Rend. Sem. Mat. Univ. Padova, Vol. 133 (2015)
DOI 10.4171/RSMUP/133-12

Automorphism-invariant modules

Adel Alahmadi (*) - Alberto Facchini (**) - Nguyen Khanh Tung (***)

Abstract - A module M is called *automorphism-invariant* if it is invariant under automorphisms of its injective envelope. In this paper, we study the endomorphism rings of automorphism-invariant modules and their injective envelopes. We investigate some cases where automorphism-invariant modules are quasi-injective and a connection between automorphism-invariant modules and boolean rings.

Mathematics Subject Classification (2010). 16D50; 16U60, 16W20.

Keywords. Automorphism-invariant modules, injective envelopes.

1. Introduction

A module M is called *automorphism-invariant* if it is invariant under automorphisms of its injective envelope, that is, if $\varphi(M) \subseteq M$ for every $\varphi \in \mathrm{Aut}(E(M))$ (equivalently, if $\varphi(M) = M$ for every $\varphi \in \mathrm{Aut}(E(M))$). In [6, Theorem 16], it was shown that automorphism-invariant modules are

(*) Indirizzo dell'A.: Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.
E-mail: analahmadi@kau.edu.sa
(**) Indirizzo dell'A.: Dipartimento di Matematica, Università di Padova, 35121 Padova, Italy.
E-mail: facchini@math.unipd.it
The second author is grateful to Università di Padova (Progetto di ricerca di Ateneo CPDA105885/10 "Differential graded categories" and Progetto ex 60% "Anelli e categorie di moduli") and Fondazione Cassa di Risparmio di Padova e Rovigo (Progetto di Eccellenza "Algebraic structures and their applications") for the financial support.
(***) Indirizzo dell'A.: Dipartimento di Matematica, Università di Padova, 35121 Padova, Italy.
E-mail: khanhtung06@yahoo.com

precisely *pseudo-injective modules*, where a module M is called *pseudo-injective* if, for any submodule A of M, every monomorphism $f: A \to M$ can be extended to an element of End(M). In this paper, we show that if M is an automorphism-invariant module, then

$$\mathrm{End}(M)/J(\mathrm{End}(M))$$

turns out to be a rationally closed subring of

$$\mathrm{End}(E(M))/J(\mathrm{End}(E(M))).$$

Both the rings $\mathrm{End}(M)/J(\mathrm{End}(M))$ and $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ are von Neumann regular [11, Proposition 1]. We consider in particular the case of automorphism-invariant modules of finite Goldie dimension or indecomposable. Notice that automorphism-invariant modules have the exchange property [11], so that indecomposable automorphism-invariant modules have a local endomorphism ring. Moreover, idempotents can be lifted modulo every right ideal both in End(M) and in End($E(M)$) [16].

We then study the connection between automorphism-invariant modules and boolean rings. The existence of such a connection was suggested to us by the results in Section 5 of [19], where Vámos considers modules whose endomorphism ring (or endomorphism ring modulo the Jacobson radical) is a boolean ring. He studies modules in which the identity endomorphism is the sum of two automorphisms. This condition is related to the existence of factors of the endomorphism ring isomorphic to the field $\mathbb{F}_2$ with two elements [12]. Notice that if M is an automorphism-invariant right R-module and End(M) has no factor isomorphic to $\mathbb{F}_2$, then M is quasi-injective [10, Theorem 3].

Every automorphism-invariant module is the direct sum of a quasi-injective module and a square-free module [6, Theorem 3]. This leads us to study, for an automorphism-invariant square-free module M, the relation between M being quasi-injective and the existence of factors isomorphic to $\mathbb{F}_2$ in End(M) and in End($E(M)$).

Throughout, all rings have identity element and modules are right unital. For a module M, $E(M)$ denotes the injective envelope of M.

2. Notation, definitions, and some properties of automorphism-invariant modules

Let R be a ring. For every pair of right R-modules M and N, let $\Delta(M, N)$ denote the set of all module morphisms $f: M \to N$ whose kernel ker (f) is an essential submodule of M. In [11, Proposition 1], it was shown that if M is an automorphism-invariant module, then the Jacobson radical $J(\mathrm{End}(M))$ of $\mathrm{End}(M)$ consists exactly of all the endomorphisms of M with essential kernel, that is, $J(\mathrm{End}(M) = \Delta(M, M)$. Moreover, $\mathrm{End}(M)/J(\mathrm{End}(M)$ is von Neumann regular and idempotents lift modulo $J(\mathrm{End}(M))$.

Recall that the Jacobson radical $\mathcal{J}$ of a preadditive category $\mathcal{C}$ is the ideal defined, for every $A, B \in \mathrm{Ob}(\mathcal{C})$, by $\mathcal{J}(A, B) := \{f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid 1_A - gf$ has a left inverse (equivalently, a two-sided inverse) for every morphism $g: B \to A$ in $\mathcal{C}\}$. Let $\mathcal{A}$ be the full subcategory of Mod-R whose objects are all automorphism-invariant right R-modules. It is easily seen that Δ is an ideal of $\mathcal{A}$ contained in the Jacobson radical $\mathcal{J}$ of $\mathcal{A}$, because if $f \in \Delta(M, N)$, then $\ker(gf)$ is essential in M for every g. Hence $gf \in J(\mathrm{End}(M))$ for every g [11, Proposition 1]. Thus for the ideals Δ and $\mathcal{J}$ of $\mathcal{A}$, we have that $\Delta \subseteq \mathcal{J}$.

Nevertheless the ideal Δ can be properly contained in $\mathcal{J}$. For instance, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are automorphism-invariant $\mathbb{Z}$-modules, the monomorphism $f: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ does not have an essential kernel, but for every morphism $g: \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ the composite mapping gf is the zero mapping. Hence gf has an essential kernel, and so gf belongs to $J(\mathrm{End}(\mathbb{Z}/2\mathbb{Z}))$. Therefore $\Delta(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \subset \mathcal{J}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.

A ring morphism $\varphi: R \to S$ is *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R. We will denote by $U(R)$ the group of units, i.e., invertible elements, of a ring R. A *rationally closed* subring of a ring S is a subring R of S such that the embedding $R \to S$ is a local morphism, that is, a subring R of S such that $U(R) = R \cap U(S)$ [4].

Recall that a module M satisfies Condition (C_1) if every submodule of M is essential in a direct summand of M. An automorphism-invariant module satisfies (C_1) if and only if it is quasi-injective [13]. Every automorphism-invariant module M satisfies Condition (C_2); that is, every submodule of M isomorphic to a direct summand of M is itself a direct summand of M [5], [6]. Moreover, every automorphism-invariant module M satisfies Condition (C_3); that is, if N_1 and N_2 are direct summands of M with $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is also a direct summand of M = [13].

THEOREM 2.1. *Let M be an automorphism-invariant module and $E(M)$ be its injective envelope. Then*

(a) *There is a canonical local morphism*

$$\varphi: \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$$

with kernel $J(\mathrm{End}(M))$, so that φ induces an embedding $\overline{\varphi}$, as a rationally closed subring, of the von Neumann regular ring

$\mathrm{End}(M)/J(\mathrm{End}(M))$ *into the von Neumann regular right self-injective ring*

$$\mathrm{End}(E(M))/J(\mathrm{End}(E(M))).$$

(b) *For every invertible element* v *of the ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$, *there exists an invertible element* u *of* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *such that* $\overline{\varphi}(u) = v$.

(c) *For every idempotent element* f *of the ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ *there exists an idempotent element* e *of* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *such that* $\overline{\varphi}(e) = f$ *if and only if the module* M *is quasi-injective.*

(d) *If* M *is quasi-injective, then* $\overline{\varphi}$ *is an isomorphism.*

PROOF. (a) For any right R-module M, the morphism

$$\varphi\colon \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$$

is defined as follows. If $f \in \mathrm{End}(M)$, let $\tilde{f}$ be an endomorphism of $E(M)$ that extends f. Then $\varphi(f) = \tilde{f} + J(\mathrm{End}(E(M)))$ [8, §4, p. 412]. It is easily seen that φ is a well-defined ring morphism. Moreover, φ is a local morphism, because if $f \in \mathrm{End}(M)$ and $\varphi(f)$ is invertible in the ring $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$, then $\tilde{f}$ is an automorphism of $E(M)$. Since M is automorphism-invariant, it follows that $\tilde{f}(M) = M$; that is, $f(M) = M$. This proves that f is onto. Moreover, $\tilde{f}$ is an automorphism of $E(M)$ implies that its restriction f is an injective endomorphism of M. Thus f is an automorphism, and the ring morphism φ is a local morphism. It follows that the injective morphism $\overline{\varphi}\colon \mathrm{End}(M)/\ker(\varphi) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ induced by φ is a local morphism as well. Moreover, $\ker(\varphi) = \Delta(M, M) = J(\mathrm{End}(M))$.

(b) If v is an invertible element of $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$, then $v = v' + J(\mathrm{End}(E(M)))$ for some element $v' \in \mathrm{End}(E(M))$, necessarily invertible. Therefore v' is an automorphism of $E(M)$. Since M is automorphism-invariant, the restriction u' of v' to M is an automorphism of M. Thus $u := u' + J(\mathrm{End}(M))$ is an invertible element of $\mathrm{End}(M)/J(\mathrm{End}(M))$ and $\overline{\varphi}(u) = v$.

(c) If M is quasi-injective, for every $f \in \mathrm{End}(E(M))$, the restriction f' of f to M is an endomorphism of M. Thus $\overline{\varphi}(f' + J(\mathrm{End}(M))) = f + J(\mathrm{End}(E(M)))$. Hence $\overline{\varphi}$ is onto, and (a) allows us the conclusion.

(d) Assume that for every idempotent element $f \in \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ there exists an idempotent element e of $\mathrm{End}(M)/J(\mathrm{End}(M))$ with $\overline{\varphi}(e) = f$. In order to show that M is quasi-injective, we will prove that it satisfies Condition (C_1). Let N be a submodule of M. We must show that N is essential in a direct summand of M. Now $E(M)$ has a direct-sum decomposition $E(M) = E(N) \oplus E$. Thus there is an idempotent $\varepsilon \in \operatorname{End}(E(M))$ with $E(N) = \varepsilon E(M)$ and $E = (1-\varepsilon)E(M)$. By hypothesis, there exists an idempotent $e \in \operatorname{End}(M)/J(\operatorname{End}(M))$ with $\overline{\varphi}(e) = \varepsilon + J(\operatorname{End}(E(M)))$. As idempotents lift modulo $J(\operatorname{End}(M))$, there is an idempotent $\varepsilon' \in \operatorname{End}(M)$ such that $e = \varepsilon' + J(\operatorname{End}(M))$. The idempotent $\varepsilon' \in \operatorname{End}(M)$ corresponds to a direct-sum decomposition $M = \varepsilon' M \oplus (1-\varepsilon')M$. This direct-sum decomposition of M induces a direct-sum decomposition $E(M) = E(\varepsilon' M) \oplus E((1-\varepsilon')M)$. Thus there is an idempotent $\varepsilon'' \in \operatorname{End}(E(M))$ with $E(\varepsilon' M) = \varepsilon'' E(M)$ and $E((1-\varepsilon')M) = (1-\varepsilon'')E(M)$. We claim that endomorphism ε'' of $E(M)$ extends the endomorphism ε' of M. To prove this claim, it suffices to show that $\varepsilon''(x) = x$ for every $x \in \varepsilon' M$ and $\varepsilon''(y) = 0$ for every $y \in (1-\varepsilon')M$. Now $\varepsilon' M \subseteq E(\varepsilon' M) = \varepsilon'' E(M)$, so that $\varepsilon''(x) = x$ for every $x \in \varepsilon' M$. Similarly $(1-\varepsilon')M \subseteq E((1-\varepsilon')M) = (1-\varepsilon'')E(M)$, so that for every $y \in (1-\varepsilon')M$ one has that $y \in (1-\varepsilon'')E(M)$. Hence $\varepsilon''(y) = 0$. This proves the claim. Thus $\overline{\varphi}(\varepsilon' + J(\operatorname{End}(M)) = \varepsilon'' + J(\operatorname{End}(E(M_R)))$. But $\overline{\varphi}(e) = \varepsilon + J(\operatorname{End}(E(M)))$ and $e = \varepsilon' + J(\operatorname{End}(M))$, so that $\overline{\varphi}(\varepsilon' + J(\operatorname{End}(M))) = \varepsilon + J(\operatorname{End}(E(M)))$. It follows that $\varepsilon'' + J(\operatorname{End}(E(M))) = \varepsilon + J(\operatorname{End}(E(M)))$; that is, $\varepsilon'' - \varepsilon \in J(\operatorname{End}(E(M)))$, so $1 - \varepsilon'' + \varepsilon$ is an automorphism of $E(M)$. As M is automorphism-invariant, we have that $(1-\varepsilon''+\varepsilon)(M) = M$. Thus $\varepsilon(M) \subseteq (1-\varepsilon''+\varepsilon)(M) + 1(M) + \varepsilon''(M) = M + M + \varepsilon'(M) = M$. It follows that ε restricts to an idempotent endomorphism of M. In particular, $\varepsilon(M)$ is a direct summand of M. Moreover, $N \subseteq E(N) \cap M = \varepsilon E(M) \cap M = \varepsilon(M)$, so that N is a submodule of $\varepsilon(M)$. It remains to show that N is essential in $\varepsilon(M)$. This follows immediately from the fact that $\varepsilon(M) \subseteq \varepsilon E(M) = E(N)$ and N is essential in $E(N)$. This proves that M satisfies Condition (C_1), and hence is quasi-injective [13].

The converse follows immediately from (d), noting that the inverse image of an idempotent via an injective morphism is necessarily idempotent. □

Notice that if $\varphi: R \to S$ is a local morphism and S is local (resp., semilocal), that is, a division ring (resp., a semisimple artinian ring) modulo the Jacobson radical, then R is local (resp., semilocal) [4, Theorem 1]. Nevertheless, for any ring R there exist a von Neumann regular right self-injective ring S and a local morphisms $\chi: R \to S$. For example, it suffices to take the first part $0 \to R_R \to E_1 \to E_2$ of an injective resolution of the R-module R_R, the von Neumann regular right self-injective ring $S := \operatorname{End}(E_1)/J(\operatorname{End}(E_1)) \times \operatorname{End}(E_2)/J(\operatorname{End}(E_2))$ and the local morphism χ defined in [8, Theorem 5.3].

PROPOSITION 2.2. *Let M be an automorphism-invariant module. Then*

(a) *If M is indecomposable, then* $\mathrm{End}(M)$ *is a local ring.*

(b) *If M has finite Goldie dimension, then every injective endomorphism of M is an automorphism of M and the endomorphism ring* $\mathrm{End}(M)$ *is a semiperfect ring.*

PROOF. (a) Automorphism-invariant modules have the exchange property [11], and indecomposable modules with the exchange property have a local endomorphism ring [7, Theorem 2.8].

(b) Let M be an automorphism-invariant module of finite Goldie dimension and let $\varphi\colon M \to M$ be an injective endomorphism of M. Then φ extends to an endomorphism $\varphi_0\colon E(M) \to E(M)$, which is necessarily injective. As M has finite Goldie dimension, φ_0 is an automorphism of $E(M)$. But M is automorphism-invariant, so $\varphi_0(M) = M$. Thus $\varphi(M) = M$, that is, the endomorphism φ is also surjective.

Finally, every module of finite Goldie dimension is a direct sum of indecomposable modules. Thus if $M = M_1 \oplus \dots \oplus M_n$ is automorphism-invariant and the M_i are indecomposable, then the modules M_i are automorphism-invariant. Hence they have a local endomorphism ring by (a). Therefore M is a direct sum of modules M_i with local endomorphism rings, so that $\mathrm{End}(M)$ is semiperfect. □

COROLLARY 2.3. *If M, N are two automorphism-invariant R-modules of finite Goldie dimensions isomorphic to submodules of each other, then M is isomorphic to N.*

PROOF. By the hypothesis, there exists two monomorphisms $f : M \to N$ and $g : N \to M$. So $fg \in \mathrm{End}(N)$ and fg is injective. Hence fg is an automorphism by Proposition 2.2(b). Thus f is onto. Since f is a monomorphism, f is an isomorphism. □

We now recall a theorem by Guil Asensio and Srivastava that will be repeatedly used in the sequel.

THEOREM 2.4 [10, Theorem 3]. *Let M be a right module such that* $\mathrm{End}(M)$ *has no factor isomorphic to $\mathbb{F}_2$. Then M is quasi-injective if and only if M is automorphism-invariant.*

PROPOSITION 2.5. *If R is a ring of odd characteristic, then every automorphism-invariant R-module is quasi-injective.*

PROOF. Suppose that R is a ring of odd characteristic n with a module M that is automorphism-invariant but not quasi-injective. By Theorem 2.4, the endomorphism ring $\mathrm{End}(M)$ has a factor $\mathrm{End}(M)/\mathcal{M}$ isomorphic to $\mathbb{F}_2$. Then $nR = 0$, so that $nM = 0$. Hence $n\mathrm{End}(M) = 0$, so that $n(\mathrm{End}(M)/\mathcal{M}) = 0$. Thus $n\mathbb{F}_2 = 0$, which is a contradiction because n is odd. $\square$

LEMMA 2.6. *Let M be an automorphism-invariant module. If $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where each M_i is a quasi-injective module, then M is quasi-injective.*

PROOF. It clearly suffices to prove the case $n = 2$. Assume that $M = M_1 \oplus M_2$ is automorphism-invariant, where M_1 and M_2 are quasi-injective. By [13, Theorem 5], M_1 is M_2-injective and M_2 is M_1-injective. Since M_1 and M_2 are quasi-injective, M is quasi-injective by [1, Propositions 16.10 and 16.13]. $\square$

PROPOSITION 2.7. *Let $M_1, M_2, \ldots, M_n$ be uniform modules. If $M := M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is automorphism-invariant, then M is quasi-injective.*

PROOF. By the previous lemma, it suffices to show that each M_i is quasi-injective. On the one hand, each M_i is uniform, and each M_i satisfies (C_1). On the other hand, each M_i is automorphism-invariant, being a direct summand of an automorphism-invariant module. By [13, Corollary 13], every M_i is quasi-injective. Now apply Lemma 2.6. $\square$

PROPOSITION 2.8. *The following conditions are equivalent for a ring R.*

(1) *Every automorphism-invariant R-module of finite Goldie dimension is quasi-injective.*
(2) *Every automorphism-invariant indecomposable R-module of finite Goldie dimension is uniform.*
(3) *Every automorphism-invariant indecomposable R-module of finite Goldie dimension is quasi-injective.*

PROOF. (1) $\Rightarrow$ (2) An automorphism-invariant indecomposable module M of finite Goldie dimension is quasi-injective by (1). Hence it satisfies Condition (C_1). Therefore M is uniform.

(2) $\Rightarrow$ (3) Let M be an automorphism-invariant indecomposable module of finite Goldie dimension. Then M is uniform by (2), and hence it satisfies Condition (C_1). By [13, Corollary 13], M is quasi-injective.

(3) $\Rightarrow$ (1) Let M be an automorphism-invariant module of finite Goldie dimension. Then $\mathrm{End}(M)$ is semilocal by Proposition 2.2(b). So $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where each M_i is an automorphism-invariant indecomposable module of finite Goldie dimension. By (3), every M_i is quasi-injective. From Lemma 2.6, it follows that M is quasi-injective. $\square$

By Proposition 2.5, every ring R of odd characteristic satisfies the equivalent conditions of Proposition 2.8.

Let $E(M)$ be the injective envelope of a module M. It is easily seen that

$$\sum_{\varphi \in \mathrm{Aut}(E(M))} \varphi(M)$$

is the smallest automorphism-invariant submodule of $E(M)$ containing M. We call it the *automorphism-invariant envelope* of M, and denote it by $AI(M)$. Clearly, a module is automorphism-invariant if and only if $M = AI(M)$.

LEMMA 2.9. *Let M, N be arbitrary R-modules. Then every monomorphism $M \to N$ extends to a monomorphism $AI(M) \to AI(N)$.*

PROOF. A monomorphism $\varphi: M \to N$ extends to a monomorphism $\varphi': E(M) \to E(N)$, which is necessarily a split monomorphism. Thus there is a direct-sum decomposition $E(N) = \varphi'(E(M)) \oplus C$ and, with respect to this direct-sum decomposition, $\varphi': E(M) \to \varphi'(E(M)) \oplus C$ can be written in matrix form as $\varphi' = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, where $\alpha: E(M) \to \varphi'(E(M))$ is an isomorphism. It suffices to show that $\varphi'(AI(M)) \subseteq AI(N)$. Let f be an automorphism of $E(M)$. Then $\begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ is an automorphism of $\varphi'(E(M)) \oplus C = E(N)$. Thus $\varphi'(f(M)) = \alpha f(M) = (\alpha f \alpha^{-1})(\alpha(M)) \subseteq \begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}(\alpha(M)) \subseteq \begin{pmatrix} \alpha f \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N) \subseteq AI(N)$.

Therefore $\varphi'(AI(M)) \subseteq AI(N)$. $\square$

3. Boolean rings

Recall that a non-zero ring R is a *boolean ring* if every element of R is idempotent. Every boolean ring is necessarily a commutative ring of characteristic 2. A ring is boolean if and only if it is isomorphic to a subring of $\mathbb{F}_2^X$, where X is a non-empty set and $\mathbb{F}_2$ is the field with two elements. If a ring R is boolean, then $E(R_R)$ is the Dedekind-McNeil completion of R, and the Dedekind-McNeil completion of R is a boolean ring, which is isomorphic, as an R-module, to the total ring of quotients of R ([2, Corollary 2] and [18, p. 79]). If R is a boolean ring, the identity is the unique automorphism of the R-module $E(R_R)$, so that every R-submodule of $E(R_R)$ is automorphism-invariant, but R_R is not quasi-injective (except for the case $R_R = E(R_R)$, that is, R is complete).

The condition “the endomorphism ring S of a module does not have maximal ideals $\mathcal{M}$ with $S/\mathcal{M} \cong \mathbb{F}_2$” has recently received a lot of attention (Theorem 2.4 and [12]).

LEMMA 3.1. *Let T be a ring and I the two-sided ideal of T generated by the subset $\{t - t^2 \mid t \in T\}$ of T. Then*

(a) *The ideal I is the smallest ideal of T with T/I a boolean ring or the zero ring.*
(b) *The ideal I is the intersection of all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$.*
(c) *The ideal I contains the Jacobson radical $J(T)$ of T.*
(d) *The kernel of every ring morphism $T \to \mathbb{F}_2$ contains I.*
(e) *I is a proper ideal of T if and only if there exists a ring morphism $T \to \mathbb{F}_2$, if and only if T has a maximal two-sided ideal $\mathcal{M}$ with $T/\mathcal{M} \cong \mathbb{F}_2$.*

PROOF. (a) is trivial.

(b) Let us check that

$$I = \bigcap_{T/\mathcal{M}\cong\mathbb{F}_2} \mathcal{M}.$$

($\subseteq$) Since I is generated by the elements $t - t^2$, it suffices to show that $t - t^2 \in \mathcal{M}$ for every $t \in T$ and every maximal two-sided ideal $\mathcal{M}$ with $T/\mathcal{M} \cong \mathbb{F}_2$. Now $\mathbb{F}_2$ is boolean, so that $T/\mathcal{M}$ is boolean, hence $t + \mathcal{M} = t^2 + \mathcal{M}$. It follows that $t - t^2 \in \mathcal{M}$.

($\supseteq$) By (a), the ring T/I is boolean. Boolean rings are isomorphic to subrings of $\mathbb{F}_2^X$ for some set X. Let $\varepsilon\colon T/I \to \mathbb{F}_2^X$ be an embedding and $\pi_x\colon \mathbb{F}_2^X \to \mathbb{F}_2$ $(x \in X)$, $p\colon T \to T/I$ be the canonical projections. Then the morphisms $\varphi_x := \pi_x\varepsilon p\colon T \to \mathbb{F}_2$ have kernels ker φ_x, which are maximal two-sided ideals of T, $T/\ker\ \varphi_x \cong \mathbb{F}_2$ and $\bigcap_{x\in X} \ker\ \varphi_x = I$. Thus $\bigcap_{T/\mathcal{M}\cong\mathbb{F}_2} \mathcal{M} \subseteq \bigcap_{x\in X} \ker\ \varphi_x = I$.

(c) By (b), I is the intersection of all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$, and all maximal two-sided ideals $\mathcal{M}$ of T with $T/\mathcal{M} \cong \mathbb{F}_2$

are maximal right ideals of T. Hence I is an intersection of maximal right ideals of T, so that $I \supseteq J(T)$.

(d) The kernel of every ring morphism $T \to \mathbb{F}_2$ is a maximal two-sided ideal of T with $T/\mathcal{M} \cong \mathbb{F}_2$. Thus (d) follows from (b).

(e) is now trivial. $\square$

Notice that -1_T is an element of T, so that $(-1_T) - (-1_T)^2 = -2 \cdot 1_T \in I$. Therefore I contains the two-sided ideal $2T$ of T. For example, for $T = \mathbb{Z}$, the ideal of $\mathbb{Z}$ generated by all $n - n^2$, where n ranges in $\mathbb{Z}$, is exactly the ideal $2\mathbb{Z}$.

Recall that two modules are said to be *orthogonal* to each other if they do not contain non-zero isomorphic submodules. The following Lemma appears in [15, Lemma 3.3] and will be repeatedly used in the sequel.

LEMMA 3.2 [15, Lemma 3.3]. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are orthogonal, then* $\mathrm{End}(M)/\Delta(M,M) \cong \mathrm{End}(M_1)/\Delta(M_1,M_1) \times \mathrm{End}(M_2)/\Delta(M_2,M_2)$. *The converse holds if M_1 and M_2 are relatively injective.*

COROLLARY 3.3. *Let $M = M_1 \oplus M_2$ be an automorphism-invariant R-module where M_1 and M_2 are orthogonal. Then* $\mathrm{End}(M)$ *has no factor isomorphic to $\mathbb{F}_2$ if and only if each* $\mathrm{End}(M_i)$ $(i = 1, 2)$ *has no factor isormophic to $\mathbb{F}_2$.*

PROOF. Let I be the two-sided ideal of $\mathrm{End}(M)$ generated by the set $\{x - x^2 \mid x \in \mathrm{End}(M)\}$. By Lemma 3.2, $\mathrm{End}(M)/\Delta(M,M) \cong \mathrm{End}(M_1)/\Delta(M_1,M_1) \times \mathrm{End}(M_2)/\Delta(M_2,M_2)$. As $\Delta(M,M) = J(\mathrm{End}(M))$ for any automorphism-invariant R-module M, it follows that $\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathrm{End}(M_1)/J(\mathrm{End}(M_1)) \times \mathrm{End}(M_2)/J(\mathrm{End}(M_2))$ in a canonical way. Thus there is a homomorphism $\mathrm{End}(M) \to \mathbb{F}_2$ if and only if there is a homomorphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$, if and only if there is a homomorphism $\mathrm{End}(M_i)/J(\mathrm{End}(M_i)) \to \mathbb{F}_2$ for an i equal to 1 or 2. The conclusion follows immediately. $\square$

Recall that a module is *square-free* if it does not contain a direct sum of two non-zero isomorphic submodules. The following important result is due to Er, Singh and Srivastava.

THEOREM 3.4 [6, Theorem 3]. *Every automorphism-invariant module M decomposes as a direct sum $M = X \oplus Y$, where X is quasi-injective, Y is a square-free module orthogonal to X, and X and Y are relatively injective modules.*

The previous Theorem reduces our study to considering automorphism-invariant square-free modules. Notice that if M is any right R-module such that $\mathrm{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$, then M is quasi-injective if and only if M is automorphism-invariant (Theorem 2.4).

LEMMA 3.5. *If M_1, M_2 are two right modules over a ring R and M_1, M_2 have isomorphic injective envelopes, which are non-zero modules, then M_1 and M_2 have non-zero isomorphic submodules.*

PROOF. Let $f: E(M_1) \to E(M_2)$ be an isomorphism. Then M_1 and $f^{-1}(M_2)$ are essential submodules of $E(M_1)$. Hence $M_1 \cap f^{-1}(M_2)$ is an essential submodule of $E(M_1)$. It follows that $M_1 \cap f^{-1}(M_2)$ is a non-zero submodule of M_1. Via the isomorphism f, we find that $f(M_1 \cap f^{-1}(M_2))$ is an essential submodule of $E(M_2)$ isomorphic to $M_1 \cap f^{-1}(M_2)$. But $f(M_1 \cap f^{-1}(M_2)) = f(M_1) \cap M_2$ is a submodule of M_2. $\square$

COROLLARY 3.6. *A module M is square-free if and only if its injective envelope $E(M)$ is square-free.*

PROOF. If M is not square-free, then it contains a submodule isomorphic to $N \oplus N$ for some non-zero module N. Hence the same holds for $E(M)$, that is, $E(M)$ is not square-free.

Conversely, assume that $E(M)$ is not square-free. Then $E(M)$ contains a submodule isomorphic to $N \oplus N$ for some non-zero module N. It follows that $E(M) = E_1 \oplus E_2 \oplus E_3$ with $E_1 \cong E_2 \neq 0$. Then $M \cap E_i$ is a non-zero essential submodule of E_i for $i = 1, 2$. In particular, $M \cap E_1$ and $M \cap E_2$ have isomorphic injective envelopes, which are non-zero modules. By Lemma 3.5, $M \cap E_1$ and $M \cap E_2$ have non-zero isomorphic submodules. Thus M is not square-free. $\square$

COROLLARY 3.7. *If M is an automorphism-invariant square-free module, then every injective endomorphism of M is an automorphism of M.*

PROOF. Let M be an automorphism-invariant square-free module and let $\varphi: M \to M$ be an injective endomorphism of M. Then φ extends to an endomorphism $\varphi_0: E(M) \to E(M)$, which is necessarily injective. Then $E(M) = \varphi_0(E(M)) \oplus C$, so that $E(M) = \varphi_0^2(E(M)) \oplus \varphi_0(C) \oplus C$ with $\varphi_0(C) \cong C$. By Corollary 3.6, $E(M)$ is square-free, so $C = 0$. This proves that φ_0 is an automorphism of $E(M)$. But M is automorphism-invariant, so $\varphi_0(M) = M$. Thus $\varphi(M) = M$, that is, the endomorphism φ of M is also surjective. $\square$

Arguing as in Corollary 2.3, we find that:

COROLLARY 3.8. *If M, N are two automorphism-invariant square-free R-modules isomorphic to submodules of each other, then M is isomorphic to N.*

COROLLARY 3.9. *Let M be an automorphism-invariant R-module. Assume that $M = M_1 \oplus M_2$, where M_1 and M_2 are orthogonal. Let E_i be an injective envelope of M_i. Then E_1 is orthogonal to E_2.*

PROOF. Assume that there exists $0 \neq N_1 \leq E_1$ and $0 \neq N_2 \leq E_2$ such that $N_1 \cong N_2$. Let E'_i be an injective envelope of N_i. Then $E_1 = E'_1 \oplus E''_1$ and $E_2 = E'_2 \oplus E''_2$ where $E'_1 \cong E'_2$. Set $E_R := E_1 \oplus E_2 = E'_1 \oplus E'_2 \oplus (E''_1 \oplus E''_2)$. Then E is an injective envelope of M. Since M is automorphism-invariant and $E'_1 \cong E'_2$, we get that $M = (M \cap E'_1) \oplus (M \cap E'_2) \oplus (M \cap (E''_1 \oplus E''_2))$ [17, Lemma 7]. We will show that $M \cap E'_1 \leq M_1$. Let $x \in M \cap E'_1$, then $x = x_1 + x_2$ where $x_i \in M_i$ and $x \in E'_1 \subseteq E_1$. Hence $x_2 = x - x_1 \in M_2 \cap E_1 \subseteq E_2 \cap E_1 = 0$. Therefore $x = x_1 \in M_1$. By a similar argument, we get that $M \cap E'_2 \leq M_2$. As $M \cap E'_i$ is essential in E'_i and E'_i is injective, E'_i is an injective envelope of $M \cap E'_i$. Moreover, $E'_1 \cong E'_2$. Hence, by Lemma 3.5, there exist non-zero submodules $P_!1 \leq M \cap E'_1 \leq M_1$ and $P_2 \leq M \cap E'_2 \leq M_2$ such that $P_1 \cong P_2$. Therefore M_1 is not orthogonal to M_2. This is a contradiction. $\square$

PROPOSITION 3.10. *Let M be an automorphism-invariant module and $E(M)$ be its injective envelope. The following conditions are equivalent:*

(a) *M is square-free.*
(b) *$E(M)$ is square-free.*
(c) *The von Neumann regular ring* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *is abelian.*
(d) *The von Neumann regular right self-injective ring* $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$ *is abelian.*

PROOF. (a) $\Leftrightarrow$ (b) has been proved in Corollary 3.6.

(b) $\Rightarrow$ (d) follows from the fact that $\Delta(E, E) = J(\mathrm{End}(M))$ for any injective module E and [15, Lemma 3.4].

(d) $\Rightarrow$ (c) follows from the fact that every subring of an abelian ring is an abelian ring and Theorem 2.1.

(c) $\Rightarrow$ (a) Assume that (c) holds. Set $S := \mathrm{End}(M)$. Suppose that M contains a direct sum $X \oplus Y$ of two isomorphic submodules. Taking the injective envelopes in $E(M)$, one finds that $E(M) = E(X) \oplus E(Y) \oplus C$. If $\varphi: X \to Y$ is an isomorphism, φ extends to an isomorphism $\psi: E(X) \to E(Y)$. Thus there is an isomorphism

$$\omega := \begin{pmatrix} 0 & \psi^{-1} & 0 \\ \psi & 0 & 0 \\ 0 & 0 & 1_C \end{pmatrix} : E(M) = E(X) \oplus E(Y) \oplus C \to E(M) = E(X) \oplus E(Y) \oplus C.$$

The automorphism ω of $E(M)$ restricts to an automorphism ω' of M because M is automorphism-invariant. From [17, Lemma 7], we know that $M = (M \cap E(X)) \oplus (M \cap E(Y)) \oplus (M \cap C)$. Thus $M = e_1M \oplus e_2M \oplus e_3M$ for orthogonal idempotents $e_i \in S$, where $e_1M = M \cap E(X)$ and $e_2M = M \cap E(Y)$. Now $\omega'(M \cap E(X)) = \omega(M \cap E(X)) = \omega(M) \cap \omega(E(X)) = M \cap E(Y)$. Thus $M \cap E(X)) \cong M \cap E(Y)$, that is, $e_1M \cong e_2M$. Applying the functor $\mathrm{Hom}(M, -)\colon \text{Mod-}R \to \text{Mod-}S$, one finds that $S_S = e_1S \oplus e_2S \oplus e_3S$ and $e_1S_S \cong e_2S_S$ [7, Theorem 4.7]. If $\overline{e_i}$ is the image of e_i in $S/J(S)$, then $S/J(S) = \overline{e_1}S/J(S) \oplus \overline{e_2}S/J(S) \oplus \overline{e_3}S/J(S)$ and $\overline{e_1}S/J(S) \cong \overline{e_2}S/J(S)$ [1, Exercise 7.2]. But $S/J(S)$ is abelian, so that $\overline{e_1}S/J(S) \cong \overline{e_2}S/J(S)$ implies $\overline{e_1}S/J(S) = \overline{e_2}S/J(S)$ [9, Theorem 3.4]. From this we get that $\overline{1 - e_1}\overline{e_2} = \overline{0}$, i.e., $\overline{e_2} = \overline{e_1e_2}$. Similarly $\overline{e_1} = \overline{e_2e_1}$. Thus $e_1 - e_2$ is an idempotent in $J(S)$, from which $e_1 = e_2$. Thus $M \cap E(X) = M \cap E(Y)$, and $X = Y = 0$. □

The next Corollary generalizes [14]. Recall that a ring is *duo* if all its right ideals and all its left ideals are two-sided ideals. A ring is *quasi-duo* if all its maximal right ideals and all its maximal left ideals are two-sided ideals.

COROLLARY 3.11. *The endomorphism ring of an automorphism-invariant square-free module is quasi-duo.*

PROOF. Let M be an automorphism-invariant square-free module. By Proposition 3.10, $\mathrm{End}(M)/J(\mathrm{End}(M))$ is an abelian von Neumann regular ring. One-sided ideals of abelian regular rings are generated by central idempotents, hence all of them are two-sided. Thus $\mathrm{End}(M)/J(\mathrm{End}(M))$ is a duo ring. The conclusion now follows from the fact that a ring S is quasi-duo if and only if $S/J(S)$ is quasi-duo. □

THEOREM 3.12. *Let M be an automorphism-invariant module and let $E(M)$ be its injective envelope.*

(a) *If M is quasi-injective and* $\mathrm{End}(M)$ *has a factor isomorphic to* $\mathbb{F}_2$, *then* $\mathrm{End}(E(M))$ *has a factor isomorphic to* $\mathbb{F}_2$.

(b) *If M has finite Goldie dimension and* $\mathrm{End}(M)$ *has a factor isomorphic to* $\mathbb{F}_2$*, then the following conditions hold.*

(i) $\mathrm{End}(E(M))$ *has a factor isomorphic to* $\mathbb{F}_2$.

(ii) $E(M)$ *has a direct-sum decomposition* $E(M) = E \oplus C$ *with E orthogonal to C, E an indecomposable R-module and* $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$.

(iii) $\mathrm{Aut}(E) = 1 + J(\mathrm{End}(E))$*, so that every automorphism of the R-module E is the identity on an essential R-submodule of E.*

(iv) *E is the injective envelope of its non-zero R-submodule* $\mathrm{ann}_E(2)$.

Proof. (a) If M is quasi-injective, the mapping

$$\overline{\varphi}\colon \mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M)))$$

is an isomorphism by Theorem 2.1(d). Thus $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$.

(i) We first consider the case of M indecomposable. If M is automorphism-invariant indecomposable, then $\mathrm{End}(M)$ is local by Proposition 2.2. If $\mathrm{End}(M)$ also has a factor isomorphic to $\mathbb{F}_2$, then

$$\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathbb{F}_2.$$

Since M is automorphism-invariant, we get that $M = N \oplus P$, where N is quasi-injective and P is square-free (Theorem 3.4). But M is indecomposable, so that either $M = N$ or $M = P$. If $M = N$ is quasi-injective, $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$ by (a). In the other case, $M = P$ is square-free, so that $\mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ is abelian. As M has finite Goldie dimension, $E(M)$ has finite Goldie dimension. Hence $\mathrm{End}(E(M))$ is semilocal. Therefore $\mathrm{End}(E(M))/J(\mathrm{End}(E(M)) \cong D_1 \times D_2 \times \ldots \times D_n$, where each D_i is a division ring. Consider the mapping $\varphi\colon \mathrm{End}(M) \to \mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ of Theorem 2.1. From $\ker\, \varphi = J(\mathrm{End}(M))$, it follows that $\mathrm{im}\,\varphi \cong \mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathbb{F}_2$. Moreover, the group of units of $\mathrm{End}(E(M))/J(\mathrm{End}(E(M))$ is contained in $\mathrm{im}(\varphi)$, because M is automorphism-invariant. Hence the group of units of

$$\mathrm{End}(E(M)/J(\mathrm{End}(E(M))$$

has one element. Since it is isomorphic to $D_1 \setminus \{0\} \times \ldots \times D_n \setminus \{0\}$, it follows that $D_i \cong \mathbb{F}_2$ for every $i = 1, \ldots, n$. So $\mathrm{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$. This concludes the proof of (i) for M indecomposable.

Now let M be an arbitrary automorphism-invariant module of finite Goldie dimension and assume that $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$.

The proof will be by induction on the Goldie dimension n of M. If $n = 1$, then M is indecomposable, and we are done. Suppose $n > 1$. Since M is automorphism-invariant, we have that $M = N \oplus P$, where N is quasi-injective, P is square-free and N, P are orthogonal. If $P = 0$, then M is quasi-injective, and we conclude by (a). If $N = 0$, then M is square-free. If M is indecomposable, we are done, as we have seen in the previous paragraph. Otherwise $M = M_1 \oplus M_2$ for suitable non-zero submodules M_1, M_2. The modules M_1, M_2 are orthogonal because M is square-free. By Corollary 3.3, either $\mathrm{End}(M_1)$ or $\mathrm{End}(M_2)$ has a factor isomorphic to $\mathbb{F}_2$. Without loss of generality, we can assume that $\mathrm{End}(M_1)$ has a factor isomorphic to $\mathbb{F}_2$. Let E_i be an injective envelope of M_i, so that $E(M) = E_1 \oplus E_2$. By the inductive hypothesis, we get that $\mathrm{End}(E_1)$ has a factor isomorphic to $\mathbb{F}_2$. Moreover, E_1, E_2 are orthogonal by Corollary 3.9. Thus $\mathrm{End}(E)$ has a factor isomorphic to $\mathbb{F}_2$ by Corollary 3.3, and we are done.

It remains to consider the case $M = N \oplus P$ with both N and P non-zero. Then $E(M) = E(N) \oplus E(P)$. Then $E(N)$ and $E(P)$ are orthogonal (Corollary 3.9), and either $\mathrm{End}(N)$ or $\mathrm{End}(P)$ has a factor isomorphic to $\mathbb{F}_2$ (Corollary 3.3). By the inductive hypothesis, $\mathrm{End}(E(N))$ or $\mathrm{End}(E(P))$ has a factor isomorphic to $\mathbb{F}_2$. The conclusion follows by Corollary 3.3.

(ii) Since M is of finite Goldie dimension, $E(M)$ decomposes as $E(M) = E_1 \oplus \dots \oplus E_n$, where the E_i are indecomposable injective R-modules. Now $\mathrm{End}(M)$ is semiperfect (Proposition 2.2(b)), hence semilocal. By the hypothesis, there exists a ring morphism $\mathrm{End}(M) \to \mathbb{F}_2$, so that there exists a ring morphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$. The semisimple artinian ring $\mathrm{End}(M)/J(\mathrm{End}(M))$ is a finite direct product of rings of matrices $M_{n_j}(D_j)$ over division rings D_j. The kernel of the ring morphism $\mathrm{End}(M)/J(\mathrm{End}(M)) \to \mathbb{F}_2$ is a maximal ideal of this finite direct product of rings of matrices $M_{n_j}(D_j)$. It follows that there exists an index j with $n_j = 1$ and $D_j \cong F_2$. Thus, in the direct-sum decomposition $E(M) = E_1 \oplus \dots \oplus E_n$, there exists an index i with $E_i \ncong E_k$ for every $k = 1, \dots n$ different from i and $\mathrm{End}(E_i)/J(\mathrm{End}(E_i)) \cong \mathbb{F}_2$. Set $E := E_i$ and $C := E_1 \oplus \dots \oplus E_{i-1} \oplus E_{i+1} \oplus \dots \oplus E_n$. In order to conclude the proof of (ii), it suffices to show that E is orthogonal to C. Assume the contrary. Then there exist isomorphic non-zero submodules A of E and B of C. Thus $E(B)$ is an indecomposable direct summand of C isomorphic to $E(A) \cong E$. By the Krull-Schmidt-Azumaya Theorem, the module $E(B)$ must be isomorphic to one of the modules $E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n$. This is a contradiction.

(iii) If $\varphi \in \mathrm{Aut}(E)$, we have that $\varphi + J(\mathrm{End}(E))$ is an invertible element in the ring $\mathrm{End}(E)/J(\mathrm{End}(E))$. But $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$, so that $\varphi + J(\mathrm{End}(E)) = 1 + J(\mathrm{End}(E))$. Thus $\varphi \in 1 + J(\mathrm{End}(E))$. This proves that

$\mathrm{Aut}(E) = 1 + J(\mathrm{End}(E))$. In particular, every automorphism of the R-module E is the identity on an essential R-submodule of E.

(iv) From $\mathrm{End}(E)/J(\mathrm{End}(E)) \cong \mathbb{F}_2$, it follows that $1_E + 1_E \in J(\mathrm{End}(E))$; that is 2 annihilates an essential submodule of E. Therefore $\mathrm{ann}_E(2)$ is a non-zero R-submodule of E. But E is uniform. □

Theorem 3.12(b) does not hold when M is not automorphism-invariant. To see this, take $R = \mathbb{Z}$ and $M = \mathbb{Z}_{\mathbb{Z}}$. Then $\mathbb{Q}_{\mathbb{Z}}$ is an injective envelope of $\mathbb{Z}_{\mathbb{Z}}$. The endomorphism ring of $\mathbb{Z}_Z$ is isomorphic to $\mathbb{Z}$. So it has a factor isomorphic to $\mathbb{F}_2$. But the endomorphism ring of $\mathbb{Q}_{\mathbb{Z}}$ has no factor isomorphic to $\mathbb{F}_2$.

Remark 3.13. Let M be any right R-module, let $E(M)$ be its injective envelope and $S := \mathrm{End}(E(M))$ be the endomorphism ring of $E(M)$, so that $E(M)$ turns out to be a S-R-bimodule. Let I be the two-sided ideal of S generated by the set $\{\, s - s^2 \mid s \in S \,\}$. Then the annihilator $\mathrm{ann}_{E(M)} I := \{\, e \in E(M) \mid Ie = 0 \,\}$ is an S-R-subbimodule of ${}_S E(M)_R$, as is easily seen. Thus there is an R-module direct-sum decomposition $E(M)_R = E_1 \oplus E_2$, where E_1 is an injective envelope $E(\mathrm{ann}_{E(M)} I)$ of $\mathrm{ann}_{E(M)} I$ in $E(M)_R$ and E_2 is a complement of E_1 in $E(M)_R$, so that no non-zero element of E_2 is annihilated by I, i.e., $e_2 \in E_2$ and $Ie_2 = 0$ imply $e_2 = 0$. Assume there are two non-zero R-submodules A_1, A_2 such that $A_1 \leq E_1$, $A_2 \leq E_2$ and $A_1 \cong A_2$. Then their injective envelopes $E(A_1), E(A_2)$ are isomorphic and each $E(A_i)$ is a direct summand of E_i. So $E(M)$ decomposes as a direct sum $E(M) = e_1 E(M) \oplus e_2 E(M) \oplus e_3 E(M)$ for orthogonal idempotents $e_i \in \mathrm{End}(E(M))$ where $e_i E(M) = E(A_i) (i = 1, 2)$. Since $E(A_1) \cong E(A_2)$, $e_1 E(M) \cong e_2 E(M)$. Applying the functor $\mathrm{Hom}(E(M), -)\colon \text{Mod-}R \to \text{Mod-}S$, one finds that $S_S = e_1 S \oplus e_2 S \oplus e_3 S$ and $e_1 S_S \cong e_2 S_S$ [7, Theorem 4.7], where $S = \mathrm{End}(E(M))$. So there exists a unit element $u \in S$ such that $e_1 = u^{-1} e_2 u$. As $e_2 \mathrm{ann}_{E(M)} I = 0$ and $e_1 = u^{-1} e_2 u$, it follows that $e_1 \mathrm{ann}_{E(M)} I = 0$. But this contradicts $e_1 \mathrm{ann}_{E(M)} I \neq 0$, because $e_1 \mathrm{ann}_{E(M)} I = E(A_1) \cap \mathrm{ann}_{E(M)} I \neq 0$. Therefore two R-modules E_1 and E_2 are orthogonal. By Lemma 3.2, $S/\Delta(E(M), E(M)) \cong S_1/\Delta(E_1, E_1) \times S_2/\Delta(E_2, E_2)$, where S_i denotes the endomorphism ring of the R-module E_i. As $\Delta(E, E) = J(\mathrm{End}(E))$ for any injective R-module E, it follows that $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$ in a canonical way. If I_i denotes the two-sided ideal of S_i generated by all $x - x^2$ with $x \in S_i$, then $I/J(S) \cong I_1/J(S_1) \times I_2/J(S_2)$.

Now consider the ring morphism $\rho\colon S \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$ that associates to any $f \in S$ its restriction $f|_{\mathrm{ann}_{E(M)} I}$ to $\mathrm{ann}_{E(M)} I$. The ring morphism ρ is well defined because $\mathrm{ann}_{E(M)} I$ is a left S-submodule of $E(M)$. The morphism ρ is clearly an onto mapping, and its kernel is $\ker \rho := \{ f \in S \mid f(\mathrm{ann}_{E(M)} I) = 0 \}$. In particular $I \subseteq \ker \rho$. Since S/I is a boolean ring, the ring $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is also boolean. Moreover, $J(S) \subseteq I \subseteq \ker \rho$, so that ρ induces a ring morphism $\overline{\rho}: S/J(S) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$. As $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$ and the elements of $S_2/J(S_2)$ are clearly mapped to 0 by $\overline{\rho}$, we get that $0 \times S_2/J(S_2) \subseteq \ker(\rho)$. Thus there is a surjective ring morphism $S_1/I_1 \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$.

From Remark 3.13, we get in particular that:

PROPOSITION 3.14. *Let M be an R-module, $S := \mathrm{End}(E(M))$ be the endomorphism ring of $E(M)$ and I be the two-sided ideal of S generated by the set $\{ s - s^2 \mid s \in S \}$.*

(a) *If* $\mathrm{ann}_{E(M)} I \neq 0$, *then* $\mathrm{End}(M)$ *has a factor isomorphic to* $\mathbb{F}_2$.
(b) *If M is automorphism-invariant and* $\mathrm{ann}_{E(M)} I$ *is an essential submodule of the R-module $E(M)$, then the ring* $\mathrm{End}(M)/J(\mathrm{End}(M))$ *is a boolean ring.*

PROOF. Compose the ring morphism $\varphi: \mathrm{End}(M) \to S/J(S)$ of Theorem 2.1 with the morphism $\overline{\rho}: S/J(S) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$ in Remark 3.13, obtaining a morphism $\overline{\rho}\varphi: \mathrm{End}(M) \to \mathrm{End}(\mathrm{ann}_{E(M)} I)$, where $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is a boolean ring. If $\mathrm{ann}_{E(M)} I \neq 0$, then $\mathrm{End}(\mathrm{ann}_{E(M)} I)$ is a non-zero boolean ring, so that there is a morphism $\mathrm{End}(\mathrm{ann}_{E(M)} I) \to \mathbb{F}_2$. Thus there is a morphism $\mathrm{End}(M) \to \mathbb{F}_2$, necessarily surjective. Hence $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$. This concludes the proof of (a).

If M is automorphism-invariant and $\mathrm{ann}_{E(M)} I$ is essential in $E(M)$, then in Remark 3.13 we have that $E(M) = E_1$, $E_2 = 0$ and $\ker \rho \subseteq \Delta(E(M), E(M)) = J(S)$. As $I \subseteq \ker \rho$ and $J(S) \subseteq I$, it follows that $I = \ker \rho = J(S)$. Thus $S/J(S) \cong \mathrm{End}(\mathrm{ann}_{E(M)} I)$ is a boolean ring. By Theorem(a) 2.1, the ring $\mathrm{End}(M)/J(\mathrm{End}(M))$ is isomorphic to a subring of the ring $\mathrm{End}(E(M))/J(\mathrm{End}(E(M))) = S/J(S)$. Thus $\mathrm{End}(M)/J(\mathrm{End}(M))$ is boolean. □

PROPOSITION 3.15. *Let M be an automorphism-invariant square-free module of finite Goldie dimension. Then M decomposes as a direct sum $M = N \oplus P$, where N is a module orthogonal to P,* $\mathrm{End}(N)$ *has no factor isomorphic to* $\mathbb{F}_2$, *and* $\mathrm{End}(P)/J(\mathrm{End}(P))$ *is isomorphic to a boolean ring* $\mathbb{F}_2^n$ *for some n.*

PROOF. The automorphism-invariant module M of finite Goldie dimension, decomposes as a direct sum $M = M_1 \oplus \dots \oplus M_t$ of indecomposable modules, necessarily automorphism-invariants. Let $e_1, \dots, e_t \in \mathrm{End}(M)$ be the orthogonal idempotents corresponding to this direct-sum decomposition of M. Then $\overline{e_1}, \dots, \overline{e_t} \in \mathrm{End}(M)/\Delta(M,M)$ are orthogonal idempotents of $\mathrm{End}(M)/\Delta(M,M)$, which is an abelian ring by Proposition 3.10. Thus the idempotents $\overline{e_1}, \dots, \overline{e_t}$ of $\mathrm{End}(M)/\Delta(M,M) = \mathrm{End}(M)/J(\mathrm{End}(M))$ are central, so that

$$\begin{aligned}&\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \\ &\qquad \cong \overline{e_1}\,\mathrm{End}(M)/J(\mathrm{End}(M))\overline{e_1} \times \dots \times \overline{e_t}\,\mathrm{End}(M)/J(\mathrm{End}(M))\overline{e_t} \cong \\ &\qquad \cong \mathrm{End}(M_1)/J(\mathrm{End}(M_1)) \times \dots \times \mathrm{End}(M_t)/J(\mathrm{End}(M_t)),\end{aligned}$$

is isomorphic to the direct product of the residue division rings $\mathrm{End}(M_i)/J(\mathrm{End}(M_i))$. Let N be the direct sum of the M_i with the residue division rings $\mathrm{End}(M_i)/J(\mathrm{End}(M_i))$ not isomorphic to $\mathbb{F}_2$ and P be the direct sum of the M_i with the residue division rings $\mathrm{End}(M_i)/J(\mathrm{End}(M_i))$ isomorphic to $\mathbb{F}_2$. Then $M = N \oplus P$, $\mathrm{End}(N)$ has no factor isomorphic to $\mathbb{F}_2$, because $\mathrm{End}(N)/J(\mathrm{End}(N))$ is a direct product of finitely many division rings not isomorphic to $\mathbb{F}_2$, and $\mathrm{End}(P)/J(\mathrm{End}(P))$ isomorphic to a direct product of finitely many copies of $\mathbb{F}_2$.

Finally, N and P are relatively injective by [13, Theorem 5]. As

$$\mathrm{End}(M)/\Delta(M,M) \cong \mathrm{End}(N)/\Delta(N,N) \times \mathrm{End}(P)/\Delta(P,P),$$

we conclude that N and P are orthogonal (Lemma 3.2). □

REFERENCES

[1] D. W. Anderson – K. R. Fuller, *Rings and Categories of Modules*, Second Edition, GTM 13, Springer-Verlag, 1992.

[2] B. Brainerd – J. Lambek, *On the ring of quotients of a boolean ring*, Canad. Math. Bull., **2** (1959), pp. 25–29.

[3] T. T. Bumby, *Modules which are isomorphic to submodules of each other*, Arch. Math., **16** (1965), pp. 184–185.

[4] R. Camps – W. Dicks, *On semilocal rings*, Israel J. Math., **81** (1993), pp. 203–211.

[5] H. Q. Dinh, *A note on pseudo-injective modules*, Comm. Algebra, **33** (2005), pp. 361–369.

[6] N. Er – S. Singh – A. K. Srivastava, *Rings and modules which are stable under automorphisms of their injective hulls*, J. Algebra, **379** (2013), pp. 223–229.

[7] A. Facchini, *Module Theory. Endomorphism rings and direct sum decompositions in some classes of modules*, Progress in Math. 167, Birkhäuser Verlag, 1998. Reprinted in Modern Birkhäuser Classics, Birkhäuser Verlag, 2010.

[8] A. Facchini – D. Herbera, *Local morphisms and modules with a semilocal endomorphism ring*, Algebr. Represent. Theory, **9** (2006), pp. 403–422.

[9] K. R. Goodearl, *Von Neumann Regular Rings*, Krieger, Malabar, 1991.

[10] P. A. Guil Asensio – A. K. Srivastava, *Additive unit representations in endomorphism rings and an extension of a result of Dickson and Fuller*, in: *Ring Theory and its Applications*, Contemp. Math. 609, Amer. Math. Soc., Providence 2014, D. V. Huynh, et al., eds., pp. 117–121.

[11] P. A. Guil Asensio – A. K. Srivastava, *Automorphism-invariant modules satisfy the exchange property*, J. Algebra, **388** (2013), pp. 101–106.

[12] D. Khurana – A. K. Srivastava, *Right self-injective rings in which every element is a sum of two units*, J. Algebra Appl., **6** (2007), pp. 281–286.

[13] T.-K. Lee – Y. Zhou, *Modules which are invariant under automorphisms of their injective hulls*, J. Algebra Appl., **12** (2013), 1250159, 9 pp.

[14] Mai Hoang Bien, *The endomorphism ring of a square-free injective module*, Acta Math. Vietnamica 39(3) (2014), Online DOI 10.1007/s40306-014-0075-y.

[15] S. H. Mohamed – B. J. Müller, *Continuous and Discrete Modules*, London Mathematical Society, Lecture Notes Series 147, Cambridge University Press, 1990.

[16] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc., **229** (1977), pp. 269–278.

[17] S. Singh – A. K. Srivastava, *Rings of invariant module type and automorphism-invariant modules*, in: *Ring Theory and its Applications*, Contemp. Math. 609, Amer. Math. Soc., Providence 2014, D. V. Huynh, et al., eds., pp. 299–311.

[18] B. Stenström, *Rings of quotients*, Springer-Verlag, 1975.

[19] P. Vámos, *2-good rings*, Quart. J. Math., **56** (2005), pp. 417–430.

Manoscritto pervenuto in redazione il 27 Dicembre 2013.