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SUMS OF ONE PRIME POWER AND TWO SQUARES OF PRIMES IN SHORT INTERVALS

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Let $k \geq 1$ be an integer. We prove that a suitable asymptotic formula for the average number of representations of integers $n = p_1^k + p_2^2 + p_3^2$, where p_1, p_2, p_3 are prime numbers, holds in intervals shorter than the ones previously known.

1. Introduction

The problem of representing an integer as a sum of a prime power and of two prime squares is classical. It is conjectured that every sufficiently large n subject to some congruence conditions can be represented as $n = p_1^k + p_2^2 + p_3^2$, where $k \geq 1$ is an integer. Let now N be a large integer and denote by $E_k(N)$ the cardinality of the set of integers not exceeding N that satisfy the necessary congruence conditions but can not be represented as the sum of a k-th prime power and two prime squares. Several results about $E_k(N)$ were obtained; the first one to prove a nontrivial estimate for $E_k(N)$ was Hua [3]. Later Schwarz [15] and several other authors further improved such an estimate; we recall the contribution of Harman and Kumchev [2], Leung and Liu [9], Li [10] and Lü [12]. Let $\varepsilon > 0$; so far the best known estimates are $E_1(N) \ll N^{1/3+\varepsilon}$ by Zhao [17], $E_2(N) \ll N^{17/20+\varepsilon}$ by Harman and Kumchev [2], $E_3(N) \ll N^{15/16+\varepsilon}$ and, for $k \geq 4$, $E_k(N) \ll N^{1-1/(4k^2)+\varepsilon}$ both by Brüdern [1]. Recently Liu and Zhang [11] further improved such results to $E_3(N) \ll N^{11/12+\varepsilon}$ and, for $k \geq 4$, $E_k(N) \ll N^{1-\theta(k)+\varepsilon}$, where $\theta(k) = \min(2^{k/2+2}; k(k+2))$ if k is even, $\theta(k) = \min(3 \cdot 2^{(k+1)/2}; 8\lceil (k+1)^2/8 \rceil)$ if k is odd and $\lceil x \rceil$ is the least integer greater than or equal to x.

Concerning short intervals $[N, N+H]$, $H = o(N)$, we recall here that Kumchev and Liu [4] proved that for every $A > 0$ we have

$$E_2(N+H) - E_2(N) \ll H (\log N)^{-A},$$

provided that $H \geq N^{7/20}$. Let now

$$r_k(n) = \sum_{p_1^k + p_2^2 + p_3^2 = n} \log p_1 \log p_2 \log p_3. \tag{1}$$

In this paper we study the average behaviour of $r_k(n)$ over short intervals $[N, N+H]$, $H = o(N)$ thus generalising our result in [6] which just deals with the case $k = 1$.

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Theorem 1. *Let $N \geq 2$, $1 \leq H \leq N$, $k \geq 1$ be integers. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4} H N^{1/k} + \mathcal{O}_k\left(H N^{1/k} \exp\left(-C \left(\frac{\log N}{\log\log N} \right)^{1/3} \right) \right)$$

as $N \to \infty$, uniformly for $N^{1-5/(6k)+\varepsilon} \leq H \leq N^{1-\varepsilon}$ for $k \geq 2$ and $N^{7/12+\varepsilon} \leq H \leq N^{1-\varepsilon}$ for $k = 1$.

It is worth remarking that the formula in Theorem 1 implies that every interval $[N, N+H]$ contains an integer which is a sum of a prime k-th power and two prime squares, where $N^{1-5/(6k)+\varepsilon} \leq H \leq N^{1-\varepsilon}$ for $k \geq 2$ and $N^{7/12+\varepsilon} \leq H \leq N^{1-\varepsilon}$ for $k = 1$. In fact, for $k = 1, 2$ Zhao's and Kumchev and Liu's estimates previously mentioned lead to better consequences than our Theorem 1, but for $k \geq 3$ our result gives nontrivial information.

Assuming that the Riemann hypothesis (RH) holds, we prove that a suitable asymptotic formula for such an average of $r_k(n)$ holds in much shorter intervals. We need the following auxiliary function: let

$$E(k) := \begin{cases} N^{3/2} \log N + H N^{3/4} (\log N)^{3/2} & \text{if } k = 1, \\ N \log N + H N^{1/4} (\log N)^2 & \text{if } k = 2, \\ N^{5/6} \log N + H N^{1/4} \log N + H^{1/2} N^{1/2} \log N & \text{if } k = 3, \\ N^{3/4+1/k} \log N & \text{if } k \geq 4. \end{cases} \tag{2}$$

In the remaining part of the paper we will use the notation $f = \infty(g)$ with the meaning of $g = o(f)$. We have the following

Theorem 2. *Assume the Riemann hypothesis (RH). Let $N \geq 2$, $1 \leq H \leq N$, $k \geq 2$ be integers. We have*

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4} H N^{1/k} + \mathcal{O}_k\left(H^2 N^{1/k-1} + H^{1/2} N^{1/2+1/(2k)} (\log N)^2 + E(k) \right)$$

as $N \to \infty$, uniformly for $\infty(N^{1-1/k} (\log N)^4) \leq H \leq o(N)$, where $E(k)$ is defined in (2).

We remark that a version for $k = 1$ of Theorem 2 was obtained in [6] and that the definition of $E(1)$ in (2) corresponds to the error term estimate given there. We further remark that the formula in Theorem 2 implies that every interval $[N, N+H]$ contains an integer which is a sum of a prime power and two prime squares, where $\infty(N^{1-1/k} (\log N)^4) \leq H = o(N)$.

The proofs of both Theorems 1 and 2 use the original Hardy–Littlewood settings of the circle method to exploit the easier main term treatment they allow (comparing with the one which would follow using Lemmas 2.3 and 2.9 of Vaughan [16]).

It is worth remarking that the expected best result using circle method techniques is $H \geq N^{1-1/k}$; so our Theorem 2, under the assumption of the Riemann hypothesis, comes very close to this bound. We also obtained similar results in [7; 8].

2. Notation and lemmas

In this section we make the necessary preparations for the application of the appropriate version of the circle method in (6) below. We rewrite the average value of r_k, defined in (1), over the short interval $[N+1, N+H]$ as the familiar integral of the product of suitable exponential sums over primes. These exponential sums have a leading term suggested by the prime number theorem. In this section, we provide several bounds for the ensuing error terms, in various average forms. These will be used throughout Sections 3 and 4. See also the comments at the beginning of each section.

Let $e(\alpha) = e^{2\pi i\alpha}$, $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $L = \log N$, $z = 1/N - 2\pi i\alpha$,

$$\tilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha) \quad \text{and} \quad \tilde{V}_\ell(\alpha) = \sum_{p=2}^{\infty} \log p \, e^{-p^\ell/N} e(p^\ell \alpha).$$

We maintain here the tilde-notation for coherence with other papers in the literature in which it is used to distinguish the infinite exponential sums over primes from the one having a finite number of summands. We remark that

$$|z|^{-1} \ll \min(N, |\alpha|^{-1}). \tag{3}$$

We further set

$$U(\alpha, H) = \sum_{m=1}^{H} e(m\alpha),$$

and, moreover, we also have the usual numerically explicit inequality

$$|U(\alpha, H)| \le \min(H; |\alpha|^{-1}), \tag{4}$$

see, e.g., on page 39 of Montgomery [13]. We list now the needed preliminary results.

Lemma 3 [5, Lemma 3]. *Let $\ell \ge 1$ be an integer. Then* $|\tilde{S}_\ell(\alpha) - \tilde{V}_\ell(\alpha)| \ll_\ell N^{1/(2\ell)}$.

Lemma 4 [6, Lemma 4]. *Let N be a positive integer and $\mu > 0$. Then*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha) \, \mathrm{d}\alpha = e^{-n/N} \frac{n^{\mu-1}}{\Gamma(\mu)} + \mathcal{O}_\mu\left(\frac{1}{n}\right),$$

where Γ denotes Euler's gamma function, uniformly for $n \ge 1$.

Lemma 5. *Let ε be an arbitrarily small positive constant, $\ell \ge 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} \left| \tilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \right|^2 \mathrm{d}\alpha \ll_\ell N^{2/\ell - 1} \exp\left(-c_1 \left(\frac{L}{\log L}\right)^{1/3}\right)$$

uniformly for $0 \le \xi < N^{-1+5/(6\ell)-\varepsilon}$. Assuming RH we get

$$\int_{-\xi}^{\xi} \left| \tilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \right|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell} \xi L^2$$

uniformly for $0 \le \xi \le \frac{1}{2}$.

Proof. It follows the line of Lemma 3 of [6] and Lemma 1 of [5]; we just correct an oversight in their proofs. Both (8) on page 49 of [6] and (6) on page 423 of [5] should read as

$$\int_{1/N}^{\xi} \left| \sum_{\rho:\, \gamma > 0} z^{-\rho/\ell} \Gamma(\rho/\ell) \right|^2 \mathrm{d}\alpha \le \sum_{k=1}^{K} \int_{\eta}^{2\eta} \left| \sum_{\rho:\, \gamma > 0} z^{-\rho/\ell} \Gamma(\rho/\ell) \right|^2 \mathrm{d}\alpha,$$

where $\eta = \eta_k = \xi/2^k$, $1/N \le \eta \le \xi/2$ and K is a suitable integer satisfying $K = \mathcal{O}(L)$. The remaining part of the proofs is left untouched. Hence such oversights do not affect the final result of Lemma 3 of [6] and Lemma 1 of [5]. □

Lemma 6 [5, Lemma 2]. *Let $\ell \geq 2$ be an integer, $f(\ell) = L^2$ if $\ell = 2$ and $f(\ell) = 1$ if $\ell > 2$. Let further $0 < \xi \leq \frac{1}{2}$. Then*

$$\int_{-\xi}^{\xi} |\tilde{S}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} L + f(\ell)$$

and

$$\int_{-\xi}^{\xi} |\tilde{V}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \xi N^{1/\ell} L + f(\ell).$$

Proof. The first part was proved in Lemma 2 of [5]. For the second part we argue analogously. We use Corollary 2 of Montgomery and Vaughan [14] with $T = \xi$, $a_r = \log(r) \exp(-r^\ell/N)$ if r is prime, $a_r = 0$ otherwise and $\lambda_r = 2\pi r^\ell$. By the prime number theorem we get

$$\int_{0}^{\xi} |\tilde{V}_\ell(\alpha)|^2 \, \mathrm{d}\alpha = \sum_p (\log p)^2 e^{-2p^\ell/N} (\xi + \mathbb{O}(\delta_p^{-1})) \ll_\ell \xi N^{1/\ell} L + \sum_p (\log p)^2 p^{1-\ell} e^{-2p^\ell/N}$$

since $\delta_r = \lambda_r - \lambda_{r-1} \gg_\ell r^{\ell-1}$. The last term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L^2$ otherwise. The second part of Lemma 6 follows. □

Lemma 7. *Let $\ell \geq 2$ be an integer, $f(\ell) = L^2$ if $\ell = 2$ and $f(\ell) = 1$ if $\ell > 2$. Let $0 < \tau < \omega \leq \frac{1}{2}$ and $\mu > 0$. Let further $I(\tau, \omega) := [-\omega, -\tau] \cup [\tau, \omega]$. Then we have*

$$\int_{I(\tau,\omega)} |\tilde{S}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|^\mu} \ll_{\ell,\mu} N^{1/\ell} L (\omega^{1-\mu} + \tau^{1-\mu} \oplus \log(\omega/\tau)) + f(\ell) \tau^{-\mu}$$

and

$$\int_{I(\tau,\omega)} |\tilde{V}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|^\mu} \ll_{\ell,\mu} N^{1/\ell} L (\omega^{1-\mu} + \tau^{1-\mu} \oplus \log(\omega/\tau)) + f(\ell) \tau^{-\mu},$$

where $\oplus$ means that such a term is present only if $\mu = 1$. Assuming further that RH holds, we also get

$$\int_{I(\tau,\omega)} \left| \tilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \right|^2 \frac{\mathrm{d}\alpha}{|\alpha|^\mu} \ll_{\ell,\mu} N^{1/\ell} L^2 (\omega^{1-\mu} + \tau^{1-\mu} \oplus \log(\omega/\tau)),$$

where $\oplus$ means that such a term is present only if $\mu = 1$.

Proof. We first work on $[\tau, \omega]$. By partial integration and Lemma 6 we get that

$$\begin{aligned}
\int_{\tau}^{\omega} |\tilde{S}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^\mu} &\ll \omega^{-\mu} \int_{-\omega}^{\omega} |\tilde{S}_\ell(\alpha)|^2 \, \mathrm{d}\alpha + \tau^{-\mu} \int_{-\tau}^{\tau} |\tilde{S}_\ell(\alpha)|^2 \, \mathrm{d}\alpha + \mu \int_{\tau}^{\omega} \left(\int_{-\xi}^{\xi} |\tilde{S}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \right) \frac{\mathrm{d}\xi}{\xi^{\mu+1}} \\
&\ll_\ell \omega^{-\mu} (\omega N^{1/\ell} L + f(\ell)) + \tau^{-\mu} (\tau N^{1/\ell} L + f(\ell)) + \mu \int_{\tau}^{\omega} \frac{\xi N^{1/\ell} L + f(\ell)}{\xi^{\mu+1}} \, \mathrm{d}\xi \\
&\ll_{\ell,\mu} N^{1/\ell} L (\omega^{1-\mu} + \tau^{1-\mu} \oplus \log(\omega/\tau)) + f(\ell) \tau^{-\mu}.
\end{aligned}$$

A similar computation proves the result in $[-\omega, -\tau]$ too. The estimate on $\tilde{V}_\ell(\alpha)$ can be obtained analogously. The estimate on $\tilde{S}_\ell(\alpha) - \Gamma(1/\ell)/(\ell z^{1/\ell})$ follows the same line but we need Lemma 5 instead of Lemma 6. □

In the following we will also need a fourth-power average of $\tilde{S}_2(\alpha)$.

Lemma 8 [6, Lemma 5]. *We have*

$$\int_{-1/2}^{1/2} |\tilde{S}_2(\alpha)|^4 \, \mathrm{d}\alpha \ll N L^2.$$

3. Proof of Theorem 1

We write the average that we want to study as an integral; see (6). We approximate $\tilde{V}_k$ by means of $\tilde{S}_k$ (see Lemma 3) and then split $\tilde{S}_k$ as main term and error term as in (7). This gives rise to the decomposition in (8). An important feature that we have to take into account is that the L^2-average for the error term provided by Lemma 5 only holds in a rather restricted neighbourhood of 0. Hence, we need a different argument to bound the contribution of the "periphery" of the interval of integration. The final error term, therefore, depends both on the width of the "major arc" around 0 and on the quality of Lemma 5. The allowed range for H is also a direct consequence of the constraint on ξ in the hypothesis of the same lemma.

Let $\varepsilon > 0$ and $H > 2B$, where

$$B = B(N, d) = \exp\biggl(d\Bigl(\frac{\log N}{\log\log N}\Bigr)^{1/3}\biggr), \tag{5}$$

where $d = d(\varepsilon) > 0$ will be chosen later. Recalling (1), we may write

$$\sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) = \int_{-1/2}^{1/2} \tilde{V}_k(\alpha)\tilde{V}_2(\alpha)^2 U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha. \tag{6}$$

We find it also convenient to set

$$E_\ell(\alpha) := \tilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}. \tag{7}$$

Letting $I(B, H) := \bigl[-\frac{1}{2}, -B/H\bigr] \cup \bigl[B/H, \frac{1}{2}\bigr]$, using the approximations given by Lemma 3 and recalling that $\Gamma\bigl(\frac{1}{2}\bigr) = \pi^{1/2}$, we can write

$$\begin{aligned}
\sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) &= \int_{-B/H}^{B/H} \frac{\pi\Gamma(1/k)}{4kz^{1+1/k}} U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-B/H}^{B/H} \frac{\Gamma(1/k)}{kz^{1/k}}\Bigl(\tilde{S}_2(\alpha)^2 - \frac{\pi}{4z}\Bigr) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-B/H}^{B/H} E_k(\alpha)\tilde{S}_2(\alpha)^2 U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-1/2}^{1/2} \tilde{V}_k(\alpha)(\tilde{V}_2(\alpha)^2 - \tilde{S}_2(\alpha)^2) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-1/2}^{1/2} \tilde{S}_2(\alpha)^2(\tilde{V}_k(\alpha) - \tilde{S}_k(\alpha)) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{I(B,H)} \tilde{S}_k(\alpha)\tilde{S}_2(\alpha)^2 U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
&= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6,
\end{aligned} \tag{8}$$

say. Now we evaluate these terms.

3.1. Evaluation of $\mathcal{J}_1$.

Using Lemma 4, (3) and

$$e^{-n/N} = e^{-1} + \mathcal{O}(H/N) \tag{9}$$

for $n \in [N+1, N+H]$, $1 \le H \le N$, we immediately get

$$(10)\qquad \mathcal{J}_1 = \frac{\pi\Gamma(1/k)}{4k\Gamma(1+1/k)} \sum_{n=N+1}^{N+H} n^{1/k} e^{-n/N} + \mathcal{O}_k\Big(\frac{H}{N}\Big) + \mathcal{O}_k\Big(\int_{B/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{2+1/k}}\Big)$$
$$= \frac{\pi}{4e} H N^{1/k} + \mathcal{O}_k\Big(H^2 N^{1/k-1} + N^{1/k} + \Big(\frac{H}{B}\Big)^{1+1/k}\Big).$$

3.2. Estimation of $\mathcal{J}_6$. Using $\tilde{S}_k(\alpha) \ll_k N^{1/k}$, (4), Lemma 7 with $\mu = 1$, $\tau = B/H$ and $\omega = \frac{1}{2}$, we obtain that

$$(11)\qquad \mathcal{J}_6 \ll_k N^{1/k} \int_{B/H}^{1/2} \frac{|\tilde{S}_2(\alpha)|^2}{\alpha} \,\mathrm{d}\alpha \ll_k N^{1/k} L^2 \Big(N^{1/2} + \frac{H}{B}\Big)$$

which, comparing with (10), is under control if $H = \infty(N^{1/2}L^2)$ and $B = \infty(L^2)$ (which is fine thanks to (5)).

3.3. Estimation of $\mathcal{J}_5$. By (4), Lemmas 3, 6 and 7 with $\mu = 1$, $\tau = 1/H$ and $\omega = \frac{1}{2}$, we get

$$(12)\qquad \mathcal{J}_5 \ll \int_{-1/2}^{1/2} |\tilde{S}_2(\alpha)|^2 |\tilde{V}_k(\alpha) - \tilde{S}_k(\alpha)| |U(-\alpha, H)| \,\mathrm{d}\alpha$$
$$\ll_k H N^{1/(2k)} \int_{-1/H}^{1/H} |\tilde{S}_2(\alpha)|^2 \,\mathrm{d}\alpha + N^{1/(2k)} \int_{1/H}^{1/2} \frac{|\tilde{S}_2(\alpha)|^2}{\alpha} \,\mathrm{d}\alpha$$
$$\ll_k H N^{1/(2k)} \Big(\frac{N^{1/2}L}{H} + L^2\Big) + N^{1/(2k)} (N^{1/2}L^2 + HL^2)$$
$$\ll_k N^{1/(2k)} (N^{1/2} + H) L^2.$$

which, comparing with (10), is under control if $H = \infty(N^{1/2-1/(2k)}L^2)$.

3.4. Estimation of $\mathcal{J}_4$. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$, Lemma 3 and $\tilde{V}_k(\alpha) \ll_k N^{1/k}$, we have

$$\tilde{V}_k(\alpha)(\tilde{V}_2(\alpha)^2 - \tilde{S}_2(\alpha)^2) \ll_k |\tilde{V}_k(\alpha)| (|\tilde{V}_2(\alpha)| |\tilde{V}_2(\alpha) - \tilde{S}_2(\alpha)| + |\tilde{V}_2(\alpha) - \tilde{S}_2(\alpha)|^2)$$
$$\ll_k N^{1/4} |\tilde{V}_k(\alpha)| |\tilde{V}_2(\alpha)| + N^{1/2+1/k}.$$

Clearly we have

$$(13)\qquad \mathcal{J}_4 \ll_k N^{1/4} \int_{-1/2}^{1/2} |\tilde{V}_k(\alpha)| |\tilde{V}_2(\alpha)| |U(-\alpha, H)| \,\mathrm{d}\alpha + N^{1/2+1/k} \int_{-1/2}^{1/2} |U(-\alpha, H)| \,\mathrm{d}\alpha = K_1 + K_2,$$

say. Using (4) we get

$$(14)\qquad \int_{-1/2}^{1/2} |U(-\alpha, H)| \,\mathrm{d}\alpha \ll \int_{-1/H}^{1/H} H \,\mathrm{d}\alpha + \int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha} \ll L$$

and hence, by (13)–(14), we can write

$$(15)\qquad K_2 \ll_k N^{1/2+1/k} L,$$

for every $k \ge 1$.

Now we estimate K_1; depending on k, we need to perform different computations.

Let $k=1$. In this case we will make use of the estimate

$$\int_{-1/2}^{1/2} |\tilde{V}_1(\alpha)|^2 \, \mathrm{d}\alpha = \sum_p (\log p)^2 e^{-2p/N} \ll NL \tag{16}$$

which immediately follows from the prime number theorem. Using the Cauchy–Schwarz inequality, (16), (4) and Lemmas 6 and 7 with $\mu=2$, $\tau=1/H$ and $\omega=\frac{1}{2}$, we obtain that

$$\begin{aligned} K_1 &\ll N^{1/4}\Big(\int_{-1/2}^{1/2} |\tilde{V}_1(\alpha)|^2 \, \mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/2}^{1/2} |\tilde{V}_2(\alpha)|^2 |U(-\alpha,H)|^2 \, \mathrm{d}\alpha\Big)^{1/2} \\ &\ll N^{3/4}L^{1/2}\Big[H^2\Big(\frac{N^{1/2}L}{H}+L^2\Big)+N^{1/2}L+HN^{1/2}L+H^2L^2\Big]^{1/2} \\ &\ll HN^{3/4}L^{3/2}+H^{1/2}NL. \end{aligned} \tag{17}$$

Hence, by (13) and (15)–(17), for $k=1$ we get

$$\mathcal{J}_4 \ll N^{3/2}L + HN^{3/4}L^{3/2}. \tag{18}$$

Let $k=2$. Using (4), Lemmas 6 and 7 with $\mu=1$, $\tau=1/H$ and $\omega=\frac{1}{2}$, we obtain that

$$\begin{aligned} K_1 &\ll N^{1/4}\int_{-1/2}^{1/2} |\tilde{V}_2(\alpha)|^2 |U(-\alpha,H)| \, \mathrm{d}\alpha \\ &\ll N^{1/4}\Big[H\Big(\frac{N^{1/2}L}{H}+L^2\Big)+N^{1/2}L+N^{1/2}L^2+HL^2\Big] \\ &\ll HN^{1/4}L^2+N^{3/4}L^2. \end{aligned} \tag{19}$$

Hence, by (13), (15) and (19), for $k=2$ we get

$$\mathcal{J}_4 \ll NL + HN^{1/4}L^2. \tag{20}$$

Let $k=3$. Using the Cauchy–Schwarz estimate, (4), Lemmas 6 and 7 with $\mu=1$, $\tau=1/H$ and $\omega=\frac{1}{2}$, we obtain that

$$\begin{aligned} K_1 &\ll N^{1/4}\Big(\int_{-1/2}^{1/2} |\tilde{V}_3(\alpha)|^2 |U(-\alpha,H)| \, \mathrm{d}\alpha\Big)^{1/2} \times \Big(\int_{-1/2}^{1/2} |\tilde{V}_2(\alpha)|^2 |U(-\alpha,H)| \, \mathrm{d}\alpha\Big)^{1/2} \\ &\ll N^{1/4}\Big[H\Big(\frac{N^{1/3}L}{H}+1\Big)+N^{1/3}L+N^{1/3}L^2+H\Big]^{1/2} \\ &\qquad\times\Big[H\Big(\frac{N^{1/2}L}{H}+L^2\Big)+N^{1/2}L+N^{1/2}L^2+HL^2\Big]^{1/2} \\ &\ll HN^{1/4}L+H^{1/2}N^{1/2}L+N^{2/3}L^2. \end{aligned} \tag{21}$$

Hence, by (13), (15) and (21), for $k=3$ we get

$$\mathcal{J}_4 \ll N^{5/6}L + HN^{1/4}L + H^{1/2}N^{1/2}L. \tag{22}$$

Let $k \ge 4$. By (13), $\tilde{V}_k(\alpha) \ll_k N^{1/k}$ and (14) we can write that

$$\mathcal{J}_4 \ll_k N^{3/4+1/k} \int_{-1/2}^{1/2} |U(-\alpha, H)| \, d\alpha \ll_k N^{3/4+1/k} L. \tag{23}$$

Summarising, from (18), (20) and (22)–(23) we can write that

$$\mathcal{J}_4 \ll_k E(k), \tag{24}$$

where $E(k)$ is defined in (2).

3.5. Estimation of $\mathcal{J}_2$. Now we estimate $\mathcal{J}_2$. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$ we obtain

$$\mathcal{J}_2 \ll_k \int_{-B/H}^{B/H} |E_2(\alpha)| \frac{|U(\alpha, H)|}{|z|^{1/2+1/k}} \, d\alpha + \int_{-B/H}^{B/H} |E_2(\alpha)|^2 \frac{|U(\alpha, H)|}{|z|^{1/k}} \, d\alpha = I_1 + I_2, \tag{25}$$

say. Using (3), (4) and Lemma 5 we obtain that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$I_2 \ll_k HN^{1/k} \int_{-B/H}^{B/H} |E_2(\alpha)|^2 \, d\alpha \ll_k HN^{1/k} \exp\Big(-c_1 \Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{26}$$

provided that $H \ge BN^{7/12+\varepsilon}$. Using the Cauchy–Schwarz inequality and arguing as for I_2 we get

$$\begin{aligned} I_1 &\ll_k H \Big(\int_{-B/H}^{B/H} |E_2(\alpha)|^2 \, d\alpha\Big)^{1/2} \Big(\int_{-B/H}^{B/H} \frac{d\alpha}{|z|^{1+2/k}}\Big)^{1/2} \\ &\ll_k HN^{1/k} \exp\Big(-\frac{c_1}{2} \Big(\frac{L}{\log L}\Big)^{1/3}\Big), \end{aligned} \tag{27}$$

provided that $H \ge BN^{7/12+\varepsilon}$. Inserting (26)–(27) into (25) we finally obtain

$$\mathcal{J}_2 \ll_k HN^{1/k} \exp\Big(-\frac{c_1}{2} \Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{28}$$

provided that $H \ge BN^{7/12+\varepsilon}$.

3.6. Estimation of $\mathcal{J}_3$. Now we estimate $\mathcal{J}_3$. By the Cauchy–Schwarz inequality, (4), Lemmas 8 and 5, we obtain that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\begin{aligned} \mathcal{J}_3 &\ll_k \Big(\int_{-1/2}^{1/2} |\tilde{S}_2(\alpha)|^4 \, d\alpha\Big)^{1/2} \Big(\int_{-B/H}^{B/H} |E_k(\alpha)|^2 |U(\alpha, H)|^2 \, d\alpha\Big)^{1/2} \\ &\ll_k HN^{1/2} L \Big(\int_{-B/H}^{B/H} |E_k(\alpha)|^2 \, d\alpha\Big)^{1/2} \\ &\ll_k HN^{1/k} \exp\Big(-\frac{c_1}{2} \Big(\frac{L}{\log L}\Big)^{1/3}\Big), \end{aligned} \tag{29}$$

provided that $H \ge BN^{1-5/(6k)+\varepsilon}$.

3.7. Final words. Let $k \geq 2$. By (8)–(12), (24) and (28)–(29) we have that, for every $\varepsilon > 0$, there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) = \frac{\pi}{4e} H N^{1/k} + \mathcal{O}_k\Big(H N^{1/k} \exp\Big(-\frac{c_1}{2} \Big(\frac{L}{\log L} \Big)^{1/3} \Big) + \frac{H}{B} N^{1/k} L^2 + E(k) \Big) \tag{30}$$

provided that $H \geq B N^{1-5/(6k)+\varepsilon}$, where $E(k)$ is defined in (2). The second error term is dominated by the first one by choosing $d = c_1$ in (5). So from now on we have $H \geq N^{1-5/(6k)+\varepsilon}$ for $k \geq 2$. The third error term in (30) is now dominated by the first.

Let $k = 1$. In this case (30) holds provided that $H \geq B N^{7/12+\varepsilon}$ and the second error term is dominated by the first one by choosing $d = c_1$ in (5). Hence, for $k = 1$, we get that $H \geq N^{7/12+\varepsilon}$. The third error term in (30) is now dominated by the first.

Summarising, for every $k \geq 1$ we can write that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) = \frac{\pi}{4e} H N^{1/k} + \mathcal{O}_k\Big(H N^{1/k} \exp\Big(-C \Big(\frac{L}{\log L} \Big)^{1/3} \Big) \Big)$$

provided that $H \geq N^{1-5/(6k)+\varepsilon}$ for $k \geq 2$ and $H \geq N^{7/12+\varepsilon}$ for $k = 1$. From (9) we get that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4} H N^{1/k} + \mathcal{O}_k\Big(H N^{1/k} \exp\Big(-C \Big(\frac{L}{\log L} \Big)^{1/3} \Big) \Big) + \mathcal{O}_k\Big(\frac{H}{N} \sum_{n=N+1}^{N+H} r_k(n) \Big)$$

provided that $H \leq N$, $H \geq N^{1-5/(6k)+\varepsilon}$ for $k \geq 2$ and $H \geq N^{7/12+\varepsilon}$ for $k = 1$. Using $e^{n/N} \leq e^2$ and (30), the last error term is $\ll_k H^2 N^{1/k-1}$. Hence we get

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4} H N^{1/k} + \mathcal{O}_k\Big(H N^{1/k} \exp\Big(-C \Big(\frac{L}{\log L} \Big)^{1/3} \Big) \Big)$$

uniformly for $N^{1-5/(6k)+\varepsilon} \leq H \leq N^{1-\varepsilon}$ if $k \geq 2$ and for $N^{7/12+\varepsilon} \leq H \leq N^{1-\varepsilon}$ if $k = 1$. Theorem 1 follows.

4. Proof of Theorem 2

We recall (6), which is again the starting point of our analysis. The setting in this case is very similar to the one we had in Section 3, but it is simpler than the previous one since Lemma 5 now applies to the whole integration interval. This is easily seen comparing (8) and (31). Furthermore, the bound provided by Lemma 5 in the conditional case is much stronger than the unconditional one, and the final error term is correspondingly stronger. The same remark applies to the lower bound for H that we obtain.

Let $k \geq 2$, $H \geq 2$, $H = o(N)$ be an integer. We recall that we set $L = \log N$ for brevity. From now on we assume that RH holds. We start again from (6) but in this conditional case we can simplify the setting. Recalling definition (7) and that $\Gamma\big(\frac{1}{2}\big) = \pi^{1/2}$, we can write

$$
\begin{aligned}
(31)\qquad \sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) = \int_{-1/2}^{1/2} \frac{\pi\Gamma(1/k)}{4kz^{1+1/k}} U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
+ \int_{-1/2}^{1/2} \frac{\Gamma(1/k)}{kz^{1/k}} \Big(\tilde{S}_2(\alpha)^2 - \frac{\pi}{4z}\Big) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
+ \int_{-1/2}^{1/2} E_k(\alpha)\tilde{S}_2(\alpha)^2 U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
+ \int_{-1/2}^{1/2} \tilde{V}_k(\alpha)(\tilde{V}_2(\alpha)^2 - \tilde{S}_2(\alpha)^2) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
+ \int_{-1/2}^{1/2} \tilde{S}_2(\alpha)^2 (\tilde{V}_k(\alpha) - \tilde{S}_k(\alpha)) U(-\alpha, H) e(-N\alpha)\,\mathrm{d}\alpha \\
= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5,
\end{aligned}
$$

say. Now we evaluate these terms.

4.1. Evaluation of $\mathcal{I}_1$. Using Lemma 4 we immediately get

$$
(32)\qquad \mathcal{I}_1 = \frac{\pi\Gamma(1/k)}{4k\Gamma(1+1/k)} \sum_{n=N+1}^{N+H} n^{1/k} e^{-n/N} + \mathcal{O}_k\Big(\frac{H}{N}\Big) = \frac{\pi}{4e} H N^{1/k} + \mathcal{O}_k(H^2 N^{1/k-1} + N^{1/k}).
$$

4.2. Estimation of $\mathcal{I}_5$. Clearly $\mathcal{I}_5 = \mathcal{J}_5$ of Section 3.3. Hence we have that

$$
(33)\qquad \mathcal{I}_5 \ll_k N^{1/(2k)}(N^{1/2} + H)L^2
$$

which, comparing with (32), is under control if $H = \infty(N^{1/2-1/(2k)}L^2)$.

4.3. Estimation of $\mathcal{I}_4$. Clearly $\mathcal{I}_4 = \mathcal{J}_4$ of Section 3.4. Hence we have that

$$
(34)\qquad \mathcal{I}_4 \ll_k E(k),
$$

where $E(k)$ is defined in (2).

4.4. Estimation of $\mathcal{I}_2$. Now we estimate $\mathcal{I}_2$. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$ we obtain

$$
(35)\qquad \mathcal{I}_2 \ll_k \int_{-1/2}^{1/2} |E_2(\alpha)| \frac{|U(\alpha, H)|}{|z|^{1/2+1/k}}\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} |E_2(\alpha)|^2 \frac{|U(\alpha, H)|}{|z|^{1/k}}\,\mathrm{d}\alpha = J_1 + J_2,
$$

say. Using (3), (4), Lemma 5 and Lemma 7 first with $\mu = 1/k$, $\tau = 1/N$, $\omega = 1/H$ and then with $\mu = 1 + 1/k$, $\tau = 1/H$, $\omega = \frac{1}{2}$, we obtain

$$
\begin{aligned}
(36)\qquad J_2 &\ll_k HN^{1/k} \int_{-1/N}^{1/N} |E_2(\alpha)|^2\,\mathrm{d}\alpha + H \int_{1/N}^{1/H} |E_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^{1/k}} + \int_{1/H}^{1/2} |E_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^{1+1/k}} \\
&\ll_k HN^{1/k-1/2}L^2 + H^{1/k}N^{1/2}L^2 \\
&\ll_k H^{1/k}N^{1/2}L^2.
\end{aligned}
$$

For J_1 we need few cases. Let $k \ge 3$. Using the Cauchy–Schwarz inequality and arguing as for J_2 (in this case Lemma 7 is used first with $\mu = 1$, $\tau = 1/N$, $\omega = 1/H$ and then with $\mu = 1$, $\tau = 1/H$, $\omega = \frac{1}{2}$), we get

$$(37)\quad J_1 \ll_k HN^{1/2+1/k}\Big(\int_{-1/N}^{1/N} \mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/N}^{1/N} |E_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} + H\Big(\int_{1/N}^{1/H} \frac{\mathrm{d}\alpha}{\alpha^{2/k}}\Big)^{1/2}\Big(\int_{1/N}^{1/H} |E_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{2+2/k}}\Big)^{1/2}\Big(\int_{1/H}^{1/2} |E_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}$$
$$\ll_k HN^{1/k-1/4}L + H^{1/2+1/k}N^{1/4}L(1+L)^{1/2} + H^{1/2+1/k}N^{1/4}L(1+L)^{1/2}$$
$$\ll_k HN^{1/k-1/4}L + H^{1/2+1/k}N^{1/4}L^{3/2}$$
$$\ll H^{1/2+1/k}N^{1/4}L^{3/2}.$$

For $k = 2$ arguing as before we get

$$(38)\qquad J_1 \ll HN^{1/4}L^2.$$

Combining (35)–(38), and assuming $H \ge N^{1/2}$, we finally obtain

$$(39)\qquad \mathcal{I}_2 \ll_k H^{1/2+1/k}N^{1/4}L^2$$

for every $k \ge 2$.

4.5. Estimation of $\mathcal{I}_3$. Now we estimate $\mathcal{I}_3$. By the Cauchy–Schwarz inequality, (4) and Lemma 8 we obtain

$$(40)\qquad \mathcal{I}_3 \ll_k \Big(\int_{-1/2}^{1/2} |\tilde{S}_2(\alpha)|^4\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/2}^{1/2} |E_k(\alpha)|^2|U(\alpha,H)|^2\,\mathrm{d}\alpha\Big)^{1/2}$$
$$\ll_k N^{1/2}L\Big(H^2\int_{-1/H}^{1/H} |E_k(\alpha)|^2\,\mathrm{d}\alpha + \int_{1/H}^{1/2} |E_k(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha^2}\Big)^{1/2}$$
$$\ll_k H^{1/2}N^{1/2+1/(2k)}L^2,$$

where in the last step we used Lemma 5 and 7 with $\mu = 2$, $\tau = 1/H$ and $\omega = \frac{1}{2}$.

4.6. Final words. Recalling (2), by (31)–(34) and (39)–(40), we can finally write for $H \ge N^{1/2}$ that

$$(41)\qquad \sum_{n=N+1}^{N+H} e^{-n/N} r_k(n) = \frac{\pi}{4e}HN^{1/k} + \mathcal{O}_k(H^2N^{1/k-1} + H^{1/2}N^{1/2+1/(2k)}L^2 + E(k))$$

which is an asymptotic formula for $\infty(N^{1-1/k}L^4) \le H \le o(N)$. From (9) we get

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4e}HN^{1/k} + \mathcal{O}_k(H^2N^{1/k-1} + H^{1/2}N^{1/2+1/(2k)}L^2 + E(k)) + \mathcal{O}_k\Big(\frac{H}{N}\sum_{n=N+1}^{N+H} r_k(n)\Big).$$

Using $e^{n/N} \le e^2$ and (41), it is easy to see that the last error term is $\ll_k H^2N^{1/k-1} + H^{3/2}N^{-1/2+1/(2k)}L^2 + HN^{-1}E(k)$. Hence we get

$$\sum_{n=N+1}^{N+H} r_k(n) = \frac{\pi}{4e}HN^{1/k} + \mathcal{O}_k(H^2N^{1/k-1} + H^{1/2}N^{1/2+1/(2k)}L^2 + E(k)),$$

uniformly for $\infty(N^{1-1/k}L^4) \le H \le o(N)$. Theorem 2 follows.

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References

[1] J. Brüdern, "A ternary problem in additive prime number theory", pp. 57–81 in *From arithmetic to zeta-functions*, Springer, 2016.

[2] G. Harman and A. Kumchev, "On sums of squares of primes II", *J. Number Theory* **130**:9 (2010), 1969–2002.

[3] L.-K. Hua, "Some results in the additive prime-number theory", *Quart. J. Math. Oxford Ser.* (2) **9**:1 (1938), 68–80.

[4] A. V. Kumchev and J. Y. Liu, "Sums of primes and squares of primes in short intervals", *Monatsh. Math.* **157**:4 (2009), 335–363.

[5] A. Languasco and A. Zaccagnini, "Short intervals asymptotic formulae for binary problems with primes and powers, II: density 1", *Monatsh. Math.* **181**:2 (2016), 419–435.

[6] A. Languasco and A. Zaccagnini, "Sum of one prime and two squares of primes in short intervals", *J. Number Theory* **159** (2016), 45–58.

[7] A. Languasco and A. Zaccagnini, "Short intervals asymptotic formulae for binary problems with primes and powers, I: density 3/2", *Ramanujan J.* **42**:2 (2017), 371–383.

[8] A. Languasco and A. Zaccagnini, "Sums of four prime cubes in short intervals", *Acta Math. Hungar.* **159**:1 (2019), 150–163.

[9] M.-C. Leung and M. C. Liu, "On generalized quadratic equations in three prime variables", *Monatsh. Math.* **115**:1-2 (1993), 133–167.

[10] T. Li, "On sums of squares of primes and a kth power of prime", *Rocky Mountain J. Math.* **42**:1 (2012), 201–222.

[11] Z. Liu and R. Zhang, "On sums of squares of primes and a k-th power of prime", *Monatsh. Math.* **188**:2 (2019), 269–285.

[12] G.-S. Lü, "Note on a result of Hua", *Adv. Math.* (*China*) **35**:3 (2006), 343–349.

[13] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics **84**, American Mathematical Society, Providence, RI, 1994.

[14] H. L. Montgomery and R. C. Vaughan, "Hilbert's inequality", *J. London Math. Soc.* (2) **8** (1974), 73–82.

[15] W. Schwarz, "Zur Darstellung von Zahlen durch Summen von Primzahlpotenzen", *J. Reine Angew. Math.* **206** (1961), 78–112.

[16] R. C. Vaughan, *The Hardy–Littlewood method*, 2nd ed., Cambridge Tracts in Mathematics **125**, Cambridge University Press, 1997.

[17] L. Zhao, "The additive problem with one prime and two squares of primes", *J. Number Theory* **135** (2014), 8–27.

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