Exponential Stability of Subspaces for Quantum Stochastic Master Equations

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Abstract

We study the stabilisation of quantum system on a subspace through reservoir engineering provided the system is continuously monitored. We show that the target subspace is almost surely invariant if and only if it is invariant for the average evolution. We show the same equivalence for the global asymptotic stabilisation towards the target subspace. We moreover prove a converse Lyapunov theorem for the average evolution. From this theorem we derive sharp bounds on the Lyapunov exponents. We show that taking into account the measurements can lead to a stability rate improvement. We make discuss explicit situations where the almost sure stability rate can be made arbitrary large while the average one stays constant.

1 Introduction

Pure quantum states play a key role in many aspects of quantum theory, and quantum dynamics in particular: eigenstates of Hamiltonians with non-degenerate spectrum are invariant, and ground states of Hamiltonians represent the zero-temperature equilibrium for the system; they are the output of measurement processes corresponding to non-degenerate observables; pure states are typically used to represent information in quantum information processing

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and communication; nonclassical correlations in quantum mechanics are best exhibited by maximally entangled states for joint systems, which are pure. This central role motivates a growing interest in characterizing evolutions that converge to classes of pure states of interest.

A similar interest lays on convergence to subspaces of Hilbert space, whether they represent energy eigenspaces, they are associated to certain excitation numbers or symmetric states, or represent the support for a quantum error-correcting code.

In order for a quantum dynamical system to converge to a pure state or a subspace irrespective of the initial state, it needs to include some interaction with its environment, namely it needs to be an *open system.* We shall focus on Markov quantum systems associated to Stochastic Master Equations (SME) and their corresponding semigroups [2, 11, 29]. This class of models emerges naturally in many quantum atomic, optical and nanomechanical systems [37, 38, 47]. It is of interest in measurement and decoherence theory [1, 15, 16, 20, 39, 41, 42, 52], and it has a central role in quantum filtering and measurement-based feedback control systems [5–8, 21, 22, 40, 44, 45, 53].

In many applications, convergence is not enough: a *fast* preparation of the target set needs to be enacted. Different ways to characterize the speed of convergence, as well as asymptotic invariant sets, have been developed for Markovian evolutions [26, 48].

In [4], a general approach to stabilisation of *diffusive* SME has been proposed, which relies as much as possible on open-loop control and resorts to feedback design only when the open-loop control cannot achieve the desired task. The motivation for this choice is twofold: on the one hand, open-loop control is easier to implement, as it does not require the taxing computational overhead of integrating the SME in real time. On the other hand, simulations showed that the open-loop controlled evolution converged exponentially. This is not completely surprising, as it is in agreement with another result of the paper: convergence *in probability* to subspaces for the SME can be proved by checking if the *mean* evolution converges to the same subspace.

In this paper we make those preliminary observations rigorous, and further develop them investigating the *speed of convergence* to the target. More precisely, we study the exponential stability of stochastic evolutions by deriving upper bounds for the Lyapunov exponent. The main results we present include:

- A proof for the equivalence of both invariance and asymptotic stability in mean and *almost surely* for general SMEs, i.e. including both diffusive and jump processes. This generalises the corresponding results in [4] to SMEs that include jump processes, while providing a stronger convergence with proofs that are both more direct and simpler. This result is of interest for applications since invariance in mean corresponds to invariance of the subspace for a corresponding semigroup evolutions, and the latter can be checked, at least in the finite-dimensional case, by using simple linear algebraic techniques [48–50].

- As a technical result, we show that the Perron-Frobenius Theorem for completely positive evolutions [28] can be used to systematically derive a *linear* Lyapunov function that shows that a subspace is GAS. The result can be seen as a *converse* Lyapunov Theorem, which is of practical interest in many situations in which one would like to prove that a given controlled dynamics converges to a target pure state, as well as to develop insights in design methods for dissipative quantum control [43, 48, 51].

- A proof of almost sure exponential stability of GAS subspaces, including a bound on the stability exponent is provided, by exploiting the converse Lyapunov result. Not only: it is shown that a better stability exponent can be obtained, by exploiting the effect of the measurement.

2 Quantum Stochastic Master Equations

This section is devoted to the presentation of the stochastic models that are used to describe finite-dimensional quantum systems undergoing indirect continuous measurement. The mathematical description of a finite-dimensional system is built on a finite-dimensional Hilbert space $\mathcal{H}$, which we assume of dimension d. Let $\mathcal{B}(\mathcal{H})$ denote the linear (bounded) operators on $\mathcal{H}$. Such operators can always be associated (up to a choice of basis) to $d \times d$ matrices. The system state is described by a *density operator* ρ, namely an element of $\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \text{ s.t. } \rho \geq 0, \mathrm{tr}(\rho) = 1\}$. The evolution model will be given in the so-called Schroedinger's picture, where states are the subjects of the evolution.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions [?]. Let $(W_j(t)), j = 0, \ldots, p$ be standard independent Wiener processes and let $(N_j(dx, dt)), j = p+1, \ldots, n$ be independent adapted Poisson point processes of intensity $dxdt$; the N_j's are independent of the Wiener processes. We assume that $(\mathcal{F}_t)$ is the natural filtration of the processes W, N and we assume also that $\mathcal{F}_\infty = \bigvee_{t>0} \mathcal{F}_t = \mathcal{F}$.

We consider a family $C_i, i = 0, \ldots, n$ of operators in $\mathcal{B}(\mathcal{H})$ and let $H \in \mathcal{B}(\mathcal{H})$ such that $H = H^*$. In order to keep the notation compact, it is convenient to introduce the following maps on states $\rho \in \mathcal{S}(\mathcal{H})$:

$$\begin{aligned}
\mathcal{L}(\rho) &= -i[H, \rho] + \sum_{i=0}^{n} \left(C_i \rho C_i^* - \frac{1}{2}\big(C_i^* C_i \rho + \rho C_i^* C_i\big) \right), \\
\mathcal{J}_i(\rho) &= C_i \rho C_i^*, \ i = 0, \ldots, n, \\
v_i(\rho) &= \mathrm{Tr}[J_i(\rho)], \ i = 0, \ldots, n, \\
\mathcal{G}_i(\rho) &= C_i \rho + \rho C_i^* - \mathrm{Tr}[(C_i + C_i^*)\rho]\rho, \ i = 0, \ldots, n.
\end{aligned} \tag{1}$$

Physically, H corresponds to the effective Hamiltonian for the system which includes its internal Hamiltonian and a perturbation (Lamb shift) induced by the interaction with its environment. The environment is typically associated to a number of quantum fields, and the interaction of the system with the latter is described by the operators C_i. The canonical stochastic processes correspond to the fluctuations of the outcome of continuous measurements performed on the fields, after their interaction with the system. Poisson processes correspond to particle counting measurements (typically, photons), whereas Wiener processes correspond to particle currents or field quadrature measurements [11, 15, 16, 21, 39, 42].

The presence of the system induce a change of statistic of these processes [11]: the measurement results and the system state become correlated via the interaction between the system and the environment. The evolution of the state $(\rho(t))$, conditional to the knowledge of the measurement outcome, is given by the following stochastic differential equation.

$$\rho(t) = \rho_0 + \int_0^t \mathcal{L}(\rho(s-))ds + \sum_{i=0}^{p} \int_0^t \mathcal{G}_i(\rho(s-))dW_i(s) + \sum_{i=p+1}^{n} \int_0^t \int_{\mathbb{R}} \left(\frac{\mathcal{J}_i(\rho(s-))}{v_i(\rho(s-))} - \rho(s-) \right) \mathbf{1}_{0<x<v_i(\rho(s-))}[N_i(dx,ds) - dxds]. \tag{2}$$

In particular, the solution of (2) is called a *quantum trajectory*. Results of existence and uniqueness of the solution of (2) can be found in [11, 12, 39, 41, 42].

From Eq. (2), one can introduce the measurement record for counting processes:

$$\hat{N}_i(t) = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<v_i(\rho(s-))} N_i(dx,ds), i = p+1, \dots, n.$$

These processes are counting processes with stochastic intensity given by

$$\int_0^t v_i(\rho(s-))ds, i = p+1, \dots, n.$$

In particular, for any $i \in \{p+1, \dots, n\}$, the process $(\hat{N}_i(t) - \int_0^t v_i(\rho(s-))ds)$ is a $(\mathcal{F}_t)$ martingale under the probability $\mathbb{P}$.

In terms of $\hat{N}_i(t)$, Eq. (2) can be written as

$$d\rho(t) = \mathcal{L}(\rho(t-))dt + \sum_{i=0}^{p} \mathcal{G}_i(\rho(t-))dW_i(t) + \sum_{i=p+1}^{n} \left(\frac{\mathcal{J}_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (d\hat{N}_i(t) - v_i(\rho(t-))dt), \tag{3}$$

which is called a SME [11], or filtering equation, in the control-oriented community [5, 6, 21, 44, 45]. The stochastic processes $(\hat{N}_i(t))$ describe discrete measurement outcomes such as particle counting. The processes $(Y_i(t)), Y_i(t) = W_i(t) + \int_0^t \mathrm{tr}[(C_i + C_i^*)\rho(s)]ds$, on the other hand, describe continuous measurement outcomes such as particle current or field quadrature measurements.

The class of evolutions captured by (2) comprises all evolution of a system (an atom or a spin) interacting with a electromagnetic field which is monitored [11, 37] as well as nanomechanical devices [38], and hence most typical models used for (measurement-based) feedback stabilization of states and subspaces of interest [5]. Similar models can also be derived for discrete-time evolutions, and have received particular attention given their applicability to new experimental setups [24, 33]. In the continuous time limit these discrete models converge weakly to solutions of SME [15, 16, 39, 41, 42]

In Eq. (2), the operator $\mathcal{L}$ is the generator of the Markov semi group associated to the stochastic model, and has the form of the generator of a semi group of completely positive, trace preserving maps [32, 36] on $\mathcal{B}(\mathcal{H})$. They represent the best description of the state evolution when the measurement record is not accessible, and can thus obtained as the expectations of (2) over the outcomes of the measurement processes. In the sequel, we shall use the following notation for the expectation of the process $(\rho(t))$:

$$\hat{\rho}(t) = \mathbb{E}[\rho(t)].$$

It is well know that

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}(\hat{\rho}(t)).$$

These generators are associated to master equations in the Markov approach of open quantum systems [2, 25], and have been extensively studied. Being linear dynamical system on a convex, positive set the study of their properties is generally simpler than studying directly the stochastic evolution. Their stability and controllability properties are discussed for example in [3, 27, 48–50]. In this work, we will exploit known results on the semigroup evolution to obtain new results on the stochastic ones.

3 Invariant and Stable Subspaces

This section is devoted to the presentation of the notion of stability we are interested in. Our aim is to study Globally Asymptotic Stable (GAS) subspaces for the SME (3). This notion is naturally linked with a decomposition of the underlying Hilbert space $\mathcal{H}$ and, with it, a corresponding block-decomposition of the matrices of interest.

Let $\mathcal{H}_S$ be a subspace of $\mathcal{H}$. Let us denote its orthogonal complement by $\mathcal{H}_R$. We thus have $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, with $\oplus$ meant as the orthogonal direct sum. We denote P_S the self-adjoint projector on $\mathcal{H}_S$ and $\mathcal{I}_S(\mathcal{H})$ the set of states

$$\mathcal{I}_S(\mathcal{H}) = \{\rho \in \mathcal{S}(\mathcal{H}) \text{ s.t. } \operatorname{tr}(P_S\rho) = 1\}.$$

Hence $\mathcal{I}_S(\mathcal{H})$ is the set of states whose support is $\mathcal{H}_S$ or a subspace of $\mathcal{H}_S$. When we are concerned with pure state preparation, we have $\mathcal{H}_S = \mathbb{C}|\phi\rangle$, with $|\phi\rangle$ the pure state to be prepared.

In the finite-dimensional case we are considering, we can use a matrix representation for all the operators involved. The definition of $\mathcal{H}_S$ and $\mathcal{H}_R$ allows for a convenient decomposition of all the matrices. Let $X \in \mathcal{B}(\mathcal{H})$, then its matrix representation can be written as

$$X = \begin{pmatrix} X_S & X_P \\ X_Q & X_R \end{pmatrix},$$

where X_S, X_R, X_P, X_Q are operators from $\mathcal{H}_S$ to $\mathcal{H}_S$, from $\mathcal{H}_R$ to $\mathcal{H}_R$, from $\mathcal{H}_R$ to $\mathcal{H}_S$ and from $\mathcal{H}_S$ to $\mathcal{H}_R$, respectively. In the rest of the paper, the indexes S, R, P, Q will refer to the same blocks as above.

We are now in position to introduce the two notions of subspace invariance for the solution of the SDE (3). Let P_S be the self adjoint projector onto the subspace $\mathcal{H}_S$, and $\|\cdot\|$ be any matrix p-norm. With a slight abuse of language[1] we say that:

Definition 1. *The subspace $\mathcal{H}_S$ is said invariant*

[1] In fact, the dynamics takes place in $\mathcal{B}(\mathcal{H})$ and not $\mathcal{H}$, but we are here interested in the invariance of the *support* of the solution $\rho(t)$.

- *in mean if*

$$\rho_0 \in \mathcal{I}_S(\mathcal{H}) \Rightarrow \; \hat{\rho}(t) \in \mathcal{I}_S(\mathcal{H}), \quad \forall t > 0.$$

- *almost surely if*

$$\rho_0 \in \mathcal{I}_S(\mathcal{H}) \Rightarrow \rho(t) \in \mathcal{I}_S(\mathcal{H}), \; \forall t > 0 \quad a.s.$$

Definition 2. *The subspace $\mathcal{H}_S$ is said globally asymptotic (GAS)*

- *in mean if $\forall \rho_0 \in \mathcal{S}(\mathcal{H})$,*

$$\lim_{t\to\infty} \|\hat{\rho}(t) - P_S\hat{\rho}(t)P_S\| = 0$$

- *almost surely if $\forall \rho_0 \in \mathcal{S}(\mathcal{H})$,*

$$\lim_{t\to\infty} \|\hat{\rho}(t) - P_S\hat{\rho}(t)P_S\| = 0 \quad a.s.$$

The invariance and GAS properties in mean are directly related to the Jordan structure and/or irreducibility of the completely positive map semi group $e^{t\mathcal{L}}$. Stability of pure states and subspaces for these maps has been discussed in [49,50]. We here recall the relevant results without proof and refer the interested reader to the original articles.

Theorem 1. *The subspace $\mathcal{H}_S$ is invariant in mean if and only if*

$$\forall j, C_{j,Q} = 0 \text{ and } iH_P - \frac{1}{2}\sum_j C^*_{j,S}C_{j,P} = 0.$$

Theorem 2. *The subspace $\mathcal{H}_S$ is GAS in mean if and only if no invariant subspace is included in $\bigcap_j \ker(C_{j,P})$.*

The rest of the section is dedicated to show that invariance and GAS properties in mean are equivalent to the corresponding properties almost surely. These results improve those of [4], in the sense that the proofs are more direct and it is shown that invariance and convergence in mean imply not only invariance and convergence in probability, but also almost surely. The proofs make use of the Lyapunov function

$$\begin{aligned} V :&\mathcal{S}(\mathcal{H}) \to [0,1] \\ &\rho \mapsto \mathrm{tr}(P_R\rho) \end{aligned}$$

with P_R the self adjoint projector on $\mathcal{H}_R$, and of the following Lemma (see also e.g. [4]).

Lemma 1.

$$V(\rho) = 0 \Leftrightarrow \rho \in \mathcal{I}_S(\mathcal{H}),$$

and the process $(V(\rho(t))$ is a positive super-martingale.

Proof. The first part and the positivity of V are obvious by definition of V and the fact that $\rho \geq 0$ for any $\rho \in \mathcal{S}(\mathcal{H})$.

For the second part, using the expression (3) we get for all $t \geq s \geq 0$

$$\mathbb{E}(V(\rho(t))|\mathcal{F}_s) = V(\rho(s)) + \int_s^t \mathbb{E}(V(\mathcal{L}(\rho(u))|\mathcal{F}_s)du.$$

Now explicit computations give

$$V(\mathcal{L}(\rho)) = \mathrm{tr}[P_R \mathcal{L}\rho] = -\sum_j \mathrm{tr}[C_{j,P}{}^* C_{j,P}\ \rho_R] \leq 0,$$

for all $\rho \in \mathcal{S}(\mathcal{H})$. This way we get for all $t \geq s \geq 0$

$$\mathbb{E}(V(\rho(t))|\mathcal{F}_s) \leq V(\rho(s))$$

which corresponds to the super-martingale property. □

We are now ready to state our equivalence results. The first concerns invariance of a subspace.

Theorem 3. *The subspace $\mathcal{H}_S$ is invariant in mean if and only if it is invariant almost surely.*

Proof. Given Lemma 1, it is sufficient to prove

$$V(\hat{\rho}(t)) = 0,\ \forall t \geq 0 \Leftrightarrow V(\rho(t)) = 0, \forall t \geq 0 \quad a.s.$$

Since V is linear, we have for all $t \geq 0$

$$V(\hat{\rho}(t)) \quad = \quad \mathbb{E}(V(\rho(t))) \tag{4}$$

and then the implication *almost surely* $\Rightarrow$ *in mean* is immediate.

For the opposite direction, let us remark that $V(\rho(t)) \geq 0$ for all $t \geq 0$. This way if we assume that $\mathbb{E}(V(\rho(t))) = V(\hat{\rho}(t)) = 0$ for all $t > 0$, it follows that $V(\rho(t)) = 0$, for all $t \geq 0$ almost surely and the result holds. □

Next we have the result regarding GAS.

Theorem 4. *The space $\mathcal{H}_S$ is GAS in mean if and only if it is GAS almost surely.*

Proof. Once again, given Lemma 1, it is sufficient to prove

$$\lim_{t\to\infty} V(\hat{\rho}(t)) = 0 \Leftrightarrow \lim_{t\to\infty} V(\rho(t)) = 0 \text{ a.s.}$$

The implication *almost surely* $\Rightarrow$ *in mean* follows from dominated convergence Theorem applied on V. Indeed, we have $\lim_{t\to\infty} V(\rho(t)) = 0$ a.s. and $V(\rho(t)) \leq 1$, for all $t \geq 0$. It follows

$$\lim_{t\to\infty} V(\hat{\rho}(t)) = \lim_{t\to\infty} \mathbb{E}(V(\rho(t))) = \mathbb{E}(\lim_{t\to\infty} V(\rho(t))) = 0.$$

The opposite direction relies on convergence for positive super-martingales. On one hand, the subspace being GAS in mean implies that $\lim_{t\to\infty} \mathbb{E}(V(\rho(t))) = 0$ for any initial state $\rho_0 \in \mathcal{S}(\mathcal{H})$. Since $V(\rho(t)) \geq 0$, this convergence corresponds to a L^1 convergence to 0. On the other hand, since $0 \leq V(\rho(t) \leq 1$ and given Lemma 1, the process $(V(\rho(t))$ is a positive bounded super martingale. It follows from a usual Theorem in martingale theory, that this process converges almost surely and in L^1 to a random variable V_∞. The uniqueness of the L^1 limit implies $V_\infty = 0$ almost surely. That completes the proof.

□

Given the two notion of GAS are equivalent, from now on we do not specify to which notion we refer when we say that a subspace is GAS.

4 A Converse Lyapunov Theorem

The focus of this paper is on the exponential stability (or equivalently, on the estimation of a Lyapunov exponent) of a stochastic evolution to a given GAS subspace: a key tool in deriving proper exponential estimation on such stability is represented by a general construction of a suitable *linear Lyapunov function* for the corresponding semigroup evolution. While typically this is not possible for linear systems, where the natural Lyapunov functions are quadratic, in this case we exploit the fact that the evolution is positive, so that a Perron-Frobenius-type results holds, and that the stable set has support on a strict subspace of $\mathcal{H}$.

We shall focus on the completely-positive semigroup dynamics. A continuous semigroup on $B(\mathcal{H})$ is completely positive if and only if its generator has the form [31, 36]:

$$\frac{d}{dt}X = \mathcal{K}(X) = G^*X + XG + \Psi(X), \tag{5}$$

where $\Psi(X)$ is a completely positive map and G is a complex $d \times d$ matrix. A positive map on $\mathcal{S}(\mathcal{H})$ is called irreducible if it does not admit nontrivial invariant subspaces or, equivalently, invariant operators are full rank. A generator is said to be irreducible if the semigroup it generates is of irreducible maps.

Recall that $\mathcal{H}_R$ is the orthogonal complement of the GAS subspace $\mathcal{H}_S$. Let $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ and $\rho_R \in \mathcal{B}(\mathcal{H}_R)$ such that $\rho_S \geq 0$ and $\rho_R \geq 0$, but not necessarily trace normalized. Define, by using the notation for the block-decomposition introduced before, the following maps:

$$\begin{aligned}\mathcal{L}_S(\rho_S) =& -i[H_S, \rho_S] + \sum_j C_{j,S}\rho_S C_{j,S}^* - \frac{1}{2}\{C_{j,S}^*C_{j,S}, \rho_S\}\\ \mathcal{L}_R(\rho_R) =& -i[H_R, \rho_R] + \sum_j C_{j,R}\rho_R C_{j,R}^* - \frac{1}{2}\{C_{j,P}^*C_{j,P} + C_{j,R}^*C_{j,R}, \rho_R\}.\end{aligned}$$

We have the following:

Proposition 1. *The family $\{e^{t\mathcal{L}_S}\}_{t\geq 0}$ is a semi group of trace preserving completely positive maps, and $\{e^{t\mathcal{L}_R}\}_{t\geq 0}$ is a semi group of trace non-increasing completely positive maps.*

Proof. Both generator have the form (5) and thus generate semigroups of completely positive maps. The maps $e^{t\mathcal{L}_S}$ are positive, thus $\rho_S \geq 0$ implies $e^{t\mathcal{L}_S}\rho_S \geq 0$. Moreover $e^{t\mathcal{L}_S}\rho_S = \rho_S + \int_0^t \mathcal{L}_S e^{s\mathcal{L}_S}\rho_S ds$ and direct calculation yields $\mathrm{tr}(\mathcal{L}_S\rho_S) = 0$ for all $\rho_S \in \mathcal{B}(\mathcal{H}_S)$. Thus $e^{t\mathcal{L}_S}$ is trace preserving for all $t \in \mathbb{R}_+$. Since the $e^{t\mathcal{L}_R}$ are positive maps, we have $\rho_R \geq 0 \Rightarrow e^{t\mathcal{L}_R}\rho_R \geq 0$. Moreover $e^{t\mathcal{L}_R}\rho_R = \rho_R + \int_0^t \mathcal{L}_R e^{s\mathcal{L}_R}\rho_R ds$ and $\mathrm{tr}(\mathcal{L}_R\rho_R) = -\sum_j \mathrm{tr}(C_{j,P}^*C_{j,P}\rho_R) \leq 0$ for all $\rho_R \in \mathcal{B}(\mathcal{H}_R)$. Thus $e^{t\mathcal{L}_R}$ is trace non-increasing for all $t \in \mathbb{R}_+$. □

The following proposition clarifies the signification of the semi groups we just defined. Recall the averaged evolution is given by

$$\hat{\rho}(t) = e^{t\mathcal{L}}\rho_0,$$

where $\mathcal{L}$ has the completely-positive form given in (1).

Proposition 2. *Assume $\mathcal{H}_S$ is invariant. If $\rho \in \mathcal{I}_S(\mathcal{H})$, then*

$$e^{t\mathcal{L}}\rho = \begin{pmatrix} e^{t\mathcal{L}_S}\rho_S & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

and any $\rho \in \mathcal{S}(\mathcal{H})$,

$$\hat{\rho}_R(t) = e^{t\mathcal{L}_R}\rho_R. \tag{7}$$

Proof. Assuming $\rho \in \mathcal{I}_S(\mathcal{H})$, the invariance of $\mathcal{H}_S$ imply $e^{t\mathcal{L}}\rho \in \mathcal{I}_S(\mathcal{H})$. From the invariance condition of Theorem 1, it follows that for any $\rho \in \mathcal{I}_S(\mathcal{H})$,

$$\mathcal{L}\rho = \begin{pmatrix} \mathcal{L}_S\rho_S & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\hat{\rho}_S(t)$ is the unique solution of $\frac{d\hat{\rho}_S(t)}{dt} = \mathcal{L}_S\hat{\rho}_S(t)$ which is $e^{t\mathcal{L}_S}\rho_S$. The invariance condition of Theorem 1 gives immediately

$$\frac{d\hat{\rho}_R(t)}{dt} = \mathcal{L}_R\hat{\rho}_R(t)$$

and the result follows from the uniqueness of the solution. □

The main result of the section, from a stability theory viewpoint, is the following:

Theorem 5 (Linear Lyapunov Function for GAS Subspaces)**.** *A subspace $\mathcal{H}_S$ is GAS if and only if there exists $V_K = (K\rho)$ such that:*

$$V_K(\rho) \geq 0, \quad \textit{with } V_K(\rho) = 0 \textit{ if and only if } \rho \in \mathcal{I}_S(\mathcal{H}); \tag{8}$$

$$V_K(\mathcal{L}(\rho)) < 0 \quad \textit{for all } \rho \notin \mathcal{I}_S(\mathcal{H}). \tag{9}$$

In showing that a *strict* Lyapunov function exists when $\mathcal{H}_S$ is GAS, we will need some preliminary results. The following technical Lemma is a weaker version of [35, Theorem 2.3]. We reproduce the proof from [35] for convenience.

Lemma 2. *Let $\mathcal{K}X = G^*X + XG + \Psi(X)$ be the generator of a semi group of completely positive maps on $\mathcal{B}(\mathbb{C}^d)$, $d \in \mathbb{N}$. If Ψ is irreducible, then $e^{t\mathcal{K}}$ is irreducible $\forall t \in \mathbb{R}_+$.*

Proof. The proof provides actually a stronger result. Namely, it shows that for any nonzero $|\phi\rangle, |\psi\rangle \in \mathbb{C}^d$, for any $t > 0$, $\langle\psi|e^{t\mathcal{K}}(|\phi\rangle\langle\phi|)\psi\rangle > 0$. This property is called positivity improving in [35].
First, from [28, Lemma 2.1], Ψ irreducible implies that

$$\langle\psi|(\mathrm{Id} + \Psi)^{d-1}(|\phi\rangle\langle\phi|)\psi\rangle > 0$$

for any nonzero $\phi, \psi \in \mathbb{C}^d$. Making an expansion of both $e^{t\Psi}$ and $(\mathrm{Id} + \Psi)^{d-1}$ one see that all the terms are positive, and all the terms in the second expansion also appear in the first one. Hence, for any $t > 0$, there exists $c > 0$ such that $e^{t\Psi} \geq c(\mathrm{Id} + \Psi)^{d-1}$. Therefore, $e^{t\Psi}$ is positivity improving.

Now notice that that $e^{t\mathcal{K}_0} : X \mapsto e^{tG^*}Xe^{tG}$ is a semi group of completely positive maps. We define the family of completely positive maps:

$$\Gamma_t(X) = e^{-t\mathcal{K}_0}e^{t\mathcal{K}}X.$$

Since for any $t > 0$ and $|\phi\rangle \in \mathbb{C}^d$, $|\phi\rangle \neq 0$, $e^{tG}|\phi\rangle \neq 0$, it remains to show that for any $t > 0$, $|\phi\rangle, |\psi\rangle \in \mathbb{C}^d$, $|\phi\rangle \neq 0, |\psi\rangle \neq 0$, $\langle\psi|\Gamma_t(|\phi\rangle\langle\phi|)\psi\rangle > 0$.

Suppose $\langle\psi|\Gamma_{t_0}(|\phi\rangle\langle\phi|)\psi\rangle = 0$ for a fixed t_0. The Dyson expansion of Γ_{t_0} is

$$\Gamma_{t_0} = \mathrm{Id} + \sum_n \int_{0<s_1<\ldots<s_n<t_0} \Psi_{s_1} \circ \cdots \circ \Psi_{s_n} ds_1 \ldots ds_n$$

where $s \to \Psi_s = e^{-s\mathcal{K}_0} \circ \Psi \circ e^{s\mathcal{K}_0}$ is a family of continuous completely positive maps. It follows

$$\langle\psi|\Gamma_{t_0}(|\phi\rangle\langle\phi|)\psi\rangle = \\ |\langle\psi|\phi\rangle|^2 + \sum_n \int_{0<s_1<\ldots<s_n<t_0} \langle\psi|\Psi_{s_1} \circ \cdots \circ \Psi_{s_n}(|\phi\rangle\langle\phi|)\psi\rangle ds_1 \ldots ds_n.$$

All the integrands are positive and continuous in $s_1, \ldots, s_n$. Hence the assumption $\langle\psi|\Gamma_{t_0}(|\phi\rangle\langle\phi|)\psi\rangle = 0$ implies $\langle\psi|\Psi_{s_1}\circ\cdots\circ\Psi_{s_n}(|\phi\rangle\langle\phi|)\psi\rangle = 0$ for all $(s_1, \ldots, s_n) \in [0, t_0]^n$. Especially $\langle\psi|\Psi^{\circ n}(|\phi\rangle\langle\phi|)\psi\rangle = 0$ for all $n \in \mathbb{N}\backslash\{0\}$. It follows that for all $t > 0$, $\langle\psi|e^{t\Psi}(|\phi\rangle\langle\phi|)\psi\rangle = 0$. This implies that either ϕ or ψ must be 0. Hence for all non zero ϕ and ψ, and for all times t, $\langle\psi|\Gamma_t(|\phi\rangle\langle\phi|)\psi\rangle > 0$ thus, setting $\psi = e^{tG}\psi'$ one obtain that for any t,

$$\langle\psi'|e^{t\mathcal{K}}(|\phi\rangle\langle\phi|)\psi'\rangle > 0.$$

The map $e^{t\mathcal{K}}$ is positivity improving and therefore irreducible. □

The next Lemma is the key one, and it builds on the Perron–Frobenius Theorem for completely positive maps [28]. Let us denote the spectral abscissa of $\mathcal{L}_R$ as:

$$\alpha_0 = \min\{-\mathrm{Re}(\lambda) \,|\, \lambda \in \mathrm{sp}(\mathcal{L}_R)\}. \tag{10}$$

Lemma 3. *For any $\epsilon > 0$ there exists $K_R > 0$ such that*

$$\mathcal{L}_R^*(K_R) \le -(\alpha_0 - \epsilon)K_R$$

where $\mathcal{L}_R^*$ is the adjoint of $\mathcal{L}_R$ with respect to the Hilbert–Schmidt inner product on $\mathcal{B}(\mathcal{H}_R)$.

Proof. By definition, for any $t \in \mathbb{R}_+$, the spectral radius of $e^{t\mathcal{L}_R}$ is $e^{-\alpha_0 t}$. If the completely positive maps of the semi group $e^{t\mathcal{L}_R}$ are irreducible the existence of $K_R > 0$ follows directly from Perron–Frobenius Theorem [28]. There exists $K_R > 0$ such that $\mathcal{L}_R^* K_R = -\alpha_0 K_R$.

We now deal with the cases where $\mathcal{L}_R$ generate a semi group of reducible completely positive maps. Let $\Psi : \mathcal{B}(\mathcal{H}_R) \to \mathcal{B}(\mathcal{H}_R)$ be an irreducible completely positive map. From Lemma 2, it follows that for all $\eta > 0$, $\mathcal{L}_\eta = \mathcal{L}_R^* + \eta\Psi$ is a generator of a semi group of irreducible completely positive maps. Let $\alpha_\eta = \min\{-\mathrm{Re}(\lambda) \,|\, \lambda \in \mathrm{sp}(\mathcal{L}_\eta)\}$. From Perron–Frobenius Theorem [28], there exists $K_\eta > 0$ such that $\mathcal{L}_\eta K_\eta = -\alpha_\eta K_\eta$.

Since $\lim_{\eta\to 0} \mathcal{L}_\eta = \mathcal{L}_R^*$, we have $\lim_{\eta\to 0} \alpha_\eta = \alpha_0$. Hence for any $\epsilon > 0$ there exists a η small enough such that $\alpha_\eta \ge \alpha_0 - \epsilon$. Thus $\mathcal{L}_R^* K_\eta = -\alpha_\eta K_\eta - \eta\Psi(K_\eta) \le -(\alpha_0 - \epsilon)K_\eta - \eta\Psi(K_\eta)$. Since Ψ is positive, $\mathcal{L}_R^* K_\eta \le -(\alpha_0 - \epsilon)K_\eta$ and the result follows setting $K_R = K_\eta$. □

In the construction of our Lyapunov function we shall need the following notation. For any operator K_R on $\mathcal{H}_R$ we extend it as an operator $\mathcal{H}$ by putting

$$K = \begin{pmatrix} 0 & 0 \\ 0 & K_R \end{pmatrix}.$$

and we introduce the function V_K defined by

$$\begin{aligned} V_K : &\mathcal{S}(\mathcal{H}) \to [0,1] \\ &\rho \mapsto \mathrm{tr}(K\rho) = \mathrm{tr}(K_R \rho_R). \end{aligned} \tag{11}$$

.

Finally, in order to get the strict negativity in Eq (9), we need the strict positivity of α_0

Lemma 4. *The subspace $\mathcal{H}_S$ is GAS, if and only if*

$$\alpha_0 > 0$$

Proof. Let us first prove that if $\mathcal{H}_S$ is GAS, we have $\alpha_0 > 0$. Assume $\alpha_0 \leq 0$. From Perron–Frobenius Theorem [28] there exists $\mu \in \mathcal{S}(\mathcal{H})$ such that $\mu_R \neq 0$ and $e^{t\mathcal{L}_R}\mu_R = e^{-t\alpha_0}\mu_R$. It follows $V(\hat{\mu}(t)) = e^{-\alpha_0 t}V(\mu) \geq V(\mu)$ for all $t \in \mathbb{R}_+$. That contradicts the GAS assumption $\lim_{t\to\infty} V(\hat{\mu}(t)) = 0$. Hence $\alpha_0 > 0$.

Concerning the other implication. Assume $\alpha_0 > 0$ and fix ϵ such that $\alpha_0 > \epsilon$, then by Lemma 3, there exist $K_R > 0$ such that $\mathcal{L}_R^* K_R \leq -(\alpha_0 - \epsilon)K_R$. Using Gronwall's inequality, we get $V_K(\hat{\rho}(t)) \leq e^{-(\alpha_0-\epsilon)t}V_K(\rho_0)$. Since $K_R > 0$, there exists a constant C such that $P_R \leq CK$, then $V(\hat{\rho}(t)) \leq Ce^{-(\alpha_0-\epsilon)t}V_K(\rho_0)$. This implies that $\mathcal{H}_S$ is GAS. □

Proof of Theorem 5. The "if" implication is a direct application of Krasovskii-LaSalle invariance principle. Let us focus on the converse implication. From Lemma 4 we can choose a strictly positive operator K_R on $\mathcal{H}_R$ fulfilling Lemma 3 with $\epsilon = \alpha_0/2$. We then clearly have $V_K(\rho) \geq 0$, and equal to zero if and only if $\rho \in \mathcal{I}_S(\mathcal{H})$ by construction. If we compute $V_K(\mathcal{L}(\rho))$, with $\rho \notin \mathcal{I}_S(\mathcal{H})$, we get:

$$\begin{aligned} V_K(\mathcal{L}(\rho)) &= \mathrm{tr}(K\mathcal{L}(\rho)) = \mathrm{tr}(K_R\mathcal{L}_R(\rho_R)) \\ &= \mathrm{tr}(\mathcal{L}_R^*(K_R)\rho_R) \\ &\leq -\alpha_0/2\, \mathrm{tr}(K_R\rho_R) \\ &< 0. \end{aligned}$$

□

Note that the Lemma 3 not only provides us with a linear Lyapunov function, but also with a first bound on the exponential stability without measurements. In terms of Lyapunov exponent for V we have:

Proposition 3.

$$\limsup_{t\to\infty} \frac{1}{t} \ln(V(\hat{\rho}(t))) \leq -\alpha_0$$

Proof. From the proof of Lemma 3, there exist K_R such that $V(\hat{\rho}(t)) \leq Ce^{-(\alpha_0-\epsilon)t}V_K(\rho_0)$. This way for all $\epsilon > 0$ and for all $t > 0$

$$\frac{1}{t}\ln(V(\hat{\rho}(t))) \leq -(\alpha_0 - \epsilon) + \frac{\ln(CV_K(\rho_0))}{t}$$

Taking first the limsup and then ϵ goes to zero we get the expected result. □

The purpose of the next section is to investigate the exponential stability by using this Lyapunov function.

5 Exponential stability

In this section we study the exponential stability of the stochastic dynamics. The key issue is an estimation of the Lyapunov exponent of V, namely an upper bound to

$$\limsup_{t\to\infty} \frac{1}{t} \ln(V(\rho(t))).$$

We address this issue from two different points of view, and provide necessary conditions leading to two different bounds for the Lyapunov exponent. In Section 6, we further discuss the significance and differences of these two bounds by studying some specific examples.

5.1 Preliminaries

We start by introducing a number of functions that will be instrumental to the derivation of the main results. For $\rho \in \mathcal{S}(\mathcal{H}), \rho_S \in \mathcal{S}(\mathcal{H}_S)$ and $\rho_R \in \mathcal{S}(\mathcal{H}_R)$, and for $j = 1, \dots, p$, define:

$$\begin{aligned} r_j(\rho) =& \mathrm{tr}[(C_j + C_j^*)\rho] \\ r_{j,S}(\rho_S) =& \mathrm{tr}[(C_{j,S} + C_{j,S}^*)\rho_S] \\ r_{j,R}(\rho_R) =& \mathrm{tr}[(C_{j,R} + C_{j,R}^*)\rho_R]. \end{aligned}$$

These play the role of the expectations of the measurement records associated to diffusive processes. On the other hand, for $j = p+1, \dots, n$,

$$\begin{aligned} v_{j,S}(\rho_S) =& \mathrm{tr}[C_{j,S}^* C_{j,S} \rho_S] \\ v_{j,R}(\rho_R) =& \mathrm{tr}[C_{j,R}^* C_{j,R} \rho_R]. \end{aligned}$$

These correspond to expectations for jump-type processes. We define the related vectors,

$$\begin{aligned} \mathbf{r}(\rho) = (r_j(\rho))_{j=1,\dots,p}, \mathbf{r}_S(\rho_S) = (r_{j,S}(\rho_S))_{j=1,\dots,p}, \mathbf{r}_R(\rho_R) = (r_{j,R}(\rho_R))_{j=1,\dots,p}, \\ \mathbf{v}(\rho) = (v_j(\rho))_{j=p+1,\dots,n}, \mathbf{v}_S(\rho_S) = (v_{j,R}(\rho_S))_{j=p+1,\dots,n} \\ \mathbf{v}_R(\rho_R) = (v_{j,R}(\rho_R))_{j=p+1,\dots,n}. \end{aligned}$$

In the following, for two vectors $\mathbf{a}, \mathbf{b}$, the division $\frac{\mathbf{a}}{\mathbf{b}}$ is meant element by elements: $\frac{\mathbf{a}}{\mathbf{b}} = (\frac{a_j}{b_j})_j$; $\mathbf{a}.\mathbf{b}$ denotes the inner product. Similarly for any function f of $\mathbb{R}$, $f(\mathbf{a}) = (f(a_j))_j$.

We next give a set of technical conditions that it is going to be useful in avoiding convergence in finite time. While not really restrictive they are essential to our proofs.

Assumption SP: $C_{j,R}^* C_{j,R} > 0$ for all $j = p+1, \dots, n$.

It particularly implies that for any $j = p+1, \dots, n$ and any $\rho \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{I}_S(\mathcal{H})$, $v_{j,R}(\rho_R) > 0$.

The following function will play an important role in the rest of the section:

$$\begin{aligned} &\alpha : \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R) \to \mathbb{R}_+ \\ &\quad (\rho, \rho_R) \mapsto \\ &\qquad \begin{cases} 0 & \text{if } \exists j = p+1, \dots, n \text{ s.t. } v_{j,R}(\rho_R) = 0 \\ \frac{1}{2}\|\mathbf{r}(\rho) - \mathbf{r}_R(\rho_R)\|^2 + (\mathbf{v}_R(\rho_R) - \mathbf{v}(\rho)).\mathbf{1} + \mathbf{v}(\rho). \ln\left(\frac{\mathbf{v}(\rho)}{\mathbf{v}_R(\rho_R)}\right) & \text{else,} \end{cases} \end{aligned}$$

with the convention $x\ln(x) = 0$ whenever $x = 0$ and $\mathbf{1} = (1)_{j=p+1,\dots,n}$. Given that definition we have:

Lemma 5. *Provided assumption* **SP** *is fulfilled, α is continuous on $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R)$ and the following minimum is well defined:*

$$\alpha_1 = \min\{\alpha(\rho, \rho_R) \,|\, \rho \in \mathcal{I}_S(\mathcal{H}), \rho_R \in \mathcal{S}(\mathcal{H}_R)\}.$$

Proof. Let introduce the set

$$A = \{(\rho, \rho_R) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R) | \exists j = p+1, \dots, n, v_{j,R}(\rho_R) = 0\}$$

The set A corresponds to the set of possible points of discontinuity for the function α. By definition $\alpha = 0$ on A. Nevertheless under the assumption **SP**, since $\mathcal{S}(\mathcal{H}_R)$ is compact, we get that

$$\min_{j=p+1,\dots,n} \min_{\rho_R \in \mathcal{S}(\mathcal{H}_R)} v_{j,R}(\rho_R) > 0.$$

It follows that A is empty and that α is continuous. Since the underlying set is compact and since α is continuous, the minimum is well defined. □

5.2 Statement of main results

Recall that

- $-\alpha_0$ is the eigenvalue of $\mathcal{L}_R$ with minimum real part,
- α_1 is given in Lemma 5,

and define

- $\alpha'_0 = \min \operatorname{spec} \left(\sum_{j=1}^n C^*_{j,P} C_{j,P} \right)$.

The main results of this section are summarized in the following Theorem.

Theorem 6. *Provided $\mathcal{H}_S$ is GAS,*

$$\limsup_{t \to +\infty} \frac{1}{t} \ln(V(\rho(t)) \le -\alpha_0 \quad a.s. \tag{12}$$

If moreover assumption **SP** *is fulfilled,*

$$\limsup_{t \to +\infty} \frac{1}{t} \ln(V(\rho(t)) \le -(\alpha'_0 + \alpha_1) \quad a.s. \tag{13}$$

Remark Note that, without measurements, from Proposition 3 we already have

$$\limsup_{t \to +\infty} \frac{1}{t} \ln(V(\hat{\rho}(t)) \le -\alpha_0$$

for the mean evolution. This way in the case where $\alpha_0 < \alpha'_0 + \alpha_1$, the above Theorem says that the subspace stability is improved in presence of measurements. Note that in general this is not always true but one can taylor experiment where this is satisfied (see Section 6). In general, as shown in the next proposition, one has $\alpha'_0 \le \alpha_0$. Then we, at least, need $\alpha_1 > 0$ for improving the stability via measurements (sufficient condition will be given in the sequel, see assumption ND below).

Proposition 4. *Assume $\mathcal{H}_S$ is GAS, then*

$$0 \le \alpha_0' \le \alpha_0.$$

Proof. First, α_0' is an element of the spectrum of a positive semi definite operator. It is thus non negative. Second, on the one hand, we have for any $\mu_R \in \mathcal{S}(\mathcal{H}_R)$, $\mathrm{tr}(\mathcal{L}_R(\mu_R)) \le -\alpha_0'$, then by Gronwall's inequality,

$$\mathrm{tr}(e^{t\mathcal{L}_R}\mu_R) \le e^{-\alpha_0' t}. \tag{14}$$

On the other hand, from completely positive maps Perron–Frobenius spectral Theorem [28], there exits $\rho_R \in \mathcal{S}(\mathcal{H}_R)$ such that $e^{t\mathcal{L}_R}\rho_R = e^{-t\alpha_0}\rho_R$. Then, applying (14) with ρ_R we get $e^{-t\alpha_0} \le e^{-t\alpha_0'}$ which gives the announced inequality. □

The following assumption gives a sufficient condition to have $\alpha_1 > 0$. It is similar to a non degeneracy condition in non demolition measurements [10,13,15,23].

> **Assumption ND.** For any $\rho_S \in \mathcal{S}(\mathcal{H}_S), \rho_R \in \mathcal{S}(\mathcal{H}_R)$, there exists $j = 1, \dots, n$ such that
>
> $$r_{j,S}(\rho_S) \ne r_{j,R}(\rho_R) \text{ if } j = 1, \dots, p,$$
>
> or
>
> $$v_{j,S}(\rho_S) \ne v_{j,R}(\rho_R) \text{ if } j = p+1, \dots, n.$$

Proposition 5. *Assume* **SP** *is fulfilled. The assumption* **ND** *is equivalent to*

$$\alpha_1 > 0.$$

Proof. First we prove **ND** $\Rightarrow \alpha_1 > 0$. From Lemma 5, since assumption **SP** is provided, α is continuous on $\mathcal{I}_S(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R)$, which is a compact set. Thus the minimum is reached for some $(\rho, \rho_R) \in \mathcal{I}_S(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R)$. Since for any $\rho \in \mathcal{I}_S(\mathcal{H})$, $\mathbf{r}(\rho) = \mathbf{r}_S(\rho_S)$ and $\mathbf{v}(\rho) = \mathbf{v}_S(\rho_S)$, it follows from assumption **ND** that there exists at least one j such that $r_{j,S}(\rho_S) \ne r_{j,R}(\rho_R)$ if $j \le p$ or $v_{j,S}(\rho_S) \ne v_{j,R}(\rho_R)$ if $j > p$. The functions $(x,y) \mapsto (x-y)^2$ and $(x,y) \mapsto y - x + x\ln(x/y)$ are positive on respectively $\mathbb{R}^2$ and $\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\}$. They vanish if and only if $x = y$. Thus from the definition of α, we get $\alpha_1 = \alpha(\rho, \rho_R) > 0$.

The backward implication is obtained by contradiction. Assume $\alpha_1 > 0$ and there exists a couple $(\rho, \rho_R) \in \mathcal{I}_S(\mathcal{H}) \times \mathcal{S}(\mathcal{H}_R)$ such that $\mathbf{r}(\rho) = \mathbf{r}_R(\rho_R)$ and $\mathbf{v}(\rho) = \mathbf{v}_R(\rho_R)$. Then $\alpha(\rho, \rho_R) = 0$ and thus $\alpha_1 = 0$ which contradicts the assumption $\alpha_1 > 0$. □

5.3 Proof of Theorem 6: Part 1

The following proposition is the key result leading to (12). We recover the stability rate bound obtained for the convergence without measurements.

Proposition 6. *Assume $\mathcal{H}_S$ is GAS. Then $\forall \epsilon > 0$,*

$$V(\rho(t)) = o(e^{-(\alpha_0 - \epsilon)t}) \quad \textit{a.s. and in} \quad L^1\textit{–norm.}$$

Proof. Fix $\epsilon > 0$. From Lemma 3, there exists $K > 0$ such that $\mathcal{L}_R^* K \le -(\alpha_0 - \frac{1}{2}\epsilon)K$. From Equation (7), we get

$$V_K(\hat{\rho}(t)) = \mathrm{tr}(e^{t\mathcal{L}_R^*}K\rho_{0,R}) \le e^{-(\alpha_0 - \frac{1}{2}\epsilon)t}V_K(\rho_0).$$

Taking expectation, we get

$$\mathbb{E}(V_K(\rho(t))e^{(\alpha_0-\epsilon)t}) \leq V_K(\rho_0)e^{-\frac{1}{2}\epsilon t}.$$

It follows that

$$\lim_{t\to\infty} \mathbb{E}(V_K(\rho(t))e^{(\alpha_0-\epsilon)t}) = 0,$$

which corresponds to the L^1 convergence.

Concerning the almost sure estimate, we first show that $(V_K(\rho(t))e^{(\alpha_0-\epsilon)t})_{t\in\mathbb{R}_+}$ is a positive super martingale. Using that $(\rho(t))$ is a Markov process and that V_K linear, for any $s \leq t$, we get

$$\begin{aligned}\mathbb{E}(V_K(\rho(t))|\mathcal{F}_s)e^{(\alpha_0-\epsilon)t} =& \mathrm{tr}[e^{(t-s)\mathcal{L}_R^*}K\rho_R(s)]e^{(\alpha_0-\epsilon)t}\\ \leq& V_K(\rho(s))e^{-(\alpha_0-\frac{1}{2}\epsilon)(t-s)}e^{(\alpha_0-\epsilon)t}\\ \leq& V_K(\rho(s))e^{(\alpha_0-\epsilon)s}e^{-\frac{1}{2}\epsilon(t-s)}\\ \leq& V_K(\rho(s))e^{(\alpha_0-\epsilon)s},\end{aligned}$$

then $(V_K(\rho(t))e^{(\alpha_0-\epsilon)t})_{t\in\mathbb{R}_+}$ is a positive super martingale. It follows that this super martingale converges almost surely to a random variable denoted by Z. Now, using the fact that L^1 convergence implies almost sure convergence for an extracted subsequence we can conclude that $Z = 0$.

Now since $K > 0$ there exists $C_K > 0$ such that $V(\rho) \leq C_K V_K(\rho)$ for any $\rho \in \mathcal{S}(\mathcal{H})$. Then

$$\lim_{t\to\infty} V(\rho(t))e^{(\alpha_0-\epsilon)t} = 0 \quad \text{a.s. and in } L^1\text{–norm.}$$

and the result is proved. □

Proof of Theorem 6 equation (12)**.** Coming back to the fact that $\lim_{t\to\infty} V(\rho(t))e^{(\alpha_0-\epsilon)t} = 0 \quad a.s$, there exist almost surely T such that for all $t \geq T$

$$V(\rho(t)) \leq e^{-(\alpha_0-\epsilon)t}.$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\ln(V(\rho(t))) \leq -(\alpha_0-\epsilon) \quad a.s$$

and taking then ϵ going to zero yields the result. □

5.4 Proof of Theorem 6: Part 2

Now for the second bound in Theorem 6, we first show that the process $(V(t))$ can be written as the solution of a Doleans Dade equation, a crucial step in obtaining the result. Before stating the result, let us introduce the following notation

$$\mathbf{W}(t) = (W_j(t))_{j=1,\dots,p} \text{ and } \hat{\mathbf{N}}(t) = (\hat{N}_j(t))_{j=p+1,\dots,n}.$$

Furthermore we will need the following process

$$\rho_{R,\mathrm{red.}}(t) = \begin{cases} \frac{\rho_R(t)}{\mathrm{tr}(\rho_R(t))} & \text{if } \mathrm{tr}(\rho_R(t)) = V(\rho(t)) \neq 0 \\ \mu_R & \text{if } \mathrm{tr}(\rho_R(t)) = V(\rho(t)) = 0, \end{cases} \tag{15}$$

where μ_R is an arbitrary state of $\mathcal{S}(\mathcal{H}_R)$.

Remark: In the above definition, the process $\rho_{R,\mathrm{red.}}(t)$ can be seen as a normalized version of the reduced state ρ_R, which we obtain by dividing by $\mathrm{tr}(\rho_R(t)) = V(\rho(t))$. However, since in general nothing ensures that $\mathrm{tr}(\rho_R(t))$ does not vanish, we introduce an arbitrary state μ_R in the case $\mathrm{tr}(\rho_R(t))$ is zero. This is only a formal construction, as the interesting cases are the ones where this quantity is never zero. In fact, by using the invariance property it is easy to see that if $V(\rho(t) = 0$ for some t, we get $V(\rho(s)) = 0$ for all $s \geq t$. In this situation the exponential stability is somewhat trivial: the Lyapunov exponent is equal to $-\infty$, as we have convergence to zero in finite time. In the situation where $(V(\rho(t)))$ does not vanish in finite time, the state μ_R does not actually play any role. The introduction of μ_R is only instrumental to a proper definition of $\rho_{R,\mathrm{red.}}(t)$, and the final result, in the cases of interest, will not rely on the choice of μ_R.

In order to simplify the notation we put

$$V(t) = V(\rho(t)), \forall t \in \mathbb{R}_+$$

Proposition 7. *The process $(V(t))$ is the unique solution of the SDE*

$$\begin{aligned} dV(t) =& V(t-)\Big\{\mathrm{tr}(\mathcal{L}_R\rho_{R,\mathrm{red.}}(t-))dt \\ &+ [\mathbf{r}_R(\rho_{R,\mathrm{red.}}(t-)) - \mathbf{r}(\rho(t-)))].d\mathbf{W}(t) \\ &+ \left(\frac{\mathbf{v}_R(\rho_{R,\mathrm{red.}}(t-))}{\mathbf{v}(\rho(t-))} - \mathbf{1}\right).[d\hat{\mathbf{N}}(t) - \mathbf{v}(\rho(t-))dt]\Big\}, \\ V(0) =& \mathrm{tr}(P_R\rho_0). \end{aligned} \tag{16}$$

This process is a Doleans–Dade exponential whose explicit expression is

$$\begin{aligned} V(t) = V(0) \prod_{j=p+1}^{n} \prod_{s\leq t} &\left(1 + \left(\frac{v_{j,R}(\rho_{R,\mathrm{red.}}(s-))}{v_j(\rho(s-))} - 1\right)\Delta\hat{N}_j(s)\right) \\ &\times \exp\Big\{\int_0^t \mathrm{tr}(\mathcal{L}_R\rho_{R,\mathrm{red.}}(s-))ds \\ &- \frac{1}{2}\|\mathbf{r}_R(\rho_{R,\mathrm{red.}}(s-)) - \mathbf{r}(\rho(s-))\|^2 ds \\ &+ \int_0^t [\mathbf{r}_R(\rho_{R,\mathrm{red.}}(s-)) - \mathbf{r}(\rho(s-))].d\mathbf{W}(s) \\ &- \int_0^t [\mathbf{v}_R(\rho_{R,\mathrm{red.}}(s-)) - \mathbf{v}(\rho(s-))].\mathbf{1}ds\Big\}. \end{aligned} \tag{17}$$

Proof. Since $(\rho(t))$ is well defined, the uniqueness of the solution of (16) and the expression (17) follows from usual argument of stochastic calculus. Now the fact that $(V(t)$ satisfies (16) is obtained by applying V on (3). Indeed, let us recall that $V(\rho) = 0$ implies $\rho_R = 0$. This way for all $t \geq 0$, we can write $\rho_R(t) = V(t)\rho_{R,\mathrm{red.}}(t)$ and applying V (which is linear) on (3), we get

$$\begin{aligned} dV(t) =&\mathrm{tr}(\mathcal{L}_R(\rho_R(t-)))dt \\ &+\sum_{j=1}^{p}(\mathrm{tr}((C_{j,R}+C_{j,R}^*)\rho_R(t-))-\mathrm{tr}((C_j+C_{j,R}^*)\rho(t))V(t-))dW_j(t) \\ &+\sum_{j=p+1}^{n}\left(\frac{\mathrm{tr}(C_{j,R}\rho_R(t-)C_{j,R})}{v_j(t-)}-V(t-)\right)[d\hat{N}_j(t)-v_j(t-)dt], \end{aligned}$$

which is the expansion of (16). □

In order to discuss the "strict" positivity of $(V(t))$, it is interesting to write the solution $(V(t))$ in the following form

$$\begin{aligned} V(t) =&V(0)+\int_0^t V(s)\Big\{\mathrm{tr}(\mathcal{L}_R(\rho_R(s-)))ds \\ &-[\mathbf{v}_R(\rho_{R,\mathrm{red.}}(s-))-\mathbf{v}(\rho(s-))].\mathbf{1}dt \\ &+[\mathbf{r}_R(\rho_{R,\mathrm{red.}}(s-))-\mathbf{r}(\rho(s-))].d\mathbf{W}(s)\Big\}, \\ &+\sum_{n=0}^{+\infty}V(T_n-)\frac{v_{j_{T_n},R}(\rho_{R,\mathrm{red.}}(T_n-))}{v_{j_{T_n}}(\rho(T_n-))}\mathbf{1}_{T_n\leq t}, \end{aligned} \tag{18}$$

where the sequence of stopping time (T_n) is defined by $T_0=0$ and

$$T_{n+1}=\inf\{t>T_n \text{ s.t. } \hat{\mathbf{N}}(t).\mathbf{1}\geq n\}.$$

Note that the independence of N_i ensures that for all n, we have $\hat{\mathbf{N}}(T_n).\mathbf{1}=1$ almost surely (two jumps can not appear at the same time)

Remark: Under the light of the expression (17), one can introduce $\tau=\inf\{t>0/V(t)=0\}$. Using strong Markov property of the couple $(\rho(t),V(t))$, one can see that $V(t)=0$ for all $t\geq\tau$. At this stage we can underline the fact that μ_R in the definition of $(\rho_{R,\mathrm{red.}}(t))$ will not play any role in our final result. On $\{\tau<\infty\}$ it is obvious that $V(t)=o(e^{-ct})$ for all $c>0$ and such situation is straightforward.

In addition of the above remark, the following corollary expresses that under assumption **SP** the event $\{\tau<\infty\}$ is of probability 0.

Corollary 1. *Assume* **SP** *is fulfilled. Then for all* $t\in\mathbb{R}_+$, $V(\rho(t))>0$ *almost surely.*

Proof. Assumption **SP** ensures there exists $c>0$ such that for all $j=p+1,\dots,n$ and any $\rho_R\in\mathcal{S}(\mathcal{H}_R)$, $v_{j,R}(\rho_R)>0$. It follows that $\frac{v_{j,R}(\rho_{R,\mathrm{red.}}(s-))}{v_j(\rho(s-))}>0$ almost surely for all $s\geq0$ and $j=p+1,\dots,n$. Thus from equation (17) or (18), one can see that $V(t)$ does not vanish when a jump occurs (that is at a time T_n). Concerning the smooth evolution (that is the diffusive evolution in between the jumps, i.e $\Delta N_i(.)=0$) one can see (from (17)) that V is an exponential and then does not vanish. □

The following lemma is a technical lemma which will be used in the next proposition (the proof is only based on an argument regarding strong law of large numbers for martingale and we do not give the detail of the proof)

Lemma 6. *Let $\mathbf{F}_W : \mathcal{S}(\mathcal{H}) \to \mathbb{R}^p$ be a bounded function. Let also $\mathbf{F}_J : \mathcal{S}(\mathcal{H}) \to \mathbb{R}^{n-p}$ be a function such that $\rho \mapsto \mathbf{v}(\rho).\mathbf{F}_J^2(\rho)$ is bounded. The processes $(M_W(t))$ and $(M_J(t))$ defined by*

$$M_W(t) = \int_0^t \mathbf{F}_W(\rho(s-)).d\mathbf{W}(s) \tag{19}$$

$$M_J(t) = \int_0^t \mathbf{F}_J(\rho(s-)).[d\hat{\mathbf{N}}(s) - \mathbf{v}(\rho(s-))ds] \tag{20}$$

are square integrable martingales that obey the strong law of large numbers:

$$\lim_{t\to\infty} \frac{1}{t} M_W(t) = 0 \tag{21}$$

$$\lim_{t\to\infty} \frac{1}{t} M_J(t) = 0 \tag{22}$$

almost surely.

In the following lemma we use a definition similar to the one of $\rho_{R,\text{red.}}$ for a reduced state $\rho_{S,\text{red.}}$ on $\mathcal{S}(\mathcal{H}_S)$:

$$\rho_{S,\text{red.}} = \begin{cases} \frac{\rho_S}{\text{tr}(\rho_S)} & \text{if } \text{tr}(\rho_S) \neq 0 \\ \mu_S & \text{if } \text{tr}(\rho_S) = 0 \end{cases} \tag{23}$$

Lemma 7. *Assume $\mathcal{H}_S$ is GAS and* **SP** *is fulfilled. Then,*

$$\lim_{t\to\infty} \alpha(\rho(t), \rho_{R,\text{red.}}(t)) - \alpha\left(\begin{pmatrix} \rho_{S,\text{red.}}(t) & 0 \\ 0 & 0 \end{pmatrix}, \rho_{R,\text{red.}}(t)\right) = 0$$

almost surely.

Proof. From the GAS property, we have

$$\lim_{t\to\infty} \rho_R(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho_P(t) = 0 \quad a.s.$$

Hence $1 - V(\rho(t))$ converges almost surely to 1 and then

$$\lim_{t\to\infty} \|\rho_S(t) - \rho_{S,\text{red.}}(t)\| = 0 \quad a.s.$$

We then have

$$\lim_{t\to\infty} \left\| \rho(t) - \begin{pmatrix} \rho_{S,\text{red.}}(t) & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0$$

almost surely. The result then follows from the continuity of α ensured by assumption **SP** and Lemma 5. □

Now we are in position to prove the last part of Theorem 6 which is

$$\limsup_{t\to\infty} \frac{1}{t} \ln(V(\rho(t))) \leq -(\alpha_0' + \alpha_1) \quad a.s.$$

Proof of Theorem 6 equation (13)**.** If $\rho_R(0) = 0$, the result is trivial. We therefore prove the result only for $\rho_R(0) \neq 0$. Since **SP** is fulfilled, Corollary 1 ensures $V(t) > 0$ for all $t \in \mathbb{R}_+$ almost surely.

Now let us introduce

$$\mathbf{F}_W(\rho) = \mathbf{r}_R(\rho_{R,\text{red.}}) - \mathbf{r}(\rho) \quad \text{and} \quad \mathbf{F}_J(\rho) = \ln\left(\frac{\mathbf{v}_R(\rho_{R,\text{red.}})}{\mathbf{v}(\rho)}\right).$$

They both fulfill the assumptions of Lemma 6. Using Itô–Lévy Lemma for the logarithm function or using the explicit expression of Proposition 7, we can express $V(t)$ as

$$V(t) = V(0)\times \exp\Big\{ \int_0^t \operatorname{tr}(\mathcal{L}_R\rho_{R,\text{red.}}(s-)) - \alpha(\rho(s-), \rho_{R,\text{red.}}(s-))ds \\ + M_W(t) + M_J(t)\Big\}.$$

where

$$M_W(t) = \int_0^t \mathbf{F}_W(\rho(s-)).d\mathbf{W}(s)$$

and

$$M_J(t) = \int_0^t \mathbf{F}_J(\rho(s-)).[d\hat{\mathbf{N}}(s) - \mathbf{v}(s-)ds].$$

are square integrable martingale. At this stage, we have

$$\begin{aligned}\frac{1}{t}\ln(V(t)) &= \frac{1}{t}\ln(V(0)) \\ &\quad+\frac{1}{t}\int_0^t \operatorname{tr}(\mathcal{L}_R\rho_{R,\text{red.}}(s-)) - \alpha(\rho(s-), \rho_{R,\text{red.}}(s-))ds \\ &\quad+\frac{1}{t}M_W(t) + \frac{1}{t}M_J(t).\end{aligned}$$

Now, the strong law of large numbers of Lemma 6 implies

$$\lim_{t\to\infty}\frac{1}{t}M_W(t) = \lim_{t\to\infty}\frac{1}{t}M_J(t) = 0 \quad a.s.$$

Obviously

$$\lim_{t\to\infty}\frac{1}{t}\ln(V(0)) = 0.$$

It remains to treat the integral term. To this end, from the definition of α_0', recall that

$$\operatorname{tr}(\mathcal{L}_R\rho_{R,\text{red.}}(s-)) \leq -\alpha_0'$$

for all $s \in \mathbb{R}_+$ almost surely. Then from Lemma 7 and the definition of α_1, we have

$$\limsup_{t\to\infty} -\alpha(\rho(t), \rho_{R,\text{red.}}(t)) \leq -\alpha_1$$

almost surely.

From the implication

$$\limsup_{t\to\infty} f(t) \leq C \Rightarrow \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds \leq C,$$

we finally obtained

$$\limsup_{t\to\infty}\frac{1}{t}\ln(V(\rho(t))) \leq -(\alpha_0' + \alpha_1).$$

□

As a final remark in this section, the following estimates are byproducts of Theorem 6.

Theorem 7. *Provided $\mathcal{H}_S$ is GAS, then $\alpha_0 > 0$, and for any $\epsilon > 0$,*

$$\left\| \rho(t) - \begin{pmatrix} \rho_S(t) & 0 \\ 0 & 0 \end{pmatrix} \right\| = o(e^{-\frac{1}{2}(\alpha_0 - \epsilon)t}) \quad a.s. \text{ and in } L^1\text{–norm.} \tag{24}$$

If moreover assumption **SP** *is fulfilled,*

$$\left\| \rho(t) - \begin{pmatrix} \rho_S(t) & 0 \\ 0 & 0 \end{pmatrix} \right\| = o(e^{-\frac{1}{2}(\alpha_0' + \alpha_1 - \epsilon)t}) \quad a.s. \tag{25}$$

Proof. One has to check that, for any state $\rho \in \mathcal{S}(\mathcal{H}_S)$,

$$\left\| \rho - \begin{pmatrix} \rho_S & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq 2 \max(2\sqrt{V(\rho)}, V(\rho)).$$

This can be proved as follows. Using the triangle inequality, we have

$$\left\| \rho - \begin{pmatrix} \rho_S & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq \|\rho_P\| + \|\rho_Q\| + \|\rho_R\|.$$

Since $\rho \geq 0$, $\rho_Q = \rho_P^*$, $\|\rho_P\| = \|\rho_Q\|$, and we have the inequality

$$\|\rho_P\|^2 \leq \|\rho_S\| \|\rho_R\|.$$

From the bound $\|\rho_S\| \leq 1$, it follows

$$\left\| \rho - \begin{pmatrix} \rho_S & 0 \\ 0 & 0 \end{pmatrix} \right\| \leq 2\|\rho_R\|^{1/2} + \|\rho_R\|.$$

Now, the almost sure results of the Theorem are a consequence of the inequality $\|\rho_R(t)\| \leq V(\rho(t))$ and of the result of Theorem 6. The L^1 norm follows from Proposition 6. □

6 Improved stability consequences

As stated earlier, one can taylor examples where $\alpha_0 < \alpha_0' + \alpha_1$. In fact, we next show that it is possible to add a measurement channel that does not modify $\mathcal{L}_S$ and $\mathcal{L}_R$, yet makes α_1 arbitrarily large. Define

$$C_{n+1} = \ell_S P_S + \ell_R P_R,$$

with P_S and P_R the self-adjoint projectors onto $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively, and $\ell_S, \ell_R \in \mathbb{C}$.

This new operator accounts for the addition of a diffusive "non demolition" measurement, distinguishing wether the state is in $\mathcal{H}_S$ or $\mathcal{H}_R$ [15,23]. It is worth noticing that this does not modifies the invariance and GAS property of $\mathcal{H}_S$.

Let us introduce the new operator-valued functions associated to the SME which includes C_{n+1}. We denote them with a "~" in order to distinguish them from the original ones. For any $\rho \in \mathcal{S}(\mathcal{H})$, we have:

$$\begin{aligned} \tilde{\mathcal{L}}(\rho) &= \mathcal{L}(\rho) + C_{n+1}\rho C_{n+1}^* - \frac{1}{2}\{C_{n+1}^* C_{n+1}, \rho\}, \\ \mathcal{G}_{n+1}(\rho) &= C_{n+1}\rho + \rho C_{n+1}^* - \operatorname{tr}(C_{n+1} + C_{n+1}^*)\rho)\rho. \end{aligned}$$

Direct computations yield $\tilde{\mathcal{L}}_S = \mathcal{L}_S$ and $\tilde{\mathcal{L}}_R = \mathcal{L}_R$. Therefore $\tilde{\alpha}_0 = \alpha_0$ and $\tilde{\alpha}'_0 = \alpha'_0$. We only expect $\tilde{\alpha}_1 \neq \alpha_1$. The new quantum trajectory $(\tilde{\rho}(t))$ is the solution of the SDE

$$\begin{aligned}\tilde{\rho}(t) =& \rho_0 + \int_0^t \tilde{\mathcal{L}}(\tilde{\rho}(s-))ds \\ &+ \sum_{j=0}^{p} \int_0^t \mathcal{G}_j(\tilde{\rho}(s-))dW_j(s) + \int_0^t \mathcal{G}_{n+1}(\tilde{\rho}(s-))dW_{n+1}(s) \\ &+ \sum_{j=p+1}^{n} \int_0^t \int_{\mathbb{R}} \left(\frac{\mathcal{J}_j(\tilde{\rho}(s-))}{v_j(\tilde{\rho}(s-))} - \tilde{\rho}(s-) \right) 1_{0<x<v_j(\tilde{\rho}(s-))} [N_j(dx, ds) - dxds]\end{aligned} \tag{26}$$

with $(W_{n+1}(t))$ a Wiener process independent of all the other Wiener and Poisson processes[2].

If we assume **SP**, then from the definition of α and the corresponding $\tilde{\alpha}$, we have

$$\tilde{\alpha}_1 = \alpha_1 + \frac{1}{2}\mathrm{Re}^2(\ell_S - \ell_R).$$

It is then clear that we can play with the value $\mathrm{Re}^2(\ell_S - \ell_R)$ to increase arbitrarily the value of α_1. Next proposition expresses this fact and follows directly from Theorem 6.

Proposition 8. *Assume $\mathcal{H}_S$ is GAS and* **SP** *is fulfilled. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \left(\mathbb{E}(V(\tilde{\rho}(t))) \right) \leq -\alpha_0$$

and for any $C > 0$ there exists $\ell_S, \ell_R \in \mathbb{C}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \ln \left(V(\tilde{\rho}(t)) \right) \leq -C \quad a.s.$$

Hence, whatever is the mean stability exponent α_0, we may have an arbitrarily large almost sure asymptotic stability exponent.

In the particular case of qubits, i.e. two dimensional systems, we have a finer result: the above inequality actually becomes an equality, showing that the above bound is, in some sense, sharp. Note that in the qubit case we choose $\mathcal{H}_S$ and $\mathcal{H}_R$ both one dimensional. They correspond to two orthogonal projective rays of $\mathcal{H}$. The quantum trajectory can then be expressed, in the orthonormal basis associated to $\mathcal{H}_S$ and $\mathcal{H}_R$, as

$$\rho(t) = \begin{pmatrix} p(t) & c(t) \\ \bar{c}(t) & 1 - p(t) \end{pmatrix}$$

for any time t. The evolution of $(\rho(t))$ is then uniquely determined by that of $(p(t))$ and $(c(t))$. In particular we have

$$V(\rho(t)) = 1 - p(t),$$

for all $t \geq 0$.

For the sake of simplicity we just focus on the case where only two diffusive measurements are involved ($n = p = 1$), associated to operators:

$$C_0 = \begin{pmatrix} 0 & \ell_P \\ 0 & 0 \end{pmatrix},\ C_1 = \begin{pmatrix} \ell_S & 0 \\ 0 & \ell_R \end{pmatrix} \text{ and } H = 0,$$

[2]We define the new filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$, similarly to the original one.

with $\ell_S, \ell_R, \ell_P \in \mathbb{C}$. This restriction is intended mainly to improve the readability of our proof, as the results extend easily to more general choices of C_0 and C_1. Also, adding more diffusive measurements or counting measurements is straightforward.

We can translate the SDE (2) to a SDE involving $(p(t), c(t))$. If $(\rho(t))$ is the solution of (2) with $p = n = 1$ and the above defined C_0 and C_1 then its corresponding process $(p(t), c(t))$ is the solution of

$$\begin{aligned} dp(t) =&(1-p(t))|\ell_P|^2 dt \\ &+ 2(1-p(t))\mathrm{Re}(\ell_P \bar{c}(t))dW_0(t) \\ &+ 2p(t)(1-p(t))\mathrm{Re}(\ell_S - \ell_R)dW_1(t) \end{aligned} \tag{27}$$

$$\begin{aligned} dc(t) =& -\frac{1}{2}|\ell_P|^2 c(t)dt - \frac{1}{2}(|\ell_S|^2 + |\ell_R|^2 - 2\ell_S\bar{\ell}_R)c(t)dt \\ &+ \big((1-p(t))\ell_P - 2c(t)\mathrm{Re}(\ell_P\bar{c}(t))\big)dW_0(t) \\ &+ \big(\ell_S c(t) + \bar{\ell}_R \bar{c}(t) - 2c(t)(p(t)\mathrm{Re}(\ell_S) + (1-p(t))\mathrm{Re}(\ell_R))\big)dW_1(t) \end{aligned} \tag{28}$$

with the initial condition $(p(0), c(0)) = (p_0, c_0)$. From equation (27) and the definitions of α_0, α_0' and α_1, we immediately have

$$\alpha_0 = \alpha_0' = |\ell_P|^2 \text{ and } \alpha_1 = 2\mathrm{Re}^2(\ell_S - \ell_R).$$

We then have the following refinement of Theorem 6.

Theorem 8. *Consider the two-dimensional system described above. Assume $\ell_P \neq 0$ and $p_0 < 1$. Then,*

$$\lim_{t\to\infty} \frac{1}{t}\ln(1-p(t)) = -(\alpha_0 + \alpha_1) \ \ a.s.$$

Proof. From Îto lemma, we have

$$\ln(1-p(t)) = \ln(1-p_0) - \alpha_0 t - \alpha_1 \int_0^t p(s)ds - \int_0^t \mathrm{Re}(\ell_P\bar{c}(s))ds + M_t$$

with M_t a square integrable martingale such that $\lim_{t\to\infty} M_t/t = 0$ almost surely. Note that we have almost sure convergences of $p(t)$ to 1 and of $c(t)$ to 0. Now adapting easily the proof of Theorem 6, since we have $\ell_P \neq 0$ we get the result. □

Hence, the stability exponent is exactly $\alpha_0 + \alpha_1$, and no better one can be found. This result proves the sharpness of our stability rate bound. The almost sure convergence towards $\mathcal{H}_S$ was already known [9], hence the new result in this case is the stability rate derivation.

We conclude this section and this article with some numerical simulations (see Figure 1) that illustrate the influence of an increased α_1 on the typical trajectories. In the case corresponding to a larger asymptotic stability rate leads to initially more erratic trajectories, yet the convergence is faster in the sense of the Lyapunov exponents: the increased stability rate makes the state almost "jump" to the target subspace, where it remains. This limit behaviour was first remarked and discussed in [14, 17–19]. Formulating and proving these observations, *i.e.* studying the limit $\alpha_1 \to \infty$, more rigorously needs further deep investigations.

TB: Do we need this paragraph or a conclusion. The references and the remark on the limit $\alpha_1 \to \infty$ may be sufficient.

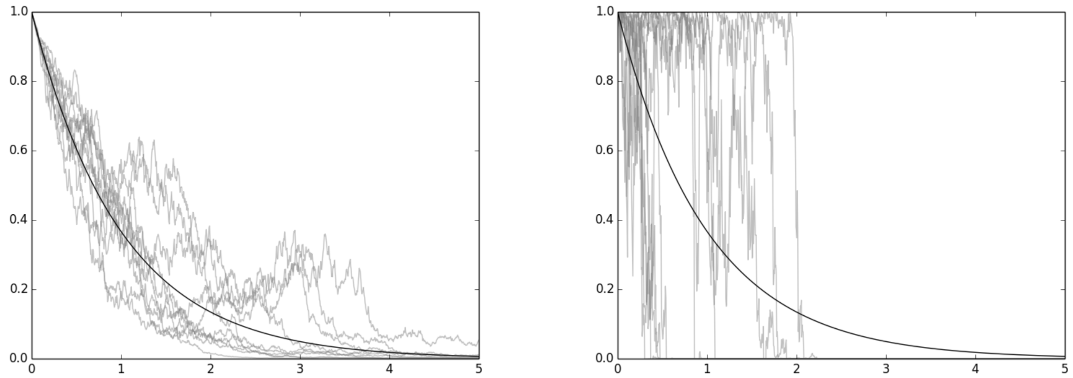

Figure 1: Numerical simulations of the evolution of $1-p(t)$. On the left $\alpha_0 = 1$ and $\alpha_1 = 1/2$. On the right $\alpha_0 = 1$ and $\alpha_1 = 8$. In each graph one gray line corresponds to a realisation and the solid black line corresponds to the average evolution. The initial condition is set to $p_0 = 0$. One can remark that when α_1 increases, the asymptotic stability increases.

CP: I do agree with Tristan regarding the last paragraph. Since we do not have investigated the problem in details; I am not sure it is relevant. In particular we do not give much more interpretations than the one given in the references.

Let us interpret this behaviour from a physical viewpoint. What the additional measurement operator is introducing is a continuous measurement of the state, extracting information regarding which one of the basis vectors it occupies. When α_1 gets large, it means that the difference of $\ell_{S,R}$ is getting larger, and so does the norm of C_{n+1}. This is equivalent to a rescaling of the variance of the noise process, and in the limit of infinitely large variance the measurement of C_{n+1} would effectively become a projective measurement, with convergence in finite time. However, the state corresponding to $\mathcal{H}_R$ is unstable for the other measurement process so, even if initially the state may tend to "jump" towards $\mathcal{H}_R$, for large yet bounded C_{n+1} it will always retain some probability of rapidly going back towards $\mathcal{H}_S$, the only stable state. Similar arguments hold for general finite dimensional systems.

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