Polynomial approximation and quadrature on geographic rectangles

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Abstract

Using some recent results on *subperiodic* trigonometric interpolation and quadrature, and the theory of admissible meshes for multivariate polynomial approximation, we study product Gaussian quadrature, hyperinterpolation and interpolation on some regions of $\mathbb{S}^d$, $d \geq 2$. Such regions include caps, zones, slices and more generally spherical rectangles defined by longitudes and (co)latitudes (geographic rectangles). We provide the corresponding Matlab codes and discuss several numerical examples on $\mathbb{S}^2$.

Key words: Algebraic cubature, geographic (spherical) rectangles, spherical caps, hyperinterpolation, interpolation, weakly admissible meshes, approximate Fekete points, discrete Leja points.

1 Introduction

In this work we study new rules for numerical cubature and define new algorithms to determine good point sets for interpolation on some regions of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ with $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$, $\|\cdot\|_2$ being the euclidean norm in $\mathbb{R}^{d+1}$.

Many cubature and interpolation point sets are known on the whole sphere. Well-known cubature sets are the so-called *spherical L-designs*, introduced by Delsarte and others [12], that provide cubature rules with a fixed algebraic

⋆ Work supported by the "ex-60%" funds and by the biennial projects CPDA124755/CPDA143275 of the University of Padova, and by the INdAM GNCS.

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degree of exactness and equal weights. Low-cardinality spherical designs (in particular close to minimal ones) are the most interesting from the computational point of view, see e.g. [37] and the references therein. Reimer in [29] and Sloan and Womersley in [31], [32], studied the so called *extremal points*, determining good points for interpolation and cubature; for a survey on this topic, see [21]. Later, Hesse and Womersley in [22] studied numerical integration over caps in $\mathbb{S}^d$ giving regularity results and a lower bound on the cardinality of rules with positive nodes and a certain degree of exactness n. Moreover, they provided rules that have $O(n^d)$ points and degree of exactness n. In particular, exploiting symmetry they presented a rule for caps of $\mathbb{S}^2$ that has $n^2/2 + \mathcal{O}(n)$ points. Using a different approach, Mhaskar showed in [26], under some mild requirements, existence of certain cubature rules having scattered data as nodes, on domains such as spherical caps and spherical collars. In [27], he generalized these results to more general compact sets of the sphere.

In [2] Beckmann and others studied integration over spherical triangles providing numerical cubature rules via certain reproducing kernels techniques.

In this paper, we study *cubature rules* of product Gaussian type on regions of $\mathbb{S}^d$ defined by longitudes and (co)latitudes ("geographic rectangles"), with caps and collars (also called zones) as special cases. In particular we will determine cubature rules that are exact on all algebraic polynomials of total degree not greater than n, by using "subperiodic" trigonometric Gaussian rules, that are rules with $n+1$ angular nodes, exact on trigonometric polynomials of degree not greater than n on subintervals of the period, $[\alpha, \beta] \subseteq [0, 2\pi]$ (see [8,9,10,11]). We show the quality of the cubature rules by numerical tests on some examples with integrands on $\mathbb{S}^2$ and $\mathbb{S}^4$.

Then, we study function approximation on such regions of the sphere. The availability of algebraic cubature formulas with positive weights, gives the possibility of constructing total-degree *hyperinterpolation* polynomials, that are ultimately truncated and discretized orthogonal polynomial expansions. Such a technique was introduced by Sloan in the seminal paper [30], and then developed in various contexts, as a valid alternative to polynomial interpolation; see, e.g., [19,20,33] and references therein. Orthogonal polynomials on the relevant regions, which are a key ingredient of hyperinterpolation, are here computed by numerical linear algebra methods (consecutive QR factorizations of weighted Vandermonde matrices).

Such a connection with *hyperinterpolation on regions* of the sphere is one of the main motivations to construct cubature formulas that are *exact* on total-degree polynomials. Indeed, concerning pure cubature some preliminary numerical experiments seem to suggest that near-exactness (say, with an error not far from machine precision) can be obtained also by product Gauss-Legendre quadrature in the angular variables, and even that a subsampling phenomenon can arise (provided that the angular intervals are sufficiently small). Such numerical observations, that go beyond the scope of the present paper, deserve in any case further deepening, as well as a comprehensive future study from both the computational and the theoretical sides.

On the other hand, the recently developed theory of subperiodic trigonometric interpolation, cf. [5], allows us to construct *Weakly Admissible Meshes* (shortened as WAMs) on geographic rectangles. The theory of WAMs, which are essentially special sequences of finite norming sets for polynomial spaces, has been introduced by Calvi and Levenberg in the seminal paper [7], and has been developed by various researchers in the last years; cf., e.g., [4,23,28]. In the present context, product-type WAMs on geographic rectangles are straightforward to compute for any degree, and can be used directly for least-squares approximation of continuous functions (near-optimal in the uniform norm). Furthermore, by the algorithms described in [34], we extract from such WAMs the so called *Approximate Fekete Points* and *Discrete Leja Points*. Both these point sets are good for polynomial interpolation, since they are asymptotically distributed as the Fekete points of the region and enjoy a slowly increasing Lebesgue constant; cf., e.g., [3].

All the Matlab codes used for the numerical experiments are available at the web site [6].

2 Some basic definitions and results

As preliminaries, it is important to give a quick glance to some well-known facts that will be important in the next sections. We will denote by $\mathbb{P}_n(\Omega)$ the space of the algebraic polynomials of total degree n in Ω. A standard parametrization of the sphere $\mathbb{S}^d$ is provided by *generalized spherical coordinates* as

$$x_k = \begin{cases} \cos(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), k = 1, \\ \sin(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), k = 2, \\ \cos(\theta_{d+2-k}) \cdot \prod_{j=1}^{d+1-k} \sin(\theta_j), k = 3, \dots, d+1 \end{cases} \tag{1}$$

with the notation $\prod_{j=1}^{0} \sin(\theta_j) \equiv 1$. Alternative choices in the range of the angles are

- $\theta_d \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$ for $k = 1, \dots, d-1$,
- $\theta_{d-1} \in [-\pi, \pi]$ and $\theta_k \in [0, \pi]$ for $k = 1, \dots, d-2, d$.

We point out that depending on the authors this parametrization may change. Independently of the choice of the range of the angles, the volume measure is expressed as

$$d\mu(\mathbf{x}) = \prod_{k=1}^{d-1} \sin^{d-k}(\theta_k) d\theta_k.$$

We will denote with $\xi = \xi(\theta_1, \dots, \theta_d)$ the transformation from generalized spherical coordinates to cartesian coordinates.

In the case $d = 2$, setting $\theta := \theta_1$, $\phi := \theta_2$, we have in particular the usual spherical coordinates transformation $\xi = \xi(\theta, \phi)$ defined by

$$\begin{aligned} x_1 &= \cos(\phi) \cdot \sin(\theta), \\ x_2 &= \sin(\phi) \cdot \sin(\theta), \\ x_3 &= \cos(\theta) \end{aligned} \tag{2}$$

with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, and volume measure $\sin(\theta)$.

The *spherical harmonics* $\mathbb{H}_k(\mathbb{S}^d)$ of (exact) degree k (cf. [1, p.133]) are widely used to determine a basis on the sphere $\mathbb{S}^d$. They are homogenous polynomials of degree k

$$p(x_1, \dots, x_{d+1}) = \sum_{b_1+\dots+b_d=k} a_{b_1,\dots,b_{d+1}} x_1^{b_1} \dots x_{d+1}^{b_{d+1}}$$

such that

$$\Delta p(x_1, \dots, x_{d+1}) = 0$$

for all $(x_1, \dots, x_{d+1}) \in \mathbb{S}^d$. It is possible to prove that the vector space $\mathbb{H}_k(\mathbb{S}^d)$ has dimension

$$\mathbf{Z}(d,k) = \begin{cases} 1, & k = 0 \\ \frac{(2k+d-1)\Gamma(k+d-1)}{\Gamma(d)\Gamma(k+1)}, & k = 1, \dots \end{cases}$$

Let $\mathbb{P}_n(\mathbb{S}^d)$ be the space of univariate polynomials whose degree is inferior or equal to n.

If $\{\mathbf{Y}_{k,1}^{(d)}, \dots, \mathbf{Y}_{k,\mathbf{Z}(d,k)}^{(d)}\}$ denotes an (arbitrary) real-valued $L_2(\mathbb{S}^d)$-orthonormal basis of $\mathbb{H}_k(\mathbb{S}^d)$ then $\{\mathbf{Y}_{k,1}^{(d)}, \dots, \mathbf{Y}_{k,\mathbf{Z}(d,k)}^{(d)}\}_{k=0,\dots,n}$ is a basis for $\mathbb{P}_n(\mathbb{S}^d) = \oplus_{k=0}^{n} \mathbb{H}_k(\mathbb{S}^d)$. It is possible to prove that

$$\sum_{k=0}^{n} \mathbf{Z}(d,k) = \frac{(2n+d)\Gamma(n+d)}{\Gamma(d+1)\Gamma(n+1)} \sim (n+1)^d.$$

Let

$$\mathbf{P}_{n,d}(t) = n!\Gamma\left(\frac{d-1}{2}\right)\sum_{k=0}^{\lfloor n/2 \rfloor}(-1)^k \frac{(1-t^2)^k t^{n-2k}}{4^k k!(n-2k)!\Gamma(k+(d-1)/2)}$$

the Legendre polynomial of degree n, and

$$\mathbf{P}_{n,d}^{(m)}(t) = \frac{d^m}{dt^m}\mathbf{P}_{n,d}(t).$$

For $d \geq 3$ the *associated Legendre* function is defined as [1, p.76]

$$\mathbf{P}_{n,d,m}(t) = \frac{(n+d-3)!}{(n+m+d-3)!}(1-t^2)^{m/2}\mathbf{P}_{n,d}^{(m)}(t),\ \ t \in [-1,1],$$

where $0 \leq m \leq n$.

The standard basis for $\mathbb{H}_n(\mathbb{S}^2)$ [1, p.133] is provided by $\{\mathbf{Y}_{l,m}^{(2)}\}_{l=0,\ldots,n,m=1,\ldots,l}$ with

$$\begin{cases} \mathbf{Y}_{l,1}^{(2)}(\xi) = c_l \mathbf{P}_l(\cos(\theta)) \\ \mathbf{Y}_{l,2m}^{(2)}(\xi) = c_{l,m}\mathbf{P}_l^{(m)}(\cos(\theta))\cos(m\phi), m = 1,\ldots,l \\ \mathbf{Y}_{l,2m+1}^{(2)}(\xi) = c_{l,m}\mathbf{P}_l^{(m)}(\cos(\theta))\sin(m\phi), m = 1,\ldots,l \end{cases} \tag{3}$$

where $\xi = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$, $\mathbf{P}_n = \mathbf{P}_{n,3}$,

$$\mathbf{P}_l^{(m)}(t) = (-1)^m(1-t^2)^{m/2}\frac{d^m}{dt^m}\mathbf{P}_l(t) = (-1)^m\frac{(l+m)!}{l!}\mathbf{P}_{l,3,m}(t),$$

and $c_l = \sqrt{\frac{2l+1}{4\pi}}$, $c_{l,m} = \sqrt{\frac{(2l+1)(l-m)!}{2\pi(l+m)!}}$.

The *normalized Legendre* function is [1, p.81]

$$\tilde{\mathbf{P}}_{n,d,m}(t) = \frac{(n+d-3)!}{n!\Gamma((d-1)/2)}\left[\frac{(2n+d-2)(n-m)!}{2^{d-2}(n+d+m-3)!}\right]^{1/2}(1-t^2)^{m/2}\mathbf{P}_{n,d}^{(m)}(t),\ \ t \in [-1,1].$$

This function is useful for determining a basis for $\mathbb{H}_k(\mathbb{S}^d)$, since starting from (3), the following theorem holds (cf. [1]),

Theorem 1 *Let $\{\mathbf{Y}_{k,j}^{(d-1)}\}_{1\leq j\leq \mathbf{Z}(d-1,k)\}}$ be an orthonormal basis for $\mathbb{H}_k(\mathbb{S}^{d-1})$, with $0 \leq k \leq n$, then an orthonormal basis of $\mathbb{H}_k(\mathbb{S}^d)$ is*

$$\{\tilde{\mathbf{P}}_{n,d+1,k}(t)\mathbf{Y}_{k,j}^{(d-1)}(\xi_{d-1}) : 1 \leq j \leq \mathbf{Z}(d-1), 0 \leq k \leq n\}$$

with $\xi_{d-1} \in \mathbb{S}^{d-1}$, $\xi_d = t\mathbf{e}_{d+1} + \sqrt{(1-t^2)}(\xi_{d-1}, 0) \in \mathbb{S}^d$, and $\{\mathbf{e}_k\}_{k=1,\ldots,d+1}$ the canonical basis of $\mathbb{R}^{d+1}$.

3 Trigonometric Gaussian quadrature on subintervals of the period

In [9] (and with a different proof in [11]), the authors introduced *subperiodic trigonometric gaussian formulas*, i.e. quadrature rules with $n+1$ nodes (angles) and positive weights, exact on the space of trigonometric polynomials of degree n, namely

$$\mathbb{T}_n([-\omega,\omega]) = \text{span}\{1, \cos(k\theta), \sin(k\theta)\}, 1 \le k \le n, \theta \in [-\omega,\omega],$$

where $0 < \omega \le \pi$.

For the reader's convenience, we report the main result of [9].

Theorem 2 *Let $\{(\xi_j, \lambda_j)\}_{1\le j\le n+1}$, be the nodes and positive weights of the algebraic gaussian rule for the weight function*

$$s(x) = \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}},\ \ x \in (-1,1).$$

Then

$$\int_{-\omega}^{\omega} f(\theta)d\theta = \sum_{j=1}^{n+1} \lambda_j f(\theta_j), \forall f \in \mathbb{T}_n([-\omega,\omega]),\ 0<\omega\le\pi,$$

where

$$\theta_j = 2\arcsin(\sin(\omega/2)\xi_j) \in (-\omega,\omega),\ \ j=1,\dots,n+1.$$

In this paper we generalize this result for symmetric weight functions w, i.e. even weight functions on symmetric intervals.

Theorem 3 *Let $w : [-\omega,\omega] \to \mathbb{R}$ be a symmetric weight function and $\{\xi_j\}_{j=1,\dots,n+1}$, $\{\lambda\}_{j=1,\dots,n+1}$ be respectively the nodes and the weights of an algebraic gaussian rule relative to the symmetric weight function*

$$\tilde{s}(x) = w(2\arcsin(\sin(\omega/2)x))\frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}}, x \in (-1,1).$$

Then

$$\int_{-\omega}^{\omega} f(\theta)w(\theta)\,d\theta = \sum_{j=1}^{n+1} \lambda_j f(\theta_j), f \in \mathbb{T}_n([-\omega,\omega]) \tag{4}$$

where

$$\theta_j = 2\arcsin(\sin(\omega/2)\xi_j) \in (-\omega,\omega),\ \ j=1,\dots,n+1.$$

Proof. Let us sort in increasing order the $n+1$ nodes of the gaussian quadrature rule relative to the weight function $\tilde{s}$, i.e. $-1 < \xi_1 < \ldots < \xi_{n+1} < 1$. It is well-known that since the rule is symmetric so are the nodes, implying that $\xi_j = -\xi_{n+2-j}$ for $j = 1,\dots,n+1$, as well as the weights, i.e. $\lambda_j = \lambda_{n+2-j}$.

We observe first that since $\sin(k\theta)$ is an odd function of θ and w is even, necessarily

$$\int_{-\omega}^{\omega} \sin(k\theta)\, w(\theta) d\theta = 0.$$

On the other side, in view of the symmetry of the nodes and the weights

$$\sum_{j=-1}^{n+1} \lambda_j \; \sin(k\theta_j) = 0,$$

i.e. the quadrature rule integrates exactly the functions $\sin(k\theta)$ for $k = 1, \ldots, n$.

Thus we must only prove that the formula integrates exactly $\cos(k\theta)$ for $k = 0, \ldots, n$.

Setting $\alpha = \sin(\omega/2)$, $\theta = \theta(x) = 2\arcsin(\alpha x)$ with $x \in [-1, 1]$ we have

$$\begin{aligned}
\int_{-\omega}^{\omega} \cos(k\theta)\, w(\theta) d\theta &= \int_{-1}^{1} \cos(2k \arcsin(\alpha x))\, w(2\arcsin(\alpha x))\, \frac{2\alpha}{\sqrt{1-\alpha^2 x^2}}\, dx \\
&= \int_{-1}^{1} \cos(2k \arcsin(\alpha x))\, \tilde{s}(x)\, dx. \qquad (5)
\end{aligned}$$

Denoting with $T_k(x) = \cos(k \arccos(x))$ the Chebyshev polynomial of degree k,

$$\begin{aligned}
\cos(2k \arcsin(\alpha x)) &= \cos\left(2k\left(\frac{\pi}{2} - \arccos(\alpha x)\right)\right) \\
&= \cos(k\pi) \cos(2k \arccos(\alpha x)) - \sin(k\pi) \sin(2k \arccos(\alpha x)) \\
&= (-1)^k T_{2k}(\alpha x), \qquad (6)
\end{aligned}$$

hence

$$\int_{-\omega}^{\omega} \cos(k\theta)\, w(\theta) d\theta = (-1)^k \int_{-1}^{1} T_{2k}(\alpha x)\, \tilde{s}(x)\, dx.$$

Consequently, in view of the exactness of the $(n+1)$ points gaussian rule (w.r.t. the weight function $\tilde{s}$) and (6),

$$\begin{aligned}
\int_{-\omega}^{\omega} \cos(k\theta)\, w(\theta) d\theta &= \int_{-1}^{1} (-1)^k T_{2k}(\alpha x)\, \tilde{s}(x)\, dx = \sum_{j=1}^{n+1} \lambda_j (-1)^k T_{2k}(\alpha \xi_j) \\
&= \sum_{j=1}^{n+1} \lambda_j \cos(2k \arcsin(\alpha \xi_j)) = \sum_{j=1}^{n+1} \lambda_j \cos(k\theta_j), \;\; k = 0, \ldots, n.
\end{aligned}$$

Since the formula is exact on a basis of $\mathbb{T}_n([-\omega, \omega])$, it is exact for every $f \in \mathbb{T}_n([-\omega, \omega])$. $\square$

This generalization of Theorem 2 will be of fundamental importance for the construction of cubature rules on some regions of the sphere.

4 Numerical cubature on some regions of $\mathbb{S}^d$

In this section we consider numerical integration on regions of the sphere determined by longitudes and (co)latitudes. Let $\mathbf{a} = \{a_k\}_{k=1,\dots,d} \in [0,\pi]^{d-1} \times [0,2\pi]$, $\mathbf{b} = \{b_k\}_{k=1,\dots,d} \in [0,\pi]^{d-1} \times [0,2\pi]$ with $a_k \le b_k$ for $k = 1,\dots,d$ and define $\Omega = \Omega(\mathbf{a},\mathbf{b})$ as the set of points $\mathbf{x} = \{x_k\}_{k=1,\dots,d+1} \in \mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ such that

$$x_k = \begin{cases} \cos(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), k = 1, \\ \sin(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j), k = 2, \\ \cos(\theta_{d+2-k}) \cdot \prod_{j=1}^{d+1-k} \sin(\theta_j), k = 3,\dots,d+1 \end{cases} \tag{7}$$

with $a_j \le \theta_j \le b_j$ for $j = 1,\dots,d$. These domains will be denominate "geographic rectangles". We observe that depending on $\mathbf{a}$, $\mathbf{b}$, several well-known subdomains of the d-sphere can be defined in this way, as collars, slices and more generally spherical rectangles defined by longitudes and latitudes.

Theorem 4 *Let* $\Omega = \Omega(\mathbf{a},\mathbf{b})$ *and* $\{\theta_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$ *and* $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$ *be respectively the nodes and the weights of a gaussian subperiodic trigonometric rule on* $[a_j,b_j]$ *w.r.t. the weight function* $w(x) = 1$*, having trigonometric degree of exactness* $n+d-j$*, for* $j = 1,\dots,d$*. Then the cubature rule*

$$S_n(f) = \sum_{j_1=1}^{n+d} \dots \sum_{j_{d-1}=1}^{n+2} \sum_{j_d=1}^{n+1} \lambda_{j_1,\dots,j_d} f(\xi_{j_1,\dots,j_d})$$

where

$$\xi_{j_1,\dots,j_d} = \xi(\theta_{j_1}^{[a_1,b_1]},\dots,\theta_{j_d}^{[a_d,b_d]})$$

$$\lambda_{j_1,\dots,j_d} = \prod_{k=1}^{d} \lambda_{j_k} \sin^{d-k}(\theta_{j_k}^{[a_k,b_k]})$$

integrates exactly in Ω *every algebraic polynomial of total degree* n*.*

Proof. Remembering that $\theta_k \in [0,\pi]$ for $k = 1,\dots,d-1$, the absolute value of the jacobian determinant of the transformation from generalized spherical coordinates to cartesian coordinates $\xi = \xi(\theta_1,\dots,\theta_d)$ is $\prod_{k=1}^{d-1} |\sin^{d-k}(\theta_k)| = \prod_{k=1}^{d-1} \sin^{d-k}(\theta_k)$, hence

$$\int_{\Omega} f(\xi) d\mu_{\mathbb{S}^d}(\xi) = \int_{a_1}^{b_1} \ldots \int_{a_d}^{b_d} f(\xi(\theta_1, \ldots, \theta_d)) \prod_{k=1}^{d-1} |\sin^{d-k}(\theta_k)| d\theta_d \ldots d\theta_1$$
$$= \int_{a_1}^{b_1} \ldots \int_{a_d}^{b_d} f(\xi(\theta_1, \ldots, \theta_d)) \prod_{k=1}^{d-1} \sin^{d-k}(\theta_k) d\theta_d \ldots d\theta_1. \quad (8)$$

If we intend to prove the exactness of the rule for all polynomials of degree n, it is sufficient to show the assert for a basis of $\mathbb{P}_n(\mathbb{R}^{d+1})$, for instance the monomial basis determined by all the multivariate algebraic polynomials

$$p_{k_1,\ldots,k_{d+1}}(x_1, \ldots, x_{d+1}) = x_1^{k_1} \cdot \ldots \cdot x_{d+1}^{k_{d+1}}$$

such that $k_1 + \ldots + k_{d+1} = n$.

We observe that for any homogeneous polynomial element of such a basis we easily have

$$\begin{aligned} & p_{k_1,\ldots,k_{d+1}}(x_1, \ldots, x_{d+1}) = x_1^{k_1} \cdot \ldots \cdot x_{d+1}^{k_{d+1}} \\ = & \left(\cos(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j) \right)^{k_1} \cdot \left(\sin(\theta_d) \cdot \prod_{j=1}^{d-1} \sin(\theta_j) \right)^{k_2} \\ & \cdot \prod_{s=3}^{d+1} \left(\cos(\theta_{d+2-s}) \cdot \prod_{j=1}^{d+1-s} \sin(\theta_j) \right)^{k_s} \\ = & \left(\cos^{k_1}(\theta_d) \cdot \sin^{k_2}(\theta_d) \right) \cdot \prod_{j=1}^{d-1} \left(\sin^{k_1+\ldots+k_{d-j+1}}(\theta_j) \cos^{k_{d-j+2}}(\theta_j) \right). \end{aligned}$$

Since $k_1 + \ldots + k_{d+1} = n$, $p_{k_1,\ldots,k_{d+1}}(\xi(\theta_1, \ldots, \theta_{d+1}))$ is a trigonometric polynomial of degree at most $k_1 + \ldots + k_{d-j+2} \leq n$ in each variable θ_j for $j = 1, \ldots, d-1$ and $k_1 + k_2$ in θ_d. Consequently $p_{k_1,\ldots,k_{d+1}}$ is a trigonometric polynomial of at most degree $n + d - k$ in the variables θ_k with $k = 1, \ldots, d$. Thus, from (8), in view of the separation of variables

$$\int_{\Omega} p_{k_1,\ldots,k_{d+1}}(\xi) d\mu_{\mathbb{S}^d}(\xi) = \prod_{j=1}^{d-1} \int_{a_j}^{b_j} \left(\sin^{k_1+\ldots+k_{d-j+1}}(\theta_j) \cos^{k_{d-j+2}}(\theta_j) \right) \sin^{d-j}(\theta_j) d\theta_j$$
$$\cdot \int_{a_d}^{b_d} \left(\cos^{k_1}(\theta_d) \cdot \sin^{k_2}(\theta_d) \right) d\theta_d. \quad (9)$$

This particular structure suggests to use tensor product formulas based on the subperiodic trigonometric gaussian rules described in Theorem 3. More precisely, denoting by $\{\theta_k^{[a_j,b_j]}\}_{k=1,\ldots,n+d-j+1}$, $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\ldots,n+d-j+1}$, $j = 1, \ldots, d-1$ the nodes of a gaussian subperiodic trigonometric rule with trigonometric degree of exactness $n + d - j$ w.r.t. the weight function $w(x) = 1$ and

$\{\theta_k^{[a_d,b_d]}\}_{k=1,\dots,n+1}$, $\{\lambda_k^{[a_d,b_d]}\}_{k=1,\dots,n+1}$, the nodes of a gaussian subperiodic trigonometric rule with trigonometric degree of exactness n w.r.t. the weight function $w(x) = 1$ we finally have

$$\int_\Omega p_{k_1,\dots,k_{d+1}}(\xi)d\mu_{\mathbb{S}^d}(\xi) = \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} p_{k_1,\dots,k_{d+1}}(\xi(\theta_1,\dots,\theta_{d+1})) \prod_{k=1}^{d-1} \sin^{d-k}(\theta_k)\, d\theta_d \dots d\theta_1$$
$$= \sum_{j_1=1}^{n+d} \dots \sum_{j_{d-1}=1}^{n+2} \sum_{j_d=1}^{n+1} \lambda_{j_1,\dots,j_d} p_{k_1,\dots,k_{d+1}}(\xi_{j_1,\dots,j_d}) \tag{10}$$

where

$$\xi_{j_1,\dots,j_d} = \xi(\theta_{j_1}^{[a_1,b_1]},\dots,\theta_{j_d}^{[a_d,b_d]})$$
$$\lambda_{j_1,\dots,j_d} = \prod_{k=1}^{d} \lambda_{j_k} \sin^{d-k}(\theta_{j_k}^{[a_k,b_k]}),$$

implying that the rule (10) has algebraic degree of exactness n. $\square$

5 Improving the cardinality of the cubature rule

In this section we improve the results of the previous section to some geographic rectangles of the sphere, e.g. the caps of $\mathbb{S}^2$. For the case $d = 2$, we obtain rules algebraic degree of exactness n, with $\mathcal{O}(n^2/2)$, as in [22]. We use here the generalized spherical coordinates defined by (7) but with $[a_{d-1}, b_{d-1}] \subseteq [-\pi, \pi]$, $[a_j, b_j] \subseteq [0, \pi]$ for $j = 1, 2, \dots, d-2, d$.

Theorem 5 *Let $\Omega = \Omega(\mathbf{a}, \mathbf{b})$. Suppose $a_d = 0$ and $b_d = \pi$ and $-\pi \leq a_{d-1} = -b_{d-1} < 0$, $[a_j, b_j] \subseteq [0, \pi]$ for $j = 1, \dots, d-2$. Let*

(1) $\{\theta_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$ and $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$, $j = 1, \dots, d-2$, be respectively the nodes and the weights of a gaussian subperiodic trigonometric rule on $[a_j, b_j]$, w.r.t. $w(x) = 1$, having trigonometric degree of exactness $n + d - j$;

(2) $\{\theta_k^{[-b_{d-1},b_{d-1}]}\}_{k=1,\dots,n+1}$ and $\{\lambda_k^{[-b_{d-1},b_{d-1}]}\}_{k=1,\dots,n+1}$ are respectively the nodes and the weights of a gaussian subperiodic trigonometric rule on $[-b_{d-1}, b_{d-1}]$, w.r.t. $w(x) = |\sin x|$, having trigonometric degree of exactness n, with $\theta_k^{[-b_{d-1},b_{d-1}]} < \theta_{k+1}^{[-b_{d-1},b_{d-1}]}$ for $k = 1 \dots, n$;

(3) $\{\theta_k^{[0,2\pi]}\}_{k=1,\dots,n+1}$ and $\{\lambda_k^{[0,2\pi]}\}_{k=1,\dots,n+1}$ be respectively the nodes and the weights of a gaussian trigonometric rule on $[0, 2\pi]$, w.r.t. $w(x) = 1$, having trigonometric degree of exactness n, with $\theta_k^{[0,2\pi]} < \theta_{k+1}^{[0,2\pi]}$ for $k = 1 \dots, n$.

Then the cubature rule

$$S_n(f) = \sum_{j_1=1}^{n+d} \cdots \sum_{j_{d-2}=1}^{n+3} \sum_{j_{d-1}=1}^{n+1} \sum_{j_d=1}^{\lceil \frac{n+1}{2} \rceil} \lambda_{j_1,\ldots,j_d} f(\xi_{j_1,\ldots,j_d})$$

where

$$\xi_{j_1,\ldots,j_d} = \xi(\theta_{j_1}^{[a_1,b_1]}, \ldots, \theta_{j_d}^{[a_d,b_d]})$$
$$\lambda_{j_1,\ldots,j_d} = \prod_{k=1}^{d} \lambda_{j_k}^{[a_k,b_k]} \cdot \prod_{k=1}^{d-2} \sin^{d-k}(\theta_{j_k}^{[a_k,b_k]})$$

integrates exactly in Ω *every algebraic polynomial* $p \in \mathbb{P}_n(\Omega)$.

Proof. Using the transformation ξ from generalized spherical to cartesian coordinates (see (7)), we have

$$\int_\Omega f(\xi) d\mu_{\mathbb{S}^d}(\xi) = \int_{a_1}^{b_1} \cdots \int_{-b_{d-1}}^{b_{d-1}} \int_0^\pi f(\xi(\theta_1,\ldots,\theta_d)) \prod_{k=1}^{d-1} |\sin^{d-k}(\theta_k)| d\theta_d \ldots d\theta_1. \tag{11}$$

Modifying the generalized spherical coordinates so that $\theta_d \in [\pi, 2\pi]$, since the absolute value of the jacobian determinant of the transformation does not change, we get

$$\int_\Omega f(\xi) d\mu_{\mathbb{S}^d}(\xi) = \int_{a_1}^{b_1} \cdots \int_{-b_{d-1}}^{b_{d-1}} \int_\pi^{2\pi} f(\xi(\theta_1,\ldots,\theta_d)) \prod_{k=1}^{d-1} |\sin^{d-k}(\theta_k)| d\theta_d \ldots d\theta_1. \tag{12}$$

Summing (11) to (12) and dividing by 2,

$$\int_\Omega f(\xi) d\mu_{\mathbb{S}^d}(\xi) = \frac{1}{2} \int_{a_1}^{b_1} \cdots \int_{-b_{d-1}}^{b_{d-1}} \int_0^{2\pi} f(\xi(\theta_1,\ldots,\theta_d)) \prod_{k=1}^{d-1} |\sin^{d-k}(\theta_k)| d\theta_d \ldots d\theta_1. \tag{13}$$

We prove the exactness of the rule, showing that the quadrature rule integrates exactly the monomial basis of $\mathbb{R}^{d+1}$, i.e. the set of homogenous polynomials $\{p_{k_1,\ldots,k_{d+1}}\}$ with $k_s \geq 0$ for all $s = 1, \ldots, d+1$ and $\sum_{j=1}^{d+1} k_j \leq n$, defined as

$$p_{k_1,\ldots,k_{d+1}}(x_1, \ldots, x_{d+1}) = \prod_{s=1}^{d+1} x_s^{k_s}. \tag{14}$$

As shown in (9), $p_{k_1,\ldots,k_{d+1}}(\xi(\theta_1,\ldots,\theta_d))$ is a trigonometric polynomial of degree at most $k_1 + \ldots + k_{d-j+1} \leq n$ in each variable θ_j. More precisely

$$p_{k_1,\ldots,k_{d+1}}(\xi(\theta_1,\ldots,\theta_d)) = \prod_{j=1}^{d-1} \left(\sin^{k_1+\ldots+k_{d-j+1}}(\theta_j) \cos^{k_{d-j+2}}(\theta_j)\right) \cdot \left(\cos^{k_1}(\theta_d) \cdot \sin^{k_2}(\theta_d)\right). \tag{15}$$

By (13) and (15), the separation of variables and the transformation $\xi = \xi(\theta_1, \ldots, \theta_d)$

$$
\begin{aligned}
\int_{\Omega} p_{k_1,\dots,k_{d+1}}(\xi) d\mu_{\mathbb{S}^d}(\xi) &= \frac{1}{2}\int_{a_1}^{b_1} \dots \int_{-b_{d-1}}^{b_{d-1}} \int_0^{2\pi} p_{k_1,\dots,k_{d+1}}(\xi(\theta_1,\dots,\theta_d)) d\theta_d \dots d\theta_1 \\
&= \frac{1}{2}\prod_{j=1}^{d-1} \int_{a_k}^{b_k} \sin^{k_1+\dots+k_{d-j+1}}(\theta_j) \cos^{k_{d-j+2}}(\theta_j) |\sin^{d-j}(\theta_j)| d\theta_j \\
&\cdot \int_0^{2\pi} \cos^{k_1}(\theta_d) \cdot \sin^{k_2}(\theta_d) d\theta_d
\end{aligned} \tag{16}
$$

Since $[a_j, b_j] \subseteq [0, \pi]$ for $j = 1, \dots, d-2$, we also have $|\sin^{d-j}(\theta_j)| = \sin^{d-j}(\theta_j)$, and consequently

$$
\begin{aligned}
\int_{\Omega} p_{k_1,\dots,k_{d+1}}(\xi) d\mu_{\mathbb{S}^d}(\xi) &= \frac{1}{2}\prod_{j=1}^{d-2} \int_{a_k}^{b_k} \sin^{k_1+\dots+k_{d-j+1}}(\theta_j) \cos^{k_{d-j+2}}(\theta_j) \sin^{d-j}(\theta_j) d\theta_j \\
&\cdot \int_{-b_{d-1}}^{b_{d-1}} \sin^{k_1+k_2}(\theta_{d-1}) \cos^{k_3}(\theta_{d-1}) |\sin(\theta_{d-1})| d\theta_{d-1} \\
&\cdot \int_0^{2\pi} \cos^{k_1}(\theta_d) \cdot \sin^{k_2}(\theta_d) d\theta_d
\end{aligned} \tag{17}
$$

This particular structure suggests as in Theorem 4 to use tensor product formulas based on the subperiodic trigonometric gaussian rules described in Theorem 2 and 3.

Denoting by $\{\theta_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$, $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$, $j = 1, \dots, d-2$ the nodes of a gaussian subperiodic trigonometric rule with trigonometric degree of exactness $n+d-j$ (w.r.t. $w(x) \equiv 1$), $\{\theta_k^{[-b_{d-1},b_{d-1}]}\}_{k=1,\dots,n+1}$, $\{\lambda_k^{[-b_{d-1},b_{d-1}]}\}_{k=1,\dots,n+1}$, the nodes of a gaussian subperiodic trigonometric rule with trigonometric degree of exctness n (w.r.t. the weight function $|\sin\theta|$) and $\{\theta_k^{[0,2\pi]}\}_{k=1,\dots,n+1}$, $\{\lambda_k^{[0,2\pi]}\}_{k=1,\dots,n+1}$, the nodes of a gaussian trigonometric rule with trigonometric degree of exactness n (w.r.t. the symmetric weight function $w(x) \equiv 1$) we finally have

$$
\int_{\Omega} p_{k_1,\dots,k_{d+1}}(\xi) d\mu_{\mathbb{S}^d}(\xi) = \frac{1}{2}\sum_{j_1=1}^{n+d} \dots \sum_{j_{d-2}=1}^{n+3} \sum_{j_{d-1}=1}^{n+1} \sum_{j_d=1}^{n+1} \lambda_{j_1,\dots,j_d} p_{k_1,\dots,k_{d+1}}(\xi_{j_1,\dots,j_d}) \tag{18}
$$

where

$$
\xi_{j_1,\dots,j_d} = \xi(\theta_{j_1}^{[a_1,b_1]}, \dots, \theta_{j_{d-2}}^{[a_{d-2},b_{d-2}]}, \theta_{j_{d-1}}^{[-b_{d-1},b_{d-1}]}, \theta_{j_d}^{[0,2\pi]})
$$

$$
\lambda_{j_1,\dots,j_d} = \prod_{k=1}^{d} \lambda_{j_k}^{[a_k,b_k]} \cdot \prod_{k=1}^{d-2} \sin^{d-k}(\theta_{j_k}^{[a_k,b_k]}).
$$

Consequently the rule has algebraic degree of exactness n, since it is exact on a basis of the space $\mathbb{P}(\mathbb{R}^{d+1})$, hence also on $\mathbb{P}(\Omega)$.

Let us suppose that n is odd. By a direct check on generalized spherical coordinates (7)

$$
\begin{aligned}
&\xi(\theta_{j_1}^{[a_1,b_1]},\ldots,\theta_{j_{d-2}}^{([a_{d-2},b_{d-2}])},\theta_{j_{d-1}}^{[-b_{d-1},b_{d-1}]},\theta_{j_d}^{[0,2\pi]})\\
=&\,\xi(\theta_{j_1}^{[a_1,b_1]},\ldots,\theta_{j_{d-2}}^{[a_{d-2},b_{d-2}]},-\theta_{j_{d-1}}^{[-b_{d-1},b_{d-1}]},\theta_{j}^{[0,2\pi]}+\pi).
\end{aligned}
\tag{19}
$$

Now observe that it is not restrictive to require that $\theta_j^{[-b_{d-1},b_{d-1}]}<\theta_{j+1}^{[-b_{d-1},b_{d-1}]}$, $\theta_j^{[0,2\pi]}<\theta_{j+1}^{[0,2\pi]}$ for $j=1,\ldots,n$ and notice that the gaussian nodes $\{\theta_j^{[0,2\pi]}\}_j$ are equispaced on the arc of the circle, i.e. $\theta_j=2\pi(j-1)/(n+1)$. Thus, if $\theta_j^{[0,2\pi]}$ is a node, so is $\theta_j^{[0,2\pi]}+\pi=\theta_{j+\frac{n+1}{2}}^{[0,2\pi]}$.

Furthermore, since the weight function $|\sin\theta_{d-1}|$ is symmetric, if $\theta_{n+2-j}^{[-b_{d-1},b_{d-1}]}$, $j=1,\ldots,n+1$ is a node so is $-\theta_{n+2-j}^{[-b_{d-1},b_{d-1}]}=\theta_j^{[-b_{d-1},b_{d-1}]}$.

Hence, from (19), and these final observations

$$
\begin{aligned}
\xi_{j_1,\ldots,j_{d-1},j_d}&=\xi(\theta_{j_1}^{[a_1,b_1]},\ldots,\theta_{j_{d-2}}^{([a_{d-2},b_{d-2}])},\theta_{j_{d-1}}^{[-b_{d-1},b_{d-1}]},\theta_{j_d}^{[0,2\pi]})\\
&=\xi(\theta_{j_1}^{[a_1,b_1]},\ldots,\theta_{j_{d-2}}^{[a_{d-2},b_{d-2}]},-\theta_{j_{d-1}}^{[-b_{d-1},b_{d-1}]},\theta_{j}^{[0,2\pi]}+\pi)\\
&=\xi(\theta_{j_1}^{[a_1,b_1]},\ldots,\theta_{j_{d-2}}^{[a_{d-2},b_{d-2}]},\theta_{n+2-j_{d-1}}^{[-b_{d-1},b_{d-1}]},\theta_{j_d+\frac{n+1}{2}}^{[0,2\pi]})\\
&=\xi_{j_1,\ldots,n+2-j_{d-1},j_d+\frac{n+1}{2}}.
\end{aligned}
\tag{20}
$$

About their weights, observe that

$$
\begin{aligned}
\lambda_{j_1,\ldots,j_{d-1},j_d}&=\prod_{k=1}^{d}\lambda_{j_k}^{[a_k,b_k]}\cdot\prod_{k=1}^{d-2}\sin^{d-k}(\theta_{j_k}^{[a_k,b_k]})\\
&=\prod_{k=1}^{d-2}\lambda_{j_k}^{[a_k,b_k]}\cdot\lambda_{n+2-j_{d-1}}^{[-b_{d-1},b_{d-1}]}\cdot\lambda_{j+\frac{n+1}{2}}^{[0,2\pi]}\cdot\prod_{k=1}^{d-2}\sin^{d-k}(\theta_{j_k}^{[a_k,b_k]})\\
&=\lambda_{j_1,\ldots,n+2-j_{d-1},j_d+\frac{n+1}{2}},
\end{aligned}
\tag{21}
$$

since $\lambda_{n+2-j}^{[-b_{d-1},b_{d-1}]}=\lambda_j^{[-b_{d-1},b_{d-1}]}$ by the symmetry of the weight function $w(x)=|\sin(x)|$ and $\lambda_j^{[0,2\pi]}=\lambda_{j+\frac{n+1}{2}}^{[0,2\pi]}$ since the gaussian rule for trigonometric polynomials on the circle has equal weights. Consequently, (20) and (21) imply that

$$
\begin{aligned}
&\sum_{j_1=1}^{n+d}\cdots\sum_{j_{d-2}=1}^{n+2}\sum_{j_{d-1}=1}^{n+1}\sum_{j_d=1}^{n+1}\lambda_{j_1,\ldots,j_d}p_{k_1,\ldots,k_{d+1}}(\xi_{j_1,\ldots,j_d})\\
=&\,2\sum_{j_1=1}^{n+d}\cdots\sum_{j_{d-2}=1}^{n+2}\sum_{j_{d-1}=1}^{n+1}\sum_{j_d=1}^{(n+1)/2}\lambda_{j_1,\ldots,j_d}p_{k_1,\ldots,k_{d+1}}(\xi_{j_1,\ldots,j_d})
\end{aligned}
\tag{22}
$$

and from (18) we finally have

$$\int_{\Omega} p_{k_1,\dots,k_{d+1}}(\xi) d\mu_{\mathbb{S}^d}(\xi) = \sum_{j_1=1}^{n+d} \dots \sum_{j_{d-2}=1}^{n+3} \sum_{j_{d-1}=1}^{n+1} \sum_{j_d=1}^{(n+1)/2} \lambda_{j_1,\dots,j_d} p_{k_1,\dots,k_{d+1}}(\xi_{j_1,\dots,j_d}). \tag{23}$$

The case in which n is an even trigonometric degree of exactness follows by the same proof of n odd, but uses a gaussian rule for trigonometric polynomials on $[0, 2\pi]$, w.r.t. $w(x) \equiv 1$, with trigonometric degree of exactness $n + 1$ (and consequently also exact on any trigonometric polynomial of degree at less or equal to n), providing

$$\int_{\Omega} p(\xi) d\mu_{\mathbb{S}^d}(\xi) = \sum_{j_1=1}^{n+d} \dots \sum_{j_{d-2}=1}^{n+3} \sum_{j_{d-1}=1}^{n+1} \sum_{j_d=1}^{(n+2)/2} \lambda_{j_1,\dots,j_d} p(\xi_{j_1,\dots,j_d}). \tag{24}$$

□

Note 1 *We observe that in Theorem 5 the intervals* $[a_k, b_k] \subseteq [0, \pi]$, $k = 1, \dots, d-2$, $a_k < b_k$ *are in general not symmetric. If* $a_k = -b_k < 0$, *by checking the proof of Theorem 5, we can substitute* $\{\theta_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$ *and* $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\dots,n+d-j+1}$ *respectively with* $\{\theta_k^{[-b_j,b_j]}\}_{k=1,\dots,n+1}$ *and* $\{\lambda_k^{[-b_j,b_j]}\}_{k=1,\dots,n+1}$ *that are the nodes and the weights of a gaussian subperiodic trigonometric rule on* $[-b_j, b_j]$, *w.r.t.* $w(x) = |\sin^{d-j}(x)|$, *having trigonometric degree of exactness* n. *As result we can determine a tensorial rule* S_n *with fewer cubature nodes, still having algebraic degree of exactness* n.

Note 2 *In the case* $d = 2$, *Theorem 5 allows us to compute a quadrature rule on the cap* $\Omega = \{\mathbf{x} \in \mathbb{S}^2 : \mathbf{x} \cdot \mathbf{z} > \cos(b_1)\}$, $\mathbf{z} = (0, 0, 1)$ *being the North-Pole,* $\cdot$ *the scalar product of* $\mathbb{R}^3$, b_1 *the radius of the cap. As shown in Lemma 3.1. of [22], this result can be easily generalized for more general caps, having center* $\mathbf{z} \neq (0, 0, 1)$, *by suitably rotating the nodes. For this region we provide a tensorial cubature rule alternative to that suggested in [22], still having about* $n^2/2$ *points.*

6 Implementation and numerical experiments

The tensorial formulas proposed in the previous sections need a clever computation of the sub-trigonometric quadrature rules, w.r.t. the weight functions $w_1(x) = 1$ and $w_2(x) = |\sin(x)|$. The case involving the Legendre weight w_1 was studied in [8], [9], where the authors also provided the Matlab routine `trigauss` that determines the nodes and the weights of the relative rule. In the present paper, also in view of Theorem 3, we consider the same problem but involving w_2, again making available to users the Matlab code `trigauss_abssin`.

As in `trigauss`, the nodes and the weights of the subperiodic trigonometric gaussian rule w.r.t. w_2 can be obtained by a technique that requires first the computation of the moments

$$m_k = \int_{-1}^{1} T_k(x)\tilde{s}(x)dx, k = 0, \ldots, 2n+1$$

where $T_k(x) = \cos(k \arccos(x))$ is the k-th Chebyshev polynomial and as indicated by Thm. 3, for $\omega \in [0, \pi]$

$$\begin{aligned}
\tilde{s}(x) &= |\sin(2\arcsin(\sin(\omega/2)x))| \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}} \\
&= 2|\sin(\arcsin(\sin(\omega/2)x))||\cos(\arcsin(\sin(\omega/2)x))| \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}} \\
&= 2|(\sin(\omega/2))x| \sqrt{1-\sin^2(\arcsin(\sin(\omega/2)x))} \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}} \\
&= 4\sin^2(\omega/2)|x|. \qquad (25)
\end{aligned}$$

Using the Chebyshev algorithm, implemented by the Matlab routine `chebyshev` (cf. [16], [17]) one determines the coefficients of the three terms recurrence and finally obtain the quadrature nodes and weights by Golub-Welsch algorithm (see for instance the Matlab codes `gauss` (cf. [16], [17]) or its fast variant for symmetric weight functions `SymmMw` [25]).

About the moment computation, we observe, that since the weight function $\tilde{s}(x) = (\sin^2(\omega/2))|x|$ is even (i.e. symmetric) and the Chebyshev polynomial $T_k(x) = \cos(k \cdot \arccos(x))$ is an odd function for odd k, thus

$$m_k = \int_{-1}^{1} T_k(x)\tilde{s}(x)dx = 0, \;\; k \text{ odd} \qquad (26)$$

For $k = 0$, a direct computation gives

$$m_0 = \int_{-1}^{1} 4(\sin^2(\omega/2))|x|dx = 8 \cdot \sin^2(\omega/2) \int_0^1 x\,dx = 4\sin^2(\omega/2). \qquad (27)$$

Furthermore, for $k = 2j$, $j = 1, \ldots, n$, since T_k is an even function for even k, $T_1(x) = x$ and

$$T_k(x) \cdot T_m(x) = \frac{1}{2}\left(T_{k+m}(x) + T_{|k-m|}(x)\right)$$

we get from (25) and $T_k(x)|x| = T_k(-x)|-x|$

$$\begin{aligned} m_k &= \int_{-1}^{1} T_k(x)\tilde{s}(x)dx = 4\sin^2(\omega/2)\int_{-1}^{1} T_k(x)|x|dx \\ &= 8\sin^2(\omega/2)\int_0^1 T_k(x)\cdot T_1(x)dx \\ &= 4\sin^2(\omega/2)\int_0^1 (T_{k+1}(x)+T_{k-1}(x))\,dx. \end{aligned} \tag{28}$$

Since

$$\int T_k(x)dx = \frac{kT_{k+1}(x)}{k^2-1} - \frac{xT_k(x)}{k-1}$$

and $T_k(1) = 1$, $T_{2j}(0) = (-1)^j$, applying the fundamental theorem of calculus, we have for $k = 2j+1$, $j = 1, \dots, n$

$$\begin{aligned} \int_0^1 T_{2j+1}(x)dx &= \frac{(2j+1)(T_{2j+2}(1) - T_{2j+2}(0))}{(2j+1)^2-1} - \frac{T_{2j}(1)}{2j} \\ &= \frac{(2j+1)(1-(-1)^{j+1})}{(2j+1)^2-1} - \frac{1}{2j} \end{aligned} \tag{29}$$

while, being $T_1(x) = x$, we directly have $\int_0^1 T_1(x)dx = 1/2$. Consequently, setting $\alpha_0 = 1/2$ and

$$\alpha_j = \int_0^1 T_{2j+1}(x)dx = \frac{(2j+1)(1-(-1)^{j+1})}{(2j+1)^2-1} - \frac{1}{2j}, j = 1, \dots, n$$

by (27), (26), (28) we conclude that

$$m_k = \begin{cases} 4\sin^2(\omega/2), k = 0, \\ 0, k \text{ odd}, \\ (4\sin^2(\omega/2))\cdot(\alpha_{k/2} + \alpha_{k/2-1}), k \text{ even}. \end{cases} \tag{30}$$

The case in which the weight function is $w(x) = |\sin^k(x)|$, $k > 1$, is more complicated, and we have computed the needed moments by the adaptive routine `quadvgk`. Starting from these observations, we have easily implemented in Matlab the tensorial rules on caps or more generally rectangles of longitude-latitude type that we have introduced in Thm. 2 and Thm. 3, cf. [6].

As numerical tests, we consider the cubature of the functions

$$
\begin{aligned}
f_1(\mathbf{x}) &= \exp\left(-x^2 - 100\,y^2 - 0.5\,z^2\right),\\
f_2(\mathbf{x}) &= \sin\left(-x^2 - 100\,y^2 - 0.5\,z^2\right),\\
f_3(\mathbf{x}) &= \max(1/4 - ((x - 1/\sqrt{5})^2 + (y - 2/\sqrt{5})^2 + (z - 2/\sqrt{5})^2), 0))^p,\ \ p = 3
\end{aligned}
$$

While in [22], Hesse and Womersley studied cubature on the cap $\mathcal{C}$ centered on the North-Pole and with radius $\pi/3$, in this paper the integration domain is the spherical rectangle $\Omega = \Omega([0,\ \pi/2; \pi/6,\ \pi/3]) \subset \mathcal{C}$.

The reference integrals were computed by formulas with high degree of exectness, and are

$$
\begin{aligned}
I(f_1) &= 2.221882314846131135 \cdot 10^{-2}\\
I(f_2) &= -4.684511626608869883 \cdot 10^{-2}\\
I(f_3) &= 1.817581787039426657 \cdot 10^{-4}
\end{aligned}
\tag{31}
$$

The first two functions belong to $C^\infty(\mathbb{S}^d)$, but present some difficulties due to different scaling w.r.t. each variable. Function f_3 is a modification of an example proposed by Hesse and Womersley, in which the parameter $p \geq 1$ controls its smoothness. In particular, it is shown in [15, Subsection 5.8.2.] that, for $p \in \mathbb{N}$, f_3 belongs to the Sobolev space $H^s(\mathbb{S}^2)$ for any $s < (2p+1)/2$. We remark that the other two examples suggested by Hesse and Womersley were easily approximated in this domain even by the rule with algebraic degree of exactness $= 5$, with an absolute error smaller than 10^{-14}.

All the numerical experiments were performed on a 2.5 Ghz Intel Core i5 computer with 8 GB of RAM and the cpu time was less than 10^{-2} seconds in all the tests.

We have also considered the computation of integrals of polynomials on the geographic rectangle $\Omega \subset \mathbb{S}^3$ defined in generalized spherical coordinates by the intervals $[0,\ \pi/3; 0,\ 2\pi/3; 0,\ 2\pi]$. Fixed a positive integer n, we approximated all I_{k_1,k_2,k_3,k_4} given by

$$
I_{k_1,k_2,k_3,k_4} := \int_\Omega x^{k_1}\, y^{k_2}\, u^{k_3}\, v^{k_4}\, d\Omega,\ \ \sum_{j=1}^{4} k_j \leq n \tag{32}
$$

using our cubature rule having degree of exactness n and $(n+3)(n+1)\lceil \frac{n+1}{2} \rceil$ points. The reference integrals have been easily computed by products of

$$
\int_{a_j}^{b_j} \cos^\alpha(\theta)\sin^\beta(\theta)d\theta
$$

for appropriate α, β (see (9)).

Results are summarized in Table 2. In the last column we have reported the number of integrals I_{k_1,k_2,k_3,k_4} computed at degree n.

All the Matlab codes used for these tests are available at [6].

7 Weakly Admissible Meshes on $\mathbb{R}^d$ or $\mathbb{C}^d$

Global polynomial approximation is a challenging topic in the multivariate setting, still with many open problems.

Few results are known about the so-called Fekete points of a multidimensional compact domain, i.e. points that maximize the Vandermonde determinant, ensuring (at most) an algebraic growth of the Lebesgue constant in the corresponding interpolation process. Using classical tools, the numerical computation of Fekete points becomes rapidly a very large scale nonlinear optimization problem.

A different approach for the computation of Fekete points is to consider a discretization of the domain, moving from the continuum to nonlinear combinatorial optimization. Good discrete models of general compact sets are provided by the so-called "Weakly Admissible Meshes", a term introduced by Calvi and Levenberg in [7].

Given a polynomial determining compact set $K \subset \mathbb{R}^d$ or $K \subset \mathbb{C}^d$ (i.e., polynomials vanishing there are identically zero), a *Weakly Admissible Mesh* (WAM) is defined in [7] is a sequence of discrete subsets $\mathbb{A}_n \subset K$ such that

$$\|p\|_K \leq C(\mathbb{A}_n)\|p\|_{\mathbb{A}_n} \ , \quad \forall p \in \mathbb{P}_n^d(K) \tag{33}$$

where both $\text{card}(\mathbb{A}_n) \geq N := \dim(\mathbb{P}_n(K))$ and $C(\mathbb{A}_n)$ grow at most *polynomially* with n (we use the notation $\|f\|_X = \sup_{x\in X} |f(x)|$ for f bounded function on the compact X). When $C(\mathbb{A}_n)$ is bounded we speak of an Admissible Mesh (AM).

Among their properties which can be considered as a recipe to construct new from known WAMs, we cite the following ones:

A: if α is an *affine* mapping and $\mathbb{A}_n$ a WAM for K, then $\alpha(\mathbb{A}_n)$ is a WAM on $\alpha(K)$ with the same constant $C(\mathbb{A}_n)$;

I: any sequence of *unisolvent* interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathbb{A}_n)$ being the Lebesgue constant itself;

P: a finite *product* of WAMs is a WAM (even for tensor-product polynomials) on the corresponding product of compacts, $C(\mathbb{A}_n)$ being the product of the corresponding constants;

U: a finite *union* of WAMs is a WAM on the corresponding union of compacts, $C(\mathbb{A}_n)$ being the maximum of the corresponding constants.

Between the many applications of WAMs in polynomial approximation, we cite

(1) *least-squares* polynomial approximation $\mathcal{L}_{\mathbb{A}_n} f$ on a WAM of a function $f \in C(K)$, is such that

$$\|f - \mathcal{L}_{\mathbb{A}_n} f\|_K \lessapprox C(\mathbb{A}_n)\sqrt{\operatorname{card}(\mathbb{A}_n)} \min\{\|f-p\|_K,\, p \in \mathbb{P}_n^d\}.$$

(2) Extraction of *Approximate Fekete Points* from WAMs [34]; they have a Lebesgue constant bounded by $NC(\mathbb{A}_n)$, and it can be proved that they are asymptotically distributed as the continuum Fekete points [3], [4].

(3) Extraction of *Discrete Leja Points* from WAMs, providing sets that *mimic* the actual Leja points but whose determination has a smaller computational cost [3].

A key feature, which is important for computational purposes, is the availability of low cardinality WAMs, so as to reduce the computational effort for determining the least-squares polynomial approximation of a function or the extraction of the Approximate Fekete Points (AFP) or Discrete Leja Points (DLP), with the techniques introduced in [34] and [3].

In the next sections we will consider the problem of determining low cardinality WAMs on geographic rectangles of the d-dimensional sphere, also showing the behaviour of the extracted AFP, DLP on two such regions of $\mathbb{S}^2$.

8 Weakly Admissible Meshes on certain regions of $\mathbb{S}^d$

We start our analysis with the following lemma (see [5], [24], [10])

Lemma 1 *Let* $\omega \in (0, \pi]$,

$$\tau_j = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right) \in (-1,1), j = 1, \ldots, 2n+1$$

be the $2n+1$ *zeros of the Chebyshev polynomial of the first kind* $T_{2n+1}(x) = \cos\left((2n+1)\arccos(x)\right)$, *and*

$$\theta_j := \theta_j(n, \omega) = 2\arcsin(\alpha\tau_j) \in [-\omega, \omega], j = 1, \ldots, 2n+1, \alpha = \sin(\omega/2).$$

Then, if L_j *is the* j*-th cardinal trigonometric polynomial w.r.t. the set* $\mathcal{A}_n([-\omega, \omega]) =$

$\{\theta_j\}_{j=1,\dots,2n+1}$, *the (trigonometric) Lebesgue constant*

$$\Lambda_n(\mathcal{A}_n([-\omega,\omega])) = \max_{\theta\in[-\omega,\omega]} \sum_{j=1} |L_j(\theta)|$$

is such that

$$\Lambda_n(\mathcal{A}_n([-\omega,\omega])) \le \frac{2}{\pi}\log(n) + \delta_n = \mathcal{O}(\log(n))$$

where $5/3 \le \delta_n \le \frac{2}{\pi}(\log(\frac{16}{\pi}) + \gamma^*) \approx 1.40379$ *with* γ^* *the Euler-Mascheroni constant.*

From property **I** of WAMs, and observing that a bivariate algebraic polynomial of total degree n on the arc $\gamma([-\omega,\omega])$ of the unit circle determined by the angles $-\omega$, ω is a trigonometric polynomial of the same degree in $[-\omega,\omega]$, it follows immediately from the previous Lemma (see also [35]) the following

Corollary 1 *The set*

$$\mathcal{M}_n^{(1)}([-\omega,\omega]) = \{\mathbf{x}\in\mathbb{S}^1 : (\cos(\theta),\sin(\theta)), \theta\in\mathcal{A}_n([-\omega,\omega])\}$$

is a WAM on the arc of the unit circle $\gamma([-\omega,\omega])$ *with constant*

$$C(\mathcal{M}_n^{(1)}([-\omega,\omega])) = \Lambda_n(\mathcal{M}_n^{(1)}([-\omega,\omega])) \le \frac{2}{\pi}\log(n) + \delta_n = \mathcal{O}(\log(n)).$$

By a suitable change of the parametrization on the unit disk, since if $p(\cdot)$ is a trigonometric polynomial of degree n so is $p(\omega+\cdot)$, we have that for $\omega = (\alpha+\beta)/2$ the set

$$\mathcal{M}_n^{(1)}([\alpha,\beta]) = \left\{\mathbf{x}\in\mathbb{S}^1 : (\cos(\omega+\theta),\sin(\omega+\theta)), \theta\in\mathcal{A}_n([-\omega,\omega])\right\} \quad (34)$$

is a WAM on the arc $\gamma([\alpha,\beta]) \in \mathbb{S}^1$ determined by the angles α, β, with constant

$$C(\mathcal{M}_n^{(1)}([\alpha,\beta])) = \Lambda_n(\mathcal{A}_n([\alpha,\beta])) = \Lambda_n(\mathcal{A}_n([-\omega,\omega])) \le \frac{2}{\pi}\log(n) + \delta_n \approx \mathcal{O}(\log(n)),$$

where $\omega = (\alpha+\beta)/2$.

We are now ready to prove the following

Theorem 6 *Let* $\Omega = \Omega(\mathbf{a},\mathbf{b}) \subseteq \mathbb{S}^d$ *be a geographic rectangle, with* $0 \le a_k < b_k \le \pi$ *for* $k = 1,\dots,d-1$ *and* $0 \le a_d < b_d \le 2\pi$. *Let*

$$\otimes_{i=1}^d \mathcal{A}_n([a_i,b_i])) = \{(x_1,\dots,x_d), x_i \in \mathcal{A}_n([a_i,b_i])), i = 1,\dots,d\}.$$

The mesh

$$\mathcal{M}_n(\Omega) = \xi(\otimes_{i=1}^d \mathcal{A}_n([a_i, b_i])) = \{\mathbf{x} \in \Omega : \mathbf{x} = \xi(\boldsymbol{\theta}), \boldsymbol{\theta} \in \otimes_{i=1}^d \mathcal{A}_n([a_i, b_i])\}$$

is a WAM on Ω *with cardinality* $(2n+1)^d$ *and constant* $C(\mathcal{M}_n) = \mathcal{O}(\log^d(n))$.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_{d+1})$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ and $\xi(\boldsymbol{\theta})$ the usual transformation from generalized spherical to cartesian coordinates. As also shown in the previous theorems, if $p \in \mathbb{P}_n(\Omega)$, then $p(\xi(\boldsymbol{\theta}))$ is a trigonometric polynomial of degree less than or equal to n in each variable θ_k, $k = 1, \ldots, d$.

Denoting with

$$\otimes_{i=1}^k \mathcal{A}_n([a_i, b_i])) = \{(x_1, \ldots, x_k), x_i \in \mathcal{A}_n([a_i, b_i])), i = 1, \ldots, k\}.$$

and with $\mathcal{M}_n(\Omega)$ the discrete subset of Ω

$$\mathcal{M}_n(\Omega) = \xi(\otimes_{i=1}^d \mathcal{A}_n([a_i, b_i])) = \{\mathbf{x} \in \Omega : \mathbf{x} = \xi(\boldsymbol{\theta}), \boldsymbol{\theta} \in \otimes_{i=1}^d \mathcal{A}_n([a_i, b_i])\}$$

we have by applying Corollary 1 d times and the definition of WAM,

$$\begin{aligned}
|p(\mathbf{x})| &= |p(\xi(\boldsymbol{\theta}))| \leq \Lambda_n(\mathcal{A}_n([a_1, b_1])) \max_{\theta_{j_1}^* \in \mathcal{A}_n([a_1,b_1])} |p(\xi(\theta_{j_1}^*, \theta_2, \ldots, \theta_d))| \\
&\leq \prod_{i=1}^2 \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{(\theta_{j_1}^*, \theta_{j_2}^*) \in \otimes_{i=1}^2 \mathcal{A}_n([a_i,b_i]))} |p(\xi(\theta_{j_1}^*, \theta_{j_2}^*, \ldots, \theta_d))| \\
&\leq \ldots \\
&\leq \prod_{i=1}^d \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{(\theta_{j_1}^*, \ldots, \theta_{j_d}^*) \in \otimes_{i=1}^d \mathcal{A}_n([a_i,b_i]))} |p(\xi(\theta_{j_1}^*, \theta_{j_2}^*, \ldots, \theta_{j_d}^*))| \\
&= \prod_{i=1}^d \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{\mathbf{x} \in \mathcal{M}_n} |p(\mathbf{x}))|. \qquad (35)
\end{aligned}$$

Since $\Lambda_n(\mathcal{A}_n([a_i, b_i])) \leq c_i \log(n)$ for some c_i, then $\mathcal{M}_n(\Omega)$ is a WAM of degree n over the domain $\Omega(\mathbf{a}, \mathbf{b})$ with constant $C(\mathcal{M}_n) = \mathcal{O}(\log^d(n))$. $\square$

For special regions, as caps, we can define WAMs with even lower cardinality, as proven by the next theorem.

Theorem 7 *Let* $\Omega = \Omega(\mathbf{a}, \mathbf{b})$. *Suppose* $a_d = 0$ *and* $b_d = \pi$ *and* $-\pi \leq a_{d-1} = -b_{d-1} < 0$, $[a_j, b_j] \in [0, \pi]$ *for* $j = 1, \ldots, d-2$. *Suppose that*

$$\mathcal{A}_n^+([-\gamma, \gamma]) = \{\theta_k \in \mathcal{A}_n([-\gamma, \gamma]), \theta_k \geq 0\}.$$

Then

$$\mathcal{M}_n(\Omega) = \otimes_{k=1}^{d-2} \mathcal{A}_n([a_k, b_k])) \times \mathcal{A}_n^+([-\gamma, \gamma]) \times \left\{ \frac{2\pi k}{2n+1} \right\}_{k=1,\ldots,2n+1} \subset \Omega$$

is a WAM of degree n in Ω with cardinality $\approx 2^{d-1}n^d$.

Proof. Let us represent the domain Ω in generalized spherical coordinates, with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in [0, \pi]^{d-2} \times [-\pi, \pi] \times [0, \pi]$, being as usual $\xi(\boldsymbol{\theta})$ the transformation into cartesian coordinates. Observe that if $p \in \mathbb{P}_n(\Omega)$ then $p(\xi(\boldsymbol{\theta}))$ is a trigonometric polynomial of degree n in each variable θ_k, $k = 1, \dots, d$. Notice also that

$$\Omega = \xi(\otimes_{k=1}^{d-2}[a_k, b_k] \times [-\gamma, \gamma] \times [0, \pi]) = \xi(\otimes_{k=1}^{d-2}[a_k, b_k] \times [-\gamma, \gamma] \times [0, 2\pi])$$

implies, $\mathcal{A}_n([0, 2\pi]) = \{2\pi k/(2n + 1)\}_{k=1,\dots,2n+1}$ being a trigonometric WAM of degree n on $[0, 2\pi]$ and $\mathcal{A}_n([a_k, b_k])$ a trigonometric WAM of degree n on $[a_k, b_k]$

$$\mathcal{M}_n(\Omega) = \xi(\otimes_{k=1}^{d-2}\mathcal{A}_n([a_k, b_k]) \times \mathcal{A}_n([-\gamma, \gamma]) \times \mathcal{A}_n([0, 2\pi])) \subset \Omega.$$

Thus,

$$\begin{aligned}
|p(\mathbf{x})| = |p(\xi(\boldsymbol{\theta}))| &\leq \Lambda_n(\mathcal{A}_n([a_1, b_1])) \max_{\theta^*_{j_1} \in \mathcal{A}_n([a_1,b_1])} |p(\xi(\theta^*_{j_1}, \theta_2, \dots, \theta_d))| \\
&\leq \prod_{i=1}^{2} \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{(\theta^*_{j_1},\theta^*_{j_2}) \in \otimes_{i=1}^{2}\mathcal{A}_n([a_i,b_i]))} |p(\xi(\theta^*_{j_1}, \theta^*_{j_2}, \dots, \theta_d))| \\
&\leq \dots \\
&\leq \prod_{i=1}^{d} \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{(\theta^*_{j_1},\dots,\theta^*_{j_d}) \in \otimes_{i=1}^{d}\mathcal{A}_n([a_i,b_i]))} |p(\xi(\theta^*_{j_1}, \theta^*_{j_2}, \dots, \theta^*_{j_d}))| \\
&= \prod_{i=1}^{d} \Lambda_n(\mathcal{A}_n([a_i, b_i])) \max_{\mathbf{x} \in \mathcal{M}_n(\Omega)} |p(\mathbf{x}))|. \qquad (36)
\end{aligned}$$

proving that $\mathcal{M}_n(\Omega)$ is a WAM of degree n in Ω.

Now observe that in view of the symmetries of the points belonging to $\mathcal{A}_n([-b_{d-1}, b_{d-1}])$ and $\left\{\frac{2\pi k}{2n+2}\right\}_{k=1,\dots,2n+1}$ we have

$$\begin{aligned}
\xi_{j_1,\dots,j_{d-1},j_d+} &:= \xi(\theta^{[a_1,b_1]}_{j_1}, \dots, \theta^{([a_{d-2},b_{d-2}])}_{j_{d-2}}, \theta^{[-b_{d-1},b_{d-1}]}_{j_{d-1}}, \theta^{[0,2\pi]}_{j_d}) \\
&= \xi(\theta^{[a_1,b_1]}_{j_1}, \dots, \theta^{[a_{d-2},b_{d-2}]}_{j_{d-2}}, -\theta^{[-b_{d-1},b_{d-1}]}_{j_{d-1}}, \theta^{[0,2\pi]}_{j} + \pi) \\
&= \xi(\theta^{[a_1,b_1]}_{j_1}, \dots, \theta^{[a_{d-2},b_{d-2}]}_{j_{d-2}}, \theta^{[-b_{d-1},b_{d-1}]}_{n+2-j_{d-1}}, \theta^{[0,2\pi]}_{j_d+\frac{n+1}{2}}) \\
&= \xi_{j_1,\dots,n+2-j_{d-1},j_d+\frac{n+1}{2}}. \qquad (37)
\end{aligned}$$

In view of the multiple occurrence of some points, the set $\mathcal{M}_n(\Omega)$ coincides with

$$\xi(\otimes_{k=1}^{d-2}\mathcal{A}_n([a_k, b_k])) \times \mathcal{A}^+_n([-\gamma, \gamma]) \times \mathcal{A}_n([0, 2\pi])) \subset \Omega,$$

that has cardinality approximatively $2^{d-1}n^d$. $\square$

9 Hyperinterpolation

Let μ be a measure over a compact domain Ω and suppose

$$\int_\Omega f(\mathbf{x})\,d\mu \approx \sum_{i=1}^{M} w_i f(\mathbf{x}_i), \tag{38}$$

is a cubature formula with nodes $\mathbf{x}_i$ and weights w_i, having algebraic degree of exactness $2n$.

If an orthonormal polynomial basis $\{\phi_j\}$ of $\mathbb{P}_n(\Omega)$ is at hand, by the cubature formula (38) we can also easily compute the discretized truncated orthogonal expansion of a function $f \in L^2_{d\mu}(\Omega)$ (i.e. such that $\int_\Omega |f|^2\,d\mu < +\infty$) as

$$\mathcal{L}_n f = \sum_{j=1}^{N} c_j\,\phi_j \tag{39}$$

where

$$c_j = \sum_{i=1}^{M} w_i\,\phi_j(\boldsymbol{x}_i) f(\boldsymbol{x}_i) \approx \int_\Omega \phi_j(\boldsymbol{x}) f(\boldsymbol{x})\,d\mu\ . \tag{40}$$

Let $W = \mathrm{diag}(\sqrt{w_1}, \ldots, \sqrt{w_M})$ and $V_\phi = (v_{i,j}) = (\phi_j(\boldsymbol{x}_i)) \in \mathbb{R}^{M\times N}$, where N is the dimension of the space $\mathbb{P}_n(\Omega)$. In view of (40), the coefficients $\{c_j\}$ can be computed in vector form as

$$(c_1, \ldots, c_N) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_M))W^2 V_\phi = (w_1 f(\boldsymbol{x}_1), \ldots, w_m f(\boldsymbol{x}_M))V_\phi\ . \tag{41}$$

By the algebraic degree of exactness of the formula, the expansion $\mathcal{L}_n f$ is a projection $L^2_{d\mu}(\Omega) \to \mathbb{P}_n(\Omega)$ (i.e. $\mathcal{L}_n$ is linear and $\mathcal{L}_n^2 = \mathcal{L}_n$), called *hyperinterpolation* (see the seminal work by Sloan [30]). Indeed, it is the orthogonal projection with respect to the discrete inner product defined by the cubature formula. In [30] the author proved, for example, that for every $f \in C(\Omega)$

$$\|\mathcal{L}_n f - f\|_{L^2_{d\mu}(\Omega)} \le 2\mu(\Omega)\,E_n(f;\Omega) \to 0\ ,\quad n \to \infty\ , \tag{42}$$

where $E_n(f;\Omega) = \inf\{\|f - p\|_\infty\ ,\ p \in \mathbb{P}_n(\Omega)\}$ is the best approximation error in $\mathbb{P}_n(\Omega)$ in the uniform norm and $\mu(\Omega) = \int_\Omega d\mu$.

Multivariate hyperinterpolation has been applied in various instances, as a valuable alternative to polynomial interpolation.

For details and applications, see e.g., [13,19,36], the survey [20] and the recent paper [33] concerning the sphere.

In this framework, upper bounds of the uniform norm of $\mathcal{L}_n : C(\Omega) \to \mathbb{P}_n(\Omega)$ (that is the operator norm with respect to $\|f\|_\Omega = \max_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|$) are available.

Denoting with $K_n(\boldsymbol{x}, \boldsymbol{y})$ the reproducing kernel of $\mathbb{P}_n(\Omega)$, with the underlying inner product [14, Ch. 3], and with

$$\psi_i(\boldsymbol{x}) = K_n(\boldsymbol{x}, \boldsymbol{x}_i) = w_i \sum_{j=1}^{N} \phi_j(\boldsymbol{x})\phi_j(\boldsymbol{x}_i) \ ,$$

from

$$\mathcal{L}_n f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \, \phi_j(\boldsymbol{x}) = \sum_{j=1}^{N} \phi_j(\boldsymbol{x}) \sum_{i=1}^{M} w_i \, \phi_j(\boldsymbol{x}_i) f(\boldsymbol{x}_i) = \sum_{i=1}^{M} f(\boldsymbol{x}_i)\psi_i(\boldsymbol{x}) \ , \quad (43)$$

one can prove that

$$\|\mathcal{L}_n\| = \sup \frac{\|\mathcal{L}_n f\|_\Omega}{\|f\|_\Omega} = \max_{\boldsymbol{x}\in\Omega} \sum_{i=1}^{M} |\psi_i(\boldsymbol{x})|. \quad (44)$$

Being $\mathcal{L}_n$ a projection on $\mathbb{P}_n(\Omega)$, we easily have

$$\|\mathcal{L}_n f - f\|_\Omega \leq (1 + \|\mathcal{L}_n\|) \, E_n(f; \Omega) \ , \quad (45)$$

while (44) provides a measure of the hyperinterpolation stability.

In the next sections, we will show examples of hyperinterpolation on regions of the sphere apparently not treated before in the numerical literature.

10 Numerical implementation

We started the computation of the hyperinterpolants by determining an orthonormal basis on the region of interest, i.e. the geographic rectangle $\Omega = \Omega(\mathbf{a}, \mathbf{b}) \subseteq \mathbb{S}^2$. In theory one could apply the Gram-Schmidt process starting from the spherical harmonics basis $\{\phi_k\}_{k=1,\ldots,N}$, but in general severe instabilities occur.

Using the positive weights formulas previously introduced on Ω, with algebraic degree of exactness $2n$, after a suitable ordering of the nodes we have

$$\int_\Omega f(\xi) d\mu \approx \sum_{i=1}^{M} \lambda_i f(\xi_i). \quad (46)$$

Let V_ϕ be the rectangular Vandermonde-like matrix

$$V_\phi = (v_{i,j}) = (\phi_j(\xi_i)) \in \mathbb{R}^{M\times N},$$

W the diagonal matrix

$$W = \mathrm{diag}(\sqrt{w_1}, \ldots, \sqrt{w_M}).$$

and $(\cdot,\cdot)_\mu$ the $L^2_\mu(\Omega)$ scalar product

$$(f,g)_\mu = \int_\Omega p(\xi)q(\xi)\,d\mu.$$

It is easy to observe that for $p,q \in \mathbb{P}_n(\Omega)$

$$(p,q)_\mu = \sum_{i=1}^{M} \sqrt{w_i}p(\xi)\sqrt{w_i}q(\xi) = \int_\Omega p(\xi)q(\xi)\,d\mu \tag{47}$$

where in the last equality we used the fact that the cubature rule has algebraic degree of exactness $2n$.

Since $\{\phi_k\}$ is a basis of $\mathbb{P}_n(\Omega)$, the Vandermonde matrix V_ϕ has full-rank, so does WV_ϕ and by the QR factorization we have

$$WV_\phi = Q_1R_1,\; Q_1 \in \mathbb{R}^{M\times N}, R_1 \in \mathbb{R}^{N\times N}$$

with Q_1 orthogonal and R_1 upper triangular and nonsingular matrix. This entails that, at least in theory, the new polynomial basis $\{\phi_k^{(1)}\}_{k=1,\ldots,N}$ defined as

$$(\phi_1^{(1)}, \ldots, \phi_N^{(1)}) = (\phi_1, \ldots, \phi_N)R_1^{-1}$$

is orthonormal since $V_{\phi^{(1)}} = V_\phi R_1^{-1}$ and the Gram matrix satisfies

$$G_{\phi^{(1)}} = (WV_{\phi^{(1)}})^T(WV_{\phi^{(1)}}) = (WV_\phi R_1^{-1})^T(WV_\phi R_1^{-1}) = Q_1^TQ_1 = I.$$

However, due to the finite precision arithmetic, when V_ϕ is severely ill conditioned, the matrix Q_1 is not numerically orthogonal. If the conditioning of V_ϕ in the 2-norm is below (or about) the reciprocal of the machine precision, by the *twice is enough* phenomenon [18], applying again the QR factorization to the matrix Q_1, i.e. $Q_1 = Q_2R_2$, we produce a matrix Q_2 orthogonal up to an error close to machine precision and the new polynomial basis

$$(\phi_1^{(2)}, \ldots, \phi_N^{(2)}) = (\phi_1, \ldots, \phi_N)R_1^{-1}R_2^{-1}$$

is numerically orthonormal with respect to the scalar product $(\cdot,\cdot)_\mu$, i.e. setting

$$V_{\phi^{(2)}} = (v_{i,j}^{(2)}) = (\phi_j^{(2)}(\xi_i)),\; G_{\phi^{(2)}} = (WV_{\phi^{(2)}})^T(WV_{\phi^{(2)}}),$$

we have that $\|G_{\phi^{(2)}} - I\|_2$ is close to machine precision.

In view of these observations, starting from the spherical harmonics basis, we computed first a new polynomial basis of $\mathbb{P}_n(\Omega)$ numerically orthonormal w.r.t. the scalar product $(\cdot,\cdot)_\mu$ and then applied numerical cubature (46) to determine the hyperinterpolation coefficients

$$c_k = \sum_{i=1}^{M} \lambda_i \phi_k^{(2)}(\xi_i) f(\xi_i)$$

so that

$$f(\xi) \approx \sum_{k=1}^{N} c_k \phi_k^{(2)}(\xi).$$

11 Numerical examples

In this chapter we compare the approximants obtained by interpolation and hyperinterpolation on some regions of the sphere. In the first case, as shown in [34] and later in [3], we interpolate on point sets obtained after the extraction of good interpolation points from WAMs already introduced.

Using the previous notations, we considered as domains the cap

$$\Omega_1 = \Omega([0,\ \pi/3; 0,\ 2\pi]) \subset \mathbb{S}^2$$

and the geographic rectangle

$$\Omega_2 = \Omega([\pi/4,\ \pi/3; \pi/8,\ \pi/4]) \subset \mathbb{S}^2.$$

In Table 3 and Table 4 we have considered the Lebesgue constant of AFP and DLP. Remembering that if $p_n \in \mathbb{P}_n(\Omega)$ interpolates a function $f \in C(\Omega)$ on a given unisolvent set $\mathcal{M} \subset \Omega$ of points for degree n then

$$\|f - p_n\| \leq (1 + \Lambda_n(\mathcal{M}))E_n(f)$$

where $E_n(f)$ is the best approximation error in $\mathbb{P}_n(\Omega)$, the results in these tables show that we can expect that the extracted points are of good quality, though not optimal.

As polynomial basis for $\mathbb{P}_n(\Omega_i)$ we used the spherical harmonics on $\mathbb{S}^2$. This detail is fundamental, since it determines the Vandermonde matrix from which we extracted the point sets.

As numerical tests we considered the following functions

$$
\begin{aligned}
f_1(x) &:= 3x^{40} + 0.01y^{40} + 100z^{40},\\
f_2(x) &:= \frac{1}{(x-0.5)^2 + (y-0.5)^2 + (z-0.2)^2},\\
f_3(x) &:= \sin(0.1x + y + 50z).
\end{aligned} \tag{48}
$$

The function f_1 is a polynomial of total degree 40, with coefficients having different scales, f_2 is a function having a singularity in the point $(0.5, 0.5, 0.2)$ (not belonging to the unit sphere), while f_3 is a trigonometric function with an argument that is linear combination of monomials having different scales in the coefficients.

We have interpolated these three test functions on the AFP, for degrees 5, 10, 15, 20, 25, 30. In Table 5 and Table 6, we provide the absolute errors of interpolants and hyperinterpolants evaluated on a fine mesh of each domain Ω_i, $i = 1, 2$. The results are very similar and report again the good quality of the interpolation sets.

In Table 7 and Table 8 we have listed the hyperinterpolation errors

$$
\frac{(\sum_{i=1}^{M} w_i(\mathcal{L}_n f(\mathbf{x}_i) - f(\mathbf{x}_i))^2)^{1/2}}{(\sum_{i=1}^{M} w_i f(\mathbf{x}_i))^{1/2}} \approx \frac{\|\mathcal{L}_n f - f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}
$$

for the functions $f = f_1$, $f = f_2$ and $f = f_3$ relatively to the degrees 5, 10, 15, 20, 25, 30.

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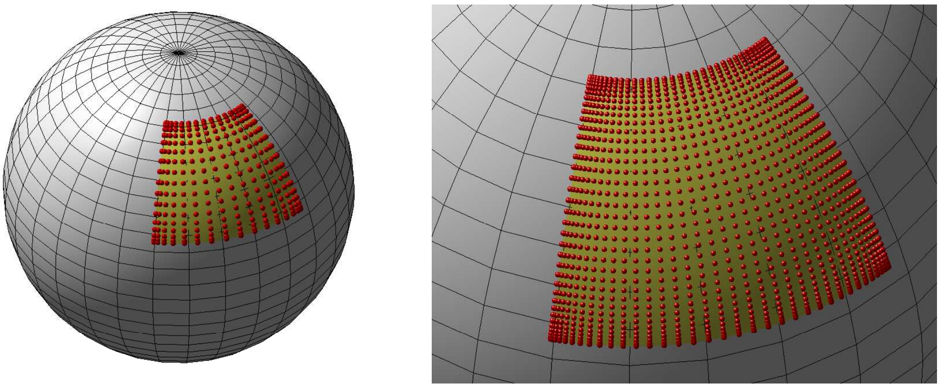

Fig. 1. Nodes of a rule on a geographic rectangle. On the left, nodes of a rule with algebraic degree of exactness $N = 15$ (272 nodes). On the right, nodes of a rule with algebraic degree of exactness $N = 35$ (1332 nodes).

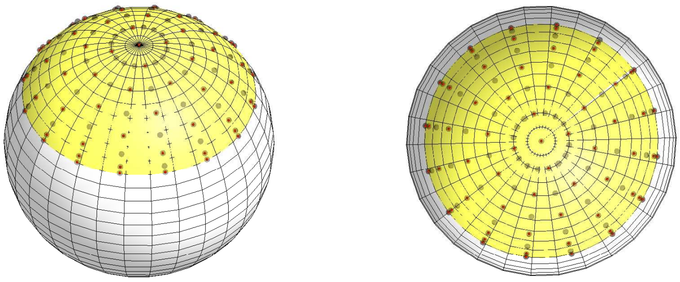

Fig. 2. Points of a WAM of degree 7 (113 points) on a cap, and related Approximated Fekete Points (64 points).

Deg.	f_1	f_2	f_3
5	$3.34e-04$	$7.38e-02$	$4.53e-06$
10	$4.89e-06$	$2.69e-02$	$5.44e-07$
15	$9.12e-09$	$5.14e-03$	$4.07e-08$
20	$1.76e-10$	$1.13e-02$	$2.43e-08$
25	$7.73e-14$	$1.13e-02$	$9.53e-09$
30	$3.33e-16$	$1.23e-03$	$2.23e-09$
35	$3.47e-17$	$2.58e-05$	$2.33e-09$
40	$1.14e-16$	$1.96e-07$	$2.82e-10$
45	$3.47e-17$	$6.94e-10$	$8.84e-10$
50	$2.08e-17$	$1.33e-12$	$5.48e-11$

Table 1
Absolute errors for degrees $5, 10, \ldots, 50$, w.r.t. the integrals on the domain of some test functions.

n	Abs.Err.	Points	Integrals
5	$4.44e-16$	144	34
10	$1.11e-16$	858	161
15	$1.67e-16$	2304	444
20	$2.78e-17$	5313	946
25	$4.16e-17$	9464	1729
30	$2.78e-17$	16368	2856
35	$8.33e-17$	24624	4389

Table 2
Maximum absolute error in the numerical approximation of integrals (32) with total degree less than or equal to n using a rule with algebraic degree of exactness $= n$.

Deg.	WAM Card	AFP/DLP Card	$\Lambda_n^{(AFP)}$	$\Lambda_n^{(DLP)}$
5	66	36	9.9	12.6
10	231	121	27.5	62.2
15	496	256	51.1	130.5
20	861	441	103.7	199.6
25	1326	676	191.3	384.5
30	1891	961	314.1	465.4

Table 3
WAM, AFP, DLP cardinality, Lebesgue constants $\Lambda_n^{(AFP)}$, $\Lambda_n^{(DLP)}$ of extracted pointset for degrees $5, 10, \ldots, 30$, on the cap Ω_1.

Deg.	WAM Card	AFP/DLP Card	$\Lambda_n^{(AFP)}$	$\Lambda_n^{(DLP)}$
5	121	36	9.3	9.2
10	441	121	49.8	55.1
15	961	256	107.5	162.0
20	1681	441	178.8	299.1
25	2601	676	262.4	423.3
30	3721	961	304.1	461.4

Table 4
WAM, AFP, DLP cardinality, Lebesgue constants $\Lambda_n^{(AFP)}$, $\Lambda_n^{(DLP)}$ of extracted pointset for degrees $5, 10, \ldots, 30$, on the geographic rectangle Ω_2.

	f_1		f_2		f_3	
Deg.	Intp.	Hyp.	Intp.	Hyp.	Intp.	Hyp.
5	$9e+00$	$1e+01$	$1e+01$	$7e-01$	$2e+00$	$2e+00$
10	$1e-01$	$3e-01$	$2e-01$	$1e-01$	$2e+00$	$3e+00$
15	$6e-03$	$3e-03$	$1e-02$	$2e-02$	$2e+00$	$7e-01$
20	$6e-05$	$2e-05$	$2e-03$	$2e-03$	$5e-02$	$2e-02$
25	$9e-07$	$6e-07$	$2e-04$	$1e-04$	$2e-02$	$8e-03$
30	$2e-08$	$5e-09$	$3e-04$	$3e-05$	$5e-04$	$2e-04$

Table 5
Absolute errors of interpolants (using AFP as point sets) and hyperinterpolants of three test functions on the cap $\Omega_1 = \Omega([0,\, \pi/3; 0,\, 2\pi])$, for degrees $5, 10, \ldots, 30$.

	f_1		f_2		f_3	
Deg.	Intp.	Hyp.	Intp.	Hyp.	Intp.	Hyp.
5	$6e-06$	$6e-06$	$3e-03$	$2e-03$	$9e-01$	$7e-01$
10	$2e-08$	$1e-08$	$2e-06$	$9e-07$	$5e-03$	$3e-03$
15	$5e-12$	$3e-12$	$1e-09$	$5e-10$	$1e-05$	$4e-06$
20	$4e-14$	$2e-14$	$3e-11$	$1e-11$	$1e-06$	$4e-07$
25	$2e-14$	$5e-15$	$2e-12$	$1e-11$	$9e-08$	$2e-08$
30	$5e-15$	$2e-15$	$2e-13$	$7e-12$	$5e-09$	$2e-09$

Table 6
Absolute errors of interpolants (using AFP as point sets) and hyperinterpolants of three test functions on the geographic rectangle $\Omega_2 = \Omega([\pi/4,\ \pi/3; \pi/8, \pi/4])$, for degrees $5, 10, \ldots, 30$.

Deg	f_1	f_2	f_3
5	$3e-06$	$6e-02$	$2e-02$
10	$2e-06$	$4e-03$	$4e-02$
15	$7e-06$	$5e-05$	$3e-02$
20	$6e-08$	$4e-05$	$3e-03$
25	$5e-10$	$6e-06$	$4e-04$
30	$1e-11$	$4e-07$	$2e-05$

Table 7
Hyperinterpolants errors of three test functions on the cap $\Omega_1 = \Omega([0,\ \pi/3; 0,\ 2\pi])$.

Deg	f_1	f_2	f_3
5	$2e-02$	$5e-05$	$2e-01$
10	$3e-05$	$2e-08$	$4e-04$
15	$5e-09$	$7e-12$	$4e-07$
20	$4e-11$	$2e-13$	$5e-08$
25	$1e-11$	$1e-13$	$3e-09$
30	$4e-12$	$1e-13$	$2e-10$

Table 8
Hyperinterpolants errors of three test functions on the geographic rectangle $\Omega_2 = \Omega([\pi/4,\ \pi/3; \pi/8,\ \pi/4])$.