

Weak limits for weighted means of orthogonal polynomials

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Abstract

This article is a first attempt to obtain weak limit formulas for weighted means of orthogonal polynomials. For this, we introduce a new mean Nevai class that guarantees the existence of a limiting measure for the weighted means. We show that for a family of measures in this mean Nevai class also the means of the Christoffel-Darboux kernels and the asymptotic distribution of the roots converge weakly to the same limiting measure. As a main example, we study the mean Nevai classes in which the limiting measure is the orthogonality measure of the ultraspherical polynomials. The respective weak limit formula can be regarded as an asymptotic weak addition formula for the corresponding class of measures.

Keywords: Orthogonal polynomials, mean weak limits, Nevai class, summation methods, ultraspherical polynomials

1. Introduction

A major result for weak limits of measures related to orthogonal polynomials on the real line is the following statement given in [14, Section 4, Theorem 14]:

Theorem 1 (Nevai 1979, [14]). *Let μ be a measure in the Nevai class $M(a, b)$ with $b > 0$. If f is μ -measurable, bounded on $\text{supp}(\mu)$ and Riemann integrable on $[-2b + a, a + 2b]$, then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) p_n^2(x) d\mu = \frac{1}{\pi} \int_{a-2b}^{a+2b} f(x) \frac{dx}{\sqrt{4b^2 - (x-a)^2}}. \quad (1)$$

The polynomials p_n of degree $n \in \mathbb{N}_0$ are the orthonormal polynomial with respect to the measure μ . They satisfy a three-term recurrence relation

$$xp_n(x) = b_{n+1}p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x)$$

with coefficients $a_n \in \mathbb{R}$ and $b_{n+1} > 0$, $n \in \mathbb{N}_0$. The Nevai class $M(a, b)$ in Theorem 1 is the set of all measures μ such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. In [18], also a converse statement is shown, namely that if $\text{supp}(\mu)$ is bounded and (1) holds true for all continuous

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functions f on $\text{supp}(\mu)$ then μ is in the Nevai class $M(a, b)$. In fact, this converse statement can be derived from the general formula

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) p_n(x) p_{n+l}(x) d\mu = \frac{1}{\pi} \int_{a-2b}^{a+2b} f(x) \frac{T_l(\frac{x-a}{2b}) dx}{\sqrt{4b^2 - (x-a)^2}}, \quad l \in \mathbb{N}_0, \quad (2)$$

which is true under the same prerequisites as in Theorem 1 (see [14, Section 4, Theorem 13] or [25, Section 4]). Here, $T_l(x)$ denote the Chebyshev polynomials of the first kind. Using formula (2) with $f(x) = x$ and $l = 0, l = 1$, we can directly verify that formula (1) implies that the recurrence coefficients satisfy $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$.

In [18, Theorem 2], Theorem 1 and the converse statement are given for a larger class of measures in which the recurrence coefficients satisfy the weaker condition $\lim_{n \rightarrow \infty} b_{2n} = b'$, $\lim_{n \rightarrow \infty} b_{2n+1} = b''$ and $\lim_{n \rightarrow \infty} a_n = a$. A similar weak limit result holds also true for orthogonal polynomials on the unit circle. The corresponding statement in this setting is Khrushchev's Theorem: the weak convergence of $|p_n(e^{i\theta})|^2 d\mu$ towards the uniform measure $\frac{d\theta}{2\pi}$ on the unit circle is equivalent to the fact that the Verblunsky coefficients satisfy the Máté-Nevai condition [19, Theorem 9.3.1].

If the assumptions of Theorem 1 are satisfied, two further related weak limits can be derived from (1) (see [14, Section 5, Lemma 1 and Theorem 3]), namely

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \frac{1}{n+1} \sum_{k=0}^n p_k^2(x) d\mu = \frac{1}{\pi} \int_{a-2b}^{a+2b} \frac{f(x) dx}{\sqrt{4b^2 - (x-a)^2}}, \quad (3)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^{n+1} f(x_{n+1,k}) = \frac{1}{\pi} \int_{a-2b}^{a+2b} \frac{f(x) dx}{\sqrt{4b^2 - (x-a)^2}}. \quad (4)$$

The limit (3) is usually referred to as weak limit of the Christoffel-Darboux kernel (or shortly CD kernel) $K(x, x) = \sum_{k=0}^n p_k^2(x)$. The values $x_{n+1,1} < x_{n+1,2} < \dots < x_{n+1,n+1}$ denote the $n+1$ roots of the polynomial p_{n+1} and the weak limit in (4) therefore gives the asymptotic distribution of the roots of the orthogonal polynomials p_{n+1} . The two sequences of measures in the formulas (3) and (4) are intimately related, see [14, Section 5, Lemma 1], [20, Theorem 2.4] [23, Lemma 1], or [24, Theorem 5.3]. Variants and generalizations of the limits (1), (3) and (4) have been intensively studied for different families of orthogonal polynomials, among others, for orthogonal polynomials on the unit circle and for classes of measures with asymptotically periodic recurrence coefficients. A general overview can be found in the book [21]. Specific variants are, for instance, discussed in [3, 4, 12, 13, 15, 25].

In this paper we want to derive and investigate analogs of the weak limits (1), (3) and (4) for weighted means of differing orthogonality measures. The motivation to study such mean weak limits originates in two works [5] and [6] in which a Landau-Pollak-Slepian type space-frequency analysis was studied for spaces of orthogonal polynomials. In the one-dimensional case given in [5], the roots of orthogonal polynomials were used to describe the spatial position of localized basis functions. The corresponding asymptotic distribution of the roots is given by the arcsine distribution (4). In the case of the unit sphere a similar description was

derived in [6]. This description however included ultraspherical and co-recursive ultraspherical polynomials with a differing parameter. In this setting, the mean asymptotic distribution of the roots of the polynomials turned out to be a uniform distribution on $[-1, 1]$. Compared to the arcsine distribution given in (4) this result came as a surprise. Our aim here is to obtain a better understanding of the differences in the two settings.

Our first goal is to introduce a new Nevai class for families of orthogonality measures that guarantees the existence of a weak limit for weighted means of the measures. For this we will shortly recapitulate some facts about regular summation methods. The extension of Theorem 1 to weighted means of orthogonal polynomials is formulated in Theorem 2. The analogs of the formulas (3) for the Christoffel-Darboux kernel and (4) for the mean asymptotic distribution of the roots are provided in Theorem 5 and Theorem 6, respectively. Finally, we will investigate some particular summation methods and the corresponding mean Nevai classes in which the limiting measure is precisely the orthogonality measure of the ultraspherical polynomials. The so obtained limit formulas can be regarded as asymptotic weak addition formulas for the underlying Nevai class.

2. Preliminaries

For non-negative integers $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, we consider a family of non-negative measures $\mu^{(k)}$ on \mathbb{R} supported on a bounded subinterval of \mathbb{R} . By $p_l^{(k)}(x)$, we denote the corresponding family of orthogonal polynomials of degree l . The polynomials $p_l^{(k)}(x)$ are normalized such that the leading coefficient is positive and that $p_l^{(k)}(x)$ are orthonormal on \mathbb{R} with respect to the inner product

$$\langle f, g \rangle^{(k)} := \int_{\mathbb{R}} f(x) \overline{g(x)} d\mu^{(k)}.$$

It is well known that the polynomials $p_l^{(k)}$ are an orthonormal basis of the Hilbert space $(L^2(\mathbb{R}, \mu^{(k)}), \langle \cdot, \cdot \rangle^{(k)})$. For a general overview on orthogonal polynomials and a multitude of their properties, we refer to the monographs [2, 8, 10, 22]. For this article, the three-term recurrence relation of the orthogonal polynomials is of major importance: setting $p_{-1}(x) = 0$ and $p_0 = \frac{1}{\sqrt{\mu^{(k)}(\mathbb{R})}}$, we have the relation

$$xp_l^{(k)}(x) = b_{l+1}^{(k)} p_{l+1}^{(k)}(x) + a_l^{(k)} p_l^{(k)}(x) + b_l^{(k)} p_{l-1}^{(k)}(x) \quad (5)$$

for $l \in \mathbb{N}_0$ with coefficients $a_l^{(k)} \in \mathbb{R}$ and $b_l^{(k)} > 0$. Introducing the Jacobi matrices

$$\mathbf{J}_l^{(k)} := \begin{pmatrix} a_0^{(k)} & b_1^{(k)} & 0 & 0 & \cdots & 0 \\ b_1^{(k)} & a_1^{(k)} & b_2^{(k)} & 0 & \cdots & 0 \\ 0 & b_2^{(k)} & a_2^{(k)} & b_3^{(k)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_{l-1}^{(k)} & a_{l-1}^{(k)} & b_l^{(k)} \\ 0 & \cdots & \cdots & 0 & b_l^{(k)} & a_l^{(k)} \end{pmatrix},$$

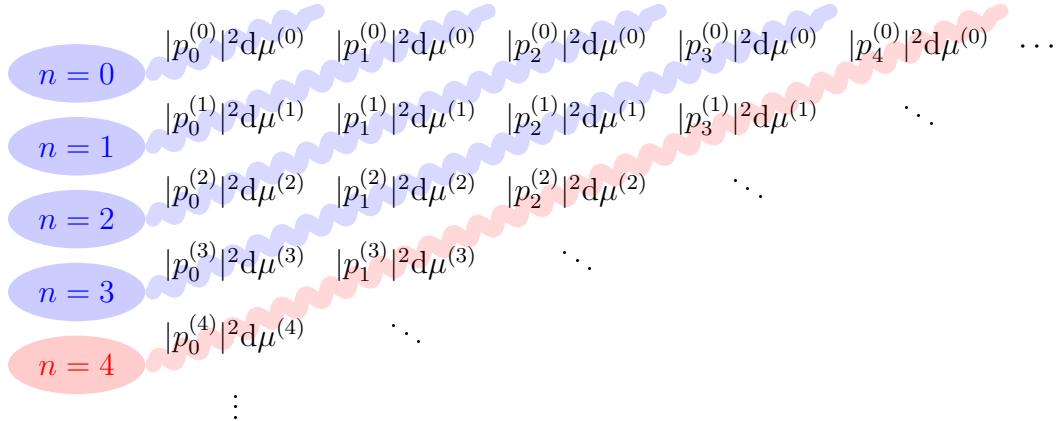


Figure 1: Graphical illustration of the summation methods. Summing over the red curve gives the measure $\bar{\mu}_4$ defined in (6), summing over all blue curves generates the measure λ_3 introduced in (13). The sum of all red and blue elements gives λ_4 .

we further have the representation

$$p_{l+1}^{(k)}(x) = \frac{1}{\sqrt{\mu^{(k)}(\mathbb{R})}} \frac{1}{b_1^{(k)} \dots b_{l+1}^{(k)}} \det \left(x \mathbf{1}_{l+1} - \mathbf{J}_l^{(k)} \right).$$

From this representation it is obvious that the roots $x_{l+1,j}^{(k)}$, $1 \leq j \leq l+1$, of $p_{l+1}^{(k)}(x)$ correspond to the $l+1$ eigenvalues of the symmetric matrix $\mathbf{J}_l^{(k)}$. To simplify our calculations, we additionally set $a_{-l}^{(k)} = 0$ if $l \in \mathbb{N}$ and $b_{-l}^{(k)} = 0$ for $l \in \mathbb{N}_0$.

3. Weak limits for weighted sums of orthogonal polynomials

For a sequence $y = (y_k)_{k \in \mathbb{N}_0}$, we consider summation methods $S = (S_n)_{n \in \mathbb{N}_0}$ given by

$$S_n(y) = \sum_{k=0}^n \sigma_{n,k} y_k.$$

Assumption 1 (Regularity of S). We assume that the weights $\sigma_{n,k}$ of the summation method S satisfy the following three conditions:

- (i) $\sigma_{n,k} \geq 0$, for all $0 \leq k \leq n$, $n \in \mathbb{N}_0$,
- (ii) $\sum_{k=0}^n \sigma_{n,k} = 1$,
- (iii) $\lim_{n \rightarrow \infty} \sigma_{n,k} = 0$ for $k \in \mathbb{N}_0$.

These three conditions imply that, according to the Silverman-Toeplitz theorem, for every convergent sequence y the sequence $(S_n y)_{n \in \mathbb{N}_0}$ converges to the same limit as y . In this work, we call the summation method S regular if the three conditions (i), (ii) and (iii) are satisfied.

Note that in the literature the non-negativity of the summation weights is usually not demanded for regularity (cf. [1, Theorem 2.3.7], [9, Section 3.2]). If negative summation weights $\sigma_{n,k}$ are allowed, Assumption 1 can be weakened by postulating (ii), (iii) together with $\sum_{k=0}^n |\sigma_{n,k}| \leq M$ for some positive constant M . In this work, we consider only non-negative weights and therefore use the conditions of Assumption 1. For a broader overview to summation methods we refer to [1] and the classic [9].

Based on a regular summation method S , we consider now for $n \geq 0$ the following mean measures. The single measures involved in the summation are illustrated in Figure 1.

$$d\bar{\mu}_n = \sum_{k=0}^n \sigma_{n,k} \left(p_k^{(n-k)}(x) \right)^2 d\mu^{(n-k)}. \quad (6)$$

Definition 1. For a regular summation method S , we say that the family of measures $\mu^{(k)}$, $k \in \mathbb{N}_0$, is in the mean Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$ if the following three conditions are satisfied for the coefficients $a_n^{(k)}$, $b_n^{(k)}$ in the three-term recurrence relation (5):

- (i) $\sup_{n,k \in \mathbb{N}_0} |a_n^{(k)}| = A < \infty$, $\sup_{n,k \in \mathbb{N}_0} |b_n^{(k)}| = B < \infty$.
- (ii) $\lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} |a_{k+l}^{(n-k)} - a_k^{(n-k)}| = 0$, $\lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} |b_{k+l}^{(n-k)} - b_k^{(n-k)}| = 0$, for all $l \in \mathbb{N}$.
- (iii) $\lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} (a_k^{(n-k)})^{l_a} (b_k^{(n-k)})^{l_b} = \Sigma_{l_a, l_b} < \infty$ for all $l_a, l_b \in \mathbb{N}_0$.

Theorem 2. Suppose that the family $\mu^{(k)}$, $k \in \mathbb{N}_0$, is in the mean Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$. Then, the sequence $\bar{\mu}_n$ converges weakly to a measure μ_Σ and for every continuous f we have

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} \int_{\mathbb{R}} f(x) (p_k^{(n-k)}(x))^2 d\mu^{(n-k)} = \int_{\mathbb{R}} f(x) d\mu_\Sigma.$$

The limiting measure μ_Σ is determined by the numbers Σ_{l_a, l_b} , $l_a \in \mathbb{N}_0$, $l_b \in 2\mathbb{N}_0$.

In order to prove Theorem 2, we can follow similar arguments as given in [14, Section 4.2, Theorem 14] or in [18, Theorem 3.1]. In the following, we will stay closer to the notion used in [18] and introduce some additional terminology.

A path ρ of length l is a sequence $(\rho_j)_{j=0}^l \in \mathbb{Z}^{l+1}$ of $l+1$ integers such that $|\rho_j - \rho_{j-1}| \leq 1$. For a path ρ , we define the weights

$$W^{(k)}(\rho) = \prod_{j=0}^{l-1} w^{(k)}(\rho_j, \rho_{j+1})$$

where

$$w^{(k)}(\rho_j, \rho_{j+1}) = \begin{cases} a_m^{(k)} & \text{if } \rho_{j+1} = \rho_j = m, \\ b_{m+1}^{(k)} & \text{if } \rho_{j+1} = \rho_j + 1 = m + 1, \\ b_m^{(k)} & \text{if } \rho_{j+1} = \rho_j - 1 = m - 1. \end{cases}$$

Recall that for negative m we set the coefficients $a_m^{(k)}$ and $b_m^{(k)}$ equal to zero, as well as the coefficient $b_0^{(k)}$. For a path ρ and $m \in \mathbb{Z}$, we further define the shifted path $T_m\rho$ by

$$T_m\rho = (\rho_j + m)_{j=0}^l.$$

By using the three-term recurrence relation (5) it is straightforward to see that

$$\int_{\mathbb{R}} x^l (p_m^{(k)}(x))^2 d\mu^{(k)} = \langle x^l p_m^{(k)}, p_m^{(k)} \rangle^{(k)} = \sum_{\rho \in Q_l} W^{(k)}(T_m\rho) \quad (7)$$

holds, where Q_l denotes the set of all paths of length l with $\rho_0 = \rho_l = 0$. The detailed elaboration for the derivation of (7) is given in [18, Proposition 3.3]. The identity (7) immediately implies the formula

$$\int_{\mathbb{R}} x^l d\bar{\mu}_n = \sum_{\rho \in Q_l} \sum_{k=0}^n \sigma_{n,k} W^{(n-k)}(T_k\rho) \quad (8)$$

for the moments of the mean measure $\bar{\mu}_n$. It allows us to prove Theorem 2.

Proof. The assumption (i) in the Definition 1 of the Nevai class implies that the mean measures $\bar{\mu}_n$, $n \in \mathbb{N}_0$, are all compactly supported in $[-2B - A, A + 2B]$. Thus, weak convergence of the measures $\bar{\mu}_n$ is equivalent to the convergence of the moments

$$\int_{\mathbb{R}} x^l d\bar{\mu}_n.$$

By (8), we only have to show that for every path $\rho \in Q_l$ the sums

$$\sum_{k=0}^n \sigma_{n,k} W^{(n-k)}(T_k\rho)$$

are converging. For this, we refine our look on the paths $\rho \in Q_l$. For $\rho \in Q_l$ with length l we set $l_a = \#\{j : \rho_{j+1} = \rho_j\}$ and $l_b = \#\{j : \rho_{j+1} \neq \rho_j\}$ such that $l = l_a + l_b$ holds. Then, the sum above can be written as

$$\sum_{k=0}^n \sigma_{n,k} \prod_{j=0}^{l_a-1} a_{k+r_{j,a}}^{(n-k)} \prod_{j=0}^{l_b-1} b_{k+r_{j,b}}^{(n-k)},$$

where $r_{j,a}$ and $r_{j,b}$ are integers between $-l/2$ and $l/2$ that depend on the chosen path $\rho \in Q_l$. Further, since $\rho \in Q_l$, l_b must be an even integer. Because of the assumptions (i), (ii) and (iii) in the Definition 1 of the Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$, we obtain as $n \rightarrow \infty$ the identity

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} \prod_{j=0}^{l_a-1} a_{k+r_{j,1}}^{(n-k)} \prod_{j=0}^{l_b-1} b_{k+r_{j,2}}^{(n-k)} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} (a_k^{(n-k)})^{l_a} (b_k^{(n-k)})^{l_b} = \Sigma_{l_a, l_b}.$$

In this formula, the condition (iii) guarantees the existence of the limit in the second equality, whereas the conditions (i) and (ii) guarantee that the first equality holds true. Therefore, starting from the identity (8) we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} x^l d\bar{\mu}_n = \sum_{\rho \in Q_l} \Sigma_{l_a, l_b},$$

and thus the statement of the theorem. In particular, the last identity implies that the limiting measure μ_{Σ} solely depends on the values Σ_{l_a, l_b} , $l_a \in \mathbb{N}$, $l_b \in 2\mathbb{N}_0$. \square

Remark 1. The assumption (i) in Definition 1 implies that the measures $\mu^{(k)}$, the means $\bar{\mu}_n$ and the limiting measure μ_{Σ} are all compactly supported in $[-2B - A, A + 2B]$.

We give a first example of such a mean weak limit. In view of Theorem 1 for measures in the Nevai class $M(a, b)$ the outcome is not yet surprising.

Definition 2. We say that the family of measures $\mu^{(k)}$, $k \in \mathbb{N}_0$, is in the uniform Nevai class $M^{(U)}(a, b)$ if the coefficients $a_n^{(k)}$, $b_n^{(k)}$ in the three-term recurrence relation (5) are uniformly in the same Nevai class $M(a, b)$, i.e.

$$\limsup_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} |a_n^{(k)} - a| = 0, \quad \limsup_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} |b_n^{(k)} - b| = 0. \quad (9)$$

We denote the characteristic function of an interval $I \subset \mathbb{R}$ by χ_I .

Corollary 3. Assume that the family $\mu^{(k)}$, $k \in \mathbb{N}_0$, is in the uniform Nevai class $M^{(U)}(a, b)$. Then, for every regular summation method S the family $\mu^{(k)}$ is in the mean Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$ with the values Σ_{l_a, l_b} given by $\Sigma_{l_a, l_b} = a^{l_a} b^{l_b}$. In this case, μ_{Σ} is explicitly given by the equilibrium measure

$$d\mu_{\Sigma} = \frac{1}{\pi} \frac{\chi_{[a-2b, a+2b]}(x)}{\sqrt{4b^2 - (x-a)^2}} dx.$$

In particular, the measure μ_{Σ} depends only on the limits a and b and is independent of the summation method S .

Proof. It is straightforward to check that the uniformity of the limits in (9) together with the fact that the summation method S is regular according to Assumption 1 implies all three conditions in Definition 1, and also that $\Sigma_{l_a, l_b} = a^{l_a} b^{l_b}$. By Theorem 2, we therefore get a limiting measure μ_{Σ} which only depends on the limits a and b of the class. Further, to get the explicit form of the equilibrium measure μ_{Σ} , it is sufficient to derive it for one particular summation method and for one particular family of measures in $M^{(U)}(a, b)$. For this, we consider a measure μ in the Nevai class $M(a, b)$ and set $\mu^{(k)} = \mu$ for all $k \in \mathbb{N}_0$. Then, the family $\mu^{(k)}$ is in $M^{(U)}(a, b)$. As a summation method S we consider the identity

scheme given by $\sigma_{n,k} = \delta_{nk}$, where δ_{nk} denotes the usual Kronecker delta. Obviously, this summation method is regular. For this construction, we explicitly get

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k} \int_{\mathbb{R}} f(x) (p_k^{(n-k)}(x))^2 d\mu^{(n-k)} &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) (p_n(x))^2 d\mu \\ &= \frac{1}{\pi} \int_{a-2b}^{a+2b} \frac{f(x)}{\sqrt{4b^2 - (x-a)^2}} dx, \end{aligned}$$

where the last equality follows by Theorem 1. □

4. Weak limits for weighted sums of Christoffel-Darboux kernels

We are now interested in weak limits related to the family $K_n^{(k)}(x, y)$, $k \in \mathbb{N}_0$, of Christoffel-Darboux kernels. They are defined as

$$K_n^{(k)}(x, y) = \sum_{l=0}^n p_l^{(k)}(x) p_l^{(k)}(y), \quad n \in \mathbb{N}_0.$$

By considering averages of the measures $K_k^{(n-k)}(x, x) d\mu^{(n-k)}$, $k = 0, \dots, n$, the weights $\sigma_{n,k}$ of the summation method should in principal depend on n and on the parameter $n - k$ of the measure $\mu^{(n-k)}$. Therefore, in this section we restrict ourselves to the following class of summation methods S .

Definition 3. Let $(\sigma_n)_{n \in \mathbb{N}_0}$ and $(\tau_n)_{n \in \mathbb{N}_0}$ be two non-negative sequences. We call the summation method S a Nörlund method if the weights $\sigma_{n,k}$ are given by

$$\sigma_{n,k} = \tau_n \sigma_{n-k}, \quad n \in \mathbb{N}_0, \quad 0 \leq k \leq n. \quad (10)$$

We call a Nörlund method S regular if the three conditions of Assumption 1 are satisfied.

Remark 2. For a regular Nörlund method S with a fixed sequence $(\sigma_n)_{n \in \mathbb{N}_0}$ the sequence $(\tau_n)_{n \in \mathbb{N}_0}$ is uniquely determined by

$$\tau_n = \left(\sum_{k=0}^n \sigma_{n-k} \right)^{-1} > 0. \quad (11)$$

The fact that S is regular implies further that $\sigma_0 > 0$. Therefore, we can always normalize a regular Nörlund method such that $\sigma_0 = 1$, $\tau_0 = 1$ and $0 < \tau_n \leq 1$ for $n \geq 1$. For more properties on Nörlund methods we refer to [1, Section 3.3].

For a regular Nörlund method S based on the sequences $(\sigma_n)_{n \in \mathbb{N}_0}$ and $\tau_n = (\sum_{k=0}^n \sigma_{n-k})^{-1}$ we can define the normalization constant

$$N_n = \tau_n \sum_{k=0}^n \frac{1}{\tau_k} \geq 1.$$

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This gives us the possibility to introduce a new summation method T based on the positive weights

$$\tau_{n,k} = \frac{\tau_n}{N_n} \frac{1}{\tau_k}. \quad (12)$$

Lemma 4. *If S is a regular Nörlund method determined by a given sequence $(\sigma_n)_{n \in \mathbb{N}_0}$ in (10), then the summation method T given by (12) is also regular.*

Proof. The first two conditions (i) and (ii) of Assumption 1 are automatically satisfied by the construction of the summation method T . Therefore, we only have to show (iii), i.e. that for fixed k the positive sequence $\tau_{n,k} = \frac{\tau_n}{N_n} \frac{1}{\tau_k}$ is converging to zero as $n \rightarrow \infty$. We know by (11) that the sequence (τ_n) is positive and monotonically decreasing and, thus, convergent. If $(\tau_n) \rightarrow 0$ then also $\frac{\tau_n}{N_n}$ converges to 0 as $n \rightarrow \infty$. If $(\tau_n) \rightarrow c > 0$, then $N_n \rightarrow \infty$ as $n \rightarrow \infty$. Thus, also in this second case the sequence $\frac{\tau_n}{N_n}$ tends to zero. \square

Remark 3. The summation method T is a Riesz summation method according to the definition given in [1, Definition 3.2.2]. Since (τ_n) is monotonically decreasing, the maximum of $\tau_{n,k}$ for fixed n is attained at $k = n$. In particular, we have the inequalities $0 < \tau_{n,k} \leq \frac{1}{N_n}$. Since the summation method T is regular, we therefore get the following estimates for the normalization constant N_n :

$$1 \leq N_n \leq n + 1.$$

We investigate now the following means related to the kernels $K_k^{(n-k)}(x, y)$:

$$d\lambda_n = \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} K_k^{(n-k)}(x, x) d\mu^{(n-k)}. \quad (13)$$

Theorem 5. *Let S be a regular Nörlund method and let $\mu^{(k)}$, $k \in \mathbb{N}_0$, be in the mean Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$. Then, the sequence λ_n converges weakly to the limiting measure μ_Σ given in Theorem 2, i.e.*

$$\text{w-lim}_{n \rightarrow \infty} \lambda_n = \text{w-lim}_{n \rightarrow \infty} \bar{\mu}_n = \mu_\Sigma.$$

In particular, for every continuous function f we have

$$\lim_{n \rightarrow \infty} \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} \int_{\mathbb{R}} f(x) K_k^{(n-k)}(x, x) d\mu^{(n-k)} = \int_{\mathbb{R}} f(x) d\mu_\Sigma.$$

Proof. Since S is a regular Nörlund method, we can decompose the weights $\sigma_{n,k}$ and rearrange the sums in the definition (13) of the measure λ_n . In this way we get

$$\begin{aligned} d\lambda_n &= \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} K_k^{(n-k)}(x, x) d\mu^{(n-k)} = \frac{\tau_n}{N_n} \sum_{k=0}^n \sigma_{n-k} \sum_{j=0}^k |p_j^{(n-k)}(x)|^2 d\mu^{(n-k)} \\ &= \frac{\tau_n}{N_n} \sum_{k=0}^n \sum_{j=0}^k \sigma_{k-j} |p_j^{(k-j)}(x)|^2 d\mu^{(k-j)} = \frac{\tau_n}{N_n} \sum_{k=0}^n \frac{1}{\tau_k} \sum_{j=0}^k \sigma_{k,j} |p_j^{(k-j)}(x)|^2 d\mu^{(k-j)} \\ &= \sum_{k=0}^n \tau_{n,k} d\bar{\mu}_k. \end{aligned}$$

By Theorem 2, we know that $\text{w-lim}_{n \rightarrow \infty} \bar{\mu}_n = \mu_\Sigma$. By Lemma 4, the Riesz summation method T is regular and, thus, preserves the weak limit μ_Σ , i.e.

$$\text{w-lim}_{n \rightarrow \infty} \lambda_n = \text{w-lim}_{n \rightarrow \infty} \sum_{k=0}^n \tau_{n,k} \bar{\mu}_k = \text{w-lim}_{n \rightarrow \infty} \bar{\mu}_n = \mu_\Sigma.$$

□

Now, let $x_{n,j}^{(k)}$ denote the j th smallest root of the orthogonal polynomial $p_n^{(k)}$ and δ_x be the Dirac point measure supported at $x \in \mathbb{R}$. We study the limiting properties of the following family of discrete measures:

$$\nu_{n+1} = \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} \sum_{j=1}^{k+1} \delta_{x_{k+1,j}^{(n-k)}}. \quad (14)$$

Theorem 6. *Let S be a regular Nörlund method and assume that $N_n \rightarrow \infty$. Let the family $\mu^{(k)}$, $k \in \mathbb{N}_0$, be in the mean Nevai class $M^{(S)}(\Sigma_{l_a, l_b})$. Then, the sequence ν_{n+1} converges also weakly to the limiting measure μ_Σ given in Theorem 2. In particular, we have*

$$\text{w-lim}_{n \rightarrow \infty} \nu_{n+1} = \text{w-lim}_{n \rightarrow \infty} \lambda_n = \text{w-lim}_{n \rightarrow \infty} \bar{\mu}_n = \mu_\Sigma$$

and for every continuous function f we get

$$\lim_{n \rightarrow \infty} \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} \sum_{j=1}^{k+1} f(x_{k+1,j}^{(n-k)}) = \int_{\mathbb{R}} f(x) d\mu_\Sigma.$$

Proof. We show that ν_{n+1} and λ_n have the same weak limits. The statement follows then from Theorem 5 and Theorem 2. For this, we use explicit bounds for the difference of the measures $K_k^{(n-k)}(x, x) d\mu^{(n-k)}$ and $\sum_{j=1}^{k+1} \delta_{x_{k+1,j}^{(n-k)}}$: according to [20, Proposition 2.3] (see also [17] for an alternative proof) and the fact that all measures $\mu^{(n-k)}$ are compactly supported in $[-A - 2B, A + 2B]$, we have for all $l \in \mathbb{N}_0$ and $n, k \in \mathbb{N}_0$, $n \geq k$, the estimate

$$\left| \int_{\mathbb{R}} x^l K_k^{(n-k)}(x, x) d\mu^{(n-k)} - \sum_{j=1}^{k+1} (x_{k+1,j}^{(n-k)})^l \right| \leq 2l(A + 2B)^l.$$

Since S is a regular summation method, this directly gives

$$\left| \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} \int_{\mathbb{R}} x^l K_k^{(n-k)}(x, x) d\mu^{(n-k)} - \frac{1}{N_n} \sum_{k=0}^n \sigma_{n,k} \sum_{j=1}^{k+1} (x_{k+1,j}^{(n-k)})^l \right| \leq \frac{2l(A + 2B)^l}{N_n}.$$

Since we assume that N_n diverges, $\nu_{n+1} - \lambda_n$ converges weakly to the zero measure for all polynomials and, thus, for all continuous functions on $[-A - 2B, A + 2B]$. The existence of the weak limit $\text{w-lim}_{n \rightarrow \infty} \lambda_n = \mu_\Sigma$ is already established by Theorem 5 such that $\text{w-lim}_{n \rightarrow \infty} \nu_{n+1} = \text{w-lim}_{n \rightarrow \infty} \lambda_n = \mu_\Sigma$. □

By Corollary 3, the limiting measure μ_Σ for families of polynomials in the uniform Nevai class $M^{(U)}(a, b)$ is given by the arcsine distribution. The preceding theorems now give us also the following result.

Corollary 7. *Assume that the family $\mu^{(k)}$ is in the uniform Nevai class $M^{(U)}(a, b)$. Then, for any regular Nörlund method S with $N_n \rightarrow \infty$ the measures λ_n and ν_{n+1} converge weakly to the equilibrium measure*

$$d\mu_\Sigma = \frac{1}{\pi} \frac{\chi_{[a-2b, a+2b]}(x)}{\sqrt{4b^2 - (x-a)^2}} dx.$$

5. Weak limits related to ultraspherical polynomials

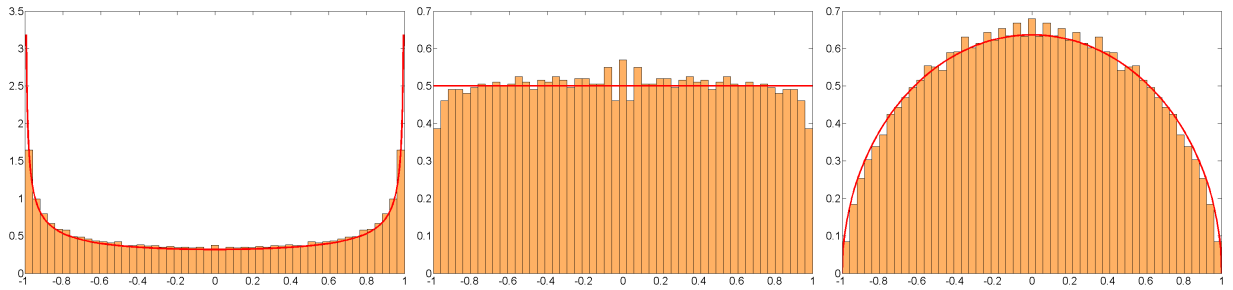


Figure 2: Graphical illustration of the mean measure ν_{n+1} , $n = 100$, given in definition (14). The mean distribution ν_{101} of the roots for families of polynomials in three different Nevai classes is shown via orange histograms. Left: the used family $p_n^{(k)} = p_n^{(0, 2\pi+1)}$ is in the uniform Nevai class $M^{(U)}(0, \frac{1}{2})$, the limiting measure (red) is the arcsine distribution derived in Corollary 7. Center: the family $p_n^{(k)} = p_n^{(k, 2\pi+1)}$ is in the mean Nevai class $M^{(C,1)}(\Sigma_{l_a, l_b})$ considered in Corollary 9. The limiting measure (red) is the uniform distribution on $[-1, 1]$. Right: the ultraspherical family $p_n^{(k)} = p_n^{(k, 2\pi+1)}$ is also in the mean Nevai class $M^{(C,2)}(\Sigma_{l_a, l_b})$ considered in Theorem 8. The corresponding limiting measure (red) is $\frac{2}{\pi} \sqrt{1-x^2} dx$.

As a more concrete example of mean weak limits we consider families of ultraspherical polynomials. Using an additional fixed parameter $\lambda > -\frac{1}{2}$, we consider the family of ultraspherical polynomials $p_n^{(k, \lambda)}(x)$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, orthogonal on $[-1, 1]$ with respect to the measure $d\mu^{(k, \lambda)}(x) = (1-x^2)^{k+\lambda-\frac{1}{2}} dx$. The recurrence coefficients of the polynomials $p_n^{(k, \lambda)}(x)$ are explicitly given by (see [8, p. 29])

$$a_k^{(n-k, \lambda)} = 0, \quad b_k^{(n-k, \lambda)} = \frac{1}{2} \sqrt{\frac{k(2n-k+2\lambda-1)}{(n+\lambda-1)(n+\lambda)}}.$$

Since the values $a_k^{(n-k, \lambda)} = 0$ are all zero, we have $\Sigma_{l_a, l_b} = 0$ if $l_a \geq 1$. Therefore only the values Σ_{0, l_b} , $l_b \in 2\mathbb{N}$ are relevant in the determination of the limiting measure μ_Σ . We list some useful summation methods for the ultraspherical polynomials.

5.1. Arithmetic mean. The first summation method is the standard arithmetic mean given by the weights $\sigma_{n, k}^{(C, 1)} = \frac{1}{n+1}$. The arithmetic mean is a regular Nörlund method and

can be decomposed as $\sigma_{n,k}^{(C,1)} = \tau_n^{(C,1)} \sigma_{n-k}^{(C,1)}$ with $\tau_n^{(C,1)} = \frac{1}{n+1}$ and $\sigma_{n-k}^{(C,1)} = 1$. The normalizing constant $N_n^{(C,1)}$ is given by

$$N_n^{(C,1)} = \frac{1}{n+1} \sum_{k=0}^n (k+1) = \frac{n+2}{2}.$$

By using the definition of the Riemann integral and the explicit formulas for $a_k^{(n-k,\lambda)}$, $b_k^{(n-k,\lambda)}$, we can compute the values Σ_{0,l_b} , $l_b \in 2\mathbb{N}$, for the family $\mu^{(k,\lambda)}$:

$$\begin{aligned} \Sigma_{0,l_b} &= \lim_{n \rightarrow \infty} 2^{-l_b} \sum_{k=0}^n \frac{1}{n+1} \frac{k^{\frac{l_b}{2}} (2n-k+2\lambda-1)^{\frac{l_b}{2}}}{(n+\lambda-1)^{\frac{l_b}{2}} (n+\lambda)^{\frac{l_b}{2}}} \\ &= 2^{-l_b} \lim_{n \rightarrow \infty} \left(\frac{n^2}{(n+\lambda-1)(n+\lambda)} \right)^{\frac{l_b}{2}} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n+1} \left(\frac{k}{n} \right)^{\frac{l_b}{2}} \left(2 - \frac{k}{n} + \frac{2\lambda-1}{n} \right)^{\frac{l_b}{2}} \\ &= 2^{-l_b} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(\frac{k}{n} \right)^{\frac{l_b}{2}} \left(2 - \frac{k}{n} \right)^{\frac{l_b}{2}} \\ &= 2^{-l_b} \int_0^1 (x(2-x))^{\frac{l_b}{2}} dx = 2^{-l_b-1} \int_{-1}^1 (1-x^2)^{\frac{l_b}{2}} dx = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{l_b+2}{2})}{2^{l_b} \Gamma(\frac{l_b+3}{2})}. \end{aligned}$$

The final identity is a standard formulation in terms of the Gamma function.

5.2. Legendre summation. As a second summation method, we consider a summation method related to the addition formula for the Legendre polynomials. The weights for this summation method are defined by $\sigma_{n,k}^{(L)} = \tau_n^{(L)} \sigma_{n-k}^{(L)}$ with

$$\tau_n^{(L)} = \frac{1}{2n+1} \quad \text{and} \quad \sigma_k^{(L)} = \begin{cases} 1, & k=0 \\ 2, & k>0. \end{cases}$$

Exactly as the arithmetic mean, also this summation is a regular Nörlund method. For the normalizing constant $N_n^{(L)}$ we get

$$N_n^{(L)} = \frac{1}{2n+1} \left(\sum_{k=0}^n (2k+1) \right) = \frac{(n+1)^2}{2n+1}.$$

In the same way as for the arithmetic mean, we get the following limits Σ_{0,l_b} for $l_b \in 2\mathbb{N}$:

$$\Sigma_{0,l_b} = \lim_{n \rightarrow \infty} 2^{-l_b} \sum_{k=1}^n \frac{2}{2n+1} \frac{k^{\frac{l_b}{2}} (2n-k+2\lambda-1)^{\frac{l_b}{2}}}{(n+\lambda-1)^{\frac{l_b}{2}} (n+\lambda)^{\frac{l_b}{2}}} = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{l_b+2}{2})}{2^{l_b} \Gamma(\frac{l_b+3}{2})}.$$

5.3. Cesàro (C, α) summation. A generalization of the arithmetic mean is Cesàro (C, α) summation. For $\alpha > 0$, this summation method is regular with the weights $\sigma_{n,k}^{(C,\alpha)} = \tau_n^{(C,\alpha)} \sigma_{n-k}^{(C,\alpha)}$ given by

$$\tau_n^{(C,\alpha)} = \frac{1}{\binom{n+\alpha}{n}} \quad \text{and} \quad \sigma_k^{(C,\alpha)} = \binom{k+\alpha-1}{k}.$$

For the normalizing constant $N_n^{(C,\alpha)}$ we get

$$N_n^{(C,\alpha)} = \frac{\binom{n+\alpha+1}{n}}{\binom{n+\alpha}{n}} = \frac{n+\alpha+1}{\alpha+1}.$$

We calculate now the limits Σ_{0,l_b} , $l_b \in 2\mathbb{N}$, for the family of ultraspherical measures $d\mu^{(k,\lambda)}(x) = (1-x^2)^{k+\lambda-\frac{1}{2}}d(x)$:

$$\begin{aligned} \Sigma_{0,l_b} &= \lim_{n \rightarrow \infty} 2^{-l_b} \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} k^{\frac{l_b}{2}} (2n-k+2\lambda-1)^{\frac{l_b}{2}} \\ &= 2^{-l_b} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n} \left(1 - \frac{k}{n}\right)^{\alpha-1} \left(\frac{k}{n}\right)^{\frac{l_b}{2}} \left(2 - \frac{k}{n}\right)^{\frac{l_b}{2}} \\ &= 2^{-l_b} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} (x(2-x))^{\frac{l_b}{2}} dx = 2^{-l_b} \frac{\alpha}{2} \int_{-1}^1 |x|^{\alpha-1} (1-x^2)^{\frac{l_b}{2}} dx \\ &= \frac{\Gamma(\frac{\alpha+2}{2})}{2^{l_b}} \frac{\Gamma(\frac{l_b+2}{2})}{\Gamma(\frac{\alpha+l_b+2}{2})} = 2^{-l_b} \left(\frac{\alpha+l_b}{2}\right)^{-1}. \end{aligned}$$

5.4. Gegenbauer summation. A direct generalization of the Legendre mean is the Gegenbauer summation (G, ν) for $\nu > 0$. The weights $\sigma_{n,k}^{(G,\nu)} = \tau_n^{(G,\nu)} \sigma_{n-k}^{(G,\nu)}$ are in this case given by

$$\tau_n^{(G,\nu)} = \frac{\Gamma(2\nu+1)\Gamma(n+1)}{(2n+2\nu)\Gamma(n+2\nu)} \quad \text{and} \quad \sigma_k^{(G,\nu)} = \frac{(2k+2\nu-1)\Gamma(k+2\nu-1)}{\Gamma(2\nu)\Gamma(k+1)}.$$

This formula is well-defined for all $n, k \in \mathbb{N}_0$ and $\nu > 0$, $\nu \neq \frac{1}{2}$. In the case $\nu = \frac{1}{2}$ we define the weights by taking the limit $\nu \rightarrow \frac{1}{2}$. In this way, we obtain $\sigma_k^{(G,1/2)} = \sigma_k^{(L)}$, i.e. the Gegenbauer summation for $\nu = \frac{1}{2}$ corresponds to the Legendre summation discussed before. For the normalizing constant $N_n^{(G,\nu)}$ we have

$$N_n^{(G,\nu)} = \frac{(2n+2\nu+1)(n+2\nu)}{(2n+2\nu)(2\nu+1)}.$$

The identities for $\tau_n^{(G,\nu)}$ and $N_n^{(G,\nu)}$ can be deduced easily from a relation of the Gegenbauer summation to the Cesàro means. Namely, for the sequences $\sigma_k^{(G,\nu)}$ we have the relations

$$\sigma_k^{(G,\nu)} = \begin{cases} \sigma_k^{(C,2\nu)}, & k = 0, \\ \sigma_k^{(C,2\nu)} + \sigma_{k-1}^{(C,2\nu)}, & k > 0. \end{cases} \quad (15)$$

This immediately implies also for the sequence τ_n the relation

$$\tau_n^{(G,\nu)} = \begin{cases} \tau_n^{(C,2\nu)}, & n = 0, \\ \left(\frac{1}{\tau_n^{(C,2\nu)}} + \frac{1}{\tau_{n-1}^{(C,2\nu)}} \right)^{-1}, & n > 0. \end{cases} \quad (16)$$

Using the relations (15) and (16) gives the above stated explicit formulas for $\tau_n^{(G,\nu)}$ and $N_n^{(G,\nu)}$, and, even more, we obtain the limits Σ_{0,l_b} for the family of ultraspherical measures from the corresponding limits of the Cesàro means:

$$\Sigma_{0,l_b} = 2^{-l_b} \left(\frac{\nu + \frac{l_b}{2}}{\nu} \right)^{-1}, \quad \text{for } \nu > 0 \text{ and } l_b \in 2\mathbb{N}.$$

Note that in the case $\nu = \frac{1}{2}$ we obtain precisely the derived formula for the Legendre summation. There is a relation of the weights $\sigma_{n,k}^{(G,\lambda)}$, $\lambda > 0$, with the addition formula of the ultraspherical polynomials $p_k^{(n-k,\lambda)}$. Applying the general addition formula [7, 3.15.1, (19)] or [16, (18.18.8)] to the orthonormal ultraspherical polynomials $p_k^{(n-k,\lambda)}$, we obtain the following special variant of the addition formula:

$$\frac{1}{m^{(\lambda)}} = \sum_{k=0}^n \sigma_{n,k}^{(G,\lambda)} (1-x^2)^{n-k} |p_k^{(n-k,\lambda)}(x)|^2 \quad (17)$$

where $m^{(\lambda)} = \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} dx = \pi 2^{1-2\lambda} \binom{2\lambda-1}{\lambda}$ and $x \in [-1, 1]$. In the Legendre case $\lambda = \frac{1}{2}$, a simplified version of the addition formula can be obtained from [16, (18.18.9)]:

$$\frac{2n+1}{2} = |p_n^{(0,\frac{1}{2})}(x)|^2 + 2 \sum_{k=1}^n (1-x^2)^k |p_{n-k}^{(k,\frac{1}{2})}(x)|^2. \quad (18)$$

The formula (18) can alternatively be regarded as a version of the addition theorem for spherical harmonics. The two given formulas are also a special case of a more general addition formula for Jacobi polynomials, see [11]. Both formulas turn out to be very useful when deriving explicit identities for the limiting measure in the ultraspherical case.

5.5. Weak limits related to Cesàro means of ultraspherical polynomials.

Theorem 8. *Suppose that the family $\mu^{(k)}$, $k \in \mathbb{N}_0$ is in the mean Nevai class $M^{(C,\alpha)}(\Sigma_{l_a,l_b})$ with $\alpha > 0$ and limits Σ_{l_a,l_b} given by*

$$\Sigma_{l_a,l_b} = \delta_{0,l_a} 2^{-l_b} \left(\frac{\frac{\alpha+l_b}{2}}{\frac{\alpha}{2}} \right)^{-1}. \quad (19)$$

Then, the limiting measure μ_Σ is given as

$$d\mu_\Sigma = \frac{1}{m^{(\frac{\alpha}{2})}} \chi_{[-1,1]}(x) (1-x^2)^{\frac{\alpha-1}{2}} dx$$

with the normalization $m^{(\frac{\alpha}{2})} = \int_{-1}^1 (1-x^2)^{\frac{\alpha-1}{2}} dx = \pi 2^{1-\alpha} \binom{\alpha-1}{\frac{\alpha}{2}}$. In particular, if f is a

continuous function on $[-1, 1]$, we have the following weak limits:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} \int_{-1}^1 f(x) |p_k^{(n-k)}(x)|^2 d\mu^{(n-k)}(x) = \frac{1}{m^{(\frac{\alpha}{2})}} \int_{-1}^1 f(x) (1-x^2)^{\frac{\alpha-1}{2}} dx, \quad (20)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha+1}{n}} \int_{-1}^1 f(x) K_k^{(n-k)}(x, x) d\mu^{(n-k)}(x) = \frac{1}{m^{(\frac{\alpha}{2})}} \int_{-1}^1 f(x) (1-x^2)^{\frac{\alpha-1}{2}} dx, \quad (21)$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha+1}{n}} \sum_{j=1}^{k+1} f(x_{k+1,j}^{(n-k)}) = \frac{1}{m^{(\frac{\alpha}{2})}} \int_{-1}^1 f(x) (1-x^2)^{\frac{\alpha-1}{2}} dx. \quad (22)$$

Proof. By the description given in Section 5.3, the Cesàro mean (C, α) is a regular Nörlund summation method. We further know that $N_n^{(C, \alpha)} \rightarrow \infty$ as $n \rightarrow \infty$. Then, by the Theorems 5, 6 and 2, we have a limiting measure μ_Σ determined by the numbers Σ_{l_a, l_b} given in (19) such that

$$\text{w-lim}_{n \rightarrow \infty} \nu_{n+1} = \text{w-lim}_{n \rightarrow \infty} \lambda_n = \text{w-lim}_{n \rightarrow \infty} \bar{\mu}_n = \mu_\Sigma.$$

In particular the limits in (20), (21) and (22) are identical and it only remains to determine the explicit form of the limiting measure μ_Σ . To obtain this formula, we consider the slightly modified Gegenbauer summation method given in Section 5.4. For the family of measures $d\mu^{(k, \lambda)}(x) = (1-x^2)^{k+\lambda-\frac{1}{2}} dx$, $k \in \mathbb{N}_0$, $\lambda > 0$, The Gegenbauer summation method $(G, \frac{\alpha}{2})$ gives the same values Σ_{l_a, l_b} as the Cesàro mean (C, α) and, therefore by Theorem 2, also the limiting measures μ_Σ are identical. Using in addition the addition formula (17) related to the ultraspherical polynomials $p_k^{(n-k, \frac{\alpha}{2})}(x)$ we obtain the following identities:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} \int_{-1}^1 f(x) |p_k^{(n-k)}(x)|^2 d\mu^{(n-k)}(x) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \sigma_{n,k}^{(C, \alpha)} \int_{-1}^1 f(x) |p_k^{(n-k, \frac{\alpha}{2})}(x)|^2 d\mu^{(n-k, \frac{\alpha}{2})}(x) \\ &= \lim_{n \rightarrow \infty} \int_{-1}^1 f(x) \sum_{k=0}^n \sigma_{n,k}^{(G, \frac{\alpha}{2})} |p_k^{(n-k, \frac{\alpha}{2})}(x)|^2 d\mu^{(n-k, \frac{\alpha}{2})}(x) \\ &= \lim_{n \rightarrow \infty} \int_{-1}^1 f(x) \frac{1}{m^{(\frac{\alpha}{2})}} (1-x^2)^{\frac{\alpha-1}{2}} dx = \frac{1}{m^{(\frac{\alpha}{2})}} \int_{-1}^1 f(x) (1-x^2)^{\frac{\alpha-1}{2}} dx. \end{aligned}$$

For $\alpha = 1$ we used the respective addition formula (18) of the Legendre case. \square

Remark 4. From the details in the proof of Theorem 8 we see that the statements of Theorem 8 hold also true if we replace the Cesàro summation method (C, α) with the Gegenbauer summation method $(G, \frac{\alpha}{2})$. The limit identity (20) can be regarded as an asymptotic weak addition formula for the class of measures $M^{(C, \alpha)}(\Sigma_{l_a, l_b})$. It therefore makes sense to denote the class $M^{(C, \alpha)}(\Sigma_{l_a, l_b})$ (or also $M^{(G, \frac{\alpha}{2})}(\Sigma_{l_a, l_b})$) given in Theorem 8 as ultraspherical mean Nevai class.

We consider some particular examples and extensions of Theorem 8.

Example 1. • The family $d\mu^{(k)} = (1-x^2)^k(1-x)^{\lambda_1}(1+x)^{\lambda_2}dx$ with fixed parameters $\lambda_1, \lambda_2 > -1$ is in the Nevai class $M^{(C,\alpha)}(\Sigma_{l_a,l_b})$ with the values Σ_{l_a,l_b} given in (19). As for the ultraspherical polynomials, this can be checked directly using the three-term recurrence relation of the Jacobi polynomials (given, for example, in [8, p. 29]). Thus, Theorem 8 yields the weak limits (20), (21) and (22) for this family of measures.

- We consider the family $d\mu^{(k)} = (1-x^2)^{k+\lambda-\frac{1}{2}}e^{(2\theta-\pi)t}|\Gamma(\lambda+k+it)|^2dx$ of Pollaczek measures with $x = \cos \theta$, $t = \frac{ax+b}{\sqrt{1-x^2}}$ and fixed parameters $\lambda > 0$, $a \geq |b|$, see [2, Chapter VI, 5]. This family of Pollaczek measures also satisfies the conditions of Theorem 8. The values Σ_{l_a,l_b} can be calculated directly from the three-term recurrence coefficients of the Pollaczek polynomials in the same way as the calculations were carried out in Section 5.3 for the ultraspherical polynomials. In fact, for $a = b = 0$ the Pollaczek polynomials correspond to the ultraspherical polynomials.

In the special case $\alpha = 1$ we get the following weak limit for the arithmetic mean. It can be regarded as an extended variant of the limit relations derived for co-recursive ultraspherical polynomials in [6].

Corollary 9. *Let the family $\mu^{(k)}$ be in the mean Nevai class $M^{(C,1)}(\Sigma_{l_a,l_b})$ with the values Σ_{l_a,l_b} given by*

$$\Sigma_{l_a,l_b} = \delta_{0,l_a} \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{l_b+2}{2})}{2^{l_b} \Gamma(\frac{l_b+3}{2})}. \quad (23)$$

Then, the limiting measure μ_Σ is the uniform probability measure on $[-1, 1]$ and for every continuous function f on $[-1, 1]$ we have

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n+1} \int_{-1}^1 f(x) |p_k^{(n-k)}(x)|^2 d\mu^{(n-k)}(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=1}^{k+1} \frac{2f(x_{k+1,j}^{(n-k)})}{(n+1)(n+2)} = \frac{1}{2} \int_{-1}^1 f(x) dx. \quad (24)$$

The linear function $T : \mathbb{R} \rightarrow \mathbb{R}$, $Ty = 2by+a$ maps the interval $[-1, 1]$ onto $[a-2b, a+2b]$. Using this linear map, we can transfer Theorem 8 easily to an arbitrary interval $[a-2b, a+2b]$.

Corollary 10. *Suppose that the family $\mu^{(k)}$ is in the mean Nevai class $M^{(C,\alpha)}(\Sigma_{l_a,l_b})$ with $\alpha > 0$ and limits Σ_{l_a,l_b} given by*

$$\Sigma_{l_a,l_b} = a^{l_a} b^{l_b} \left(\frac{\alpha+l_b}{2} \right)^{-1}. \quad (25)$$

Then, the limiting measure μ_Σ is given by

$$d\mu_\Sigma = \frac{1}{2\pi b^\alpha \left(\frac{\alpha-1}{2}\right)} \chi_{[a-2b, a+2b]}(x) (b^2 - (x-a)^2)^{\frac{\alpha-1}{2}} dx.$$

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