VI International Conference on Particle-based Methods – Fundamentals and Applications
PARTICLES 2019
E. Oñate, M. Bischoff, D.R.J. Owen P. Wriggers, , & T. Zohdi (Eds)

EVALUATION OF DYNAMIC EXPLICIT MPM FORMULATIONS FOR UNSATURATED SOILS

F. CECCATO[1], V. GIRARDI[1], A. YERRO[2] AND P. SIMONINI[1]

[1] Department of Civil Environmental and Architectural Engineering (DICEA)
University of Padua
Via Ognissanti 39, 35219, Padua, Italy
e-mail: francesca.ceccato@dicea.unipd.it
[2] Department of Civil and Environmental Engineering
Virginia Polytechnic Institute and State University
Blacksburg, VA, United States

Key words: MPM, Unsaturated soil, infiltration

Abstract. Many applications in geohazards prevention involve large deformations of unsaturated soils, e.g. rainfall induced landslides, embankment collapses due to wetting etc. These phenomena can be investigated with multiphase implementations of the Material Point Method (MPM) able to account for the behaviour of unsaturated soils. This paper compares two formulations: (i) a fully coupled three-phase formulation (3P) in which the governing equations are derived from the momentum balance and the mass balance equations of solid, liquid and gas phase assuming non-zero gas pressure, the primary unknowns are the absolute accelerations of the phases (a_S–a_L–a_G formulation); (ii) a simplified approach that neglects the momentum balance equation of the gas (2P_s). Potentialities and limitations of these approaches are highlighted considering a 1D infiltration problem. Despite the introduced simplifications, the simplified formulation gives reasonably good results in many engineering cases.

1 INTRODUCTION

Many natural hazards involve large deformations of unsaturated soils, e.g. rainfall induced landslides, embankment collapses due to wetting etc. These phenomena can be investigated with multiphase implementations of the Material Point Method (MPM) able to account for the behavior of unsaturated soils.

Recently, Yerro et al. [1] proposed a single-point three-phase (SP-3P) MPM formulation in which the governing equations are derived from the momentum balance and the mass balance equations of solid, liquid and gas phase assuming non-zero gas pressure. This approach takes into account the relative accelerations and relative velocities of the pore fluids and the primary unknowns are the absolute accelerations of the phases (a_S–a_L–a_G formulation). The formulation is lagrangian for the solid phase; the material points (MPs) move with the kinematic of the solid skeleton, but carry the information of all phases (single-point approach).

In contrast, Bandara et al. [2] and Wang et al. [3] used a simplified approach, which neglects the momentum and the mass balance equations of the gas, thus reducing the computational cost. The formulation proposed in Wang et al. is an extension of the two-phase formulation developed in [4] for saturated soils. The governing equations are derived from the dynamic equilibrium of the liquid phase and the mixture and the primary unknowns are the absolute accelerations of the soild and the liquid (a_S–a_L formulation). In Section 2.2 this formulation is introduced showing that can be derived as a simplified version of the one presented in Yerro et al. [1]. In Bandara et al. the relative acceleration of the liquid with respect to the solid skeleton is neglected and the primary unknowns are the absolute acceleration of the solid skeleton and the relative velocity of the fluid.

The simplifying assumptions introduced in [2,3] are reasonable in many engineering cases, but sometimes deviations from the full three-phase formulation can be relevant. This paper highlights the differences between these approaches, considering a 1D infiltration example simulated with an internal version of the software Anura3D (www.anura3D.eu) (Section 3).

2 MPM FORMULATIONS FOR UNSATURATED SOILS

The unsaturated porous media are assumed to be a combination of three different phases (ph): solid (S), liquid (L) and gas (G). The solid phase is made by solid grains that constitutes the solid skeleton of the media; meanwhile the liquid and the gas phases fill the voids. The fluids are a mixture of two components (c): water (w) and dry air (a) (Fig. 1). The mass fraction of a component in a phase are defined as:

$$\omega_{ph}^{c} = \frac{m_{ph}^{c}}{m_{ph}} \tag{1}$$

The total mass of a component is:

$$m^{c} = \sum_{ph} m_{ph}^{c} \tag{2}$$

The volumetric concentration ratio of solid, liquid, and gas are defined respectively as n_S, n_L, and n_G, moreover $n_S + n_L + n_G = 1$, $n = 1 - n_S = n_L + n_G$=porosity. Note that in unsaturated soils, the concentration ratio of porous fluids (n_L and n_G) is controlled by the degree of saturation $S_{ph} = V_{ph}/V_{void}$ ($S_L = 1 - S_G$) as $n_L = nS_L$ and $n_G = nS_G$.

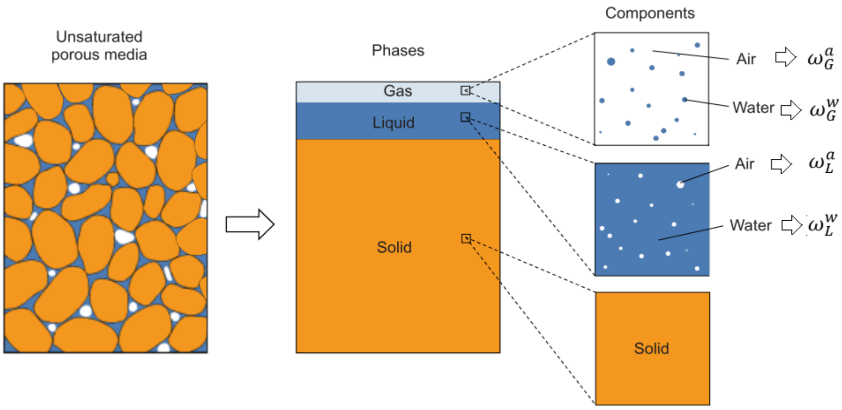


Figure 1 Definition of phases and components in an unsaturated medium (after [1])

2.1 Three-phase (3P)

The 3P-SP formulation [1] considers one set of MPs that represent a partially saturated porous media. Each MP carries information of the three phases interacting in the continuum (i.e., solid skeleton "S", liquid "L, and gas "G"), and while it provides a Lagrangian description of the solid (MPs move according to the solid), the fluid phases filling the voids are represented by means of a Eulerian approach.

Three main governing equations are posed at the nodes of the computational grid: the dynamic linear momentum balances of the gas phase (Eq. 3), liquid phase (Eq. 4), and mixture (Eq. 5), being the accelerations of each phase ($\boldsymbol{a}_G$, $\boldsymbol{a}_L$, and $\boldsymbol{a}_S$) the primary unknowns of the system.

$$\rho_G \boldsymbol{a}_G = \nabla p_G - \boldsymbol{f}_G^d + \rho_G \boldsymbol{b} \tag{3}$$

$$\rho_L \boldsymbol{a}_L = \nabla p_L - \boldsymbol{f}_L^d + \rho_L \boldsymbol{b} \tag{4}$$

$$n_S \rho_S \boldsymbol{a}_S + n_L \rho_L \boldsymbol{a}_L + n_G \rho_G \boldsymbol{a}_G = \nabla \cdot \boldsymbol{\sigma} + \rho_m \boldsymbol{b} \tag{5}$$

where the density of the mixture is $\rho_m = n_S \rho_S + n_L \rho_L + n_G \rho_G$, the liquid and gas pressures are p_L and p_G respectively, and $\boldsymbol{\sigma}$ is the total stress tensor. One assumption is that the liquid and gas seepages are assumed laminar in slow velocity regime; hence, drag forces ($\boldsymbol{f}_G^d$ and $\boldsymbol{f}_L^d$) fulfill Darcy's law.

As usual in MPM, Eq. 3, 4, and 5 are discretized in space by means of the Galerkin method and solved in time with a semi-explicit time discretization scheme.

The model enables mass exchange between fluid phases, in order to account for dissolved air in the liquid and water vapour in the gas, and the mass balance equations are formulated for each component (i.e. solid, water "w", air "a"):

$$\sum_{ph} \left[\frac{\partial}{\partial t} \left(\frac{m_{ph}^c}{V} \right) + \nabla \cdot \boldsymbol{j}_{\boldsymbol{ph}}^{\boldsymbol{c}} \right] = 0 \tag{6}$$

Where V is the volume of the mixture and $\boldsymbol{j}_{\boldsymbol{ph}}^{\boldsymbol{c}}$ is the flux of the component in the phase. Yerro et. al [1] show that the mass balance equations of the solid, water, and air, can be manipulated, leading to the following expressions:

$$\frac{Dn}{Dt} = n \nabla \cdot \boldsymbol{v}_S \tag{7}$$

$$\begin{aligned} n \frac{\partial(\omega_L^w \rho_L S_L + \omega_G^w \rho_G S_G)}{\partial p_L} \dot{p}_L + n \frac{\partial(\omega_L^w \rho_L S_L + \omega_G^w \rho_G S_G)}{\partial p_G} \dot{p}_G \\ = -\boldsymbol{\nabla} \cdot [\omega_G^w n \rho_G S_G (\boldsymbol{v_G} - \boldsymbol{v_S})] - \boldsymbol{\nabla} \cdot [\omega_L^w n \rho_L S_L (\boldsymbol{v_L} - \boldsymbol{v_S})] \\ - (\omega_G^w \rho_G S_G + \omega_L^w \rho_L S_L) \boldsymbol{\nabla} \cdot \boldsymbol{v_S} - \boldsymbol{\nabla} \cdot \boldsymbol{i}_{\boldsymbol{G}}^{\boldsymbol{w}} \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial(\omega_L^a \rho_L S_L + \omega_G^a \rho_G S_G)}{\partial p_L} \dot{p}_L + n \frac{\partial(\omega_L^a \rho_L S_L + \omega_G^a \rho_G S_G)}{\partial p_G} \dot{p}_G \\ = -\boldsymbol{\nabla} \cdot [\omega_G^a n \rho_G S_G (\boldsymbol{v_G} - \boldsymbol{v_S})] - \boldsymbol{\nabla} \cdot [\omega_L^a n \rho_L S_L (\boldsymbol{v_L} - \boldsymbol{v_S})] \\ - (\omega_G^a \rho_G S_G + \omega_L^a \rho_L S_L) \boldsymbol{\nabla} \cdot \boldsymbol{v_S} - \boldsymbol{\nabla} \cdot \boldsymbol{i}_{\boldsymbol{L}}^{\boldsymbol{a}} \end{aligned} \tag{9}$$

Where $\boldsymbol{i}_{\boldsymbol{ph}}^{\boldsymbol{c}}$ is the diffusive flux, modelled by the Fick's law. Note that the previous expressions can be simplified when the liquid and gas phases are considered as simply water ($\omega_L^a = 0$; $\boldsymbol{i}_{\boldsymbol{G}}^{\boldsymbol{w}} = \boldsymbol{0}$) and dry air ($\omega_G^w = 0$; $\boldsymbol{i}_{\boldsymbol{L}}^{\boldsymbol{a}} = \boldsymbol{0}$); the examples in Section 3 are solved under these hypothesis..

The mass balance equations (Eq. 7, 8, 9) are posed at the MPs and solved in terms of changes in porosity, liquid pressure and gas pressure. In this formulation, the solid mass remains constant thorough the calculation, and the mass balance of the solid phase is automatically fulfilled. However, the conservation of the fluid masses is totally controlled by the accuracy in which the mass balance equations are solved.

The constitutive stresses controlling the unsaturated soil behavior, net stress ($\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - p_G \boldsymbol{I}$) and suction ($s = p_G - p_L$), are updated at the MPs by considering a constitutive equation. Finally, the degree of saturation and the hydraulic permeability are updated taking into account the corresponding hydraulic constitutive equations, i.e., the soil-water retention curve, and the Hillel expression, respectively as introduced in Section 2.3.

2.2 Two-phase with suction (2P_s)

The governing equations of the two-phase formulation with suction effect (2P_s) are derived in this section highlighting the additional hypothesis introduced with respect to the 3P formulation explained in Section 2.1.

Assuming that:

1) no air is dissolved in liquid ($\omega_L^a = 0$, $\omega_G^a = 1$) and no water vapour is present in the gas phase ($\omega_G^w = 0$, $\omega_L^w = 1$),
2) no diffusion fluxes of air in the liquid phase ($\boldsymbol{i}_{\boldsymbol{L}}^{\boldsymbol{a}} \approx 0$) and water in the gas ($\boldsymbol{i}_{\boldsymbol{G}}^{\boldsymbol{w}} \approx 0$)
3) gas density is negligible compared to the other phases ($\rho_G \approx 0$),
4) the gas pressure is constant and equal to 0,
5) the gradient of the product $n\rho_L S_L$ is negligible, i.e. $\boldsymbol{\nabla}(n\rho_L S_L) \approx \boldsymbol{0}$

the momentum balance equation and the mass balance equation of the gas can be neglected, while the momentum balance of the mixture reduces to Eq. 10.

$$n_S \rho_S \boldsymbol{a}_S + n_L \rho_L \boldsymbol{a}_L = \nabla \cdot \boldsymbol{\sigma} + \rho_m \boldsymbol{b} \tag{10}$$

where the mixture density is $\rho_m = n_S \rho_S + n_L \rho_L$.

The mass balance equation of the liquid is rewritten as:

$$n \frac{D(\rho_L S_L)}{D p_L} \dot{p}_L = -(1-n)(\rho_L S_L) \nabla \cdot \boldsymbol{v_S} - n(\rho_L S_L) \nabla \cdot \boldsymbol{v_L} \tag{11}$$

The left hand side of Eq. 11 can be reformulated by

1) introducing the constitutive equation for the water: $\dot{\rho}_L = -\rho_L / K_L \, \dot{p}_L$, K_L=liquid bulk modulus
2) considering that S_L is a function of p_L which leads to Eq. 12.

$$\left[-\frac{S_L n}{K_L} + n \frac{\partial S_L}{\partial p_L} \right] \dot{p}_L = n S_L \nabla \cdot \boldsymbol{v_L} + (1-n) S_L \nabla \cdot \boldsymbol{v_S} \tag{12}$$

The derivative of the degree of saturation with respect to suction is given by the soil-water retention curve (Section 2.3).

Note that, due to the introduced simplifications, the difference between the 3P and the 2P_s formulation increases when

1) gas density increases
2) suction increases
3) gas pressure is not zero (i.e. atmospheric pressure)
4) degree of saturation decreases
5) derivative of the degree of saturation increases

The effect of these simplifications will be more clear with the infiltration examples reported in Section 3.

2.3 Hydraulic constitutive equations

The well know Van Genuchten soil-water retention curve (SWRC) [5] can be used with assumed constant parameters λ and p_0.

$$S_L = S_{min} + \left[1 + \left(\frac{s}{p_0}\right)^{\frac{1}{1-\lambda}}\right]^{-\lambda} (S_{max} - S_{min}) \tag{13}$$

Alternatively, an approximated linear law (Eq.14) can be used to compute S_L.

$$S_L = 1 - a_v \cdot s \tag{14}$$

Furthermore, since the hydraulic conductivity parameter is susceptible of variable water content in soil unsaturated zones, the seepage process should be modelled with permeability laws function of S_L. An example of this kind of law is the Hillel expression [6] as function of the saturated hydraulic conductivity and an exponent r, which assumes values between 2 and 4.

$$k_L = k_{sat} \cdot {S_L}^r \tag{15}$$

3 NUMERICAL EXAMPLES

In order to compare the formulations presented in the previous section a one-dimensional infiltration problem, similar to the one presented in [7], is considered here. A 1m column has an initial suction of s_0=500kPa; for t>0, zero suction is applied at the head of the column and the suction begins to decrease with time.

The equation that represents the movement of water in unsaturated soils is the Richards equation. However, because of the nonlinearities of soil hydraulic parameters (for instance, permeability depends on degree of saturation, and degree of saturation depends on fluid pressures), it is very difficult to obtain an analytical solution to describe the unsaturated flow. In order to derive an analytical solution the following assumptions are introduced [8]:

- vertical liquid flow,
- deformability of the solid skeleton and the solid grains are neglected,
- neither water vapour nor dissolved air are considered in the gas and liquid phases respectively,
- validity of the Darcy's law and constant permeability,
- barotropic behaviour of the liquid,
- linearized water retention curve, i.e. Eq. 14

The mathematical expression that describes the one-dimensional vertical water flow within an unsaturated soil can be derived from the mass balance equation of the liquid (Eq. 11).

Under the aforementioned assumptions it reduces to Eq. 16

$$\left[\frac{nS_L}{K_L} + na_v\right]\frac{\partial p_L}{\partial t} = \frac{k}{\rho_L g}\frac{\partial^2 p_L}{\partial y^2} \quad (16)$$

Which can be written as

$$\frac{\partial p_L}{\partial t} = C_i \frac{\partial^2 p_L}{\partial y^2} \quad (17)$$

This expression is the diffusion equation, where y is the infiltration direction and C_i corresponds to

$$C_i = \frac{k}{n\rho_L g\left(\frac{S_L}{K_L} + a_s\right)} \quad (18)$$

Assuming that the variation of C_i is small in the considered process, a dimensionless time T can be defined as function of C_i and the column height h as

$$T = \frac{C_i t}{h^2} \quad (19)$$

The analytical solution that comes out applying the boundary conditions previously described, is the following:

$$\frac{s}{s_0} = \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{2j-1}\cos\left[(2j-1)\frac{\pi}{2}\frac{y}{h}\right]e^{-(2j-1)^2\frac{\pi^2}{4}\frac{C_i t}{h^2}} \quad (20)$$

being s suction ($s = p_G - p_L$), s_0 initial suction, h the infiltration length, and t time.

Note that the previous equation is equivalent to the one-dimensional consolidation problem in saturated media, the well-known Terzaghi's solution.

Figure 2 compares the numerical results with the analytical solution for different values of a_v assuming s_0=500kPa. Material parameters are summarized in Table 1. In this conditions the degree of saturation varies between 0.5 and 1 for the case a_v=1.0e-3 kPa^{-1} and between 0.995 and 1 for the case a_v=1.0e-5 kPa^{-1}. When S_L is close to 1 and the derivative $\partial S_L/\partial s$ is small, the additional terms comparing in the 3P formulation are small, thus the two formulations give the same results. When reducing the initial suction to 5kPa in case a_v=1e-3kPa^{-1}, a good agreement between the two formulations is recovered as the suction varies only between 0.995 and 1.

Table 1. Material Parameters of the infiltration problem (* parameters not used in 2P_s)

Solid density	2700 kg/m^3	Gas bulk modulus*	10 kPa
Liquid density	1000 kg/m^3	Intrinsic permeability liquid	$5\cdot10^{-11}$m^2/s
Gas density*	10 kg/m^3	Intrinsic permeability gas*	$5\cdot10^{-11}$m^2/s
Porosity	0.3	Liquid viscosity	10^{-6} kPa s
Liquid bulk modulus	80000 kPa	Gas viscosity*	$2\cdot10^{-8}$ kPa s

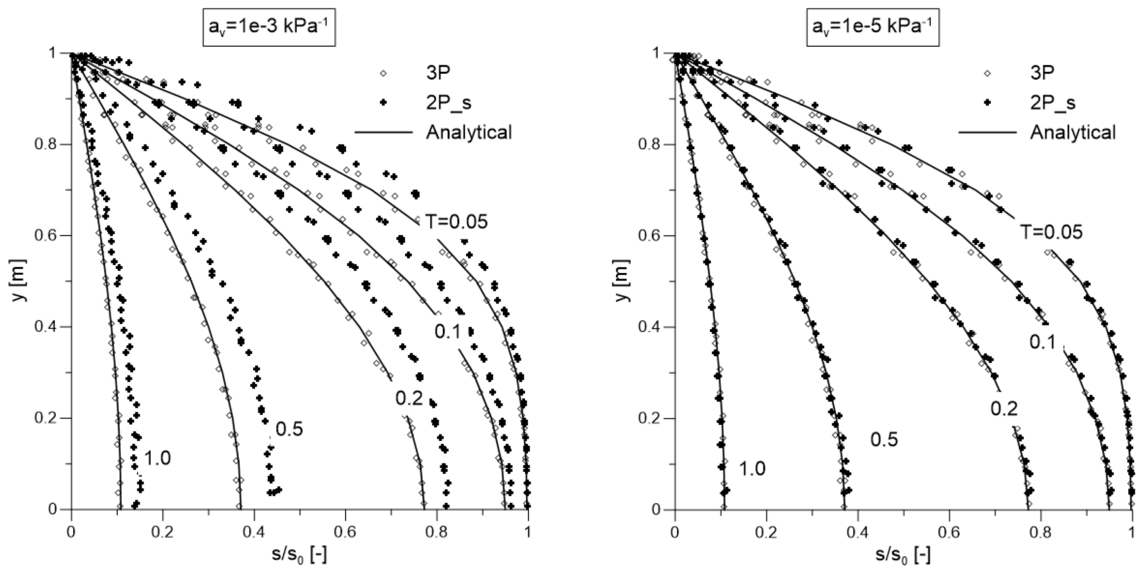


Figure 2: Numerical results of normalized suction evolution along depth; comparison between 2P_s and 3P for a_v=1e-3kPa^{-1} and a_v=1e-5kPa^{-1} in case s_0=500kPa

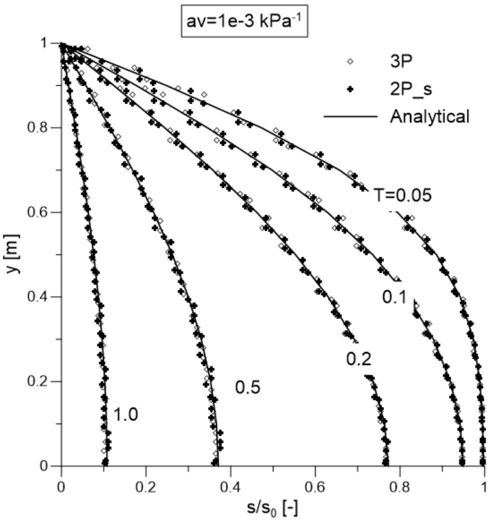


Figure 3: Numerical results of suction evolution along depth y, comparison between 2P_s and 3P for a_v=1e-3kPa^{-1} in case s_0=5kPa

The simulations are now repeated using the Van Genuchten water retention curve and the Hillel law. Two sets of parameters have been used (Tab. 2, Fig. 4), which are close to typical values for clay (soil A) and sand (soil B).

Assuming an initial suction corresponding to an initial degree of saturation of 0.85 the 3P and the 2P_s gives very similar results (Fig. 5). The normalized time T in Figure 5 is computed for a value of $C_{i,ref}$ calculated with the values of k, S_L, and $a_v = \partial S_L / \partial p_L$ at the beginning of the simulation.

These conditions are probably the most frequent in many geohazard problems such as dam and levee stability or shallow landslides.

When running these simulations in a common laptop computer, the 2P_s is 6 times faster than the 3P, thus the use of the 2P_s can be more convenient for problems when S_L is relatively close to 1 and varies in a narrow range.

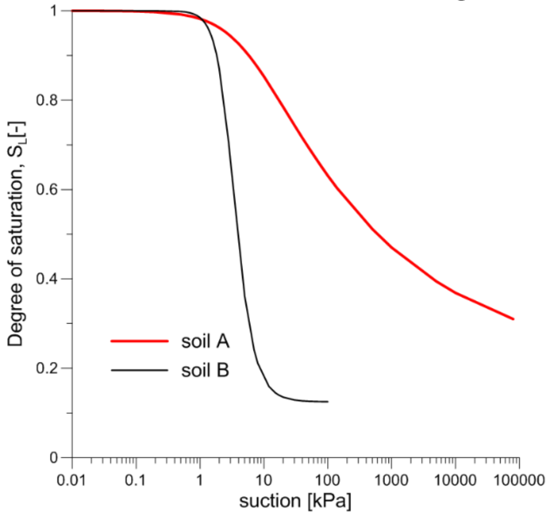
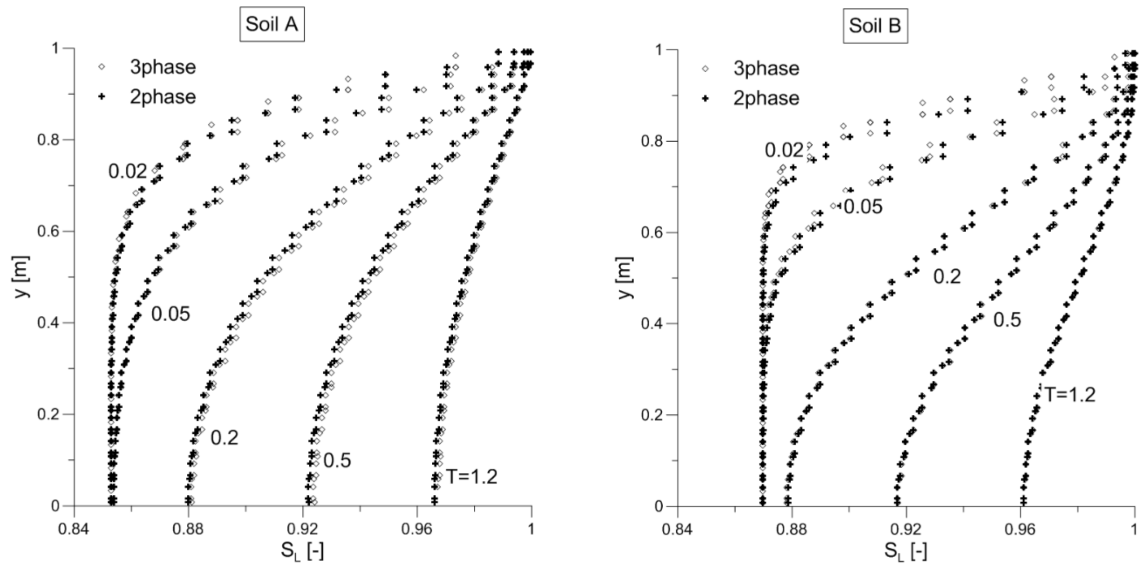


Figure 4: SWRC for parameters in Table 2

Table 2: Parameters of Van Genuchten SWRC

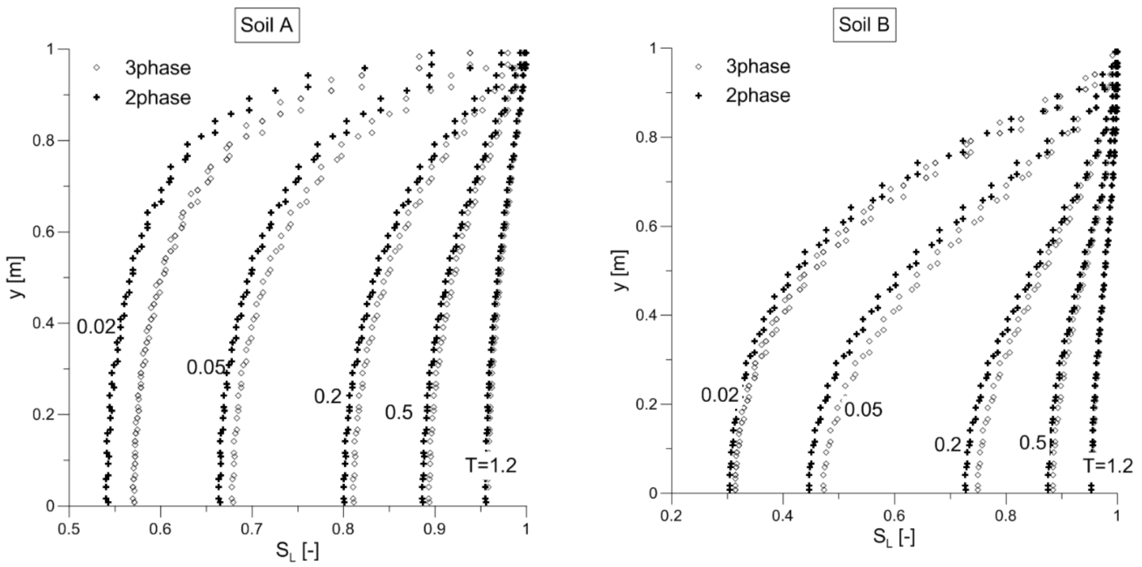
	Soil A	Soil B
S_{min}	0.2	0.125
S_{max}	1	1
p_0	5	3
λ	0.17	0.7



(a) Soil A, s_0=10kPa (b) Soil B, s_0=2kPa

Figure 5: Comparison between MPM results using different hydraulic conductivity curves: evolution of the degree of saturation with depth for soil A (a) and B (b).

Increasing the suction to values corresponding to an effective degree of saturation $S_e = \frac{S_L - S_{min}}{S_{max} - S_{min}} = 0.13$ the results obtained by the two formulations are slightly different, especially for the lower values of S_L.



(a) Soil A, s_0=80000kPa

(b) Soil B, s_0=7kPa

Figure 6 Comparison between MPM results using different hydraulic conductivity curves: evolution of the degree of saturation with depth for soil A (a) and B (b).

4 CONCLUSIONS

This paper briefly illustrates two mathematical formulations for unsaturated soils recently implemented in the MPM code Anura3D, namely the full three-phase formulation and the simplified two-phase formulation with suction effect. The results obtained with 3P and 2P_s are compared for different material parameters in a one-dimensional infiltration case showing that the differences increases when

1) suction increases
2) degree of saturation decreases
3) derivative of the degree of saturation increases, i.e. the SWRC is relatively steep like in Soil B of Fig. 4

In many real cases under analysis for geohazard assessment the differences between 3P and 2P_s are negligible, thus the simplified formulation can be used to reduce significantly the computational cost.

5 REFERENCES

[1] Yerro A, Alonso EE, Pinyol NM. The material point method for unsaturated soils. *Geotechnique*, **65**, 201–17 , 2015.

[2] Bandara SS, Ferrari A, Laloui L. Modelling landslides in unsaturated slopes subjected to rainfall infiltration using material point method. *Int. J. Numer. Anal. Methods Geomech. (published online)*, , 2016.

[3] Wang B, Vardon PJ, Hicks MA. Rainfall-induced slope collapse with coupled material point method. *Eng. Geol.*, **239**, 1–12 , 2018.

[4] Jassim I, Stolle D, Vermeer P. Two-phase dynamic analysis by material point method. *Int. J. Numer. Anal. Methods Geomech. Anal. methods Geomech.*, **37**, 2502–22 , 2013.

[5] Van Genuchten M. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil Sci. Soc. Am. J.*, **44**, 892–8 , 1980.

[6] Hillel D. *Soil and water – physical principles and processes*. Academic press, London (UK), (1971).

[7] Yerro A, Alonso E, Pinyol N. Modelling large deformation problems in unsaturated soils, E-UNSAT 2016, 2016, pp.1–6.

[8] Alonso EE, Lloret A. Evolution in time of the reliability of slopes in partially saturated soils, International conference on applications of statistics and probability in soil and structural engineering, 1983, pp.1363–76.