Technical Note

Extended Theory of Hydraulic Hysteresis in Open-Channel Flow

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Abstract: The occurrence of hysteresis in a supercritical, open-channel flow approaching an obstacle has been recognized and investigated both experimentally and theoretically over the last few decades. However, the available theory and experimental investigations in the literature do not include the case when subcritical flow, controlled from downstream, can establish across the obstacle. The present work fills this gap by proposing a new theory that includes this occurrence and shows that two different steady flow states can establish for the same obstacle geometry and flow conditions—one with supercritical to subcritical transition far downstream from the obstacle, and the other with supercritical to subcritical transition far upstream from the obstacle. The proposed, more general theory includes the existing theory as a special case. Finally, two specific examples are illustrated and discussed, i.e., the case of flow over a raised bed hump, and the case of flow through a channel contraction. **DOI: 10.1061/(ASCE)HY.1943-7900.0001342.**

Introduction

For a supercritical, open-channel flow approaching an obstacle, there is a range of values of the geometric and flow parameters such that two steady flow states are possible and stable, with the state that actually establishes being determined by the past history of the flow, i.e., by the way in which flow conditions have evolved up to the current conditions. In these conditions, the flow is said to have a hysteretic behavior.

The occurrence of hysteresis in a supercritical flow approaching an obstacle has been recognized and investigated both experimentally and theoretically over the last few decades. The case of a sill in a channel of constant width is the first and most studied type of obstacle (Abecasis and Quintela 1964; Mehrotra 1974; Muskatirovic and Batinic 1977; Pratt 1983; Baines 1984; Austria 1987; Lawrence 1987; Baines and Whitehead 2003). Defina and Susin (2003) demonstrated both theoretically and experimentally the occurrence of hysteresis in the flow under a sluice gate. More recently, the hysteretic behavior of flow approaching a channel contraction has been demonstrated by Akers and Bokhove (2008) and Defina and Viero (2010). All these studies have shown that the flow configuration that establishes across an obstacle depends not only on the incoming flow characteristics and on the geometry and size of the obstacle, but also on the past history of the flow.

A criterion to identify hysteresis occurrence for a steady, open-channel flow has been proposed by Defina and Susin (2003), in which one-dimensional flow equations were used for energy and momentum conservation, showing that these equations can confidently be used to predict hydraulic hysteresis, at least for the case of negligibly small energy dissipation (Muskatirovic and Batinic 1977; Austria 1987; Baines and Whitehead 2003; Akers and Bokhove 2008).

In the preceding theoretical and experimental studies, two different stable states were identified, with the incoming supercritical flow that can either remain supercritical across the obstacle [Fig. 1(a)] or it can undergo a supercritical to subcritical transition upstream of the obstacle, followed by a subcritical to supercritical transition at the obstacle [Fig. 1(b)]. Accordingly, it was implicitly assumed that hysteresis only depends on the obstacle characteristics and on the upstream flow conditions.

Experimental investigations carried out in this study proved the occurrence of hysteresis also when the flow through the obstacle is affected by a downstream subcritical flow that can extend upstream across the obstacle. In this case, subcritical to supercritical transition at the obstacle does not occur, but still two different steady flow states can establish—one with supercritical to subcritical transition far downstream from the obstacle [Fig. 1(a)], the other with supercritical to subcritical transition far upstream from the obstacle [Fig. 1(c)]. For example, Fig. 2 shows, for the same flow conditions away from the obstacle, two steady stable states for the case of a raised bed hump (discussed subsequently) with subcritical flow downstream from the obstacle.

The occurrence of hysteresis in such conditions has never been reported in the literature, nor it can be predicted by the available theory of hysteresis. A new theory of hysteresis in open-channel flows is proposed in this paper, which includes the previous one and extends it to the case in which a downstream subcritical flow affects the flow through the obstacle. Besides the practical interest (e.g., in the design of energy dissipators and stilling basins), the importance of identifying the conditions so that this dual behavior can establish, also stems from the possibility of using the results of the present work to check the ability of numerical models in correctly predicting the flow in the presence of an obstacle (e.g., Catella and Bechi 2006; Jaafar and Merkley 2010; Viero et al. 2013a; Cozzolino et al. 2015).

In the following sections, the new theory is outlined and, for illustration purposes, is applied to two specific obstacles, namely, a raised bed hump and a channel contraction.

Theoretical Approach

To derive the extended theory of hydraulic hysteresis for steady open-channel flow, which includes the case when a downstream

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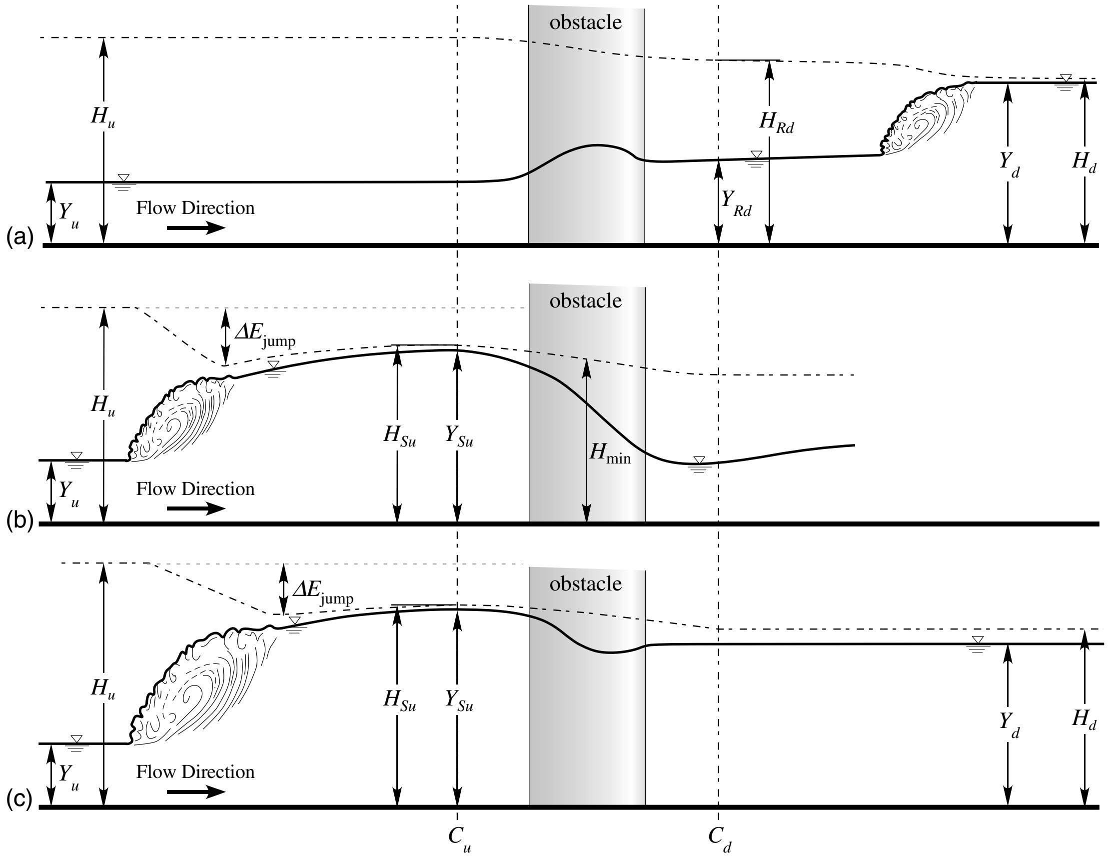

Fig. 1. Steady flow regimes in the flow through an obstacle; the specific energy of the supercritical incoming flow, H_u, is greater than the minimum specific energy required to pass the obstacle: (a) supercritical smooth flow configuration with a jump far downstream from the obstacle; (b) supercritical to subcritical transition upstream of the obstacle and critical conditions at the obstacle; (c) subcritical smooth flow configuration with a jump far upstream from the obstacle

subcritical flow affects the flow through the obstacle, the one-dimensional hydraulic approach is used, friction and bed slope are neglected, and hydrostatic pressure is assumed away from the obstacle.

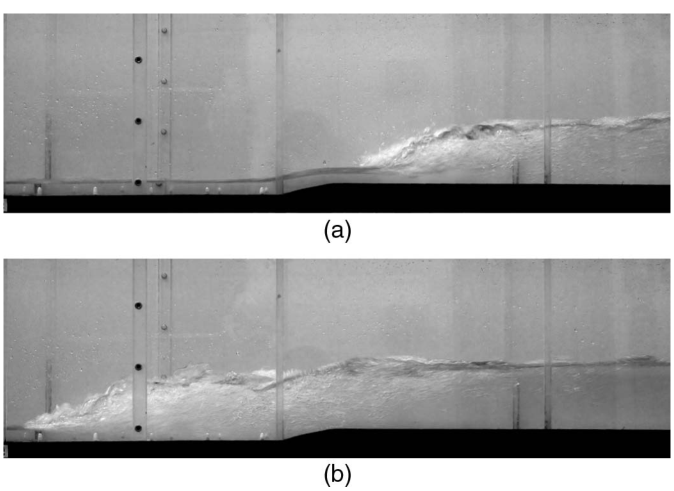

Fig. 2. Steady stable states in a 30 cm wide laboratory flume; bottom slope is 0.02, raised bed hump is 2 cm high, flow rate is 15 L/s; upstream Froude number is $\mathsf{F}_u \cong 4$, downstream Froude number is $\mathsf{F}_d \cong 0.4$: (a) supercritical; (b) subcritical smooth flow across the bed hump

In the mathematical developments, H denotes the specific energy, i.e., the flow energy per unit weight, relative to the channel bottom; subscripts u and d denote the characteristics of the upstream supercritical flow and downstream subcritical flow, respectively, and subscripts Su and Rd denote the characteristics of the upstream subcritical (slow) flow and downstream supercritical (rapid) flow, respectively (Fig. 2).

For hysteresis to occur, the incoming supercritical flow must have enough energy to pass the obstacle without undergoing transition (Defina and Susin 2003). Contrarily, the problem has one trivial solution, with supercritical to subcritical transition (i.e., a hydraulic jump) upstream of the obstacle (Castro-Orgaz and Hager 2009).

If the subcritical downstream flow has enough energy to pass the obstacle

$$H_d \geq H_d^{\min} \tag{1}$$

it can affect the upstream flow field. In this case, the possible flow configurations across the obstacle are shown in Fig. 1, and they are referred to as supercritical smooth flow configuration [Fig. 1(a)] and subcritical smooth flow configuration [Fig. 1(c)].

The hysteresis domain, i.e., the flow and geometry conditions for which both the preceding configurations can establish and are stable, is outlined in the following.

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Lower Boundary of the Hysteresis Domain

The lower boundary of the hysteresis domain is determined by the necessary condition that supercritical to subcritical transition occurs downstream from the obstacle [Fig. 2(a)]. To fulfill the aforementioned condition, two different constraints must be satisfied. On one hand, as in Defina and Susin (2003), the specific energy H_u of the supercritical incoming flow must be large enough so that the flow can pass the obstacle without transition

$$H_u \geq H_u^{\min} \tag{2}$$

On the other hand, the supercritical flow downstream from the obstacle must be strong enough to backstop the downstream subcritical flow. Downstream from the obstacle, the supercritical flow has a specific energy $H_{Rd} = H_u - \Delta H_u$, and ΔH_u is the loss of specific energy owing to the channel geometry variation and/or energy dissipation, and a water depth Y_{Rd} (Valiani and Caleffi 2008). To ensure that the supercritical flow can actually establish downstream from the obstacle, the momentum of the supercritical flow at section C_d must be greater than the momentum of the downstream subcritical flow, i.e., Y_{Rd} must be smaller than the conjugate depth, Y_d^{conj}, of the downstream subcritical flow. This is equivalent to imposing $H_{Rd} \geq H_d^{\text{conj}}$, with H_d^{conj} as the specific energy at the section C_d when water depth is Y_d^{conj}. Therefore, this condition can be written as

$$H_u \geq H_d^{\text{conj}} + \Delta H_u \tag{3}$$

On the whole, the lower boundary of the hysteresis domain is

$$H_u \geq \max\{H_u^{\min}, H_d^{\text{conj}} + \Delta H_u\} \tag{4}$$

Although trivial, it is interesting to estimate the absolute lower boundary of the hysteresis domain, i.e., the minimum energy of the supercritical incoming flow, for given downstream subcritical flow, below which hysteresis cannot occur, no matter the shape and size of the obstacle. This threshold condition is found when the obstacle is negligibly small so that (1) Eq. (2) is certainly satisfied, and (2) $\Delta H_u = 0$. With this, Eq. (4) reduces to $H_u \geq H_d^{\text{conj}}$ or, equivalently, to

$$Y_u \leq Y_d^{\text{conj}} \quad \text{or} \quad Y_d \leq Y_u^{\text{conj}} \tag{5}$$

The continuity equation between section C_u and section C_d can be written as

$$\frac{Y_d}{Y_u} = \frac{\mathsf{F}_u^{2/3}}{\mathsf{F}_d^{2/3}} \tag{6}$$

where F_j $(j = u, d)$ represents the Froude number. Using Eq. (6) and the conjugate depth relationship

$$2Y_j^{\text{conj}} = R_j Y_j \tag{7}$$

with

$$R_j = -1 + \sqrt{1 + 8\mathsf{F}_j^2} \tag{8}$$

Eq. (5) can be rearranged to read

$$\mathsf{F}_d \geq \mathsf{F}_u (2/R_u)^{3/2} \quad \text{or} \quad \mathsf{F}_u \geq \mathsf{F}_d (2/R_d)^{3/2} \tag{9}$$

which represents the absolute lower boundary of the hysteresis domain. As per Eq. (9), at moderately small F_d (e.g., $\mathsf{F}_d <$ 0.1–0.2), the incoming flow must have an unbelievably high F_u (i.e., $\mathsf{F}_u >$ 10–40) for hysteresis can occur. This is possibly one reason why hysteresis is seldom observed in the presence of a downstream subcritical flow with low Froude number.

Upper Boundary of the Hysteresis Domain

The upper boundary of the hysteresis domain is determined by the necessary condition that supercritical to subcritical transition occurs upstream of the obstacle [Fig. 2(b)].

Upstream of the obstacle, the subcritical flow has a water depth Y_{Su} such that the specific energy is $H_{Su} = H_d + \Delta H_d$, in which ΔH_d is the loss of specific energy resulting from channel geometry variation and/or energy dissipation. To ensure that the subcritical flow can actually establish upstream of the obstacle, the momentum of the subcritical flow just upstream of the obstacle must be greater than or equal to the momentum of the incoming supercritical flow, i.e., Y_{Su} must be greater than the conjugate depth, Y_u^{conj} of the incoming supercritical flow. This is equivalent to imposing that the specific energy H_{Su}^{conj}, at the section C_u upstream of the obstacle (Fig. 2), computed using the water depth Y_{Su}^{conj}, be greater than the specific energy of the incoming flow. Therefore, the upper boundary of the hysteresis domain is

$$H_u \leq H_{Su}^{\text{conj}} \tag{10}$$

If Eq. (1) is not satisfied, i.e., in the absence of downstream subcritical flow or when the downstream subcritical flow has not enough energy to propagate upstream of the obstacle, one has to prescribe the critical flow condition, $H_d = H_d^{\text{crit}}$, at section C_d. In this case, the two possible stable states are those of Figs. 1(a and b).

Hysteresis Domain

On the whole, the hysteresis (or double solution) domain is defined by the lower boundary Eq. (4) and upper boundary Eq. (10), as

$$\underbrace{\max\{H_u^{\min}, H_d^{\text{conj}} + \Delta H_u\}}_{\text{lower boundary}} \leq H_u \leq \underbrace{H_{Su}^{\text{conj}}}_{\text{upper boundary}} \tag{11}$$

The preceding constraint actually identifies the hysteresis region provided that, in the absence of downstream subcritical flow or when the downstream subcritical flow has not enough energy to propagate upstream of the obstacle, $\mathsf{F}_d = 1$ is conventionally assumed.

When the energy of the incoming flow is within the aforementioned interval, then the history of the flow plays a crucial role, because it determines the flow configuration that actually establishes across the obstruction.

Importantly, the present theory includes the theory proposed by Defina and Susin (2003). To show this, first consider the upper boundary of the hysteresis region, $H_u = H_{Su}^{\text{conj}}$. This condition is equivalent to $Y_u = Y_{Su}^{\text{conj}}$, i.e., the flow depth of the supercritical and subcritical flow at section C_u are related to each other through the conjugate depth equation. Therefore, $H_{Su} = H_u - \Delta E_{\text{jump}}$, and hence

$$H_{Su} = H_{Su}^{\text{conj}} - \Delta E_{\text{jump}} \tag{12}$$

At the limit condition, when subcritical to supercritical transition occurs at the obstacle, H_{Su} turns out to be the minimum energy that the incoming flow must have to pass the obstacle, i.e., $H_{Su} = H_{Su}^{\min}$. With this, Eq. (12) can be rewritten as

$$H_{Su}^{\min} + \Delta E_{\text{jump}} = H_{Su}^{\text{conj}} \tag{13}$$

Now consider the lower boundary of the hysteresis region. In the absence of downstream subcritical flow, as stated previously, assume $\mathsf{F}_d = 1$, so that $H_d^{\text{conj}} = H_d = H_d^{\text{crit}}$ is the minimum specific energy of the flow. Accordingly, $H_d^{\text{conj}} + \Delta H_u$ turns out to be always smaller than or equal to $H_u^{\min}$, and hence

$$H_u = H_u^{\min} \tag{14}$$

Eqs. (13) and (14) show that the present theory reduces to that proposed by Defina and Susin (2003), that applies when critical flow conditions establishes at the obstacle or, equivalently, when downstream subcritical flow does not affect the flow through the obstacle.

Momentum Approach

Within the assumption that both upstream and downstream flow have enough energy to pass the obstacle without undergoing transition, the existence of a double solution domain can also be inferred by approaching the problem in terms of momentum, M, rather than specific energy balance equation. Applying the momentum equation between section C_u and section C_d yields

$$M_u - S_u = M_{Rd} \tag{15}$$

when considering the supercritical flow, and

$$M_{Su} - S_d = M_d \tag{16}$$

when considering the subcritical flow through the obstacle; S_u and S_d = resistances opposed by the obstruction when it is approached by supercritical or subcritical flow, respectively. So that supercritical flow can establish downstream from the obstacle, the following constraint must be satisfied:

$$M_{Rd} \geq M_d \tag{17}$$

which, with Eqs. (15) and (16), can be written as

$$M_u \geq M_{Su} - S_d + S_u \tag{18}$$

In addition, the subcritical flow can establish upstream of the obstacle only if

$$M_{Su} \geq M_u \tag{19}$$

Therefore, the dual solution domain is identified by

$$M_{Su} - S_d + S_u \leq M_u \leq M_{Su} \tag{20}$$

Eq. (20) shows that if the obstacle does not experience any resistance then, within the previously stated hypotheses, no double solution can establish, or more precisely, one double solution can exist if, and only if, $M_u = M_{Su} = M_d = M_{Rd}$.

Additionally, a dual solution domain exists only if $S_d > S_u$, i.e., if the obstacle produces a greater resistance on the subcritical flow than on the supercritical flow. This is often the case, because the resistance is mainly attributable to pressure forces acting on the flow direction, and the pressure increases with increasing water depth (e.g., Henderson 1966).

Interestingly, Eq. (20) also suggests that, if the channel geometry is the same upstream and downstream from the obstacle (which is the case of, e.g., a bridge pier), then a double solution domain can be found only if the energy dissipated by the flow through the obstacle is considered.

However, close and across an obstacle, streamlines curvature is often severe and pressure is far from being hydrostatic, so that S_d and S_u can hardly be estimated. For this reason, the use of Eq. (11) is preferred.

For illustration purposes, the proposed theory is now applied to two specific obstacles, namely a raised bed hump and a channel contraction. Eq. (11) is specified for each obstacle, and the hysteresis domain is expressed in terms of the fundamental flow parameters and geometrical characteristics of the obstruction. Both examples use a rectangular channel and neglect bed slope and bed friction.

Applications

Flow over a Raised Bed Hump

In the following, q is the flow rate per unit width (which, in this case, is the same upstream and downstream from the obstacle), g is gravity, a is the hump height, ΔE_u is the energy dissipated by the incoming flow in passing the obstacle (Fig. 3), and $H_d^{\text{crit}} = 3/2\sqrt[3]{q^2/g}$ is the critical (i.e., minimum) specific energy. For the case of flow over a raised bed hump, the minimum specific energy the upstream supercritical flow must have to flow over the step without undergoing transition is $H_u^{\min} = H_d^{\text{crit}} + \Delta H_u$. Hence $H_u^{\min}$ turns out to be always less than or equal to $H_d^{\text{conj}} + \Delta H_u$. Therefore, the lower boundary of the hysteresis domain, identified by the left inequality of Eq. (11), can be written as

$$\underbrace{Y_u\left(1+\frac{\mathsf{F}_u^2}{2}\right)}_{H_u} \geq \underbrace{Y_d^{\text{conj}}\left[1+\frac{q^2}{2g(Y_d^{\text{conj}})^3}\right]}_{H_d^{\text{conj}}} + \underbrace{a + \Delta E_u}_{\Delta H_u} \tag{21}$$

Recalling that $Y_d^{\text{conj}} = Y_d R_d/2$, Eq. (21) with Eq. (6), can be rewritten as

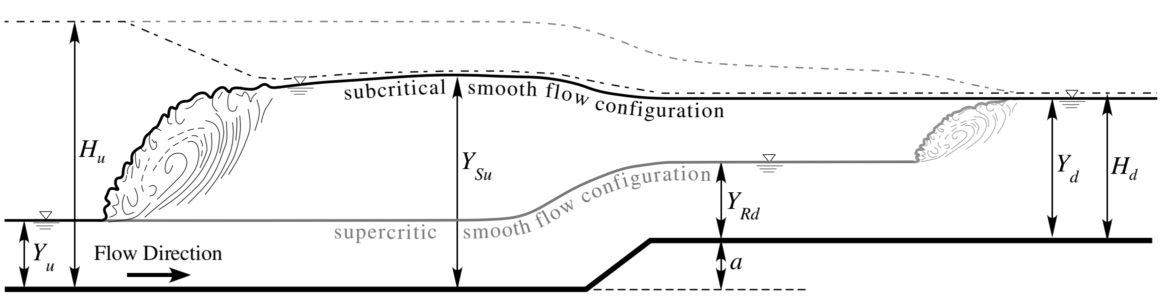

Fig. 3. Supercritical (gray lines) and subcritical (black lines) flow over a raised bed hump; gray and black dash-dotted lines denote the corresponding specific energy, H

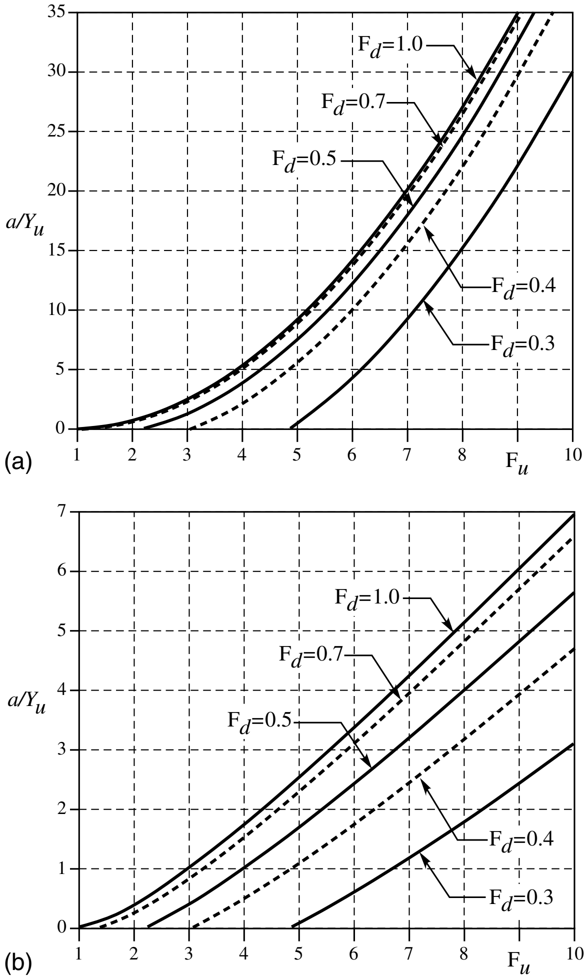

Fig. 4. Flow over a raised bed hump for $\mathsf{F}_d = 0.3, 0.4, 0.5, 0.7$, and 1: (a) lower boundary; (b) upper boundary of the hysteresis domain

$$\frac{a}{Y_u} \le 1 + \frac{\mathsf{F}_u^2}{2} - \frac{4\mathsf{F}_d^2 + R_d^3}{2R_d^2}\left(\frac{\mathsf{F}_u}{\mathsf{F}_d}\right)^{2/3} - \frac{\Delta E_u}{Y_u} \tag{22}$$

The behavior of the endpoint of Eq. (22), when $\Delta E_u = 0$, is plotted in Fig. 4(a) for some values of the downstream Froude number. The points $(\mathsf{F}_d, \mathsf{F}_u)$ when $a/Y_u = 0$ fall onto the curve given by Eq. (9). The lower boundary, in terms of the incoming flow energy, becomes an upper boundary for the nondimensional step height a/Y_u and vice versa.

The upper boundary of the hysteresis domain is expressed by the right inequality of Eq. (11). This condition can also be written as

$$H_{Su} \ge H_u^{\text{conj}} \tag{23}$$

where H_u^{conj} = specific energy at the cross section, C_u, when water depth is Y_u^{conj}

$$H_u^{\text{conj}} = Y_u^{\text{conj}}\left[1 + \frac{\mathsf{F}_u^2}{2}\left(\frac{Y_u}{Y_u^{\text{conj}}}\right)^3\right] \tag{24}$$

Because $H_{Su} = H_d + \Delta H_d$, Eq. (23) with Eq. (6), and after some algebra, can be written as

$$\frac{a}{Y_u} \ge \frac{4\mathsf{F}_u^2 + R_u^3}{2R_u^2} - \frac{\mathsf{F}_u^{2/3}}{\mathsf{F}_d^{2/3}}\left(1 + \frac{\mathsf{F}_d^2}{2}\right) - \frac{\Delta E_d}{Y_u} \tag{25}$$

The behavior of Eq. (25), when $\Delta E_d = 0$, is plotted in Fig. 4(b) for some values of the downstream Froude number. Fig. 4 shows that the hysteresis domain becomes progressively insensitive to F_d when F_d itself approaches 1, i.e., when the momentum of the downstream subcritical flow approaches its minimum value.

The hysteresis domain is then given by the constraint

$$\frac{4\mathsf{F}_u^2 + R_u^3}{2R_u^2} - \frac{\mathsf{F}_u^{2/3}}{\mathsf{F}_d^{2/3}}\left(1 + \frac{\mathsf{F}_d^2}{2}\right) - \frac{\Delta E_d}{Y_u} \le \frac{a}{Y_u}$$
$$\le 1 + \frac{\mathsf{F}_u^2}{2} - \frac{4\mathsf{F}_d^2 + R_d^3}{2R_d^2}\left(\frac{\mathsf{F}_u}{\mathsf{F}_d}\right)^{2/3} - \frac{\Delta E_u}{Y_u} \tag{26}$$

This is plotted in the $(\mathsf{F}_u, a/Y_u)$ plane for the case $\mathsf{F}_d = 0.5$ in Fig. 5(a) and in the $(\mathsf{F}_d, a/Y_u)$ plane for the case $\mathsf{F}_u = 4$ in Fig. 5(b); $\Delta E_u = \Delta E_d = 0$ are assumed in both cases.

Even at moderately small F_u, the amplitude of the hysteresis region, which is defined as the difference, Δa, between the step height at the lower and upper boundary, is considerably wide.

However, the amplitude of the hysteresis region is strongly affected by energy dissipation. The effect of energy dissipation is to move both the boundaries toward smaller values of the relative step height a/Y_u; moreover, because ΔE_u turns to be much greater

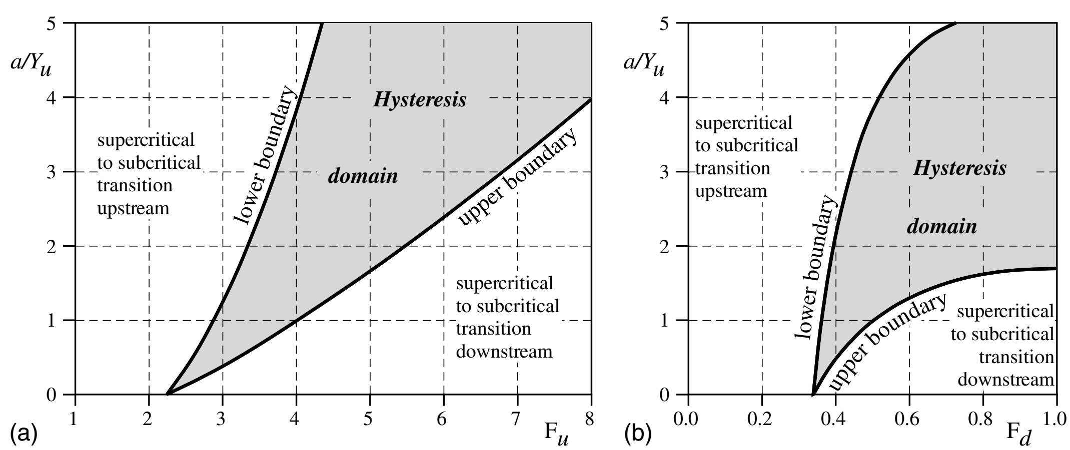

Fig. 5. Flow over a raised bed hump when $\Delta E_u = \Delta E_d = 0$ and the hysteresis domain as a function of (a) upstream Froude number, F_u, for $\mathsf{F}_d = 0.5$; (b) downstream Froude number, F_d, for $\mathsf{F}_u = 4$

than ΔE_d, the hysteresis region reduces its amplitude with increasing energy dissipation. The amount of dissipated energy, and hence the amplitude of the hysteresis region, mainly depends on the severity of the ramp (e.g., Defina and Susin 2006, Figs. 4 and 6); for an abrupt step, the two boundaries may come so near each other, that the hysteresis domain almost vanishes (e.g., Defina and Susin 2006, Fig. 6).

In addition, the amplitude of the hysteresis region can also reduce because of the cyclic instabilities that affect hydraulic jumps on nonflat bottom (Long et al. 1991; Mossa 1999; Mossa et al. 2003; Defina et al. 2008).

Finally, for given F_d, a complete hysteresis cycle can be accomplished by only changing the Froude number of the upstream supercritical flow. In other words, a stationary hydraulic jump downstream from the obstacle can be moved upstream of the bed hump by reducing F_u to less than the lower boundary, and can be pushed downstream from the obstacle by increasing F_u up to the upper boundary. On the contrary, for a given F_u and when a/Y is sufficiently large [e.g., a/Y greater than approximately 2 in Fig. 5(b)], a stationary hydraulic jump downstream from the bed hump can be moved upstream of the obstacle by reducing F_d, but cannot be pushed downstream the obstacle anymore by changing F_d.

Flow through a Channel Contraction

The one-dimensional theory to predict the flow through a channel contraction should be used with due care. Strengths and weaknesses of this simplified approach are discussed after deriving the boundaries of the hysteresis domain.

As for the raised hump, even in this case, $H_u^{\min}$ turns out to be always greater than or equal to $H_d^{\mathrm{conj}} + \Delta H_u$, so that the lower boundary of the hysteresis domain, expressed by the left inequality of Eq. (11) can be written as

$$\underbrace{Y_u\left(1+\frac{\mathsf{F}_u^2}{2}\right)}_{H_u} \geq \underbrace{Y_d^{\mathrm{conj}}\left[1+\frac{q_d^2}{2g(Y_d^{\mathrm{conj}})^3}\right]}_{H_d^{\mathrm{conj}}} + \underbrace{\Delta E_u}_{\Delta H_u} \tag{27}$$

where q_d = flow rate per unit width downstream from the obstacle; and ΔE_u = energy dissipated by the incoming flow through the obstacle.

In this case, the continuity equation gives

$$\frac{Y_d}{Y_u} = \left(\frac{B\mathsf{F}_u}{b\mathsf{F}_d}\right)^{2/3} \tag{28}$$

with B and b = upstream and downstream channel width, respectively. Eq. (27), with Eq. (28), can be rewritten as

$$\frac{b}{B} \geq \frac{\mathsf{F}_u}{\mathsf{F}_d}\left[\left(1+\frac{\mathsf{F}_u^2}{2}-\frac{\Delta E_u}{Y_u}\right)\frac{2R_d^2}{4\mathsf{F}_d^2+R_d^3}\right]^{-3/2} \tag{29}$$

The behavior of the endpoint of Eq. (29), i.e., the lower boundary of the hysteresis region, is plotted in Fig. 6(a) for some values of the downstream Froude number and when $\Delta E_u = 0$.

The upper boundary of the hysteresis domain is expressed by the right side of Eq. (11) or, equivalently, by Eq. (23). Because $H_{Su} = H_d + \Delta E_d$, with ΔE_d as energy dissipated in the contraction, Eq. (23) can be written as

$$H_d + \Delta E_d \geq Y_u^{\mathrm{conj}}\left[1+\frac{\mathsf{F}_u^2}{2}\left(\frac{Y_u}{Y_u^{\mathrm{conj}}}\right)^3\right] \tag{30}$$

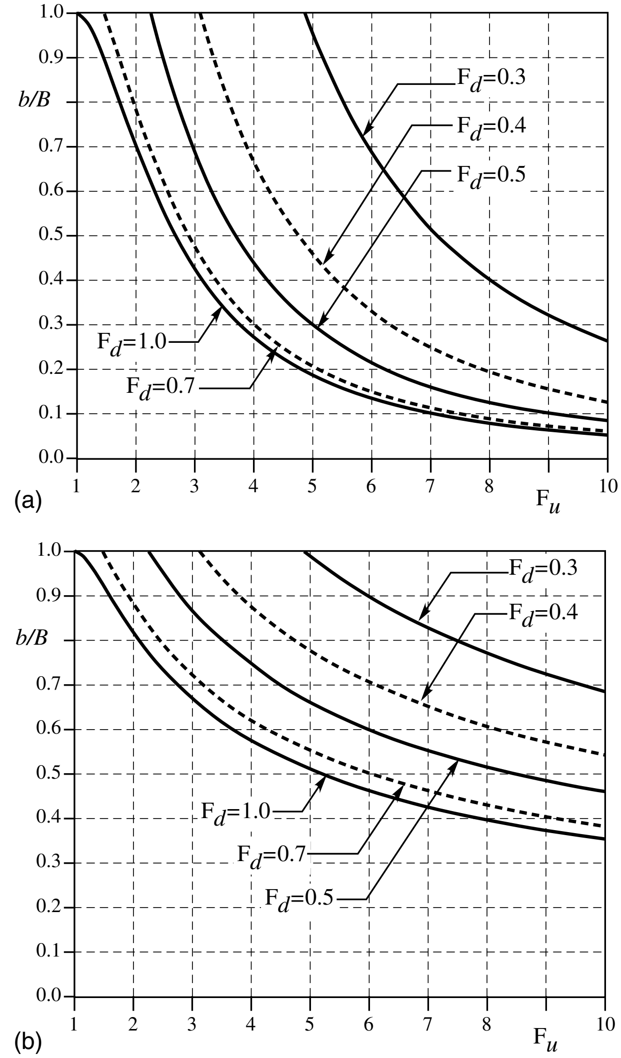

Fig. 6. Flow in a channel contraction for $\mathsf{F}_d = 0.3$, 0.4, 0.5, 0.7, and 1: (a) lower boundary; (b) upper boundary of the hysteresis domain

Eq. (30), after some algebra, and recalling Eq. (28), is rewritten as

$$\frac{b}{B} \leq \frac{\mathsf{F}_u}{\mathsf{F}_d}\left(1+\frac{\mathsf{F}_d^2}{2}\right)^{3/2}\left(\frac{4\mathsf{F}_u^2+R_u^3}{2R_u^2}-\frac{\Delta E_d}{Y_u}\right)^{-3/2} \tag{31}$$

The behavior of the endpoint of Eq. (31), i.e., the upper boundary of the hysteresis region, is plotted in Fig. 6(b) for some values of the downstream Froude number and when $\Delta E_d = 0$.

As for the raised bed hump, both the boundaries of the hysteresis domain become progressively insensitive to F_d when F_d is close to 1, i.e., when the momentum of the downstream subcritical flow approaches its minimum value.

The hysteresis domain, as predicted by the one-dimensional theory, is then given by the constraint

$$\frac{\mathsf{F}_u}{\mathsf{F}_d}\left[\left(1+\frac{\mathsf{F}_u^2}{2}-\frac{\Delta E_u}{Y_u}\right)\frac{2R_d^2}{4\mathsf{F}_d^2+R_d^3}\right]^{-3/2} \leq \frac{b}{B} \leq \frac{\mathsf{F}_u}{\mathsf{F}_d}\left(1+\frac{\mathsf{F}_d^2}{2}\right)^{3/2}\left(\frac{4\mathsf{F}_u^2+R_u^3}{2R_u^2}-\frac{\Delta E_d}{Y_u}\right)^{-3/2} \tag{32}$$

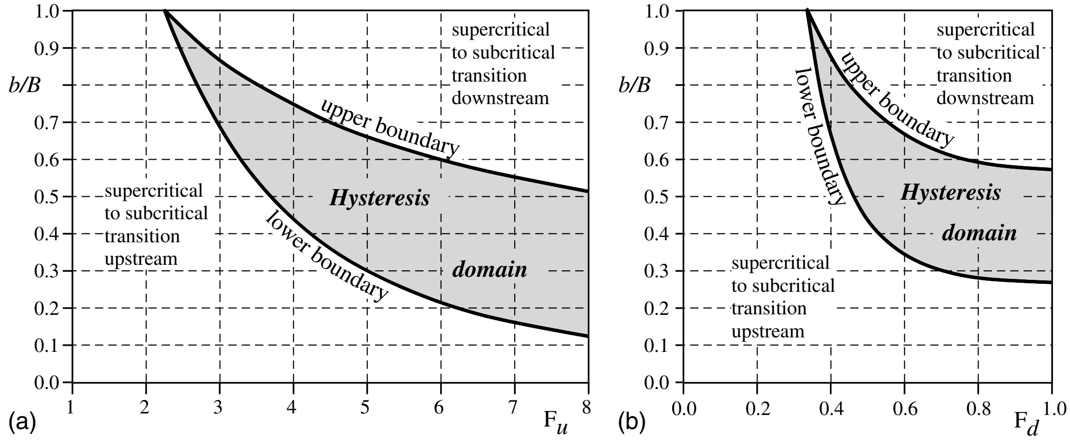

Fig. 7. Flow in a channel contraction when $\Delta E_u = \Delta E_d = 0$ and the hysteresis domain as a function of (a) upstream Froude number, F_u, for $\mathsf{F}_d = 0.5$; (b) downstream Froude number, F_d, for $\mathsf{F}_u = 4$

which is plotted in the $(\mathsf{F}_u, a/Y_u)$ plane for the case $\mathsf{F}_d = 0.5$ in Fig. 7(a), and in the $(\mathsf{F}_d, a/Y_u)$ plane for the case $\mathsf{F}_u = 4$ in Fig. 7(b).

Similarly to the raised bed hump, a complete hysteresis cycle cannot be accomplished by only changing the Froude number of the downstream subcritical flow when b/B is within a specific range that depends on F_u (e.g., $0.27 < b/B < 0.58$ for $\mathsf{F}_u = 4$, as shown in Fig. 7).

The one-dimensional approach is able to capture the key aspects of the hysteresis phenomenon in a channel contraction and can provide, at least at the leading order of approximation, quantitative information on the boundaries of the hysteresis domain. However, it must be stressed that the hysteresis domain plotted in Fig. 7 is the ideal one, because it is estimated by neglecting energy dissipation. Similarly to the raised bed hump, energy dissipation acts to reduce the amplitude of the hysteresis domain. Not less important, the flow in the contraction has a distinct two-dimensional character with steady, oblique shock waves and localized patches of subcritical flow (Akers and Bokhove 2008; Defina and Viero 2010; Viero et al. 2013b), and the flow is also strongly affected by three-dimensional phenomena related to free surface slope and curvature (i.e., nonhydrostatic pressure). All these effects move both the boundaries of the hysteresis region toward greater [Akers and Bokhove 2008, Fig. 6(a)] or smaller (Defina and Viero 2010, Fig. 14) values of the ratio b/B, possibly depending on the flow depth-to-contraction width ratio, Y/b (Defina and Susin 2006, Fig. 21).

Conclusions

In this paper, the hysteretic behavior of a steady, supercritical open-channel flow approaching an obstacle was examined for the case when subcritical flow can establish close downstream from the obstacle. A simple, one-dimensional theoretical approach to predict conditions for the occurrence of hydraulic hysteresis and to evaluate the boundaries of the hysteresis domain was proposed. The proposed theory was shown to reduce to the theory proposed by Defina and Susin (2003) when the subcritical flow downstream from the obstacle cannot affect the upstream flow, or when it cannot establish at all.

For the cases of flow over a raised bed hump and through a channel contraction, the amplitude of the hysteresis region is large so that hysteresis is likely to occur in many practical cases, even at moderately small values of the Froude number of the incoming supercritical flow. However, energy dissipation in the flow through an obstacle has a large impact on the shape and size of the hysteresis domain. Hence, energy dissipation has to be carefully evaluated on a case-by-case basis, because it strongly depends on the geometry of the obstruction.

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