

INVARIABLE GENERATION OF PROSOLUBLE GROUPS

BY

ELOISA DETOMI AND ANDREA LUCCHINI

*Eloisa Detomi and Andrea Lucchini,
Università degli Studi di Padova,
Dipartimento di Matematica,
Via Trieste 63, 35121 Padova, Italy*

ABSTRACT

A group G is *invariably generated* by a subset S of G if $G = \langle s^{g(s)} \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. Answering two questions posed by Kantor, Lubotzky and Shalev in [8], we prove that the free prosoluble group of rank $d \geq 2$ cannot be invariably generated by a finite set of elements, while the free solvable profinite group of rank d and derived length l is invariably generated by precisely $l(d - 1) + 1$ elements.

1. Introduction

Following [2] we say that a subset S of a group G invariably generates G if $G = \langle s^{g(s)} \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. We also say that a group G is invariably generated (IG for short) if G is invariably generated by some subset S of G ; when S can be chosen to be finite, we say that G is FIG. A group G is IG if and only if it cannot be covered by a union of conjugates of a proper subgroup, which amount to saying that in every transitive permutation representation of G on a set with more than one element there is a fixed-point-free element. Using this characterization, Wiegold [13] proved that the free group on two (or more) letters is not IG. Kantor, Lubotzky and Shalev studied invariable generation in finite and infinite groups. For example in [7] they proved that every finite group G is invariably generated by at most $\log_2 |G|$ elements. In [8] they studied invariable generation of infinite groups, with emphasis on linear groups, proving that a finitely generated linear group is FIG if and only if it is virtually soluble.

Let G be a profinite group. Then generation and invariable generation in G are interpreted topologically. Just as every finite group is IG, every profinite group G is also IG. Indeed every proper subgroup of a profinite group G is contained in a maximal open subgroup M , and, since M has finite index, G cannot coincide with the union $\cup_{g \in G} M^g$. On the other hand, finitely generated profinite groups are not necessarily FIG. In fact by [7, Proposition 2.5], there exist 2-generated finite groups H with $d_I(H)$ (the minimal number of invariable generators) arbitrarily large. This implies that the free profinite of rank $d \geq 2$ is not FIG. In [8] the following questions are asked: Are finitely generated prosoluble groups FIG? Are finitely generated soluble profinite groups FIG?

We prove that the first question has in general a negative answer:

THEOREM 1: *The free prosoluble group of rank $d \geq 2$ is not FIG.*

We will deduce Theorem 1 from the following result (see Theorem 8). Let G be a finite 2-generated soluble group and let p be the smallest prime divisor of $|G|$. Then either $d_I(G) \geq p$ or there exists a prime $q > p$ such that $d_I(G) < d_I(C_q \wr G)$, where $C_q \wr G$ is the wreath product with respect to the regular permutation representation of G .

In contrast, the second question has a positive answer. More precisely we can adapt the arguments used in the proof of Theorem 1 to show:

THEOREM 2: *Let F be the free soluble profinite group of rank d and derived length l . Then $d_I(F) = l(d - 1) + 1$.*

Denote by $d(G)$ the smallest cardinality of a generating set of a finitely generate profinite group G . Clearly if G is pronilpotent, then $d(G) = d_I(G)$. More precisely, by [7, Proposition 2.4] a finitely generated profinite group G is pronilpotent if and only if every generating set of G invariably generates G . But what can we say about the difference $d_I(G) - d(G)$ when G is a prosupersoluble group? In this case $G/\text{Frat}(G)$ is metabelian, so Theorem 2 implies that $d_I(G) - d(G) \leq d(G) - 1$. Although supersolubility is a quite strong property and in particular a metabelian group is not in general supersoluble, the previous estimate is sharp.

THEOREM 3: *Let F be the free prosupersoluble group of rank d . Then $d_I(F) = 2d - 1$.*

2. Preliminaries

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The textbooks [11] and [14] provide a good introduction to the theory of profinite groups. In the context of profinite groups, generation and invariable generation are interpreted topologically. By a standard argument (see e.g. [14, Proposition 4.2.1]) it can be proved that a profinite group G is invariably generated by d elements if and only if G/N is invariably generated by d elements for every open normal subgroup N of G . Therefore in the following we will mainly work on finite groups.

If G is a finite soluble group, the minimal number of generators for G can be computed in term of the structure of G -modules of the chief factors of G with the following formula due to Gaschütz [4].

PROPOSITION 4: *Let G be a finite soluble group. For every irreducible G -module V define $r_G(V) = \dim_{\text{End}_G(V)} V$, set $\theta_G(V) = 0$ if V is a trivial G -module, and $\theta_G(V) = 1$ otherwise, and let $\delta_G(V)$ be the number of chief factors G -isomorphic to V and complemented in an arbitrary chief series of G . Then*

$$d(G) = \max_V \left(\theta_G(V) + \left\lceil \frac{\delta_G(V)}{r_G(V)} \right\rceil \right)$$

where V ranges over the set of non G -isomorphic complemented chief factors of G and $\lceil x \rceil$ denotes the smallest integer greater or equal to x .

There is no similar formula for the minimal size of the invariable generating sets. The best result in this direction is a criterion we gave in [1] to decide whether an invariable generating set of a group G can be lifted to an extension over an abelian normal subgroup. To formulate this result, we need to recall some notation from [1].

Let G be a finite group acting irreducibly on an elementary abelian finite p -group V . For a positive integer u we consider the semidirect product $V^u \rtimes G$: unless otherwise stated, we assume that the action of G is diagonal on V^u , that is, G acts in the same way on each of the u direct factors. In [1, Proposition 8] we proved the following.

PROPOSITION 5: *Suppose G acts faithfully and irreducibly on V and $H^1(G, V) = 0$. Assume that g_1, \dots, g_d invariably generate G . There exist some elements $w_1, \dots, w_d \in V^u$ such that $g_1w_1, g_2w_2, \dots, g_dw_d$ invariably generate $V^u \rtimes G$ if and only if*

$$u \leq \sum_{i=1}^d \dim_{\text{End}_G(V)} C_V(g_i).$$

The assumption $H^1(G, V) = 0$ in the case of soluble groups is assured by the following unpublished result by Gaschütz (see [12, Lemma 1]).

LEMMA 6: *Let $G \neq 1$ be a finite soluble group and let V be an irreducible G -module. Then $H^1(G, V) = 0$.*

In the following we will use this straightforward consequence of Proposition 5.

COROLLARY 7: *Let $G \neq 1$ be a finite soluble group and let V be an irreducible G -module. Assume that x_1, \dots, x_d invariably generate $V^u \rtimes G$, where $x_i = v_i g_i$ with $v_i \in V^u$ and $g_i \in G$. Then g_1, \dots, g_d invariably generate G and*

$$u \leq \sum_{i=1}^d \dim_{\text{End}_{G/C_G(V)}(V)} C_V(g_i).$$

Proof. Clearly, g_1, \dots, g_d invariably generate G . Denote by \bar{g}_i the image of g_i in the quotient group $G/C_G(V)$. By Lemma 6 and Proposition 5 we have

$$u \leq \sum_{i=1}^d \dim_{\text{End}_{G/C_G(V)}(V)} C_V(\bar{g}_i).$$

Since $\dim C_V(\bar{g}_i) = \dim C_V(g_i)$, the result follows. ■

3. Proof of Theorem 1

If G is a finite group, $\pi(G)$ is the set of primes dividing the order of G .

THEOREM 8: *Let G be a 2-generated finite soluble group. Either $d_I(G) \geq \min \pi(G)$ or there exists a finite soluble group H having G as an epimorphic image and such that*

- $d(H) = 2$;
- $d_I(H) > d_I(G)$;

- $\min \pi(H) = \min \pi(G)$.

Proof. By Dirichlet’s theorem on primes in arithmetic progressions, there exists a prime q such that the exponent of G divides $q - 1$. Let \mathbb{F} be the field of order q . By a result of Brauer (see e.g. [3, B 5.21]) \mathbb{F} is a splitting field for G so

$$V := \mathbb{F}G = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$$

where the V_j are absolutely irreducible $\mathbb{F}G$ -modules no two of which are G -isomorphic, and $n_j = \dim_{\mathbb{F}} V_j$. Consider the semidirect product $H = V \rtimes G$; note that H is isomorphic to $C_q \wr G$ with respect to the regular permutation representation of G . By [9, Corollary 2.4], as C_q and G have coprime orders, $d(C_q \wr G) = \max(d(G), d(C_q) + 1) = 2$.

Clearly $d_I(G) \leq d_I(H)$. Assume $d_I(G) = d_I(H) = d$. By Corollary 7 applied to each homomorphic image $V_j^{n_j} \rtimes G$, it follows that there exists an invariable generating set g_1, \dots, g_d of G such that, for any j

$$n_j \leq \sum_{i=1}^d \dim_{\mathbb{F}} C_{V_j}(g_i).$$

Multiplying by n_j we get

$$n_j^2 \leq \sum_{i=1}^d n_j \dim_{\mathbb{F}} C_{V_j}(g_i).$$

It follows that:

$$|G| = \sum_{j=1, \dots, r} n_j^2 \leq \sum_{\substack{i=1, \dots, d \\ j=1, \dots, r}} n_j \dim_{\mathbb{F}} C_{V_j}(g_i) = \sum_{i=1, \dots, d} \dim_{\mathbb{F}} C_{\mathbb{F}G}(g_i).$$

On the other hand, by Lemma 9 below,

$$\dim_{\mathbb{F}} C_{\mathbb{F}G}(g_i) = \frac{|G|}{|g_i|}$$

and therefore

$$1 \leq \sum_{i=1}^d \frac{1}{|g_i|}.$$

Since $d = d_I(G)$ we have $g_i \neq 1$ for every i , hence $|g_i| \geq p = \min \pi(G)$. Therefore

$$1 \leq \sum_{i=1}^d \frac{1}{|g_i|} \leq \frac{d}{p}$$

which implies that $p \leq d$, as required. ■

LEMMA 9: *If $g \in G$, then $\dim_{\mathbb{F}} C_{\mathbb{F}G}(g) = |G : \langle g \rangle|$.*

Proof. Let t_1, \dots, t_r be a left transversal of $\langle g \rangle$ in G . Assume that $x \in C_{\mathbb{F}G}(g)$. As every element of G can be uniquely written in the form $t_i g^j$, we can write $x = \sum_{i,j} a_{t_i g^j} t_i g^j$, where $a_{t_i g^j} \in \mathbb{F}$, and, since $xg = x$, we have in particular

$$a_{t_i g^j} = a_{t_i g^{j+1}}$$

for every i and j . Hence $x = \sum_i b_i t_i (1 + g + \dots + g^{|g|-1})$, for some $b_i \in \mathbb{F}$. Conversely, every \mathbb{F} -linear combination of the elements $t_i (1 + g + \dots + g^{|g|-1})$ is centralized by g . In other words the elements $t_i (1 + g + \dots + g^{|g|-1})$, $1 \leq i \leq r$, are a basis for $C_{\mathbb{F}G}(g)$. ■

COROLLARY 10: *For every $d \in \mathbb{N}$, there exists a finite 2-generated soluble group G with $d_I(G) \geq d$.*

Proof. Let p be a prime number with $d \leq p$ and consider the set Ω_p of the finite 2-generated soluble groups whose order is divisible by no prime smaller than p . Assume by contradiction, that $d_I(G) < d$ for every $G \in \Omega_p$ and let G^* be a group in Ω_p such that $d_I(G^*) = \max_{G \in \Omega_p} d_I(G)$. Since $d_I(G^*) \leq d$ and $d \leq p$, by the Theorem 8 there exists H in Ω_p with $d_I(G^*) < d_I(H)$, and this contradicts the maximality of $d_I(G^*)$. ■

Proof of Theorem 1. Let F be the d -generated free prosoluble group, with $d \geq 2$. Assume that F is FIG. In particular $d_I(H) \leq d_I(F)$ for every 2-generated finite soluble group H , but this contradicts Corollary 10. ■

4. Proof of Theorem 2

We need, as a preliminary result, a formula for the minimal number of generators of a G -module.

LEMMA 11: *Let G be a finite group. Assume that A is a direct product*

$$A = A_1^{n_1} \times \dots \times A_r^{n_r}$$

where, for each i , A_i is a finite elementary abelian p_i -group for a prime number p_i , A_i is an irreducible $\mathbb{F}_{p_i}G$ -module and A_i is not G -isomorphic to A_j for $i \neq j$. Then the minimal number of elements needed to generate A as G -module is

$$d_G(A) = \max_{i \in \{1, \dots, r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right),$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x .

Proof. If J_i is the Jacobson radical of $\mathbb{F}_{p_i}G$, then $\mathbb{F}_{p_i}G/J_i$ is semisimple and Artinian, hence we can apply the Wedderburn-Artin theorem (see e.g. [6, Lemma 1.11, Theorems 1.14 and 3.3]) and we conclude that A_i occurs precisely $\dim_{\text{End}_G(A_i)}(A_i) = r_G(A_i)$ times in $\mathbb{F}_{p_i}G/J_i$. Then, by [5, Lemma 7.12], A can be generated, as G -module, by

$$d_G(A) = \max_{i \in \{1, \dots, r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right)$$

elements. ■

PROPOSITION 12: *Let G be a finite soluble d -generated group of derived length l . Then $d_I(G) \leq l(d - 1) + 1$.*

Proof. The proof is by induction on l . If $l = 1$, then G is abelian and $d_I(G) = d(G) \leq d = 1(d - 1) + 1$.

Assume $l > 1$ and let A be the last non-trivial term of the derived series of G . Then $dl(G/A) = l - 1$. Since $d_I(G) = d_I(G/\text{Frat}(G))$, without loss of generality we can assume $\text{Frat}(G) = 1$. Then A is a direct product of complemented minimal normal subgroups of G and we can write

$$A = A_1^{n_1} \times \dots \times A_r^{n_r}$$

where each A_i is an elementary abelian p_i -group, for a prime number p_i , A_i is an irreducible $\mathbb{F}_{p_i}G$ -module and A_i is not G -isomorphic to A_j for $i \neq j$. Therefore by Lemma 11

$$(4.1) \quad d_G(A) = \max_{i \in \{1, \dots, r\}} \left(\left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right).$$

On the other hand, by Proposition 4,

$$(4.2) \quad d \geq d(G) = \max_V \left(\theta_G(V) + \left\lceil \frac{\delta_G(V)}{r_G(V)} \right\rceil \right)$$

where V ranges over the set of non G -isomorphic complemented chief factors of G . Note that $\theta_G(A_i) = 1$ for every i . Indeed, if we assume that A_i is a trivial G -module, then, as $\text{Frat}(G) = 1$, we have $G = A_i \times H$ for a complement H of A_i in G . Hence $G' = H'$ and G' does not contain A_i , contradicting the fact that A_i is a subgroup of the last term of the derived series of G .

Since $n_i \leq \delta_G(A_i)$, by equations 4.1 and 4.2 we deduce that

$$d \geq \max_{i \in \{1, \dots, r\}} \left(1 + \left\lceil \frac{n_i}{r_G(A_i)} \right\rceil \right) = 1 + d_G(A)$$

hence $d_G(A) \leq d - 1$. Let a_1, \dots, a_{d-1} be a set of generators for A as G -module and let g_1, \dots, g_t be invariable generators for G modulo A with $t = d_I(G/A)$. Then it is straightforward to check that the elements

$$g_1, \dots, g_t, a_1, \dots, a_{d-1}$$

invariably generate G , hence

$$d_I(G) \leq t + (d - 1) = d_I(G/A) + (d - 1).$$

Since $dl(G/A) = l - 1$, by inductive hypothesis we have that

$$d_I(G/A) \leq (l - 1)(d - 1) + 1,$$

and we conclude that

$$d_I(G) \leq (l - 1)(d - 1) + 1 + (d - 1) = l(d - 1) + 1,$$

as required. ■

Denote by $dl(G)$ the derived length of a soluble group G . It follows from the previous proposition, that if G is a finitely generated solvable profinite group, then $d_I(G) \leq dl(G)(d(G) - 1) + 1$. In order to complete the proof of Theorem 2 it suffices to prove the following result:

THEOREM 13: *Let d be a positive integer and let p be a prime number. For every positive integer $l < \frac{p-1}{d-1} + 1$ there exists a finite soluble group G_l such that*

- $p = \min \pi(G_l)$,
- $dl(G_l) = l$,
- $d(G_l) = d$,
- $d_I(G_l) = l(d - 1) + 1$.

Proof. We prove the theorem by induction on l . If $l = 1$, then we can take $G_1 = C_p^d$. So suppose that a group G_l , with the desired properties, has been constructed for $l < \frac{p-1}{d-1}$. As in the proof of Theorem 8, if we take a prime q

such that the exponent of G_l divides $q - 1$ and we consider the field \mathbb{F} be the field of order q , then

$$V := \mathbb{F}G_l = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$$

where the V_j are absolutely irreducible $\mathbb{F}G$ -modules no two of which are G -isomorphic, and $n_j = \dim_{\mathbb{F}} V_j$. Consider the semidirect product $G_{l+1} = V^{d-1} \rtimes G_l$. It can be easily seen that $dl(G_{l+1}) = dl(G_l) + 1 = l + 1$ and that G_{l+1} is isomorphic to the wreath product $C_q^{d-1} \wr G_l$ with respect to the regular permutation representation of G_l . In particular, by [9, Corollary 2.4], as C_q^{d-1} and G_l have coprime orders,

$$d(G_{l+1}) = d(C_q^{d-1} \wr G_l) = \max(d(G_l), d(C_q^{d-1}) + 1) = d.$$

Now let $t = d_I(G_{l+1})$ and suppose that w_1g_1, \dots, w_tg_t , with $w_i \in V^{d-1}$ and $g_i \in G_l$, invariably generate G_{l+1} . By Corollary 7, for any $j \in \{1, \dots, t\}$

$$(d - 1)n_j \leq \sum_{i=1}^t \dim_{\mathbb{F}} C_{V_j}(g_i).$$

As in the proof of Theorem 8, this implies

$$(4.3) \quad d - 1 \leq \sum_{i=1}^t \frac{\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)}{|G_l|}.$$

Notice that g_1, \dots, g_t must invariably generate G_l so $t \geq d_I(G_l) = l(d - 1) + 1$ and in particular we may assume $g_i \neq 1$ for every $i \leq l(d - 1) + 1$. Therefore, by Lemma 9,

$$\frac{\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)}{|G_l|} \leq \frac{1}{p} \quad \text{if } i \leq l(d - 1) + 1.$$

Since the trivial bound $\dim_{\mathbb{F}} C_{\mathbb{F}G_l}(g_i)/|G_l| \leq 1$ holds for all $i = l(d - 1) + 2, \dots, t$, it follows from (4.3) that

$$d - 1 \leq \frac{l(d - 1) + 1}{p} + t - l(d - 1) - 1$$

i.e.

$$t \geq \left\lceil (l + 1)(d - 1) + 1 - \frac{l(d - 1) + 1}{p} \right\rceil.$$

Since we are assuming $l < \frac{p-1}{d-1}$, we have $\frac{l(d-1)+1}{p} < 1$ and consequently $d_I(G_{l+1}) = t \geq (l + 1)(d - 1) + 1$. On the other hand, since $dl(G_l) = l + 1$, by Proposition 12 we have $d_I(G_{l+1}) \leq (l + 1)(d - 1) + 1$ and therefore the equality $d_I(G_{l+1}) = (l + 1)(d - 1) + 1$ has been proved. \blacksquare

5. Proof of Theorem 3

PROPOSITION 14: *For every $d \in \mathbb{N}$ there exists a finite supersoluble group G such that $d(G) = d$ and $d_I(G) \geq 2d - 1$.*

Proof. Let $K = C_2^d$. There are $\alpha := 2^d - 1$ different epimorphisms $\sigma_1, \dots, \sigma_\alpha$ from K to C_2 ($\sigma_i : K \rightarrow C_2$ is uniquely determined by $M_i = \ker \sigma_i$, a $(d - 1)$ -dimensional subspace of K). To any i , there corresponds a K -module V_i defined as follows: $V_i \cong C_3$ and $v_i^k = v_i$ if $k \in M_i$, $v_i^k = v_i^2$ otherwise. Let $W_i = V_i^{d-1}$ and consider $G = \left(\prod_{1 \leq i \leq \alpha} W_i\right) \rtimes K$. The group G is supersoluble and, by Proposition 4, it is easy to see that $d(G) = d$. Now assume that g_1, \dots, g_r invariably generate G . We write $g_i = (w_{i1}, \dots, w_{i\alpha})k_i$ with $k_i \in K$ and $w_{ij} \in W_j$. In particular k_1, \dots, k_r generate K and, up to reordering the elements g_1, \dots, g_r , we can assume that the first d -elements k_1, \dots, k_d are a basis for K . Let $M = \langle k_1^{-1}k_2, \dots, k_{d-1}^{-1}k_d \rangle$. It can be easily checked that M is a maximal subgroup of K , so $M = M_j$ for some $j \in \{1, \dots, \alpha\}$. Moreover $k_i \notin M_j$ for every $i \in \{1, \dots, d\}$, in particular $C_{V_j}(k_i) = 0$ for every $i \in \{1, \dots, d\}$. On the other hand $w_{1j}k_1, \dots, w_{rj}k_r$ invariably generate G , so, by Corollary 7,

$$d - 1 \leq \sum_{1 \leq i \leq r} \dim_{\mathbb{F}_3} C_{V_j}(k_i) = \sum_{d+1 \leq i \leq r} \dim_{\mathbb{F}_3} C_{V_j}(k_i) \leq r - d.$$

Hence $r \geq 2d - 1$. ■

Proof of Theorem 3. Let F be the free pro-supersoluble group of rank $d \geq 2$. By Proposition 14, there exists a finite supersoluble d -generated group G such that $d_I(G) \geq 2d - 1$. Hence $d_I(F) \geq 2d - 1$.

To prove the converse, since $d_I(F) = d_I(F/\text{Frat}(F))$, it suffices to consider $G = F/\text{Frat } F$. By [10, Proposition 3.3], G' is abelian hence $dl(G) \leq 2$ and it follows from Proposition 12 that $d_I(G) \leq 2d - 1$. Therefore $d_I(F) = 2d - 1$. ■

References

1. E. Detomi, A. Lucchini, Invariable generation with elements of coprime prime-power order, arXiv:1409.0997
2. J. D. Dixon, Random sets which invariably generate the symmetric group, *Discrete Math.* 105 (1992) 25-39.
3. K. Doerk, T. Hawkes, Finite soluble groups, de Gruyter Expositions in Mathematics (4) Walter de Gruyter & Co. Berlin, 1992.
4. W. Gaschütz, Die Eulersche Funktion endlicher auflösbarer Gruppen, *Illinois J. Math.* 3 (1959) 469-476.

5. K. Gruenberg, Relation modules of finite groups, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 25. American Mathematical Society, Providence, R.I., 1976.
6. T. W. Hungerford, Algebra, Reprint of the 1974 original, Graduate Texts in Mathematics 73, Springer-Verlag, New York, Berlin, 1980.
7. W. M. Kantor, A. Lubotzky, A. Shalev, Invariable generation and the Chebotarev invariant of a finite group, *J. Algebra* 348 (2011), 302–314.
8. W. M. Kantor, A. Lubotzky, A. Shalev, Invariable generation of infinite group, arXiv:1407.4631
9. A. Lucchini, Generating wreath products, *Arch. Math. (Basel)* 62 (1994), no. 6, 481–490.
10. B. C. Oltikar and Luis Ribes, On prosupersolvable groups, *Pacific J. Math.* 77 (1978), no. 1, 183–188.
11. L. Ribes, L., P. Zalesskii, Profinite groups. A Series of Modern Surveys in Mathematics, 40. Springer-Verlag, Berlin (2000).
12. U. Stambach, Cohomological characterisations of finite solvable and nilpotent groups, *J. Pure Appl. Algebra* 11 (1977/78), no. 1–3, 293–301.
13. J. Wiegold, Transitive groups with fixed-point-free permutations, *Arch. Math. (Basel)* 27 (1976), 473–475.
14. J.S Wilson, Profinite Groups, Clarendon Press, Oxford (1998).