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Schrödinger spectra and the effective Hamiltonian of weak KAM theory on the flat torus

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In this paper we investigate the link between the spectrum of some periodic Schrödinger type operators and the effective Hamiltonian of the weak KAM theory. We show that the extension of some local quasimodes is linked to the localization of the Schrödinger spectrum. Such a result provides additional information with respect to the well known Bohr-Sommerfeld quantization rules, here in a more general setting than the integrable or quasi-integrable ones. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4960741]

I. INTRODUCTION

For $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ and $V \in C^\infty(\mathbb{T}^n)$, we study the eigenvalue equation

$$-\frac{1}{2}\hbar^2\Delta_x\psi_\hbar(x) + V(x)\psi_\hbar(x) = E_\hbar\psi_\hbar(x) \tag{1}$$

in connection to weak KAM solutions of negative type for the Hamilton-Jacobi equation,

$$\frac{1}{2}|P + \nabla_x S(P,x)|^2 + V(x) = \bar{H}(P), \quad P \in \mathbb{R}^n, \tag{2}$$

where $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is the effective Hamiltonian of the weak KAM theory (see for example Refs. 4, 6, 7, and 14). We remind that $P \mapsto \bar{H}(P)$ is a convex function and that $\bar{H}(P) \geq \max_{y\in\mathbb{T}^n} V(y)$ for all $P \in \mathbb{R}^n$, $\bar{H}(0) = \max_{y\in\mathbb{T}^n} V(y)$. This is given by the well known formula (see Ref. 4)

$$\bar{H}(P) = \inf_{v\in C^1(\mathbb{T}^n)} \sup_{y\in\mathbb{T}^n} \frac{1}{2}|P + \nabla_x v(y)|^2 + V(y). \tag{3}$$

In this setting, all the weak KAM solutions of negative type for (2) coincide with viscosity solutions (see Ref. 7). As we recall in Section II B, these functions are Lipschitz continuous and semiconcave with linear modulus on $\mathbb{T}^n$ and this ensures the second order differentiability for $\mathcal{L}^n$ - a.e. $x \in \mathbb{T}^n$. Moreover, by denoting $\Sigma(S)$ as the singular set of S (i.e., the set of $x \in \mathbb{T}^n$ where S is not differentiable) it is known (see Ref. 14) that $\mathbb{T}^n \setminus \overline{\Sigma(S)}$ is an open and dense subset of $\mathbb{T}^n$ and S is $C^{1,1}_{\rm loc}$ on such a domain.

The spectrum of $-\frac{1}{2}\hbar^2\Delta_x + V : W^{2,2}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is bounded from below and discrete, as we recall in Section II A. In what follows, we will write $\mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V) = \{E_{\hbar,\alpha}\}_{\alpha\in\mathbb{N}}$.

The first result of the paper is about the determination of some WKB - type wave functions in a low regularity setting providing local quasimodes of order $O(\hbar)$ and $O(\hbar^\infty)$ linked to the solutions of the Equation (2).

Theorem 1.1. *Let $V \in C^\infty(\mathbb{T}^n)$ and let $S(P,\cdot) : \mathbb{T}^n \to \mathbb{R}$ be a solution of (2). For $0 < \hbar \leq 1$, $P \in \mathbb{R}^n$ and $\Omega \subset\subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$ we define*

$$u_{P,\hbar}(x) := (\mathrm{Vol}(\Omega))^{-1}\, e^{i(P\cdot x + S(P,x))/\hbar}, \quad x \in \Omega, \tag{4}$$

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and $\Phi(\Omega) := \{\varphi_\hbar \in L^2(\Omega) \,|\, \exists\, \Delta_x\varphi_\hbar(x) \text{ for } \mathcal{L}^n - a.e.\, x \in \Omega,\, \Delta_x\varphi_\hbar \in L^2(\Omega)\}$. *Then,* $u_{P,\hbar}$ *is* $L^2(\Omega)$ *- normalized,* $u_{P,\hbar} \in \Phi(\Omega)$ *and*

$$-\frac{1}{2}\hbar^2\Delta_x u_{P,\hbar} + V(x)u_{P,\hbar} = \bar{H}(P)u_{P,\hbar} + O(\hbar), \tag{5}$$

where the remainder is estimated in $L^2(\Omega)$ *as* $\frac{1}{2}\|\Delta_x S(P,\cdot)\|_{L^\infty(\Omega)}\,\hbar$.

Let $\delta\mathcal{E}(\hbar) := \inf_{\alpha\in\mathbb{N}}(2\pi\hbar)^{-n}(E_{\hbar,\alpha+1} - E_{\hbar,\alpha})$ *and let* $0 < \hbar \leq \hbar_0$ *be such that* $\pi\,\delta\mathcal{E}(\hbar)\,e^{-1/\hbar} \leq 1$. *Fix* $x_0 \in \mathbb{T}^n \setminus \overline{\Sigma(S)}$ *and* $0 < \rho(x_0, P) \leq 1$ *such that* $B_{2\rho}(x_0) \subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$. *Define* $r := \pi\,\delta\mathcal{E}(\hbar)e^{-1/\hbar}$ $\rho(x_0,P)\,(1 + \|\nabla^2 S(P,\cdot)\|_{L^2(B_\rho(x_0))})^{-1}$, $S_0(P,x) := S(P,x_0) + \nabla_x S(P,x_0)\cdot(x - x_0)$,

$$\varphi_{P,\hbar}(x) := (\mathrm{Vol}(B_r))^{-1/2}\,e^{i(P\cdot x + S_0(P,x))/\hbar}, \quad x \in B_r(x_0). \tag{6}$$

Then, $\varphi_{P,\hbar}$ *is* $L^2(B_r(x_0))$ *- normalized,* $\varphi_{P,\hbar} \in \Phi(B_r(x_0))$. *The function* $u_{P,\hbar}(x) := \mathrm{Vol}(B_r(x_0))^{-1}$ $e^{i(P\cdot x + S(P,x))/\hbar}$ *fulfills* $\|\varphi_{P,\hbar} - u_{P,\hbar}\|_{L^2(B_r)} \leq 2\pi\,\delta\mathcal{E}(\hbar)\,\hbar^{-1}e^{-1/\hbar}$. *Moreover,*

$$-\frac{1}{2}\hbar^2\Delta_x\varphi_{P,\hbar} + V(x)\varphi_{P,\hbar} = \bar{H}(P)\varphi_{P,\hbar} + O(\hbar^\infty), \tag{7}$$

where the remainder is estimated in $L^2(B_r)$ *as* $\|\nabla V\|_{C^0(\mathbb{T}^n)}\,\pi\,\delta\mathcal{E}(\hbar)\,e^{-1/\hbar}$.

In Lemma 3.1 we prove that for any fixed $0 < \hbar \leq 1$ the set of energies

$$\mathcal{E} := \{\bar{H}(P) \,|\, P \in \ell\cdot\hbar\,\mathbb{Z}^n;\; 0 < \ell \leq 1\} \tag{8}$$

fulfills $\overline{\mathcal{E} \cap [\max_{y\in\mathbb{T}^n} V(y), \lambda]} = [\max_{y\in\mathbb{T}^n} V(y), \lambda]$ for any $\lambda > \max_{y\in\mathbb{T}^n} V(y)$. In Theorem 1.2 we select $P \in \ell\cdot\hbar\,\mathbb{Z}^n$ and energies in $\mathcal{E}$ to get the Schrödinger spectrum up to $O(\hbar^\infty)$ terms.

We recall that the determination of the series of smooth WKB-wave functions or Lagrangian distributions given by oscillatory integrals which are $O(\hbar^N)$-energy quasimodes on $\mathbb{T}^n$ works in the case of KAM or integrable settings (see for example Refs. 1, 10, and 11 and the references therein).

Moreover, the well known Weyl's law for our class of Schrödinger operators (see for example Chaps. 1.5 and 10.4 in Ref. 9) provides the asymptotics as $\hbar \to 0^+$ for the number of eigenvalues

$$\sharp\,(a,b)\;\cap\;\mathrm{Spec}\Big(-\frac{1}{2}\hbar^2\Delta_x + V\Big) \simeq (2\pi\hbar)^{-n}\Big(\mathrm{Vol}(a < \frac{1}{2}|p|^2 + V < b) + O(1)\Big) \tag{9}$$

for any $\hbar$-independent interval $(a,b) \subset [\min_{y\in\mathbb{T}^n} V(y), +\infty)$. Notice also that $\delta\mathcal{E}(\hbar) := \inf_{\alpha\in\mathbb{N}}(2\pi\hbar)^{-n}$ $(E_{\hbar,\alpha+1} - E_{\hbar,\alpha})$ is uniformly bounded with respect to $0 < \hbar \leq 1$ since in the opposite case the Weyl's law would be not satisfied.

The second result of the paper deals with a characterization of the points of the spectrum bigger than the energy value $\max_{y\in\mathbb{T}^n} V(y)$, and this target is obtained by the use of the above quasimodes. This is the content of the next

Theorem 1.2. *Let* $\varphi_{P,\hbar}$ *be as in Thm. 1.1. The following statements are equivalent:*

(a) $E_\hbar \in (\max_{y\in\mathbb{T}^n} V(y), b) \cap \mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V)$

(b) *Let* $E_\hbar \in \mathbb{R}$, $0 < \ell \leq 1$ *and let* $P \in \ell\cdot\hbar\,\mathbb{Z}^n$ *be running over a finite set of vectors satisfying* $|E_\hbar - \bar{H}(P)| \leq \delta\mathcal{E}(\hbar)\,e^{-1/\hbar}$. *Any nonvanishing function in*

$$\Big\{\varphi_\hbar := \sum_P c_P\,\varphi_{P,\hbar}\Big\} \subset \Phi(B_r(x_0)) \tag{10}$$

plus a remainder $\widetilde{\varphi}_\hbar$ *in* $\Phi(B_r(x_0))$ *estimated as* $O(\hbar^\infty)$ *can be extended to* $\psi_\hbar \in W^{2,2}(\mathbb{T}^n)$ *such that*

$$-\frac{1}{2}\hbar^2\Delta_x\psi_\hbar + V(x)\psi_\hbar = E_\hbar\psi_\hbar. \tag{11}$$

In particular,

$$\|\widetilde{\varphi}_\hbar\|_{L^2(B_r(x_0))} \leq (2\pi\hbar)^{-n}\Big(1 + \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\,\pi\Big)\,e^{-1/\hbar}\,\|\varphi_\hbar\|_{L^2(B_r(x_0))}. \tag{12}$$

We now provide some remarks about the above result. To begin with, we underline that the series (10) is convergent since the involved vectors P belong to a finite set. Moreover, the point **(b)** can be rewritten as

$$\psi_\hbar\Big|_{B_r(x_0)} = \sum_P c_P\, \varphi_{P,\hbar} + \widetilde{\varphi}_\hbar \tag{13}$$

for some eigenfunction $\psi_\hbar$ with eigenvalue $E_\hbar$ and where the sum is taken over a finite set of points $P \in \ell \cdot \hbar\, \mathbb{Z}^n$ satisfying $|E_\hbar - \bar{H}(P)| \leq \delta \mathcal{E}(\hbar)\, e^{-1/\hbar}$, for a set of linearly independent $\varphi_{P,\hbar}$ and $c_P \in \mathbb{C}$ not depending on $\hbar$. Notice that such vectors P are in fact spectral invariants for the Schrödinger operator, which generalize the points of the lattice arising thanks to the Bohr-Sommerfeld quantization rules, see (20). About the set of all the wave functions $\varphi_{P,\hbar}$, we do not have information about the maximum number of linearly independent ones. Anyway, in view of Theorem 1.2 if $\bar{H}(P)$ takes energy values in the spectrum mod. $O(\hbar^\infty)$ then the set (10) is concentrated in the finite dimensional space of eigenfunctions plus error terms $O(\hbar^\infty)$ with the estimate shown in (12). Observe that we cannot say whether or not the property (13) is fulfilled for all the eigenfunctions $\psi_\hbar \in W^{2,2}(\mathbb{T}^n)$. Moreover, we stress that in some cases the existence of eigenfunctions which take, locally over a ball $B_r(x_0)$, the form (13) up to an error $O(\hbar^{N-n})$ is a consequence of $O(\hbar^N)$ - quasimodes on $\mathbb{T}^n$ given by smooth WKB-type wave functions. Indeed, by assuming that $V(x) = \sum_{i=1}^n V_i(x_i)$ then the Hamilton-Jacobi equation is solved, for any fixed P such that $\bar{H}(P)$ is above $\max_{y\in\mathbb{T}^n} V(y)$, by the unique smooth function $S(P,x) = \sum_{i=1}^n \int_0^{x_i} \sqrt{2(\bar{H}(P) - V_i(y))}dy - P_i \cdot x_i$. Moreover, under typical conditions of the KAM theorem, the existence of smooth $S(P,x)$ is still guaranteed. Whence, for a suitable subset of vectors $P \in \hbar\,\mathbb{Z}^n$ (related in addition to non-resonance conditions), the series,

$$\psi_\hbar^{(N)}(x) := \sum_{j=0}^N \hbar^j\, a_j(P,x)\, e^{i(P\cdot x + S(P,x))/\hbar}, \quad x \in \mathbb{T}^n, \tag{14}$$

obtained by smooth amplitudes a_j solving suitable continuity type equations, provide $O(\hbar^{N+1})$ - quasimodes in $C^\infty(\mathbb{T}^n)$, see for example Ref. 1, for quasienergies $\sum_{j=0}^N \hbar^j E_j$ and where $E_0 = \bar{H}(P)$. We observe that the restriction of (14) on $B_r(x_0)$ can be rewritten as

$$\psi_\hbar^{(N)}(x) = \sum_{j=0}^N (\hbar^j\, a_j(P,x_0) + O(\hbar^\infty))\, e^{i(P\cdot x + S(P,x))/\hbar}, \quad x \in B_r(x_0) \subset \mathbb{T}^n, \tag{15}$$

thanks to the smoothness of a_j and where the above $O(\hbar^\infty)$-remainder is computed in $W^{2,2}(U_0)$-norm. This function is $O(\hbar^\infty)$ close to

$$\sum_{j=0}^N \hbar^j\, a_j(P,x_0) e^{i(P\cdot x + S(P,x))/\hbar}, \quad x \in B_r(x_0) \subset \mathbb{T}^n, \tag{16}$$

which belongs, when rescaled by $\mathrm{Vol}(B_r(x_0))^{-1}$, to the type of wave functions in Thm. 1.1. Notice that (15) and (16) hold in a small neighbourhood of an arbitrary fixed point $x_0 \in \mathbb{T}^n$ since here S is smooth. We also underline that, if $E_{\hbar,\alpha+1} - E_{\hbar,\alpha} \simeq C_\alpha\, \hbar^n$, then some simple arguments (see for example Ref. 1) show that for any $O(\hbar^N)$-quasimode $\psi_\hbar^{(N)} \in W^{2,2}(\mathbb{T}^n)$ with $N > n$, there exists at least one eigenfunction $\psi_\hbar \in W^{2,2}(\mathbb{T}^n)$ belonging to the eigenvalue $E_\hbar$ such that $\|\psi_\hbar^{(N)} - \psi_\hbar\|_{L^2(\mathbb{T}^n)} = O(\hbar^{N-n})$, whence also $L^2(B_r(x_0))$—close to (16). We stress that the quasienergies $E_\hbar^{(N)} := \sum_{j=0}^N \hbar^j E_j$ are linked to this point of the spectrum by $E_\hbar \in (E_\hbar^{(N)} - O(\hbar^N), E_\hbar^{(N)} + O(\hbar^N))$. Thus, in this construction of quasimodes,

$$E_\hbar = \bar{H}(P) + O(\hbar), \tag{17}$$

which is in fact a weaker relationship, with respect to our approach, between the effective Hamiltonian and the Schrödinger spectrum. We also underline that, without further assumptions which are additional with respect to integrably or KAM settings, it cannot be proved (in our knowledge) that the series (14) is $L^2(\mathbb{T}^n)$-convergent as $N \to +\infty$.

For Schrödinger operators $-\frac{1}{2}\hbar^2\Delta_x + V(x)$, where $V(x) = \sum_{i=1}^n V_i(x_i)$, the well known Bohr-Sommerfeld quantization rules (see for example Ref. 12) read

$$\frac{1}{2\pi}\oint p_i\, dx_i = \Big(k_i - \frac{1}{4}\mu_i + O(\hbar)\Big)\hbar, \quad k_i \in \mathbb{Z}, \tag{18}$$

where the integration is taken over the closed trajectory which lies on the one dimensional hypersurface $\{\frac{1}{2}|p_i|^2 + V_i(x_i) = E_i\}$, which in fact coincides with a Lagrangian torus $\Lambda_i \subset \mathbb{T}\times\mathbb{R}$ and μ_i is the related Maslov cocycle. This provides the exact conditions to recover the points of the spectrum. Indeed,

$$P_i = \frac{1}{2\pi}\oint p_i\, dx_i \tag{19}$$

and thus the points of the spectrum of $-\frac{1}{2}\hbar^2\Delta_x + V(x)$ above $\max V$ (hence in the region of regular values of the momentum map $(H_1,\ldots,\ H_n)$ where $H_i = \frac{1}{2}|p_i|^2 + V_i(x_i)$) take the form

$$E_\hbar = \bar{H}\Big(k\hbar - \frac{1}{4}\mu\hbar + O(\hbar^2)\Big) + O(\hbar^\infty), \tag{20}$$

where $\bar{H}(P) = \sum_i \bar{H}_i(P_i)$ and $\bar{H}_i(P_i) = \inf_v \sup_y \frac{1}{2}|P_i + \nabla_x v(y)|^2 + V_i(y)$, $k = (k_1,\ldots,\ k_n) \in \mathbb{Z}^n$, $\mu = (\mu_1,\ldots,\ \mu_n) \in \mathbb{Z}^n$.

On the one hand, we underline that our approach is based on the selection of $P \in \ell\cdot\hbar\ \mathbb{Z}^n$ such that $|E_\hbar - \bar{H}(P)| \leq \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}$, whence it is the converse viewpoint with respect to (20). On the other hand, the equalities (18) are linked (see Prop. 5.1 in Ref. 12) to the possibility to glue microlocal solutions of the eigenvalue equations

$$-\frac{1}{2}\hbar^2\frac{d^2}{dz^2}\psi_\hbar^i(z) + V_i(z)\psi_\hbar^i(z) = E_\hbar^i\psi_\hbar^i(z) + O(\hbar^\infty), \quad z \in \mathbb{T},$$

on small open domains Ω in $\Lambda_i \subset \mathbb{T}\times\mathbb{R}$ at energy $E_\hbar^i$. In our paper, this is replaced by the possibility to extend the local quasimodes $\varphi_{P,\hbar}$ defined on $B_r(x_0) \subset \mathbb{T}^n$ to global eigenfunctions on $\mathbb{T}^n$. In fact, the sets Graph$(P + \nabla_x S)$ generalize $\Lambda_1 \times\cdots\times \Lambda_n$ but they are not smooth enough in order to provide (in our knowledge) a direct generalization of the Bohr-Sommerfeld approach. Thus, we underline the open problem to find a more general "quantization rule" for P in place of (18), which could be related to the extension of $\varphi_{P,\hbar}$ shown in point (b) of Theorem 1.2.

To conclude the Introduction, in the following we remind some of the several results in the literature that exhibit the construction and semiclassical study of energy eigenfunctions or quasimodes linked to weak KAM tori, KAM tori, and Lagrangian tori within the phase space.

In Ref. 5 the author suggests a quantum analogue for weak KAM theory and for Mather's minimization principle for Lagrangian dynamics. It is rigorously constructed from the eigenfunctions of a certain non-selfadjoint operator a candidate for wave function which is a minimizer, and recovered aspects of weak KAM theory in the asymptotics of the classical limit. Regarding such wave functions as a quasimode, an $O(\hbar)$ - error estimate is provided. In Ref. 3 some attempts are still shown to devise quantum analogues of certain aspects of Mather's theory of minimizing measures. This target allows the construction of WKB energy quasimodes with $O(\hbar)$ - error estimate.

In the paper[10] the aim is to study the relationship between the effective Nekhoroshev stability for near-integrable Hamiltonian systems and the semiclassical asymptotics for Schrödinger operators. In particular, for a given real analytic Hamiltonian close to a completely integrable one and a Cantor set defined through a Diophantine condition, a family of invariant tori is shown with frequencies which is Gevrey smooth. A symplectic normal form of the Hamiltonian is also obtained in a neighborhood of the union of invariant tori which can be viewed as a Birkhoff normal form around all invariant tori. This result leads to effective stability of a quasiperiodic motion. As an application, some energy quasimodes which are associated with a family of KAM tori are shown with exponentially small error terms. The well-known approach of WKB construction of energy quasimodes in the hypotheses of KAM theorem is also used in the paper[1] where the authors show that the quantum tunneling between phase space KAM tori is suppressed in the semiclassical limit, and that the distribution of quantum group velocities converges to the distribution of the (classical) asymptotic velocities up to an error term.

In Ref. 11 the author studies the possible concentration in phase space of a sequence of eigenfunctions or quasimodes of an operator whose principal symbol has completely integrable Hamilton flow. Since the semiclassical wave front set of such a sequence is invariant under such flow, this may allow concentration of the semiclassical wave front set along a single closed orbit if all frequencies of the flow are rationally related. It is shown that, subject to some non-degeneracy hypotheses, this concentration may not occur. In the two-dimensional case, it is shown that semiclassical wave front set must fill out an entire Lagrangian torus.

II. PRELIMINARIES AND SETTINGS

A. Schrödinger eigenvalues and eigenfunctions

We consider $V \in C^\infty(\mathbb{T}^n)$ and the class of Schrödinger operators $\widehat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V$ with $\widehat{H}_\hbar : W^{2,2}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$. By the resolvent set $\rho(\widehat{H}_\hbar)$ we mean the set of $\lambda \in \mathbb{C}$ such that $\overline{\mathrm{Ran}(\widehat{H}_\hbar - \lambda\,\mathrm{id})} = L^2(\mathbb{T}^n)$, $\widehat{H}_\hbar - \lambda\,\mathrm{id} : W^{2,2}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is injective and the related resolvent operator $R_\lambda := (\widehat{H}_\hbar - \lambda\,\mathrm{id})^{-1}$ acting on $R_\lambda : \mathrm{Ran}(\widehat{H}_\hbar - \lambda\,\mathrm{id}) \to W^{2,2}(\mathbb{T}^n)$ is L^2 - bounded. Thus, the spectrum reads $\mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V) := \mathbb{C} \setminus \rho(\widehat{H}_\hbar)$. The pointwise part of the spectrum is given by the subset of $\lambda \in \mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V)$ such that $\widehat{H}_\hbar - \lambda\,\mathrm{id}$ is not injective, and the discrete part is the subset of the pointwise part such that the related eigenspaces are finite dimensional. In fact, for this class of operators the set $\mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V)$ coincides with its discrete part.

The so-called Weyl's law (see for example Chaps. 1.5 and 10.4.1 in Ref. 9) provides the asymptotics as $\hbar \to 0^+$ for the number of the eigenvalues

$$\sharp\,(a,b) \,\cap\, \mathrm{Spec}\Big(-\frac{1}{2}\hbar^2\Delta_x + V\Big) \simeq (2\pi\hbar)^{-n}\Big(\mathrm{Vol}(a < \frac{1}{2}|p|^2 + V < b) + O(1)\Big).$$

The operator $\widehat{H}_\hbar$ is selfadjoint on the Hilbert space $W^{2,2}(\mathbb{T}^n)$ with respect to $L^2(\mathbb{T}^n)$-scalar product and there exists a complete orthonormal set of eigenfunctions in $W^{2,2}(\mathbb{T}^n)$, see Chap. XIII-16 in Ref. 13.

B. Weak solutions of Hamilton-Jacobi equation

Here we deal with viscosity solutions of the Hamilton-Jacobi equation on $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ (see for example Refs. 2, 4, 6, 7, and 14)

$$\frac{1}{2}|P + \nabla_x S(P,x)|^2 + V(x) = \bar{H}(P), \quad P \in \mathbb{R}^n. \tag{21}$$

The function $\bar{H}(P)$ is called the effective Hamiltonian and can be expressed by

$$\bar{H}(P) = \inf_{v \in C^1(\mathbb{T}^n;\mathbb{R})} \sup_{x\in\mathbb{T}^n} \frac{1}{2}|P + \nabla_x v(x)|^2 + V(x) \tag{22}$$

which is a convex and superlinear function of $P \in \mathbb{R}^n$, see for example Refs. 4 and 6. For the targets of our paper, which are related to Schrödinger eigenfunctions, we will consider the case $P \in \ell \cdot \hbar\,\mathbb{Z}^n$ with $0 < \ell, \hbar \leq 1$.

As showed in Thm. 7.6.2 of Ref. 7, all the Lipschitz continuous weak KAM solutions of negative type coincide with the viscosity solutions. Thanks to Thm 5.3.6 in Ref. 2, any viscosity solution $S(P,\cdot)$ is locally semiconcave with linear modulus in $\mathbb{T}^n$. For an open set $A \subset \mathbb{R}^n$, we recall that $u : A \to \mathbb{R}$ is called semiconcave with linear modulus if

$$u(x+h) + u(x-h) - 2u(x) \leq C|x|^2 \tag{23}$$

for all $x, h \in \mathbb{R}^n$ such that $[x-h, x+h] \in A$. The value $C > 0$ is called the semiconcavity constant. The function u is called locally semiconcave in A if it is semiconcave for every compact subset of A. Now, recalling Theorem 2.3.1 in Ref. 2 it follows that $S(P,\cdot)$ is twice differentiable $\mathcal{L}^n$ - a.e. $x \in \mathbb{T}^n$ and that $\nabla_x S(P,\cdot) \in BV_{\mathrm{loc}}(\mathbb{T}^n;\mathbb{R}^n)$. Alternatively, one can apply Proposition 1.1.3 in Ref. 2, i.e., any S semiconcave can be written as $S = S_1 + S_2$ where S_1 is C^2 and S_2 is concave, and Theorem

7.1 in Ref. 8 about the well posedness of the second order differentiability of convex functions here applied for $-S_2$. We also recall that, thanks to Thm. 4.9.2 in Ref. 7, the map $x \mapsto \nabla_x S(P,x)$ is continuous on $\mathrm{dom}(\nabla_x S) := \{x \in \mathbb{T}^n \mid \exists\, \nabla_x S(P,x)\}$, and

$$\overline{\mathrm{Graph}(P + \nabla S(P,\cdot))} \subset \{(x,p) \in \mathbb{T}^n \times \mathbb{R}^n \mid H(x,p) = \bar{H}(P)\}. \tag{24}$$

In addition to the above regularity results, by denoting $\Sigma(S)$ as the singular set of S (namely, the set of $x \in \mathbb{T}^n$ where S is not differentiable) it has been proved (see Ref. 14) that $\mathbb{T}^n \setminus \overline{\Sigma(S)}$ is an open and dense subset of $\mathbb{T}^n$ and furthermore that S is $C^{1,1}_{\mathrm{loc}}$ on $\mathbb{T}^n \setminus \overline{\Sigma(S)}$.

About the notion of the Aubry set in cotangent bundle of $\mathbb{T}^n$ we remind that

$$\mathcal{A}^*_P = \bigcap_{S \in \mathcal{S}^\mp} \{(x, P + \nabla_x S(P,x)) \mid x \in \mathbb{T}^n \text{ s.t. } \exists\, \nabla_x S(P,x)\} \tag{25}$$

with intersection taken over all Lipschitz continuous weak KAM solutions $\mathcal{S}^\mp$ of negative (or positive) type of the Hamilton-Jacobi equation (21), see Ref. 7. We recall that $\mathcal{A}^*_P$ is a compact set which is invariant under the Hamiltonian flow of H.

In our paper we do not exhibit the study of the semiclassical localization (in the sense of Wigner measures and the semiclassical wave front set) of the eigenfunctions arising in Theorem 1.2; an open problem is to prove this type of result with respect to the Aubry set.

In the following we provide an useful result by using the second order regularity of the map $x \mapsto S(P,x)$ and its linear approximation.

Lemma 2.1. Let $S(P,\cdot) : \mathbb{T}^n \to \mathbb{R}$ be a viscosity solution of (21) where $P \in \mathbb{R}^n$. Let $\Omega \subset\subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$. Then,

$$\|\Delta_x S(P,\cdot)\|_{L^\infty(\Omega)} < +\infty. \tag{26}$$

Let $x_0 \in \mathbb{T}^n \setminus \overline{\Sigma(S)}$ and $0 < \rho(x_0,P) \leq 1$ be such that $B_{2\rho}(x_0) \subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$. Define $r := \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar} \rho(x_0,P)\,(1 + \|\nabla^2 S(P,\cdot)\|_{L^2(B_\rho(x_0))})^{-1}$, $S_0(P,x) := S(P,x_0) + \nabla_x S(P,x_0)\cdot(x - x_0)$. Let $0 < \hbar_0 \leq 1$ be such that $\pi\delta\mathcal{E}(\hbar)e^{-1/\hbar} \leq 1$ for any $0 < \hbar \leq \hbar_0$. Then, we have $\Delta_x S_0(P,x) = 0\ \forall\, x \in B_r(x_0)$ and

$$\|S(P,\cdot) - S_0(P,\cdot)\|_{L^\infty(B_r(x_0))} \leq \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}. \tag{27}$$

Moreover,

$$\Big\| \frac{1}{2}|P + \nabla_x S_0(P,\cdot)|^2 + V(\cdot) - \bar{H}(P) \Big\|_{L^\infty(B_r(x_0))} \leq \|\nabla_x V\|_{C^0(\mathbb{T}^n)} \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}. \tag{28}$$

Proof. We remind that $\mathbb{T}^n \setminus \overline{\Sigma(S)}$ is an open and dense subset of $\mathbb{T}^n$. We can fix $x_0 \in \mathbb{T}^n \setminus \overline{\Sigma(S)}$, and find $0 < \rho(x_0,P) \leq 1$ be such that $B_{2\rho}(x_0) \subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$. Thus, $\overline{B_\rho(x_0)} \subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$. Now we underline that since $\delta\mathcal{E}(\hbar)$ is uniformly bounded then there exists an interval $0 < \hbar \leq \hbar_0$ such that $\pi\delta\mathcal{E}(\hbar)e^{-1/\hbar} \leq 1$. This gives $r \leq \rho(x_0,P)$ and hence $\overline{B_r(x_0)} \subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$.

Since any S is $C^{1,1}_{\mathrm{loc}}$ on $\mathbb{T}^n \setminus \overline{\Sigma(S)}$ then it is twice differentiable $\mathcal{L}^n$ - a.e. and thus for any $\Omega \subset\subset \mathbb{T}^n \setminus \overline{\Sigma(S)}$ we have $\|\nabla^2_x S(P,\cdot)\|_{L^\infty(\Omega)} < +\infty$ and $\|\Delta_x S(P,\cdot)\|_{L^\infty(\Omega)} < +\infty$, and also $\|\nabla^2_x S(P,\cdot)\|_{L^\infty(B_r(x_0))} < +\infty$. By the equality

$$S(P,x) - S_0(P,x) = \int_0^1 \int_0^\lambda \nabla^2_x S(P, x\cdot\mu + x_0(1-\mu))\, d\mu\, d\lambda\, (x - x_0)\cdot(x - x_0), \tag{29}$$

written for $x \in B_r(x_0)$, we have

$$\begin{aligned} |S(P,x) - S_0(P,x)| &\leq \sup_{0\leq\mu\leq 1} |\nabla^2_x S(P, x\cdot\mu + x_0(1-\mu))|\, r^2 \leq \|\nabla^2_x S(P,\cdot)\|_{L^\infty(B_r(x_0))}\, r^2 \\ &\leq \|\nabla^2_x S(P,\cdot)\|_{L^\infty(B_r(x_0))}\, r. \end{aligned} \tag{30}$$

The setting of r gives

$$|S(P,x) - S_0(P,x)| \leq \|\nabla^2_x S(P,\cdot)\|_{L^\infty(B_r(x_0))}\, \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar} \rho(x_0,P)\,(1 + \|\nabla^2 S(P,\cdot)\|_{L^2(B_\rho(x_0))})^{-1}. \tag{31}$$

Since $\|\nabla_x^2 S(P,\cdot)\|_{L^\infty(B_\rho(x_0))} \geq \|\nabla_x^2 S(P,\cdot)\|_{L^\infty(B_r(x_0))}$ and $0 < \rho(x_0,P) \leq 1$ then

$$|S(P,x) - S_0(P,x)| \leq \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}. \tag{32}$$

The expression

$$\frac{1}{2}|P + \nabla_x S_0(P,x)|^2 + V(x) - \bar{H}(P) = \frac{1}{2}|P + \nabla_x S(P,x_0)|^2 + V(x) - \bar{H}(P) \tag{33}$$

reads

$$\frac{1}{2}|P + \nabla_x S(P,x_0)|^2 + V(x_0) - \bar{H}(P) + V(x) - V(x_0) = V(x) - V(x_0). \tag{34}$$

To conclude, for $x \in B_r(x_0)$

$$|V(x) - V(x_0)| \leq \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, r \leq \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}. \tag{35}$$

□

III. MAIN RESULTS

Lemma 3.1. Let $P \mapsto \bar{H}(P)$ given by (22). Then, for any fixed $0 < \hbar \leq 1$, the set of energies

$$\mathcal{E} := \{\bar{H}(P) \mid P \in \ell \cdot \hbar\, \mathbb{Z}^n;\ 0 < \ell \leq 1\} \tag{36}$$

fulfills

$$\overline{\mathcal{E} \cap [\max_{y\in\mathbb{T}^n} V(y), \lambda]} = [\max_{y\in\mathbb{T}^n} V(y), \lambda] \tag{37}$$

for any $\lambda > \max_{y\in\mathbb{T}^n} V(y)$.

Proof. We recall that $P \longmapsto \bar{H}(P)$ is a convex and superlinear function on $\mathbb{R}^n$ (see Prop. 2.2 in Ref. 6). Thus, the convexity implies the continuity of $\bar{H}$ and the superlinearity (i.e., $\lim_{|P|\to+\infty} \bar{H}(P)/|P| = +\infty$) implies that $\bar{H}$ is unbounded. Moreover,

$$\bar{H}(P) := \inf_{v\in C^1(\mathbb{T}^n;\mathbb{R})} \sup_{x\in\mathbb{T}^n} \frac{1}{2}|P + \nabla_x v(x)|^2 + V(x) \tag{38}$$

$$\geq \inf_{v\in C^1(\mathbb{T}^n;\mathbb{R})} \sup_{x\in\mathbb{T}^n} V(x) = \max_{x\in\mathbb{T}^n} V(x). \tag{39}$$

To conclude, we observe that $\{P \in \ell \cdot \hbar\, \mathbb{Z}^n \mid 0 < \ell \leq 1\}$ is dense in $\mathbb{R}^n$. The statement directly follows. □

Lemma 3.2. Let $P \mapsto \bar{H}(P)$ given by (22), and $E_\hbar \in [\max_{y\in\mathbb{T}^n} V(y) + \epsilon, b) \cap \mathrm{Spec}(-\frac{1}{2}\hbar^2\Delta_x + V)$. Then, any set

$$\{P \in \ell \cdot \hbar\, \mathbb{Z}^n \mid 0 < \ell \leq 1; \quad |\bar{H}(P) - E_\hbar| \leq \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}\} \tag{40}$$

is not empty and bounded.

Proof. In view of (37), and since we assume $E_\hbar > \max_{y\in\mathbb{T}^n} V(y)$ it follows that

$$\{P \in \ell \cdot \hbar\, \mathbb{Z}^n \mid 0 < \ell \leq 1; \quad E_\hbar - \delta\mathcal{E}(\hbar)\, e^{-1/\hbar} \leq \bar{H}(P) \leq E_\hbar + \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}\} \tag{41}$$

is not empty. Moreover, since $\bar{H}$ is superlinear (as recalled in the previous lemma) any sublevel is bounded. □

Lemma 3.3. Let $\Omega \subset \mathbb{T}^n$ be an open set, $\varphi \in L^2(\Omega)$ and $(\Psi_\alpha)_{\alpha\in\mathbb{N}}$ be a complete orthonormal set in $L^2(\mathbb{T}^n)$. Then,

$$\varphi = \sum_{\alpha=0}^{\infty} \langle \varphi, \Psi_\alpha|_\Omega \rangle_{L^2(\Omega)} \Psi_\alpha|_\Omega \tag{42}$$

and the set $(\Psi_\alpha|_\Omega)_{\alpha\in\mathbb{N}}$ contains a subset $(\Psi_{\alpha(\beta)}|_\Omega)_{\beta\in\mathbb{N}}$ of linearly independent functions which is a basis of $L^2(\Omega)$.

Proof. Any $\varphi \in L^2(\Omega)$ can be extended to $\widetilde{\varphi} \in L^2(\mathbb{T}^n)$ by requiring $\widetilde{\varphi}(x) = \varphi(x)$ when $x \in \Omega$ and $\widetilde{\varphi}(x) = 0$ otherwise. Whence,

$$\widetilde{\varphi} = \sum_{\alpha=0}^{\infty} \langle \widetilde{\varphi}, \Psi_\alpha \rangle_{L^2(\mathbb{T}^n)} \Psi_\alpha \tag{43}$$

is $L^2(\mathbb{T}^n)$-convergent. Now observe that the map $\psi \mapsto \mathcal{X}_\Omega(\psi) := \mathcal{X}_\Omega \cdot \psi$, where $\mathcal{X}_\Omega$ is the characteristic function of Ω, is linear and $L^2(\mathbb{T}^n)$-bounded operator. Thus, it is a continuous map and it follows the equality

$$\mathcal{X}_\Omega(\widetilde{\varphi}) = \sum_{\alpha=0}^{\infty} \langle \widetilde{\varphi}, \Psi_\alpha \rangle_{L^2(\mathbb{T}^n)} \mathcal{X}_\Omega(\Psi_\alpha). \tag{44}$$

However, it is easily shown that $\langle \widetilde{\varphi}, \Psi_\alpha \rangle_{L^2(\mathbb{T}^n)} = \langle \varphi, \Psi_\alpha|_\Omega \rangle_{L^2(\Omega)}$ and $\mathcal{X}_\Omega(\widetilde{\varphi}) = \widetilde{\varphi}$. This gives,

$$\widetilde{\varphi} = \sum_{\alpha=0}^{\infty} \langle \varphi, \Psi_\alpha|_\Omega \rangle_{L^2(\Omega)} \mathcal{X}_\Omega(\Psi_\alpha). \tag{45}$$

Since any term of the series is zero outside Ω then it follows the $L^2(\Omega)$-convergence, and this directly provides (42). The second statement of the lemma follows from the arbitrary choice of φ. □

Proposition 3.4. Let $\varphi_{P,\hbar}$ and $U_0 := B_r(x_0)$ given by Thm 1.1, and fix the wave functions $(\Psi_{\alpha,\hbar})_{\alpha\in\mathbb{N}}$ as a complete orthonormal set in $L^2(\mathbb{T}^n)$ given by eigenfunctions in $W^{2,2}(\mathbb{T}^n)$ of the Schrödinger operator. Then,

$$\Delta_x \varphi_{P,\hbar} = \sum_{\alpha\in\mathbb{N}} \langle \varphi_{P,\hbar}, \Psi_{\alpha,\hbar} \rangle_{L^2(U_0)} \Delta_x \Psi_{\alpha,\hbar}|_{U_0}. \tag{46}$$

Proof. We remind that $\varphi_{P,\hbar}(x) := (\mathrm{Vol}(B_r(x_0)))^{-1/2} e^{i(P\cdot x + S_0(P,x))/\hbar}$ where $S_0(P,x) := S(P,x_0) + \nabla_x S(P,x_0)\cdot(x - x_0)$. Thus, $\varphi_{P,\hbar} \in W^{2,2}(B_r(x_0))$. Moreover, we also recall that $\Delta_x : W^{2,2}(B_r(x_0)) \to L^2(B_r(x_0))$ is a continuous linear map.

By applying Lemma 3.3, we have

$$\varphi_{P,\hbar} = \lim_{N\to\infty} \sum_{\alpha=0}^{N} \langle \varphi_{P,\hbar}, \Psi_{\alpha,\hbar} \rangle_{L^2(U_0)} \Psi_{\alpha,\hbar}|_{U_0} \tag{47}$$

in $L^2(U_0)$. Hence, by the continuity of the Laplacian we directly obtain

$$\Delta_x \varphi_{P,\hbar} = \lim_{N\to\infty} \sum_{\alpha=1}^{N} \langle \varphi_{P,\hbar}, \Psi_{\alpha,\hbar} \rangle_{L^2(U_0)} \Delta_x \Psi_{\alpha,\hbar}|_{U_0}. \tag{48}$$

□

Proof of Theorem 1. To begin, we prove that $u_{P,\hbar}$ is in $L^2(\Omega)$ and it is normalized.

$$\int_\Omega |u_{P,\hbar}(x)|^2 dx = \int_\Omega \left|(\mathrm{Vol}(\Omega))^{-1/2} e^{i(P\cdot x + S(P,x))/\hbar}\right|^2 dx = \int_\Omega \mathrm{Vol}(\Omega)^{-1} dx = 1. \tag{49}$$

Let S_0 be as in Lemma 2.1 and $\varphi_{P,\hbar}(x) := (\mathrm{Vol}(B_r))^{-1/2} e^{i(P\cdot x + S_0(P,x))/\hbar}$. The normalization of $\varphi_{P,\hbar}$ in $L^2(B_r(x_0))$ is easily proved in the same way. We now look at $u_{P,\hbar}$ when normalized on $\Omega = B_r(x_0)$ and look at an estimate for

$$\int_{B_r} |\varphi_{P,\hbar}(x) - u_{P,\hbar}(x)|^2 dx = 2\,\mathrm{Vol}(B_r)^{-1} \int_{B_r} 1 - \cos([S(P,x) - S_0(P,x)]/\hbar)\, dx \tag{50}$$

$$\leq 2\mathrm{Vol}(B_r)^{-1}\,\mathrm{Vol}(B_r)\, \|S(P,\cdot) - S_0(P,\cdot)\|_{L^\infty(B_r(x_0))}\, \hbar^{-1}. \tag{51}$$

Recalling the estimate (27) we have

$$\int_{B_r} |\varphi_{P,\hbar}(x) - u_{P,\hbar}(x)|^2 dx \leq 2\pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}\, \hbar^{-1}. \tag{52}$$

We now prove that, for any $\hbar > 0$, it holds $u_{P,\hbar} \in \Phi(\Omega)$. Indeed,

$$-\frac{1}{2}\hbar^2 \Delta_x u_{P,\hbar}(x) = \Big(\frac{|P + \nabla_x S(P,x)|^2}{2} - \frac{\hbar}{2}\Delta_x S(P,x)\Big) u_{P,\hbar}(x). \tag{53}$$

Thus,

$$\hbar^2 \|\Delta_x u_{P,\hbar}\|_{L^2(\Omega)} \tag{54}$$

$$\leq \|P + \nabla_x S(P,\cdot)\|^2_{L^\infty(\Omega)} \|u_{P,\hbar}\|_{L^2(\Omega)} + \hbar \|\Delta_x S(P,x)\|_{L^\infty(\Omega)} \|u_{P,\hbar}\|_{L^2(\Omega)} \tag{55}$$

$$\leq \|P + \nabla_x S(P,\cdot)\|^2_{L^\infty(\Omega)} + \hbar \|\Delta_x S(P,x)\|_{L^\infty(\Omega)} < +\infty. \tag{56}$$

In the same way we prove that $\varphi_{P,\hbar} \in \Phi(B_r(x_0))$, indeed

$$\hbar^2 \|\Delta_x \varphi_{P,\hbar}\|_{L^2(B_r(x_0))} \tag{57}$$

$$\leq \|P + \nabla_x S_0(P,\cdot)\|^2_{L^\infty(B_r(x_0))} \|\varphi_{P,\hbar}\|_{L^2(B_r(x_0))} + \hbar \|\Delta_x S_0(P,x)\|_{L^\infty(B_r(x_0))} \|\varphi_{P,\hbar}\|_{L^2(B_r(x_0))}$$

$$\leq \|P + \nabla_x S_0(P,\cdot)\|^2_{L^\infty(B_r(x_0))} = |P + \nabla_x S(P,x_0)|^2 < +\infty. \tag{58}$$

We now look at

$$-\frac{1}{2}\hbar^2 \Delta_x \varphi_{P,\hbar}(x) + V(x)\varphi_{P,\hbar}(x) - \bar{H}(P)\varphi_{P,\hbar}(x) \tag{59}$$

$$= \Big(\frac{|P + \nabla_x S_0(P,x)|^2}{2} + V(x) - \bar{H}(P) - \frac{\hbar}{2}\Delta_x S_0(P,x)\Big)\varphi_{P,\hbar}(x) \tag{60}$$

$$= \Big(\frac{|P + \nabla_x S_0(P,x)|^2}{2} + V(x) - \bar{H}(P)\Big)\varphi_{P,\hbar}(x). \tag{61}$$

By applying again the $L^2(B_r(x_0))$-normalization of $\varphi_{P,\hbar}$ and the estimate (28) shown in Lemma 2.1 we obtain the estimate

$$\Big\| -\frac{1}{2}\hbar^2 \Delta_x \varphi_{P,\hbar} + V(x)\varphi_{P,\hbar} - \bar{H}(P)\varphi_{P,\hbar}\Big\|_{L^2(B_r(x_0))} \leq \|\nabla_x V\|_{L^2(B_r(x_0))} \pi\, \delta \mathcal{E}(\hbar) e^{-1/\hbar}. \tag{62}$$

To conclude,

$$-\frac{1}{2}\hbar^2 \Delta_x u_{P,\hbar} + V(x) u_{P,\hbar} - \bar{H}(P) u_{P,\hbar} \tag{63}$$

$$= \Big(\frac{|P + \nabla_x S(P,x)|^2}{2} + V(x) - \bar{H}(P) - \frac{\hbar}{2}\Delta_x S(P,x)\Big) u_{P,\hbar}(x) \tag{64}$$

$$= -\frac{\hbar}{2}\Delta_x S(P,x) u_{P,\hbar}(x) \tag{65}$$

and the $L^\infty(\Omega)$ - estimate of (65) given by $\hbar \|\Delta_x S(P,\cdot)\|_{L^\infty(\Omega)}/2$ is an $O(\hbar)$ - remainder. □

Proof of Theorem 2. The point (ii) implies (i).

To prove the converse, let us now assume that the point (i) holds true. We denote $(\Psi_{\alpha,\hbar})_{\alpha\in\mathbb{N}}$ a complete orthonormal set in $L^2(\mathbb{T}^n)$ given by eigenfunctions in $W^{2,2}(\mathbb{T}^n)$ of the Schrödinger operator. Whence

$$-\frac{1}{2}\hbar^2 \Delta_x \Psi_{\alpha,\hbar} + V(x)\Psi_{\alpha,\hbar} = E_{\alpha,\hbar}\Psi_{\alpha,\hbar},$$

where

$$\Psi_{\alpha,\hbar} \in W^{2,2}(\mathbb{T}^n)$$

and $\langle \Psi_{\alpha,\hbar}, \Psi_{j,\hbar}\rangle_{L^2(\mathbb{T}^n)} = \delta_{\alpha j}$. We remind that all the eigenspaces are finite dimensional and thus $\alpha \mapsto E_{\alpha,\hbar}$ is a constant map on bounded subsets of $\mathbb{N}$ whose cardinality is the dimension of the eigenspaces. In what follows, we denote $U_0 := B_r(x_0)$.

By Lemma 3.3 the family $(\Psi_{\alpha,\hbar}|_{U_0})_{\alpha\in\mathbb{N}}$ generates $L^2(U_0)$. For the sake of simplicity, we assume that $\Psi_{\alpha,\hbar}|_{U_0}$ are all linearly independent and spanning $L^2(U_0)$; in fact as shown in Lemma 3.3

there always exists a subsequence $\Psi_{\alpha(\beta),\hbar}|_{U_0}$ of these functions which are linearly independent and generating $L^2(U_0)$.

We now fix an arbitrary $\varphi_\hbar \in \mathrm{Span}(\varphi_{P,\hbar})$, namely,

$$\varphi_\hbar(x) := \sum_P c_P\, \varphi_{P,\hbar}(x), \quad x \in U_0, \tag{66}$$

where the sum is taken over a finite set of points $P \in \ell \cdot \hbar\, \mathbb{Z}^n$ satisfying $|E_\hbar - \bar{H}(P)| \le \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}$, such that $\varphi_{P,\hbar}$ are linearly independent, and $c_P \in \mathbb{C}$ not depending on $\hbar$. This implies that the sum in (66) is convergent in $L^2(U_0)$ to a nonvanishing wave function, and moreover $\widehat{H}_\hbar := -\frac{1}{2}\hbar^2\Delta_x + V$ works as

$$\widehat{H}_\hbar\, \varphi_\hbar = \sum_P c_P\, \widehat{H}_\hbar\, \varphi_{P,\hbar}. \tag{67}$$

Recalling Theorem 1.1, it follows

$$\widehat{H}_\hbar \varphi_\hbar = \sum_P c_P \left(\bar{H}(P)\varphi_{P,\hbar} - \frac{\hbar}{2}\Delta_x S \cdot \varphi_{P,\hbar}\right) \tag{68}$$

$$= \bar{H}(P) \sum_P c_P\, \varphi_{P,\hbar} - \sum_P c_P\, (V(x) - V(x_0))\varphi_{P,\hbar} \tag{69}$$

$$= \bar{H}(P)\varphi_\hbar - \sum_P c_P\, (V(x) - V(x_0))\varphi_{P,\hbar}. \tag{70}$$

We now decompose,

$$\varphi_\hbar(x) = \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar}(x), \quad x \in U_0, \tag{71}$$

where the series is $L^2(U_0)$ - convergent thanks to Lemma 3.3. By (68) and the above equality it follows

$$\widehat{H}_\hbar \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar} = \bar{H}(P) \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar} \tag{72}$$

$$- \sum_P c_P\, (V(x) - V(x_0))\varphi_{P,\hbar}. \tag{73}$$

By applying Lemma 3.4,

$$-\frac{1}{2}\hbar^2\Delta_x\varphi_\hbar = -\frac{1}{2}\hbar^2 \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Delta_x \Psi_{\alpha,\hbar} \tag{74}$$

and since $\varphi \mapsto V\varphi$ is L^2 - bounded (hence continuous) on $L^2(U_0)$,

$$V(x)\varphi_\hbar = \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} V(x)\Psi_{\alpha,\hbar}. \tag{75}$$

Thus, the equality (72) can be rewritten as

$$\sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2} \widehat{H}_\hbar\, \Psi_{\alpha,\hbar} = \bar{H}(P) \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar} \tag{76}$$

$$- \sum_P c_P (V(x) - V(x_0))\varphi_{P,\hbar}. \tag{77}$$

Namely,

$$\sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (E_{\alpha,\hbar} - \bar{H}(P))\, \Psi_{\alpha,\hbar} = -\sum_P c_P (V(x) - V(x_0))\varphi_{P,\hbar}. \tag{78}$$

Recalling that P are selected in such a way $|E_\hbar - \bar{H}(P)| \le \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}$, we look at

$$\sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (E_{\alpha,\hbar} - E_\hbar)\, \Psi_{\alpha,\hbar} \tag{79}$$

$$= - \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (E_\hbar - \bar{H}(P))\, \Psi_{\alpha,\hbar} \tag{80}$$

$$- \sum_P c_P (V(x) - V(x_0)) \varphi_{P,\hbar}.$$

By denoting as $R_\hbar$ the right-hand side of (80), it easily follows that

$$\|R_\hbar\|_{L^2(U_0)} \le \delta\mathcal{E}(\hbar)\, e^{-1/\hbar} \|\varphi_\hbar\|_{L^2(U_0)} + \|V(\cdot) - V(x_0)\|_{L^\infty(U_0)} \|\varphi_\hbar\|_{L^2(U_0)}. \tag{81}$$

In view of Lemma 2.1 and thanks to the setting $r \le \pi\, \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}$,

$$\|R_\hbar\|_{L^2(U_0)} \le \left(\delta\mathcal{E}(\hbar)\, e^{-1/\hbar} + \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, \pi\, \delta\mathcal{E}(\hbar) e^{-1/\hbar}\right) \|\varphi_\hbar\|_{L^2(U_0)} \tag{82}$$

$$= \left(1 + \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, \pi\right) \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}\, \|\varphi_\hbar\|_{L^2(U_0)} \tag{83}$$

where we underline that $\|\varphi_\hbar\|_{L^2(U_0)} \le \sum_P |c_P|$ and that the sum is supposed to be finite.

The above equality now reads

$$\sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (E_{\alpha,\hbar} - E_\hbar)\, \Psi_{\alpha,\hbar} = R_\hbar \tag{84}$$

or equivalently

$$\sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (2\pi\hbar)^{-n} (E_{\alpha,\hbar} - E_\hbar)\, \Psi_{\alpha,\hbar} = (2\pi\hbar)^{-n}\, R_\hbar. \tag{85}$$

Notice that the above series is in fact computed for the eigenfunctions $\Psi_{\alpha,\hbar}$ (restricted to U_0) with eigenvalues $E_{\alpha,\hbar} \ne E_\hbar$. Let us denote with Π the vector space generated by all such eigenfunctions and define $T : L^2(U_0) \to \Pi$ the linear operator

$$T(\varphi_\hbar)(x) := \sum_{\alpha\in\mathbb{N}} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} (2\pi\hbar)^{-n} (E_{\alpha,\hbar} - E_\hbar)\, \Psi_{\alpha,\hbar}(x), \quad x \in U_0. \tag{86}$$

If $B := \{\alpha \in \mathbb{N} \mid E_{\alpha,\hbar} \ne E_\hbar\}$ and

$$\widetilde{\varphi}_\hbar(x) := - \sum_{\alpha\in B} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)}\, \Psi_{\alpha,\hbar}(x), \quad x \in U_0 \tag{87}$$

then $\widetilde{\varphi}_\hbar \in \Pi$ and $T(\varphi_\hbar) = -T(\widetilde{\varphi}_\hbar)$. Notice that $T : \Pi \to \Pi$ is a diagonal operator with respect to the basis $\Psi_{\alpha,\hbar}|_{U_0}$ of Π, with non-vanishing eigenvalues $(2\pi\hbar)^{-n}(E_{\alpha,\hbar} - E_\hbar) \ne 0$. As a consequence, T is invertible on Π. Hence, (85) becomes

$$- T^{-1}(T(\widetilde{\varphi}_\hbar)) = (2\pi\hbar)^{-n}\, T^{-1}(R_\hbar), \tag{88}$$

which means that

$$- \widetilde{\varphi}_\hbar = (2\pi\hbar)^{-n}\, T^{-1}(R_\hbar). \tag{89}$$

It is easily proved that

$$\|T^{-1}\|_{\Pi\to\Pi} \le \sup_{\alpha\in\mathbb{N}} (2\pi\hbar)^n\, (E_{\hbar,\alpha+1} - E_{\hbar,\alpha})^{-1} \tag{90}$$

$$\le [\inf_{\alpha\in\mathbb{N}} (2\pi\hbar)^{-n}\, (E_{\hbar,\alpha+1} - E_{\hbar,\alpha})]^{-1} = \delta\mathcal{E}(\hbar)^{-1} < +\infty. \tag{91}$$

By applying (83), (89) and (91) we are now in the position to provide the estimate

$$\|\widetilde{\varphi}_\hbar\|_{L^2(U_0)} \le (2\pi\hbar)^{-n}\, \delta\mathcal{E}(\hbar)^{-1} \left(1 + \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, \pi\right) \delta\mathcal{E}(\hbar)\, e^{-1/\hbar}\, \|\varphi_\hbar\|_{L^2(U_0)}$$

$$\le (2\pi\hbar)^{-n} \left(1 + \|\nabla_x V\|_{C^0(\mathbb{T}^n)}\, \pi\right) e^{-1/\hbar}\, \|\varphi_\hbar\|_{L^2(U_0)}. \tag{92}$$

As a consequence, the following decomposition holds true:

$$\varphi_\hbar(x) + \widetilde{\varphi}_\hbar(x) = \sum_{\alpha\in A} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar}(x), \quad x \in U_0, \tag{93}$$

where $A := \mathbb{N} \setminus B = \{\alpha \in \mathbb{N} \mid E_{\alpha,\hbar} = E_\hbar\}$ is a finite set. Now define

$$\psi_\hbar(x) := \sum_{\alpha\in A} \langle \varphi_\hbar, \Psi_{\alpha,\hbar}\rangle_{L^2(U_0)} \Psi_{\alpha,\hbar}(x), \quad x \in \mathbb{T}^n, \tag{94}$$

which is a Schrödinger eigenfunction in $W^{2,2}(\mathbb{T}^n)$ (not necessarily normalized on $\mathbb{T}^n$) for the eigenvalue $E_\hbar$. □

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