

Traversing the Schrödinger Bridge strait: Robert Fortet's marvelous proof redux

Montacer Essid and Michele Pavon

May 9, 2018

Abstract

In the early 1930's, Erwin Schrödinger, motivated by his quest for a more classical formulation of quantum mechanics, posed a large deviation problem for a cloud of independent Brownian particles. He showed that the solution to the problem could be obtained through a system of two linear equations with nonlinear coupling at the boundary (*Schrödinger system*). Existence and uniqueness for such a system, which represents a sort of bottleneck for the problem, was first established by R. Fortet in 1938/40 under rather general assumptions by proving convergence of an ingenious but complex approximation method. It is the first proof of what are nowadays called Sinkhorn-type algorithms in the much more challenging continuous case. Schrödinger bridges are also an early example of the maximum entropy approach and have been more recently recognized as a regularization of the important Optimal Mass Transport problem.

Unfortunately, Fortet's contribution is by and large ignored in contemporary literature. This is likely due to the complexity of his approach coupled with an idiosyncratic exposition style and to missing details and steps in the proofs. Nevertheless, Fortet's approach maintains its importance to this day as it provides the only existing algorithmic proof under rather mild assumptions. It can be adapted, in principle, to other relevant problems such as the regularized Wasserstein barycenter problem. It is the purpose of this paper to remedy this situation by rewriting the bulk of his paper with all the missing passages and in a transparent fashion so as to make it fully available to the scientific community. We consider the problem in \mathbb{R}^d rather than \mathbb{R} and use as much as possible his notation to facilitate comparison.

1 Introduction

In 1931/21, Erwin Schrödinger showed that the solution to a hot gas experiment (large deviations problem) could be reduced to establishing existence and uniqueness of a pair of positive functions $(\varphi, \hat{\varphi})$ satisfying what was later named the Schrödinger system, see (14) below. This is a system of two linear PDE's with nonlinear coupling at the boundary. Besides Schrödinger's original motivation, this problem features two more: The first is a *maximum entropy principle in statistical inference*, namely choosing a posterior distribution so as to make the fewest number of assumptions about what is beyond the available information. This inference method has been noticeably developed over the years by Jaynes, Burg, Dempster and Csiszár [27, 28, 4, 5, 20, 13, 14, 15]. The second, more recent, is regularization of the Optimal Mass Transport problem [31, 32, 33, 30, 29, 9] providing an effective computational approach to the latter, see e.g. [16, 2, 11, 10].

The first proof of existence and uniqueness for the Schrödinger system was provided in 1938/40 by the French analyst Robert Fortet [24, 25]. Subsequent significant contributions are due to Beurlin (1960), Jamison (1975) and Föllmer (1988). Fortet's proof is algorithmic, being based on a complex iterative scheme. It represents also the first proof, in the much more challenging continuous setting, of convergence of a procedure (called *iterative proportional fitting* (IPF)) proposed by Deming and Stephan [19] (1940) for contingency tables. In the latter discrete setting, the first convergence proof

was provided in a special case some twenty five years after Fortet and Deming-Stephan by R. Sinkhorn [39] who was unaware of their work. These iterative schemes are nowadays often called *Sinkhorn-type algorithms* or *Iterative Bregman projections*, cf. e.g. [16, 2, 12]. Unfortunately, in spite of its importance, Fortet’s contribution has by and large sunk into oblivion. This is arguably due to the complexity of his approach, to the unconventional organization of the paper and to a number of gaps in his arguments. Nonetheless, to this day, Fortet’s existence result is the central one as it is based on the convergence of an algorithm under rather weak assumptions and does not require a kernel bounded away from zero. Other proofs in the continuous setting [3, 26, 23], [30, Section 2] are non constructive except [10]. The latter proof, however, assumes compactly supported marginal distributions. Finally, Fortet’s approach may, in principle, be tailored to attack other significant problems such as the regularized Wasserstein barycenter problem, see e.g. [2] for the discrete version.

The purpose of this paper is to make his fundamental contribution fully available to the scientific community. To achieve this, we review, elaborate upon and generalize to \mathbb{R}^d Fortet’s proof of existence and uniqueness for the Schrödinger system. We systematically fill in all the missing steps and provide thorough explanations of the rationale behind different articulations of his approach, but keep as much as possible his original notation to make comparison simpler. Nevertheless, we have chosen to reorganize the paper to improve its readability since, for instance, Fortet often presents the proof before the statement of the result. Finally, our original work, completing a sketchy proof, or proving Fortet’s claims or making explicit what is implicit in [25], appears in a sequence of Propositions, Observations and one Claim (all not present in [25]) to make it easily identifiable.

The paper is organized as follows: In the remains of this section, we provide a concise introduction to the Schrödinger bridge problem which is not present in [25]. We include, for the benefit of the reader, Schrödinger’s original motivation, elements of the transformation of the large deviation problem into a maximum entropy problem and a derivation of the Schrödinger system. Section 2 features Fortet’s statement of the problem and his basic assumptions. Section 3 is devoted to his first existence theorem. In Section 4, a special case of his second existence theorem is stated and his uniqueness result is proved.

1.1 The hot gas Gedankenexperiment

In 1931-32, Erwin Schrödinger considered the following thought experiment [37, 38]: A cloud of N independent Brownian particles is evolving in time. Suppose that at $t = 0$ the empirical distribution is $\rho_0(x)dx$ and at $t = 1$ it is $\rho_1(x)dx$. If N is large, say of the order of Avogadro’s number, we expect, by the law of large numbers,

$$\rho_1(y) \approx \int_{\mathbb{R}^3} p(0, x, 1, y) \rho_0(x) dx,$$

where

$$p(s, y, t, x) = [2\pi(t - s)]^{-\frac{d}{2}} \exp \left[-\frac{|x - y|^2}{2(t - s)} \right], \quad s < t \quad (1)$$

is the transition density of the Wiener process. If this is not the case, the particles have been transported in an unlikely way. But of the many unlikely ways in which this could have happened, which one is the most likely? In modern probabilistic terms, this is a problem of *large deviations of the empirical distribution* as observed by Föllmer [23]. The area of large deviations is concerned with the probabilities of very rare events. Thanks to Sanov’s theorem [36], Schrödinger’s problem can be turned into a maximum entropy problem for distributions on trajectories. Let $\Omega = C([0, 1]; \mathbb{R}^d)$ be the space of \mathbb{R}^d valued continuous functions and let X^1, X^2, \dots be i.i.d. Brownian evolutions on $[0, 1]$ with values in \mathbb{R}^d (X_i is distributed according to the Wiener measure W on $C([0, 1]; \mathbb{R}^d)$). The *empirical distribution* μ_N associated to X^1, X^2, \dots, X^N is defined by

$$\mu_N(\omega) := \frac{1}{N} \sum_{i=1}^N \delta_{X^i}(\omega), \quad \omega \in \Omega. \quad (2)$$

Notice that (2) defines a map from Ω to the space \mathcal{D} of probability distributions on $C([0, 1]; \mathbb{R}^d)$. Hence, if E is a subset of \mathcal{D} , it makes sense to consider $\mathbb{P}(\omega : \mu_N(\omega) \in E)$. By the ergodic theorem, see e.g. [21, Theorem A.9.3.], the distributions μ_N converge weakly¹ to W as N tends to infinity. Hence, if $W \notin E$, we must have $\mathbb{P}(\omega : \mu_N(\omega) \in E) \searrow 0$. Large deviation theory, see e.g. [21, ?], provides us with a much finer result: Such a decay is *exponential* and the exponent may be characterized solving a *maximum entropy problem*. Indeed, in our setting, let $E = \mathcal{D}(\rho_0, \rho_1)$, namely distributions on $C([0, 1]; \mathbb{R}^d)$ having marginal densities ρ_0 and ρ_1 at times $t = 0$ and $t = 1$, respectively. Then, Sanov's theorem, roughly asserts that if the "prior" W does not have the required marginals, the probability of observing an empirical distribution μ_N in $\mathcal{D}(\rho_0, \rho_1)$ decays according to

$$\mathbb{P} \left(\frac{1}{N} \sum_{i=1}^N \delta_{X^i} \in \mathcal{D}(\rho_0, \rho_1) \right) \sim \exp [-N \inf \{ \mathbb{D}(P \| W); P \in \mathcal{D}(\rho_0, \rho_1) \}],$$

where

$$\mathbb{D}(P \| W) = \begin{cases} \mathbb{E}_P \left(\log \frac{dP}{dW} \right), & \text{if } P \ll W \\ +\infty & \text{otherwise} \end{cases}.$$

is the relative entropy functional or Kullback-Leibler divergence between P and W . Thus, the most likely random evolution between two given marginals is the solution of the Schrödinger Bridge Problem:

Problem 1.

$$\text{Minimize } \mathbb{D}(P \| W) \text{ over } P \in \mathcal{D}(\rho_0, \rho_1). \quad (3)$$

The optimal solution is called the *Schrödinger bridge* between ρ_0 and ρ_1 over W , and its marginal flow (ρ_t) is the *entropic interpolation*.

Let $P \in \mathcal{D}$ be a finite-energy diffusion, namely under P the canonical coordinate process $X_t(\omega) = \omega(t)$ has a (forward) Ito differential

$$dX_t = \beta_t dt + dW_t \quad (4)$$

where β_t is adapted to $\{\mathcal{F}_t^-\}$ (\mathcal{F}_t^- is the σ -algebra of events observable up to time t) and

$$\mathbb{E}_P \left[\int_0^1 \|\beta_t\|^2 dt \right] < \infty. \quad (5)$$

Let

$$P_x^y = P[\cdot | X_0 = x, X_1 = y], \quad W_x^y = W[\cdot | X_0 = x, X_1 = y]$$

be the disintegrations of P and W with respect to the initial and final positions. Let also π and π^W be the joint initial-final time distributions under P and W , respectively. Then, we have the following decomposition of the relative entropy [23]

$$\begin{aligned} \mathbb{D}(P \| W) &= E_P \left[\log \frac{dP}{dW} \right] = \\ &= \iint \left[\log \frac{\pi(x, y)}{\pi^W(x, y)} \right] \pi(x, y) dx dy + \iint \left(\log \frac{dP_x^y}{dW_x^y} \right) dP_x^y \pi(x, y) dx dy. \end{aligned} \quad (6)$$

Both terms are nonnegative. We can make the second zero by choosing $P_x^y = W_x^y$. Thus, the problem reduces to the static one

¹Let \mathcal{V} be a metric space and $\mathcal{D}(\mathcal{V})$ be the set of probability measures defined on $\mathcal{B}(\mathcal{V})$, the Borel σ -field of \mathcal{V} . We say that a sequence $\{P_N\}$ of elements of $\mathcal{D}(\mathcal{V})$ converges weakly to $P \in \mathcal{D}(\mathcal{V})$, and write $P_N \rightrightarrows P$, if $\int_{\mathcal{V}} f dP_N \rightarrow \int_{\mathcal{V}} f dP$ for every bounded, continuous function f on \mathcal{V} .

Problem 2. Minimize over densities π on $\mathbb{R}^d \times \mathbb{R}^d$ the index

$$\mathbb{D}(\pi \|\pi^W) = \iint \left[\log \frac{\pi(x, y)}{\pi^W(x, y)} \right] \pi(x, y) dx dy \quad (7)$$

subject to the (linear) constraints

$$\int \pi(x, y) dy = \rho_0(x), \quad \int \pi(x, y) dx = \rho_1(y). \quad (8)$$

If π^* solves the above problem, then

$$P^*(\cdot) = \int_{\mathbb{R}^d \times \mathbb{R}^d} W_{xy}(\cdot) \pi^*(x, y) dx dy,$$

solves Problem 1.

Consider now the case when the prior is W_ϵ , namely Wiener measure with variance ϵ , so that

$$p(0, x, 1, y) = [2\pi\epsilon]^{-\frac{d}{2}} \exp \left[-\frac{|x-y|^2}{2\epsilon} \right].$$

Using $\pi^{W_\epsilon}(x, y) = \rho_0^W(x) p(0, x; 1, y)$ and the fact that the quantity

$$\iint [\log \rho_0^W(x)] \pi(x, y) dx dy = \int [\log \rho_0^W(x)] \rho_0(x) dx$$

is independent of π satisfying (8), we get

$$\begin{aligned} \mathbb{D}(\pi \|\pi^{W_\epsilon}) &= - \iint [\log \pi^W(x, y)] \pi(x, y) dx dy + \iint [\log \pi(x, y)] \pi(x, y) dx dy \\ &= \iint \frac{|x-y|^2}{2\epsilon} \pi(x, y) dx dy - \mathcal{S}(\pi) + C, \end{aligned} \quad (9)$$

where \mathcal{S} is the differential entropy and C does not depend on π . Thus, Problem 2 of minimizing $\mathbb{D}(\pi \|\pi^{W_\epsilon})$ over $\Pi(\rho_0, \rho_1)$, namely the ‘‘couplings’’ of ρ_0 and ρ_1 ² is equivalent to

$$\inf_{\pi \in \Pi(\rho_0, \rho_1)} \int \frac{|x-y|^2}{2} \pi(x, y) dx dy + \epsilon \int \pi(x, y) \log \pi(x, y) dx dy, \quad (10)$$

namely a regularization of Optimal Mass Transport (OMT) [40] with quadratic cost function obtained by subtracting a term proportional to the entropy.

1.2 Derivation of the Schrödinger system

We outline the derivation of the Schrödinger system for the sake of continuity in exposition. Two good surveys on Schrödinger Bridges are [41, 30]. The Lagrangian function for Problem 2 has the form

$$\begin{aligned} \mathcal{L}(\pi; \lambda, \mu) &= \iint \left[\log \frac{\pi(x, y)}{\pi^W(x, y)} \right] \pi(x, y) dx dy \\ &+ \int \lambda(x) \left[\int \pi(x, y) dy - \rho_0(x) \right] + \int \mu(y) \left[\int \pi(x, y) dx - \rho_1(y) \right]. \end{aligned}$$

Setting the first variation with respect to π equal to zero, we get the (sufficient) optimality condition

$$1 + \log \pi^*(x, y) - \log p(0, x, 1, y) - \log \rho_0^W(x) + \lambda(x) + \mu(y) = 0,$$

²Probability densities on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals ρ_0 and ρ_1 .

where we have used the expression $\pi^W(x, y) = \rho_0^W(x)p(0, x, 1, y)$ with p as in (1). We get

$$\begin{aligned}\frac{\pi^*(x, y)}{p(0, x, 1, y)} &= \exp[\log \rho_0^W(x) - 1 - \lambda(x) - \mu(y)] \\ &= \exp[\log \rho_0^W(x) - 1 - \lambda(x)] \exp[-\mu(y)].\end{aligned}$$

Hence, the ratio $\pi^*(x, y)/p(0, x, 1, y)$ factors into a function of x times a function of y . Denoting these by $\hat{\varphi}(x)$ and $\varphi(y)$, respectively, we can then write the optimal $\pi^*(\cdot, \cdot)$ in the form

$$\pi^*(x, y) = \hat{\varphi}(x)p(0, x, 1, y)\varphi(y), \quad (11)$$

where φ and $\hat{\varphi}$ must satisfy

$$\hat{\varphi}(x) \int p(0, x, 1, y)\varphi(y)dy = \rho_0(x), \quad (12)$$

$$\varphi(y) \int p(0, x, 1, y)\hat{\varphi}(x)dx = \rho_1(y). \quad (13)$$

Let us define $\hat{\varphi}(0, x) = \hat{\varphi}(x)$, $\varphi(1, y) = \varphi(y)$ and

$$\hat{\varphi}(1, y) := \int p(0, x, 1, y)\hat{\varphi}(0, x)dx, \quad \varphi(0, x) := \int p(0, x, 1, y)\varphi(1, y).$$

Then, (12)-(13) can be replaced by the system

$$\hat{\varphi}(1, y) = \int p(0, x, 1, y)\hat{\varphi}(0, x)dx, \quad (14a)$$

$$\varphi(0, x) = \int p(0, x, 1, y)\varphi(1, y)dy, \quad (14b)$$

$$\varphi(0, x) \cdot \hat{\varphi}(0, x) = \rho_0(x), \quad (14c)$$

$$\varphi(1, y) \cdot \hat{\varphi}(1, y) = \rho_1(y). \quad (14d)$$

The arguments leading to (14) apply to the much more general case where the prior measure on path space is not Wiener measure but any finite energy diffusion measure \bar{P} [23]. In that case, $p(0, x, 1, y)$ is the transition density of \bar{P} . As already said, the question of existence and uniqueness of positive functions $\hat{\varphi}$, φ satisfying (14), left open by Schrödinger, is a highly nontrivial one and was settled in various degrees of generality by Fortet, Beurlin, Jamison and Föllmer [25, 3, 26, 23]. The pair $(\varphi, \hat{\varphi})$ is unique up to multiplication of φ by a positive constant c and division of $\hat{\varphi}$ by the same constant. A proof based on convergence of an iterative scheme in Hilbert's projective metric (convergence of rays in a suitable cone) was provided in [10] in the case when both marginals have compact support.

At each time t , the marginal ρ_t factorizes as

$$\rho_t(x) = \varphi(t, x) \cdot \hat{\varphi}(t, x). \quad (15)$$

Schrödinger saw “Merkwürdige Analogien zur Quantenmechanik, die mir sehr des Hindenkens wert erscheinen”³ Indeed (15) resembles Born's relation

$$\rho_t(x) = \psi(t, x) \cdot \bar{\psi}(t, x)$$

with ψ and $\bar{\psi}$ satisfying two adjoint equations like φ and $\hat{\varphi}$. Moreover, the solution of Problem 1 exhibits the following remarkable *reversibility property*: Swapping the two marginal densities ρ_0 and ρ_1 , the new solution is simply the time reversal of the previous one, cf. the title “On the reversal of natural laws” of [37].

³Remarkable analogies to quantum mechanics which appear to me very worth of reflection.

We mention, for the benefit of the reader, that there exist also dynamic versions of the problem such as stochastic control formulations originating with [17, 18, 35]. These formulations are particularly relevant in applications where the prior distribution on paths is associated to the uncontrolled (*free*) evolution of a dynamical system, see e.g [6, 7, 8] and in image morphing/interpolation [10, Subsection 5.3]. The stochastic control problems leads directly to a fluid dynamic formulation, see [30, 9]. The latter can be viewed as a regularization of the Benamou-Brenier dynamic formulation of Optimal Mass Transport [1].

2 Fortet's statement of the problem

Let $d \in \mathbb{N}^*$. Define by $\mathcal{B}(\mathcal{J})$ the Borel σ -algebra of $\mathcal{J} \subseteq \mathbb{R}^d$, and m the Lebesgue measure on \mathcal{J} . Almost everywhere (a.e.) will always be intended with respect to m . In this paper, measurable functions with respects to the Borel σ -algebra on their corresponding interval of definition will simply be referred to as measurable. Moreover, all properties concerning measures of sets will (tacitly) refer to their Lebesgue measure. From here on, we shall try to adhere to Fortet's notation as much as possible. In particular, with respect to the notation employed in Section 1, the following changes are made: The two marginal densities $\rho_0(x)$ and $\rho_1(y)$ are replaced by $\omega_1(x)$ and $\omega_2(y)$, respectively. The kernel (transition density) $p(0, x, 1, y)$ is replaced by $g(x, y)$. Finally, the pair $(\hat{\varphi}(x), \varphi(y))$ is replaced by the pair $(\varphi(x), \psi(y))$.

Let $\mathcal{J}^1, \mathcal{J}^2 \subseteq \mathbb{R}^d$ be closed sets, but not necessarily bounded.

Let $\omega_1 : \mathcal{J}^1 \rightarrow \mathbb{R}, \omega_2 : \mathcal{J}^2 \rightarrow \mathbb{R}$ and $g : \mathcal{J}^1 \times \mathcal{J}^2 \rightarrow \mathbb{R}$ satisfying the **assumptions (H)**:

(H.i) $g(x, y) \geq 0, \quad \forall x \in \mathcal{J}^1, \forall y \in \mathcal{J}^2;$

(H.ii) $\omega_1(x) \geq 0, \omega_2(y) \geq 0, \quad \forall x \in \mathcal{J}^1, \forall y \in \mathcal{J}^2;$

(H.iii) $\int_{\mathcal{J}^1} \omega_1(x) dx = \int_{\mathcal{J}^2} \omega_2(y) dy = 1;$

(H.iv) g is continuous;

(H.v) There exists $\Sigma > 0$ such that $g(x, y) < \Sigma, \forall x \in \mathcal{J}^1, y \in \mathcal{J}^2;$

(H.vi) $\forall x \in \mathcal{J}^1, y \mapsto g(x, y)$ vanishes only on a set of measure 0 in $\mathcal{J}^2;$

(H.vii) $\forall y \in \mathcal{J}^2, x \mapsto g(x, y)$ vanishes only on a set of measure 0 in $\mathcal{J}^1;$

(H.viii) ω_1 and ω_2 are continuous.

Notice that in Fortet's paper, (H.i)-(H.iii) are denoted Hypothesis I [25, p.83], whereas hypotheses (H.iv)-(H.viii) are called Hypothesis II a) and b) [25, p.85].

We are seeking a solution (φ, ψ) of the following Schrödinger system of equations (S):

$$\begin{cases} \varphi(x) \int_{\mathcal{J}^2} g(x, y) \psi(y) dy = \omega_1(x), \\ \psi(y) \int_{\mathcal{J}^1} g(x, y) \varphi(x) dx = \omega_2(y), \end{cases} \quad (\text{S})$$

cf. system (12)-(13).

3 First existence theorem

3.1 Theorem I

Theorem I. [25, p.96] *Assume (H), as well as the condition:*

$$\int_{\mathfrak{J}^2} \frac{\omega_2(y)}{\left[\int_{\mathfrak{J}^1} g(z, y) \omega_1(z) dz \right]} dy < +\infty \quad (\star)$$

Then:

- i) System (S) admits a solution (φ, ψ) ;
- ii) φ is measurable, non-negative and continuous;
- iii) φ vanishes only for all values $x \in \mathfrak{J}^1$ such that $\omega_1(x) = 0$;
- iv) ψ is measurable and non-negative;
- v) ψ vanishes only for almost every $y \in \mathfrak{J}^2$ such that $\omega_2(y) = 0$.

3.2 Application: the Bernstein case

Consider the case where $\mathfrak{J}^1 = \mathfrak{J}^2 = \mathbb{R}$ ($d = 1$), and we have gaussian marginals and transition kernel:

$$\omega_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-x^2/2\sigma_1^2}, \quad \omega_2(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-y^2/2\sigma_2^2}, \quad g(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-x)^2/2\sigma^2}$$

for $\sigma_1, \sigma_2, \sigma > 0$.

Then the integrand in (\star) is:

$$\frac{\omega_2(y)}{\left[\int_{\mathbb{R}} g(z, y) \omega_1(z) dz \right]} = \frac{\sqrt{\sigma^2 + \sigma_1^2}}{\sigma_2^2} e^{-y^2 \frac{\sigma^2 + \sigma_1^2 - \sigma_2^2}{\sigma_2(\sigma_1 + \sigma)}}$$

which is integrable if and only if $\sigma^2 + \sigma_1^2 - \sigma_2^2 > 0$. If $\sigma_1 \geq \sigma_2$, this is true and one can apply Theorem I. If it is not the case, exchange the roles of ω_1 and ω_2 to satisfy condition (\star) , and apply the theorem. Hence up to exchanging the marginals, one can always show existence and uniqueness of a solution to the system (S) in the Bernstein case.

Consider now the case $\mathfrak{J}^1 = \mathfrak{J}^2 = \mathbb{R}^d$, $d > 1$, and

$$\omega_i(x) = \frac{1}{(2\pi|\Sigma_i|)^{d/2}} e^{-x^T \Sigma_i^{-1} x/2}, \quad i = 1, 2, \quad g(x, y) = \frac{1}{(2\pi|\Sigma|)^{d/2}} e^{-(y-x)^T \Sigma^{-1} (y-x)/2}$$

for some symmetric, positive definite matrices $\Sigma, \Sigma_1, \Sigma_2$. Then

$$\frac{\omega_2(y)}{\left[\int_{\mathbb{R}^d} g(z, y) \omega_1(z) dz \right]} = \frac{|\Sigma + \Sigma_1|^{d/2}}{|\Sigma_2|^{d/2}} e^{-y^T [\Sigma_2^{-1} - (\Sigma + \Sigma_1)^{-1}] y/2}$$

which is integrable if and only if the eigenvalues of $\Sigma_2^{-1} - (\Sigma + \Sigma_1)^{-1}$ have positive real part. Hence on \mathbb{R}^d , a sufficient condition for the existence and uniqueness of a solution to the system (S) is that the eigenvalues of $\Sigma_2^{-1} - (\Sigma + \Sigma_1)^{-1}$ or $\Sigma_1^{-1} - (\Sigma + \Sigma_2)^{-1}$ have positive real part.

3.3 Proof of Theorem I

Assume (H) and (\star) true.

The proof introduced by Fortet heavily relies on various monotonicity properties of an iterative scheme. The architecture of the proof is as follows:

- Step 1) The problem is first reduced to proving an equivalent statement;
- Step 2) A proper functional space for the iteration scheme is defined;
- Step 3) The iteration scheme is introduced. Its monotonicity properties are established;
- Step 4) Two separate cases are identified. In the first case, the iteration scheme converges in a finite number of steps. The existence of a fixed point solution to the problem is then deduced;
- Step 5) In the second case, the existence of a fixed point solution to the problem is also proved.

3.3.1 Step 1: Preliminary reduction [25, pp. 86-87]

Note that system (S) is equivalent to the following system:

$$\left\{ \begin{array}{l} (S'1) \quad \varphi(x) = \frac{\omega_1(x)}{\int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathcal{J}^1} g(z, y) \varphi(z) dz \right]} dy} \\ (S'2) \quad \psi(y) = \frac{\omega_2(y)}{\int_{\mathcal{J}^1} g(x, y) \varphi(x) dx} \end{array} \right. \quad (S')$$

It suffices to find a solution φ of (S'1) to get ψ from (S'2), and hence solve (S'). Consider instead the solution of the equation

$$h(x) = \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{h(z)} dz \right]} dy \quad (16)$$

which we shall formally write as

$$h = \Omega(h) \quad (1')$$

Every solution of (16) which isn't a.e. zero or infinite yields a solution φ of (S'1) by:

$$\varphi(x) = \frac{\omega_1(x)}{h(x)} \quad (17)$$

Note that (17) does not define $\varphi(x)$ for values of $x \in \mathcal{J}^1$ such that $\omega_1(x) = h(x) = 0$. We shall show, however, that there exists a solution h such that $h(x) > 0$ everywhere. Thus, we shall devote our attention to finding a solution h to equation (16) or, equivalently, to finding a fixed point of the map Ω . The proof relies on an iterative scheme and thus requires introducing a suitable functional space to study the iteration. We introduce the space of functions of class (C) as:

Definition 1. (Step 2) [Function of class (C)] [25, p.87] $H : \mathcal{J}^1 \mapsto \mathbb{R}$ is a function of class (C) if:

- i) H is measurable;

ii) There exists $c > 0$ such that for every $x \in \mathfrak{J}^1$, we have:

$$H(x) \geq c;$$

iii) For almost every $x \in \mathfrak{J}^1$,

$$H(x) < +\infty.$$

Functions of class (C) are a natural inputs for the map Ω as the following result shows.

Remark II. [25, p.89]

1. $H \equiv 1$ is of class (C).
2. If H_1 is of class (C), and H_2 is measurable, finite a.e., and $H_1 \leq H_2$ everywhere, then H_2 is of class (C).
3. If H_2 is of class (C), and H_1 is measurable, $c < H_1$ everywhere for some $c > 0$, and $H_1 \leq H_2$ almost everywhere, then H_1 is of class (C)
4. If H_1 and H_2 are of class (C), then $\max(H_1, H_2)$ and $\min(H_1, H_2)$ are of class (C).⁴

The following properties are never explicitly stated in [25].

Proposition 1 (Properties of Ω). *The map Ω defined in (1') is isotone on functions of class (C), meaning that if H, H' are of class (C) such that*

$$H \leq H' \quad \text{a.e.},$$

then

$$\Omega(H) \leq \Omega(H')$$

everywhere. Moreover, $\forall c > 0$ and H of class (C) one has $\Omega(cH) = c \Omega(H)$, namely Ω is positively homogeneous of degree one.

Proof. Suppose $H \leq H'$ a.e. Then,

$$\frac{\omega_1}{H} \geq \frac{\omega_1}{H'} \quad \text{a.e.}$$

By non-negativity of all the involved quantities, we get

$$\int_{\mathfrak{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathfrak{J}^1} g(z, y) \frac{\omega_1(z)}{H(z)} dz \right]} dy \leq \int_{\mathfrak{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathfrak{J}^1} g(z, y) \frac{\omega_1(z)}{H'(z)} dz \right]} dy$$

for every $x \in \mathfrak{J}^1$. The second property is evident. □

3.3.2 Lemma for functions of class (C)

Unfortunately, class (C) is not invariant under map Ω , since the image of a class (C) function might not admit a positive lower bound. Images of class (C) functions under Ω are however ‘nearly’ of class (C), which is part of the content of his Lemma [25, p.89] (notice that we added point (iv) below which is not in the original statement):

Lemma ([25], p.89). *Let H be a function of class (C). Define $A = \{x \in \mathfrak{J}^1 | \omega_1(x) > 0\}$.*

Let $H' = \Omega(H)$

Then:

⁴In this paper, the maximum or minimum of two functions will always be taken pointwise.

i) H' is measurable;

ii) For all compact sets $\mathcal{K} \subseteq \mathfrak{J}^1$, there exists a constant $c > 0$, depending on \mathcal{K} , such that

$$c < H'(x), \quad \forall x \in \mathcal{K};$$

iii) $H'(x) < +\infty$ for almost every $x \in A$;

iv)

$$\int_{\mathfrak{J}^1} \frac{H'(x)}{H(x)} \omega_1(x) dx = 1;$$

v) If we have moreover $H'(x) \leq H(x)$ or $H'(x) \geq H(x)$ for almost every $x \in A$, then $H'(x) = H(x)$ for almost every $x \in A$.

Proof. Let H be a function of class (C). In particular, there exists $c > 0$ such that $c < H$ everywhere. Consider two sequences of compact sets $\mathfrak{J}_1^1, \dots, \mathfrak{J}_n^1, \dots, \mathfrak{J}_1^2, \dots, \mathfrak{J}_n^2, \dots$ such that:

$$\begin{cases} \mathfrak{J}_n^1 \subseteq \mathfrak{J}_{n+1}^1, & \mathfrak{J}_n^2 \subseteq \mathfrak{J}_{n+1}^2, & \forall n \in \mathbb{N}^* \\ \mathfrak{J}_n^1 \uparrow \mathfrak{J}^1, & \mathfrak{J}_n^2 \uparrow \mathfrak{J}^2, & \text{as } n \rightarrow +\infty \end{cases}$$

Define $\forall y \in \mathfrak{J}^2, \forall n \in \mathbb{N}^*$

$$G_n(H, y) = \int_{\mathfrak{J}_n^1} g(z, y) \frac{\omega_1(z)}{H(z)} dz$$

First, $G_n(H, \cdot)$ is well defined since $0 < c < H$ and \mathfrak{J}_n^1 is bounded.

Second, $G_n(H, \cdot) > 0$ at least for n large enough from (H.i)-(H.iii) and (H.vii).

Third, $G_n(H, \cdot)$ is continuous by (H.iv) and the fact that \mathfrak{J}_n^1 is bounded.

Besides, $G_n(H, y)$ is a non-decreasing sequence in n , and from (H.iii),(H.v) we have:

$$G_n(H, y) \leq \frac{\Sigma}{c} \int_{\mathfrak{J}_n^1} \omega_1(z) dz \leq \frac{\Sigma}{c}$$

Which implies that $G_n(H, \cdot)$ is uniformly bounded from above in n . Hence by monotone convergence theorem, it admits a pointwise limit

$$G(H, y) \equiv \int_{\mathfrak{J}^1} g(z, y) \frac{\omega_1(z)}{H(z)} dz = \lim_{n \rightarrow +\infty} G_n(H, y)$$

that is a measurable function in y , finite everywhere, and positive by monotonicity.

We actually have better than positivity:

Claim ([25], p.88). For any compact $K \subseteq \mathfrak{J}^2$, there exists a constant $\alpha_K > 0$ such that

$$G(H, y) > \alpha_K > 0, \quad \forall y \in K$$

Proof. By monotonicity of the sequence $(G_n(H, y))_n$, it suffices to show this property on some $G_n(H, y)$ for some $n \in \mathbb{N}^*$.

We are thus seeking to prove that for any compact $K \subseteq \mathfrak{J}^2$, there exists some $n \in \mathbb{N}^*$, and a constant $\alpha_{K,n} > 0$ such that for any $y \in K$,

$$G_n(H, y) > \alpha_{K,n} > 0$$

We will proceed to a proof by contradiction.

Choose such a K . Assume that for all $n, k \in \mathbb{N}^*$, we can find some $y_k \in K$ where

$$G_n(H, y_k) < \frac{1}{k}$$

Choose n_0 large enough such that $\frac{\omega_1}{H} > 0$ a.e. on a set $I' \subseteq \mathfrak{J}_{n_0}^1$, of positive measure. Such an n_0 and I' exist since

$$\mathfrak{J}_n^1 \uparrow \mathfrak{J}^1, \quad \int_{\mathfrak{J}^1} \omega_1(z) dz = 1,$$

and H , being of class (C), is a.e. finite. According to our assumption, for any k , there exists $y_k \in K$ such that

$$G_{n_0}(H, y_k) = \int_{\mathfrak{J}_{n_0}^1} g(z, y_k) \frac{\omega_1(z)}{H(z)} dz < \frac{1}{k}.$$

As $k \rightarrow +\infty$, y_k converges to a limit $y \in K$, up to extracting a subsequence, since K is compact. Moreover, $H \geq c$ by Definition 1, and hence

$$0 \leq g(z, y_k) \frac{\omega_1(z)}{H(z)} < \frac{\Sigma}{c} \omega_1(z), \quad \forall k$$

which is integrable by (H.iii). By the dominated convergence theorem, one can pass to the limit inside the integral $G_{n_0}(H, y_k)$ as $k \rightarrow +\infty$ and deduce from the continuity of g that:

$$\int_{\mathfrak{J}_{n_0}^1} g(z, y) \frac{\omega_1(z)}{H(z)} dz = 0$$

By non-negativity of the integrand, for such a y , we have:

$$g(z, y) \frac{\omega_1(z)}{H(z)} = 0, \quad \text{for almost every } z \in \mathfrak{J}_{n_0}^1$$

This is in particular true for almost every $z \in I' \subseteq \mathfrak{J}_{n_0}^1$.

Recall that for almost every $z \in I'$, $\frac{\omega_1(z)}{H(z)} > 0$.

This implies that

$$g(z, y) = 0, \quad \text{for almost every } z \in I'$$

This contradicts (H.vii) since I' has positive measure, and concludes the proof of the claim. \diamond

We can then conclude that $G(H, y) > \alpha_m > 0 \forall m \in \mathbb{N}^*, y \in \mathfrak{J}_m^2$ thanks to the monotonicity of the sequence of $G_n(H, y)$. We can define for $n \in \mathbb{N}^*$ large enough, $x \in \mathfrak{J}^1$:

$$H'_{|n}(x) = \int_{\mathfrak{J}_n^2} g(x, y) \frac{\omega_2(y)}{G(H, y)} dy. \quad (18)$$

This integral is well defined and finite since we showed that $G(H, y) > \alpha_n > 0$ for $y \in \mathfrak{J}_n^2$, is continuous by (H.iv) and non-decreasing in n . We can thus set

$$H'(x) = \lim_{n \rightarrow +\infty} H'_{|n}(x)$$

to be the pointwise limit (potentially infinite) for every $x \in \mathfrak{J}^1$. H' is measurable, positive and bounded from below by a positive constant on any compact $\mathcal{K} \subseteq \mathfrak{J}^1$. The proof of the validity of these

⁵In Fortet's paper, $H'_{|n}$ is denoted H'_n [25, p.88]. Unfortunately, the same notation is later used for another quantity [25, p.90].

properties for H' follows the very same pattern as that for $G(H, y)$. This proves i) and ii). To prove iii), iv) and v), define:

$$F(x, y) = g(x, y) \frac{\omega_2(y)}{G(H, y)} \frac{\omega_1(x)}{H(x)}.$$

$F(x, y)$ is measurable, non-negative, and bounded for $x \in \mathfrak{J}_q^1, y \in \mathfrak{J}_p^2$, for any $p, q \in \mathbb{N}^*$. This because g is bounded from above, $G(H, \cdot)$ and H are bounded from below by positive constants, and ω_1, ω_2 are continuous on these compact sets. We then define

$$I_{p,q} = \iint_{\mathfrak{J}_q^1 \times \mathfrak{J}_p^2} F(x, y) dx dy \quad (19)$$

$$= \int_{\mathfrak{J}_p^2} \omega_2(y) \frac{G_q(H, y)}{G(H, y)} dy = \int_{\mathfrak{J}^2} \omega_2(y) \mathbb{1}_{\mathfrak{J}_p^2}(y) \frac{G_q(H, y)}{G(H, y)} dy \quad (20)$$

$$= \int_{\mathfrak{J}_q^1} \omega_1(x) \frac{H'_p(x)}{H(x)} dx = \int_{\mathfrak{J}^1} \omega_1(x) \mathbb{1}_{\mathfrak{J}_q^1}(x) \frac{H'_p(x)}{H(x)} dx \quad (21)$$

where we used Fubini-Tonelli's theorem to exchange the order of integration, and we denoted $\mathbb{1}_A$ the indicator function of the set A . Furthermore, the monotonicity (in the sense of inclusion) of the sets $\mathfrak{J}_q^1, \mathfrak{J}_p^2$ and monotonicity of the sequences $H'_p(x), G_q(H, y)$ implies the monotonicity of the functions $H'_p \mathbb{1}_{\mathfrak{J}_q^1}$ and $G_q(H, y) \mathbb{1}_{\mathfrak{J}_p^2}$, respectively in p and q . One can then use the Beppo-Levi monotone convergence theorem to take limits as p and $q \rightarrow \infty$ inside the integrals in (20), (21) to infer first from (20) that

$$\lim_{p \rightarrow +\infty} \lim_{q \rightarrow +\infty} I_{p,q} = \int_{\mathfrak{J}^2} \omega_2(y) \lim_{p \rightarrow +\infty} \mathbb{1}_{\mathfrak{J}_p^2}(y) dy = 1.$$

It then follows from (21) that:

$$\lim_{p \rightarrow +\infty} \lim_{q \rightarrow +\infty} I_{p,q} = \int_{\mathfrak{J}^1} \omega_1(x) \lim_{p \rightarrow +\infty} \frac{H'_p(x)}{H(x)} dx = 1$$

which gives

$$\int_{\mathfrak{J}^1} \frac{H'(x)}{H(x)} \omega_1(x) dx = 1 \quad (22)$$

Recalling that $A = \{x \in \mathfrak{J}^1 | \omega_1(x) > 0\}$, we derive from (22) that H' is finite a.e. on A , otherwise the integral in (22) would be infinite. This establishes iii) and iv). Finally, assume that for almost every $x \in A$ one has either

$$H'(x) \leq H(x), \quad \text{or} \quad H'(x) \geq H(x)$$

Then (22) allows us to conclude that $H' = H$ a.e. on A , otherwise we would contradict the fact that ω_1 integrates to one. This establishes v), and completes the proof of the lemma. \square

Remark I. [25, p.89] *The lemma remains valid if we only assume that H is measurable but only bounded from below by 0, as long as we can guarantee that the integral $G(H, y)$ remains finite a.e. in y . We can even allow $G(H, y)$ to be infinite for values of y where $\omega_2(y) = 0$.*

The above lemma allows us to extract sufficient information on $H' = \Omega(H)$ in order to proceed to the iteration scheme, and prove the first existence result Theorem I.

3.3.3 Step 3: Iterative procedure

Starting from $H_1 \equiv 1$, one would like to proceed to successive iterations of Ω by setting $H_{n+1} = \Omega(H_n)$, and show convergence. As illustrated by the Lemma, if H is of class (C), then $\Omega(H)$ is not necessarily of class (C). Thus, there is no guarantee of obtaining an a.e. finite function if one applies the map Ω

one more time. Moreover, one has to guarantee the convergence of such an iteration scheme. Fortet therefore introduces a **truncation** procedure between two successive iterations of Ω that takes care of these issues. The approximation scheme reads [25, p.90]:

$$\begin{aligned} H_1 &\equiv 1, & H'_1 &= \Omega(H_1), & H''_1 &= \min(H_1, H'_1) \\ H_n &= \max\left(H''_{n-1}, \frac{1}{n}\right), & H'_n &= \Omega(H_n), & H''_n &= \min(H_1, H'_n), \quad \forall n \geq 2 \end{aligned} \quad (\text{AS})$$

The max step guarantees that H_n always remains in the class (C), and hence we can apply Ω in the iteration scheme. The vanishing lower bound will lead to a fixed point of Ω which is not necessarily of class (C). As for the min step, it is needed, in particular, to guarantee the monotonicity of the scheme.

Observation 1. *Note that condition (\star) , as well as assumption (H.v), guarantees the (everywhere) finiteness of $H'_1 = \Omega(H_1)$, since*

$$H'_1(x) = \Omega(H_1)(x) = \int_{\mathcal{Y}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathcal{Y}^1} g(z, y)\omega_1(z)dz\right]} dy < \Sigma \int_{\mathcal{Y}^2} \frac{\omega_2(y)}{\left[\int_{\mathcal{Y}^1} g(z, y)\omega_1(z)dz\right]} dy < +\infty.$$

The following result is stated, but not proven, on [25, p.90].

Proposition 2 (Monotonicity of the scheme (AS)). *For H_n, H'_n defined by the scheme (AS), one has $\forall n \in \mathbb{N}^*$:*

$$H_{n+1} \leq H_n, \quad H'_{n+1} \leq H'_n$$

everywhere.

Proof. By the monotonicity property of Ω in Proposition 1, it suffices to show that $H_{n+1} \leq H_n$ to deduce that $H'_{n+1} \leq H'_n$, since by definition $H'_n = \Omega(H_n)$, $\forall n \in \mathbb{N}^*$. We prove $H_{n+1} \leq H_n$ by induction. For $n = 1$:

$$H''_1 = \min(H_1, \Omega(H_1)) = \begin{cases} 1 & \text{if } \Omega(H_1) \geq 1, \\ \Omega(H_1) & \text{if } \Omega(H_1) \leq 1. \end{cases}$$

Thus

$$H_2 = \max\left(H''_1, \frac{1}{2}\right) = \begin{cases} 1 & \text{if } \Omega(H_1) \geq 1 \\ \min(\Omega(H_1), \frac{1}{2}) & \text{if } \Omega(H_1) \leq 1 \end{cases} \leq 1 = H_1,$$

which proves the initialization step of the induction. Let us now assume that the property is true for some $n \in \mathbb{N}^*$, namely we have

$$H_{n+1} \leq H_n$$

pointwise. Then, by the monotonicity of Ω (Proposition 1), we have that $H'_{n+1} \leq H'_n$, and thus

$$H''_{n+1} = \max(H'_{n+1}, H_1) \leq \max(H'_n, H_1) = H''_n$$

Since we also have $\frac{1}{n+2} < \frac{1}{n+1}$, we can infer that

$$H_{n+2} = \min\left(H''_{n+1}, \frac{1}{n+2}\right) \leq \min\left(H''_n, \frac{1}{n+1}\right) = H_{n+1}$$

which concludes the proof by induction. \square

Observation 2. *Since $H_1 \equiv 1$, each H_n is finite everywhere. In addition, **Observation 1** and **Proposition 2** also show that each H'_n is finite everywhere.*

The monotonicity of Proposition 2 will be crucial to establishing existence of a fixed point for (1'). When iterating (AS), we distinguish two separate cases which lead to different fixed points:

3.3.4 First case [25, Section 2, p. 86]

In this case, we assume that, as we iterate following the approximation scheme (AS), there exists some $n_0 \in \mathbb{N}^*$ such that a.e. one has

$$H'_{n_0} \leq H_1. \quad (23)$$

We shall show, using the Lemma, that $\Omega(H'_{n_0})$ is a solution to equation (16) (and that H'_{n_0} is ‘nearly’ a solution). We first need to show that $\Omega(H'_{n_0})$ is well defined. This will be accomplished by approximating H'_{n_0} as shown below. First of all, notice that (23) together with the definition of H''_{n_0} in scheme (AS) yields

$$H''_{n_0} = H'_{n_0}. \quad (24)$$

Let us define

$$K_p = \max\left(H'_{n_0}, \frac{1}{p}\right), \quad p \in \mathbb{N}^*. \quad (25)$$

Although H'_{n_0} may not be of class (C), it follows from the Lemma that K_p is of class (C) since we have the uniform lower bound $\frac{1}{p} > 0$. Furthermore, as $p \rightarrow +\infty$, $K_p \rightarrow H'_{n_0}$ pointwise. Set

$$K'_p = \Omega(K_p).$$

Note that $K_{p+1} \leq K_p$. By Proposition 1, the sequence of $K'_p = \Omega(K_p)$ is also decreasing in p . By the non-negativity of K'_p , the sequence $\{K'_p\}$ admits a pointwise limit K' which is measurable and non-negative:

$$K' = \lim_{p \rightarrow +\infty} K'_p = \lim_{p \rightarrow +\infty} \Omega(K_p). \quad (26)$$

Recalling that Ω was defined as an integral operator, we can then use Beppo-Levi monotone convergence theorem to get from the monotonicity of the sequence of K_p that

$$\lim_{p \rightarrow +\infty} \Omega(K_p) = \Omega\left(\lim_{p \rightarrow +\infty} K_p\right) = \Omega(H'_{n_0}).$$

Putting this together with (26), we finally get

$$K' = \Omega(H'_{n_0}). \quad (27)$$

To show that K' is a solution of (1'), we first need the following result whose statement and sketch of the proof can be found on [25, p.91].

Proposition 3.

$$\int_{\mathfrak{I}^1} \frac{K'(x)}{H'_{n_0}(x)} \omega_1(x) dx = 1$$

Proof. From the scheme (AS), we know that

$$H_{n_0} \geq \frac{1}{n_0} = \frac{1}{n_0} H_1$$

Using both properties of Proposition 1, we get

$$H'_{n_0} \geq \frac{H'_1}{n_0}.$$

By the definition of K_p (25), we now get:

$$K_p \geq H'_{n_0} \geq \frac{H'_1}{n_0}. \quad (28)$$

Furthermore, since we assumed that $H'_{n_0} \leq 1$ a.e., one also has by the definition of K_p (25) that $K_p \leq H_1 = 1$ a.e.. This implies, by Proposition 1 that $K'_p \leq H'_1$ everywhere. Plugging the latter inequality in (28) yields that $\forall p \in \mathbb{N}^*$,

$$\frac{K'_p}{K_p} \leq n_0. \quad (29)$$

This implies that, taking the limit for $p \rightarrow +\infty$, we also have

$$\frac{K'}{H'_{n_0}} \leq n_0.$$

Since K_p is of class (C), Lemma iv) yields

$$\int_{\mathfrak{J}^1} \frac{K'_p(x)}{K_p(x)} \omega_1(x) dx = 1$$

By (29), the integrand is uniformly bounded in p . By (H.iii), the measure is finite. We conclude by the bounded convergence theorem that

$$\int_{\mathfrak{J}^1} \frac{K'(x)}{H'_{n_0}(x)} \omega_1(x) dx = 1$$

which concludes the proof. \square

Lastly, we shall also need the following result whose statement and sketch of the proof can also be found on [25, p.91].

Proposition 4. *We have*

$$K' \leq H'_{n_0}$$

everywhere on \mathfrak{J}^1 .

Proof. First of all, notice that for $p > n_0 + 1$, one has from the scheme (AS), from (24) and from the definition of (25) K_p :

$$H_{n_0+1} = \max\left(H''_{n_0}, \frac{1}{n_0+1}\right) = \max\left(H'_{n_0}, \frac{1}{n_0+1}\right) = K_{n_0+1}.$$

By monotonicity of the sequence of K_p , we also have that, for $p > n_0 + 1$, $K_{n_0+1} \geq K_p$ everywhere. This together with the above equality then gives for $p > n_0 + 1$:

$$H_{n_0+1} \geq K_p.$$

Applying Ω to both sides of the above inequality and using again Proposition 1, we get

$$K'_p \leq H'_{n_0+1}$$

Since $K'_p \geq K'$ and $H'_{n_0+1} \leq H'_{n_0}$ (Proposition 2), we finally obtain

$$K' \leq H'_{n_0}.$$

\square

We now employ Propositions 3 and 4 to complete the first case: On $A = \{x \in \mathfrak{J}^1 | \omega_1(x) > 0\}$, we must have a.e.

$$K' = H'_{n_0}. \quad (30)$$

Recalling that $K' = \Omega(H'_{n_0})$ (see(27)), we conclude from (30) that

$$\Omega(H'_{n_0}) = H'_{n_0}, \quad \text{a.e. on } A.$$

We proceed to show that actually this equality holds on all of \mathcal{J}^1 . Indeed, by (30), for every $y \in \mathcal{J}^2$:

$$\begin{aligned} G(K', y) &= \int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{K'(z)} dz = \int_A g(z, y) \frac{\omega_1(z)}{K'(z)} dz \\ &= \int_A g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz = \int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz \\ &= G(H'_{n_0}, y). \end{aligned}$$

It follows in view of (27), that for every $x \in \mathcal{J}^1$:

$$\Omega(K')(x) = \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{G(K', y)} dy = \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{G(H'_{n_0}, y)} dy = \Omega(H'_{n_0})(x) = K'(x).$$

Thus, K' as a fixed point of the map Ω . This concludes the proof of the first case. The following bounds for K' are merely stated on [25, p.91].

Proposition 5. *For every $x \in \mathcal{J}^1$*

$$0 < K'(x) \leq 1 \tag{31}$$

Proof. By assumption (23), $H'_{n_0} \leq 1$ a.e which implies $H''_{n_0} = H'_{n_0}$ (24). Hence

$$H_{n_0+1} = \max\left(H''_{n_0}, \frac{1}{n_0+1}\right) = \max\left(H'_{n_0}, \frac{1}{n_0+1}\right) = K_{n_0+1}$$

by definition (25) of K_p . Applying the map Ω and using Proposition 1, we get

$$H'_{n_0} = \Omega(H_{n_0}) \geq \Omega(H_{n_0+1}) = \Omega(K_{n_0+1}) = K'_{n_0+1}$$

Since the sequence of K'_p monotonically decreases to K' , we then conclude that

$$1 \geq H'_{n_0} \geq K'_{n_0+1} \geq K'.$$

Thus, $K' \leq 1$ everywhere. To prove $K' > 0$, recall that by (27)

$$K'(x) = \Omega(H'_{n_0})(x) = \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz\right]} dy$$

H'_{n_0} is not necessarily of class (C). In particular, we do not have an *a priori* positive lower bound. Thus we cannot apply the Lemma to prove the statement as we cannot *a priori* guarantee that

$$\int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz < +\infty, \quad \text{a.e. in } y.$$

Notice instead that since $H'_{n_0} = H''_{n_0}$, we get from the scheme (AS):

$$H_{n_0+1} \geq H''_{n_0} = H'_{n_0}$$

By Proposition 2, $H_{n_0} \geq H_{n_0+1}$ and thus $H_{n_0} \geq H'_{n_0}$ everywhere. Since H_{n_0} is of class (C), we have by the Lemma v) that $H_{n_0} = H'_{n_0}$ a.e. on $A = \{x \in \mathcal{J}^1 | \omega_1(x) > 0\}$. In particular, there exists a

constant $c > 0$ such that for a.e. $x \in A$, $H'_{n_0}(x) \geq c$. It follows that

$$\begin{aligned} \int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz &= \int_A g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz + \int_{\mathcal{J}^1 \setminus A} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz \\ &= \int_A g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz, \quad \text{since } \omega_1 = 0 \text{ on } \mathcal{J}^1 \setminus A \text{ and } H'_{n_0}(z) > 0 \\ &\leq \frac{1}{c} \int_A g(z, y) \omega_1(z) dz, \quad \text{since } H'_{n_0} \geq c \text{ on } A \\ &\leq \frac{\Sigma}{c}, \quad \text{from (H.iii), (H.v)} \end{aligned}$$

We conclude that, for all $x \in \mathcal{J}^1$, we have

$$\begin{aligned} K'(x) &= \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_{\mathcal{J}^1} g(z, y) \frac{\omega_1(z)}{H'_{n_0}(z)} dz \right]} dy \\ &\geq \frac{c}{\Sigma} \int_{\mathcal{J}^2} g(x, y) \omega_2(y) dy > 0, \end{aligned}$$

where the last inequality follows from (H.i), (H.iii), (H.v). \square

3.3.5 Second case [25, Section 2, p. 92]

Contrary to the first case, assume now that $\forall n \in \mathbb{N}^*$, there exists a positive measure set J_n on which $H'_n > H_1$. Define by H and H' the respective limits of the sequences H_n and H'_n . By Proposition 2, nonnegativity of the sequences and Observation 2, these limits exist, are measurable and finite. We shall show that H' is a fixed point of the map Ω . First notice that the sequence of J_n 's is monotonically decreasing:

Proposition 6 (Monotonicity of J_n). *We have $\forall n \in \mathbb{N}^*$:*

$$J_{n+1} \subseteq J_n$$

Proof. Let $n \in \mathbb{N}^*$, $x \in J_{n+1}$. Then $H'_{n+1}(x) > H_1(x)$. By Proposition 2, $H'_n(x) \geq H'_{n+1}(x) > H_1(x)$. Thus, $x \in J_n$. \square

We can then define

$$J = \lim_{n \rightarrow +\infty} J_n = \bigcap_{n \in \mathbb{N}^*} J_n.$$

One has moreover ⁶ the following inequality which is stated on [25, p.92].

Proposition 7.

$$H \leq H'$$

everywhere.

Proof. By the scheme (AS) and Proposition 2, the nonnegative sequence of H''_n is also decreasing. Hence, it admits a limit H'' . By definition, $H_n = \max(H''_{n-1}, \frac{1}{n})$. Thus, the limits must be equal $H = H''$. Since $H''_n = \min(H_1, H'_n) \leq H'_n$, we get, passing to the limit, that $H \leq H'$. \square

The following result shows that H' cannot vanish, otherwise we would fall back in the first case⁷.

⁶Fortet seems to imply by this proposition that H and H' cannot vanish at a point without vanishing everywhere. Although this is true for H' , see Proposition 8 below, it does not imply the same property for H .

⁷The statement can be found on [25, p.92]. The proof there provided, however, appears to be incorrect as it does not make use of hypothesis (\star) confusing H'_n of the iteration (AS) with H'_n (also denoted by H'_n by Fortet) defined in (18).

Proposition 8. *Assume that there exists some $x_0 \in \mathfrak{J}^1$ such that $H'(x_0) = 0$. Then the sequence H'_n converges uniformly to 0 on \mathfrak{J}^1 . In particular, $H' \equiv 0$.*

Proof. Assume that there exists some $x_0 \in \mathfrak{J}^1$ such that $H'(x_0) = 0$. By definition of H' , this implies that the sequence $H'_n(x_0)$ converges to 0, i.e.:

$$H'_n(x_0) = \int_{\mathfrak{J}^2} g(x_0, y) \frac{\omega_2(y)}{G(H_n, y)} dy \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

This implies that

$$\text{The measure } \frac{\omega_2(y)}{G(H_n, y)} dy \text{ converges weakly to 0 on } \mathfrak{J}^2. \quad (32)$$

The proof of the above statement can be found in the Appendix A. Now pick any $x \in \mathfrak{J}^1$. We know from **Observation 2** that $H'_n(x)$ is finite. In particular, approximating $\mathfrak{J}^2 \supseteq \dots \supseteq \mathfrak{J}_q^2 \supseteq \dots \supseteq \mathfrak{J}_1^2$ by compact sets \mathfrak{J}_q^2 , one can write for $q \in \mathbb{N}^*$:

$$H'_n(x) = \int_{\mathfrak{J}_q^2} g(x, y) \frac{\omega_2(y)}{G(H_n, y)} dy + \int_{\mathfrak{J}^2 \setminus \mathfrak{J}_q^2} g(x, y) \frac{\omega_2(y)}{G(H_n, y)} dy. \quad (33)$$

By boundedness of \mathfrak{J}_q^2 and (32), the first integral

$$\int_{\mathfrak{J}_q^2} g(x, y) \frac{\omega_2(y)}{G(H_n, y)} dy \leq \Sigma \int_{\mathfrak{J}^2} \frac{\omega_2(y)}{G(H_n, y)} dy$$

converges uniformly in x to 0 as $n \rightarrow +\infty$. As for the second integral, notice that $H_n \leq H_1$ from **Proposition 2**. Hence

$$\int_{\mathfrak{J}^2 \setminus \mathfrak{J}_q^2} g(x, y) \frac{\omega_2(y)}{G(H_n, y)} dy \leq \int_{\mathfrak{J}^2 \setminus \mathfrak{J}_q^2} g(x, y) \frac{\omega_2(y)}{G(H_1, y)} dy \leq \Sigma \int_{\mathfrak{J}^2 \setminus \mathfrak{J}_q^2} \frac{\omega_2(y)}{G(H_1, y)} dy$$

which can be made, uniformly in x , arbitrarily small when $q \rightarrow +\infty$, by absolute continuity of the measure $\frac{\omega_2(y)}{G(H_1, y)}$ with respects to the Lebesgue measure, thanks to condition (\star) . We therefore conclude the uniform convergence of the sequence of H'_n to 0. \square

It follows from **Proposition 8** that if H' vanishes at one point, H'_n converges uniformly to 0. In that case, for n large enough, we would have for every x , $H'(x) \leq 1 = H_1$. We would namely be in the first case. We can then conclude that, in this second case, we necessarily have $H' > 0$ everywhere. To prove that H' satisfies (1'), we shall show that, although we do not have $H'_n \leq 1$ a.e. for some n , this holds for the limit H' . The rest of the proof will then be similar to the first case provided we can show that the set J has zero measure. This is stated, followed by a very sketchy proof by contradiction, on [25, p.93].

Proposition 9. *The set $J = \lim_{n \rightarrow +\infty} J_n$ has measure 0.*

Proof. Assume that it is not the case. Then one has for $x \in J$, $H'(x) > H_1(x) = 1$. The scheme (AS) thus yields $H''(x) = \min(H_1(x), H'(x)) = 1$, and hence

$$H(x) = \max(H''(x), 0) = 1 < H'(x), \quad \forall x \in J \quad (34)$$

Similarly, for $x \in \mathfrak{J}^1 \setminus J$, one has from the approximation scheme $H''(x) = \min(H_1(x), H'(x)) = H'(x)$, and hence

$$H(x) = \max(H''(x), 0) = H'(x), \quad \forall x \in \mathfrak{J}^1 \setminus J \quad (35)$$

From (34), (35) and the fact that J has positive measure, it follows that:

$$\int_{\mathfrak{J}^1} \frac{H'(x)}{H(x)} \omega_1(x) dx = \int_J \frac{H'(x)}{H(x)} \omega_1(x) dx + \int_{\mathfrak{J}^1 \setminus J} \frac{H'(x)}{H(x)} \omega_1(x) dx > \int_{\mathfrak{J}^1} \omega_1(x) dx = 1. \quad (36)$$

Recall now that for $n \in \mathbb{N}^*$, H_n is of class (C), and hence, by Lemma iv) we have that:

$$\int_{\mathfrak{J}^1} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx = 1. \quad (37)$$

The strategy consists in passing to the limit in the above equation and derive a contradiction with (36). However, passing to the limit is delicate, hence we consider the following decomposition:

$$1 = \int_{\mathfrak{J}^1} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx = \int_{\mathfrak{J}^1 \setminus J_{n-1}} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx + \int_{J_{n-1}} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx \quad (38)$$

$$= \int_{\mathfrak{J}^1 \setminus J_{n-1}} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx + \int_{J_{n-1}} H'_n(x) \omega_1(x) dx \quad (39)$$

since again by the scheme, one has for $x \in J_{n-1}$: $H''_{n-1}(x) = 1$ and, consequently, $H_n(x) = 1$. Now notice that by monotonicity of J_n 's (Proposition 6) and of H'_n (Proposition 2), one has:

$$\int_{J_{n-1}} H'_n(x) \omega_1(x) dx \geq \int_J H'_n(x) \omega_1(x) dx \geq \int_J H'(x) \omega_1(x) dx = \int_J \frac{H'(x)}{H(x)} \omega_1(x) dx \quad (40)$$

where the last equality holds because of (34).

Let us now focus on the second integral:

$$\int_{\mathfrak{J}^1 \setminus J_{n-1}} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx = \int_{\mathfrak{J}^1} \mathbb{1}_{\mathfrak{J}^1 \setminus J_{n-1}}(x) \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx$$

Notice that for $x \in \mathfrak{J}^1 \setminus J_{n-1}$, we have $H'_{n-1}(x) \leq 1$. We then get from the scheme (AS) that $H''_{n-1}(x) = H'_{n-1}(x)$. It follows that

$$\text{either } H_n(x) = H'_{n-1}(x), \quad \text{or} \quad H_n(x) = \frac{1}{n} \text{ in the case } H'_{n-1}(x) \leq \frac{1}{n}$$

In any case one has

$$H_n(x) \geq H'_{n-1}(x).$$

By Proposition 2, $H'_{n-1}(x) \geq H'_n(x)$. We get that

$$\frac{H'_n(x)}{H_n(x)} \leq 1, \quad \forall x \in \mathfrak{J}^1 \setminus J_{n-1}.$$

The bounded convergence theorem allows us to conclude that

$$\lim_{n \rightarrow +\infty} \int_{\mathfrak{J}^1 \setminus J_{n-1}} \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx = \int_{\mathfrak{J}^1} \lim_{n \rightarrow +\infty} \mathbb{1}_{\mathfrak{J}^1 \setminus J_{n-1}}(x) \frac{H'_n(x)}{H_n(x)} \omega_1(x) dx = \int_{\mathfrak{J}^1 \setminus J} \frac{H'(x)}{H(x)} \omega_1(x) dx. \quad (41)$$

Using (40) and (41) into (39), one gets when passing to the limit in n that

$$1 \geq \int_J \frac{H'(x)}{H(x)} \omega_1(x) dx + \int_{\mathfrak{J}^1 \setminus J} \frac{H'(x)}{H(x)} \omega_1(x) dx$$

which contradicts (36). \square

Now that we know that J is of measure 0, we are ready to show that H' is indeed a solution of (1'). Since we have $H'_n = \Omega(H_n)$, Beppo-Levi's monotone convergence theorem implies that $H' = \Omega(H)$. Since J is of measure 0, we have that $H' \leq H_1 = 1$ a.e.. By the definition $H''_n = \min(H_1, H'_n)$, passing to the limit we then get $H'' = H'$ a.e.. This, together with $H_n = \max(H''_{n-1}, \frac{1}{n})$, also gives $H = H'' = H'$ a.e. . We conclude that everywhere:

$$H' = \Omega(H) = \Omega(H'),$$

which proves that H' is solution to (1').

3.3.6 Conclusion

To summarize, in both cases we found a measurable solution h of (1') ($h = K'$ in the first case, $h = H'$ in the second case) such that we have everywhere

$$0 < h \leq 1.$$

Moreover, we have continuity of the solution. This is stated with a sketch of the proof on [25, p.95].

Proposition 10. *h is continuous on \mathfrak{J}^1 .*

Proof. Recall the definition

$$G(H, y) \equiv \int_{\mathfrak{J}^1} g(z, y) \frac{\omega_1(z)}{H(z)} dz$$

Since $h \leq 1 = H_1$ everywhere in x , we get $\frac{\omega_2(y)}{G(h, y)} \leq \frac{\omega_2(y)}{G(H_1, y)}$ everywhere in y . Then, for $x_1, x_2 \in \mathfrak{J}^1$,

$$|h(x_1) - h(x_2)| \leq \int_{\mathfrak{J}^2} |g(x_1, y) - g(x_2, y)| \frac{\omega_2(y)}{G(h, y)} dy \leq \int_{\mathfrak{J}^2} |g(x_1, y) - g(x_2, y)| \frac{\omega_2(y)}{G(H_1, y)} dy$$

From (H.v), $|g(x_1, y) - g(x_2, y)| \frac{\omega_2(y)}{G(H_1, y)} \leq 2\Sigma \frac{\omega_2(y)}{G(H_1, y)}$, which is integrable by (*). Thus one can use the dominated convergence theorem to deduce that

$$\lim_{x_2 \rightarrow x_1} |h(x_1) - h(x_2)| \leq \int_{\mathfrak{J}^2} \lim_{x_2 \rightarrow x_1} |g(x_1, y) - g(x_2, y)| \frac{\omega_2(y)}{G(H_1, y)} dy = 0$$

from the continuity of g . □

We now reformulate the existence results and the properties of h in terms of the original variables (φ, ψ) . Since $0 < h \leq 1$ everywhere, equation (17) defines a proper measurable function φ on \mathfrak{J}^1 , non-negative and only vanishing for values x where $\omega_1(x) = 0$, which is moreover continuous from (H.viii), Proposition 10 and the fact that $h > 0$. This proves Theorem I.ii),iii) and the existence of φ solution of (S'1). Given such a φ , one can define a measurable solution ψ from (S'2). By property $\varphi \geq 0$, (H.i), (H.ii), we have that $\psi \geq 0$, which proves Theorem I.i),iv). It remains to establish Theorem I.v). Let $A' = \{y \in \mathfrak{J}^2 | \omega_2(y) > 0\}$, and $A'' = \{y \in A' | \psi(y) = 0\} \subseteq A'$. The goal is to show that A'' has measure 0. To this end, we compute:

$$\int_{\mathfrak{J}^2} g(x, y) \psi(y) dy = \int_{A' \setminus A''} g(x, y) \psi(y) dy$$

since by (S'2), $\psi = 0$ outside of A' , and by definition of A'' , $\psi = 0$ on A'' . We can then multiply the above equation by $\varphi(x)$ and integrate over \mathfrak{J}^1 . Since all functions involved are non-negative and

measurable, one can decide the order of integration by Fubini-Tonnelli. On the one hand, we have

$$\begin{aligned} \int_{\mathfrak{J}^1} \varphi(x) \left[\int_{A' \setminus A''} g(x, y) \psi(y) dy \right] dx &= \int_{\mathfrak{J}^1} \varphi(x) \left[\int_{\mathfrak{J}^2} g(x, y) \psi(y) dy \right] dx \\ &= \int_{\mathfrak{J}^2} \psi(y) \left[\int_{\mathfrak{J}^1} \varphi(x) g(x, y) dx \right] dy \\ &= \int_{\mathfrak{J}^2} \omega_2(y) dy, \quad \text{from (S'2)} \\ &= 1 \end{aligned}$$

On the other hand, we get:

$$\begin{aligned} \int_{\mathfrak{J}^1} \varphi(x) \left[\int_{A' \setminus A''} g(x, y) \psi(y) dy \right] dx &= \int_{A' \setminus A''} \psi(y) \left[\int_{\mathfrak{J}^1} \varphi(x) g(x, y) dx \right] dy \\ &= \int_{A' \setminus A''} \omega_2(y) dy, \quad \text{from (S'2)} \end{aligned}$$

We deduce that

$$\int_{A' \setminus A''} \omega_2(y) dy = \int_{A'} \omega_2(y) dy$$

which is only possible if A'' has measure 0, since $\omega_2 > 0$ on $A'' \subset A'$. This concludes the proof of Theorem I. \square

4 Second existence theorem and uniqueness theorem

In [25, Section 3, pp. 97-102], Fortet proceeds to derive an existence theorem for System (S) still under hypotheses (H.i)-(H.viii) but without assuming the integrability condition (\star). The latter condition is replaced by the assumption that the kernel function $g(x, y)$ be of class (B) [25, p. 97]. The latter property appears in general hard to check. We have therefore decided to present only a special case of the second existence theorem where this property can be readily verified.

Theorem II. [25, p. 101] *Suppose $\mathfrak{J}^1 = \mathfrak{J}^2 = \mathbb{R}$ and that $g(x, y) = U(x - y)$ only depends on the difference $t = x - y$. Assume, moreover, that for t sufficiently large $U(t)$ is non increasing and for t sufficiently small it is non decreasing or viceversa. Assume, finally, (H.i)-(H.viii). Then system (S) admits a solution $(\varphi(x), \psi(y))$. The function $\varphi(x)$ is zero for the values x for which $\omega_1(x)$ is zero. On the complement, φ is strictly positive and continuous. The non negative function $\varphi(y)$ is measurable and equal to zero, up to a zero measure set, only for the values y where $\omega_2(y) = 0$.*

Observation 2. *Notice that this theorem applies to the important case where $g(x, y) = p(0, x, 1, y)$ the heat kernel (1) and arbitrary continuous densities $\omega_1(x)$ and $\omega_2(y)$ with support equal to the real line.*

Sketch of the Proof of Theorem II, pp.98-101. 1. A continuous, positive function ρ is introduced which satisfies, in particular, the following property:

$$\int_{\mathfrak{J}^1} \frac{\omega_1(x) \rho(x)}{\left[\int_{\mathfrak{J}^2} g(x, z) \omega_2(z) dz \right]} dx < +\infty.$$

2. By Theorem I and by construction of ρ , the system

$$\begin{cases} \bar{\varphi}(x) \int_{\mathcal{J}^2} g(x, y) \bar{\psi}(y) dy = \omega_1(x) \rho(x), \\ \bar{\psi}(y) \int_{\mathcal{J}^1} g(x, y) \bar{\varphi}(x) dx = \omega_2(y), \end{cases}$$

admits a solution $(\bar{\varphi}, \bar{\psi})$.

3. The same techniques as in the Lemma and the proof of Theorem I permit to show that there exists a fixed point for the operator $\bar{\Omega}$ defined on functions of class (C) by:

$$\bar{\Omega}(H)(x) = \frac{\bar{\varphi}(x)}{\omega_1(x)} \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2(y)}{\left[\int_A g(z, y) \frac{\bar{\varphi}(z)}{H(z)} dz \right]} dy.$$

The fixed point \bar{h} , which is not necessarily of class (C), enjoys properties similar to the fixed point of Ω defined in (1').

4. Set

$$\begin{cases} \varphi(x) = \frac{\bar{\varphi}(x)}{\bar{h}(x)}, & x \in A, \\ \varphi(x) = 0, & x \in \mathcal{J}^1 \setminus A. \end{cases}$$

Then φ is a solution of (S'1). The other function ψ can then be recovered from (S'2).

The assumptions of Theorem II are used to show that the various integrals in this proof are well defined. \square

Fortet defines as a nonnegative (positive in French) solution of (S) to be a pair of nonnegative functions $(\varphi(x), \psi(y))$ satisfying (S) and the following properties: They are a.e. finite, and different from zero (up to a zero measure set) for the values where $\omega_1 \neq 0$ and $\omega_2 \neq 0$, respectively. Moreover, under hypotheses (H.i)-(H.viii), $\varphi(x)$ is zero at the same time as ω_1 and ψ is zero at the same time as ω_2 . The proof of the following uniqueness theorem [25, pp.102-104] has been slightly reformulated and completed.

Theorem III. [25, p. 104] *Assume (H.i)-(H.viii). Let (φ_1, ψ_1) and (φ_2, ψ_2) be two nonnegative and measurable solutions of system (S). Then, there exists a positive constant c such that*

$$\frac{\varphi_1(x)}{\varphi_2(x)} \equiv c \equiv \frac{\psi_1(y)}{\psi_2(y)}. \quad (42)$$

Proof. Let $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ be two solutions of System (S). According to Theorem I or II, φ_1 and φ_2 are positive and finite on the support A_1 of ω_1 . Hence, there exists a value of $x_0 \in A_1$ such that

$$0 < \varphi_1(x_0) < +\infty, \quad 0 < \varphi_2(x_0) < +\infty.$$

Recall that if (φ_2, ψ_2) is a solution of System (S), then so is $(\tilde{\varphi}_2, \tilde{\psi}_2) = (k\varphi_2, \frac{1}{k}\psi_2)$ for $k \neq 0$. Setting $k = \frac{\varphi_1(x_0)}{\varphi_2(x_0)}$, one has that

$$\tilde{\varphi}_2(x_0) = \frac{\varphi_1(x_0)}{\varphi_2(x_0)} \varphi_2(x_0) = \varphi_1(x_0).$$

Thus, without loss of generality, one can always pick two solutions $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ where the φ agree at one point $x_0 \in \mathcal{J}^1$, so that:

$$0 < \varphi_1(x_0) = \varphi_2(x_0) < +\infty.$$

Let $A_1 = \{x \in \mathcal{J}^1 | \omega_1(x) > 0\}$. On A_1 , we define

$$h_1(x) = \frac{\omega_1(x)}{\varphi_1(x)}, \quad h_2(x) = \frac{\omega_2(x)}{\varphi_2(x)}.$$

Then, h_1 and h_2 are two distinct solutions of equation (16). Let, as before,

$$G(H, y) = \int_{A_1} g(z, y) \frac{\omega_1(z)}{H(z)} dz.$$

Then, on $A_2 = \{y \in \mathcal{J}^2 | \omega_2(y) > 0\}$, we have

$$\psi_1(y) = \frac{\omega_2(y)}{G(h_1, y)}, \quad \psi_2(y) = \frac{\omega_2(y)}{G(h_2, y)}.$$

From this we deduce that $G(h_1, y)$ and $G(h_2, y)$ are a.e. finite on A_2 . Let $h(x) = \max(h_1, h_2)$. Then $G(h, y)$ is a.e. finite on A_2 and

$$0 < h(x_0) = h_1(x_0) = h_2(x_0) < +\infty.$$

Let us set

$$h'(x) = \int_{\mathcal{J}^2} g(x, y) \frac{\omega_2 dy}{G(h, y)}.$$

By the same argument used to prove Lemma iv), it follows that

$$\int_{A_1} \frac{h'}{h} \omega_1 dx = 1. \quad (43)$$

Since $h \geq h_1$, it follows from Proposition 1 that also $h' \geq h_1$. Similarly, $h \geq h_2$ implies $h' \geq h_2$. We infer that $h' \geq h$. It then follows from (43) that, a.e. on A_1 ,

$$h'(x) = h(x).$$

From $h \geq h_1$, it follows that $G(h, y) \leq G(h_1, y)$. Moreover,

$$\int_{\mathcal{J}^2} g(x_0, y) \frac{\omega_2(y) dy}{G(h, y)} = h(x_0) = h_1(x_0) = \int_{\mathcal{J}^2} g(x_0, y) \frac{\omega_2(y) dy}{G(h_1, y)}.$$

Thus, a.e. on A_2 , we have

$$G(h, y) = G(h_1, y).$$

We conclude that everywhere on \mathcal{J}^1 we have $h = h_1$. Similarly, we get $h = h_2$ and, finally, $h_1 = h_2$ everywhere. \square

The following remark is on [25, p.104]:

Remark I. *All the results of this paper hold with minor modifications of the statements if one merely assumes that ω_1 and ω_2 are measurable integrable functions.*

Acknowledgments

The authors thank Robert V. Kohn for useful suggestions. The second named author would also like to thank the Courant Institute of Mathematical Sciences of the New York University for the hospitality during the time this paper was written.

A Proof of (32) from Theorem I

Let $Z \subset \mathfrak{J}^2$ be the set of $\{y \in \mathfrak{J}^2 | g(x_0, y) = 0\}$.

Define $Z_k = \{y \in \mathfrak{J}^2 | g(x_0, y) < \frac{1}{k}\}$ for $k \in \mathbb{N}^*$. We have $Z_{k+1} \subset Z_k$, and $Z_k \downarrow Z$ as $k \rightarrow +\infty$.

By assumption (H.vi) we know that Z has Lebesgue measure 0. From the continuity of g (H.iv), we also know that Z is closed.

Hence $m(Z_k) \rightarrow 0$ as $k \rightarrow +\infty$.

Denote by $\mathfrak{J}_k^2 = \mathfrak{J}^2 \setminus Z_k$. Then we have $\mathfrak{J}_k^2 \subset \mathfrak{J}_{k+1}^2$ and $\mathfrak{J}_k^2 \uparrow \mathfrak{J}^2 \setminus Z$ as $k \rightarrow +\infty$.

Since

$$H'_n(x_0) = \int_{\mathfrak{J}^2} g(x_0, y) \frac{\omega_2(y)}{G(H_n, y)} dy \rightarrow 0, \quad \text{as } n \rightarrow +\infty$$

$\forall \epsilon > 0$, we have for n large enough:

$$\int_{\mathfrak{J}^2} g(x_0, y) \frac{\omega_2(y)}{G(H_n, y)} dy < \epsilon$$

Fix $\epsilon > 0, k \in \mathbb{N}^*$. We then have for n large enough:

$$0 \leq \int_{\mathfrak{J}_k^2} g(x_0, y) \frac{\omega_2(y)}{G(H_n, y)} dy + \int_{\mathfrak{J}^2 \setminus \mathfrak{J}_k^2} g(x_0, y) \frac{\omega_2(y)}{G(H_n, y)} dy < \epsilon$$

and in particular, by non-negativity, the first integral yields:

$$0 \leq \int_{\mathfrak{J}_k^2} \frac{\omega_2(y)}{G(H_n, y)} dy < k\epsilon$$

This implies that the measure $\frac{\omega_2(y)}{G(H_n, y)} dy$ converges weakly to 0 on \mathfrak{J}_k^2 . Indeed, it is the case when evaluated on any step function with support included in \mathfrak{J}_k^2 , and step functions are dense in the family of bounded continuous functions.

We would like the measure $\frac{\omega_2(y)}{G(H_n, y)} dy$ to converge to 0 for any step function which support I is included in \mathfrak{J}^2 , and not merely on \mathfrak{J}_k^2 .

Pick a subset $I \subset \mathfrak{J}^2$, and consider:

$$\int_{\mathfrak{J}^2} \mathbb{1}_I(y) \frac{\omega_2(y)}{G(H_n, y)} dy = \int_{\mathfrak{J}_k^2 \cap I} \frac{\omega_2(y)}{G(H_n, y)} dy + \int_{(\mathfrak{J}_k^2 \cap I)^c} \frac{\omega_2(y)}{G(H_n, y)} dy$$

The first integral converges to 0 as $n \rightarrow +\infty$, since the measure $\frac{\omega_2(y)}{G(H_n, y)} dy$ converges weakly to 0 on \mathfrak{J}_k^2 .

As for the second integral, we have that $H_n \leq H_1$, so $\frac{\omega_2(y)}{G(H_n, y)} \leq \frac{\omega_2(y)}{G(H_1, y)}$ which implies:

$$\int_{(\mathfrak{J}_k^2 \cap I)^c} \frac{\omega_2(y)}{G(H_n, y)} dy \leq \int_{(\mathfrak{J}_k^2 \cap I)^c} \frac{\omega_2(y)}{G(H_1, y)} dy \leq \int_{Z_k} \frac{\omega_2(y)}{G(H_1, y)} dy$$

where the last inequality comes from $(\mathfrak{J}_k^2 \cap I)^c \subset Z_k$.

Condition (\star) states that $\int_{\mathfrak{J}^2} \frac{\omega_2(y)}{G(H_1, y)} dy < +\infty$, thus we know that the measure $\frac{\omega_2(y)}{G(H_1, y)} dy$ is absolutely continuous with respects to the Lebesgue measure m on \mathfrak{J}^2 . This implies that the second integral converges to 0, as $k \rightarrow +\infty$ since $m(Z_k) \rightarrow 0$.

Hence, for any $I \subset \mathfrak{J}^2$, $\int_{\mathfrak{J}^2} \mathbb{1}_I(y) \frac{\omega_2(y)}{G(H_n, y)} dy \rightarrow 0$ as $n \rightarrow +\infty$. Since step functions are dense in the family of bounded continuous functions, we conclude that the measure $\frac{\omega_2(y)}{G(H_n, y)} dy$ converges weakly to 0 on \mathfrak{J}^2 as $n \rightarrow +\infty$.

References

- [1] J. Benamou and Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numerische Mathematik*, **84**, no. 3, 2000, 375-393.
- [2] J. Benamou, G. Carlier, M. Cuturi, L. Nenna and G. Peyré, Iterative Bregman projections for regularized transportation problems, *SIAM Journal on Scientific Computing*, **37** (2), (2015), A1111-A1138.
- [3] A. Beurling, An automorphism of product measures, *Ann. Math.* **72** (1960), 189-200.
- [4] J. P. Burg, Maximum entropy spectral analysis, in *Proc.37th Meet.Society of Exploration Geophysicists*, 1967. Reprinted in *Modern Spectrum Analysis*, D. G. Childers, Ed. New York: IEEE Press, 1978. pp. 34-41.
- [5] J. Burg, D. Luenberger, and D. Wenger, Estimation of Structured Covariance Matrices, *Proceedings of the IEEE*, **70**, pp. 963-974, 1982.
- [6] Y. Chen, T.T. Georgiou and M. Pavon, Optimal steering of a linear stochastic system to a final probability distribution, Part I, *IEEE Trans. Aut. Control*, **61**, Issue 5, 1158-1169, 2016.
- [7] Y. Chen, T.T. Georgiou and M. Pavon, Optimal steering of a linear stochastic system to a final probability distribution, Part II, *IEEE Trans. Aut. Control*, **61**, Issue 5, 1170-1180, 2016.
- [8] Y. Chen, T.T. Georgiou and M. Pavon, Fast cooling for a system of stochastic oscillators, *J. Math. Phys.*, **56**, n.11, 113302, 2015.
- [9] Y. Chen, T.T. Georgiou and M. Pavon, On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint, *J. Optim. Theory and Applic.*, **169** (2), 671-691, 2016.
- [10] Y. Chen, T.T. Georgiou and M. Pavon, Entropic and displacement interpolation: a computational approach using the Hilbert metric, *SIAM Journal on Applied Mathematics*, **76** (6), 2375-2396, 2016.
- [11] Y. Chen, T.T. Georgiou and M. Pavon, Optimal transport over a linear dynamical system, *IEEE Trans. Aut. Control*, **62**, n. 5, 2137-2152, 2017.
- [12] L. Chizat, G. Peyr, B. Schmitzer and F.-X. Vialard, Scaling algorithms for unbalanced optimal transport problems, ArXiv e-prints, [arXiv:1607.05816v2](https://arxiv.org/abs/1607.05816v2), *Mathematics of Computation*, in press.
- [13] I. Csiszár, I-divergence geometry of probability distributions and minimization problems, *Annals of Probability*, **3**, pp. 146-158, 1975.
- [14] I. Csiszár, Sanov property, generalized I-projections, and a conditional limit theorem, *Annals of Probability*, **12**, pp. 768-793, 1984.
- [15] I. Csiszár, Why least squares and maximum entropy? An axiomatic approach to inference for linear inverse problems, *The Annals of Statistics*, **19**(4): 2032-2066, 1991.
- [16] M.Cuturi, Sinkhorn Distances: Lightspeed Computation of Optimal Transport, *Advances in Neural Information Processing Systems*, 2292-2300, 2013.
- [17] P. Dai Pra, A stochastic control approach to reciprocal diffusion processes, *Applied Mathematics and Optimization*, **23** (1), 1991, 313-329.
- [18] P.Dai Pra and M.Pavon, On the Markov processes of Schroedinger, the Feynman-Kac formula and stochastic control, in *Realization and Modeling in System Theory - Proc. 1989 MTNS Conf.*, M.A.Kaashoek, J.H. van Schuppen, A.C.M. Ran Eds., Birkaeuser, Boston, 1990, 497- 504.

- [19] W. E. Deming and F. F. Stephan, On a least squares adjustment of a sampled frequency table when the expected marginal totals are known, *Ann. Math. Statist.*, **11**, (1940), 427-444
- [20] A. P. Dempster, Covariance selection, *Biometrics*, **28**,157-175, 1972.
- [21] R. S. Ellis, Entropy, *Large deviations and statistical mechanics*, Springer-Verlag, New York, 1985.
- [22] H. Föllmer, in *Stochastic Processes - Mathematics and Physics* , Lecture Notes in Mathematics (Springer-Verlag, New York,1986), Vol. 1158, pp. 119-129.
- [23] H. Föllmer, Random fields and diffusion processes, in: *École d'Ètè de Probabilitès de Saint-Flour XV-XVII*, edited by P. L. Hennequin, Lecture Notes in Mathematics, Springer-Verlag, New York, 1988, vol.1362,102-203.
- [24] R. Fortet, Résolution d'un système d'equations de M. Schrödinger, *Comptes Rendus*, **206** (1938), 721-723.
- [25] R. Fortet, Résolution d'un système d'equations de M. Schrödinger, *J. Math. Pure Appl.* IX (1940), 83-105.
- [26] B. Jamison, The Markov processes of Schrödinger, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **32** (1975), 323-331.
- [27] E. T. Jaynes, Information Theory and Statistical Mechanics, *Physical Review Series II*, **106** (4): 62017630, 1957. doi:10.1103/PhysRev.106.620. MR87305, and Information Theory and Statistical Mechanics II, *Physical Review Series II*, **108** (2): 17117190, 1957. doi:10.1103/PhysRev.108.171. MR96414.
- [28] E. T. Jaynes. On the rationale of maximum-entropy methods. *Proceedings of the IEEE*, 70(9):939–952, Sept. 1982.
- [29] C. Léonard, From the Schrödinger problem to the Monge-Kantorovich problem, *J. Funct. Anal.*, 2012, **262**, 1879-1920.
- [30] C. Léonard, A survey of the Schroedinger problem and some of its connections with optimal transport, *Discrete Contin. Dyn. Syst. A*, 2014, **34** (4): 1533-1574.
- [31] T. Mikami, Monge's problem with a quadratic cost by the zero-noise limit of h-path processes, *Probab. Theory Relat. Fields*, **129**, (2004), 245-260.
- [32] T. Mikami and M. Thieullen, Duality theorem for the stochastic optimal control problem., *Stoch. Proc. Appl.*, 116, 1815-1835 (2006).
- [33] T. Mikami and M. Thieullen, Optimal Transportation Problem by Stochastic Optimal Control, *SIAM Journal of Control and Optimization*, **47**, N. 3, 1127-1139 (2008).
- [34] E. Nelson, *Dynamical Theories of Brownian Motion*, Princeton University Press, Princeton, 1967.
- [35] M.Pavon and A.Wakolbinger, On free energy, stochastic control, and Schroedinger processes, Modeling, Estimation and Control of Systems with Uncertainty, G.B. Di Masi, A.Gombani, A.Kurzanski Eds., Birkauer, Boston, 1991, 334-348.
- [36] I. S. Sanov, On the probability of large deviations of random magnitudes (in Russian), *Mat. Sb. N. S.*, **42** (84) (1957) 111744. Select. Transl. Math. Statist. Probab., 1, 213-244 (1961).
- [37] E. Schrödinger, Über die Umkehrung der Naturgesetze, *Sitzungsberichte der Preuss Akad. Wissen. Berlin, Phys. Math. Klasse* (1931), 144-153.

- [38] E. Schrödinger, Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique, *Ann. Inst. H. Poincaré* **2**, 269 (1932).
- [39] R. Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *Ann. Math. Statist.*, **35** (1964), 876-879.
- [40] C. Villani, *Topics in optimal transportation*, AMS, 2003, vol. 58.
- [41] A. Wakolbinger, Schrödinger Bridges from 1931 to 1991, in: E. Cabaña et al. (eds) , *Proc. of the 4th Latin American Congress in Probability and Mathematical Statistics*, Mexico City 1990, *Contribuciones en probabilidad y estadística matemática* 3 (1992) , 61-79.