Any additive functor $F : \mathcal{C}^\circ \to \mathcal{A}b$ can uniquely be extended to an additive functor $\overline{F} : (\text{proj-}\mathcal{C})^\circ \to \mathcal{A}b$, thus Mod-$\mathcal{C}$ is equivalent to Mod-proj-$\mathcal{C}$.

B.4. Coherent Functors. In [3] Auslander introduced the study of *coherent functors* in the category Mod-$\mathcal{A}$ with $\mathcal{A}$ an abelian category (a "genetic" introduction to this theme can be found in [26]). In the same collection Freyd [22] introduced the study of the *Freyd category* of *finitely presented functors*[6] associated to a projectively complete category $\mathcal{P}$. These theories, and the related vocabulary, are widely

[2] i.e., additive category in which every section has a cokernel, or equivalently, every retraction has a kernel

[3] Some authors define this a locally small category in order to underline that its homomorphism form a set. The wider notion of category, which permits to consider also homomorphisms which do not form a set, is very convenient once working with localization procedures.

[4] It also called Karoubian by some authors.

[5] It is also called Cauchy complete in [52], or amenable by [22].

[6] Freyd's work [22] has been further investigated and developed by Beligiannis in his very inspiring paper [8] to which we refer (see also [3]).

inspired by the theory of finitely presented and coherent modules over a ring R which is also the easiest case (pre-additive category with a single object see B.2).

The basic idea is that whatever one knows on finitely presented (resp. coherent) modules over a ring has its counterpart for finitely presented (resp. coherent) functors in Mod-$\mathcal{C}$ replacing the role of projective finitely generated modules by representable functors in Mod-$\mathcal{C}$ (since they are the projective compact objects of this category). It is well known ([11, Ch.I], [10, §1.5]) that, given a ring R, right coherent modules coh-R form a full abelian subcategory of all right R modules Mod-R, while finitely presented modules fp-R form a full projectively complete subcategory of Mod-R admitting cokernels. The category fp-R is an abelian subcategory of Mod-R if and only if the ring R is right coherent. In that case coherent and finitely presented modules coincide: coh-R = fp-R (these theorems go back to Henri Cartan). In general finitely generated modules form a full projectively complete subcategory fg-R of Mod-R which is an abelian subcategory if and only if the ring R is right Noetherian, in this case coh-R = fp-R = fg-R.

Since Mod-$\mathcal{C}$ and Mod-proj($\mathcal{C}$) are equivalent, from now on, given any pre-additive category we will pass to its projective completion $\mathcal{P} := \mathrm{proj}(\mathcal{C})$.

Definition B.5. An object $F \in$ Mod-$\mathcal{P}$ is called *finitely generated* if there exists an epimorphism $\mathcal{P}_X \twoheadrightarrow F$ with $X \in \mathcal{P}$. An object $F \in$ Mod-$\mathcal{P}$ is called *finitely presented* if it fits into an exact sequence in Mod-$\mathcal{P}$: $\mathcal{P}_{X_1} \longrightarrow \mathcal{P}_{X_2} \longrightarrow F \longrightarrow 0$ with $X_i \in \mathcal{P}$ for $i = 1, 2$. A finitely generated F is called *coherent* if any subobject $G \hookrightarrow F$ finitely generated is finitely presented. Hence any finitely generated subfunctor of a coherent functor is coherent. We will denote by **fg-**$\mathcal{P}$, resp. **fp-**$\mathcal{P}$, resp. **coh-**$\mathcal{P}$ the full subcategory of Mod-$\mathcal{P}$ whose objects are the finitely generated, resp. finitely presented, resp. coherent functors. Following Beligiannis [8, Def. 3.1] the categories fp-$\mathcal{P}$ and ($\mathcal{P}$-fp)$^{\circ}$ are called the *Freyd categories* of $\mathcal{P}$.

We obtain the following commutative diagram of fully faithful functors:

(9)
$$\begin{tikzcd} & \mathcal{P} \arrow[d, hook, "P_{\mathcal{P}}"'] \arrow[drr, "Y_{\mathcal{P}}"] & & \\ \text{coh-}\mathcal{P} \arrow[r, hook] & \text{fp-}\mathcal{P} \arrow[r, hook] & \text{fg-}\mathcal{P} \arrow[r, hook] & \text{Mod-}\mathcal{P} \end{tikzcd}$$

where by definition $P_{\mathcal{P}}$ is the Yoneda functor whose codomain is restricted to finitely presented functors. (The class of natural transformations between finitely generated functors is a set since, if $\mathcal{P}_X \twoheadrightarrow F$ and $\mathcal{P}_Y \twoheadrightarrow G$, any morphism $\alpha : F \to G$ can be lifted to a morphism $\mathcal{P}_X \to \mathcal{P}_Y \in \mathcal{P}(X, Y)$).

B.6. Given $\mathcal{P}$ a projectively complete category, Freyd proved in [22] that fp-$\mathcal{P}$ is projectively complete, it has cokernels and an object F is projective in fp-$\mathcal{P}$ (i.e., for any epimorphism $p : G_1 \twoheadrightarrow G_2$ in fp-$\mathcal{P}$ the map fp-$\mathcal{P}(F, G_1) \to$ fp-$\mathcal{P}(F, G_2)$ is surjective) if and only if $F \cong \mathcal{P}_X$.

Given $\mathcal{C}$ be a pre-additive category. A family of objects $\mathcal{G}$ is called a *generating family* if, for any non zero morphism $f : C \to D$ in $\mathcal{C}$, there exists a morphism $h : G \to C$ with G in $\mathcal{G}$, such that $f \circ h \neq 0$. A co-generating family of $\mathcal{C}$ is a generating family of $\mathcal{C}^{\circ}$. Hence if $\mathcal{C}$ is projectively complete, it is a generating (resp. co-generating) family of projective (resp. injective) objects for fp-$\mathcal{C}$ (resp. ($\mathcal{C}$-fp)$^{\circ}$). Moreover (fp-$\mathcal{P}, P_{\mathcal{P}}$) is "universal" between the projectively complete categories with cokernels "containing an image" of $\mathcal{P}$ ([8]).

Definition B.7. ([22, p. 103], [8, §4]). A projectively complete category $\mathcal{P}$ is called *right (resp. left) coherent* if for any $X \in \mathcal{P}$ the functor $\mathcal{P}_X$ (resp. ${}_X\mathcal{P}$) is coherent. $\mathcal{P}$ is called *coherent* [7] if it is both left and right coherent. A pre-additive

[7]We remark that the notion of coherent additive category has nothing to do with the one proposed by Peter Johnstone for a general category.

category $\mathcal{C}$ is called (resp. right, resp. left) coherent if and only if the category $\mathrm{proj}(\mathcal{C})$ is (resp. right, resp. left) coherent.

This statement, which is probably originally due to H. Cartan, is proposed in its version for a ring R, as an exercise in Bourbaki [11, §2 Exer. 11] and explained in great detail in [10, §1.5]. Here we propose its translation in the language of pre-additive categories (since $\text{Mod-}(\mathcal{C}^\circ) = \mathcal{C}\text{-Mod}$, passing to the opposite category, one can recover the previous results for left modules).

Proposition B.8. *The category* coh-$\mathcal{C}$ *is closed under extension in* Mod-$\mathcal{C}$*. Moreover* coh-$\mathcal{C}$ *is an abelian category and the canonical functor* coh-$\mathcal{C} \to$ Mod-$\mathcal{C}$ *is exact.*

Definition B.9. Let $A \xrightarrow{f} B$ be a morphism in an additive category $\mathcal{B}$. A *weak kernel*[8] of f is a map $K \xrightarrow{i} A$ such that $fi = 0$ and, for any $X \xrightarrow{j} A$ with $fj = 0$, there exists, possibly many, $X \xrightarrow{\alpha} K$ such that $i\alpha = j$. The category $\mathcal{B}$ has *weakly pull-back squares* if, given any pair $f_i : X_i \to Y$ with $i = 1, 2$, there exists an object Z with the dashed arrows such that any commutative diagram of this type can be completed with (a not necessarily unique) dotted arrow:

$$\begin{tikzcd} W \arrow[r, dotted] \arrow[rr, bend left] \arrow[dr] & Z \arrow[r, dashed, "g_1"] \arrow[d, dashed, "g_2"'] & X_1 \arrow[d, "f_1"] \\ & X_2 \arrow[r, "f_2"] & Y. \end{tikzcd} \tag{10}$$

One can define dually the notions of weak cokernel and weak push-out.

Proposition B.10. *(*[8, Prop. 4.5]*). Let $\mathcal{P}$ be a projectively complete category. The following are equivalent:*

(1) *$\mathcal{P}$ is right coherent;*
(2) *$\mathcal{P}$ admits weak kernels;*
(3) fp-$\mathcal{P}$ = coh-$\mathcal{P}$ *is an abelian exact full subcategory of* Mod-$\mathcal{P}$ *whose projective objects are exactly the representable functors in $\mathcal{P}$.*

Moreover:

- *$\mathcal{P}$ has kernels iff* fp-$\mathcal{P}$ = coh-$\mathcal{P}$ *is abelian with* $\mathrm{gl.dim}(\text{coh-}\mathcal{P}) \leq 2$*;*
- fp-$\mathcal{P}$ = coh-$\mathcal{P}$ *is abelian with* $\mathrm{gl.dim}(\text{fp-}\mathcal{P}) = 0$ *iff $\mathcal{P} \simeq$* coh-$\mathcal{P}$ *is semisimple;*
- $\mathrm{gl.dim}(\text{coh-}\mathcal{P}) = 1$ *iff $\mathcal{P}$ is not abelian semisimple but for any morphism f in $\mathcal{P}$ we have that* $\mathrm{Ker}(f)$ *is split monic.*

Appendix C. t-structures

C.1. Horthogonal classes and t-structures. Let $\mathcal{C}$ be a pre-additive category and $\mathcal{U} \subseteq \mathcal{C}$; we set $\mathcal{U}^\perp = \{C \in \mathcal{C} \mid \mathcal{C}(U, C) = 0 \ \forall U \in \mathcal{U}\}$ and ${}^\perp\mathcal{U} = \{C \in \mathcal{C} \mid \mathcal{C}(C, U) = 0 \ \forall U \in \mathcal{U}\}$.

Given $\mathcal{C}$ a triangulated category, we will denote by $[1]$ its suspension functor, by $[n]$ its n^{th}-iterated by $X \to Y \to Z \xrightarrow{+}$ a distinguished triangle. We will denote by $\mathrm{Hom}^n_{\mathcal{C}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y[n])$. When we say that $\mathcal{U}$ is a subcategory of $\mathcal{C}$, we always mean that $\mathcal{U}$ is a full subcategory closed under isomorphisms, finite direct sums and direct summands. Given $\mathcal{U}, \mathcal{V}$ full subcategories of $\mathcal{C}$, $\mathcal{U} \star \mathcal{V}$ is the full subcategory of $\mathcal{C}$ consisting of objects X which may be included in a distinguished triangle $U \to X \to V \xrightarrow{+}$ in $\mathcal{C}$, with $U \in \mathcal{U}$ and $V \in \mathcal{V}$; $\mathcal{U}$ is called *extension closed* if $\mathcal{U} \star \mathcal{U} = \mathcal{U}$. By the octahedral axiom $(\mathcal{U} \star \mathcal{V}) \star \mathcal{W} = \mathcal{U} \star (\mathcal{V} \star \mathcal{W})$ ([27]). In general

[8] Freyd introduced in [22, p. 99] the notion of weak kernel which permits to define the notion of *weak pull-back square*. In [42, Ch. 6, 6.1.1] Neeman independently introduced the notion of *homotopy pull-back square* which coincides with Freyd weak pull-back square.

$\mathcal{U} \star \mathcal{V}$ is not idempotently complete but it is if the subcategories are orthogonal $\mathcal{C}(\mathcal{U}, \mathcal{V}) = 0$ ([27, Prop. 2.1]).

C.2. Notation. Given $\mathcal{P}$ be a projectively complete category. We denote by: $[\cdots \to L \to \overset{\bullet}{M} \to N \cdots]$ the complex in $C(\mathcal{P})$ whose element M is placed in degree zero; $[X^i \to X^{i+1} \to \cdots \to X^{i+n}]$ the complex in degrees i to $i+n$ ($n \in \mathbb{N}$) whose remains terms are 0 and by $X^{\geq n}$ (resp. $X^{\leq n}$) the complex which coincides with $X^\bullet$ in degrees greater than (resp. less than) or equal to n and is zero otherwise.

We refer to [7] for the notion of *t-structure*. We denote by $\mathcal{D} := (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t-structure in a triangulated category $\mathcal{C}$, by $\delta^{\leq n}$ (resp. $\delta^{\geq n}$) the truncation functor and by $\mathcal{H}_\mathcal{D} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the heart of the t-structure which is an abelian category. The truncation functors induce the t-cohomological functors $\mathrm{H}^i_\mathcal{D} \colon \mathcal{C} \to \mathcal{H}_\mathcal{D}$, $i \in \mathbb{Z}$, with $\mathrm{H}^0_\mathcal{D}(X) := \delta^{\geq 0}\delta^{\leq 0}(X)$ and $\mathrm{H}^i_\mathcal{D}(X) := \mathrm{H}^0_\mathcal{D}(X[i])$. We will denote by $\mathcal{D}^{[a,b]} = \mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b} \subseteq \mathcal{C}$ with $a \leq b$ in $\mathbb{Z}$ ($\mathcal{D}^{[a,a]} = \mathcal{H}_\mathcal{D}[-a]$).

One can attach to any thick subcategory[9] $\mathcal{N}$ of $\mathcal{C}$ (see [36, 4.5 and 4.6]) its multiplicative system (compatible with the triangulation) $\Sigma(\mathcal{N})$ containing all the morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ fitting in a distinguished triangle $X \xrightarrow{f} Y \to Z \xrightarrow{+}$ with $Z \in \mathcal{N}$. The quotient category $\mathcal{C}/\mathcal{N} := \mathcal{C}[\Sigma(\mathcal{N})^{-1}]$ (which could be not locally small) is endowed with the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ such that by [36, Prop. 4.6.2]:

(1) $\mathcal{C}/\mathcal{N}$ carries a unique triangulated structure such that Q is exact;
(2) a morphism in $\mathcal{C}$ is annihilated by Q if and only if it factors through an object in $\mathcal{N}$ and moreover $\mathcal{N} = \operatorname{Ker} Q$ (since it is thick);
(3) every exact functor $\mathcal{C} \to \mathcal{U}$ annihilating $\mathcal{N}$ factors uniquely through Q via an exact functor $\mathcal{C}/\mathcal{N} \to \mathcal{U}$.

Proposition C.3. *([32, Prop. 13.2.6]) Let $\mathcal{E}$ a full additive cogenerating (resp. generating) subcategory of an abelian category $\mathcal{A}$ such that there exists $d > 0$ such that, for any exact sequence $Y_d \to \cdots \to Y_1 \to Y \to 0$ (resp. $0 \to Y_1 \to \cdots \to Y_d$) with $Y_j \in \mathcal{E}$, we have $Y \in \mathcal{E}$. Hence $\frac{K(\mathcal{E})}{K(\mathcal{E}) \cap \mathcal{N}} \xrightarrow{\simeq} D(\mathcal{A})$.*

Lemma C.4. [49, Lem. 1.2.17] *Let $\mathcal{N}$ be a thick subcategory on a triangulated category $\mathcal{C}$ endowed with a t-structure $\mathcal{D}$. The pair $(Q(\mathcal{D}^{\leq 0}), Q(\mathcal{D}^{\geq 0}))$ is a t-structure on $\mathcal{C}/\mathcal{N}$ if and only if for any distinguished triangle $X_1 \to X_0 \to N \xrightarrow{+1}$ with $X_1 \in \mathcal{D}^{\geq 1}$, $X_0 \in \mathcal{D}^{\leq 0}$ and $N \in \mathcal{N}$ we have $X_1, X_0 \in \mathcal{N}$.*

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