ON THE MAXIMAL OPERATOR OF A GENERAL ORNSTEIN–UHLENBECK SEMIGROUP

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Abstract. If Q is a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts, we consider the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$ with covariance Q and drift matrix B. Our main result is that the associated maximal operator is of weak type $(1,1)$ with respect to the invariant measure. The proof has a geometric gist and hinges on the "forbidden zones method" previously introduced by the third author. For large values of the time parameter, we also prove a refinement of this result, in the spirit of a conjecture due to Talagrand.

1. Introduction

Let Q be a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts; here $n \geq 1$. We first introduce the covariance matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \qquad t \in (0, +\infty]. \tag{1.1}$$

Observe that both Q_t and Q_∞ are well defined, symmetric and positive definite. Then we define the family of normalized Gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(x) = (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_t^{-1} x, x\rangle} dx\,, \qquad t \in (0, +\infty].$$

On the space $\mathcal{C}_b(\mathbb{R}^n)$ of bounded continuous functions, we consider the Ornstein–Uhlenbeck semigroup $\left(\mathcal{H}_t\right)_{t>0}$, explicitly given by Kolmogorov's formula

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y)\,, \quad x \in \mathbb{R}^n\,. \tag{1.2}$$

The Gaussian measure γ_∞ is the unique invariant measure of the semigroup $\mathcal{H}_t$. We are interested in the maximal operator defined as

$$\mathcal{H}_* f(x) = \sup_{t>0} \left|\mathcal{H}_t f(x)\right|.$$

Under the above assumptions for B and Q, our main result will be the following.

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Theorem 1.1. *The Ornstein–Uhlenbeck maximal operator* $\mathcal{H}_*$ *is of weak type* $(1,1)$ *with respect to the invariant measure* γ_∞*, with an operator quasinorm that depends only on the dimension and the matrices* Q *and* B*.*

In other words, the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : \mathcal{H}_* f(x) > \alpha\} \leq \frac{C}{\alpha} \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0, \tag{1.3}$$

holds for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$.

The history of $\mathcal{H}_*$ is quite long and started with the first attempts to prove that $\mathcal{H}_*$ maps the L^p space into L^p. When $\left(\mathcal{H}_t\right)_{t>0}$ is symmetric, i.e., when each operator $\mathcal{H}_t$ is self-adjoint on $L^2(\gamma_\infty)$, then $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for $1 < p \leq \infty$, as a consequence of the general Littlewood–Paley–Stein theory for symmetric semigroups of contractions on L^p spaces [17, Ch. III].

It is easy to see that the maximal operator is unbounded on $L^1(\gamma_\infty)$. This led, about fifty years ago, to the study of the weak type $(1,1)$ of $\mathcal{H}_*$. The first positive result is due to B. Muckenhoupt [14], who proved an estimate like (1.3) in the one-dimensional case with $Q = I$ and $B = -I$. The analogous question in the higher-dimensional case was an open problem until 1983, when the third author [16] proved the weak type $(1,1)$ in any finite dimension. Other proofs are due to Menárguez, Pérez and Soria [12] (see also [11, 15]) and to Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea [8]. Moreover, a different proof of the weak type $(1,1)$ of $\mathcal{H}_*$, based on a covering lemma halfway between covering results by Besicovitch and Wiener, was given by Aimar, Forzani and Scotto [1].

In [4] the present authors recently considered a normal Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$, that is, we assumed that $\mathcal{H}_t$ is for each $t > 0$ a normal operator on $L^2(\gamma_\infty)$. Under this extra assumption, we proved that the associated maximal operator is of weak type $(1,1)$ with respect to the invariant measure γ_∞. This extends some earlier work in the non-symmetric framework by Mauceri and Noselli [10], who proved some ten years ago that, if $Q = I$ and $B = \lambda(R - I)$ for some positive λ and a real skew-symmetric matrix R generating a periodic group, then the maximal operator $\mathcal{H}_*$ is of weak type $(1,1)$.

In this paper we go beyond the hypothesis of normality, which underlies the results in [4] and [10]. In Theorem 1.1 we prove the estimate (1.3) under only the aforementioned spectral assumptions on B and Q. The proof has a geometric core and strongly relies on the *ad hoc* technique developed by the third author in [16].

Since the maximal operator $\mathcal{H}_*$ is trivially bounded from L^∞ to L^∞, we obtain by interpolation the following corollary.

Corollary 1.2. *The Ornstein–Uhlenbeck maximal operator* $\mathcal{H}_*$ *is bounded on* $L^p(\gamma_\infty)$ *for all* $p > 1$*.*

This result improves Theorem 4.2 in [10], where the L^p boundedness of $\mathcal{H}_*$ is proved for all $p > 1$ in the normal framework and under the additional assumption that the infinitesimal generator of $\left(\mathcal{H}_t\right)_{t>0}$ is a sectorial operator of angle less than $\pi/2$.

A question related to the Ornstein–Uhlenbeck semigroup and the weak type $(1,1)$ inequality was recently addressed by Ball, Barthe, Bednorz, Oleszkiewicz and Wolff [2]. Inspired by a conjecture formulated by Talagrand in a slightly different context [18], they conjectured the following, in the standard case $Q = I$ and $B = -I$: For each fixed $t > 0$, there exists a function $\psi_t = \psi_t(\alpha)$, satisfying

$$\lim_{\alpha \to +\infty} \psi_t(\alpha) = 0$$

and

$$\gamma_\infty\{x \in \mathbb{R}^n : |\mathcal{H}_t f(x)| > \alpha\} \le C \, \frac{\psi_t(\alpha)}{\alpha} \tag{1.4}$$

for all large $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$. In [2] this conjecture is proved with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$ in dimension 1 and with $\psi_t(\alpha) = C(n,t) \log\log \alpha/\sqrt{\log \alpha}$ as $n > 1$; in the latter case the constant tends to ∞ with the dimension. Then Eldan and Lee [6] improved the result in [2] for $n > 1$, proving (1.4) with $\psi_t(\alpha) = C(t)\,(\log\log \alpha)^4/\sqrt{\log \alpha}$, where the constant $C(t)$ is independent of the dimension. Finally Lehec [9], revisiting the argument in [6], proved the conjecture in any dimension with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$, which turns out to be sharp. All the results in [2, 6, 9] are established for $Q = I$ and $B = -I$.

In analogy with these results, we prove in Proposition 6.1 that the maximal operator with t large, associated to a general Ornstein–Uhlenbeck semigroup, satisfies

$$\gamma_\infty \left\{ x \in \mathbb{R}^n : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \le C \, \frac{\psi(\alpha)}{\alpha} \tag{1.5}$$

for $\alpha > 0$ large and for all normalized functions $f \in L^1(\gamma_\infty)$. Here $\psi(\alpha) = 1/\sqrt{\log \alpha}$ and $C = C(n, Q, B)$, and this estimate is shown to be sharp. It cannot be extended to $\mathcal{H}_*$, since the maximal operator corresponding to small values of t only satisfies an inequality with $\psi(\alpha) = 1$.

In this paper we focus our attention on the Ornstein–Uhlenbeck maximal function in $\mathbb{R}^n$. In view of possible applications to stochastic analysis and to SPDE's, it would be very interesting to investigate the case of the infinite-dimensional Ornstein-Uhlenbeck maximal operator as well (see [5, 19, 3] for an introduction to the infinite-dimensional setting). The Riesz transforms associated to a general Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$ will be considered in a forthcoming paper.

The scheme of the paper is as follows. In Section 2 we introduce the Mehler kernel $K_t(x,u)$, that is, the integral kernel of $\mathcal{H}_t$. Some estimates for the norm and the determinant of Q_t and related matrices are provided in Section 3. As a consequence, we obtain precise bounds for the Mehler kernel. In Section 4 we consider the relevant geometric features of the problem; in particular, we introduce in Subsection 4.1 a system of polar-like coordinates. We also express Lebesgue measure in terms of these coordinates. Sections 5, 6, 7 and 8 are devoted to the proof of Theorem 1.1. First, Section 5 introduces some preliminary simplifications of the proof; in particular, we reduce most of the problem to an ellipsoidal annulus. In Section 6 we consider the supremum in the definition of the maximal operator taken only over $t > 1$ and prove the sharpened version (1.5) of (1.3). Section 7 is devoted to the case of small t under an additional local condition. Finally, in Section 8 we treat the remaining case and conclude the proof of Theorem 1.1, by proving the estimate (1.3) for small t under a global assumption.

In the following, we use the "variable constant convention", according to which the symbols $c > 0$ and $C < \infty$ will denote constants which are not necessarily equal at different occurrences. They all depend only on the dimension and on Q and B. For any two nonnegative quantities a and b we write $a \lesssim b$ instead of $a \le Cb$ and $a \gtrsim b$ instead of $a \ge cb$. The symbol $a \simeq b$ means that both $a \lesssim b$ and $a \gtrsim b$ hold.

By $\mathbb{N}$ we mean the set of all nonnegative integers. If A is an $n \times n$ matrix, we write $\|A\|$ for its operator norm on $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$.

2. The Mehler Kernel

For $t > 0$, the difference

$$Q_\infty - Q_t = \int_t^\infty e^{sB} Q e^{sB^*} ds \tag{2.1}$$

is a symmetric and strictly positive definite matrix. So is the matrix

$$Q_t^{-1} - Q_\infty^{-1} = Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}, \tag{2.2}$$

and we can define

$$D_t = (Q_t^{-1} - Q_\infty^{-1})^{-1} Q_t^{-1} e^{tB}. \tag{2.3}$$

Then formula (1.2), the definition of the Gaussian measure and some elementary computations yield

$$\begin{aligned}
\mathcal{H}_t f(x) &= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(e^{tB}x - y) \exp\left[-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right] dy \\
&= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(u) \exp\left[-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x - u), e^{tB}x - u\rangle\right] du \\
&= \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\langle Q_t^{-1} e^{tB}x, e^{tB}x\rangle\right] \\
&\qquad \times \exp\left[-\frac{1}{2}\langle Q_t^{-1} e^{tB}x, (Q_\infty^{-1} - Q_t^{-1})^{-1} Q_t^{-1} e^{tB}x\rangle\right] \\
&\qquad\qquad \times \int f(u) \exp\left[\frac{1}{2}\langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t x)\,,\, u - D_t x\rangle\right] d\gamma_\infty(u),
\end{aligned}$$

that is,

$$\begin{aligned}
\mathcal{H}_t^{Q,B} f(x) = \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} &\exp\left[\frac{1}{2}\langle Q_t^{-1} e^{tB}x, D_t x - e^{tB}x\rangle\right] \\
&\times \int f(u) \exp\left[\frac{1}{2}\langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t x)\,,\, u - D_t x\rangle\right] d\gamma_\infty(u)\,,
\end{aligned} \tag{2.4}$$

where we repeatedly used the fact that $Q_\infty^{-1} - Q_t^{-1}$ is symmetric. We now express the matrix D_t in various ways.

Lemma 2.1. *For all $x \in \mathbb{R}^n$ and $t > 0$ we have*
(i) $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$;
(ii) $D_t = e^{tB} + Q_t e^{-tB^*} Q_\infty^{-1}$.

Proof. (i) Formulae (2.1) and (1.1) imply

$$Q_\infty - Q_t = e^{tB} Q_\infty e^{tB^*} \tag{2.5}$$

(see also [13, formula (2.1)]). From (2.3) and (2.2) it follows that

$$D_t = Q_\infty (Q_\infty - Q_t)^{-1} e^{tB},$$

and combining this with (2.5) we arrive at (i).
(ii) Multiplying (2.5) by $e^{-tB^*} Q_\infty^{-1}$ from the right, we obtain

$$Q_\infty e^{-tB^*} Q_\infty^{-1} - Q_t e^{-tB^*} Q_\infty^{-1} = e^{tB},$$

and (ii) now follows from (i). □

By means of (i) in this lemma, we can define D_t for all $t \in \mathbb{R}$, and they will form a one-parameter group of matrices.

Now (ii) in Lemma 2.1 yields

$$\langle Q_t^{-1} e^{tB} x, D_t x - e^{tB} x\rangle = \langle Q_t^{-1} e^{tB} x, Q_t e^{-tB^*} Q_\infty^{-1} x\rangle = \langle Q_\infty^{-1} x, x\rangle.$$

Thus (2.4) may be rewritten as

$$\mathcal{H}_t f(x) = \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where K_t denotes the Mehler kernel, given by

$$K_t(x,u) = \Big(\frac{\det Q_\infty}{\det Q_t}\Big)^{1/2} \exp\big(R(x)\big) \\ \times \exp\Big[-\frac{1}{2}\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x)\,,\, u - D_t x\big\rangle\Big] \tag{2.6}$$

for $x, u \in \mathbb{R}^n$. Here we introduced the quadratic form

$$R(x) = \frac{1}{2}\big\langle Q_\infty^{-1} x, x\big\rangle, \qquad x \in \mathbb{R}^n.$$

3. Some auxiliary results

In this section we collect some preliminary bounds, which will be essential ingredients in the proof of the weak type $(1,1)$ for the maximal operator $\mathcal{H}_*$.

Lemma 3.1. *For $s > 0$ the matrices D_s and $D_{-s} = D_s^{-1}$ satisfy*

$$\|D_s\| \lesssim e^{Cs} \qquad \textit{and} \qquad \|D_{-s}\| \lesssim e^{-cs}. \tag{3.1}$$

Proof. First we prove estimates for $\|e^{sB^*}\|$ and $\|e^{-sB^*}\|$. They can be obtained by means of a Jordan decomposition of sB^*, that is, writing sB^* as the sum of a complex diagonal matrix and a triangular, nilpotent matrix, and these two terms will commute. Another possibility is to use standard theory of strongly continuous semigroups, see [7, Theorem 3.14 and Theorem 5.5 in Chapter 1]. Both arguments rely on the fact that the eigenvalues of B have negative real parts. The result will be

$$\|e^{-sB^*}\| \lesssim e^{Cs} \qquad \text{and} \qquad \|e^{sB^*}\| \lesssim e^{-cs}, \qquad s > 0. \tag{3.2}$$

Finally, (3.2) implies (3.1) for $D_s = Q_\infty e^{-sB^*} Q_\infty^{-1}$ and $D_{-s} = Q_\infty e^{sB^*} Q_\infty^{-1}$. □

In the following lemma, we collect estimates of some basic quantities related to the matrices Q_t.

Lemma 3.2. *For all $t > 0$ we have*

(i) $\det Q_t \simeq (\min(1,t))^n$;
(ii) $\|Q_t^{-1}\| \simeq (\min(1,t))^{-1}$;
(iii) $\|Q_\infty - Q_t\| \lesssim e^{-ct}$;
(iv) $\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1}\, e^{-ct}$;
(v) $\| \left(Q_t^{-1} - Q_\infty^{-1}\right)^{-1/2} \| \lesssim t^{1/2}\, e^{Ct}$.

Proof. (i) and (ii) Using (3.2), we see that for each $t > 0$ and for all $v \in \mathbb{R}^n$

$$\begin{aligned}
\langle Q_t v, v\rangle &= \left\langle \int_0^t e^{sB} Q e^{sB^*} v\, ds, v \right\rangle = \int_0^t \langle Q^{1/2} e^{sB^*} v, Q^{1/2} e^{sB^*} v\rangle ds \\
&= \int_0^t \left|Q^{1/2} e^{sB^*} v\right|^2 ds \simeq \int_0^t \left|e^{sB^*} v\right|^2 ds \\
&\lesssim \int_0^t e^{-cs} ds\, |v|^2 \simeq \min(1,t)\, |v|^2.
\end{aligned}$$

Since $\| \left(e^{sB^*}\right)^{-1} \| = \|e^{-sB^*}\| \lesssim e^{Cs}$, there is also a lower estimate

$$\int_0^t \left|e^{sB^*} v\right|^2 ds \gtrsim \int_0^t e^{-Cs} ds\, |v|^2 \simeq \min(1,t)|v|^2.$$

Thus any eigenvalue of Q_t has order of magnitude $\min(1,t)$, and (i) and (ii) follow.
(iii) From the definition of Q_t and (3.2), we get

$$\|Q_\infty - Q_t\| = \left\| \int_t^\infty e^{sB} Q e^{sB^*} ds \right\| \lesssim e^{-ct}.$$

(iv) Using now (ii) and (iii), we have

$$\begin{aligned}
\|Q_t^{-1} - Q_\infty^{-1}\| = \|Q_t^{-1}(Q_\infty - Q_t) Q_\infty^{-1}\| &\lesssim \|Q_t^{-1}\|\, \|Q_\infty - Q_t\| \\
&\lesssim (\min(1,t))^{-1}\, e^{-ct} \lesssim t^{-1}\, e^{-ct}.
\end{aligned}$$

(v) Since $\|A^{1/2}\| = \|A\|^{1/2}$ for any symmetric positive definite matrix A, we consider $(Q_t^{-1} - Q_\infty^{-1})^{-1}$, which can be rewritten as

$$(Q_t^{-1} - Q_\infty^{-1})^{-1} = (Q_\infty^{-1}(Q_\infty - Q_t) Q_t^{-1})^{-1} = Q_t (Q_\infty - Q_t)^{-1} Q_\infty. \tag{3.3}$$

It follows from (2.5) that $(Q_\infty - Q_t)^{-1} = e^{-tB^*} Q_\infty^{-1} e^{-tB}$, so that

$$\|(Q_\infty - Q_t)^{-1}\| \lesssim e^{Ct},$$

as a consequence of (3.2). Inserting this and the simple estimate $\|Q_t\| \lesssim t$ in (3.3), we obtain $\|(Q_t^{-1} - Q_\infty^{-1})^{-1}\| \lesssim t e^{Ct}$, and (v) follows. □

Proposition 3.3. *For $t \geq 1$ and $w \in \mathbb{R}^n$, we have*

$$\langle (Q_t^{-1} - Q_\infty^{-1}) D_t w,\, D_t w\rangle \simeq |w|^2.$$

Proof. By (2.3) and Lemma 2.1 (i) we have

$$\begin{aligned}\langle (Q_t^{-1} - Q_\infty^{-1}) D_t w,\, D_t w\rangle =& \langle Q_t^{-1} e^{tB} w\,,\, Q_\infty e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ =& \langle Q_\infty Q_t^{-1} e^{tB} w\,,\, e^{-tB^*} Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Since $Q_\infty Q_t^{-1} = I + (Q_\infty - Q_t) Q_t^{-1}$, this leads to

$$\begin{aligned}&\langle (Q_t^{-1} - Q_\infty^{-1}) D_t w,\, D_t w\rangle \\ &\quad = \langle e^{tB} w\,,\, e^{-tB^*} Q_\infty^{-1}\, w\rangle + \langle (Q_\infty - Q_t) Q_t^{-1} e^{tB} w\,,\, e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ &\qquad = \langle Q_\infty^{-1} w, w\rangle + \langle e^{-tB} (Q_\infty - Q_t) Q_t^{-1} e^{tB} w\,,\, Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Using (2.1) and the definition of Q_∞, we observe that the last term here can be written as

$$\begin{aligned}&\left\langle \int_t^\infty e^{(s-t)B} Q e^{(s-t)B^*}\, ds\, e^{tB^*}\, Q_t^{-1} e^{tB} w\,,\, Q_\infty^{-1}\, w \right\rangle \\ &= \big\langle Q_\infty\, e^{tB^*}\, Q_t^{-1} e^{tB} w\ ,\ Q_\infty^{-1}\, w \big\rangle \\ &= \big\langle\, e^{tB^*}\, Q_t^{-1} e^{tB} w\ ,\ w \big\rangle \\ &= \ \big| Q_t^{-1/2} e^{tB} w \big|^2.\end{aligned}$$

Since $\big|Q_t^{-1/2} e^{tB} w\big|^2 \lesssim e^{-ct}|w|^2$ for $t \geq 1$, the claim of the proposition follows if t is large enough. In the opposite case $1 < t < C$, we apply Lemma 3.2 (v) to conclude that

$$\langle (Q_t^{-1} - Q_\infty^{-1}) D_t w,\, D_t w\rangle \gtrsim e^{-Ct} |D_t w|^2 \sim |w|^2.$$

The converse inequality is clear, and the claim follows again. □

We can now write the estimates for the kernel K_t which we will use later. If $t > 1$, we combine (2.6) with Proposition 3.3 and write $u - D_t\, x = D_t(D_{-t}\, u - x)$. Because of Lemma 3.2 (i), the result will be

$$\begin{aligned}&\exp(R(x))\, \exp\big(-C\, \big|D_{-t} u - x\big|^2\big) \\ &\qquad \lesssim K_t(x, u) \lesssim \exp(R(x))\, \exp\big(-c\, \big|D_{-t}\, u - x\big|^2\big), \qquad t > 1. \quad (3.4)\end{aligned}$$

For $t \leq 1$ we use Lemma 3.2 (v) to see that

$$\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x)\,,\, u - D_t x\rangle = |(Q_t^{-1} - Q_\infty^{-1})^{1/2}(u - D_t x)|^2 \gtrsim t^{-1} |u - D_t x|^2,$$

Then (2.6) and Lemma 3.2 (i) imply

$$K_t(x,u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c\,\frac{|u-D_t x|^2}{t}\right), \qquad t \le 1. \tag{3.5}$$

4. Geometric aspects of the problem

4.1. **A system of adapted polar coordinates.** We first need a technical lemma.

Lemma 4.1. *For all x in $\mathbb{R}^n$ and $s \in \mathbb{R}$, we have*

$$\langle B^* Q_\infty^{-1} x, x\rangle = -\frac{1}{2}\,|Q^{1/2}\, Q_\infty^{-1} x|^2; \tag{4.1}$$

$$\frac{\partial}{\partial s} D_s x = -Q_\infty B^*\, Q_\infty^{-1} D_s x = -Q_\infty e^{-sB^*} B^* Q_\infty^{-1} x; \tag{4.2}$$

$$\frac{\partial}{\partial s} R\big(D_s x\big) = \frac{1}{2}\,\big|Q^{1/2} Q_\infty^{-1} D_s x\big|^2 \simeq \big|D_s x\big|^2. \tag{4.3}$$

Proof. To prove (4.1), we use the definition of Q_∞ to write for any $z \in \mathbb{R}^n$

$$\begin{aligned}
\langle B^* z, Q_\infty z\rangle &= \int_0^\infty \langle B^* z, e^{sB}\, Q\, e^{sB^*} z\rangle\, ds \\
&= \int_0^\infty \langle e^{sB^*}\, B^* z,\, Q\, e^{sB^*} z\rangle\, ds \\
&= \frac{1}{2} \int_0^\infty \frac{d}{ds} \langle e^{sB^*}\, z,\, Q\, e^{sB^*} z\rangle\, ds \\
&= -\frac{1}{2}\, |Q^{1/2}\, z|^2.
\end{aligned}$$

Setting $z = Q_\infty^{-1} x$, we get (4.1).

Further, (4.2) easily follows if we observe that

$$\frac{\partial}{\partial s} D_s x = \frac{\partial}{\partial s}\big(Q_\infty e^{-sB^*} Q_\infty^{-1} x\big) = -Q_\infty B^*\, Q_\infty^{-1} Q_\infty e^{-sB^*} Q_\infty^{-1} x = -Q_\infty B^*\, Q_\infty^{-1} D_s x.$$

Finally, we get by means of (4.2) and (4.1)

$$\begin{aligned}
\frac{\partial}{\partial s} R\,(D_s x) &= \frac{1}{2}\frac{\partial}{\partial s} \langle Q_\infty^{-1/2} D_s x, Q_\infty^{-1/2} D_s x\rangle \\
&= -\langle Q_\infty^{-1/2} Q_\infty B^* Q_\infty^{-1} D_s x, Q_\infty^{-1/2} D_s x\rangle \\
&= \frac{1}{2}\big|Q^{1/2} Q_\infty^{-1} D_s x\big|^2,
\end{aligned}$$

and (4.3) is verified. □

Fix now $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n :\, R(x) = \beta\}\,.$$

As a consequence of (4.3), the map $s \mapsto R(D_s z)$ is strictly increasing for each $0 \neq z \in \mathbb{R}^n$. Hence any $x \in \mathbb{R}^n$, $x \neq 0$, can be written uniquely as

$$x = D_s \tilde{x}\,, \tag{4.4}$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We consider s and $\tilde{x}$ as the polar coordinates of x. Our estimates in what follows will be uniform in β.

Next, we write Lebesgue measure in terms of these polar coordinates. A normal vector to the surface E_β at the point $\tilde{x} \in E_\beta$ is $\mathbf{N}(\tilde{x}) = Q_\infty^{-1}\tilde{x}$, and the tangent hyperplane at $\tilde{x}$ is $\mathbf{N}(\tilde{x})^\perp$. For $s > 0$ the tangent hyperplane of the surface $D_s E_\beta = \{D_s\tilde{x} : \tilde{x} \in E_\beta\}$ at the point $D_s\tilde{x}$ is $D_s(\mathbf{N}(\tilde{x})^\perp)$, and a normal to $D_s E_\beta$ at the same point is $w = (D_s^{-1})^*(\mathbf{N}(\tilde{x})) = D_{-s}^* Q_\infty^{-1}\tilde{x} = Q_\infty^{-1} e^{sB}\tilde{x}$.

The scalar product of w and the tangent of the curve $s \mapsto D_s\tilde{x}$ at the point $D_s\tilde{x}$ is, because of (4.2) and (4.1),

$$\left\langle \frac{\partial}{\partial s} D_s\tilde{x}, w \right\rangle \tag{4.5}$$
$$= -\langle Q_\infty e^{-sB^*} B^* Q_\infty^{-1}\tilde{x},\, Q_\infty^{-1} e^{sB}\tilde{x}\rangle = -\langle B^* Q_\infty^{-1}\tilde{x}, \tilde{x}\rangle = \frac{1}{2}\,|Q^{1/2}\,Q_\infty^{-1}\tilde{x}|^2 > 0.$$

Thus the curve $s \mapsto D_s\tilde{x}$ is transversal to each surface $D_s E_\beta$. Let dS_s denote the area measure of $D_s E_\beta$. Then Lebesgue measure is given in terms of our polar coordinates by

$$dx = H(s,\tilde{x})\, dS_s(D_s\tilde{x})\, ds, \tag{4.6}$$

where

$$H(s,\tilde{x}) = \left\langle \frac{\partial}{\partial s} D_s\tilde{x}, \frac{w}{|w|} \right\rangle = \frac{|Q^{1/2}\,Q_\infty^{-1}\tilde{x}|^2}{2\,|Q_\infty^{-1} e^{sB}\tilde{x}|}.$$

To see how dS_s varies with s, we take a continuous function $\varphi = \varphi(\tilde{x})$ on E_β and extend it to $\mathbb{R}^n \setminus \{0\}$ by writing $\varphi(D_s\tilde{x}) = \varphi(\tilde{x})$. For any $t > 0$ and small $\varepsilon > 0$, we define the shell

$$\Omega_{t,\varepsilon} = \{D_s\tilde{x} : t < s < t + \varepsilon,\ \tilde{x} \in E_\beta\}.$$

Then $\Omega_{t,\varepsilon}$ is the image under D_t of $\Omega_{0,\varepsilon}$, and the Jacobian of this map is $\det D_t = e^{-t\,\mathrm{tr}\,B}$. Thus

$$\int_{\Omega_{t,\varepsilon}} \varphi(x)\, dx = e^{-t\,\mathrm{tr}\,B} \int_{\Omega_{0,\varepsilon}} \varphi(D_t x)\, dx,$$

which we can rewrite as

$$\int_{t<s<t+\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s,\tilde{x})\, dS_s(D_s\tilde{x})\, ds$$
$$= e^{-t\,\mathrm{tr}\,B} \int_{0<s<\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s,\tilde{x})\, dS_s(D_s\tilde{x})\, ds.$$

Now we divide by ε and let $\varepsilon \to 0$, getting

$$\int_{E_\beta} \varphi(\tilde{x})\, H(t,\tilde{x})\, dS_t(D_t\tilde{x}) = e^{-t\,\mathrm{tr}\,B} \int_{E_\beta} \varphi(\tilde{x})\, H(0,\tilde{x})\, dS_0(\tilde{x}).$$

Since this holds for any φ, it follows that

$$dS_t(D_t\tilde{x}) = e^{-t\,\mathrm{tr}\,B}\, \frac{H(0,\tilde{x})}{H(t,\tilde{x})}\, dS_0(\tilde{x}).$$

Together with (4.6), this implies the following result.

Proposition 4.2. *The Lebesgue measure in* $\mathbb{R}^n$ *is given in terms of polar coordinates* $(t, \tilde{x})$ *by*

$$dx = e^{-t\,\mathrm{tr}\,B}\, \frac{|Q^{1/2}\, Q_\infty^{-1} \tilde{x}|^2}{2\, |Q_\infty^{-1} \tilde{x}|}\, dS_0(\tilde{x})\, dt\,.$$

We also need estimates of the distance between two points in terms of the polar coordinates.

Lemma 4.3. *Fix* $\beta > 0$. *Let* $x^{(0)},\ x^{(1)} \in \mathbb{R}^n \setminus \{0\}$ *and assume* $R(x^{(0)}) > \beta/2$. *Write*

$$x^{(0)} = D_{s^{(0)}}(\tilde{x}^{(0)}) \qquad \text{and} \qquad x^{(1)} = D_{s^{(1)}}(\tilde{x}^{(1)})$$

with $s^{(0)},\ s^{(1)} \in \mathbb{R}$ *and* $\tilde{x}^{(0)},\ \tilde{x}^{(1)} \in E_\beta$.

(i) *Then*

$$\big|x^{(0)} - x^{(1)}\big| \gtrsim c\, \big|\tilde{x}^{(0)} - \tilde{x}^{(1)}\big|. \tag{4.7}$$

(ii) *If also* $s^{(1)} \ge 0$, *then*

$$\big|x^{(0)} - x^{(1)}\big| \gtrsim c\, \sqrt{\beta}\, |s^{(0)} - s^{(1)}|. \tag{4.8}$$

Proof. Let $\Gamma : [0, 1] \to \mathbb{R}^n \setminus \{0\}$ be a differentiable curve with $\Gamma(0) = x^{(0)}$ and $\Gamma(1) = x^{(1)}$. It suffices to bound the length of any such curve from below by the right-hand sides of (4.7) and (4.8).

For each $\tau \in [0, 1]$, we write

$$\Gamma(\tau) = D_{s(\tau)}\, \tilde{x}(\tau),$$

with $\tilde{x}(\tau) \in E_\beta$ and $\tilde{x}(i) = \tilde{x}^{(i)}$, $s(i) = s^{(i)}$ for $i = 0, 1$. Thus

$$\Gamma'(\tau) \;\; = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=s(\tau)}\, \tilde{x}(\tau) + D_{s(\tau)} \tilde{x}'(\tau).$$

The group property of D_s implies that

$$\frac{\partial}{\partial s} D_s\Big|_{s=s(\tau)} = D_{s(\tau)} \frac{\partial}{\partial s} D_s\Big|_{s=0},$$

and so

$$\Gamma'(\tau) \;\; = D_{s(\tau)} v,$$

with

$$v \;\; = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=0}\, \tilde{x}(\tau) + \tilde{x}'(\tau).$$

The vector $\tilde{x}'(\tau)$ is tangent to E_β and so orthogonal to $\mathbf{N}(\tilde{x})$. Then (4.5) (with $s = 0$) and the triangle inequality on the unit sphere imply that the angle between $\frac{\partial}{\partial s} D_s\big|_{s=0} \tilde{x}(\tau)$ and $\tilde{x}'(\tau)$ is larger than some positive constant. It follows that

$$|v|^2 \gtrsim |s'(\tau)|^2 \left|\frac{\partial}{\partial s}\, D_s\Big|_{s=0}\, \tilde{x}(\tau)\right|^2 + \big|\tilde{x}'(\tau)\big|^2 \gtrsim |s'(\tau)|^2\, \beta + \big|\tilde{x}'(\tau)\big|^2, \tag{4.10}$$

where we also used the fact that, by (4.2),

$$\Big|\frac{\partial}{\partial s}\,D_s\Big|_{s=0}\,\tilde x(\tau)\Big| \simeq |\tilde x(\tau)| \simeq \sqrt{\beta}.$$

Since

$$|v| \;= \big|D_{-s(\tau)}\Gamma'(\tau)\big| \le \big\|D_{-s(\tau)}\big\|\,\big|\Gamma'(\tau)\big| \lesssim e^{-C\min(s(\tau),0)}\big|\Gamma'(\tau)\big|$$

because of Lemma 3.1, we obtain from (4.10)

$$\big|\Gamma'(\tau)\big| \gtrsim e^{C\min(s(\tau),0)}\,\big(\sqrt{\beta}\,|s'(\tau)| + \big|\tilde x'(\tau)\big|\big). \tag{4.11}$$

Next, we derive a lower bound for $s(0)$; assume first that $s(0) < 0$. The assumption $R(x^{(0)}) > \beta/2$ implies, together with Lemma 3.1,

$$\beta/2 \le R(D_{s(0)}\,\tilde x^{(0)}) \lesssim \big|D_{s(0)}\,\tilde x^{(0)}\big|^2 \lesssim e^{c\,s(0)}\big|\tilde x^{(0)}\big|^2 \simeq e^{c\,s(0)}\beta.$$

It follows that

$$s(0) > -\tilde s,$$

for some $\tilde s$ with $0 < \tilde s < C$, and this obviously holds also without the assumption $s(0) < 0$.

Assume now that $s(\tau) > -2\tilde s$ for all $\tau \in [0,1]$. Then (4.11) implies

$$\big|\Gamma'(\tau)\big| \gtrsim \sqrt{\beta}\,|s'(\tau)|$$

and

$$\big|\Gamma'(\tau)\big| \gtrsim |\tilde x'(\tau)|.$$

Integrating these estimates with respect to τ in $[0,1]$, we immediately see that the length of Γ is bounded below by the right-hand sides of (4.7) and (4.8).

If instead $s(\tau) \le -2\tilde s$ for some $\tau \in [0,1]$, we can proceed as in the proof of Lemma 4.2 in [4]. More precisely, since the image $s([0,1])$ contains the interval $[-2\tilde s, \max(s(0), s(1))]$, we can find a closed subinterval I of $[0,1]$ whose image $s(I)$ is exactly the interval $[-2\tilde s, \max(s(0), s(1))]$. Thus we may control the length of Γ, in the light of (4.11), by

$$\int_0^1 \big|\Gamma'(\tau)\big|\,d\tau \ge \int_I \big|\Gamma'(\tau)\big|\,d\tau \gtrsim \sqrt{\beta}\,\int_I |s'(\tau)|\,d\tau \ge \sqrt{\beta}\,\big(\max{(s(0),s(1))} \;+\; 2\tilde s\big).$$

Here

$$\sqrt{\beta}\,\big(\max{(s(0),s(1))} \;+\; 2\tilde s\big) \gtrsim \sqrt{\beta} \gtrsim \operatorname{diam} E_\beta \ge \big|\tilde x^{(0)} - \tilde x^{(1)}\big|,$$

and (4.7) follows. Under the additional hypotheses of (b), we have

$$\max{(s(0),s(1))} \;+\; 2\tilde s \ge |s(0) - s(1)|,$$

which implies (4.8). □

4.2. **The Gaussian measure of a tube.** We fix a large $\beta > 0$. Define for $x^{(1)} \in E_\beta$ and $a > 0$ the set

$$\Omega = \left\{x \in E_\beta : \left|x - x^{(1)}\right| < a\right\}.$$

This is a spherical cap of the ellipsoid E_β, centered at $x^{(1)}$. Observe that $|x| \simeq \sqrt{\beta}$ for $x \in \Omega$, and that the area of Ω is $|\Omega| \simeq \min\left(a^{n-1}, \beta^{(n-1)/2}\right)$. Then consider the tube

$$Z = \{D_s \tilde{x} : s \geq 0,\ \tilde{x} \in \Omega\}. \tag{4.12}$$

Lemma 4.4. *There exists a constant C such that $\beta > C$ implies that the Gaussian measure of the tube Z fulfills*

$$\gamma_\infty(Z) \lesssim \frac{a^{n-1}}{\sqrt{\beta}}\, e^{-\beta}.$$

Proof. Proposition 4.2 yields, since $H(0, \tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\beta}$,

$$\gamma_\infty(Z) \simeq \int_0^\infty e^{-s \operatorname{tr} B}\, e^{-R(D_s\tilde{x})} \int_\Omega H(0, \tilde{x})\, dS(\tilde{x})\, ds \lesssim \sqrt{\beta}\, a^{n-1} \int_0^\infty e^{-s \operatorname{tr} B} e^{-R(D_s\tilde{x})}\, ds.$$

By (4.3) we have

$$R(D_s\tilde{x}) - R(\tilde{x}) \simeq \int_0^s \left|D_{s'}\tilde{x}\right|^2 ds \gtrsim s|\tilde{x}|^2 \simeq s\beta,$$

which implies

$$\gamma_\infty(Z) \lesssim \sqrt{\beta}\ a^{n-1}\, e^{-\beta} \int_0^\infty e^{-s \operatorname{tr} B}\, e^{-cs\beta}\ ds.$$

Assuming β large enough, one has $c\beta > -2 \operatorname{tr} B$, and then the last integral is finite and no larger than C/β. The lemma follows. □

5. Some simplifications

In this section, we introduce some preliminary simplifications and reductions in the proof of (1.3), i.e., of Theorem 1.1.

(1) We may assume that f is nonnegative and normalized in the sense that

$$\|f\|_{L^1(\gamma_\infty)} = 1,$$

since this involves no loss of generality.

(2) We may assume that our fixed α is large, $\alpha > C$, since otherwise (1.3) is trivial.

(3) In many cases, we may restrict x in (1.3) to the ellipsoidal annulus

$$\mathcal{E} = \left\{x \in \mathbb{R}^n : \frac{1}{2}\log\alpha \leq R(x) \leq 2\log\alpha\right\}.$$

To begin with, we can always forget the unbounded component of the complement of $\mathcal{E}$, since

$$\gamma_\infty\{x \in \mathbb{R}^n : R(x) > 2\log\alpha\} \tag{5.1}$$

$$\lesssim \int_{R(x)>2\log\alpha} \exp(-R(x))dx \lesssim (2\log\alpha)^{(n-2)/2} \exp(-2\log\alpha) \lesssim \frac{1}{\alpha}.$$

(4) When $t > 1$, we may forget also the inner region where $R(x) < \frac{1}{2}\log\alpha$. Indeed, from (3.4) we get, if $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ with $R(x) < \frac{1}{2}\log\alpha$,

$$K_t(x,u) \lesssim e^{R(x)} < \sqrt{\alpha} \leq \alpha,$$

since α is large. In other words, for any $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$

$$R(x) < \frac{1}{2}\log\alpha \qquad \Rightarrow \qquad K_t(x,u) \lesssim \alpha, \tag{5.2}$$

for all $t > 1$.
Replacing α by $C\alpha$ for some C, we see from (5.1) and (5.2) that we can assume $x \in \mathcal{E}$ in the proof of (1.3), when the supremum of the maximal operator is taken only over $t > 1$.

Before introducing the last simplification, we need to define a global region

$$G = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |x-u| > \frac{1}{1+|x|} \right\}$$

and a local region

$$L = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |x-u| \leq \frac{1}{1+|x|} \right\}.$$

(5) When $t \leq 1$ and $(x,u) \in G$, we shall see that (5.2) is still valid, and it is again enough to consider $x \in \mathcal{E}$.

To prove this, we need a lemma which will also be useful later.

Lemma 5.1. *If $(x,u) \in G$ and $0 < t \leq 1$, then*

$$\frac{1}{(1+|x|)^2} \lesssim t^2|x|^2 + |u - D_t x|^2.$$

Proof. From the definition of G we have

$$\begin{aligned}
\frac{1}{1+|x|} &\leq |x-u| \\
&\lesssim |x - D_t x| + |D_t x - u| \\
&= |Q_\infty\big(Q_\infty^{-1}x - e^{-tB^*}Q_\infty^{-1}x\big)| + |u - D_t x| \\
&\lesssim |(I - e^{-tB^*})Q_\infty^{-1}x| + |u - D_t x| \\
&\lesssim t|x| + |u - D_t x|.
\end{aligned}$$

The lemma follows. □

To verify now (5.2) in the global region with $t \leq 1$, we recall from (3.5) that

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\Big(- c\,\frac{|u - D_t x|^2}{t}\Big).$$

It follows from Lemma 5.1 that

$$t^2 \gtrsim \frac{1}{(1+|x|)^4} \qquad \text{or} \qquad \frac{|u-D_t x|^2}{t} \gtrsim \frac{1}{(1+|x|)^2 t}.$$

The first inequality here implies that

$$K_t(x,u) \lesssim e^{R(x)}\,(1+|x|)^n \lesssim e^{2R(x)},$$

and (5.2) follows. If the second inequality holds, we have

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-\frac{c}{(1+|x|)^2 t}\right) \lesssim e^{R(x)}\,(1+|x|)^n,$$

and we get the same estimate. Thus (5.2) is verified.

Finally, let

$$\mathcal{H}_*^G f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_G(x,u)\,f(u)\,d\gamma_\infty(u) \right|,$$

and

$$\mathcal{H}_*^L f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_L(x,u)\,f(u)\,d\gamma_\infty(u) \right|.$$

6. The case of large t

In this section, we consider the supremum in the definition of the maximal operator taken only over $t>1$, and we prove (1.5).

Proposition 6.1. *For all functions $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$,*

$$\gamma_\infty \left\{ x : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}}, \qquad \alpha > 2. \tag{6.1}$$

In particular, the maximal operator

$$\sup_{t>1} |\mathcal{H}_t f(x)|$$

is of weak type $(1,1)$ with respect to the invariant measure γ_∞.

Proof. We can assume that $f \ge 0$. Looking at the arguments in Section 5, items (3) and (4), we see that is suffices to consider points $x \in \mathcal{E}$. For both x and u we use the coordinates introduced in (4.4) with $\beta = \log\alpha$, that is,

$$x = D_s \tilde{x}, \qquad u = D_{s'}\tilde{u},$$

where $\tilde{x}, \tilde{u} \in E_{\log\alpha}$ and $s, s' \in \mathbb{R}$.

From (3.4) we have

$$K_t(x,u) \lesssim \exp(R(x)) \exp\left(-c\left|D_{-t}u - x\right|^2\right)$$

for $t>1$ and $x,u \in \mathbb{R}^n$. Since $x \in \mathcal{E}$ and $D_{-t}u = D_{-t}\,D_{s'}\tilde{u} = D_{s'-t}\tilde{u}$, we can apply Lemma 4.3 (i), getting

$$\left|D_{-t}u - x\right| \gtrsim \left|\tilde{x} - \tilde{u}\right|,$$

so that

$$\int K_t(x,u) f(u)\, d\gamma_\infty(u) \lesssim \exp\big(R(D_s\tilde{x})\big) \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u).$$

In view of (4.3), the right-hand side here is strictly increasing in s, and therefore the inequality

$$\exp\big(R(D_s\tilde{x})\big) \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u) > \alpha \tag{6.2}$$

holds if and only if $s > s_\alpha(\tilde{x})$ for some function $\tilde{x} \mapsto s_\alpha(\tilde{x})$, with equality for $s = s_\alpha(\tilde{x})$. Since $\alpha > 2$ and $\|f\|_{L^1(\gamma_\infty)} = 1$, it follows that $s_\alpha(\tilde{x}) > 0$.

For some C, the set of points $x \in \mathcal{E}$ where the supremum in (6.1) is larger than $C\alpha$ is contained in the set $\mathcal{A}(\alpha)$ of points $D_s\tilde{x} \in \mathcal{E}$ fulfilling (6.2). We use Proposition 4.2 to estimate the γ_∞ measure of this set. Observe that $H(0,\tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\log\alpha}$ and that $D_s\tilde{x} \in \mathcal{E}$ implies $s \lesssim 1$, so that also $e^{-s\,\mathrm{tr}\,B} \lesssim 1$. We get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)\cap\mathcal{E}) &= \int_{\mathcal{A}(\alpha)\cap\mathcal{E}} e^{-R(x)} dx \\
&\lesssim \sqrt{\log\alpha} \int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{x})}^{C} e^{-R(D_s\tilde{x})}\, dS(\tilde{x})\, ds \\
&\lesssim \sqrt{\log\alpha} \int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{x})}^{+\infty} \exp\big(-R(D_{s_\alpha(\tilde{x})}\tilde{x}) - c\log\alpha\,(s - s_\alpha(\tilde{x}))\big)\, ds\, dS(\tilde{x}),
\end{aligned}$$

where the last inequality follows from (4.3), since $|D_s\tilde{x}|^2 \gtrsim |\tilde{x}|^2 \simeq \log\alpha$. Integrating in s, we obtain

$$\gamma_\infty(\mathcal{A}(\alpha)\cap\mathcal{E}) \lesssim \frac{1}{\sqrt{\log\alpha}} \int_{E_{\log\alpha}} \exp\big(-R(D_{s_\alpha(\tilde{x})}\tilde{x})\big)\, dS(\tilde{x}).$$

Now combine this estimate with the case of equality in (6.2) and change the order of integration, to get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)\cap\mathcal{E}) &\lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \int\int_{E_{\log\alpha}} \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big) dS(\tilde{x})\, f(u)\, d\gamma_\infty(u) \\
&\lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u)\, d\gamma_\infty(u)\,,
\end{aligned}$$

which proves Proposition 6.1. □

Finally, in analogy with [9], we show that the factor $1/\sqrt{\log\alpha}$ in (6.1) is sharp.

Proposition 6.2. *For any $t > 1$ and any large α, there exists a function f, normalized in $L^1(\gamma_\infty)$ and such that*

$$\gamma_\infty\{x : |\mathcal{H}_t f(x)| > \alpha\} \simeq \frac{1}{\alpha\sqrt{\log\alpha}}\,.$$

Proof. Take a point z with $R(z) = \log\alpha$, and let f be (an approximation of) a Dirac measure at the point $u = D_t z$. Then, as a consequence of (3.4), $K_t(x,u) \simeq$

$\exp(R(x))$ in the ball $B(D_{-t}u, 1) = B(z, 1)$. We then have $\mathcal{H}_t f(x) = K_t(x, u) \gtrsim \alpha$ in the set $\mathcal{B} = \{x \in B(z, 1) : R(x) > R(z)\}$, whose measure is

$$\gamma_\infty(\mathcal{B}) \simeq e^{-R(z)} \frac{1}{\sqrt{R(z)}} = \frac{1}{\alpha \sqrt{\log \alpha}}.$$

□

7. The local case for small t

Proposition 7.1. *If $(x, u) \in L$ and $0 < t \le 1$, then*

$$\big|K_t(x, u)\big| \lesssim \frac{\exp\big(R(x)\big)}{t^{n/2}} \exp\Big(-c\,\frac{|u-x|^2}{t}\Big).$$

Proof. In view of (3.5), it is enough to show that

$$\frac{|u - D_t x|^2}{t} \ge \frac{|u-x|^2}{t} - C. \tag{7.1}$$

We write

$$\begin{aligned}|u - D_t\, x|^2 = |u - x + x - D_t\, x|^2 &= |u-x|^2 + 2\langle u - x, x - D_t\, x\rangle + |x - D_t\, x|^2 \\ &\ge |u-x|^2 - 2|u-x|\,|x - D_t\, x|.\end{aligned}$$

But

$$|u-x|\,|x - D_t\, x| = |u-x|\,|Q_\infty(I - e^{-tB^*})Q_\infty^{-1}x| \lesssim |u-x|\,t\,|x| \le t$$

since $(x, u) \in L$, and (7.1) follows. □

Proposition 7.2. *The maximal operator $\mathcal{H}_*^L$ is of weak type $(1,1)$ with respect to the invariant measure γ_∞.*

Proof. The proof is standard, since Proposition 7.1 implies

$$\mathcal{H}_*^L f(x) \lesssim \sup_{0<t\le 1} \frac{\exp\big(R(x)\big)}{t^{n/2}} \int \exp\Big(-c\,\frac{|x-u|^2}{t}\Big)\,\chi_L(x,u)\,f(u)\,d\gamma_\infty(u).$$

The supremum here defines an operator of weak type $(1,1)$ with respect to the Lebesgue measure in $\mathbb{R}^n$. From this the proposition follows, cf. [8, Section 3]. □

8. The global case for small t

In this section, we conclude the proof of Theorem 1.1.

Proposition 8.1. *The maximal operator $\mathcal{H}_*^G$ is of weak type $(1,1)$ with respect to the invariant measure γ_∞.*

Proof. For $m \in \mathbb{N}$ and $0 < t \leq 1$, we introduce regions $\mathcal{S}_t^m$. If $m > 0$, we let

$$\mathcal{S}_t^m = \left\{ (x, u) \in G : 2^{m-1}\sqrt{t} < |u - D_t x| \leq 2^m \sqrt{t} \right\}.$$

If $m = 0$, we replace the condition $2^{m-1}\sqrt{t} < |u - D_t x| \leq 2^m \sqrt{t}$ by $|u - D_t x| \leq \sqrt{t}$. Note that for any fixed $t \in (0, 1]$ these sets form a partition of G.

In the set $\mathcal{S}_t^m$ we have, because of (3.5),

$$K_t(x, u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c2^{2m}\right).$$

Then setting

$$\mathcal{K}_t^m(x, u) = \frac{\exp(R(x))}{t^{n/2}} \chi_{\mathcal{S}_t^m}(x, u), \tag{8.1}$$

one has, for all $(x, u) \in G$ and $0 < t < 1$,

$$K_t(x, u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) \mathcal{K}_t^m(x, u).$$

Hence, it suffices to prove that for $m = 0, 1, \dots$ and $f \geq 0$ normalized in $L^1(\gamma_\infty)$

$$\gamma_\infty \left\{ x \in \mathcal{E} : \sup_{0<t\leq 1} \int \mathcal{K}_t^m(x, u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\} \lesssim \frac{2^{Cm}}{\alpha}, \tag{8.2}$$

for large α, since this will allow summing in m in the space $L^{1,\infty}$.

Fix $m \in \mathbb{N}$. Then $(x, u) \in S_t^m$, $t \in (0, 1]$ implies $|u - D_t x| \leq 2^m \sqrt{t}$. Now Lemma 5.1 leads to

$$1 \lesssim (1 + |x|)^4 t^2 + (1 + |x|)^2\, 2^{2m}\, t \leq ((1 + |x|)^2\, 2^{2m}\, t)^2 + (1 + |x|)^2\, 2^{2m}\, t.$$

Consequently,

$$(1 + |x|)^2\, 2^{2m}\, t \gtrsim 1 \tag{8.3}$$

as soon as there exists a point u with $\mathcal{K}_t^m(x, u) \neq 0$, and then $t \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\alpha, m) > 0$. Hence the supremum in (8.2) can as well be taken over $\varepsilon \leq t \leq 1$, and this supremum is a continuous function of $x \in \mathcal{E}$.

To prove (8.2), the idea, which goes back to [16], is to construct a finite sequence of pairwise disjoint balls $\left(\mathcal{B}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$ and a finite sequence of sets $\left(\mathcal{Z}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$, called forbidden zones. These zones will together cover the level set in (8.2). We will show that

$$\left\{ x \in \mathcal{E} : \sup_{\varepsilon\leq t\leq 1} \int \mathcal{K}_t^m(x, u)\, f(u)\, d\gamma_\infty(u) \geq \alpha \right\} \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}, \tag{8.4}$$

and that for each ℓ

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u). \tag{8.5}$$

Since the $\mathcal{B}^{(\ell)}$ will be pairwise disjoint, we could then conclude

$$\gamma_\infty\Big(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\Big) \lesssim \frac{2^{Cm}}{\alpha} \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha}.$$

This would imply (8.2) and so complete the proof of Proposition 8.1.

The sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ will be introduced by means of a sequence of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$, which we define by recursion. To find the first point $x^{(1)}$, consider the minimum of the quadratic form $R(x)$ in the compact set

$$\mathcal{A}_1(\alpha) = \left\{ x \in \mathcal{E} : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty \ge \alpha \right\}.$$

Should this set be empty, (8.2) is immediate. By continuity, this minimum is attained at some point $x^{(1)}$ of the set.

We now describe the recursion to construct $x^{(\ell)}$ for $\ell \ge 2$. Like $x^{(1)}$, these points will satisfy

$$\sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x^{(\ell)},u)\, f(u)\, d\gamma_\infty \ge \alpha.$$

Once an $x^{(\ell)}$, $\ell \ge 1$, is defined, we can thus by continuity choose $t_\ell \in [\varepsilon, 1]$ such that

$$\int \mathcal{K}_{t_\ell}^m(x^{(\ell)},u)\, f(u)\, d\gamma_\infty \ge \alpha. \tag{8.6}$$

Using this t_ℓ, we associate with $x^{(\ell)}$ the tube

$$\mathcal{Z}^{(\ell)} = \left\{ D_s \eta \in \mathbb{R}^n : \; s \ge 0,\; R(\eta) = R(x^{(\ell)}),\; |\eta - x^{(\ell)}| < A\, 2^{3m} \sqrt{t_\ell} \right\},$$

Here the constant $A > 0$ is to be determined, depending only on n, Q and B.

All the $x^{(\ell)}$ will be minimizing points. To avoid having them too close to one another, we will not allow $x^{(\ell)}$ to be in any $\mathcal{Z}^{(\ell')}$ with $\ell' < \ell$. More precisely, assuming $x^{(1)}, \dots, x^{(\ell)}$ already defined, we will choose $x^{(\ell+1)}$ as a minimizing point of $R(x)$ in the set

$$\mathcal{A}_{\ell+1}(\alpha) = \left\{ x \in \mathcal{E} \setminus \bigcup_{\ell'=1}^{\ell} \mathcal{Z}^{(\ell')} : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \ge \alpha \right\}, \tag{8.7}$$

provided this set is nonempty. But if $\mathcal{A}_{\ell+1}(\alpha)$ is empty, the process stops with $\ell_0 = \ell$ and (8.4) follows. We will soon see that this actually occurs for some ℓ.

Now assume that $\mathcal{A}_{\ell+1}(\alpha) \ne \emptyset$. In order to assure that a minimizing point exists, we must verify that $\mathcal{A}_{\ell+1}(\alpha)$ is closed and thus compact, although the $\mathcal{Z}^{(\ell')}$ are not open. To do so, observe that for $1 \le \ell' \le \ell$, the minimizing property of $x^{(\ell')}$ means that there is no point in $\mathcal{A}_{\ell'}(\alpha)$ with $R(x) < R(x^{(\ell')})$. Thus we have the inclusions

$$\mathcal{A}_{\ell+1}(\alpha) \subset \mathcal{A}_{\ell'}(\alpha) \subset \left\{ x : R(x) \ge R(x^{(\ell')}) \right\}, \qquad 1 \le \ell' \le \ell.$$

It follows that

$$\mathcal{A}_{\ell+1}(\alpha) = \mathcal{A}_{\ell+1}(\alpha) \cap \bigcap_{1 \le \ell' \le \ell} \{x : R(x) \ge R(x^{(\ell')})\} =$$

$$\bigcap_{\ell'=1}^{\ell} \left\{ x \in \mathcal{E} \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')}),\ \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \geq \alpha \right\}.$$

The sets $\{x \in \mathcal{E} \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')})\}$ are closed in view of the choice of $\mathcal{Z}^{(\ell')}$. This makes $\mathcal{A}_{\ell+1}(\alpha)$ compact, and a minimizing point $x^{(\ell+1)}$ can be chosen. Thus the recursion is well defined.

We observe that (8.3) applies to t_ℓ and $x^{(\ell)}$, so that

$$|x^{(\ell)}|^2\, 2^{2m}\, t_\ell \gtrsim 1. \tag{8.8}$$

Further, we define balls

$$\mathcal{B}^{(\ell)} = \{u \in \mathbb{R}^n : |u - D_{t_\ell} x^{(\ell)}| \leq 2^m \sqrt{t_\ell}\,\}.$$

Because of (8.1) and the definitions of $\mathcal{K}_t^m$ and $\mathcal{S}_t^m$, the inequality (8.6) implies

$$\alpha \leq \frac{\exp\big(R(x^{(\ell)})\big)}{t_\ell^{n/2}} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u). \tag{8.9}$$

We now verify that the sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ have the required properties. The proof follows the lines of the proof of Lemma 6.2 in [4], with only slight modifications.

Lemma 8.2. *The collection of balls $\mathcal{B}^{(\ell)}$ is pairwise disjoint.*

Proof. Two balls $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$ with $\ell < \ell'$ will be disjoint if

$$\big|D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}} x^{(\ell')}\big| > 2^m(\sqrt{t_\ell} + \sqrt{t_{\ell'}}). \tag{8.10}$$

By means of the coordinates from Subsection 4.1 with $\beta = R(x^{(\ell)})$, we write

$$x^{(\ell')} = D_s \tilde{x}^{(\ell')}$$

for some $\tilde{x}^{(\ell')}$ with $R(\tilde{x}^{(\ell')}) = R(x^{(\ell)})$ and some $s \in \mathbb{R}$. Note that $s \geq 0$, because $R(x^{(\ell')}) \geq R(x^{(\ell)})$. Since $x^{(\ell')}$ does not belong to the forbidden zone $\mathcal{Z}^{(\ell)}$, we must have

$$|\tilde{x}^{(\ell')} - x^{(\ell)}| \geq A 2^{3m} \sqrt{t_\ell}. \tag{8.11}$$

We first assume that $t_{\ell'} \geq M\, 2^{4m}\, t_\ell$, for some $M \geq 2$ to be chosen. Lemma 4.3 (ii) implies

$$\big|D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}} x^{(\ell')}\big| = \big|D_{t_\ell} x^{(\ell)} - D_{t_{\ell'}+s} \tilde{x}^{(\ell')}\big| \gtrsim |x^{(\ell)}|\,(t_{\ell'} + s - t_\ell) \gtrsim |x^{(\ell)}|\, t_{\ell'}.$$

Using our assumption and then (8.8), we get

$$|x^{(\ell)}|\, t_{\ell'} \gtrsim |x^{(\ell)}|\, \sqrt{M}\, 2^{2m} \sqrt{t_\ell}\, \sqrt{t_{\ell'}} \gtrsim \sqrt{M}\, 2^m \sqrt{t_{\ell'}} \simeq \sqrt{M}\, 2^m\, (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Fixing M suitably large, we obtain (8.10) from the last two formulae.

It remains to consider the case when $t_{\ell'} < M\, 2^{4m}\, t_\ell$. Then

$$\sqrt{t_\ell} > \frac{2^{-2m-1}}{\sqrt{M}} (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Applying this to (8.11), we obtain (8.10) by choosing A so that $A/\sqrt{M}$ is large enough. □

We next verify that the sequence $(x^{(\ell)})$ is finite. For $\ell < \ell'$, we have (8.11), as in the preceding proof . Then Lemma 4.3 (i) implies

$$\left|x^{(\ell')} - x^{(\ell)}\right| \gtrsim A\, 2^{3m}\sqrt{t_\ell}.$$

Since $t_\ell \geq \varepsilon$, we see that the distance $\left|x^{(\ell')} - x^{(\ell)}\right|$ is bounded below by a positive constant. But all the $x^{(\ell)}$ are contained in the bounded set $\mathcal{E}$, so they are finite in number. Thus the set considered in (8.7) must be empty for some ℓ, and the recursion stops. This implies (8.4).

We finally prove (8.5) . Observe that the forbidden zone $\mathcal{Z}^{(\ell)}$ is a tube as defined in (4.12), with $a = A\, 2^{3m}\sqrt{t_\ell}$ and $\beta = R(x^{(\ell)})$. This value of β is large since $x^{(\ell)} \in \mathcal{E}$, and thus we can apply Lemma 4.4 to obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{\left(A2^{3m}\sqrt{t_\ell}\right)^{n-1}}{\sqrt{R(x^{(\ell)})}} \exp\left(-R(x^{(\ell)})\right).$$

We bound the exponential here by means of (8.9) and observe that $R(x^{(\ell)}) \sim |x^{(\ell)}|^2$, getting

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{1}{\alpha |x^{(\ell)}| \sqrt{t_\ell}} \left(A2^{3m}\right)^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty(u).$$

As a consequence of (8.8), we obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^m}{\alpha} \left(A2^{3m}\right)^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u),$$

proving (8.5). This concludes the proof of Proposition 8.1. □

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