Submitted to Rev. Mat. Iberoam., 1–30

Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type III. Semisimple classes in $\mathbf{PSL}_n(q)$

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Abstract. We show that Nichols algebras of most simple Yetter-Drinfeld modules over the projective special linear group over a finite field, corresponding to semisimple orbits, have infinite dimension. We introduce a new criterium to determine when a conjugacy class collapses and prove that for infinitely many pairs (n, q), any finite-dimensional pointed Hopf algebra H with $G(H) \simeq \mathbf{PSL}_n(q)$ or $\mathbf{SL}_n(q)$ is isomorphic to a group algebra.

That is not dead which can eternal lie.
And with strange aeons even death may die.

Abdul Alhazred

1. Introduction

1.1.

This is the third paper of a series devoted to finite-dimensional pointed Hopf algebras over $\mathbb{C}$ with group of group-likes isomorphic to a finite simple group of Lie type. An Introduction to the whole series is in Part I [ACG1]. Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$ and $\mathbb{F}_q$ the field with q elements. In this paper we consider Nichols algebras associated to semisimple conjugacy classes in $\mathbf{PSL}_n(q)$; we first show that any semisimple class $\mathcal{O}$ lying in a large family *collapses* [AFGV1, Definition 2.2], that is, the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \mathbf{q})$ is infinite for every finite faithful 2-cocycle $\mathbf{q}$. In previous work [ACG1, ACG2, AFGV1, AFGV2] we attacked the question of the collapse of conjugacy classes in various groups using the criteria of type D and F, which are based on results on Nichols algebras of Yetter-Drinfeld modules over finite groups [AHS, CH, HS]. Here the criterium

Mathematics Subject Classification (2010): 16T05.
Keywords: Nichols algebra; Hopf algebra; rack; finite group of Lie type; conjugacy class.

of type C, conjectured some years ago by the authors of [AFGV1, HS] but carried into effect only now thanks to the remarkable classification in [HV], is added to the panoply. See Theorem 2.9. Note that there are racks of type C which are not of type D or F, see for example Lemmata 2.12 and 2.19.

If $n = 2$, we assume that $q \neq 2, 3, 4, 5, 9$ to avoid coincidences with cases treated elsewhere, see [ACG1, Subsection 1.2]. Indeed $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$, $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$ are not simple and $\mathbf{PSL}_2(4) \simeq \mathbf{PSL}_2(5) \simeq \mathbb{A}_5$ and $\mathbf{PSL}_2(9) \simeq \mathbb{A}_6$.

Recall that a conjugacy class $\mathcal{O}$ is said to collapse if the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \mathbf{q})$ is infinite for every finite faithful 2-cocycle $\mathbf{q}$. Our first main result says

Theorem 1.1. *Let $\mathcal{O}$ be a semisimple conjugacy class in $\mathbf{PSL}_n(q)$. If either $n > 2$ and $\mathcal{O}$ is not irreducible or $n = 2$, $q \neq 2, 3, 4, 5, 9$ and $\mathcal{O}$ is not listed in Table 1, then it collapses.*

Table 1: Kthulhu semisimple classes in $\mathbf{PSL}_2(q)$.

q	class	Remark
7	involutions	kthulhu
even and not a square	irreducible order 3	sober
all	irreducible, order > 3	sober

Proof. If $n > 2$ and $\mathcal{O}$ is not irreducible, then Proposition 3.18 applies. If $n = 2$, then the result follows by Proposition 4.2. □

1.2.

In the first two papers of the series, we dealt with unipotent classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_n(q)$. The outcome is that most non-semisimple conjugacy classes collapse, and yet unpublished results on other finite simple groups of Lie type convey to the idea that this is the case in general. On the contrary, we see in the present paper that a semisimple irreducible conjugacy class for $n = 2, 3$ does not satisfy the criteria of types C, D or F for a rack to collapse; it appears to us that this would be true for general n. An intuitive explanation might be as follows. If G is a finite (almost) simple group, then there exists a conjugacy class $\mathcal{C}$ of G so that if $x \in G - \{e\}$, then the probability that x and a random element of $\mathcal{C}$ generate G is at least 1/10; in particular G can be generated by a pair of elements in $\mathcal{C}$. See [BGK, GK, GM] and references therein; for G of Lie type, $\mathcal{C}$ is semisimple. Of course this does not prove that generically the conjugacy classes do not collapse– see [AFGV1, AFGV2] for alternating and sporadic groups– but it might be an indication of the plausibility of our guess.

1.3.

The fact that large families of conjugacy classes of finite simple groups do not collapse shows the limits of the criteria of types C, D or F, and urges for the computation of their second cohomology groups and the determination of the corresponding Nichols algebras. For cocycles coming from Yetter-Drinfeld modules over $\mathbf{PSL}_n(q)$, abelian techniques can be applied. This type of techniques were applied in [FGV1, FGV2] for the study of Nichols algebras over $\mathbf{SL}_2(q)$ and $\mathbf{PSL}_2(q)$, respectively; it was proved there that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any irreducible representation ρ of $C_{\mathbf{PSL}_2(q)}(x)$ with $x \in \mathcal{O}$ and q even. For q odd, a list of the open cases was given. We deal with these open cases when $q \equiv 1 \mod 4$, see Theorem 4.4, applying the criterium of type C. In general, the use of the criteria and the abelian techniques, together with [ACG1, Theorem 1.4], yield the following result.

Let $n = 2^a b$, $q = 1 + 2^c d$, where $a, c \in \mathbb{N}_0$; $b, d \in \mathbb{N}$ with $(b, 2) = (d, 2) = 1$ and let $\mathcal{G}$ be the set of all pairs (n, q) with $n \in \mathbb{N}$ and $q = p^m$, p prime, such that one of the following conditions holds:

(i) $n > 3$ is odd;

(ii) $n > 3$ and q is even;

(iii) $n = 3$ and $q > 2$;

(iv) $0 < a < c$, $n > 2$;

(v) $a = c > 1$.

(vi) $n = 2$ and $q \equiv 1 \mod 4$.

Recall that a finite group G *collapses* when every finite-dimensional pointed Hopf algebra H with $G(H) \simeq G$ is isomorphic to $\mathbb{C}G$.

Theorem 1.2. *Let $\boldsymbol{G} = \mathbf{PSL}_n(q)$, $x \in \boldsymbol{G}$ and $\mathcal{O}_x$ its conjugacy class.*

(a) *If $(n, q) \in \mathcal{G}$, then $\boldsymbol{G}$ collapses.*

(b) *Let $n = 3$, $q = 2$. If $\mathcal{O}_x$ is not regular unipotent, then $\dim \mathfrak{B}(\mathcal{O}_x, \rho) = \infty$ for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.*

(c) *Let $n = 2$, $q > 3$. Then $\dim \mathfrak{B}(\mathcal{O}_x, \rho) = \infty$ for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$, with a possible exception when x is semisimple irreducible, image of $\mathtt{x} = \left(\begin{smallmatrix} a & \zeta b \\ b & a \end{smallmatrix}\right) \in \mathbf{SL}_2(q)$ for some a, $b \neq 0$, $\zeta \in \mathbb{F}_q^\times - \mathbb{F}_q^2$ and $q \equiv 3 \mod 4$.*

Proof. If $n = 2$ we may assume that $q > 5$ and $q \neq 9$, otherwise [AFGV1, Theorem 1.2] applies. Let $\mathcal{O}$ be a conjugacy class in $\boldsymbol{G}$ and $x \in \mathcal{O}$. If x is not semisimple, then by Proposition 2.13 it collapses unless $\mathcal{O}$ is unipotent and is listed in Table 4. If it is unipotent and $n = 2$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for all $\rho \in \operatorname{Irr}_{C_{\boldsymbol{G}}(x)}$ by [FGV2, Theorem 1.6]. Thus, the only open case is when $n = 3$, $q = 2$ and $\mathcal{O}$ is regular unipotent; see Proposition 5.7. Assume now that $\mathcal{O}$ is semisimple. For $n = 2$, this is Theorem 4.4. For $n > 2$, if $\mathcal{O}$ is not irreducible, then it collapses by Theorem 1.1, while if $\mathcal{O}$ is irreducible, then Lemma 5.6 applies. $\square$

As a by-product of our arguments we obtain the following result.

Theorem 1.3. *Assume that either* (i), (ii) *or* (iii) *in the list defining* $\mathcal{G}$ *hold. Then* $\mathbf{SL}_n(q)$ *collapses.*

Proof. Let $\mathcal{O}$ be a conjugacy class in $\mathbf{SL}_n(q)$. If $\mathcal{O}$ is not semisimple or it is semisimple and not irreducible, then $\mathcal{O}$ collapses by the proofs of Propositions 2.13 and 3.18, and the results in Section 2. If $\mathcal{O}$ is semisimple irreducible, then under the assumptions any $x \in \mathcal{O}$ has odd order. Thus, by Remark 3.1 (b), Lemma 4.1 (b)(iii) applies. □

Notation

We denote the cardinal of a set X by $|X|$. If $k < \ell$ are positive integers, then we set $\mathbb{I}_{k,\ell} = \{i \in \mathbb{N} : k \leq i \leq \ell\}$ and simply $\mathbb{I}_\ell = \mathbb{I}_{1,\ell}$. We denote by $\mathbb{G}'_\ell$ the set of non-trivial ℓ-th roots of unity in $\mathbb{C}$.

Let G be a group; $N < G$, respectively $N \lhd G$, means that N is a subgroup, respectively a normal subgroup, of G. The set of isomorphism classes of irreducible representations of G is denoted by $\operatorname{Irr} G$. If G acts on a set X and $x \in X$ we denote by $\mathcal{O}_x^G$ the orbit of x for this action. In particular, if $x \in G$, then $\mathcal{O}_x^G$ indicates the conjugacy class of x in G. The centralizer, respectively the normalizer, of $x \in G$ is denoted by $C_G(x)$, respectively $N_G(x)$; the inner automorphism defined by conjugation by x is denoted by $\operatorname{Ad} x$. If $\mathcal{F} \in \operatorname{Aut} G$, then $G^{\mathcal{F}}$ is the subgroup of points fixed by $\mathcal{F}$. The standard Frobenius morphism will be indicated by F_q.

The algebraic closure of $\mathbb{F}_q$ is denoted $\Bbbk = \overline{\mathbb{F}_q}$. For $a, b \in \mathbb{N}$ we will set $(a)_b = b^{a-1} + \cdots + 1$. We recall that for $a, b, c \in \mathbb{N}$ there holds

$$((a)_b, b-1) = (a, b-1), \quad ((a)_b, (c)_b) = ((a,c))_b, \quad (ac)_b = (a)_b (c)_{b^a}. \tag{1.1}$$

Let G be a group and V a Yetter-Drinfeld module over G with comodule map δ. We denote by $V_g = \{v \in V : \ \delta(v) = g \otimes v\}$ the set of g-homogeneous elements for $g \in G$ and by $\operatorname{supp} V = \{g \in G : \ V_g \neq 0\}$ the support of V. Recall that the category ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ of Yetter-Drinfeld modules over G is braided, with braiding $c_{V,W}(v \otimes w) = g \cdot w \otimes v$ for $v \in V_g$, $w \in W$, and $V, W \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$.

2. Preliminaries on racks

2.1. Racks

A rack is a non-empty set X with a self-distributive operation $\triangleright : X \times X \to X$ such that $\varphi_x := x \triangleright \_\_$ is a bijection for every $x \in X$. We assume that all racks in this paper are finite, unless explicitly stated and also that are crossed sets, namely that

$$x \triangleright x = x, \qquad x \triangleright y = y \implies y \triangleright x = x, \qquad \forall x, y \in X.$$

The main example of a rack is a conjugacy class $\mathcal{O}$ in a finite group G with the operation $x \triangleright y = xyx^{-1}$, $x, y \in \mathcal{O}$. We say that a rack X is *abelian* if $x \triangleright y = y$, for all $x, y \in X$; thus any subset of an abelian group is an abelian rack.

If X is a rack, the inner group of the rack is $\mathrm{Inn}X := \langle \varphi_x, x \in X \rangle < \mathbb{S}_X$. If $X = \mathcal{O}$ a conjugacy class, then $\mathrm{Inn}X = \langle \mathrm{Ad}(y), y \in \mathcal{O} \rangle$.

A rack X is *of type D* if it has a decomposable subrack $Y = R \coprod S$ with elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$ [AFGV1, Definition 3.5]. If $\mathcal{O}$ is a finite conjugacy class in G, then the following are equivalent:

1. The rack $\mathcal{O}$ is of type D.
2. There exist r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$ and $(rs)^2 \neq (sr)^2$.

A rack X is *of type F* if it has a family of subracks $(R_a)_{a \in \mathbb{I}_4}$ and elements $r_a \in R_a$, $a \in \mathbb{I}_4$, such that $R_a \triangleright R_b = R_b$, for $a, b \in \mathbb{I}_4$, and $R_a \cap R_b = \emptyset$, $r_a \triangleright r_b \neq r_b$ for $a \neq b \in \mathbb{I}_4$ [ACG1, Definition 2.4]. If $\mathcal{O}$ is a finite conjugacy class in G, then the following are equivalent:

1. The rack $\mathcal{O}$ is of type F.
2. There exist $r_a \in \mathcal{O}$, $a \in \mathbb{I}_4$, such that $\mathcal{O}_{r_a}^{\langle r_a : a \in \mathbb{I}_4 \rangle} \neq \mathcal{O}_{r_b}^{\langle r_a : a \in \mathbb{I}_4 \rangle}$ and $r_a r_b \neq r_b r_a$, $a \neq b \in \mathbb{I}_4$.

A rack X of type D, respectively F, collapses [AFGV1, Theorem 3.6], respectively [ACG1, Theorem 2.8]. A rack is *cthulhu*, respectively *sober*, when it is neither of type D nor of type F, respectively if every subrack is either abelian or indecomposable. Clearly, sober implies cthulhu. Let $\pi : X \to Y$ be a surjective morphism of racks. If Y is of type D, respectively F, then so is X; hence X cthulhu implies Y cthulhu.

The following result extends the isogeny argument [ACG1, Lemma 1.2].

Lemma 2.1. *Let G be a group, $x \in G$, $N \lhd G$, $\boldsymbol{G} = G/N$ and $\pi : G \to \boldsymbol{G}$ the natural projection. Then the restriction $\pi_{\mathcal{O}} : \mathcal{O}_x^G \to \mathcal{O}_{\pi(x)}^{\boldsymbol{G}}$ is surjective. Assume that G is finite. If $\mathcal{O}_x^G$ is ctuhlhu, then so is $\mathcal{O}_{\pi(x)}^{\boldsymbol{G}}$. Assume that $N < Z(G)$ and let $N^{[x]} := \{c \in N : cx \in \mathcal{O}_x^G\}$. Then $N^{[x]} < N$ and $\pi_{\mathcal{O}}^{-1}(\pi(y))$ has exactly $|N^{[x]}|$ elements. Thus, $\pi_{\mathcal{O}}$ is injective if and only if $N^{[x]}$ is trivial.*

Proof. If $\pi(y) \in \mathcal{O}_{\pi(x)}^{\boldsymbol{G}}$, then there is $g \in G$ such that $\pi(y) = \pi(g)\pi(x)\pi(g)^{-1} = \pi(gxg^{-1}) \in \pi(\mathcal{O}_x^G)$. That is, $\pi_{\mathcal{O}}$ is surjective. It is easy to see that $N^{[x]}$ is a subgroup of N, provided that $N < Z(G)$. Let $g \in G$. If $h \in G$ satisfies $\pi(hxh^{-1}) = \pi(gxg^{-1})$, then there is $c \in N$ such that $hxh^{-1} = cgxg^{-1} = gcxg^{-1}$, hence $cx = g^{-1}hxh^{-1}g \in \mathcal{O}_x^G$. Conversely, if $c \in N^{[x]}$, say $cx = uxu^{-1} \in \mathcal{O}_x^G$ for some $u \in G$, then $(gu)x(gu)^{-1} = cgxg^{-1}$. □

2.2. Racks of type C

We now translate the main result of [HV] to the context of racks, yielding a new criterium. First we recall:

Theorem 2.2. [HV, Theorem 2.1] *Let G be a non-abelian group and V and W be two simple Yetter-Drinfeld modules over G such that G is generated by the support of $V \oplus W$, $\dim V \leq \dim W$ and $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W) \neq 0$. Then the following are equivalent:*

(a) $\dim \mathfrak{B}(V \oplus W) < \infty$.

(b) *G, V and W are as in* [HV, Theorem 2.1].

In particular, $(\dim V, \dim W)$ *belongs to*

$$\{(1,3),(1,4),(2,2),(2,3),(2,4)\}. \qquad (2.1)$$ □

The theorem above motivates the following definition.

Definition 2.3. A rack X is *of type C* when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that

$$r \triangleright s \neq s, \qquad (2.2)$$

$$R = \mathcal{O}_r^{\mathrm{Inn}\, Y}, \qquad S = \mathcal{O}_s^{\mathrm{Inn}\, Y}, \qquad (2.3)$$

$$\min\{|R|,|S|\} > 2 \quad \text{or} \quad \max\{|R|,|S|\} > 4. \qquad (2.4)$$

Remark 2.4. Since X is a crossed set, (2.2) implies that $s \triangleright r \neq r$, hence $|R| \neq 1$ and $|S| \neq 1$. That is, (2.4) says that either $|R| \neq 2$ or $|S| > 4$.

Assume that R is indecomposable and $Y = R \coprod S$ a decomposable rack. Then $R = \mathcal{O}_r^{\mathrm{Inn}\, R} = \mathcal{O}_r^{\mathrm{Inn}\, Y}$ by [AG, Lemma 1.15]. The formulation (2.3) is more flexible, see Lemma 2.10. On the other hand, racks with 2 elements are not indecomposable. Thus, in presence of (2.2), the hypothesis *R and S are indecomposable* implies both (2.3) and (2.4).

Remark 2.5. Let H be a finite group, $r \in H$ and $s \in \mathcal{O}_r^H$. If $R := \mathcal{O}_r^{\langle r,s\rangle} \neq S := \mathcal{O}_s^{\langle r,s\rangle}$, then $Y := R \coprod S$ is a decomposable subrack of $\mathcal{O}_r^H$ that satisfies (2.3), because $\langle Y\rangle = \langle r,s\rangle$ so $R = \mathcal{O}_r^{\langle Y\rangle} = \mathcal{O}_r^{\mathrm{Inn}\, Y}$.

Conversely, any subrack decomposition $Y = R \coprod S \subset \mathcal{O}_r^H$ with $r \in R$ and $s \in S$ implies $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ and (2.3) is verified for $R' := \mathcal{O}_r^{\langle r,s\rangle}$, $S' := \mathcal{O}_s^{\langle r,s\rangle}$.

Remark 2.6. Let H be a finite group, $x_1, \dots, x_n \in H$ and $L = \langle x_1, \dots, x_n\rangle$. If $Y = \mathcal{O}_{x_1}^L \cup \dots \cup \mathcal{O}_{x_n}^L$, then the conjugacy classes $\mathcal{O}_{x_i}^{\mathrm{Inn}\, Y}$ and $\mathcal{O}_{x_i}^L$ are equal for all $1 \leq i \leq n$ and $Y = \mathcal{O}_{x_1}^{\mathrm{Inn}\, Y} \cup \dots \cup \mathcal{O}_{x_n}^{\mathrm{Inn}\, Y}$. Indeed, $\langle Y\rangle = L$, and consequently $\mathcal{O}_{x_i}^{\mathrm{Inn}\, Y} = \mathcal{O}_{x_i}^{\langle \mathrm{Ad}(y),\, y \in Y\rangle} = \mathcal{O}_{x_i}^L$ for all $1 \leq i \leq n$.

The following lemma will help us to determine when a rack is of type C.

Lemma 2.7. *Let H be a finite group, $r \in H$ and $s \in \mathcal{O}_r^H$ such that* (2.2) *holds.*

(a) *If $|\mathcal{O}_r^{\langle r,s\rangle}| = 2$, then $s^{-1} \triangleright r = s \triangleright r$, i. e., $s^2 r = r s^2$.*

(b) *If $\operatorname{ord} r = \operatorname{ord} s$ is odd, then $\mathcal{O}_r^H$ is of type C if and only if $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$.*

Proof. (a) Indeed, $s^{-1} \triangleright r \neq r \neq s \triangleright r$ and $s^{-1} \triangleright r, r, s \triangleright r \in \mathcal{O}_r^{\langle r,s\rangle}$. Thus $|\mathcal{O}_r^{\langle r,s\rangle}| = 2$ implies $s^{-1} \triangleright r = s \triangleright r$. (b) If $\operatorname{ord} s$ is odd, then (2.2) forces $s^2 r \neq r s^2$ and $r^2 s \neq s r^2$, hence (2.4); while (2.3) holds by Remark 2.5. □

We describe now the criterium of type C in group-theoretical terms.

Lemma 2.8. *Let $\mathcal{O}$ be a conjugacy class in a finite group G. $\mathcal{O}$ is of type C if and only if there are $H < G$, $r, s \in H \cap \mathcal{O}$ such that*

$$rs \neq sr; \tag{2.5}$$

$$\mathcal{O}_r^H \neq \mathcal{O}_s^H; \tag{2.6}$$

$$H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle; \tag{2.7}$$

$$\min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2 \quad \text{or} \quad \max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4. \tag{2.8}$$

Proof. If $\mathcal{O}$ is of type C with decomposable subrack $Y = R \coprod S$, then we take $H = \langle Y \rangle$. Conversely, if H, r and s satisfy (2.5) and (2.6), then we take $R = \mathcal{O}_r^H$, $S = \mathcal{O}_s^H$ and $Y = R \coprod S$. Thus, $\langle Y \rangle = H$ by (2.7) hence (2.3) is satisfied. Finally, (2.4) follows from (2.8). □

Theorem 2.9. *A rack X of type C collapses.*

Proof. Let G be a finite group and $M \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ such that X is isomorphic to a subrack of $\operatorname{supp} M$. We will check that $\mathfrak{B}(M)$ has infinite dimension. This implies that X collapses by [AFGV1, Lemma 2.3].

Let $Y = R \coprod S$ be as in Definition 2.3. Let $K = \langle Y \rangle \leq G$. Then $M_Y := \oplus_{y \in Y} M_y \in {}^{\mathbb{C}K}_{\mathbb{C}K}\mathcal{YD}$, with $M_R := \oplus_{x \in R} M_x$ and $M_S := \oplus_{z \in S} M_z$ being Yetter-Drinfeld submodules of M_Y. By (2.3), $R = \mathcal{O}_r^K$, $S = \mathcal{O}_s^K$. Let V, respectively W, be a simple Yetter-Drinfeld submodule of M_R, respectively M_S. Then $\operatorname{supp} V = R$ (since $\operatorname{supp} V$ is stable under the conjugation of K), $\operatorname{supp} W = S$ and $\operatorname{supp}(V \oplus W) = Y$, that generates K. Now $(\operatorname{id} - c_{W,V} c_{V,W})(V \otimes W) \neq 0$ by (2.2). Without loss of generality, we may assume that $\dim V \leq \dim W$. Now $\dim V \geq |R| > 2$ or $\dim W \geq |S| > 4$, by (2.4). Hence $(\dim V, \dim W)$ does not belong to the set (2.1). Thus $\dim \mathfrak{B}(V \oplus W) = \infty$ by Theorem 2.2 and a fortiori $\dim \mathfrak{B}(M) = \infty$. □

The criterium of type C is very flexible as the following Lemma shows; it also means that the classification of simple racks of type C is crucial.

Lemma 2.10. *If a rack Z contains a subrack of type C, respectively projects onto a rack of type C, then Z is of type C.*

Proof. The first statement is obvious. Let $\pi : Z \to X$ be a surjective morphism of racks with X of type C and let $Y = R \coprod S \subset X$ be as in Definition 2.3 with $|R| \leq |S|$; in particular, $|R| > 2$ or $|S| > 4$. Fix $\widetilde{r}, \widetilde{s} \in Z$ such that $\pi(\widetilde{r}) = r$, $\pi(\widetilde{s}) = s$. Define recursively

$$\begin{aligned}
&R_1 = \pi^{-1}(R), && S_1 = \pi^{-1}(S), && Y_1 = \pi^{-1}(Y), && K_1 = \langle \varphi_y, y \in Y_1 \rangle \leq \operatorname{Inn} Z,\\
&R_2 = \mathcal{O}_{\widetilde{r}}^{K_1}, && S_2 = \mathcal{O}_{\widetilde{s}}^{K_1}, && Y_2 = R_2 \coprod S_2, && K_2 = \langle \varphi_y, y \in Y_2 \rangle \leq \operatorname{Inn} Z;\\
&R_j = \mathcal{O}_{\widetilde{r}}^{K_{j-1}}, && S_j = \mathcal{O}_{\widetilde{s}}^{K_{j-1}}, && Y_j = R_j \coprod S_j, && K_j = \langle \varphi_y, y \in Y_j \rangle \leq \operatorname{Inn} Z.
\end{aligned}$$

Notice that $R_1 \supseteq R_2 \supseteq \dots$ and $S_1 \supseteq S_2 \supseteq \dots$, hence $Y_i = R_i \coprod S_i$ is a rack decomposition. Now the sequence $Y_1 \supseteq Y_2 \supseteq \dots \supseteq Y_i \supseteq Y_{i+1} \supseteq \dots$ stabilizes because Z is finite. Let $i \in \mathbb{N}$ such that $Y_i = Y_{i-1}$; then $\widetilde{R} := R_i = R_{i-1} = \mathcal{O}_{\widetilde{r}}^{K_{i-1}}$ and $\widetilde{S} := S_i = S_{i-1} = \mathcal{O}_{\widetilde{s}}^{K_{i-1}}$. Thus $\widetilde{Y} := \widetilde{R} \coprod \widetilde{S}$ is a subrack of Z that satisfies (2.2) and (2.3). We claim now that $\pi(Y_j) = Y$ for all $j \in \mathbb{N}$; hence $|R_j| \geq |R| > 2$ or $|S_j| \geq |S| > 4$, proving (2.4) for $\widetilde{Y}$.

Indeed, $\pi(R_1) = R$ because π is surjective. Assume that $\pi(Y_j) = Y$; hence $\pi(R_j) = R$ and $\pi(S_j) = S$. Let $t \in R$. There exist $y_1, \dots, y_h \in Y$ such that $y_1 \triangleright (y_2 \triangleright \dots \triangleright (y_h \triangleright r) \dots) = t$ by (2.4) for Y. Pick $\widetilde{y}_1, \dots, \widetilde{y}_h \in Y_j$ such that $\pi(\widetilde{y}_\ell) = y_\ell$, $\ell \in \mathbb{I}_h$. Then

$$\widetilde{y}_1 \triangleright (\widetilde{y}_2 \triangleright \dots \triangleright (\widetilde{y}_h \triangleright \widetilde{r}) \dots) \in \mathcal{O}_{\widetilde{r}}^{K_j} = R_{j+1}, \qquad \text{hence}$$
$$\pi(\widetilde{y}_1 \triangleright (\widetilde{y}_2 \triangleright \dots \triangleright (\widetilde{y}_h \triangleright \widetilde{r}) \dots) = y_1 \triangleright (y_2 \triangleright \dots \triangleright (y_h \triangleright r) \dots) = t \in \pi(R_{j+1}).$$

□

2.3. Kthulhu racks

Definition 2.11. A rack is *kthulhu* if it is neither of type D nor of type F, nor of type C; i. e. cthulhu and not of type C. A rack is *austere* if every subrack generated by two elements is either abelian or indecomposable. Clearly, sober implies austere and austere implies kthulhu.

Let $\pi : X \to Y$ be a surjective morphism of racks. By Lemma 2.10 and previous results, X kthulhu implies Y kthulhu.

In [ACG1] and [ACG2] we proved that the non-semisimple classes in $\mathbf{PSL}_n(q)$ that are not listed in Table 2 and the unipotent classes in $\mathbf{PSp}_{2n}(q)$ that are not listed in Table 3 are either of type D or F. In this Section we determine which ones are of type C.

Table 2: Unipotent classes in $\mathbf{PSL}_n(q)$ not of type D.

n	type	q	Remark	kthulhu
2	(2)	even or not a square	sober [ACG1, Lemma 3.5]	yes
3	(3)	2	sober, [ACG1, Lemma 3.7 (b)]	yes
	$(2,1)$	2	cthulhu, [ACG1, Lemma 3.7 (a)]	
		even ≥ 4	cthulhu, [ACG1, Prop. 3.13, 3.16]	type C
4	$(2,1,1)$	2	cthulhu, [ACG1, Lemma 3.12]	
		even ≥ 4	not type D, [ACG1, Prop. 3.13] open for type F	Lemma 2.12

We recall that by the isogeny argument in [ACG1, Lemma 1.2], for unipotent classes we can work in $G = \mathbf{SL}_n(q)$ and $G = \mathbf{Sp}_{2n}(q)$.

Table 3: Cthulhu unipotent classes in $\mathbf{PSp}_{2n}(q)$.

n	type	q	Remark	kthulhu
≥ 2	$W(1)^a \oplus V(2)$	even	cthulhu [ACG2, Lemma 4.22]	yes Lemma 2.14
	$(1^{r_1}, 2)$	odd, 9 or not a square		
3	$W(1) \oplus W(2)$	2	cthulhu [ACG2, Lemma 4.25]	type C Lemma 2.17
2	$W(2)$	even	cthulhu [ACG2, Lemma 4.26]	yes Lemma 2.14
	$(2, 2)$	3	one class cthulhu [ACG2, Lemma 4.5]	type C Lemma 2.15
	$V(2)^2$	2	cthulhu [ACG2, Lemma 4.24]	type C Lemma 2.16

Lemma 2.12. *Let $G = \mathbf{SL}_n(q)$ with $n \geq 3$ and q even. Then any unipotent conjugacy class $\mathcal{O}$ with associated partition $(2, 1^{n-2})$ is of type C.*

Proof. By Lemma 2.10, it is enough to prove the assertion for $G = \mathbf{SL}_3(2)$. Denote by α_1 and α_2 the positive simple roots and let $x = x_{\alpha_1+\alpha_2}(1) = \left(\begin{smallmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{smallmatrix}\right)$ and $z = x_{-\alpha_2}(1) = \left(\begin{smallmatrix} 1&0&0\\ 0&1&0\\ 0&1&1 \end{smallmatrix}\right)$. Then x, y and z are conjugated in $\mathbf{SL}_3(2)$ with $y = v \triangleright x$, $z = w \triangleright x$ and $v = \left(\begin{smallmatrix} 0&1&0\\ 1&0&1\\ 1&0&0 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{smallmatrix}\right)$. Denote by H the subgroup of G generated by x, y and z. As x, y and z lie in the F_q-stable parabolic subgroup $\mathbb{P}$ of $\mathbf{SL}_n(\Bbbk)$ with F_q-stable Levi factor $\mathbb{L}$ with root system $\{\pm\alpha_2\}$, it follows that $H \subset \mathbb{P}^{F_q} \subsetneq G$. Moreover, since x is in the unipotent radical of $\mathbb{P}$, which is normal, and $y \in \mathbb{L}^{F_q}$ we have that $\mathcal{O}^H_x \neq \mathcal{O}^H_y$ and a direct computation shows that $\mathcal{O}^H_x = \{x, \left(\begin{smallmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&1&1\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right)\}$ and $\mathcal{O}^H_y = \mathcal{O}^H_z$. Thus $|\mathcal{O}^H_x| = 3$. The class $\mathcal{O}^H_y$ contains y, z and $y \triangleright z = {}^t z \neq z, y$. The result follows by Remark 2.6 and Lemma 2.8. □

As a consequence of the lemma above, Theorem 2.9 and [ACG1, Theorem 1.3] we obtain the following.

Proposition 2.13. *Let $x \in \boldsymbol{G}$ and pick $\mathtt{x} \in \mathbf{SL}_n(q)$ such that $\pi(\mathtt{x}) = x$, with Jordan decomposition $\mathtt{x} = \mathtt{x}_s\mathtt{x}_u$. Assume that $\mathtt{x}_u \neq e$. Then either $\mathcal{O} = \mathcal{O}^{\boldsymbol{G}}_x$ collapses or else $\mathtt{x}_s$ is central and $\mathcal{O}$ is a unipotent class listed in Table 4.* □

For the rest of the Section $G = \mathbf{Sp}_{2n}(q)$, $\boldsymbol{G} = \mathbf{PSp}_{2n}(q)$, $n \geq 2$ and $u \in G$ is a unipotent element. Recall that $\mathcal{C}(G, u)$ denotes the set of G-conjugacy classes contained in $\mathcal{O}^{\mathbb{G}}_u$. For unexplained notation see [ACG2, 4.2.1].

Lemma 2.14. *Let $G = \mathbf{Sp}_{2n}(q)$ and $\mathcal{O}$ a conjugacy class corresponding to*

(a) *$W(1)^a \oplus V(2)$ or $W(2)$ if q is even, or*

Table 4: Kthulhu unipotent classes in $\mathbf{PSL}_n(q)$.

n	type	q	Remark
2	(2)	even or not a square	sober, [ACG1, Lemma 3.5]
3	(3)	2	sober, [ACG1, Lemma 3.7] (b)

(b) *a partition* $(2, 1^{n-2})$ *if* q *is odd.*

Then $\mathcal{O}$ *is austere, hence kthulhu.*

Proof. By the proof of [ACG2, Lemma 4.26] there exists an automorphism of $\mathbf{Sp}_4(q)$ mapping each class labeled by $W(2)$ to a class corresponding to $W(1)^2 \oplus V(2)$. On the other hand, if $\mathcal{O}$ is labeled by $W(1)^a \oplus V(2)$ or by $(2, 1^{n-2})$, then it is austere by [ACG2, Lemma 4.22]. □

Lemma 2.15. *Let* $G = \mathbf{Sp}_4(3)$. *Then the conjugacy class* $\mathcal{O}$ *of type* $(2,2)$ *is of type C.*

Proof. There are two classes of type $(2, 2)$, represented by $w = \begin{pmatrix} 1 & & & -1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $z = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. By [ACG2, Lemma 4.5], $\mathcal{O}_w^G$ is of type D and $\mathcal{O}_z^G$ is chutlhu. We show that the latter is of type C.

Let $y = \begin{pmatrix} 2 & & & 2 \\ & 1 & 0 & \\ & 1 & 1 & \\ 1 & & & \end{pmatrix}$, then $y \in \mathcal{O}_z^G$ since $y = vzv^{-1}$ with $v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathbf{Sp}_4(3)$. Let $H = \langle z, y \rangle \subset G$. Since $\operatorname{ord}(z) = 3$, by Lemma 2.7 (b) it is enough to prove that $\mathcal{O}_z^H \neq \mathcal{O}_y^H$. Let $\mathbb{M}$ be the F_q-stable subgroup of $\mathbf{Sp}_4(\Bbbk)$ of matrices $\begin{pmatrix} a & 0 & b \\ 0 & M & 0 \\ c & 0 & d \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M \in \mathbf{SL}_2(\Bbbk)$. Clearly, $\mathbb{M} \simeq \mathbf{SL}_2(\Bbbk) \times \mathbf{SL}_2(\Bbbk)$ and $H \subseteq \mathbb{M}^{F_q} \simeq \mathbf{SL}_2(3) \times \mathbf{SL}_2(3)$.

Assume $y = AzA^{-1}$ with $A \in H$. Then, there exist $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M \in \mathbf{SL}_2(3)$ such that $A = \begin{pmatrix} a & 0 & b \\ 0 & M & 0 \\ c & 0 & d \end{pmatrix}$. But then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and this implies that $a = c \neq 0$, $d = 2a + b$ and consequently $ad - bc = 2$, a contradiction. Thus, z and y are not conjugated in H. □

Now we show that the remaining cthulhu classes in Table 3 are of type C.

Lemma 2.16. *Assume* $G = \mathbf{Sp}_4(2)$ *and* $\mathcal{O}$ *is of the form* $V(2)^2$. *Then* $\mathcal{O}$ *is of type C.*

Proof. By [LS, Theorem 6.21], there is only one class in $\mathcal{C}(G, u)$ which is cthulhu by [ACG2, Lemma 4.24]. Since $\mathbb{G}^{F_q} = \mathbf{Sp}_4(2) \simeq \mathbb{S}_6$ and $\mathcal{O}$ corresponds to the partition $(1^2, 2^2)$, we show that the latter class in $\mathbb{S}_6$ is of type C. Let $x = (1,2)(3,4)$, $y = (3,6)(4,5)$ and $z = (1,6)(2,5)$ and denote by H the subgroup generated by x, y and z. Since they are all even permutations, $H \subseteq \mathbb{A}_6 \subsetneq \mathbb{S}_6$. Further, a direct computation shows that $\mathcal{O}_x^H = \{x, (1,2)(5,6), (3,4)(5,6)\}$, with $y \triangleright x = (1,2)(5,6)$,

$z \triangleright x = (3,4),(5,6)$ and $z \triangleright (y \triangleright x) = x \triangleright (y \triangleright x) = y \triangleright x$, $y \triangleright (z \triangleright x) = x \triangleright (z \triangleright x) = z \triangleright x$. Further, since $(z \triangleright y) \triangleright y = z = (y \triangleright z) \triangleright y$, we have $\mathcal{O}_y^H = \mathcal{O}_z^H$ and $\mathcal{O}_y^H = \{y, z, (13)(24), (35)(46), (15)(26), (14)(23)\}$. Hence, $|\mathcal{O}_x^H| = 3$, $|\mathcal{O}_y^H| = 6$, $\mathcal{O}_x^H \neq \mathcal{O}_y^H$ and the result follows by Remark 2.6 and Lemma 2.8. □

Lemma 2.17. *Let $G = \mathbf{Sp}_6(2)$ and assume $\mathcal{O}$ is of the form $W(1) \oplus W(2)$. Then $\mathcal{C}(G,u)$ consists of only one class $\mathcal{O}$ which is of type C.*

Proof. By [ACG2, Lemma 4.25], $\mathcal{C}(G,u)$ consists of only one cthulhu class, represented by $u = x_{\alpha_1}(1) = \mathrm{id}_6 + e_{1,2} + 2e_{5,4}$.

Let $\mathtt{J} = \left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{smallmatrix}\right)$. Recall that there is a natural embedding $\iota\colon \mathbf{SL}_3(q) \to G$ given by $A \mapsto \mathrm{diag}(A, \mathtt{J}^t A^{-1} \mathtt{J})$. By Lemma 2.12, for $x = \mathrm{id}_3 + e_{12}$ the class $\mathcal{O}_x^{\mathbf{SL}_3(q)}$ is of type C. Since $\iota(x) = x_{\alpha_1}(1)$ we have the statement. □

The next result follows putting together the results in this section and [ACG2, Theorem 1.1].

Proposition 2.18. *Let $\mathcal{O}$ be a unipotent conjugacy class in $\mathbf{PSp}_{2n}(q)$. If $\mathcal{O}$ is not listed in Table 5, then it collapses.* □

Table 5: Kthulhu unipotent classes in $\mathbf{PSp}_{2n}(q)$.

n	type	q	Remark
≥ 2	$W(1)^a \oplus V(2)$	even	austere
	$(1^{r_1}, 2)$	odd, 9 or not a square	Lemma 2.14
2	$W(2)$	even	

We end this section with an example of a conjugacy class in $\mathbb{S}_4$ that is cthulhu and of type C. This is the so-called *cube rack*, i.e., the class $\mathcal{O}_{(3)}^{\mathbb{S}_4}$ of 3-cycles, which is a union of two so-called *tetrahedral racks*, i.e., conjugacy classes of 3-cycles in $\mathbb{A}_4$.

Lemma 2.19. *The cube rack is cthulhu but not kthulhu.*

Proof. The cube rack is cthulhu by [ACG1, Lemma 2.12]. Further, it is of type C, since as racks $\mathcal{O}_{(3)}^{\mathbb{S}_4} = \mathcal{O}_{(123)}^{\mathbb{A}_4} \coprod \mathcal{O}_{(132)}^{\mathbb{A}_4}$, where $\mathcal{O}_{(123)}^{\mathbb{A}_4} \simeq \mathcal{O}_{(132)}^{\mathbb{A}_4}$ are noncommuting indecomposable subracks of size 4. □

3. Preliminaries on semisimple classes in $\mathbf{PSL}_n(q)$

From now on, $G = \mathbf{SL}_n(q)$, $\boldsymbol{G} = \mathbf{PSL}_n(q)$ and $\pi : G \to \boldsymbol{G}$ is the natural projection.

3.1. Semisimple irreducible conjugacy classes

Let $\mathtt{S} \in \mathbf{GL}_n(q)$ be semisimple and $\chi_{\mathtt{S}} = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + (-1)^n \in \mathbb{F}_q[X]$ its characteristic polynomial. It is well-known, see e. g. [ACG1, Remark 4.1], that $\mathcal{O}_{\mathtt{S}}^G = \mathcal{O}_{\mathtt{S}}^{\mathbf{GL}_n(q)} \cap G$. Hence, for $\mathtt{S} \in \mathbf{SL}_n(q)$:

$$\mathcal{O}_{\mathtt{S}}^G = \{\mathtt{R} \in \mathbf{SL}_n(q) : \mathtt{R} \text{ is semisimple and } \chi_{\mathtt{R}} = \chi_{\mathtt{S}}\}. \tag{3.1}$$

Assume that $\mathtt{S} \in G$ is irreducible, that is, $\chi_{\mathtt{S}}$ is irreducible. Thus

$$\mathcal{O}_{\mathtt{S}}^G = \{\mathtt{R} \in \mathbf{SL}_n(q) : \chi_{\mathtt{R}} = \chi_{\mathtt{S}}\}. \tag{3.2}$$

Hence, the irreducible semisimple conjugacy classes in G are parametrized by the monic irreducible polynomials of degree n with constant term equal to $(-1)^n$, and they can be represented by the companion matrix.

Recall [AFGV1, Definition 3.8] that g and its class $\mathcal{O}$ are

- *real* when $g^{-1} \in \mathcal{O}$;
- *quasi-real* when g is not an involution and there is $j \in \mathbb{Z}$ such that $g \neq g^j \in \mathcal{O}$.

Remark 3.1. We collect some standard facts about $\mathcal{O}_{\mathtt{S}}^G$.

(a). If $\mathtt{S} \in G$ is irreducible, then the subalgebra of matrices in $M_n(q)$ commuting with $\mathtt{S}$ is isomorphic to $\mathbb{F}_{q^n}$. Hence $C_{\mathbf{SL}_n(q)}(\mathtt{S}) \simeq \mathbb{Z}/(n)_q$.

(b). Let $\mathtt{S} \in G$ be semisimple. Then $\chi_{\mathtt{S}} = \chi_{\mathtt{S}^q}$, hence $\mathtt{S}^q \in \mathcal{O}_{\mathtt{S}}^G$, but $\mathtt{S} \neq \mathtt{S}^q$ unless it is diagonalizable over $\mathbb{F}_q$. If $\eta \in \Bbbk$ is an eigenvalue of $\mathtt{S}$, then η^{q^l} is again so, for $l = 1, \dots, n-1$. They are all distinct if and only if $\mathtt{S}$ is irreducible. In this case, $\operatorname{ord} \mathtt{S}$ divides $(n)_q$. If $q+1 = \operatorname{ord} \mathtt{S}$, then the conjugacy class $\mathcal{O}_{\mathtt{S}}^G$ is *real*. It is *quasi-real* if it is not an involution nor diagonalizable over $\mathbb{F}_q$.

(c). If $\mathtt{S} \in \mathbf{GL}_n(q)$ is semisimple irreducible such that $\mathtt{S}^q = \lambda\mathtt{S}$ for some $\lambda \in \mathbb{F}_q$, then λ is a primitive n-th root of 1. In particular $n|(q-1)$. Indeed, $\mathtt{S}$ and $\mathtt{S}^q$ are conjugate, so $\det(\mathtt{S}) = \det(\mathtt{S}^q) = \lambda^n \det(\mathtt{S})$ whence $\lambda^n = 1$. In addition, we have $\mathtt{S}^{q^j} = \lambda^j\mathtt{S}$ for $j \in \mathbb{I}_{n-1}$. Since all such matrices are distinct, we have the claim.

(d). If $\lambda \in \mathbb{F}_q$, $\lambda^n = 1$, then

$$\chi_{\lambda\mathtt{S}} = X^n + a_{n-1}\lambda X^{n-1} + \cdots + a_j\lambda^{n-j}X^j + \cdots + a_1\lambda^{n-1}X + (-1)^n.$$

Hence, for $\mathtt{S} \in \mathbf{GL}_n(q)$ semisimple, $\lambda\mathtt{S} \in \mathcal{O}_{\mathtt{S}}^G$ if and only if $a_j(1-\lambda^{n-j}) = 0$ for every $j \in \mathbb{I}_{n-1}$. By (c) if $\mathtt{S}$ is irreducible, then the characteristic polynomial of $\mathtt{S}$ is $X^n + (-1)^n \det \mathtt{S}$.

(e). Assume that $q = t^h$ and $\mathtt{S} \in \mathbf{SL}_n(t) < \mathbf{SL}_n(q)$, with characteristic polynomial $\chi_{\mathtt{S},t}$. Then $\chi_{\mathtt{S},q} = \chi_{\mathtt{S},t}$ (because they are determinants of the same matrix) and $\mathcal{O}_{\mathtt{S}}^G \cap \mathbf{SL}_n(t) = \mathcal{O}_{\mathtt{S}}^{\mathbf{SL}_n(t)}$ by (3.1).

The following lemma will be useful in the sequel.

Lemma 3.2. *Let $n > 2$ and let $x \in \boldsymbol{G}$ be an irreducible semisimple element. Then $x^{q^i} \neq x^{q^j}$ for every $i \not\equiv j \mod n$. In particular, x is quasi-real.*

Proof. As $x^{q^n} = x$, it is enough to prove that $x^{q^j} \neq x$ for every $j \in \mathbb{I}_{n-1}$. Assume $x^{q^j} = x$. We may assume that $j \mid n$. Indeed if $x^{q^j-1} = 1$, then

$$\operatorname{ord} x \mid ((q^j - 1), (n)_q) = ((q-1)(j)_q, (n)_q) = (j,n)_q \left(q-1, \frac{(n)_q}{(j,n)_q}\right)$$

and the latter divides $q^{(j,n)} - 1$. Let $n = jk$ and let $\mathtt{x} \in \mathbf{SL}_n(q)$ such that $\pi(\mathtt{x}) = x$. Then $\mathtt{x}^{q^j} = \lambda\mathtt{x}$ for some $\lambda \in \mathbb{F}_q$ and $\mathtt{x}^{q^{\ell j}} = \lambda^\ell x$ for every ℓ. In particular, $\mathtt{x} = \mathtt{x}^{q^n} = \lambda^k \mathtt{x}$ so $\lambda^k = 1$. In addition, as all such powers of $\mathtt{x}$ are distinct, λ is a primitive k-th root of 1, so $k \mid (q-1)$.

Let η be an eigenvalue of $\mathtt{x}$. Then $\eta^{q^j} = \lambda\eta$. Thus, the eigenvalues of $\mathtt{x}$ are $\lambda^t \eta^{q^i}$ for $t \in \mathbb{I}_{k-1}$ and $i \in \mathbb{I}_{j-1}$. Therefore $1 = \det \mathtt{x} = \eta^{(j)_q} \lambda^{j\binom{k}{2}}$. Taking the $(q-1)$-th power we have $\eta^{q^j-1} \lambda^{(q-1)j\binom{k}{2}} = 1$. If q is odd then $(q-1)/2 \in \mathbb{Z}$; if instead q is even, then k is odd, so $(k-1)/2 \in \mathbb{Z}$. Therefore $\eta^{q^j-1} = 1$, so $\eta \in \mathbb{F}_{q^j}$, hence $j = n$ by irreducibility of $\chi_{\mathtt{x}}$. □

Remark 3.3. If $\mathtt{S} \in \mathbf{SL}_2(q)$ is semisimple irreducible such that $\mathtt{S}^q = \lambda\mathtt{S}$, then $q \equiv 3 \mod 4$. Indeed, by Remark 3.1 (d) its minimal polynomial is $X^2 + 1$ which is irreducible only if $q \equiv 3 \mod 4$. Thus, a semisimple irreducible element $\pi(\mathtt{S})$ in $\mathbf{PSL}_2(q)$ is quasi-real unless $q \equiv 3 \mod 4$ and $\chi_{\mathtt{S}} = X^2 + 1$.

3.2. Subgroups of $\mathbf{PSL}_2(q)$

In this subsection $\boldsymbol{G} = \mathbf{PSL}_2(q)$. We stress that we assume $q \neq 2, 3, 4, 5, 9$ to avoid coincidences with cases treated elsewhere, see [ACG1, Subsection 1.2]. We recall Dickson's classification of all subgroups of $\boldsymbol{G}$. Let $d = (2, q-1)$.

Theorem 3.4. [Su, Theorems 6.25, p. 412; 6.26, p. 414] *A subgroup of* $\mathbf{PSL}_2(q)$, $q = p^m$ *is isomorphic to one of the following groups.*

(a) *The dihedral groups of order* $2(q \pm 1)/d$ *and their subgroups. There are always such subgroups.*

(b) *A group* H *of order* $q(q-1)/d$ *and its subgroups. It has a normal* p*-Sylow subgroup* Q *that is elementary abelian and the quotient* H/Q *is cyclic of order* $(q-1)/d$*. There are always such subgroups.*

(c) $\mathbb{A}_4$*, and there are such subgroups except when* $p = 2$ *and* m *is odd.*

(d) $\mathbb{S}_4$*, and there are such subgroups if and only if* $q^2 \equiv 1 \mod 16$.

(e) $\mathbb{A}_5$*, and there are such subgroups if and only if* $q(q^2 - 1) \equiv 0 \mod 5$.

(f) $\mathbf{PSL}_2(t)$ *for some* t *such that* $q = t^h$, $h \in \mathbb{N}$*. There are always such subgroups.*

(g) $\mathbf{PGL}_2(t)$ *for some* t *such that* $q = t^h$, $h \in \mathbb{N}$*. If* q *is odd, then there are such subgroups if and only if* h *is even and* $q = t^h$*. Note that for* q *even, this reduces to case* (f). □

For further use we record the following consequence of Theorem 3.4. If q is odd, then involutions in G are semisimple. By looking at the possible eigenvalues of such an element we see that there is only one class of non-trivial involutions in $\boldsymbol{G}$ for q odd.

Corollary 3.5. *Let $\mathcal{O}$ be the conjugacy class of non-trivial involutions in $\boldsymbol{G}$. If $q > 7$ is odd, then $\mathcal{O}$ is of type D; while $\mathcal{O}$ is kthulhu, when $q = 7$.*

Proof. Assume that $q > 7$ is odd. Recall that the case $q = 9$ is excluded. By Theorem 3.4 (a), $\boldsymbol{G}$ contains dihedral subgroups D_1 and D_2 of order $q-1$ and $q+1$ respectively. If $q \equiv 1 \mod 4$, resp. $q \equiv 3 \mod 4$, then $\mathcal{O} \cap D_1$, respectively $\mathcal{O} \cap D_2$, is of type D by [AFGaV1, Lemma 2.1].

Assume that $q = 7$. Let Y be a subrack of $\mathcal{O}$ and $K = \langle Y \rangle$. We claim that Y admits no decomposition as in neither of the definitions of type D, F nor C. If $K = \boldsymbol{G}$, then $Y = \mathcal{O}$ (because Y is a union of K-conjugacy classes) which is indecomposable. By inspection, the possible proper subgroups of $\boldsymbol{G}$ containing an involution are $\mathbb{S}_3$, $\mathbb{A}_4$, $\mathbb{S}_4$, or $\mathbb{D}_4$. If $K = \mathbb{S}_3$, then Y is indecomposable. The involutions of $\mathbb{A}_4$ generate the 2-Sylow subgroup, thus K could not be $\mathbb{A}_4$. Assume $K = \mathbb{S}_4$. If $u = (12)(34)$, then $\mathcal{O}_u^{\mathbb{A}_4} = \mathcal{O}_u^{\mathbb{S}_4}$ does not generate K. If $u = (12)$, then $\mathcal{O}_u^{\mathbb{S}_4}$ is indecomposable by [AG, Proposition 3.2 (2)]. Finally, there are 3 classes of involutions in $\mathbb{D}_4$, say $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$; here $\mathcal{C}_3 \subset Z(\mathbb{D}_4)$ has one element, while $\mathcal{C}_i = \{x_i, y_i\}$, $i \in \mathbb{I}_2$ are abelian, and $x_1 \triangleright x_2 = y_2$. Thus no Y generating $\mathbb{D}_4$ admits a decomposition as required. Hence $\mathcal{O}$ is kthulhu but not sober. □

3.3. Subgroups of $\mathbf{PSL}_3(q)$

In this subsection we recall some classification results about the subgroups of $\boldsymbol{G} = \mathbf{PSL}_3(q)$ and $\mathbf{PSU}_3(q)$. The classification of the subgroups of $\boldsymbol{G}$ for q odd was obtained by Mitchell in 1911, whereas the classification of maximal subgroups in $\boldsymbol{G}$ for q even was achieved by Hartley in 1925. We set $d = (q-1, 3)$. If q is even, $d = 1$ exactly when q is not a square.

Theorem 3.6. [B, Theorems 1.1, 7.1]

Let K be a subgroup of $\mathbf{PSL}_3(q)$ with $q = p^m$ and p an odd prime.

(I) *Assume that K has no non-trivial normal elementary abelian subgroup. Then K is isomorphic to one of the following groups.*

(a) $\mathbf{PSL}_3(t)$ *for some $t = p^a$ such that $q = t^h$, $h \in \mathbb{N}$.*

(b) $\mathbf{PSU}_3(t)$ *for some $t = p^a$ such that $2a \mid m$.*

(c) *If $t = p^a$ satisfies $t \equiv 1 \mod 3$ and $3a \mid m$, then there is a subgroup containing the subgroup of type* (a) *with index 3.*

(d) *If $t = p^a$ satisfies $t \equiv 2 \mod 3$ and $6a \mid m$, then there is a subgroup containing the subgroup of type* (b) *with index 3.*

(e) $\mathbf{PSL}_2(t)$ *or* $\mathbf{PGL}_2(t)$ *for some $t = p^a \neq 3$ such that $q = t^h$, $h \in \mathbb{N}$.*

(f) $\mathbf{PSL}_2(5)$, *when* $q \equiv \pm 1 \mod 10$.

(g) $\mathbf{PSL}_2(7)$, *when* $q^3 \equiv 1 \mod 7$.

(h) $\mathbb{A}_6$, $\mathbb{A}_7$ *or a group containing* $\mathbb{A}_6$ *with index 2, when* $p = 5$ *and* m *is even.*

(i) $\mathbb{A}_6$, *when* $q \equiv 1$ *or* $19 \mod 30$.

There are always such subgroups under the indicated restrictions.

(II) *Assume that* K *has a non-trivial normal elementary abelian subgroup. Then one of the following happens:*

(j) K *has a cyclic* p*-regular normal subgroup of index* ≤ 3.

(k) K *has a diagonal normal subgroup* L *such that* K/L *is isomorphic to a subgroup of* $\mathbb{S}_3$.

(l) K *has a normal elementary-abelian* p*-subgroup* H *such that* K/H *is isomorphic to a subgroup of* $\mathbf{GL}_2(q)$. *We include the case* $H = \{1\}$.

(m) K *has a normal abelian subgroup* H *of type* $(3, 3)$, *with* K/H *isomorphic to a subgroup of* $\mathbf{SL}_2(3)$. *All subgroups of* $\mathbf{SL}_2(3)$ *do occur in this context. This happens when* $q \equiv 1 \mod 9$.

(n) K *has a normal abelian subgroup* H *of type* $(3, 3)$, *with* K/H *isomorphic to a subgroup of the quaternion group* Q *of order 8. All subgroups of* Q *do occur in this context. This happens when* $q \equiv 1 \mod 3$, $q \not\equiv 1 \mod 9$. □

The following theorem gives the classification of the maximal subgroups of $\boldsymbol{G}$ for q even.

Theorem 3.7. [Ha, Theorem 8, Summary] *Let* M *be a maximal subgroup of* $\mathbf{PSL}_3(q)$, *with* $q = 2^m$. *Then* M *is one of the following:*

(a) *A subgroup of order* $q^3(q+1)(q-1)^2/d$ *or* $6(q-1)^2/d$.

(b) *The normalizer of a maximal torus of order* $(3)_q/d$. *The torus has index* 3 *in* M.

(c) $\mathbf{PSL}_3(2^k)$, *where* m/k *is prime.*

(d) *A group containing* $\mathbf{PSL}_3(2^{2a})$ *as a normal subgroup of index* 3. *This happens if* $m = 6a$.

(e) $\mathbf{PSU}_3(t)$. *This happens when* $q = t^2$ *is square.*

(f) *A group containing* $\mathbf{PSU}_3(2^a)$ *as a normal subgroup of index* 3. *This happens if* a *is odd and* $m = 6a$.

(g) *A group isomorphic to* $\mathbb{A}_6$. *This happens when* $q = 4$. □

For inductive arguments we will also need the classification of maximal subgroups for $\mathbf{PSU}_3(q)$, for q even, also due to Hartley.

Theorem 3.8. [Ha] *Let $q = 2^m$. Let M be a maximal subgroup of $\mathbf{PSU}_3(q)$. Then M is one of the following subgroups, where $e := (3, q+1)$:*

(a) *A subgroup of order $q^3(q+1)(q-1)/e$, $q(q+1)^2(q-1)/e$ or $6(q+1)^2/e$.*

(b) *The normalizer of a maximal torus of order $3(q^2-q+1)/e$. The torus has index 3 in M.*

(c) $\mathbf{PSU}_3(2^l)$, *where m/l is an odd prime. If $l = 1$ this group has order 72.*

(d) *A group containing $\mathbf{PSU}_3(2^l)$ as a normal subgroup of index 3. This happens if l is odd and $m = 3l$. If $l = 1$ this group has order 216.*

(e) *A group of order 36. This happens when $q = 4$.* □

3.4. Reduction to the irreducible case

In this subsection we prove that all non-irreducible semisimple conjugacy classes in $\boldsymbol{G} = \mathbf{PSL}_n(q)$ collapse. The proof is split in several lemmata in which we use different criteria for a rack to collapse. Recall that $G = \mathbf{SL}_n(q)$ and $\pi : G \to \boldsymbol{G}$ is the natural projection.

We begin with the case where $\mathcal{O}$ is a conjugacy class of a diagonal and non-central element.

Lemma 3.9. *If $T \in G$ is diagonal but not central, then $\mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ collapses.*

Proof. Assume first that $n > 2$. Say that $T = \operatorname{diag}(a_1 \dots, a_n)$ with $a_j \in \mathbb{F}_q^\times$; since T is not central, at least two of the a_j's are different, so $q > 2$. Assume that $a_1 \neq a_2$. and consider the following subsets of $\mathcal{O}^G_T$:

$$X_1 = \left\{ r_c := \begin{pmatrix} a_1 & c & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} : \ c \in \mathbb{F}_q \right\},$$
$$X_2 = \left\{ s_f := \begin{pmatrix} a_2 & f & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} : \ f \in \mathbb{F}_q \right\}.$$

Since

$$\begin{pmatrix} d & c \\ 0 & e \end{pmatrix} \triangleright \begin{pmatrix} e & f \\ 0 & d \end{pmatrix} = \begin{pmatrix} e & de^{-1}(c+f)-c \\ 0 & d \end{pmatrix}, \tag{3.3}$$

$Y = X_1 \coprod X_2$ is a decomposable subrack of $\mathcal{O}^{\mathbf{SL}_n(q)}_T$, cf. (3.1). Set $r = r_1$ and $s = s_0$, so that $r \triangleright s \neq s$. Then $r_c \triangleright s = s_{(a_1 a_2^{-1} - 1)c}$, hence $Y \triangleright s = X_2 = \mathcal{O}^{\langle Y \rangle}_s$ by (3.3); similarly, $Y \triangleright r = X_1 = \mathcal{O}^{\langle Y \rangle}_r$. Also $|X_1| = |X_2| = q > 2$. Hence $\mathcal{O}^G_T$ is of type C and since $\pi_{|Y}$ is injective, $\mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ is of type C.

Assume next that $n = 2$, so that $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, with $a^2 \neq 1$. Let

$$X_a = \{r_c := \left(\begin{smallmatrix} a & c \\ 0 & a^{-1} \end{smallmatrix}\right) : \ c \in \mathbb{F}_q\}, \qquad X_{a^{-1}} = \left\{s_f := \begin{pmatrix} a^{-1} & f \\ 0 & a \end{pmatrix} : \ f \in \mathbb{F}_q\right\}.$$

Then $Y = X_a \coprod X_{a^{-1}}$ is a decomposable subrack of $\mathcal{O}_T^{\mathbf{SL}_2(q)}$, by (3.3); and $r_c \triangleright s_f \neq s_f$ if and only if $c + f \neq 0$; respectively, $r_c \triangleright (s_f \triangleright (r_c \triangleright s_f)) \neq s_f$ if and only if $2(c + f) \neq 0$.

Assume that q is odd. Then $\mathcal{O}_T^{\mathbf{SL}_2(q)}$ is of type D. If $a^4 \neq 1$, then $\pi_{|Y}$ is injective, hence $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type D. If $a^4 = 1$, then $\pi(T)$ is an involution and $q \neq 7$, hence $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type D by Corollary 3.5.

Assume that q is even, hence $\boldsymbol{G} = G$. Set $r = r_1$ and $s = s_0$; thus $r \triangleright s \neq s$. Now $r_c \triangleright s = s_{c+ca^2}$ by (3.3), hence $Y \triangleright s = X_{a^{-1}}$; similarly, $Y \triangleright r = X_a$. Also $|X_{a^{\pm 1}}| = q > 2$. Hence $\mathcal{O}_T^G$ is of type C, and the claim follows. □

Lemma 3.10. *Let $T \in \mathbf{SL}_n(q)$ semisimple not diagonal, with at least one eigenvalue. Then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C.*

Proof. By hypothesis, we may assume that either $T = \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$, where $B \in \mathbf{GL}_e(q)$ is semisimple irreducible and not diagonal, C is semisimple and $a = (\det B \det C)^{-1}$, or else $T = \left(\begin{smallmatrix} a & 0 \\ 0 & B \end{smallmatrix}\right)$, where $B \in \mathbf{GL}_e(q)$, is semisimple irreducible not diagonal, $e = n - 1$, and $a = (\det B)^{-1}$. In both cases $B \neq B^q \in \mathcal{O}_B^{\mathbf{GL}_e(q)}$ and $\det B^q = (\det B)^q = \det B$. We treat only the first possibility, the second being analogous. Let

$$X_1 = \left\{r_v := \begin{pmatrix} a & v & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} : v \in \mathbb{F}_q^e\right\}, \qquad X_2 = \left\{s_w := \begin{pmatrix} a & w & 0 \\ 0 & B^q & 0 \\ 0 & 0 & C \end{pmatrix} : w \in \mathbb{F}_q^e\right\}.$$

Then $r_v \triangleright s_w = s_{(a(w-v)+vB^q)B^{-1}}$; hence $Y = X_1 \coprod X_2$ is a decomposable subrack of $\mathcal{O}_T^{\mathbf{SL}_n(q)}$, cf. (3.1). If $s = s_0$, then $r_v \triangleright s = s_{v(-a+B^q)B^{-1}}$, thus $Y \triangleright s = X_2$ because $-a + B^q$ is invertible by hypothesis. Similarly, $Y \triangleright r = X_1$. Also, if $v \neq 0$ and $r = r_v$, then $r \triangleright s \neq s$. Since $|X_1| = |X_2| = q^e > 2$, $\mathcal{O}_T^G$ is of type C. Since $\pi_{|Y}$ is injective, $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C. □

Next we treat the cases where the non-irreducible semisimple element has no eigenalues in $\mathbb{F}_q$.

Lemma 3.11. *Let $T \in \mathbf{SL}_n(q)$ semisimple not irreducible, with at least* 3 *irreducible blocks and no eigenvalues in $\mathbb{F}_q$. Then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C.*

Proof. We may assume that $T = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$, where $A \in \mathbf{GL}_d(q)$, $B \in \mathbf{GL}_e(q)$ and $C \in \mathbf{GL}_f(q)$ are semisimple not diagonal, A and B are irreducible, $d + e + f = n$ and $\det A \det B \det C = 1$.

As in the previous proof, $B \neq B^q \in \mathcal{O}_B^{\mathbf{SL}_e(q)}$ and $\det B^q = \det B$. Let

$$R = \left\{\begin{pmatrix} E & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} : \ E \in \mathcal{O}_A^{\mathbf{SL}_d(q)}\right\}, \qquad S = \left\{\begin{pmatrix} F & 0 & 0 \\ 0 & B^q & 0 \\ 0 & 0 & C \end{pmatrix} : \ F \in \mathcal{O}_A^{\mathbf{SL}_d(q)}\right\},$$

Then $Y = R \coprod S$ is a decomposable subrack of $\mathcal{O}_T^{\mathbf{SL}_n(q)}$. A normal subgroup of $\mathbf{GL}_d(q)$ is either central or contains $\mathbf{SL}_d(q)$ [D, p. 40]. Let $K = \langle \mathcal{O}_A^{\mathbf{SL}_d(q)} \rangle \lhd \mathbf{GL}_d(q)$. Since A is not central, $\mathbf{SL}_d(q) \leq K$ and K is not abelian. Therefore there are x, y in $\mathcal{O}_A^{\mathbf{GL}_d(q)}$ such that $x \triangleright y \neq y$. By (3.1), $\mathcal{O}_A^{\mathbf{GL}_d(q)} = \mathcal{O}_A^{\mathbf{SL}_d(q)}$. Then $r := \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix}\right)$ and $s := \left(\begin{smallmatrix} y & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix}\right)$ satisfy $r \triangleright s \neq s$. By the same reason, $|R| = |S| = |\mathcal{O}_T^{\mathbf{SL}_d(q)}| > 2$. Finally, $\mathbf{GL}_d(q) \triangleright y \supseteq K \triangleright y \supseteq \mathbf{SL}_d(q) \triangleright y = \mathbf{GL}_d(q) \triangleright y$. Hence (2.3) holds and $\mathcal{O}_T^G$ is of type C. Since $\pi_{|Y}$ is injective, $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C. □

Now we prove that the conjugacy class of a non-irreducible non-diagonal semisimple element with two irreducible blocks and no eigenvalues in $\mathbb{F}_q$ collapses. Again, we split the proof in several lemmata, depending on the relation between the blocks, n and q.

Lemma 3.12. *Let $T \in \mathbf{SL}_n(q) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ semisimple, with $A \in GL_d(q)$ and $B \in \mathbf{GL}_e(q)$ irreducible and without eigenvalues in $\mathbb{F}_q$. If $B^q \notin Z(\mathbf{SL}_e(q))B$, or else $A^q \notin Z(\mathbf{SL}_d(q))A$, then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C.*

Proof. Up to interchanging the role of A and B we may assume that $B^q \notin Z(\mathbf{SL}_e(q))B$. Arguing as in the previous proof, we take

$$R = \left\{ \left(\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right) : \ C \in \mathcal{O}_A^{\mathbf{SL}_d(q)} \right\}, \qquad S = \left\{ \left(\begin{smallmatrix} D & 0 \\ 0 & B^q \end{smallmatrix}\right) : \ D \in \mathcal{O}_A^{\mathbf{SL}_d(q)} \right\}.$$

Then $Y = R \coprod S$ is a decomposable subrack of $\mathcal{O}_T^{\mathbf{SL}_n(q)}$ and for x, y in $\mathcal{O}_A^{\mathbf{SL}_d(q)}$ such that $x \triangleright y \neq y$, the elements $r := \left(\begin{smallmatrix} x & 0 \\ 0 & B \end{smallmatrix}\right)$ and $s := \left(\begin{smallmatrix} y & 0 \\ 0 & B^q \end{smallmatrix}\right)$ satisfy $r \triangleright s \neq s$. The same argument as in the proof of Lemma 3.11 shows that $|R| = |S| = |\mathcal{O}_A^{\mathbf{SL}_d(q)}| > 2$ and that (2.3) holds for $\mathcal{O}_T^G$. We prove that $\pi_{|Y}$ is injective. Clearly $\pi_{|R}$ and $\pi_{|S}$ are injective. If $\pi\left(\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right) = \pi\left(\begin{smallmatrix} D & 0 \\ 0 & B^q \end{smallmatrix}\right)$ for some $C, D \in \mathcal{O}_A^{\mathbf{SL}_d(q)}$, then there would be a $\lambda \in \mathbb{F}_q$ such that $B^q = \lambda B$. Computing the determinant we get $\lambda^e = 1$ and therefore $\lambda \operatorname{id} \in Z(\mathbf{SL}_e(q))$ contradicting our hypothesis. □

Lemma 3.13. *Let T, A, B, d, e be as in Lemma 3.12 with A and B irreducible and without eigenvalues in $\mathbb{F}_q$ and let $l := (q-1, d, e)$. If $l \neq d$ or $l \neq e$, then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C.*

Proof. Up to interchanging the role of A and B we may assume that $l \neq e$. Arguing as in the proof of Lemma 3.12, with same Y, we conclude that $\mathcal{O}_T^{\mathbf{SL}_n(q)}$ is of type C. We show that $\pi_{|Y}$ is injective. If $\pi\left(\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right) = \pi\left(\begin{smallmatrix} D & 0 \\ 0 & B^q \end{smallmatrix}\right)$ for some $C, D \in \mathcal{O}_A^{\mathbf{SL}_d(q)}$, then there would be $\lambda \in \mathbb{F}_q$ such that $B^q = \lambda B$, $D = \lambda C$. Computing the determinants we get $\lambda^d = \lambda^e = 1$ and therefore $\lambda^l = 1$. By Remark 3.1 (c), λ would be a primitive e-th root of 1, contradicting our hypothesis. □

Lemma 3.14. *Let $T \in \mathbf{SL}_n(q) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ be semisimple, with $A, B \in \mathbf{GL}_d(q)$ irreducible with no eigenvalues in $\mathbb{F}_q$. If $B \notin Z(\mathbf{SL}_d(q))\mathcal{O}_A^{\mathbf{SL}_d(q)}$, then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type C.*

Proof. Since A and B are irreducible, then each of them lies in a maximal torus of $\mathbf{GL}_d(q)$ of order q^d-1. All such tori are conjugate in $\mathbf{GL}_d(q)$ [MaT, Proposition 25.1]. So, up to replacing B by $B' \in \mathcal{O}_B^{\mathbf{GL}_d(q)} = \mathcal{O}_B^{\mathbf{SL}_d(q)}$ in the same torus as A, we can assume that A and B commute. We set

$$R = \left\{ \left(\begin{smallmatrix} D & 0 \\ 0 & B \end{smallmatrix}\right) : \ D \in \mathcal{O}_A^{\mathbf{SL}_d(q)} \right\}, \qquad S = \left\{ \left(\begin{smallmatrix} E & 0 \\ 0 & A \end{smallmatrix}\right) : \ E \in \mathcal{O}_B^{\mathbf{SL}_d(q)} \right\}.$$

Then $Y = R \coprod S$ is a decomposable subrack of $\mathcal{O}_T^{\mathbf{GL}_n(q)}$. The same argument as above gives $|R|, |S| > 2$ and (2.3) for $\mathcal{O}_T^G$. Therefore $\mathcal{O}_T^G$ is of type C if we can find noncommuting $D \in \mathcal{O}_A^{\mathbf{SL}_d(q)}$ and $E \in \mathcal{O}_B^{\mathbf{SL}_d(q)}$. Let $D \in \mathcal{O}_A^{\mathbf{SL}_d(q)}$ be in a maximal torus S_D. Since D is irreducible, it has distinct eigenvalues so its centralizer is S_D. Therefore it is enough to choose $E \notin S_D$. If $B \notin S_D$ we take $E = B$. Assume $B \in S_D$ and let $x \in \mathbf{SL}_d(q)$ such that $x \notin N_{\mathbf{SL}_d(q)}(S_D)$. Then $E := xBx^{-1} \in xS_Dx^{-1} \neq S_D$. As every semisimple element with distinct eigenvalues lies in a unique maximal torus [Hu, 2.3], $E \notin S_D$. Injectivity of $\pi_{|Y}$ yields the statement. □

We are left with the analysis of $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ for $T \in \mathbf{SL}_n(q) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ semisimple, with $A, B \in \mathbf{GL}_d(q)$ irreducible, $B \in Z(\mathbf{SL}_d(q))\mathcal{O}_A^{\mathbf{GL}_d(q)}$, $A^q \in Z(\mathbf{SL}_d(q))A$, $\det(A)\det(B) = 1$, $d > 1$ and $d \mid (q-1)$. It follows from Remark 3.1 (c) that under these assumptions $Z(\mathbf{SL}_d(q))\mathcal{O}_A^{\mathbf{GL}_d(q)} = \mathcal{O}_A^{\mathbf{GL}_d(q)}$ so $B \in \mathcal{O}_A^{\mathbf{GL}_d(q)}$, $\det(A) = \det(B) = \pm 1$ and the characteristic polynomial of A is

$$\chi_A = X^d + (-1)^d \det(A).$$

This polynomial is irreducible only if d is even, $\det(A) = 1$ and -1 is not a square in $\mathbb{F}_q$, i.e., $q \equiv 3 \mod 4$. We analyze this situation, studying separately the cases $d > 2$ and $d = 2$.

Lemma 3.15. *Let $T \in \mathbf{SL}_{2d}(q) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ be semisimple, with $A \in \mathbf{SL}_d(q)$ irreducible, $d > 2$, $A^q = \mu A$, $d|q-1$, d even and μ a primitive d-th root of 1. Then $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type D.*

Proof. It is always possible to find $a \neq b \in \mathbb{I}_{d-1}$ such that, $\mu^{2(a+b)} \neq 1$. The matrices $x = \operatorname{diag}(A, \mu^a A)$ and $y = \operatorname{diag}(A, \mu^b A)$ lie in $\mathcal{O}_T^{\mathbf{SL}_{2d}(q)}$. We set

$$R = \left\{ X = \begin{pmatrix} A & X' \\ 0 & \mu^a A \end{pmatrix} : \ X \in \mathcal{O}_T^{\mathbf{SL}_{2d}(q)} \right\},$$
$$S = \left\{ Z = \begin{pmatrix} A & Z' \\ 0 & \mu^b A \end{pmatrix} : \ Z \in \mathcal{O}_T^{\mathbf{SL}_{2d}(q)} \right\}.$$

Then $Y := R \coprod S$ is a decomposable subrack. Let

$$r = \begin{pmatrix} \mathrm{id}_d & \mathrm{id}_d \\ 0 & \mathrm{id}_d \end{pmatrix} \triangleright x = \begin{pmatrix} A & (\mu^a - 1)A \\ 0 & \mu^a A \end{pmatrix} \in R$$

and $s := y \in S$. A direct computation shows that $(rs)^2 \neq (sr)^2$. Since for our choice of a and b the map $\pi_{|Y}$ is injective, $\mathcal{O}_{\pi(T)}^{\boldsymbol{G}}$ is of type D. □

Lemma 3.16. *Let $T \in \mathbf{SL}_4(q) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ be semisimple, with $A \in \mathbf{GL}_2(q)$ irreducible, $A^q = -A$, $q \neq 3$. Then $\mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ is of type D.*

Proof. By Remark 3.1 (c), the characteristic polynomial of A is necessarily X^2+1 so $q \equiv 3 \mod 4$, and $\pi(T)$ is an involution. Let ζ be a generator of $\mathbb{F}_q^{\times}$. By [FaV, Lemma 2.5], $\mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ is of type D if and only if there exist $r, s \in \mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ such that $\operatorname{ord}(rs) > 4$ is even. Let

$$\mathtt{r} := \begin{pmatrix} 0 & -\zeta & 0 & 0 \\ \zeta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathtt{s} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

As $\mathtt{r}$ and $\mathtt{s}$ are semisimple matrices with characteristic polynomial equal to $(X^2+1)^2$, they lie in $\mathcal{O}^{\mathbf{SL}_4(q)}_T$. In addition,

$$\mathtt{rs} = \begin{pmatrix} 0 & 0 & -\zeta & 0 \\ 0 & 0 & 0 & \zeta^{-1} \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (\mathtt{rs})^2 = \operatorname{diag}(\zeta, \zeta^{-1}, \zeta, \zeta^{-1}),$$

so, for $r := \pi(\mathtt{r})$, $s := \pi(\mathtt{s})$ we have $\operatorname{ord}(rs) = 2\operatorname{ord}(\zeta^2) = q - 1 > 4$, as $q \neq 3$, $q \equiv 3 \mod 4$. □

Lemma 3.17. *Let $T \in \mathbf{SL}_4(3) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ be semisimple, with $A = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\mathcal{O}^{\boldsymbol{G}}_{\pi(T)}$ is of type D.*

Proof. Let

$$v := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathtt{s} := v \triangleright T = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathcal{O}^{\mathbf{SL}_4(q)}_{\pi(T)}.$$

A direct computation shows that $|T\mathtt{s}| = 12$, so setting $r := \pi(T)$ and $s := \pi(\mathtt{s})$ we have $\operatorname{ord}(rs) \in \{6, 12\}$. We apply [FaV, Lemma 2.5]. □

Summarizing, we have the following result.

Proposition 3.18. *Assume $T \in \mathbf{SL}_n(q)$ is semisimple and not irreducible, then $\mathcal{O}^{\mathbf{PSL}_n(q)}_T$ collapses.* □

Proof. If T is has at least one eigenvalue and it is not central, then the claim follows by Lemmata 3.9, 3.10. Assume T has no eigenvalues in $\mathbb{F}_q$. If T has at least 3 irreducible blocks, we apply Lemma 3.11, and if T has exactly 2 irreducible blocks, then the assertion is a consequence of Lemmata 3.12, 3.13, 3.14 3.15, 3.16 and 3.17. □

3.5. General results on irreducible semisimple classes

In this subsection we set $\boldsymbol{G} = \mathbf{PSL}_n(q)$. In this subsection we prove that for certain subgroups $K \leq \boldsymbol{G}$ and $x \in \boldsymbol{G}$ semisimple, we have that $\mathcal{O}^{\boldsymbol{G}}_x \cap K = \mathcal{O}^K_x$. These results will be used in the sequel.

Lemma 3.19. *Let $K = \mathbf{PSL}_n(t) \leq \boldsymbol{G}$ for $t = p^a$ and $a|m$ with $(t-1,n) = (q-1,n)$. If $x \in K$ is semisimple, then $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \mathcal{O}_x^K$.*

Proof. Let $\mathtt{x} \in \mathbf{SL}_n(t)$ such that $\pi(\mathtt{x}) = x$. By Remark 3.1 (e), $\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(\Bbbk)} \cap \mathbf{SL}_n(t) = \mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(t)}$. A fortiori, $\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(q)} \cap \mathbf{SL}_n(t) = \mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(t)}$. Let $y = \pi(\mathtt{y}) \in \mathcal{O}_x^{\boldsymbol{G}} \cap K$, with $\mathtt{y} \in \mathbf{SL}_n(t)$. Since $(t-1,n) = (q-1,n)$, we have that $Z(\mathbf{SL}_n(q)) = Z(\mathbf{SL}_n(t))$ and then, for some $z \in Z(\mathbf{SL}_n(q))$ there holds $\mathtt{y} \in z(\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(q)} \cap \mathbf{SL}_n(t)) = z\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(t)}$, so $y \in \mathcal{O}_x^K$. □

Lemma 3.20. *Let $K = \mathbf{PSL}_n(t) \leq \boldsymbol{G}$ for $t = p^a$ and $a|m$. Assume that $((n)_t, n) = 1$. If $x \in K$ is semisimple irreducible, then $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \mathcal{O}_x^K$.*

Proof. Let $\mathtt{x} \in \mathbf{SL}_n(t)$ such that $\pi(\mathtt{x}) = x$ and let $\mathtt{y} \in \mathbf{SL}_n(t)$ with $y = \pi(\mathtt{y}) \in \mathcal{O}_{\mathtt{x}}^{\boldsymbol{G}} \cap K$. So $\mathtt{y} \in (z\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_n(q)}) \cap \mathbf{SL}_n(t)$ for some $z\,\mathrm{id} \in Z(\mathbf{SL}_n(q))$. Therefore it is enough to prove that $z = 1$. Since $\operatorname{ord}\mathtt{x}$ and $\operatorname{ord}\mathtt{y} = \operatorname{ord} z\mathtt{x}$ divide $(n)_t$, we have $z^{(n)_t} = 1$, whence the statement. □

Lemma 3.21. *Let $K = \mathbf{PSU}_n(t) \leq \boldsymbol{G}$ for $t = p^a$ and $2a|m$. If $(t+1,n) = (q-1,n)$, then $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \mathcal{O}_x^K$ for every semisimple $x \in K$.*

Proof. Let $y = \pi(\mathtt{y}) \in \mathcal{O}_x^{\mathbf{PSL}_n(q)} \cap K$, with $\mathtt{y} \in \mathbf{SU}_n(t)$. For some $z \in Z(\mathbf{SL}_n(q))$ and $\mathtt{x} \in \mathbf{SU}_n(t)$ with $\pi(\mathtt{x}) = x$ we have $\mathtt{y} \in \mathcal{O}_{z\mathtt{x}}^{\mathbf{SL}_n(q)} \cap \mathbf{SU}_n(t)$. Under our assumptions $Z(\mathbf{SL}_n(q)) \simeq \mathbb{Z}/(q-1,n) = \mathbb{Z}/(t+1,n) \simeq Z(\mathbf{SU}_n(t))$, so $z\mathtt{x} \in \mathbf{SU}_n(t)$ and it is semisimple. The centralizer of $\mathtt{x}$ in $\mathbf{SL}_n(\Bbbk)$ is connected, so $\mathcal{O}_{z\mathtt{x}}^{\mathbf{SL}_n(\Bbbk)} \cap \mathbf{SU}_n(t) = \mathcal{O}_{z\mathtt{x}}^{\mathbf{SU}_n(t)}$. Thus, $\mathtt{y} \in \mathcal{O}_{z\mathtt{x}}^{\mathbf{SU}_n(t)}$, and $y \in \mathcal{O}_x^K$. □

4. Finite-dimensional pointed Hopf algebras over $\mathbf{PSL}_2(q)$

4.1. Abelian techniques

Let G be a finite group, $\mathcal{O}$ a conjugacy class of G, $g \in \mathcal{O}$ and $(\rho, V) \in \operatorname{Irr} C_G(g)$. The abelian techniques are those used to conclude that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ from the consideration of abelian subracks and via the classification of braided vector spaces of diagonal type with finite-dimensional Nichols algebra [H].

Lemma 4.1. *Assume that* $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$.

(a) [AZ] *If g is real, then $\rho(g) = -1$. In particular,* $\operatorname{ord} g$ *is even.*

(b) [AF, FGV1] *If g is quasi-real, with $j \in \mathbb{Z}$ such that $g \neq g^j \in \mathcal{O}$, then:*

  (i) *If $\deg \rho > 1$, then $\rho(g) = -1$ and g has even order.*

  (ii) *If $\deg \rho = 1$, then $\rho(g) = -1$ and g has even order or $\rho(g) \in \mathbb{G}'_3$.*

  (iii) *If $g^{j^2} \neq g$, then $\rho(g) = -1$.* □

4.2. Semisimple classes in $\mathbf{PSL}_2(q)$

We recall some basic facts.

• There exist two (conjugacy classes of) maximal tori in $\mathbf{SL}_2(q)$: the split torus $T_1 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), a \in \mathbb{F}_q^\times\}$ of order $q-1$ and the non-split torus T_2 of order $q+1$. Every non-central $x \in \mathbf{SL}_2(q)$ semisimple is conjugated to an element of either T_1 or T_2. Two elements $x, y \in T_i$ are conjugated if and only if $x = y^{\pm 1}$. Both T_1 and T_2 are cyclic. So this is the situation for $\mathbf{PSL}_2(q) = \mathbf{SL}_2(q)$ when q is even. For uniformity of the notation, we set $\mathbf{T}_i := T_i$, $i \in \mathbb{I}_2$, when q is even.

• Suppose q is odd. There exist two conjugacy classes of maximal tori in $\boldsymbol{G}$, namely the images $\mathbf{T}_1$ of the split torus T_1, of order $\frac{q-1}{2}$; and $\mathbf{T}_2$ of the non-split torus T_2, of order $\frac{q+1}{2}$. Every $x \in \boldsymbol{G}$ semisimple is conjugated to an element of either $\mathbf{T}_1$ or $\mathbf{T}_2$. Two elements $x, y \in \mathbf{T}_i$ are conjugated if and only if $x = y^{\pm 1}$. As a consequence, there is exactly one conjugacy class of involutions, at most one semisimple conjugacy class of elements of order 3, at most one conjugacy class of elements of order 4, etc.

Proposition 4.2. *Let $\mathcal{O}$ be a semisimple conjugacy class in $\mathbf{PSL}_2(q)$. If $\mathcal{O}$ is not listed in Table 1, then it collapses.*

Proof. If either x is conjugated to an element in $\mathbf{T}_1$ or else x is an involution but $q \neq 7$, then $\mathcal{O}_x^{\boldsymbol{G}}$ collapses by Lemma 3.9 and Corollary 3.5 (there are no semisimple involutions when q is even). Assume in the rest of the proof that $x \in \boldsymbol{G}$ is conjugated to an element in $\mathbf{T}_2$ and $\operatorname{ord} x > 2$.

Suppose first that $\operatorname{ord} x = 3$. Then, necessarily $q \equiv 2 \mod 3$. If moreover q is odd or a square, then $\boldsymbol{G}$ contains a subgroup isomorphic to $\mathbb{A}_4$, by Theorem 3.4, case (c). Since $\boldsymbol{G}$ contains only one conjugacy class of elements of order 3, we have $Y := \mathcal{O}_x^{\boldsymbol{G}} \cap \mathbb{A}_4 = \mathcal{O}_{(123)}^{\mathbb{A}_4} \coprod \mathcal{O}_{(132)}^{\mathbb{A}_4}$. Thus Y is the cube rack, which is of type C by Lema 2.19, whence it collapses.

Now let X be a subrack of $\mathcal{O}_x^{\boldsymbol{G}}$ and $K = \langle X \rangle$; X is a union of K-orbits. We divide the proof with respect to the classification given in Theorem 3.4.

If K is as in case (a), then $K \leq \mathbb{D}_{\frac{q+1}{d}}$ and X is abelian. Clearly, K could not be as in case (b) because $\operatorname{ord} x \nmid$ the order of such group.

Suppose that $K = \mathbf{PSL}_2(t)$ for some t such that $q = t^h$, $h \in \mathbb{N}$, case (f). Assume first that $t \neq 2, 3$. Let $y, z \in X$. Then $\mathcal{O}_y^K$ does not intersect any split torus of $\mathbf{PSL}_2(t)$, since otherwise it would intersect $\mathbf{T}_1$. Let $\mathfrak{T}$ be a non-split torus of $\mathbf{PSL}_2(t)$; then $\mathcal{O}_y^K$ intersects $\mathfrak{T}$. Also we may assume that either $\mathfrak{T} \subset \mathbf{T}_2$ or $\mathfrak{T} \subset \mathbf{T}_1$ (only when h is even). In the first possibility,

$$\emptyset \neq \mathcal{O}_y^K \cap \mathfrak{T} \overset{\diamondsuit}{=} \mathcal{O}_y^{\boldsymbol{G}} \cap \mathbf{T}_2 = \mathcal{O}_z^{\boldsymbol{G}} \cap \mathbf{T}_2 \overset{\diamondsuit}{=} \mathcal{O}_z^K \cap \mathfrak{T}.$$

Here in $\diamondsuit$ we use that a conjugacy class intersects a torus in $\{x^{\pm 1}\}$. Hence $\mathcal{O}_y^K = \mathcal{O}_z^K = X$ is indecomposable by [AG, Lemmata 1.9 & 1.15]. The second possibility is analogous. Now assume that $K = \mathbf{PSL}_2(2) = \mathbb{S}_3$. Then $\mathcal{O} \cap K$ consists of 3-cycles, hence it is abelian. Finally, $K = \mathbf{PSL}_2(3) = \mathbb{A}_4$ is excluded because $\mathcal{O} \cap K$ consists of involutions.

Assume that $K = \mathbf{PGL}_2(t)$ for some t such that $q = t^h$, case (g); we may suppose that q odd, and then $h = 2k \in \mathbb{N}$ should be even. If $x \in K$ is semisimple, then $\operatorname{ord} x$ divides $|\mathbf{PGL}_2(t)| = t(t^2-1)$, hence $\operatorname{ord} x$ does not divide $q+1 = t^{2k}+1$ and x is split in $\mathbf{PSL}_2(q)$. In other words, $X \cap K = \emptyset$.

Hence, the only possible cases where X might not be sober are when $K \simeq \mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{S}_5$, that is cases (c), (d) and (e), respectively. From the previous considerations, the next statement follows at once.

Case 1. If $\operatorname{ord} x > 5$, then $\mathcal{O}_x^{\boldsymbol{G}}$ is sober.

We next analyze the low order cases.

Case 2. If $\operatorname{ord} x = 3$ and q is even and not a square, X is sober.

If q is even and not a square, then cases (c) and (d) are not possible since $\boldsymbol{G}$ does not contain a subgroup isomorphic to $\mathbb{A}_4$. If $K \simeq \mathbb{A}_5$, case (e), then $X = \mathcal{O}_3^{\mathbb{A}_5}$ is indecomposable.

Case 3. If $\operatorname{ord} x = 4$, then $\mathcal{O}_x^{\boldsymbol{G}}$ is sober.

Cases (c) and (e) are clearly excluded. If $K \simeq \mathbb{S}_4$, case (d), then $X = \mathcal{O}_4^4$ is indecomposable by [AG, Lemmata 1.9,1.15] (as $\mathcal{O}_4^4$ generates $\mathbb{S}_4$ which is centerless).

Case 4. The conjugacy classes of elements of order 5 in $\boldsymbol{G}$ are sober.

Cases (c) and (d) are clearly excluded. Assume that $K \simeq \mathbb{A}_5$, case (e). There are two conjugacy classes $\mathcal{O}_1$ and $\mathcal{O}_2$ of elements of order 5 in $\mathbb{A}_5$ and each of them is real, i. e. stable under inversion. Suppose that $\mathcal{O}_1 \subset X$ and pick $x \in \mathcal{O}_1$. Then $\mathcal{O}_x^{\boldsymbol{G}} = \mathcal{O}_{x^4}^{\boldsymbol{G}}$ and $\mathcal{O}_{x^2}^{\boldsymbol{G}} = \mathcal{O}_{x^3}^{\boldsymbol{G}}$ are the two different conjugacy classes of elements of order 5 in $\boldsymbol{G}$. If $X = \mathcal{O}_1 \coprod \mathcal{O}_2$, then x^2 would belong to $\mathcal{O}_x^{\boldsymbol{G}}$, a contradiction. Hence $X = \mathcal{O}_1$ is indecomposable. □

4.3. Finite-dimensional pointed Hopf algebras over $\mathbf{PSL}_2(q)$

Nichols algebras associated to conjugacy classes in $\mathbf{PSL}_2(q)$ were previously studied in [FGV2]. Using abelian techniques, it was proved that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any $(\rho, V) \in \operatorname{Irr} C_{\mathbf{PSL}_2(q)}(x)$ with $x \in \mathcal{O}$ and q even, [FGV1, Proposition 3.1]. For q odd, a list of the open cases was given [FGV2, Theorem 1.6]. Here we discard the case when $q \equiv 1 \mod 4$ by the criterium of type C. First, the conjugacy class of involutions in $\mathbf{PSL}_2(7)$ is kthulhu, Proposition 4.2. The associated Nichols algebras over $\mathbf{PSL}_2(7)$ are dealt with the next lemma.

Lemma 4.3. [FGV2, Prop. 4.3] *Let $x \in \mathbf{PSL}_2(7)$ be an involution. Then $\dim \mathfrak{B}(\mathcal{O}_x, \rho) = \infty$, for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.* □

Theorem 4.4. *Let $\mathcal{O}$ be a semisimple conjugacy class in $\mathbf{PSL}_2(q)$. If $\mathcal{O}$ is not a semisimple irreducible conjugacy class represented in $\mathbf{SL}_2(q)$ by $\mathtt{x} = \left(\begin{smallmatrix} a & \zeta b \\ b & a \end{smallmatrix}\right)$ with $ab \neq 0$, $\zeta \in \mathbb{F}_q^\times - \mathbb{F}_q^2$ and $q \equiv 3 \mod 4$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.*

Proof. If q is even, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, for every $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$ by [FGV1, Proposition 3.1], since in this case $\mathbf{PSL}_2(q) = \mathbf{SL}_2(q)$. Assume q is odd. If $q \equiv 1 \mod 4$, the open cases in [FGV2, Theorem 1.6] were given by split semisimple classes which collapse by Proposition 4.2. □

Remark 4.5. For $q > 3$ and $q \equiv 3 \mod 4$ there are $\frac{q-3}{4}$ semisimple irreducible conjugacy classes in $\mathbf{PSL}_2(q)$ of size $q(q-1)$, [FGV2, Table 3].

5. Finite-dimensional pointed Hopf algebras over $\mathbf{PSL}_n(q)$

In this last section we study Hopf algebras over $\mathbf{PSL}_n(q)$, using the previous results and abelian techniques. We begin by studying irreducible semisimple conjugacy classes in $\mathbf{PSL}_3(q)$.

5.1. Semisimple classes in $\mathbf{PSL}_3(q)$

In this subsection $\boldsymbol{G} = \mathbf{PSL}_3(q)$. Let x be an irreducible semisimple element in $\boldsymbol{G}$. We show in Propositions 5.4 and 5.5 that $\mathcal{O}_x^{\boldsymbol{G}}$ is austere, hence kthulhu.

Let $\mathtt{x} \in \mathbf{SL}_3(q)$ such that $x = \pi(\mathtt{x})$. If x is semisimple and irreducible, then for every eigenvalue η of $\mathtt{x}$ we have $\eta^{q^2} \neq \eta \neq \eta^q$, $\eta^{(3)_q} = 1$, and $\mathtt{x}^{(3)_q} = 1$, i.e., $\mathtt{x}$ lies in a maximal torus of order $(3)_q$. Since $Z(\mathbf{SL}_3(q))$ lies in all such tori, x lies in the maximal torus of $\boldsymbol{G}$ of order $\frac{(3)_q}{(3,q-1)}$.

If x is semisimple and not irreducible, then $\mathtt{x}$ lies in a maximal torus whose exponent is either q^2-1 or $q-1$, whence x lies in a maximal torus of $\boldsymbol{G}$ of order $\frac{q^2-1}{(3,q-1)}$ or $\frac{q-1}{(3,q-1)}$.

We need first the following technical lemma.

Lemma 5.1. *Let $\ell > 1$ be odd. If $q \equiv 1 \mod \ell$, then $\left(\ell^2, (\ell)_q\right) = \ell$.*

Proof. Let $q = 1 + a\ell$. We show that $(\ell)_q \equiv \ell \mod \ell^2$.

$$\begin{aligned}(\ell)_q &= \textstyle\sum_{j=0}^{\ell-1}(1+a\ell)^j = \sum_{j=0}^{\ell-1}\sum_{i=0}^{j}\binom{j}{i}a^i\ell^i = \sum_{i=0}^{\ell-1}a^i\ell^i\sum_{j=i}^{\ell-1}\binom{j}{i} \\ &\equiv \textstyle\sum_{j=0}^{\ell-1}\binom{j}{0} + a\ell\sum_{j=1}^{\ell-1}\binom{j}{1} \mod \ell^2 \\ &\equiv \ell + a\ell\binom{\ell}{2} \mod \ell^2 \equiv \ell \mod \ell^2.\end{aligned}$$

Hence, $((\ell)_q, \ell^2) = (\ell, \ell^2) = \ell$. □

Remark 5.2. Let $x = \pi(\mathtt{x}) \in \boldsymbol{G}$, for some semisimple irreducible $\mathtt{x} \in \mathbf{SL}_3(q)$.

(a) We claim that $(\operatorname{ord} x, 6) = 1$ and $\operatorname{ord} x \neq 5$. Indeed, $\operatorname{ord} \mathtt{x}$ divides $(3)_q$ which is always odd. In addition, $(3)_q$ is divisible by 3 only if $q \equiv 1 \mod 3$ and by Lemma 5.1, it is never divisible by 9. Looking at all the possible values of q modulo 5 it is easily verified that $5 \not| (3)_q$.

(b) Let $H \leq K \leq \boldsymbol{G}$, with $[K : H] \leq 3$. Then, if $x \in K$, we have $x \in H$. Indeed left multiplication by x induces a permutation of the coclasses of H in K, which has order ≤ 3. By (a), $x = e$, i. e., $xH = H$.

(c) If x^k is not irreducible, then $x^k = e$. In fact x^k is semisimple and therefore it lies in a maximal torus of $\boldsymbol{G}$. The statement follows because $\big((3)_q, (q^2-1)\big) = (q-1, 3)$ and $\left(\frac{(3)_q}{(3,q-1)}, \frac{(q^2-1)}{(3,q-1)}\right) = 1$.

Lemma 5.3. *Let $x = \pi(\mathtt{x}) \in \boldsymbol{G}$ be semisimple and irreducible. Assume $q = t^{3l}$. Then $\mathcal{O}_x^{\boldsymbol{G}} \cap \mathbf{PSL}_3(t) = \emptyset$.*

Proof. Let $\mathtt{y} \in \mathbf{SL}_3(t)$. If its characteristic polynomial is not irreducible over $\mathbb{F}_t$, then $\pi(\mathtt{y}) \notin \mathcal{O}_x^{\boldsymbol{G}}$. If it is irreducible, then its roots are in $\mathbb{F}_{t^3} \subset \mathbb{F}_q$, so $\pi(\mathtt{y}) \notin \mathcal{O}_x^{\boldsymbol{G}}$. □

We prove now that $\mathcal{O}_x^{\boldsymbol{G}}$ is austere when q is odd.

Proposition 5.4. *Assume that q is odd. Let $K \leq \boldsymbol{G}$ and x be an irreducible semisimple element in $\boldsymbol{G}$. Then one of the following holds:*

(i) *$K \cap \mathcal{O}_x^{\boldsymbol{G}} = \emptyset$;*

(ii) *$K \cap \mathcal{O}_x^{\boldsymbol{G}}$ is abelian;*

(iii) *there is $y \in K \cap \mathcal{O}_x^{\boldsymbol{G}}$ such that $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \mathcal{O}_y^K$.*

In particular $\mathcal{O}_x^{\boldsymbol{G}}$ is austere, hence kthulhu.

Proof. We proceed by inspection of the different subgroups of $\boldsymbol{G}$ as listed in Theorem 3.6.

If K is as in case (a), then $K = \mathbf{PSL}_3(t)$ for some $t = p^a$. In this case, we have that if $y \in K \cap \mathcal{O}_x^{\boldsymbol{G}}$, then $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \mathcal{O}_y^K$. Indeed, if $(q-1, 3) = (t-1, 3)$, we apply Lemma 3.19. If $(q-1, 3) \neq (t-1, 3)$, then $(q-1, 3) = 3$ and $(t-1, 3) = 1$, so $(3, (3)_t) = 1$ and Lemma 3.20 applies.

If K is as in case (b), then $K = \mathbf{PSU}_3(t)$ for some $t = p^a$. Again, we have that if $y \in K \cap \mathcal{O}_x^{\boldsymbol{G}}$, then $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \mathcal{O}_y^K$. For, if $(q-1, 3) = (t+1, 3)$, then we apply Lemma 3.21. If on the other hand, $(q-1, 3) \neq (t+1, 3)$, then $(q-1, 3) = 3$ and $(t+1, 3) = 1$, $p \neq 3$ and $t \equiv 1 \mod 3$. Without loss of generality we assume $x = \pi(\mathtt{x})$ for some $\mathtt{x} \in \mathbf{SU}_3(t)$. Let $y = \pi(\mathtt{y}) \in \mathcal{O}_x^{\boldsymbol{G}} \cap K$, with $\mathtt{y}, \in \mathbf{SU}_3(t)$. Then, for some $\zeta \in \mathbb{F}_q$ with $\zeta^3 = 1$ there holds $\mathtt{y} \in \mathcal{O}_{\zeta\mathtt{x}}^{\mathbf{SL}_3(q)} \cap \mathbf{SU}_3(t)$. There are three conjugacy classes of maximal tori in $\mathbf{SU}_3(t)$, with exponent $t+1$, t^2-1, which both divide $q-1$, and $t^2 - t + 1$. Since y and x are irreducible, both $\operatorname{ord}(\zeta\mathtt{x}) = \operatorname{ord}(\mathtt{y})$ and $\operatorname{ord}(\mathtt{x})$ divide $t^2 - t + 1$. But then, $\zeta = \zeta^{t^2-t+1} = 1$. Hence, $\mathtt{y} \in \mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_3(q)} \cap \mathbf{SU}_3(t) = \mathcal{O}_{\mathtt{x}}^{\mathbf{SU}_3(t)}$ where equality follows because the centralizer of a semisimple element in $\mathbf{SL}_3(\Bbbk)$ is connected. Thus, $y \in \mathcal{O}_x^K$.

If K is as in case (c), then there is a subgroup containing $\mathbf{PSL}_3(t)$ with index 3 and $t \equiv 1 \mod 3$. In this case, $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \emptyset$. Indeed, if $y \in \mathcal{O}_x^{\boldsymbol{G}}$, then by Lemma 5.3, $y \notin \mathbf{PSL}_3(t)$, hence $y \notin K$ by Remark 5.2 (b).

If K is as in case (d), then there is a subgroup containing $\mathbf{PSU}_3(t)$ with index 3 and $t \equiv 2 \mod 3$. As before, it follows that $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \emptyset$. If $y \in \mathcal{O}_x^{\boldsymbol{G}}$, then by Lemma 5.3, $y \notin \mathbf{PSU}_3(t) \subset \mathbf{PSL}_3(t^2)$. Thus, by Remark 5.2 (b), $y \notin K$.

If K is as in case (e), then $K = \mathbf{PSL}_2(t)$ or $\mathbf{PGL}_2(t)$ for some $t = p^a \neq 3$. In this case, $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \emptyset$. Indeed, the order of K divides $t(t^2-1)$ which in turn divides $t(q^2-1)$. If $x \in K$ were irreducible, then $\operatorname{ord}(x)$ would be a divisor of $(t(q^2-1), (3)_q) = (3, q-1)$. Hence, Remark 5.2 (a) applies.

If K is as in cases (f), (g), (h), (i), then the order of any element in K lies in $\{2,3,4,5,7\}$. By Remark 5.2 (a), if $y \in K \cap \mathcal{O}_x^{\boldsymbol{G}}$, then $\operatorname{ord} y = 7$ and K is either $\mathbb{A}_7$ or $\mathbf{PSL}_2(7)$. In addition, 7 divides $(3)_q$ only if $q \equiv 2 \mod 7$ or $q \equiv 4 \mod 7$. For both choices of K there are exactly two classes of elements of order 7, one containing $\{y, y^2, y^4\} = \{y, y^q, y^{q^2}\}$, and the other containing y^3, y^5, y^6. We show that $\mathcal{O}_y^K = \mathcal{O}_y^{\boldsymbol{G}} \cap K$. Assume this is not the case. Then, $y^3 \in \mathcal{O}_y^{\boldsymbol{G}}$, so $x^3 \in \mathcal{O}_x^{\boldsymbol{G}}$. If $x = \pi(\mathtt{x})$, then the only powers of $\mathtt{x}$ lying in $\mathcal{O}_{\mathtt{x}}^{\mathbf{SL}_3(q)}$ are $\mathtt{x}, \mathtt{x}^q$ and $\mathtt{x}^{q^2}$. By looking at the order of $\mathtt{x}$ (which can be 7 or 21), and of $\mathtt{x}^3$ (which is always 7) we necessarily have $\mathtt{x}^3 \in \mathcal{O}_{z\mathtt{x}}^{\mathbf{SL}_3(q)}$ for some $1 \neq z \in Z(\mathbf{SL}_3(q))$, a third root of 1, with $\mathtt{x}^7 = z^{-1}$, and $q \equiv 1 \mod 3$. This is impossible. Indeed, let ζ, ζ^q and ζ^{q^2} be the eigenvalues of $\mathtt{x}$. They are primitive 21-th roots of 1. Then, the eigenvalues of $\mathtt{x}^3$ are ζ^3, ζ^{3q} and ζ^{3q^2}, whereas the eigenvalues of $z\mathtt{x}$ are $z\zeta, z\zeta^q$ and $z\zeta^{q^2}$. A direct verification using that $q \equiv 2 \mod 7$ or $q \equiv 4 \mod 7$ shows that ζ^3 cannot be equal to any of $z\zeta, z\zeta^q$ and $z\zeta^{q^2}$.

If K is as in case (j), then K contains a cyclic p-regular normal subgroup $H = \langle h \rangle$ of index ≤ 3, which is contained in a maximal torus S. Let $y \in K$. If y is irreducible in $\boldsymbol{G}$, then by Remark 5.2 (a), (b), $y \in H$ and H is contained in the torus S of order $\frac{(3)_q}{(3,(3)_q)}$. Therefore, $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \mathcal{O}_x^{\boldsymbol{G}} \cap H$ is abelian.

If K is as in case (k), then K contains a diagonal normal subgroup L such that K/L is isomorphic to a subgroup of $\mathbb{S}_3$. In this case, we also have that $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \emptyset$. Assume on the contrary that $x \in K$. The order of its coclass xL in K/L is $1, 2$, or 3. It cannot be 1 because x is irreducible hence not diagonal, and it cannot be 2 nor 3 by Remark 5.2 (a).

If K is as in case (l), then K contains a normal abelian p-subgroup H such that K/H is isomorphic to a subgroup of $\mathbf{GL}_2(q)$. In particular, $|K|$ divides $p^N|\mathbf{GL}_2(q)| = p^N q(q-1)(q^2-1)$, for some $N > 0$. But since we have that $\left(p^N(q-1)(q^2-1), \frac{(3)_q}{(3,q-1)}\right) = 1$, it follows that $K \cap \mathcal{O}_x^{\boldsymbol{G}} = \emptyset$.

If K is as in case (m), then K contains a normal abelian subgroup H of type $(3,3)$ such that K/H is isomorphic to a subgroup of $\mathbf{SL}_2(3)$. We show that $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \emptyset$. Assume that $y \in K$ is semisimple and irreducible. We look at the coclass yH in K/H. Then $\operatorname{ord} yH$ is either $1, 2, 3$, or 4, whence by Remark 5.2 (a) we have $y \in H$, a 3-group, which is impossible.

Finally, if K is as in case (n), then K contains a normal abelian subgroup H of type $(3,3)$ such that K/H is isomorphic to a subgroup of the quaternion group Q of order 8. Hence, $\mathcal{O}_x^{\boldsymbol{G}} \cap K = \emptyset$ by Remark 5.2 (a). □

We prove now that $\mathcal{O}_x^{\boldsymbol{G}}$ is austere when q is even.

Proposition 5.5. *Assume that $q = 2^m$. Let x be an irreducible semisimple element in $\boldsymbol{G}$. Then, for $y \in \mathcal{O}_x^{\boldsymbol{G}}$ we have either $xy = yx$ or $\mathcal{O}_x^{\langle x,y\rangle} = \mathcal{O}_y^{\langle x,y\rangle}$. In particular $\mathcal{O}_x^{\boldsymbol{G}}$ is austere, hence kthulhu.*

Proof. Let $x, y \in \mathcal{O}_x^{\boldsymbol{G}}$, with $xy \neq yx$. If $K := \langle x, y\rangle \neq \boldsymbol{G}$, then K lies in a proper maximal subgroup M_1 of $\boldsymbol{G}$. We analyse the different possibilities for M_1 listed in Theorem 3.7, from which we adopt notation.

Since x is a semisimple irreducible element, $\operatorname{ord} x$ is coprime with the order of the groups in (a), thus this case is not possible. Further, $\operatorname{ord} x$ is different from the order of any element in (g) by Remark 5.2 (a). If M_1 were as in (b), then by Remark 5.2 (b), x and y would lie in the same maximal torus, and they would commute. Hence, the possible cases are (c), (d), (e) and (f), that is M_1 is isomorphic to $\mathbf{PSL}_3(t)$ or $\mathbf{PSU}_3(t)$ for some t, or contains one of them with index 3. We analyze these cases further.

Claim 1. M_1 is either $\mathbf{PSL}_3(t)$, where q is a prime power of t and $q \neq t^3$, or $\mathbf{PSU}_3(t)$. The latter may occur only if m is even.

If $x, y \in \mathbf{PGL}_3(t)$ with $q = t^3$, then by Remark 5.2 (b), $x, y \in \mathbf{PSL}_3(t)$, which is impossible by Lemma 5.3, so case (d) is excluded. Similarly, if $x, y \in \mathbf{PGU}_3(t)$, with $q = t^6$, then $x, y \in \mathbf{PSU}_3(t) \leq \mathbf{PSL}_3(t^2)$, impossible by Lemma 5.3. Thus, case (f) is also excluded.

Now we proceed inductively looking at the maximal subgroups of M_1 as above.

Claim 2. K is either $\mathbf{PSL}_3(t')$, for some $t' = 2^b$ and $b|m$, $3b \not| m$, or $\mathbf{PSU}_3(t')$, for $t' = 2^c$ and $c|2m$, $3c \not| 2m$.

If $K = M_1$ the claim is trivial. Otherwise, $K \leq M_2$ where M_2 is a maximal subgroup of M_1. If $M_1 = \mathbf{PSL}_3(t)$ we argue as in Claim 1. If $M_1 = \mathbf{PSU}_3(t)$, then we claim that $M_2 = \mathbf{PSU}_3(t')$ for t an odd prime power of t'. We analyse the different possibilities for M_2 listed in Theorem 3.8, from which we adopt notation. The groups as in (a) or (e) are discarded because their order is coprime with $\operatorname{ord} x$. The groups as in (b) may not occur by Remark 5.2 (b) and the noncommutativity of x and y. If x would lie in a subgroup as in (d), by Lemma 5.3 it would lie in $\mathbf{PSU}_3(2^l) \leq \mathbf{PSL}_3(2^{2l})$ with $t = 2^a$, $6l = 2a|m$. By Lemma 5.3 we have a contradiction. Thus, the only possible case is (e) and we have the claim.

Claim 3. $\mathcal{O}_x^K = \mathcal{O}_x^{\boldsymbol{G}} \cap K = \mathcal{O}_y^K$.

Let us observe that x, y are again irreducible in K. If $K = \mathbf{PSL}_3(t)$ for some t the argument in the proof of Proposition 5.4 case (a) gives the claim. If $K = \mathbf{PSU}_3(t)$, we argue as in the proof of Proposition 5.4 case (b). $\square$

5.2. Finite-dimensional pointed Hopf algebras over $\mathbf{PSL}_n(q)$

In this last subsection we assume that $\boldsymbol{G} = \mathbf{PSL}_n(q)$, $n > 2$. We show that for infinitely many pairs (n, q), the group $\mathbf{PSL}_n(q)$ collapses.

Let $n = 2^a b$, $q = 1 + 2^c d$, where $a, c \in \mathbb{N}_0$ $b, d \in \mathbb{N}$ with $(b, 2) = (d, 2) = 1$ and let $\mathcal{G}_{ss}$ be the set of pairs (n, q) with $n \in \mathbb{N}$, $n > 2$ and $q = p^m$, p a prime, such that one of the following hold:

(a) n is odd;

(b) q is even;

(c) $0 < a < c$;

(d) $a = c > 1$.

A direct computation similar to the one in the proof of Lemma 5.1 shows that if $(n, q) \in \mathcal{G}_{ss}$, then $\frac{(n)_q}{(q-1,n)}$ is odd.

Lemma 5.6. *Let $(n, q) \in \mathcal{G}_{ss}$ and let $x \in \mathbf{PSL}_n(q)$ be an irreducible semisimple element. Then,* $\dim \mathfrak{B}(\mathcal{O}_x^{\boldsymbol{G}}, \rho) = \infty$ *for every* $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.

Proof. By Lemma 3.2, $x \neq x^q$, $x \neq x^{q^2}$ and $x^q, x^{q^2} \in \mathcal{O}_x^{\boldsymbol{G}}$. In addition the order of the maximal torus containing x is $\frac{(n)_q}{(q-1,n)}$ which is odd under our assumptions. Then, Lemma 4.1 (b) (iii) applies. □

Proposition 5.7. *Let $\mathcal{O}$ be a conjugacy class in $\boldsymbol{G} = \mathbf{PSL}_3(2)$. If $\mathcal{O}$ is not unipotent of type* (3)*, then* $\dim \mathfrak{B}(\mathcal{O}_x^{\boldsymbol{G}}, \rho) = \infty$ *for every* $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$.

Proof. This follows from Propositions 2.13, 3.18 and Lemma 5.6. □

Remark 5.8. Assume $n = 2^a b$ and $q = 1+2^c d$ with $(b, 2) = (d, 2) = 1$ and $a > c > 0$ or else $a = c = 1$. Let x be an irreducible semisimple element in $\boldsymbol{G}$. If $\operatorname{ord} x$ is odd, then $\dim \mathfrak{B}(\mathcal{O}_x, \rho) = \infty$ for all $\rho \in \operatorname{Irr} C_{\boldsymbol{G}}(x)$ by Lemmata 3.2 and 4.1 (b) (iii). Hence, the only potentially non-collapsing classes in $\boldsymbol{G}$ are the classes of semisimple irreducible elements of even order. In this case such elements always exist: for instance, any generator of a maximal torus S of order $\frac{(n)_q}{(q-1,n)}$, which is even under our assumptions. Any semisimple irreducible element is conjugate to an element in S.

Let $x = \pi(\mathtt{x})$ be an irreducible semisimple element in S. Its centralizer is

$$C_{\boldsymbol{G}}(x) = \{\pi(\mathtt{y}) \in \boldsymbol{G} \mid \mathtt{y}\mathtt{x}\mathtt{y}^{-1} = \lambda\mathtt{x},\ \lambda \in \mathbb{F}_q\}.$$

If for some $\mathtt{y} \in G$ there holds $\mathtt{y}\mathtt{x}\mathtt{y}^{-1} = \lambda\mathtt{x}$, then $\lambda\mathtt{x} \in \mathcal{O} \cap C_G(\mathtt{x}) = \{\mathtt{x}^{q^i},\ i \in \mathbb{I}_{0,n-1}\}$. By Lemma 3.2 this implies $\lambda = 1$ and consequently $\mathtt{y} \in C_G(\mathtt{x})$. Thus, $C_{\boldsymbol{G}}(x) = \pi(C_G(\mathtt{x})) = S$ is cyclic, so all its irreducible representations are 1-dimensional. By Lemma 4.1 (b) (iii), if $\dim \mathfrak{B}(\mathcal{O}_x, \rho) < \infty$, then $\rho(x) = -1$. In this case the study of abelian subracks of $\mathcal{O}$ is not effective for determining the dimension of the associated Nichols algebras.

Acknowledgements

We thank Robert Guralnick for conversations on subgroups generated by two elements in finite simple groups and Jay Taylor for interesting discussions on semisimple classes.

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This work was partially supported by ANPCyT-Foncyt, CONICET, Secyt (UNC), the bilateral agreement between the Universities of Córdoba and Padova, the Visiting Scientist Program of the University of Padova, the Grant CPDA 1258 18/12 of the University of Padova and an AMIDILA-ERASMUS Academic Staff Fellowship.