Forum Math. **20** (2008), 99–143
DOI 10.1515/FORUM.2008.005

**Forum Mathematicum**


# A partial solution of the isoperimetric problem for the Heisenberg group

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(Communicated by Giorgio Talenti)

**Abstract.** We provide a solution to the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$ when the competing sets belong to a restricted class of $C^2$ graphs. Within this restricted class we characterize the isoperimetric profiles as the bubble sets (1.5) (modulo nonisotropic dilations and left-translations). We also compute the isoperimetric constant.

2000 Mathematics Subject Classification: 53C17, 49K20.

## Contents



## 1 Introduction

The classical isoperimetric problem states that among all measurable sets with assigned volume the ball minimizes the perimeter. This is the content of the celebrated isoperimetric inequality, see [DG3],

$$(1.1) \qquad |E|^{(n-1)/n} \le C_n P(E),$$

which holds for all measurable sets $E \subset \mathbb{R}^n$ with constant $C_n = n\sqrt{\pi}/\Gamma(n/2+1)^{1/n}$. In (1.1), $P(E)$ denotes the perimeter in the sense of De Giorgi, see [DG1], [DG2], i.e., the total variation of the indicator function of $E$. Equality holds in (1.1) if and only if (up to negligible sets) $E = B(x,R) = \{y \in \mathbb{R}^n \mid |y-x| < R\}$, a Euclidean ball. It is

First author supported in part by NSF grants DMS-0002801 and CAREER DMS-0239771.
Second author supported in part by NSF Grants DMS-0070492 and DMS-0300477.

well known that (1.1) is equivalent to the geometric Sobolev inequality for $BV$ functions, see [FR]. An analogous "isoperimetric inequality" was proved in [GN] in the general setting of a Carnot-Carathéodory space, and such inequality was used, among other things, to establish a geometric embedding for horizontal $BV$ functions, similar to Fleming and Rishel's one. However, the question of the optimal configurations in such isoperimetric inequality was left open.

The aim of this paper is to bring a partial solution to this open problem in the Heisenberg group $\mathbb{H}^n$. We recall that $\mathbb{H}^n$ is the simplest and perhaps most important prototype of a class of nilpotent Lie groups, called Carnot groups, which play a fundamental role in analysis and geometry, see [Ca], [Ch], [H], [St], [Be], [Gro1], [Gro2], [E1], [E2], [E3], [DGN2]. Its underlying manifold is $\mathbb{R}^{2n+1}$ with non-commutative group law

$$(1.2) \qquad gg' = (x,y,t)(x',y',t') = \left(x+x', y+y', t+t' + \frac{1}{2}(\langle x, y'\rangle - \langle x', y\rangle)\right),$$

where we have let $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$. If $L_g(g') = gg'$ denotes the operator of left-translation, let $(L_g)_*$ indicate its differential. The Heisenberg algebra admits the decomposition $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{2n} \times \{0\}$, and $V_2 = \{0\} \times \mathbb{R}$. Identifying $\mathfrak{h}_n$ with the space of left-invariant vector fields on $\mathbb{H}^n$, one easily recognizes that a basis for $\mathfrak{h}_n$ is given by the $2n+1$ vector fields

$$(1.3) \qquad \begin{cases} (L_g)_*\left(\frac{\partial}{\partial x_i}\right) \overset{def}{=} X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}, \\ (L_g)_*\left(\frac{\partial}{\partial y_i}\right) \overset{def}{=} X_{n+i} = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}, \\ (L_g)_*\left(\frac{\partial}{\partial t}\right) \overset{def}{=} T = \frac{\partial}{\partial t}, \end{cases}$$

and that the only non-trivial commutation relation is

$$(1.4) \qquad [X_i, X_{n+j}] = T\delta_{ij}, \quad i, j = 1, \dots, n.$$

In (1.3) we have identified the standard basis $\{e_1, \dots, e_{2n}, e_{2n+1}\}$ of $\mathbb{R}^{2n+1}$ with the system of (constant) vector fields $\{\partial/\partial x_1, \dots, \partial/\partial y_n, \partial/\partial t\}$. Because of (1.4) we have $[V_1, V_1] = V_2$, $[V_1, V_2] = \{0\}$, thus $\mathbb{H}^n$ is a graded nilpotent Lie group of step $r = 2$. Lebesgue measure $dg = dz\,dt$ is a bi-invariant Haar measure on $\mathbb{H}^n$. If we denote by $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$ the non-isotropic dilations associated with the grading of the Lie algebra, then $d(\delta_\lambda g) = \lambda^Q\, dg$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

In what follows we denote by $P_H(E; \mathbb{H}^n)$ the intrinsic, or $H$-perimeter of $E \subset \mathbb{H}^n$ associated with the bracket-generating system $X = \{X_1, \dots, X_{2n}\}$. Such notion will be recalled in Section 2. To state our theorem we let $\mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n \mid t > 0\}$, $\mathbb{H}^n_- = \{(z,t) \in \mathbb{H}^n \mid t < 0\}$, and consider the collection

$$\mathscr{E} = \{E \subset \mathbb{H}^n \mid E \text{ satisfies (i)–(iii)}\},$$

where

(i) $|E \cap \mathbb{H}^n_+| = |E \cap \mathbb{H}^n_-|$;

(ii) there exist $R > 0$, and functions $u, v : \bar{B}(0,R) \to [0,\infty)$, with $u, v \in C^2(B(0,R)) \cap C(\bar{B}(0,R))$, $u = v = 0$ on $\partial B(0,R)$, and such that

$$\partial E \cap \mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n_+ \mid |z| < R, t = u(z)\},$$

$$\partial E \cap \mathbb{H}^n_- = \{(z,t) \in \mathbb{H}^n_- \mid |z| < R, t = -v(z)\}.$$

(iii) $\{z \in B(0,R) \mid u(z) = 0\} \cap \{z \in B(0,R) \mid v(z) = 0\} = \emptyset$.

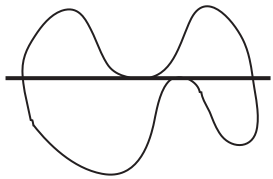


Fig. 1.1. $E \in \mathscr{E}$

We note explicitly that condition (iii) serves to guarantee that every $E \in \mathscr{E}$ is a piecewise $C^2$ domain in $\mathbb{H}^n$ (with possible discontinuities in the derivatives only on that part of $E$ which intersects the hyperplane $t = 0$). We also stress that the upper and lower portions of a set $E \in \mathscr{E}$ can be described by possibly different $C^2$ graphs, and that, besides $C^2$ smoothness, and the fact that their common domain is a ball, no additional assumption is made on the functions $u$ and $v$. For instance, we do not require a priori that $u$ and/or $v$ are spherically symmetric. Here is our main result.

**Theorem 1.1.** *Let $V > 0$, and define the number $R > 0$ by*

$$R = \left( \frac{(Q-2)\Gamma\left(\frac{Q+2}{2}\right)\Gamma\left(\frac{Q-2}{2}\right)}{\pi^{(Q-1)/2}\Gamma\left(\frac{Q+1}{2}\right)} \right)^{1/Q} V^{1/Q}.$$

*Given such $R$, then the variational problem*

$$\min_{E \in \mathscr{E}, |E| = V} P_H(E; \mathbb{H}^n)$$

*has a unique solution $E_R = \delta_R(E_o) \in \mathscr{E}$, where $\partial E_o$ is described by the graph $t = \pm u_o(z)$, with*

$$(1.5) \qquad u_o(z) \overset{def}{=} \left\{ \frac{\pi}{8} + \frac{|z|}{4}\sqrt{1 - |z|^2} - \frac{1}{4}\sin^{-1}(|z|) \right\}, \quad |z| \le 1.$$

*The sign $\pm$ depends on whether one considers $\partial E_o \cap \mathbb{H}^n_+$, or $\partial E_o \cap \mathbb{H}^n_-$. Finally, the boundary $\partial E_R = \delta_R(\partial E_o)$ of the bounded open set $E_R$ is only of class $C^2$, but not of class $C^3$, near its two characteristic points $\left(0, \pm\frac{\pi R^2}{8}\right)$, it is $C^\infty$ away from them, and $S_R = \partial E_R$ has positive constant $H$-mean curvature given by*

$$\mathscr{H} = \frac{Q-2}{R}.$$

**Remark 1.2.** We notice explicitly that the function $u_o$ in (1.5) can also be expressed as follows

$$u_o(z) = \frac{1}{2}\int_{\sin^{-1}(|z|)}^{\pi/2} \sin^2 \tau \, d\tau.$$

**Remark 1.3.** We emphasize that, as the reader will recognize, for our proof of the existence of a global minimizer it suffices to assume that the two functions $u$ and $v$ in the definition of the sets of the class $\mathscr{E}$ are $C^{1,1}_{\text{loc}}(B(0,R))$. It is an open question whether $u, v \in C^1(B(0,R))$ is enough. This is possible thanks to a sharp result of Balogh concerning the size of the characteristic set, see Theorem 3.9 below. In our proof of the uniqueness of the global minimizer, instead, it is convenient to work under the hypothesis of $C^2$ smoothness. However, with little extra care, it should be possible to relax it to $C^{1,1}_{\text{loc}}$.

For the notion of $H$-mean curvature of a $C^2$ hypersurface $\mathscr{S} \subset \mathbb{H}^n$ we refer the reader to Definition 3.2 in Section 3. This notion of horizontal mean curvature, which is of course central to the present study, was introduced in [DGN4]. Its geometric interpretation is that, in the neighborhood of a non-characteristic point $g \in \mathscr{S}$, it coincides with the standard Riemannian mean curvature of the $2n-1$-dimensional submersed manifold obtained by intersecting the hypersurface $\mathscr{S}$ with the fiber of the horizontal subbundle $H_g\mathbb{H}^n$, see also [DGN3] where a related notion of Gaussian curvature was introduced. A seemingly different notion, based on the Riemannian regularization of the sub-Riemannian metric of $\mathbb{H}^n$, was proposed in [Pa], but the two are in fact equivalent, see [DGN4]. From Theorem 1.1 we obtain the following isoperimetric inequality for the horizontal perimeter.

**Theorem 1.4.** *Let $\mathscr{E}$ be as above, and denote by $\tilde{\mathscr{E}}$ the class of sets of the type $\delta_\lambda L_g(E)$, for some $E \in \mathscr{E}$, $\lambda > 0$ and $g \in \mathbb{H}^n$, then the following isoperimetric inequality holds*

$$|E|^{(Q-1)/Q} \le C_Q P_H(E;\mathbb{H}^n), \quad E \in \tilde{\mathscr{E}}, \tag{1.6}$$

*where*

$$C_Q = \frac{(Q-1)\Gamma\left(\frac{Q}{2}\right)^{2/Q}}{Q^{(Q-1)/Q}(Q-2)\Gamma\left(\frac{Q+1}{2}\right)^{1/Q}\pi^{(Q-1)/2Q}},$$

*with equality if and only if for some* $\lambda > 0$ *and* $g \in \mathbb{H}^n$ *one has* $E = L_g \delta_\lambda(E_o)$, *where* $E_o$ *is given by* (1.5).

Fig. 1.1 gives a representation of the isoperimetric set $E_o$ in Theorem 1.1 in the special case $n = 1$. We note that the invariance of the isoperimetric quotient with respect to the group left-translations $L_g$ and dilations $\delta_\lambda$ is guaranteed by Propositions 2.11 and 2.12.

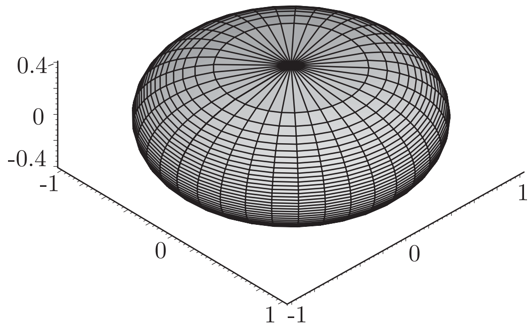


Fig. 1.2. Isoperimetric set in $\mathbb{H}^1$ with $R = 1$

A remarkable property of the isoperimetric sets is that, similarly to their Riemannian predecessors, they have constant $H$-mean curvature. It is tempting, and also natural, to conjecture that the set $E_o$ described by (1.5), along with its left-translated and dilated, exaust *all* the isoperimetric sets in $\mathbb{H}^n$ (for the definition of such sets, see Definition 1.6 below). By this we mean that Theorem 1.4 continues to be valid when one replaces the class $\tilde{\mathscr{E}}$ with that of all measurable sets $E \subset \mathbb{H}^n$ with locally finite $H$-perimeter. At the moment, this remains a challenging open problem. In this connection, another interesting conjecture is as follows: *Let* $\mathscr{S} \subset \mathbb{H}^n$ *be a* $C^2$, *compact oriented hypersurface. Suppose that for some* $\alpha > 0$

$$(1.7) \qquad \mathscr{H} \equiv \alpha \quad on\ \mathscr{S}.$$

*Is it true that, up to a left translation, if we denote by* $\mathscr{S}^+ = \mathscr{S} \cap \mathbb{H}^n_+$, $\mathscr{S}^- = \mathscr{S} \cap \mathbb{H}^n_-$, *then* $\mathscr{S}^+$, $\mathscr{S}^-$ *are respectively described by*

$$(1.8) \qquad t = \pm \left\{ \frac{1}{4}|z|\sqrt{R^2 - |z|^2} - \frac{R^2}{4}\tan^{-1}\left( \frac{|z|}{\sqrt{R^2 - |z|^2}} \right) + \frac{\pi R^2}{8} \right\}, \qquad |z| \le R,$$

*where* $R = (Q-2)/\alpha$? Concerning this conjecture we remark that Theorem 1.1 provides evidence in favor of it. As it is well known, the Euclidean counterpart of it is contained in the celebrated *soap bubble* theorem of A. D. Alexandrov [A]. We mention that, after this paper was completed, we have received an interesting preprint from Ritoré and Rosales [RR2] in which, among other results, the authors prove the above soap bubble conjecture in the first Heisenberg group $\mathbb{H}^1$.

To put the above results in a broader perspective we recall that in any Carnot group a general scale invariant isoperimetric inequality is available. In fact, using the results in [CDG], [GN] one can prove the following theorem, see Theorem 2.9 in Section 2.

**Theorem 1.5.** *Let $\boldsymbol{G}$ be a Carnot group with homogeneous dimension $Q$. There exists a constant $C_{\text{iso}}(\boldsymbol{G}) > 0$ such that, for every $H$-Caccioppoli set $E \subset \boldsymbol{G}$, one has*

$$|E|^{(Q-1)/Q} \le C_{\text{iso}}(\boldsymbol{G}) P_H(E; \boldsymbol{G}).$$

A measurable set $E \subset \boldsymbol{G}$ is called a $H$-Caccioppoli set if $P_H(E;\omega) < \infty$ for any $\omega \subset\subset \boldsymbol{G}$. Theorem 1.5 generalizes an earlier result of Pansu [P1], who proved a related inequality for the first Heisenberg group $\mathbb{H}^1$, but with the $H$-perimeter in the right-hand side replaced by the 3-dimensional Hausdorff measure $\mathscr{H}^3$ in $\mathbb{H}^1$ constructed with the Carnot-Carathéodory distance associated with the horizontal subbundle $H\mathbb{H}^1$ defined by $\{X_1, X_2\}$ in (1.3). One should keep in mind that the homogeneous dimension of $\mathbb{H}^1$ is $Q = 4$, so $3 = Q - 1$, which explains the appearance of $\mathscr{H}^3$ in Pansu's result. It should also be said that some authors attribute to Pansu [P2] the conjecture that the isoperimetric sets in $\mathbb{H}^1$ have the form (1.5). We mention that other isoperimetric and Fleming-Rishel type Gagliardo-Nirenberg inequalities have been obtained by several authors at several times, see [Va1], [Va2], [VSC], [CS], [BM], [FGW], [MaSC]. We now introduce the following definition.

**Definition 1.6.** Given a Carnot group $\boldsymbol{G}$ with homogeneous dimension $Q$ we define the *isoperimetric constant* of $\boldsymbol{G}$ as

$$\alpha_{\text{iso}}(\boldsymbol{G}) = \inf_{E \subset \boldsymbol{G}} \frac{P_H(E; \boldsymbol{G})}{|E|^{(Q-1)/Q}},$$

where the infimum is taken on all $H$-Caccioppoli sets $E$ such that $0 < |E| < \infty$. If a measurable set $E_o$ is such that

$$\alpha_{\text{iso}}(\boldsymbol{G}) = \frac{P_H(E_o; \boldsymbol{G})}{|E_o|^{(Q-1)/Q}},$$

then we call it an *isoperimetric set in $\boldsymbol{G}$*.

We stress that, thanks to Theorem 1.5, the isoperimetric constant is strictly positive. It should also be observed that, using the representation formula for the $H$-perimeter

$$P_H(E; \boldsymbol{G}) = \int_{\partial E} \frac{W}{|\boldsymbol{N}|} dH_{N-1}, \tag{1.9}$$

valid for any bounded open set $E \subset \boldsymbol{G}$ of class $C^1$, with Riemannian outer normal $\boldsymbol{N}$ and angle function $W = \sqrt{p_1^2 + \cdots + p_m^2}$ (see Lemma 2.8, and (3.1), (3.2)), one immediately recognizes that, since for any $\omega \subset\subset \boldsymbol{G}$ one has $W \le C(\omega)|\boldsymbol{N}|$, then $P_H(E; \boldsymbol{G}) \le CH_{N-1}(\partial E) < \infty$. As a consequence, $\alpha_{\mathrm{iso}}(\boldsymbol{G}) < \infty$ as well. What is not obvious instead is the existence of isoperimetric sets. In this regard, one has the following basic result proved in [LR].

**Theorem 1.7.** *Let $\boldsymbol{G}$ be a Carnot group, then there exists a bounded $H$-Caccioppoli set $F_o$ such that*

$$P_H(F_o; \boldsymbol{G}) = \alpha_{\mathrm{iso}}(\boldsymbol{G})|F_o|^{(Q-1)/Q}.$$

*The equality continues to be valid if one replaces $F_o$ by $L_{g_o} \circ \delta_\lambda(F_o)$, for any $\lambda > 0$, $g_o \in \boldsymbol{G}$.*

Of course, this result leaves open the fundamental question of the classification of such sets. We stress that, in the generality of Theorem 1.5, this problem is presently totally out of reach. When $\boldsymbol{G} = \mathbb{H}^n$, however, Theorems 1.1 and 1.4 provide some basic progress in this direction. Our main contribution is to use direct methods of the Calculus of Variations to prove that the critical point (1.8) is a global minimizer in the class $\mathscr{E}$. Furthermore, such global minimizer is unique (modulo left-translations and dilations) in such class. These results follow from some delicate properties of convexity, and strict convexity at the global minimizer, of the $H$-perimeter functional subject to a volume constraint.

In connection with our work, we mention that several authors have recently studied the isoperimetric problem in $\mathbb{H}^n$, but under the restriction that the class of competitors be $C^2$ smooth and cylindrically symmetric, i.e., spherical symmetry about the $t$-axis of the graph of the competing sets. For instance, in the recent interesting work [BC], for the first Heisenberg group $\mathbb{H}^1$, the authors prove that the flow by $H$-mean curvature of a $C^2$ surface which is convex, and which is described by $t = \pm f(|z|)$, with $f' < 0$, converges to the sets (1.5). Notice, however, that $f$ is spherically symmetric, convex, and that it is assumed that the upper and lower part of the surface are described by the same strictly decreasing function $f$. We also mention the paper [Pa] in which the author, still for $\mathbb{H}^1$, heuristically derives the surface described by (1.5) by imposing the condition of constant $H$-mean curvature among all $C^2$ surfaces which can be described by $t = \pm f(|z|)$. Recently, Hladky and Pauls in [HP] have proposed a general geometric framework, which they call Vertically Rigid manifolds, and which encompasses the class of Carnot groups, in which they study the isoperimetric and the minimal surface problems. In this setting they introduce a notion of horizontal mean curvature, and they show, in particular, that remarkably the isoperimetric sets have constant horizontal mean curvature. In the paper [LM] the authors prove, among other interesting results, that the $u_o$ in our Theorem 1.1 is a critical point (but not the unique global minimizer) of the $H$-perimeter, when the class of competitors is restricted to $C^2$ domains, with defining function of the type $t = \pm f(|z|)$. A similar result has been also obtained in the interesting recent preprint

[RR1], which also contains a classification of the Delaunay type surfaces in $\mathbb{H}^n$. In this connection, we also mention the earlier paper [To], in which the author describes the Delaunay type surfaces of revolution in $\mathbb{H}^1$, heuristically computes the special solutions (1.5), and shows that standard Schwarz symmetrization does not work in the Heisenberg group. In [FMP] the authors gave a complete classification of the constant mean curvature surfaces (including minimal) which are invariant with respect to 1-dimensional closed subgroups of $\mathrm{Iso}_0(H_3, g)$. We also mention the paper [Mo1], in which the author proved that the Carnot-Carathéodory ball in $\mathbb{H}^n$ is not an isoperimetric set. Subsequently, in [Mo2] he proved that, as a consequence of this fact, a generalization of the Brunn-Minkowski inequality to $\mathbb{H}^n$ fails. Finally, in their interesting paper [MoM] the authors have established the isoperimetric inequality for the Baouendi-Grushin vector fields $X_1 = \partial_x$, $X_2 = |x|^\alpha \partial_t$, $\alpha > 0$, in the plane $(x, t)$, and explicitly computed the isoperimetric profiles. In the special case $\alpha = 1$, such profiles are identical (up to a normalization of the vector fields) to our $u_o$ in Theorem 1.1, see Remark 1.2 above.

**Acknowledgment**[1]**.** For the first Heisenberg group $\mathbb{H}^1$, and under the assumption that the isoperimetric profile be of class $C^2$ and of the type $t = f(|z|)$, the idea of using calculus of variations to explicitly determine $f(|z|)$, first came about in computations that Giorgio Talenti and the second named author carried in a set of unpublished notes in Oberwolfach in 1995. We would like to thank G. Talenti for his initial contribution to the present study.

## 2 Isoperimetric inequalities in Carnot groups

The appropriateness of the notion of $H$-perimeter in Carnot-Carathéodory geometry is witnessed by the isoperimetric inequalities. Similarly to their Euclidean counterpart, these inequalities play a fundamental role in the development of geometric measure theory. Theorem 1.5 represents a sub-Riemannian analogue of the classical global isoperimetric inequality. Such result can be extracted from the isoperimetric inequalities obtained in [CDG] and [GN], but it is not explicitly stated in either paper. Since a proof of Theorem 1.5 is not readily available in the literature, for completeness we present it in this section.

Given a Carnot group $\boldsymbol{G}$, its Lie algebra $\mathfrak{g}$ satisfies the properties $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, where $[V_1, V_j] = V_{j+1}$, $j = 1, \ldots, r-1$, and $[V_1, V_r] = \{0\}$. If $m_j = \dim V_j$,

[1] The results in this paper were presented by the second named author in the lecture: "Remarks on the best constant in the isoperimetric inequality for the Heisenberg group and surfaces of constant mean curvature", Analysis seminar, University of Arkansas, April 12, 2001, (http://comp.uark.edu/~lanzani/schedule.html), by the third named author at the international meeting on "Subelliptic equations and sub-Riemannian geometry", Arkansas, March 2003, and by the first named author in the lecture "Hypersurfaces of minimal type in sub-Riemannian geometry", Seventh New Mexico Analysis Seminar, University of New Mexico, October 2004.

$j = 1, \dots, r$, then the homogeneous dimension of $\boldsymbol{G}$ is defined by $Q = m_1 + 2m_2 + \cdots + rm_r$. The non-isotropic dilations associated with the grading of $\mathfrak{g}$ are given by $\Delta_\lambda(\xi_1 + \cdots + \xi_r) = \lambda\xi_1 + \cdots + \lambda^r\xi_r$. Via the exponential mapping $\exp : \mathfrak{g} \to \boldsymbol{G}$, which is a global diffeomorphism onto, such dilations induce a one-parameter group of dilations on $\boldsymbol{G}$ as follows $\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g)$. The push forward through exp of the standard Lebesgue measure on $\mathfrak{g}$ is a bi-invariant Haar measure on $\boldsymbol{G}$. We will denote it by $dg$. Clearly, $d(\delta_\lambda g) = \lambda^Q\, dg$. For simplicity, we let $m = m_1$. We fix some orthonormal basis $\{e_1, \dots, e_m\}, \dots, \{e_{r,1}, \dots, e_{r,m_r}\}$, of the layers $V_1, \dots, V_r$, and consider the corresponding left-invariant vector fields on $\boldsymbol{G}$ defined by $X_1(g) = (L_g)_*(e_1), \dots, X_m(g) = (L_g)_*(e_m), \dots, X_{r,1}(g) = (L_g)_*(e_{r,1}), \dots, X_{r,m_r}(g) = (L_g)_*(e_{r,m_r})$. We will assume that $\boldsymbol{G}$ is endowed with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ with respect to which these vector fields constitute and orthonormal basis. No other inner product will be used in this paper. We denote by $\boldsymbol{HG} \subset \boldsymbol{TG}$ the subbundle of the tangent bundle generated by $\{X_1, \dots, X_m\}$. We next recall the notion of $H$-perimeter, see e.g. [CDG]. Given an open set $\Omega \subset \boldsymbol{G}$, we let

$$\mathscr{F}(\Omega) = \left\{ \zeta = \sum_{i=1}^m \zeta_i X_i \in \Gamma_0^1(\Omega, \boldsymbol{HG}) \mid |\zeta|_\infty = \sup_\Omega |\zeta| = \sup_\Omega \left( \sum_{i=1}^m \zeta_i^2 \right)^{1/2} \le 1 \right\},$$

where we say that $\zeta \in \Gamma_0^1(\Omega, \boldsymbol{HG})$ if $X_j\zeta_i \in C_0(\Omega)$ for $i, j = 1, \dots, m$. Given $\zeta \in \Gamma_0^1(\Omega, \boldsymbol{HG})$ we define

$$\operatorname{div}_H \zeta = \sum_{i=1}^m X_i \zeta_i.$$

For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, the $H$-variation of $u$ with respect to $\Omega$ is defined by

$$\operatorname{Var}_H(u; \Omega) = \sup_{\zeta \in \mathscr{F}(\Omega)} \int_{\boldsymbol{G}} u \operatorname{div}_H \zeta \, dg.$$

We say that $u \in L^1(\Omega)$ has bounded $H$-variation in $\Omega$ if $\operatorname{Var}_H(u; \Omega) < \infty$. The space $BV_H(\Omega)$ of functions with bounded $H$-variation in $\Omega$, endowed with the norm

$$\|u\|_{BV_H(\Omega)} = \|u\|_{L^1(\Omega)} + \operatorname{Var}_H(u; \Omega),$$

is a Banach space. A fundamental property of the space $BV_H$ is the following special case of the compactness Theorem 1.28 proved in [GN].

**Theorem 2.1.** *Let $\Omega \subset \boldsymbol{G}$ be a (PS) (Poincaré-Sobolev) domain. The embedding*

$$i : BV_H(\Omega) \hookrightarrow L^q(\Omega)$$

*is compact for any* $1 \le q < Q/(Q-1)$.

We now recall a special case of Theorem 1.4 in [CDG].

**Theorem 2.2.** *Let $\boldsymbol{G}$ be a Carnot group with homogeneous dimension $Q$. There exists a constant $C(\boldsymbol{G}) > 0$, such that for every $g_o \in \boldsymbol{G}$, $0 < R < R_o$, one has for every $C^1$ domain $E \subset \bar{E} \subset B(g_o, R)$*

$$|E|^{(Q-1)/Q} \leq C P_H(E; B(g_o, R)).$$

To prove Theorem 1.5 we need to extend Theorem 2.2 from bounded $C^1$ domains to arbitrary sets having locally finite $H$-perimeter. That such extension be possible is due in part to the following approximation result for functions in the space $BV_H$, which is contained in Theorem 1.14 in [GN], see also [FSS1].

**Theorem 2.3.** *Let $\Omega \subset \boldsymbol{G}$ be open, where $\boldsymbol{G}$ is a Carnot group. For every $u \in BV_H(\Omega)$ there exists a sequence $\{u_k\}_{k \in N}$ in $C^\infty(\Omega)$ such that*

$$u_k \to u \quad \text{in } L^1(\Omega) \text{ as } k \to \infty, \tag{2.1}$$

$$\lim_{k \to \infty} \mathrm{Var}_H(u_k; \Omega) = \mathrm{Var}_H(u; \Omega). \tag{2.2}$$

We next introduce the notion of $H$-perimeter.

**Definition 2.4.** Let $E \subset \boldsymbol{G}$ be a measurable set, $\Omega$ be an open set. The *$H$-perimeter* of $E$ with respect to $\Omega$ is defined by

$$P_H(E; \Omega) = \mathrm{Var}_H(\chi_E; \Omega),$$

where $\chi_E$ denotes the indicator function of $E$. We say that $E$ is a *$H$-Caccioppoli set* if $\chi_E \in BV_H(\Omega)$ for every $\Omega \subset\subset \boldsymbol{G}$.

The reader will notice that when the step of the group $\boldsymbol{G}$ is $r = 1$, and therefore $\boldsymbol{G}$ is Abelian, the space $BV_H$ coincides with the space $BV$ introduced by De Giorgi, see [DG1], [DG2], [DCP], and thereby in such setting the Definition 2.4 coincides with his notion of perimeter. A fundamental rectifiability theorem á la De Giorgi for $H$-Caccioppoli sets has been established, first for the Heisenberg group $\mathbb{H}^n$, and then for every Carnot group of step $r = 2$, in the papers [FSS2], [FSS3], [FSS4]. We will need the following simple fact.

**Lemma 2.5.** *Let $R_o > 0$ be given and consider a $H$-Caccioppoli set $E \subset \bar{E} \subset B(e, R_o)$, then*

$$P_H(E, B(e, R_o)) = P_H(E, \boldsymbol{G}). \tag{2.3}$$

*Proof.* This can be easily seen as follows. Clearly, one has trivially $P_H(E, B(e, R_o)) \leq P_H(E, \boldsymbol{G})$. To establish the opposite inequality, let $r_o < R_o$ be such that $E \subset B(e, r_o)$, and pick $f \in C_0^\infty(B(e, R_o))$ be such that $0 \le f \le 1$, and $f \equiv 1$ on $\bar{B}(e, r_o)$. If $\zeta \in \mathscr{F}(\boldsymbol{G})$, then it is clear that $f\zeta \in \Gamma_0^1(B(e, R_o); H\boldsymbol{G})$, and that $\|f\zeta\|_{L^\infty(B(e,R_o))} \le 1$, i.e., $f\zeta \in \mathscr{F}(B(e, R_o))$. We have

$$\begin{aligned}\int_{\boldsymbol{G}} \chi_E \operatorname{div}_H \zeta \, dg &= \int_{B(e,R_o)} \chi_E f \operatorname{div}_H \zeta \, dg \\ &= \int_{B(e,R_o)} \chi_E \operatorname{div}_H (f\zeta) \, dg - \int_{B(e,R_o)} \chi_E \langle \nabla_H f, \zeta \rangle \, dg \\ &= \int_{B(e,R_o)} \chi_E \operatorname{div}_H (f\zeta) \, dg \le P_H(E, B(e, R_o)).\end{aligned}$$

Taking the supremum over all $\zeta \in \mathscr{F}(\boldsymbol{G}; H\boldsymbol{G})$ we reach the conclusion $P_H(E, B(e, R_o)) \ge P_H(E, \boldsymbol{G})$, thus obtaining (2.3). □

In the next result we extend the isoperimetric inequality from $C^1$ to bounded $H$-Caccioppoli sets.

**Theorem 2.6.** *Let $\boldsymbol{G}$ be a Carnot group with homogeneous dimension $Q$. There exists a constant $C_{\mathrm{iso}}(\boldsymbol{G}) > 0$ such that for every bounded $H$-Caccioppoli set $E \subset \boldsymbol{G}$ one has*

$$|E|^{(Q-1)/Q} \le C_{\mathrm{iso}}(\boldsymbol{G}) P_H(E; \boldsymbol{G}).$$

*Proof.* In [CDG] it was proved that Theorem 2.2 implies the following Sobolev inequality of Gagliardo-Nirenberg type: for every $u \in C_0^1(B(g_o, R))$

$$\left\{ \int_{B(g_o,R)} |u|^{Q/(Q-1)} \, dg \right\}^{(Q-1)/Q} \le C \frac{R}{|B(g_o, R)|^{1/Q}} \int_{B(g_o,R)} |\nabla_H u| \, dg. \tag{2.4}$$

If now $u \in BV_H(B(g_o, R))$, with $\operatorname{supp} u \subset B(g_o, R)$, then by Theorem 2.3 there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \in C_0^\infty(B(g_o, R))$ such that $u_k \to u$ in $L^1(B(g_o, R))$, and $\operatorname{Var}_H(u_k; B(g_o, R)) \to \operatorname{Var}_H(u; B(g_o, R))$, as $k \to \infty$. Passing to a subsequence, we can assume that $u_k(g) \to u(g)$, for $dg$-a.e. $g \in B(g_o, R)$. Applying (2.4) to $u_k$ and passing to the limit we infer from the theorem of Fatou

$$\left\{ \int_{B(g_o,R)} |u|^{Q/(Q-1)} \, dg \right\}^{(Q-1)/Q} \le C \frac{R}{|B(g_o, R)|^{1/Q}} \operatorname{Var}_H(u; B(g_o, R)),$$

for every $u \in BV_H(B(g_o, R))$, with $\operatorname{supp} u \subset B(g_o, R)$. If now $E \subset \bar{E} \subset B(g_o, R)$ is a $H$-Caccioppoli set, then taking $u = \chi_E$ in the latter inequality we obtain

$$|E|^{(Q-1)/Q} \le C \frac{R}{|B(g_o, R)|^{1/Q}} P_H(E; B(e, R_o)).$$

At this point, to reach the desired conclusion we only need to use Lemma 2.5 and observe that $|B(g_o, R)| = R^Q |B(e, 1)|$. We thus obtain the conclusion with $C_{iso}(\boldsymbol{G}) = C|B(e, 1)|^{-1/Q}$. $\square$

The following is a basic consequence of Theorem 2.6.

**Theorem 2.7.** *Let* $\boldsymbol{G}$ *be a Carnot group with homogeneous dimension* $Q$. *With* $C_{iso}(\boldsymbol{G})$ *equal to the constant in Theorem 2.2, one has for any bounded* $H$*-Caccioppoli set*

$$|E|^{(Q-1)/Q} \le C_{iso}(\boldsymbol{G}) P_H(E; \boldsymbol{G}).$$

To establish Theorem 1.5 we next prove that one can remove from Theorem 2.7, without altering the constant $C_{iso}(\boldsymbol{G})$, the restriction that the $H$-Caccioppoli set be bounded. We recall a useful representation formula. In what follows $N$ indicates the topological dimension of $\boldsymbol{G}$, and $H_{N-1}$ the $(N-1)$-dimensional Hausdorff measure constructed with the Riemannian distance of $\boldsymbol{G}$.

**Lemma 2.8.** *Let* $\Omega \subset \boldsymbol{G}$ *be an open set and* $E \subset \boldsymbol{G}$ *be a* $C^1$ *bounded domain. One has*

$$P_H(E; \Omega) = \int_{\Omega \cap \partial E} \frac{|\boldsymbol{N}_H|}{|\boldsymbol{N}|} \, dH_{N-1},$$

*where* $\boldsymbol{N}_H = \sum_{j=1}^m \langle \boldsymbol{N}, X_j \rangle X_j$ *is the projection onto* $H\boldsymbol{G}$ *of the Riemannian normal* $\boldsymbol{N}$ *exterior to E. In particular, when*

$$E = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}, \tag{2.5}$$

*with* $\phi \in C^1(\boldsymbol{G})$, *and* $|\nabla \phi| \ge \alpha > 0$ *in a neighborhood of* $\partial E$, *then* $\boldsymbol{N} = \nabla \phi$, *and therefore* $|\boldsymbol{N}_H| = |\nabla_H \phi|$. *When* $\Omega = \boldsymbol{G}$ *we thus obtain in particular*

$$P_H(E; \boldsymbol{G}) = \int_{\partial E} \frac{|\nabla_H \phi|}{|\nabla \phi|} \, dH_{N-1}. \tag{2.6}$$

For the proof of this lemma we refer the reader to [CDG]. For a detailed study of the perimeter measure in Lemma 2.8 and (2.6), we refer the reader to [DGN1], [DGN2] and [CG]. We can finally provide the proof of Theorem 1.5.

**Theorem 2.9.** *Let* $\boldsymbol{G}$ *be a Carnot group with homogeneous dimension* $Q$. *With the same constant* $C_{iso}(\boldsymbol{G}) > 0$ *as in Theorem 2.7, for every* $H$*-Caccioppoli set* $E \subset \boldsymbol{G}$ *one has*

$$|E|^{(Q-1)/Q} \le C_{iso}(\boldsymbol{G}) P_H(E; \boldsymbol{G}).$$

*Proof.* In view of Theorem 2.7 we only need to consider the case of an unbounded $H$-Caccioppoli set $E$. If $P_H(E; \boldsymbol{G}) = +\infty$ there is nothing to prove, so we assume that

$P_H(E; \boldsymbol{G}) < +\infty$ and $|E| < +\infty$. We consider the $C^\infty$ $H$-balls $B_H(e, R) = \{g \in \boldsymbol{G} \mid \rho(g) < R\}$, generated by the pseudo-distance $\rho = \rho_e = \Gamma(\cdot, e)^{1/(2-Q)} \in C^\infty(\boldsymbol{G} \setminus \{e\}) \cap C(\boldsymbol{G})$, where $\Gamma(\cdot, e) \in C^\infty(\boldsymbol{G} \setminus \{e\})$ is the fundamental solution with singularity at the identity for the sub-Laplacian $\Delta_H = \sum_{j=1}^m X_j^2$ (the reader should notice that any other smooth gauge would do). For any $R > 0$ we have

$$(2.7) \qquad P_H(E \cap B_H(e, R); \boldsymbol{G}) \le P_H(E; B_H(e, R)) + P_H(B_H(e, R); E).$$

Here, when we write $P_H(B_H(e, R); E)$ we mean the standard measure theoretic extension of the $H$-perimeter from open sets to Borel sets, see for instance [Z]. Thanks to the smoothness of $B_H(e, R)$ we have from Lemma 2.8

$$P_H(B_H(e, R); E) = \int_{\partial B_H(e, R) \cap E} \frac{|\boldsymbol{N}_H|}{|\boldsymbol{N}|} \, dH_{N-1} = \int_{\partial B_H(e, R) \cap E} \frac{|\nabla_H \rho|}{|\nabla \rho|} \, dH_{N-1}.$$

Recalling that $\Gamma(\cdot, e)$ is homogeneous of degree $2 - Q$, see [F1], [F2], and therefore $\rho$ is homogeneous of degree one, we infer that for some constant $C(\boldsymbol{G}) > 0$,

$$(2.8) \qquad |\nabla_H \rho| \le C(\boldsymbol{G}).$$

This gives

$$(2.9) \qquad P_H(B_H(e, R); E) \le C(\boldsymbol{G}) \int_{\partial B_H(e, R) \cap E} \frac{dH_{N-1}}{|\nabla \rho|}.$$

By Federer co-area formula [Fe], we obtain

$$\infty > |E| = \int_{\boldsymbol{G}} \chi_E \, dg = \int_0^\infty \int_{\partial B_H(e, t) \cap E} \frac{dH_{N-1}}{|\nabla \rho|} \, dt,$$

therefore there exists a sequence $R_k \nearrow \infty$ such that

$$(2.10) \qquad \int_{\partial B_H(e, R_k) \cap E} \frac{dH_{N-1}}{|\nabla \rho|} \xrightarrow[k \to \infty]{} 0.$$

Using (2.10) in (2.9) we find

$$(2.11) \qquad \lim_{k \to \infty} P_H(B_H(e, R_k); E) = 0.$$

From (2.7), (2.11), we conclude

$$(2.12) \qquad \limsup_{k \to \infty} P_H(E \cap B_H(e, R_k); \boldsymbol{G}) \le P_H(E; \boldsymbol{G}).$$

We next apply Theorem 2.7 to the bounded $H$-Caccioppoli set $E \cap B_H(e, R_k)$ obtaining

$$|E \cap B_H(e, R_k)|^{(Q-1)/Q} \leq C_{\text{iso}}(\boldsymbol{G}) P_H(E \cap B_H(e, R_k); \boldsymbol{G}).$$

Letting $k \to \infty$ in the latter inequality, from (2.12), and from the relation

$$\lim_{k \to \infty} |E \cap B_H(e, R_k)|^{(Q-1)/Q} = |E|^{(Q-1)/Q},$$

we conclude that

$$|E|^{(Q-1)/Q} \leq C_{\text{iso}}(\boldsymbol{G}) P_H(E; \boldsymbol{G}).$$

This completes the proof. □

We close this section with two basic properties of the $H$-perimeter which clearly play a role also in Theorem 1.4.

**Proposition 2.10.** *In a Carnot group* $\boldsymbol{G}$ *one has for every measurable set* $E \subset \boldsymbol{G}$ *and every* $r > 0$

$$P_H(E; \boldsymbol{G}) = r^{Q-1} P_H(\delta_{1/r} E; \boldsymbol{G}). \tag{2.13}$$

*Proof.* Let $E \subset \boldsymbol{G}$ be a measurable set. If $\zeta \in C_0^1(\boldsymbol{G}, H\boldsymbol{G})$, then the divergence theorem, and a rescaling, give

$$\int_E \operatorname{div}_H \zeta \, dg = \int_E \sum_{j=1}^m X_j \zeta_j \, dg = r^Q \int_{E_r} \sum_{j=1}^m X_j \zeta_j(\delta_r g) \, dg, \tag{2.14}$$

where we have let $E_r = \delta_{1/r}(E) = \{g \in \boldsymbol{G} \mid \delta_r g \in E\}$. Since

$$X_j(\zeta_j \circ \delta_r) = r(X_j \zeta_j \circ \delta_r),$$

we conclude

$$\int_E \sum_{j=1}^m X_j \zeta_j \, dg = r^{Q-1} \int_E \sum_{j=1}^m X_j(\zeta_j \circ \delta_r) \, dg. \tag{2.15}$$

Formula (2.15) implies the conclusion. □

Proposition 2.10 asserts that the $H$-perimeter scales appropriately with respect to the non-isotropic group dilations. Since on the other hand one has $|\delta_{1/r} E| = r^{-Q}|E|$, we easily obtain the following important scale invariance of the isoperimetric quotient.

**Proposition 2.11.** *For any $H$-Caccioppoli set in a Carnot group $\boldsymbol{G}$ one has*

$$(2.16)\qquad \frac{P_H(E;\boldsymbol{G})}{|E|^{(Q-1)/Q}} = \frac{P_H(\delta_{1/r}E;\boldsymbol{G})}{|\delta_{1/r}E|^{(Q-1)/Q}}, \quad r > 0.$$

Another equally important fact, which is however a trivial consequence of the left-invariance on the vector fields $X_1, \dots, X_m$, and of the definition of $H$-perimeter, is the translation invariance of the isoperimetric quotient.

**Proposition 2.12.** *For any $H$-Caccioppoli set in a Carnot group $\boldsymbol{G}$ one has*

$$(2.17)\qquad \frac{P_H(L_{g_o}(E);\boldsymbol{G})}{|L_{g_o}(E)|^{(Q-1)/Q}} = \frac{P_H(E;\boldsymbol{G})}{|E|^{(Q-1)/Q}}, \quad g_o \in \boldsymbol{G},$$

*where $L_{g_o}g = g_o g$ is the left-translation on the group.*

## 3 Partial solution of the isoperimetric problem in $\mathbb{H}^n$

The objective of this section is proving Theorems 1.1 and 1.4. This will be accomplished in several steps. First, we introduce the relevant notions and establish some geometric properties of the $H$-perimeter that are relevant to the isoperimetric profiles. Next, we collect some results from convex analysis and calculus of variations. Finally, we proceed to proving Theorems 1.1 and 1.4. In what follows we adopt the classical non-parametric point of view, see for instance [MM], according to which a $C^2$ hypersurface $\mathscr{S} \subset \boldsymbol{G}$ locally coincides with the zero set of a real function. Thus, for every $g_0 \in \mathscr{S}$ there exists an open set $\mathcal{O} \subset \boldsymbol{G}$ and a function $\phi \in C^2(\mathcal{O})$ such that: (i) $|\nabla\phi(g)| \neq 0$ for every $g \in \mathcal{O}$; (ii) $\mathscr{S} \cap \mathcal{O} = \{g \in \mathcal{O} \mid \phi(g) = 0\}$. We will always assume that $\mathscr{S}$ is oriented in such a way that for every $g \in \mathscr{S}$ one has

$$\begin{aligned}\boldsymbol{N}(g) &= \nabla\phi(g)\\ &= X_1\phi(g)X_1 + \cdots + X_m\phi(g)X_m + \cdots + X_{r,1}\phi(g)X_{r,1}\\ &\quad + \cdots + X_{r,m_r}\phi(g)X_{r,m_r}.\end{aligned}$$

To justify the second equality the reader should bear in mind that we have endowed $\boldsymbol{G}$ with a left-invariant Riemannian metric with respect to which $\{X_1, \dots, X_m, \dots, X_{r,m_r}\}$ constitute an orthonormal basis. Given a surface $\mathscr{S} \subset \boldsymbol{G}$, we let

$$(3.1)\qquad p_i = \langle \boldsymbol{N}, X_i\rangle, \quad i = 1, \dots, m,$$

and define the angle function

$$(3.2)\qquad W = \sqrt{p_1^2 + \cdots + p_m^2}.$$

The motivation for the name comes from the fact that, if $U \angle V$ denotes the angle between two vector fields $U$, $V$ on $\boldsymbol{G}$, then

$$(3.3)\qquad \cos(\boldsymbol{v}_H \angle \boldsymbol{N}) = \frac{\langle \boldsymbol{v}_H, \boldsymbol{N}\rangle}{|\boldsymbol{N}|} = \frac{W}{|\boldsymbol{N}|}.$$

The characteristic locus of $\mathscr{S}$ is the closed set

$$\Sigma = \{g \in \mathscr{S} \mid W(g) = 0\} = \{g \in \mathscr{S} \mid H_g\boldsymbol{G} \subset T_g\mathscr{S}\}.$$

We recall that is was proved in [B], [Ma] that $\mathscr{H}^{Q-1}(\Sigma) = 0$, where $\mathscr{H}^s$ denotes the $s$-dimensional Hausdorff measure associated with the Carnot-Carathéodory distance of $\boldsymbol{G}$, and $Q$ indicates the homogeneous dimension of $\boldsymbol{G}$. We also recall the earlier result of Derridj [De1], [De2], which states that when $\mathscr{S}$ is $C^\infty$ the standard surface measure of $\Sigma$ vanishes. Later on in this section we will need a result from [B], see Theorem 3.9 below.

On the set $\mathscr{S}\backslash\Sigma$ we define the *horizontal Gauss map* by

$$(3.4)\qquad \boldsymbol{v}_H = \bar{p}_1 X_1 + \cdots + \bar{p}_m X_m,$$

where we have let

$$(3.5)\qquad \bar{p}_1 = \frac{p_1}{W}, \ldots, \bar{p}_m = \frac{p_m}{W}, \quad \text{so that } |\boldsymbol{v}_H|^2 = \bar{p}_1^2 + \cdots + \bar{p}_m^2 \equiv 1 \quad \text{on } \mathscr{S}\backslash\Sigma.$$

Given a point $g_0 \in \mathscr{S}\backslash\Sigma$, the horizontal tangent space of $\mathscr{S}$ at $g_0$ is defined by

$$T_{H,g_0}(\mathscr{S}) = \{\boldsymbol{v} \in H_{g_0}\boldsymbol{G} \mid \langle \boldsymbol{v}, \boldsymbol{v}_H(g_0)\rangle = 0\}.$$

For instance, when $\boldsymbol{G} = \mathbb{H}^1$, then a basis for $T_{H,g_0}(\mathscr{S})$ is given by the single vector field

$$(3.6)\qquad \boldsymbol{v}_H^\perp = \bar{p}_2 X_1 - \bar{p}_1 X_2.$$

Given a function $u \in C^1(\mathscr{S})$ one clearly has $\delta_H u(g_0) \in T_{H,g_0}(\mathscr{S})$. We next recall some basic definitions from [DGN4].

Let $\nabla^H$ denote the horizontal Levi-Civita connection introduced in [DGN4]. Let $\mathscr{S} \subset \boldsymbol{G}$ be a $C^2$ hypersurface. Inspired by the Riemannian situation we introduce a notion of horizontal second fundamental on $\mathscr{S}$ as follows.

**Definition 3.1.** Let $\mathscr{S} \subset \boldsymbol{G}$ be a $C^2$ hypersurface, with $\Sigma = \emptyset$, then we define a tensor field of type $(0,2)$ on $T_H\mathscr{S}$, as follows: for every $X, Y \in C^1(\mathscr{S}; T_H\mathscr{S})$

$$(3.7)\qquad II^{H,\mathscr{S}}(X,Y) = \langle \nabla_X^H Y, \boldsymbol{v}_H\rangle \boldsymbol{v}_H.$$

We call $II^{H,\mathscr{S}}(\cdot,\cdot)$ the *horizontal second fundamental form* of $\mathscr{S}$. We also define $\mathscr{A}^{H,\mathscr{S}} : T_H\mathscr{S} \to T_H\mathscr{S}$ by letting for every $g \in \mathscr{S}$ and $\boldsymbol{u}, \boldsymbol{v} \in T_{H,g}$

(3.8) $\quad \langle \mathscr{A}^{H,\mathscr{S}} \boldsymbol{u}, \boldsymbol{v} \rangle = -\langle II^{H,\mathscr{S}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{\nu}_H \rangle = -\langle \nabla^H_X Y, \boldsymbol{\nu}_H \rangle,$

where $X, Y \in C^1(\mathscr{S}, T_H\mathscr{S})$ are such that $X_g = \boldsymbol{u}$, $Y_g = \boldsymbol{v}$. We call the endomorphism $\mathscr{A}^{H,\mathscr{S}} : T_{H,g}\mathscr{S} \to T_{H,g}\mathscr{S}$ the *horizontal shape operator*. If $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m-1}$ denotes a local orthonormal frame for $T_H\mathscr{S}$, then the matrix of the horizontal shape operator with respect to the basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m-1}$ is given by the $(m-1) \times (m-1)$ matrix $[-\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{\nu}_H \rangle]_{i,j=1,\dots,m-1}$.

By the horizontal Koszul identity in [DGN4], one easily verifies that

$$\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{\nu}_H \rangle = -\langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{\nu}_H, \boldsymbol{e}_j \rangle.$$

Using Definition 3.1 one recognizes that

(3.9) $\quad II^{H,\mathscr{S}}(X, Y) - II^{H,\mathscr{S}}(Y, X) = \langle [X, Y]^H, \boldsymbol{\nu}_H \rangle \boldsymbol{\nu}_H,$

and therefore, unlike its Riemannian counterpart, the horizontal second fundamental form of $\mathscr{S}$ is not necessarily symmetric. This depends on the fact that, if $X, Y \in C^1(\mathscr{S}; HT\mathscr{S})$, then it is not necessarily true that the projection of $[X, Y]$ onto the horizontal bundle $H\mathbb{H}^n$, $[X, Y]^H$, belongs to $C^1(\mathscr{S}; T_H\mathscr{S})$.

**Definition 3.2.** We define the *horizontal principal curvatures* as the real eigenvalues $\kappa_1, \dots, \kappa_{m-1}$ of the symmetrized operator

$$\mathscr{A}^{H,\mathscr{S}}_{\text{sym}} = \frac{1}{2}\{\mathscr{A}^{H,\mathscr{S}} + (\mathscr{A}^{H,\mathscr{S}})^t\},$$

The $H$-mean curvature of $\mathscr{S}$ at a non-characteristic point $g_0 \in \mathscr{S}$ is defined as

$$\mathscr{H} = -\text{trace}\, \mathscr{A}^{H,\mathscr{S}}_{\text{sym}} = \sum_{i=1}^{m-1} \kappa_i = \sum_{i=1}^{m-1} \langle \nabla^H_{\boldsymbol{e}_i} \boldsymbol{e}_i, \boldsymbol{\nu}_H \rangle.$$

If $g_0$ is characteristic, then we let

$$\mathscr{H}(g_0) = \lim_{g \to g_0,\, g \in \mathscr{S} \setminus \Sigma} \mathscr{H}(g),$$

provided that such limit exists, finite or infinite. We do not define the $H$-mean curvature at those points $g_0 \in \Sigma$ at which the limit does not exist. Finally, we call $\vec{\mathscr{H}} = \mathscr{H} \boldsymbol{\nu}_H$ the *H-mean curvature vector*.

Hereafter, when we say that a function $u$ belongs to the class $C^k(\mathscr{S})$, we mean that $u \in C(\mathscr{S})$ and that for every $g_0 \in \mathscr{S}$, there exist an open set $\mathcal{O} \subset \mathbb{H}^1$, such that $u$ coincides with the restriction to $\mathscr{S} \cap \mathcal{O}$ of a function in $C^k(\mathcal{O})$. The tangential horizontal gradient of a function $u \in C^1(\mathscr{S})$ is defined as follows

$$(3.10)\quad \nabla^{H,\mathscr{S}} u = \nabla_H u - \langle \nabla_H u, \boldsymbol{\nu}_H \rangle \boldsymbol{\nu}_H.$$

The definition of $\nabla^{H,\mathscr{S}} u$ is well-posed since it is noted in [DGN4] that it only depends on the values of $u$ on $\mathscr{S}$. Since $|\boldsymbol{\nu}_H| \equiv 1$ on $\mathscr{S} \backslash \Sigma$, we clearly have $\langle \nabla^{H,\mathscr{S}} u, \boldsymbol{\nu}_H \rangle = 0$, and therefore

$$(3.11)\quad |\nabla^{H,\mathscr{S}} u|^2 = |\nabla_H u|^2 - \langle \nabla_H u, \boldsymbol{\nu}_H \rangle^2.$$

**Definition 3.3.** We say that a $C^2$ hypersurface $\mathscr{S}$ has *constant $H$-mean curvature* if $\mathscr{H}$ is globally defined on $\mathscr{S}$, and $\mathscr{H} \equiv$ *const*. We say that $\mathscr{S}$ is *$H$-minimal* if $\mathscr{H} \equiv 0$.

Minimal surfaces have been recently studied in [Pa], [GP], [CHMY], [CH], [DGN5], [DGNP], [BSV]. The last two papers contain also a complete solution of the Bernstein type problem for the Heisenberg group $\mathbb{H}^1$. The following result is taken from [DGN4].

**Proposition 3.4.** *The $H$-mean curvature coincides with the function*

$$(3.12)\quad \mathscr{H} = \sum_{i=1}^{m} \nabla^{H,\mathscr{S}} \bar{p}_i = \sum_{i=1}^{m} X_i \bar{p}_i.$$

For instance, when $\boldsymbol{G} = \mathbb{H}^1$, then according to Proposition 3.4, the $H$-mean curvature of $\mathscr{S}$ is given by

$$(3.13)\quad \mathscr{H} = \sum_{i=1}^{2} \nabla_i^{H,\mathscr{S}} \nu_{H,i} = \nabla_1^{H,\mathscr{S}}(\bar{p}_1) + \nabla_2^{H,\mathscr{S}}(\bar{p}_2) = X_1\bar{p}_1 + X_2\bar{p}_2, \quad \text{on } \mathscr{S}\backslash\Sigma.$$

In this situation, given a $C^2$ surface $\mathscr{S} \subset \mathbb{H}^1$, there is only one horizontal principal curvature $\kappa_1(g_0)$ at every $g_0 \in \mathscr{S}\backslash\Sigma$. Since in view of (3.6) the vector $\boldsymbol{\nu}_H^{\perp}(g_0)$ constitutes an orthonormal basis of $T_{H,g_0}(\mathscr{S})$, according to Definition 3.1 we have

$$\kappa_1(g_0) = II_H(\boldsymbol{\nu}_H^{\perp}(g_0), \boldsymbol{\nu}_H^{\perp}(g_0)).$$

One can verify, see [DGN4], that the right-hand side of the latter equation equals $-\mathscr{H}(g_0)$. We recall one more result concerning the $H$-mean curvature that will be useful in the proof of Proposition 3.28. Details can be found in [DGN4].

**Proposition 3.5.** *Suppose $\mathscr{S} \subset \mathbb{H}^n$ is a level set of a function $\phi$ that takes the form*

$$\phi(z,t) = t - u\left(\frac{|z|^2}{4}\right),$$

*for some $C^2$ function $u : [0,\infty) \to \mathbb{R}$. For every point point $g = (z,t) \in \mathscr{S}$ such that $z \neq 0$ the $H$-mean curvature at $g$ is given by*

$$(3.14)\quad \mathscr{H} = -\frac{2su''(s) + (Q-3)u'(s)(1+u'(s)^2)}{2\sqrt{s}(1+u'(s)^2)^{3/2}}, \quad s = \frac{|z|^2}{4}.$$

In Proposition 3.5 the hypothesis $z \neq 0$ is justified by the fact that, under the given assumptions, if $\mathscr{S}$ intersects the $t$-axis in $\mathbb{H}^n$, then the points of intersections are necessarily characteristic for $\mathscr{S}$.

Hereafter in this paper, we restrict our attention to $\boldsymbol{G} = \mathbb{H}^n$. In Definition 3.3, following the classical tradition, we have called a hypersurface $H$-minimal if its $H$-mean curvature vanishes identically. However, in the classical setting the measure theoretic definition of minimality is also based on the notion of local minimizer of the area functional. In the paper [DGN4] we have proved that there is a corresponding sub-Riemannian counterpart of such interpretation based on appropriate first and second variation formulas for the $H$-perimeter. For instance, the following first variation formula holds in the Heisenberg group $\mathbb{H}^1$.

**Theorem 3.6.** *Let $\mathscr{S} \subset \mathbb{H}^1$ be an oriented $C^2$ surface, then the first variation of the $H$-perimeter with respect to the deformation*

$$(3.15)\quad J_\lambda(g) = g + \lambda\mathscr{X}(g) = g + \lambda(a(g)X_1 + b(g)X_2 + k(g)T), \quad g = (x, y, t) \in \mathscr{S},$$

*is given by*

$$(3.16)\quad \frac{d}{d\lambda}P_H(\mathscr{S}^\lambda)|_{\lambda=0} = \int_{\mathscr{S}} \mathscr{H}\frac{\cos(\mathscr{X}\angle\boldsymbol{N})}{\cos(\boldsymbol{v}_H\angle\boldsymbol{N})}|\mathscr{X}|\,d\sigma_H,$$

*where $\angle$ denotes the angle between vectors in the inner product $\langle\cdot,\cdot\rangle$. In particular, $\mathscr{S}$ is stationary with respect to* (3.15) *if and only if it is $H$-minimal.*

Versions of Theorem 3.6 have also been obtained independently by other people. An approach based on motion by $H$-mean curvature can be found in [BC]. When $a = \bar{p}h$, $b = \bar{q}h$, and $h \in C_0^\infty(\mathscr{S}\backslash\Sigma)$, then a proof based on CR-geometry can be found in [CHMY]. A Riemannian geometry proof, valid in any $\mathbb{H}^n$, can be found in [RR1].

In what follows we set

$$\mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n \mid t > 0\}, \quad \mathbb{H}^n_- = \{(z,t) \in \mathbb{H}^n \mid t < 0\}.$$

Consider a domain $\Omega \subset \mathbb{R}^{2n}$ and a $C^1$ function $u : \Omega \to [0,\infty)$. We assume that $E \subset \mathbb{H}^n$ is a $C^1$ domain for which

$$E \cap \mathbb{H}^n_+ = \{(z,t) \in \mathbb{H}^n \mid z \in \Omega, 0 < t < u(z)\}.$$

The reader should notice that, since $u > 0$ in $\Omega$, the graph of $u$ is not allowed to have flat parts. For $z = (x,y) \in \mathbb{R}^{2n}$, we set $z^\perp = (y, -x)$. Indicating with $\phi(z,t) = t - u(z)$ the defining function of $E \cap \mathbb{H}^n_+$, a simple computation gives

$$(3.17)\quad |\nabla_H \phi| = \sqrt{\left|\nabla_x u + \frac{y}{2}\right|^2 + \left|\nabla_y u - \frac{x}{2}\right|^2} = \left|\nabla_z u + \frac{z^\perp}{2}\right|.$$

The reader should be aware that in the latter equation, the norm in the left-hand side comes from the Riemannian inner product on $T\mathbb{H}^n \cong \mathbb{H}^n$, whereas the norm in the right-hand side is simply the Euclidean norm in $\mathbb{R}^{2n}$. Invoking the representation formula (2.6) for the $H$-perimeter, which presently gives

$$P_H(E; \mathbb{H}^n_+) = \int_{\partial E \cap \mathbb{H}^n_+} \frac{|\nabla_H \phi|}{|\nabla \phi|} dH_{2n},$$

and keeping in mind that, see (3.17), $|\nabla \phi| = \sqrt{1 + |\nabla_H \phi|^2}$, and that $dH_{2n} = \sqrt{1 + |\nabla_H \phi|^2}\, dz$, we obtain

$$(3.18)\quad P_H(E; \mathbb{H}^n_+) = \int_\Omega \left|\nabla_z u + \frac{z^\perp}{2}\right| dz = \int_\Omega \sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}\, dz.$$

When $F \subset \mathbb{H}^n$ is a closed set we define

$$P_H(E; F) = \inf_{F \subset \Omega, \Omega \text{ open}} P_H(E; \Omega).$$

Let now $u \in C^1(\Omega)$, $u \geq 0$, then using the latter formula we obtain the following generalization of (3.18)

$$(3.19)\quad P_H(E; \overline{\mathbb{H}^n_+}) = \int_\Omega \left|\nabla_z u + \frac{z^\perp}{2}\right| dz = \int_\Omega \sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}\, dz.$$

The reader should notice that, unlike (3.18), in equation (3.19) we allow the graph of $u$ to have flat parts, i.e., subsets of $\Omega$ in which the function $u$ vanishes.

In what follows, we recall an invariance property of the $H$-perimeter which plays a role in the proof of Theorem 1.1. Consider the map $\mathcal{O} : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$\mathcal{O}(x, y, t) = (y, x, -t).$$

It is obvious that $\mathcal{O}$ preserves Lebesgue measure (which is a bi-invariant Haar measure on $\mathbb{H}^n$). In fact, the map $\mathcal{O}$ is a group and Lie algebra automorphism of $\mathbb{H}^n$. Such map is called *inversion* in [F3], p. 20. Using the properties of the map $\mathcal{O}$ and a standard contradiction argument, one can easily prove the following result.

**Theorem 3.7.** *Let $E \subset \mathbb{H}^n$ be a bounded open set such that $\partial E \cap \mathbb{H}^n_+$ and $\partial E \cap \mathbb{H}^n_-$ are $C^1$ hypersurfaces, and assume that $E$ satisfies the following condition: there exists $R > 0$ such that*

(3.20) $E \cap \{t = 0\} = B(0, R)$.

*Suppose E is an isoperimetric set satisfying* $|E \cap \mathbb{H}^n_+| = |E \cap \mathbb{H}^n_-| = |E|/2$, *then*

$$P_H(E; \overline{\mathbb{H}^n_+}) = P_H(E; \overline{\mathbb{H}^n_-}).$$

We now introduce the relevant functional class for our problem. The space of competing functions $\mathscr{D}$ is defined as follows. Consider the vector space $\mathscr{V} = C_0(\mathbb{R}^{2n})$.

**Definition 3.8.** We let

$$(3.21) \quad \mathscr{D} = \{u \in \mathscr{V} \mid \text{there exists } R > 0 \text{ such that } u \geq 0 \text{ in } B(0, R),$$
$$\bar{B}(0, R) = \bigcap \{B(0, R + \rho) \mid \operatorname{supp}(u) \subset B(0, R + \rho)\},$$
$$u \in C^{1,1}_{\mathrm{loc}}(B(0, R)) \cap W^{1,1}(B(0, R))\}.$$

We note explicitly that, as a consequence of Definition (3.8), if $u \in \mathscr{D}$ and $R$ is as in (3.21), we have $u = 0$ on $\partial B(0, R)$. Furthermore, the functions in $\mathscr{D}$ are allowed to have large sets of zeros, i.e., their graph is allowed to touch the hyperplane $t = 0$ in sets of large measure. We remark that $\mathscr{D}$ is not a vector space, nor it is a convex subset of $\mathscr{V}$. We mention that the requirement $u \in C^{1,1}_{\mathrm{loc}}(B(0, R))$ in the definition of the class $\mathscr{D}$, is justified by the following considerations. When we compute the Euler-Lagrange equation of the $H$-perimeter functional (3.18) we need to know that, with $\Omega = B(0, R)$, the set $\left\{z = (x, y) \in \Omega \subset \mathbb{R}^{2n} \mid \left|\nabla_z u(z) + \frac{z^\perp}{2}\right| = 0\right\}$, which is the projection of the characteristic set of the graph of $u$ onto $\mathbb{R}^{2n} \times \{0\}$, has vanishing $2n$-dimensional Lebesgue measure. This is guaranteed by the following sharp result of Z. Balogh (see Theorem 3.1 in [B]) provided that $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, but it fails in general for $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for every $0 < \alpha < 1$.

**Theorem 3.9.** *Let* $\Omega = B(0, R) \subset \mathbb{R}^{2n}$ *and consider* $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, *then* $|\mathscr{A}(u)| = 0$, *where* $\mathscr{A}(u) = \{z \in \Omega \mid \nabla_z u(z) + z^\perp/2 = 0\}$, *and* $|E|$ *denotes the* $2n$*-dimensional Lebesgue measure of* $E$ *in* $\mathbb{R}^{2n}$. *If instead* $u \in C^2(\Omega)$, *then the Euclidean dimension of* $\mathscr{A}(u)$ *is* $\leq n$.

Following classical ideas from the Calculus of Variation, we next introduce the admissible variations for the problem at hand, see [GH] and [Tr].

**Definition 3.10.** Given $u \in \mathscr{D}$, we say that $\phi \in \mathscr{V}$, with $\operatorname{supp} \phi \subseteq \operatorname{supp} u$, is *$\mathscr{D}$-admissible at u* if $u + \lambda\phi \in \mathscr{D}$ for all $\lambda \in \mathbb{R}$ sufficiently small.

Now, for $u \in \mathscr{D}$ we let

$$(3.22) \quad \mathscr{G}[u] = \int_{\operatorname{supp}(u)} u(z)\, dz = \int_{B(0, R)} u(z)\, dz.$$

With (3.18) in mind, we define for such $u$

$$\mathscr{F}[u] = \int_{\mathrm{supp}(u)} \sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}. \tag{3.23}$$

In the class of $C^1$ graphs over $\mathbb{R}^{2n} \times \{0\}$, the isoperimetric problem consists in minimizing the functional $\mathscr{F}[u]$, subject to the constraint that $\mathscr{G}[u] = V$, where $V > 0$ is given and $B(0, R)$ is replaced by an a priori unknown domain $\Omega$. We emphasize that finding the section of the isoperimetric profile with the hyperplane $\{t = 0\}$, i.e., finding the domain $\Omega$, constitutes here part of the problem. Because of the lack of an obvious symmetrization procedure, this seems a difficult question at the moment. To make further progress we restrict the class of domains $E$ by imposing that their section with the hyperplane $\{t = 0\}$ be a ball, i.e., we assume that, given $E \in \mathscr{E}$, there exists $R = R(E) > 0$ such that $\Omega = B(0, R)$. Under this hypothesis, we can appeal to Theorem 3.7. The latter implies that it suffices to solve the following variational problem: *given $V > 0$, find $R_o > 0$ and $u_o \in \mathscr{D}$ with* $\mathrm{supp}(u_o) = B(0, R_o)$ *for which the following holds*

$$\mathscr{F}[u_o] = \min\{\mathscr{F}[u] \mid u \in \mathscr{D}\} \quad \text{and} \quad \mathscr{G}[u_o] = \frac{V}{2}. \tag{3.24}$$

To reduce the problem (3.24) to one without constraint, we will apply the following standard version of the Lagrange multiplier theorem (see, e.g., Proposition 2.3 in [Tr]).

**Proposition 3.11.** *Let $\mathscr{D}$ be a subset of a normed vector space $\mathscr{V}$, and consider functionals $\mathscr{F}, \mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_k$ defined on $\mathscr{D}$. Suppose there exist constants $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, and $u_o \in \mathscr{D}$, such that $u_o$ minimizes (uniquely)*

$$\mathscr{F} + \lambda_1 \mathscr{G}_1 + \lambda_2 \mathscr{G}_2 + \cdots + \lambda_k \mathscr{G}_k \tag{3.25}$$

*on $\mathscr{D}$, then $u_o$ minimizes $\mathscr{F}$ (uniquely) when restricted to the set*

$$\{u \in \mathscr{D} \mid \mathscr{G}_j[u] = \mathscr{G}_j[u_o], j = 1, \ldots, k\}.$$

**Remark 3.12.** The procedure of applying the above proposition to solving a problem of the type

$$\begin{cases} \text{minimize} \quad \{\mathscr{F}[u] \mid u \in \mathscr{D}\}, \\ \text{subject to the constraints} \quad \mathscr{G}_1[u] = V_1, \ldots, \mathscr{G}_k[u] = V_k, \end{cases}$$

consists of two main steps. First, one needs to show that constants $\lambda_1, \ldots, \lambda_k$ and a $u_o \in \mathscr{D}$ can be found in such a way that $u_o$ solves the Euler-Lagrange equation of (3.25), and $u_o$ satisfies $\mathscr{G}_1[u_o] = V_1, \ldots, \mathscr{G}_k[u_o] = V_k$. Finally, one proves that the solution $u_o$ of the Euler-Lagrange equation is indeed a minimizer of (3.25). If the functional involved possesses appropriate convexity properties, then one can show that such minimizer $u_o$ is unique.

We then proceed with the first step outlined in the Remark 3.12. In what follows, with $z \in \mathbb{R}^{2n}$, $u \in \mathbb{R}$, and $p = (p_1, p_2) \in \mathbb{R}^{2n}$, we let

$$(3.26) \quad \begin{cases} f(z,u,p) = f(z,p) = \sqrt{\left|p_1 + \frac{y}{2}\right|^2 + \left|p_2 - \frac{x}{2}\right|^2} = \left|p + \frac{z^\perp}{2}\right|, \\ g(z,u,p) = g(u) = u, \\ h(z,u,p) = f(z,p) + \lambda g(u). \end{cases}$$

The constrained variational problem (3.24) is then equivalent to the following one without constraint (provided the parameter $\lambda$ is appropriately chosen): *to minimize the functional*

$$(3.27) \quad \mathscr{F}[u] = \int_{\text{supp}(u)} h(z, u(z), \nabla_z u(z))\, dz = \int_{\text{supp}(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} dz,$$

*over the set* $\mathscr{D}$ *defined in* (3.21). We easily recognize that the Euler-Lagrange equation of (3.27) is

$$(3.28) \quad \text{div}_z \left[ \frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}} \right] = \lambda.$$

**Remark 3.13.** Before proceeding we note explicitly that, if $u \in C^2(\Omega)$, and we consider the $C^2$ hypersurface $\mathscr{S} = \{(z,t) \in \mathbb{H}^n \mid z \in \Omega, t = u(z)\}$, indicating with $\Sigma$ its characteristic set, then $g = (z,t) \notin \Sigma$ if and only if $|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle \neq 0$. In this situation, using Proposition 3.4, it can be recognized that, at every $g \notin \Sigma$, the quantity in the left-hand side of (3.28) represents the $H$-mean curvature $\mathscr{H}$ of $\mathscr{S}$.

As we have said, solving (3.28) on an arbitrary domain of $\Omega \subset \mathbb{R}^{2n}$ is a difficult task. However, when $\Omega$ is a ball in $\mathbb{R}^{2n}$, the equation (3.28) admits a remarkable family of spherically symmetric solutions. We note explicitly that for a graph $t = u(z)$ with spherical symmetry in $z$, the only characteristic points can occur at the intersection of the graph with the $t$-axis.

**Theorem 3.14.** *Given* $R > 0$, *for every*

$$(3.29) \quad -\frac{Q-2}{R} \leq \lambda < 0,$$

*the equation* (3.28), *with the Dirichlet condition* $u = 0$ *on* $\partial\Omega$, *where* $\Omega = B(0,R) = \{z \in \mathbb{R}^{2n} \mid |z| < R\}$, *admits the spherically symmetric solution* $u_{R,\lambda} \in \mathscr{D} \cap C^2(\Omega \backslash \{0\})$, *with*

$$(3.30)\quad u_{R,\lambda}(z) = C_{R,\lambda} + \frac{|z|}{4\lambda}\sqrt{(Q-2)^2 - (\lambda|z|)^2} - \frac{(Q-2)^2}{4\lambda^2}\sin^{-1}\left(\frac{\lambda|z|}{Q-2}\right),$$

*and*

$$(3.31)\quad C_{R,\lambda} = -\frac{R}{4\lambda}\sqrt{(Q-2)^2 - (\lambda R)^2} - \frac{(Q-2)^2}{4\lambda^2}\sin^{-1}\left(\frac{\lambda R}{Q-2}\right).$$

*Proof.* We look for a spherically symmetric solution in the form $u(z) = \bar{u}(|z|^2/4)$, for some function $\bar{u} \in C^2((0, R^2/4)) \cap C([0, R^2/4])$, with $\bar{u}(R^2/4) = 0$. The equation (3.28) becomes

$$(3.32)\quad \operatorname{div}_z\left[\frac{\bar{u}'(|z|^2/4)z + z^{\perp}}{|z|\sqrt{1 + \bar{u}'(|z|^2/4)}}\right] = \lambda, \quad \text{in } B(0,R)\backslash\{0\}.$$

Since

$$\operatorname{div}_z\left[\frac{z^{\perp}}{|z|\sqrt{1 + \bar{u}'(|z|^2/4)}}\right] = 0,$$

we obtain that (3.32) reduces to the equation

$$(3.33)\quad \operatorname{div}_z\left[\frac{\bar{u}'(|z|^2/4)z}{|z|\sqrt{1 + \bar{u}'(|z|^2/4)}}\right] = \lambda.$$

The transformation

$$(3.34)\quad F(r) \overset{def}{=} \frac{\bar{u}'\left(\frac{r^2}{4}\right)}{r\sqrt{1 + \left(\bar{u}'\left(\frac{r^2}{4}\right)\right)^2}},$$

turns the nonlinear equation (3.33) into the following linear one

$$F'(r) + \frac{2n}{r}F(r) = \frac{\lambda}{r},$$

which is equivalent to

$$(r^{2n}F)' = \lambda r^{2n-1}.$$

We note that

$$|r^{2n}F(r)| \le r^{2n-1}, \quad \text{for } 0 < r \le \frac{R^2}{4},$$

therefore we conclude that $\lim_{r\to 0} r^{2n}F(r) = 0$. We can thus easily integrate the above ode, obtaining $F(r) \equiv \lambda/2n$. Setting $s = r^2/4$ in the latter identity one obtains from (3.34)

$$\frac{\bar{u}'(s)}{\sqrt{1+(\bar{u}'(s))^2}} = \frac{\lambda}{n}\sqrt{s} = \frac{2\lambda}{Q-2}\sqrt{s}. \tag{3.35}$$

Excluding the case of $H$-minimal surfaces (corresponding to $\lambda = 0$), equation (3.35) gives

$$\frac{(\bar{u}')^2}{1+(\bar{u}')^2} = \alpha^2 s, \tag{3.36}$$

with

$$\alpha = \frac{2\lambda}{Q-2}. \tag{3.37}$$

This in turn implies

$$\bar{u}'(s) = \pm\sqrt{\frac{s}{\beta^2 - s}}, \quad \text{where } \beta = \frac{1}{\alpha}. \tag{3.38}$$

At this point, an observation must be made. We cannot choose the sign in (3.38) arbitrarily. In fact, equation (3.35) implies that $\bar{u}'$ does not change sign on the interval $[0, R^2/4]$, and one has $\bar{u}' > 0$, or $\bar{u}' < 0$, according to whether $\alpha > 0$ or $\alpha < 0$. On the other hand, if the '+' branch of the square root were chosen in (3.38), then $\bar{u}$ would be increasing and, since $\bar{u} \geq 0$ on $(0, R^2/4)$, it would be thus impossible to fulfill the boundary condition $\bar{u}(R^2/4) = 0$.

These considerations show that it must be $\bar{u}' < 0$ on $(0, R^2/4)$. We then have to take $\alpha < 0$ (hence $\beta < 0$ as well), and therefore $\lambda < 0$. Equation (3.38) thus becomes

$$\bar{u}'(s) = -\sqrt{\frac{s}{\beta^2 - s}}, \quad 0 \leq s < \frac{R^2}{4}. \tag{3.39}$$

We stress that, thanks to the assumption (3.29), and to (3.37), we have that if

$$0 \leq s < \frac{R^2}{4} = \frac{(Q-2)^2}{4\lambda^2} = \frac{1}{\alpha^2} = \beta^2,$$

then the function $\bar{u}'$ given by (3.39) is smooth on the interval $[0, R^2/4)$, and satisfies

$$\lim_{s\to(R^2/4)^-} \bar{u}'(s) = -\infty.$$

Integrating (3.39) by standard calculus techniques we find for $s \in [0, R^2/4]$

$$\text{(3.40)} \quad \bar{u}(s) = \sqrt{s(\beta^2 - s)} - \beta^2 \tan^{-1}\left(\sqrt{\frac{s}{\beta^2 - s}}\right) + C$$

$$= C + \sqrt{s(\beta^2 - s)} + \beta^2 \sin^{-1}\left(\frac{\sqrt{s}}{\beta}\right).$$

Recalling that $\alpha = \beta^{-1}$, and the equation (3.37), if we impose the condition $\bar{u}(R^2/4) = 0$, we obtain the solution

$$\text{(3.41)} \quad \bar{u}(s) = C_{R,\lambda} + \frac{\sqrt{s}}{2\lambda}\sqrt{(Q-2)^2 - 4\lambda^2 s} + \frac{(Q-2)^2}{4\lambda^2}\sin^{-1}\left(\frac{2\lambda\sqrt{s}}{Q-2}\right),$$

where $C_{R,\lambda}$ is given by (3.31). Setting $u_{R,\lambda}(z) = \bar{u}(|z|^2/4)$, we finally obtain (3.30) from (3.41). We are finally left with proving that such a $u_{R,\lambda}$ belongs to the class $\mathscr{D}$. The membership $u_{R,\lambda} \in \mathscr{D}$ is equivalent to proving that the function $s \to \bar{u}(s^2/4)$ is of class $C^1$ in the open interval $(-R, R)$, and that furthermore $\nabla u_{R,\lambda} \in C^{0,1}(\Omega)$. For the first part, from (3.41) it is clear that we only need to check the continuity of $\bar{u}'$ at $s = 0$. Since the function is even this amounts to proving that $\bar{u}'(s) \to 0$ as $s \to 0$. But this is obvious in view of (3.39). Finally, we have

$$|\nabla u_{R,\lambda}(z) - \nabla u_{R,\lambda}(0)| = \left|\bar{u}'\left(\frac{|z|^2}{4}\right)\right| \le C|z|,$$

which shows that $\nabla u_{R,\lambda} \in C^{0,1}_{\text{loc}}(\Omega)$. □

In the next Proposition 3.15 we complete the analysis of the regularity of the functions $u_{R,\lambda}$. It suffices to consider the upper half of the "normalized" candidate isoperimetric profile $E_o \subset \mathbb{H}^n$, where $\partial E_o$ is the graph of the function $t = u_o(z)$, with $u_o = u_{1,\lambda}$ and $\lambda = -(Q-2)$. The characteristic locus of $E_o$ is given by the two points in $\mathbb{H}^n$

$$\Sigma = \left\{\left(0, 0, \pm\frac{\pi}{8}\right)\right\}.$$

Unlike its Euclidean counterpart, the hypersurface surface $\partial E_o$ is not $C^\infty$ at the characteristic points $\left(0, 0, \pm\frac{\pi}{8}\right)$.

**Proposition 3.15.** *The hypersurface $S_o = \partial E_o \subset \mathbb{H}^n$ is $C^2$, but not $C^3$, near its characteristic locus $\Sigma$. However, $S_o$ is $C^\infty$ (in fact, real-analytic) away from $\Sigma$.*

*Proof.* First, we show that near the characteristic points $\left(0, 0, \pm\frac{\pi}{8}\right)$ the function $u_o(z)$ given by (1.5) is only of class $C^2$, but not of class $C^3$. To see this we let

$$u_1(s) = \frac{\pi}{8} + \frac{s}{4}\sqrt{1-s^2} - \frac{1}{4}\sin^{-1}(s), \quad 0 \le s \le 1,$$

and note that $u_o(z) = u_1(|z|)$ for $0 \le |z| \le 1$. Therefore, the regularity of $u_o$ at $|z| = 0$ is equivalent to verifying up to what order of derivatives $n$ one has

$$\lim_{s\to 0^+} u_+^{(n)}(s) = \lim_{s\to 0^-} u_-^{(n)}(s)$$

where $u_+(s) = u_1(s)$ and $u_-(s) = u_1(-s)$. It is easy to compute

$$-u_-'(s) = u_+'(s) = -\frac{1}{2}\frac{s^2}{\sqrt{1-s^2}}, \quad -u_-''(s) = u_+''(s) = -\frac{1}{2}\frac{s(s^2-2)}{(s^2-1)\sqrt{1-s^2}},$$

$$-u_-^{(3)}(s) = u_+^{(3)}(s) = -\frac{1}{2}\frac{2+s^2}{(s^2-1)^2\sqrt{1-s^2}}.$$

We clearly have

$$\lim_{s\to 0^-} u_-^{(n)} = \lim_{s\to 0^+} u_+^{(n)} \text{ for } n = 0,1,2 \quad \text{whereas } \lim_{s\to 0^-} u_-^{(3)} = 1 \text{ and } \lim_{s\to 0^+} u_+^{(3)} = -1.$$

This shows the function $t = u_o(z)$ is only $C^2$, but not $C^3$, near $z = 0$. Next, we investigate the regularity of $\partial E_o$ near $|z| = 1$, that is, at the points where the upper and lower branches that form $\partial E_o$ meet. To this end, we observe that $\partial E_o$ can also be generated by rotating around the $t$-axis the curve in the $(x_1, t)$-plane whose trace is

$$\{(x_1, t) \mid t^2 = u_1(x_1)^2, 0 \le x_1 \le 1\}.$$

It suffices to show that this curve is smooth ($C^\infty$) across the $x_1$ axis. To this end we compute the derivatives of $u_1$. It is easy to see by induction that for $n \ge 3$

$$(3.42) \quad u_1^{(n)}(x_1) = (-1)^n C_n \frac{P_{n-1}(x_1)}{(x_1^2-1)^{n-1}\sqrt{1-x_1^2}},$$

where $C_n > 0$ is a constant depending only on $n$, and $P_{n-1}(x_1)$ is a polynomial in $x_1$ of degree $n-2$. The $n$-th derivatives of the function $-u_1(x_1)$ clearly takes the same form, but with a negative sign. Letting $s \to 1^-$ in (3.42) we see that

$$\frac{d^n}{dx_1^n}u_1, \frac{d^n}{dx_1^n}(-u_1) \to \pm\infty, \quad \text{(depending on whether } n \text{ is odd or even).}$$

This implies that the curve with equation $t^2 = u_1(x_1)^2$ is smooth across the $x_1$-axis. □

From Theorem 3.14 and Proposition 3.15, we immediately obtain the following interesting consequence.

**Theorem 3.16.** *Let* $V > 0$ *be given, and define* $R = R(V) > 0$ *by the formula*

$$(3.43)\quad R = \left( \frac{(Q-2)\Gamma\left(\frac{Q+2}{2}\right)\Gamma\left(\frac{Q-1}{2}\right)}{\pi^{(Q-1)/2}\Gamma\left(\frac{Q+1}{2}\right)} \right)^{1/Q} V^{1/Q}.$$

*With such choice of* $R$, *let* $\Omega = B(0,R) = \{z \in \mathbb{R}^{2n} \mid |z| < R\}$. *If we take*

$$(3.44)\quad \lambda = -\frac{Q-2}{R},$$

*then the equation* (3.28), *with the Dirichlet condition* $u = 0$ *on* $\partial\Omega$, *admits the spherically symmetric solution* $u_R \in \mathscr{D} \cap C^2(\Omega)$, *where*

$$(3.45)\quad u_R(z) = \frac{\pi R^2}{8} + \frac{|z|}{4}\sqrt{R^2 - |z|^2} - \frac{R^2}{4}\sin^{-1}\left(\frac{|z|}{R}\right).$$

*Furthermore, such* $u_R$ *satisfies the condition*

$$(3.46)\quad \int_\Omega u_R(z)\,dz = \frac{V}{2}.$$

*Proof.* The first part of the theorem, up to formula (3.45), is a direct consequence of Theorem 3.14. We only need to prove (3.46). In this respect, keeping in mind the definition (3.43), it will suffice to prove that

$$(3.47)\quad \int_\Omega u_R(z)\,dz = \frac{\pi^{(Q-1)/2}\Gamma\left(\frac{Q+1}{2}\right)}{2(Q-2)\Gamma\left(\frac{Q+2}{2}\right)\Gamma\left(\frac{Q-1}{2}\right)} R^Q.$$

To establish (3.47) we note explicitly that $u_R(z) = \bar{u}(|z|^2/4)$, where

$$(3.48)\quad \bar{u}(s) = \frac{\pi R^2}{8} + \frac{1}{2}\sqrt{s(R^2 - 4s)} - \frac{R^2}{4}\sin^{-1}\left(\frac{2\sqrt{s}}{R}\right).$$

One has therefore

$$\int_\Omega u_R(z)\,dz = \int_{|z|<R} \bar{u}(|z|^2/4)\,dz = \sigma_{2n-1}\int_0^R \bar{u}(r^2/4) r^{2n}\frac{dr}{r}$$

$$= 2^{2n-1}\sigma_{2n-1}\int_0^{R^2/4} \bar{u}(s) s^{(Q-4)/2}\,ds.$$

Integrating by parts the last integral, and using the fact that $\bar{u}(R^2/4) = 0$, that $\bar{u}$ is smooth at 0, and (3.39) (in which now $\beta^2 = \frac{R^2}{4}$), we obtain

$$\int_\Omega u_R(z)\,dz = \frac{2^{2n}\sigma_{2n-1}}{Q-2}\int_0^{R^2/4} \frac{s^{(Q-1)/2}}{\sqrt{\frac{R^2}{4}-s}}\,ds \tag{3.49}$$

$$= \frac{2^{2n+1}\sigma_{2n-1}}{Q-2}\int_0^{R^2/4} s^{(Q-2)/2}\sqrt{\frac{s}{R^2-4s}}\,ds.$$

With the substitution

$$t^2 = \frac{R^2-4s}{s}, \quad ds = \frac{-2R^2 t}{(4+t^2)^2}\,dt,$$

the integral (3.49) becomes

$$\int_\Omega u_R(z)\,dz = \frac{2^Q\sigma_{2n-1}R^Q}{Q-2}\int_0^\infty \frac{1}{(4+t^2)^{(Q+2)/2}}\,dt$$

$$= \frac{\sigma_{2n-1}R^Q}{4(Q-2)}\int_{\mathbb{R}} \frac{1}{(1+t^2)^{(Q+2)/2}}\,dt.$$

Now we use the formula

$$\int_{\mathbb{R}} \frac{dt}{(1+t^2)^a} = \pi^{1/2}\frac{\Gamma\left(a-\frac{1}{2}\right)}{\Gamma(a)},$$

valid for any $a > 1/2$. We thus obtain

$$\int_\Omega u_R(z)\,dz = \frac{\sigma_{2n-1}\pi^{1/2}\Gamma\left(\frac{Q+1}{2}\right)}{4(Q-2)\Gamma\left(\frac{Q+2}{2}\right)}R^Q$$

where $\sigma_{2n-1}$ is the measure of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^{2n}$. Finally, using in the latter equality the fact that

$$\sigma_{2n-1} = \frac{2\pi^n}{\Gamma(n)} = \frac{2\pi^{(Q-2)/2}}{\Gamma\left(\frac{Q-2}{2}\right)},$$

we obtain (3.47). □

With the problem (3.24) in mind, it is convenient to rephrase part of the conclusion of Theorem 3.16 in the following way.

**Corollary 3.17.** *Let $V > 0$ be given, and for any $R > 0$ consider the function $u_R$ defined by* (3.45). *There exists $R = R(V) > 0$ (the choice of $R$ is determined by* (3.43)*) such that with $\Omega = B(0, R)$ one has with $u_o = u_R$*

$$\mathscr{G}[u_o] = \int_\Omega u_o(z)\, dz = \frac{V}{2}.$$

Although the following lemma will not be used until we come to the proof of Theorem 1.4, it is nonetheless appropriate to present it at this moment, since it complements Corollary 3.17.

**Lemma 3.18.** *Let $u_o(z)$ be given by* (3.45), *and $\Omega = \operatorname{supp}(u_o) = B(0, R)$, then*

$$\mathscr{F}[u_o] = \int_\Omega \sqrt{|\nabla_z u_o|^2 + \frac{|z|^2}{4} + \langle \nabla_z u_o, z^\perp \rangle}\, dz = \frac{\pi^{(Q-1)/2}\Gamma\left(\frac{Q-1}{2}\right)}{2\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{Q-1}{2}\right)} R^{Q-1}. \tag{3.50}$$

*Proof.* We recall that $u_o(z) = \bar{u}(|z|^2/4)$ where $\bar{u}$ is given by (3.48). One has

$$\nabla_z u_o(z) = \frac{1}{2}\bar{u}'(|z|^2/4) z,$$

and therefore

$$|\nabla_z u_o(z)|^2 + \frac{|z|^2}{4} + \langle \nabla_z u_o(z), z^\perp \rangle = \frac{|z|^2}{4}\left(1 + \bar{u}'\left(\frac{|z|^2}{4}\right)^2\right).$$

We thus obtain

$$\int_\Omega \sqrt{|\nabla_z u_o|^2 + \frac{|z|^2}{4} + \langle \nabla_z u_o, z^\perp \rangle}\, dz = \frac{1}{2}\int_{|z|<R} |z| \sqrt{1 + \bar{u}'(|z|^2/4)^2}\, dz$$

$$= \frac{\sigma_{2n-1}}{2}\int_0^R \sqrt{1 + \bar{u}'(r^2/4)^2} r^{2n+1}\frac{dr}{r} = 2^{2n-1}\sigma_{2n-1}\int_0^{R^2/4} \sqrt{1 + \bar{u}'(s)^2} s^{(Q-3)/2}\, ds.$$

Formula (3.39), in which $\beta = -R/2$, gives

$$\sqrt{1 + \bar{u}'(s)^2} = \frac{R}{\sqrt{R^2 - 4s}}.$$

Inserting this equation in the above integral we obtain

$$\int_\Omega \sqrt{|\nabla_z u_o|^2 + \frac{|z|^2}{4} + \langle \nabla_z u_o, z^\perp \rangle}\, dz = 2^{2n-1}\sigma_{2n-1} R \int_0^{R^2/4} s^{(Q-4)/2}\sqrt{\frac{s}{R^2 - 4s}}\, ds.$$

We notice that the last integral above is similar to the one in (3.49). Proceeding as in the last part of the proof of Theorem 3.16, we finally reach the conclusion. □

At this point, recalling that (3.28) represents the Euler-Lagrange equation of the unconstrained functional (3.27), and keeping (3.26) in mind, if we combine Theorem 3.16 with Corollary 3.17, and take Remark 3.12 into account, we obtain the following result.

**Theorem 3.19.** *Let $\mathscr{F}$ and $\mathscr{G}$ be the functionals*

$$\mathscr{F}[u] = \int_{\mathrm{supp}(u)} f(z, \nabla_z u(z))\, dz, \quad \mathscr{G}[u] = \int_{\mathrm{supp}(u)} g(u)\, dz,$$

*where $f$ and $g$ are defined in* (3.26). *Given $V > 0$, there exists $R = R(V) > 0$ (see* (3.43)*) such that the function $u_o = u_R$ in* (3.45) *is a critical point in $\mathscr{D}$ of the functional $\mathscr{F}[u]$ subject to the constraint $\mathscr{G}[u] = \frac{V}{2}$. This follows from the fact that $u_o$ is a critical point in $\mathscr{D}$ of the unconstrained functional $\mathscr{F}[u]$ in* (3.27).

Our next objective is to prove that the function $u_o$ in (3.45) is: 1) A global minimizer of the variational problem (3.24); 2) The unique global minimizer. We will need some basic facts from Calculus of Variations, which we now recall.

**Definition 3.20.** Let $\mathscr{V}$ be a normed vector space, and $\mathscr{D} \subset \mathscr{V}$. Given a functional $\mathscr{F} : \mathscr{D} \to \mathbb{R}$, $u \in \mathscr{D}$, and if $\phi$ is $\mathscr{D}$-admissible at $u$, one calls

$$\delta\mathscr{F}(u; \phi) \overset{def}{=} \lim_{\varepsilon \to 0} \frac{\mathscr{F}[u + \varepsilon\phi] - \mathscr{F}[u]}{\varepsilon}$$

the *Gâteaux derivative of $\mathscr{F}$* at $u$ in the direction $\phi$ if the limit exists.

**Definition 3.21.** Let $\mathscr{V}$ be a normed vector space, and $\mathscr{D} \subset \mathscr{V}$. Consider a functional $\mathscr{F} : \mathscr{D} \to \overline{\mathbb{R}}$. $\mathscr{F}$ is said to be *convex over $\mathscr{D}$* if for every $u \in \mathscr{D}$, and every $\phi \in \mathscr{V}$ such that $\phi$ is $\mathscr{D}$-admissible at $u$, and $u + \phi \in \mathscr{D}$, one has

$$\mathscr{F}[u + \phi] - \mathscr{F}[u] \geq \delta\mathscr{F}(u; \phi),$$

whenever the right-hand side is defined. We say that $\mathscr{F}$ is *strictly convex* if strict inequality holds in the above inequality except when $\phi \equiv 0$.

We have the following

**Theorem 3.22.** *Suppose $\mathscr{F}$ is convex and proper over a non-empty convex subset $\mathscr{D}^* \subset \mathscr{V}$ (i.e., $\mathscr{F} \not\equiv \infty$ over $\mathscr{D}^*$), and suppose that $u_o \in \mathscr{D}^*$ is such that $\delta\mathscr{F}(u_o; \phi) = 0$ for all $\phi$ which are $\mathscr{D}^*$-admissible at $u_o$ (that is, $u_o$ is a critical point of the functional*

*$\mathscr{F}$), then $\mathscr{F}$ has a global minimum in $u_o$. If moreover $\mathscr{F}$ is strictly convex at $u_o$, then $u_o$ is the unique element in $\mathscr{D}^*$ satisfying*

$$\mathscr{F}[u_o] = \inf\{\mathscr{F}[v] \mid v \in \mathscr{D}^*\}.$$

*Proof.* Let $u \in \mathscr{D}^*$, and $u \neq u_o$, then the convexity of $\mathscr{D}^*$ implies that $\phi = u - u_o$ is $\mathscr{D}^*$-admissible at $u_o$. From Definition 3.21 we immediately infer that

$$\mathscr{F}[u] - \mathscr{F}[u_o] = \mathscr{F}[u_o + \phi] - \mathscr{F}[u_o] \geq \delta\mathscr{F}(u_o; \phi) = 0.$$

This shows that $\mathscr{F}$ has a local minimum in $u_o$. When $\mathscr{F}$ is strictly convex at $u_o$ we obtain $\mathscr{F}[u_o + \phi] > \mathscr{F}[u_o]$, for every $\phi \in \mathscr{V}$ such that $\phi$ is $\mathscr{D}^*$-admissible at $u_o$. If $\bar{u}_o \in \mathscr{D}^*$ is another global minimizer of $\mathscr{F}$, then taking $\phi = u_o - \bar{u}_o$, we see that $\mathscr{F}[u_o] > \mathscr{F}[\bar{u}_o]$. Reversing the roles of $u_o$ and $\bar{u}_o$ we find $\mathscr{F}[u_o] = \mathscr{F}[\bar{u}_o]$. From the strict convexity of $\mathscr{F}$ at $u_o$ we conclude that it must be $u_o = \bar{u}_o$. □

Our next goal is to adapt the above results to the problem (3.24). Given $V > 0$ we consider the number $R = R(V) > 0$ defined in (3.43), and consider the fixed ball $B(0, R)$. We consider the normed vector space $\mathscr{V}(R) = \{u \in C(\bar{B}(0, R)) \mid u = 0$ on $\partial B(0, R)\}$. Let

$$\mathscr{D}(R) = \{u \in \mathscr{V}(R) \mid u \geq 0, u \in C^2(B(0, R)) \cap W^{1,1}(B(0, R)),\; \bar{B}(0, R) = \bigcap\{B(0, R + \rho) \mid \mathrm{supp}(u) \subset B(0, R + \rho)\}\}. \tag{3.51}$$

We notice that $\mathscr{D}(R)$ is a non-empty convex subset of $\mathscr{V}(R)$, and that for every $u \in \mathscr{D}(R)$ one has $u = 0$ on $\partial B(0, R)$. Let $h = h(z, u, p)$ be the function in (3.26) and consider the functional (3.27). Given $u \in \mathscr{D}(R)$ and $\phi$ which is $\mathscr{D}(R)$-admissible at $u$, in view of Theorem 3.9, we see that $\mathscr{F}$ is Gâteaux differentiable at $u$ in the direction of $\phi$, and

$$\begin{aligned} \delta\mathscr{F}(u; \phi) &= \int_{B(0,R)} \{h_u(z, u(z), \nabla u(z))\phi(z) + \langle \nabla_p h(z, u(z), \nabla u(z)), \nabla\phi(z)\rangle\}\, dz \\ &= \int_{B(0,R)} \left\{\frac{\langle \nabla_z u + z^\perp/2, \nabla_z \phi\rangle}{|\nabla_z u + z^\perp/2|} + \lambda\phi\right\} dz, \end{aligned} \tag{3.52}$$

where in the above $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^{2n}$. One has the following well-known sufficient condition for the convexity (strict convexity) of $\mathscr{F}$.

**Proposition 3.23.** *If for a.e. $z \in B(0, R)$, for all $u \in \mathscr{D}(R)$ and $p = \nabla u$, the function $h$ in the definition of $\mathscr{F}$ satisfies for every $\phi$ which is $\mathscr{D}(R)$-admissible at $u$, and every $q = \nabla\phi$,* (3.53) $$h(z, u+v, p+\phi) - h(z,u,p) \geq h_u(z,u,p)\phi + \langle \nabla_p h(z,u,p), q\rangle$$

*then $\mathscr{F}$ is convex on $\mathscr{D}(R)$. If, instead, the strict inequality holds unless $v = 0$ and $q = 0$, then $\mathscr{F}$ is strictly convex.*

*Proof.* Let $u \in \mathscr{D}(R)$, and let $\phi$ be $\mathscr{D}(R)$-admissible at $u$. Using (3.52), we obtain

$$\begin{aligned}
&\mathscr{F}[u+\phi] - \mathscr{F}[u] \\
&\quad = \int_{B(0,R)} \{h(z, u(z) + \phi(z), \nabla u(z) + \nabla\phi(z)) - h(z, u(z), \nabla u(z))\}\, dz \\
&\quad \geq \int_{B(0,R)} \{h_u(z, u(z), \nabla u(z))\phi(z) + \langle \nabla_p h(z, u(z), \nabla u(z)), \nabla\phi(z)\rangle\}\, dz \\
&\quad = \delta\mathscr{F}(u;\phi).
\end{aligned}$$

Appealing to Definition 3.21 the conclusion follows. □

Our next goal is to prove that the unconstrained functional $\mathscr{F}$ in (3.27) is convex on the convex set $\mathscr{D}(R)$. Since each one of them has an independent interest, we will provide two different proofs of this fact. The former is based on the following linear algebra lemma, which is probably well known, and whose proof we have provided for the reader's convenience.

**Lemma 3.24.** *Let $\mathbf{A} = [A_{ij}]$ be an $m \times m$ matrix with entries given by*

$$A_{ij} = \delta_{ij} - \frac{a_i a_j}{D} \quad \text{where } D = \sum_{i=1}^{m} a_i^2 \neq 0,$$

*then $\mathbf{A}$ has $\lambda = 0$ as an eigenvalue of multiplicity one, and $\lambda = 1$ as an eigenvalue of multiplicity $m - 1$.*

*Proof.* First, consider the matrix $\mathbf{I} - \mathbf{A}$, which takes the form

$$\frac{1}{D}\begin{pmatrix} a_1a_1 & a_1a_2 & a_1a_3 & \cdots & a_1a_m \\ a_2a_1 & a_2a_2 & a_2a_3 & \cdots & a_2a_m \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ a_ma_1 & a_ma_2 & a_ma_3 & \cdots & a_ma_m \end{pmatrix}.$$

It is easy to see that an equivalent row-echelon form of the matrix has the last $m-1$ rows containing all zeros, thus $\mathbf{I} - \mathbf{A}$ is a matrix of rank one. From the rank-nullity theorem we conclude that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$ of multiplicity $m - 1$. We are thus left with showing the $\lambda = 0$ is a simple eigenvalue. For this we show that $\det(\mathbf{A}) = 0$. Observe that $\det(\mathbf{A}) = D^{-m}\det(\mathbf{B})$, where

$$\mathbf{B} = \begin{pmatrix} D - a_1^2 & -a_1a_2 & -a_1a_3 & \cdots & -a_1a_m \\ -a_1a_2 & D - a_2^2 & -a_2a_3 & \cdots & -a_2a_m \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ -a_ma_1 & -a_ma_2 & -a_ma_3 & \cdots & D - a_m^2 \end{pmatrix}.$$

To continue the computation of $\det(\mathbf{B})$, we replace rows $R_j$ by $a_1R_j - a_jR_1$ for $j = 2, \dots, m$ and observe that $a_1R_j - a_jR_1$ takes the form

$$a_1R_j - a_jR_1 = [-a_jD\ 0\ \cdots 0\ a_1D\ 0 \cdots 0].$$

We then have

$$\det(\mathbf{B}) = \det(\mathbf{C}),$$

where

$$\mathbf{C} = \begin{pmatrix} D - a_1^2 & -a_1a_2 & \cdots & \cdots & \cdots & -a_1a_m \\ -a_2D & a_1D & 0 & \cdots & \cdots & 0 \\ -a_3D & 0 & a_1D & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ -a_mD & 0 & 0 & 0 & \cdots & a_1D \end{pmatrix}.$$

To compute $\det(\mathbf{C})$ we take advantage of the special structure of the matrix, and consider

$$\mathbf{C}\mathbf{C}^T = D^2 \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_m \\ -a_2 & a_2^2 & a_2a_3 & \cdots & a_2a_m \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ -a_m & a_ma_2 & a_ma_3 & \cdots & a_m^2 \end{pmatrix}.$$

We note that if either $a_2 = 0$ or $a_3 = 0$, then the matrix $\mathbf{C}\mathbf{C}^T$ has a column of zeros, and therefore its determinant vanishes. Suppose then that $a_2, a_3 \neq 0$. Replacing rows $R_2$ and $R_3$ by $R_2 + a_2R_1$ and $R_3 + a_3R_1$ respectively, we see that the new rows two and three have first entries given by $-a_2 - a_1a_2$ and $-a_3 - a_1a_2$, whereas all the remaining entries vanish. Either one of these rows is already a zero row or else, using one to eliminate the other, we obtain a row of zeros, and therefore we conclude that $\det(\mathbf{C}\mathbf{C}^T) = 0$. Hence, $\det(\mathbf{A}) = D^{-m}\det(\mathbf{B}) = D^{-m}\det(\mathbf{C}) = 0$. This completes the proof of the lemma. □

**Proposition 3.25.** *Given* $V > 0$, *let* $R = R(V) > 0$ *be as in* (3.43). *The functional* $\mathscr{F}$ *in* (3.27) *is convex on* $\mathscr{D}(R)$. *As a consequence, the function* $u_R$ *in* (3.45) *is a global minimizer of* $\mathscr{F}$ *on* $\mathscr{D}(R)$.

*Proof.* Considering the integrand $h(z,u,p) = \left|p + \frac{z^\perp}{2}\right| + \lambda u$ in the functional $\mathscr{F}$ in (3.27), we have

$$\begin{cases} h_{p_i} = \left(p_i + \frac{z_i^\perp}{2}\right) \Big/ \left|p + \frac{z^\perp}{2}\right|, \\ h_{p_i p_j} = \frac{1}{\left|p+\frac{z^\perp}{2}\right|}\left\{\delta_{ij} - \frac{\left(p_i+\frac{z_i^\perp}{2}\right)\left(p_j+\frac{z_j^\perp}{2}\right)}{\left|p+\frac{z^\perp}{2}\right|^2}\right\}, \\ h_{u,p_i} = 0, \end{cases}$$

where in the above we have let

$$z_i^\perp = \begin{cases} y_i & \text{if } 1 \le i \le n \\ -x_i & \text{if } n+1 \le i \le 2n \end{cases} \tag{3.54}$$

The hessian of $h$ with respect to the variable $(u,p) \in \mathbb{R} \times \mathbb{R}^{2n}$ now takes the form

$$\nabla^2 h(u,p) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathscr{A} & \\ 0 & & & \end{pmatrix}$$

where, aside from the multiplicative factor $1/|p + z^\perp/2|$, the block $\mathscr{A}$ takes the form of the matrix **A** in Lemma 3.24. We thus conclude that the eigenvalues of $\nabla^2 h(u,p)$ are $\lambda = 0$ (of multiplicity two) and $\lambda = 1/|p + z^\perp/2|$ of multiplicity $2n-1$. Thus, from Theorem 3.9, for a.e. $z \in B(0,R)$, the function $(u,p) \to h(z,u,p)$ is convex. This in turn implies that $\mathscr{F}$ is convex on $\mathscr{D}(R)$. From Theorems 3.19 and 3.22 we conclude that $u_R$ is a global minimizer of $\mathscr{F}$ on $\mathscr{D}(R)$. □

We next prove a slightly stronger result than Proposition 3.25, namely the convexity of the function in $\mathbb{R}^{2n}$ which defines the integrand in $\mathscr{F}$ in (3.27). The proof of this result is based on the following lemma.

**Lemma 3.26.** *Let* $\alpha \in \mathbb{R}^{2n}$ *be fixed, with* $\alpha \neq 0$, *then one has*

$$f(q) \stackrel{def}{=} |\alpha|\,|q|^2 - (|q+\alpha| - |\alpha|)\langle q, \alpha\rangle \ge 0, \quad \textit{for every } q \in \mathbb{R}^{2n}.$$

*Proof.* We observe that $f(0) = f(-\alpha) = 0$, and that $f \in C^\infty(\mathbb{R}^{2n}\setminus\{-\alpha\})$. We want to analyze the possible critical points in $\mathbb{R}^{2n}\setminus\{-\alpha\}$ of the function $f$. It is easier to reduce the problem by introducing spherical coordinates. Let $\alpha = r_0\omega_0$, with $\omega_0 \in \mathbb{S}^{2n-1}$ and $r_0 > 0$, we can consider a system of spherical coordinates in which the "north pole" coincides with $\omega_0$ and the colatitude angle $\theta$ denotes the angle formed by the vector $q \in \mathbb{R}^{2n}\backslash\{-\alpha\}$ with $\omega_0$. In such a system we let $q = r\omega$, with $\omega \in \mathbb{S}^{2n-1}$, and $r = |q|$, so that $\cos\theta = \langle\omega, \omega_0\rangle$. We observe that the function $f$ is constant on every $2n-2$ dimensional sub-sphere $\sin\theta = const$ of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$, and we want to exploit these symmetries of $f$. For $z \in \mathbb{R}^{2n}$ we let $r = r(z) = |z|$, and $\theta = \theta(z) = \cos^{-1}(\langle z/r, \omega_0\rangle)$. Writing $f(z) = f(r(z), \theta(z))$, we are thus led to consider

$$f(r,\theta) = f(q) = r_0 r^2 - (\sqrt{r_0^2 + r^2 + 2r_0 r\cos\theta} - r_0)r_0 r\cos\theta.$$

If we now set $t = r/r_0$, then we can consider the function

$$g(t,\theta) = \frac{1}{r_0^3} f(r_0 t, \theta) = t^2 - (\sqrt{1 + t^2 + 2t\cos\theta} - 1)t\cos\theta,$$

for $(t,\theta) \in Q = [0,\infty) \times [0,\pi]$, with $(t,\theta) \neq (1,\pi)$. When $t = 0$, then $q = 0$ and we have already observed that $f(0) = 0$. When $\theta = 0$, then $q = \rho\alpha$ for some $\rho \geq 0$, one readily recognizes that $f(\rho\alpha) = 0$. Finally, when $t \geq 0$ and $\theta = \pi$ we have $g(t,\pi) = 0$ if $0 \leq t \leq 1$, and $g(t,\pi) > 0$ for $t > 1$. In conclusion, we have $f(q) = 0$ for $q = \rho\alpha$ for some $\rho \geq -1$, whereas we have $f(q) > 0$ for $q = \rho\alpha$ with $\rho < -1$. We now consider the possible critical points of $f$. Using the chain rule we see that

$$\nabla f = \frac{f_r}{r} z - \frac{f_\theta}{r\sin\theta}\left(\omega_0 - \frac{\cos\theta}{r} z\right).$$

Since $\left\langle z, \omega_0 - \frac{\cos\theta}{r} z\right\rangle = 0$, we find

$$|\nabla f|^2 = f_r^2 + \frac{f_\theta^2}{r^2\sin^2\theta}\left|\omega_0 - \frac{\cos\theta}{r} z\right|^2 = f_r^2 + \frac{1}{r^2} f_\theta^2,$$

which allows us to conclude that $\nabla f$ vanishes outside of the set of points $q = \rho\alpha$ with $\rho \geq -1$, if and only if $f_r = f_\theta = 0$ at interior points of $Q$. This is equivalent to studying the interior critical points of the function $g(t,\theta)$ in $Q$. One has

$$\begin{aligned}(3.55)\quad \nabla g(t,\theta) &= \Bigg(2t - (\sqrt{1+t^2+2t\cos\theta} - 1)\cos\theta - \frac{(t+\cos\theta)t\cos\theta}{\sqrt{1+t^2+2t\cos\theta}}, \\ &\qquad (\sqrt{1+t^2+2t\cos\theta} - 1)t\sin\theta + \frac{t^2\sin\theta\cos\theta}{\sqrt{1+t^2+2t\cos\theta}}\Bigg) \\ &= (g_t, g_\theta).\end{aligned}$$

Since now $0 < \theta < \pi$ it is clear that $g_\theta = 0$ if and only if

$$(3.56)\quad (\sqrt{1+t^2+2t\cos\theta}-1) = -\frac{t\cos\theta}{(\sqrt{1+t^2+2t\cos\theta}}.$$

On the other hand, we see that $g_t = 0$ at points where (3.56) holds if and only if

$$g_t = 2t - \frac{t^2\cos\theta}{\sqrt{1+t^2+2t\cos\theta}} = 0,$$

which is equivalent to

$$(3.57)\quad 2 = \frac{t\cos\theta}{\sqrt{1+t^2+2t\cos\theta}}.$$

It is clear that if $\pi/2 < \theta < \pi$, then (3.57) has no solutions. Suppose then that $0 < \theta < \pi/2$. In this range, equation (3.57) is equivalent to

$$4 = \frac{t^2\cos^2\theta}{1+t^2+2t\cos\theta},$$

which is in turn equivalent to

$$(4-\cos^2\theta)t^2 + 8t\cos\theta + 4 = 0.$$

An easy verification which we leave to the reader shows that the latter equation has no solutions $t > 0$ in the range $0 < \theta < \pi/2$. In conclusion, the function $g(t,\theta)$, and therefore has no interior critical points. Therefore, $g(t,\theta) \geq 0$ for every $(t,\theta) \in Q$. This allows to conclude that $f(q) \geq 0$ for all $q \in \mathbb{R}^{2n}$, thus completing the proof of the lemma. □

At this point we observe that Lemma 3.26 provides an alternative proof of Proposition 3.25. It suffices in fact to consider for every $u \in \mathscr{D}(R)$ and every $\phi$ which is $\mathscr{D}(R)$-admissible at $u$, the vectors $\alpha(z) = \nabla u(z) + z^\perp/2$, $q(z) = \nabla\phi(z)$. Let $\mathscr{F}$ be given by (3.27) and recall (3.52). One has,

$$(3.58)\quad \mathscr{F}[u+\phi] - \mathscr{F}[u]$$

$$= \int_{B(0,R)} \{|\nabla_z u + z^\perp/2 + \nabla_z\phi| - |\nabla_z u + z^\perp/2| + \lambda\phi\}\, dz$$

$$= \int_{B(0,R)} \left\{ \frac{2\langle\nabla_z\phi, \nabla_z u + z^\perp/2\rangle + |\nabla_z\phi|^2}{|\nabla_z u + z^\perp/2| + |\nabla_z u + z^\perp/2 + \nabla_z\phi|} + \lambda\phi \right\} dz.$$

From Theorem 3.9 we know that there exists $Z \subset \Omega$, with $|\Omega\backslash Z| = 0$, such that $|\alpha(z)| \neq 0$ for every $z \in Z$. We intend to show that for every $z \in Z$ we have

$$(3.59)\quad \frac{2\langle q,\alpha\rangle + |q|^2}{|\alpha| + |\alpha+q|} \geq \frac{\langle q,\alpha\rangle}{|\alpha|}.$$

This would imply

$$(3.60)\quad \frac{2\langle \nabla_z\phi, \nabla_z u + z^\perp/2\rangle + |\nabla_z\phi|^2}{|\nabla_z u + z^\perp/2| + |\nabla_z u + z^\perp/2 + \nabla_z\phi|} \geq \frac{\langle \nabla_z\phi, \nabla_z u + z^\perp/2\rangle}{|\nabla_z u + z^\perp/2|},$$

which would prove that $\mathscr{F}$ is convex. For every $z \in Z$ the inequality (3.59) is easily seen to be equivalent to

$$(3.61)\quad (|q+\alpha| - |\alpha|)\langle q,\alpha\rangle \leq |q|^2|\alpha|,$$

which is true in view of Lemma 3.26. Finally, we give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We fix $V > 0$ and consider the collection of all sets $E \in \mathscr{E}$ such that $V = |E|$. We want to show that the problem of minimizing $P_H(E;\mathbb{H}^n)$ within this subclass admits a unique solution, and that the latter is given by (3.45), in which the parameter $R = R(V)$ has been chosen as in (3.43). According to condition (i) in the definition of the class $\mathscr{E}$, we have $V/2 = |E \cap \mathbb{H}^n_+|$. Still from assumption (i), and in view of Theorem 3.7, it is enough to minimize $P_H(E;\overline{\mathbb{H}^n_+})$. This is an important point. In fact, Theorem 3.7 states that, if $E$ is an isoperimetric set, i.e., if $E$ minimizes $P_H(\circ;\mathbb{H}^n)$ under the constraint $|E| = V$, then

$$(3.62)\quad P_H(E;\overline{\mathbb{H}^n_+}) = P_H(E;\overline{\mathbb{H}^n_-}).$$

This implies that the minimizer must be sought for within the class of sets $E \in \mathscr{E}$ such that $|E| = V$, and for which (3.62) holds, which is in turn equivalent to proving existence and uniqueness of a global minimizer in the class $\mathscr{D}(R)$ defined by (3.51). The existence of a global minimizer follows from Proposition 3.25, and such global minimizer is provided by the spherically symmetric function $u_R$ in (3.45). We are thus left with proving its uniqueness. The latter will follow if we can prove that for every $\mathscr{D}(R)$-admissible function $\phi$ at $u_R$ the strict inequality

$$\mathscr{F}[u_R + \phi] > \mathscr{F}[u_R]$$

holds, unless $\phi \equiv 0$. This will follow from the strict inequality in (3.60) for every $z \in Z$, with $u$ replaced by the function $u_R$ in (3.45), unless $\phi \equiv 0$ in $B(0,R)$. Such strict inequality is equivalent to proving strict inequality in (3.61) on the set $Z$, with $q(z) = \nabla\phi(z)$ and $\alpha(z) = \nabla u_R(z) + z^\perp/2$. We emphasize here that, in view of (3.45), the vector-valued function $\alpha(z)$ only vanishes at $z = 0$. Keeping in mind that $u_R \in C^2(B(0,R))$, and that, since $\phi$ is $\mathscr{D}(R)$-admissible at $u_R$, we have $\phi \in C^2(B(0,R))$, and $\phi = 0$ on $\partial B(0,R)$, an analysis of the proof of Lemma 3.26, brings to the conclusion that the desired strict inequality holds, unless either $\nabla\phi \equiv 0$, in which case we conclude $\phi \equiv 0$, or there exists a function $\rho \in C^1(B(0,R))$, with $\rho \geq -1$, and such that for every $z \in Z$

$$\nabla\phi(z) = \rho(z)\left(\nabla u_R(z) + \frac{z^\perp}{2}\right). \tag{3.63}$$

We remark explicitly that the possibility $\rho \equiv \mathit{const}$ in (3.63) is forbidden by the fact that the vector field $z \to \nabla u_R(z) + z^\perp/2$ is not conservative in $B(0,R)$. Furthermore, since the functions in both sides of (3.63) are in $C^1(B(0,R))$, the validity of the inequality for every $z \in Z$ is equivalent to its being valid on the whole $B(0,R)$.

We thus want to show that (3.63) cannot occur. To illustrate the idea, we focus on the case $n = 1$ and leave the trivial modifications to the interested reader. We argue by contradiction and suppose that (3.63) hold. This means

$$\phi_x = \rho\left(u_{R,x} + \frac{y}{2}\right), \quad \phi_y = \rho\left(u_{R,y} - \frac{x}{2}\right).$$

Since $\phi \in C^2(B(0,R))$, differentiating the first equation with respect to $y$ and the second with respect to $x$, and keeping in mind that $u_R$ is spherically symmetric (see (3.45)), from the fact that $\phi \in C^2(B(0,R))$, and therefore $\phi_{xy} = \phi_{yx}$, we infer that we must have

$$\left(\frac{x}{2} - \bar{u}'\frac{y}{2}\right)\rho_x + \left(\frac{y}{2} + \bar{u}'\frac{x}{2}\right)\rho_y + \rho = 0, \tag{3.64}$$

where, we recall, $u_R(z) = \bar{u}(|z|^2/4)$, see (3.48). We now fix a point $z_0 \in B(0,R)\backslash\{0\}$, and consider the characteristic curve starting at $z_0 = (x_0, y_0)$, $z(s) = z(s, z_0)$ of the transport equation (3.64). Letting $z(s) = (x(s), y(s))$, we know that such curve satisfies the system

$$\begin{cases} x' = \frac{x}{2} - \bar{u}'\frac{y}{2}, & x(0) = x_0, \\ y' = \frac{y}{2} + \bar{u}'\frac{x}{2}, & y(0) = y_0. \end{cases} \tag{3.65}$$

It is clear that $s \to \rho(z(s))$ satisfies the Cauchy problem

$$\frac{d}{ds}\rho(z(s)) = -\rho(z(s)), \quad \rho(z(0)) = \rho(z_0),$$

and therefore

$$\rho(z(s)) = \rho(z(s, z_0)) = \rho(z_0)e^{-s}. \tag{3.66}$$

Multiplying the first equation in (3.65) by $x$, and the second by $y$, we find

$$\frac{d}{ds}|z(s)|^2 = |z(s)|^2,$$

which gives

$$|z(s)|^2 = |z_0|^2 e^s. \tag{3.67}$$

It is clear that $-\infty < s \le 2\log(R/|z_0|)$. For every $s$ in this range, we obtain from (3.63), (3.66), and from (3.39),

$$\nabla\phi(z(s)) = \frac{\rho(z_0)e^{-s}}{2}\left(-\frac{|z(s)|}{\sqrt{R^2-|z(s)|^2}}z(s) + z(s)^{\perp}\right).$$

Using (3.67), we finally obtain

$$|\nabla\phi(z(s))|^2 = \frac{\rho(z_0)^2 e^{-2s}}{4}|z_0|^2 e^{2s}\left[\frac{|z(s)|^2}{R^2-|z(s)|^2}+1\right].$$

Letting $s \to -\infty$ in the latter equation, we reach the conclusion

$$|\nabla\phi(0)|^2 = \frac{\rho(z_0)^2|z_0|^2}{4},$$

which contradicts the continuity of $|\nabla\phi|$ at $z = 0$, unless $\rho \equiv 0$. But this would contradict our assumptions on $\rho$. We conclude that $u_R$ given by (3.45) is the unique minimizer to the variational problem (3.24) in $\mathscr{D}(R)$. □

**Remark 3.27.** We mention that an alternative proof of the uniqueness of the global minimizer $u_R$ in Theorem 1.1 could be obtained by the interesting comparison Theorem C′ on p. 163 in [CHMY].

**Proposition 3.28.** *Suppose $E \in \mathscr{E}$ is a critical point of the $H$-perimeter subject to the constraint $|E| = const$, then $S = \partial E$ has constant $H$-mean curvature. In particular, the isoperimetric set $E_o$ found in Theorem 1.1 is a set of constant positive $H$-mean curvature $\mathscr{H} = \frac{Q-2}{R}$.*

*Proof.* Let $E \in \mathscr{E}$ be given and let $u$ be the function describing $\partial E$ in $\mathbb{H}^n_+$. To prove that $\partial E$ has constant $H$-mean curvature we could appeal to Remark 3.13. Instead, we proceed directly as follows. We recall that $u(z) = \bar{u}(|z|^2/4)$ for some $C^2$ function $\bar{u}$, and the assumptions that $E$ is a critical point of the $H$-perimeter means that $\bar{u}$ satisfies (3.33). From the discussion in the proof of Theorem 3.14, the left hand side of (3.33) (that is the Euler-Lagrange equation) becomes

$$rF'(r) + (Q-2)F(r)$$

where $F(r)$ is given by (3.34). A simple computation gives

$$F'(r) = \frac{r^2\bar{u}''(r^2/4) - 2\bar{u}'(r^2/4)(1+\bar{u}'(r^2/4)^2)}{2r^2(1+\bar{u}'(r^2/4)^2)^{3/2}},$$

and therefore we have

$$rF'(r) + (Q-2)F(r) = \frac{2(Q-3)\bar{u}'(r^2/4)(1+\bar{u}'(r^2/4)^2) + r^2\bar{u}''(r^2/4)}{2r(1+\bar{u}'(r^2/4)^2)^{3/2}}. \tag{3.68}$$

Rewriting the Euler-Lagrange equation (3.33) for such functions $u$ (or $\bar{u}$) we have

$$\frac{2(Q-3)\bar{u}'(r^2/4)(1+\bar{u}'(r^2/4)^2) + r^2\bar{u}''(r^2/4)}{2r(1+\bar{u}'(r^2/4)^2)^{3/2}} = \lambda \tag{3.69}$$

where $\lambda$ is of course a constant. We make a change of notation by letting $s = r^2/4$ in (3.69), we found

$$\frac{(Q-3)\bar{u}'(s)(1+\bar{u}'(s)^2) + 2s\bar{u}''(s)}{2\sqrt{s}(1+\bar{u}'(s)^2)^{3/2}} = \lambda. \tag{3.70}$$

Comparing (3.70) with (3.14), we infer that the $H$-mean curvature of such surfaces is

$$\mathscr{H} = -\frac{(Q-3)\bar{u}'(s)(1+\bar{u}'(s)^2) + 2s\bar{u}''(s)}{2\sqrt{s}(1+\bar{u}'(s)^2)^{3/2}} = -\lambda.$$

If the set $E_o$ is described by $u_R(z)$, where $u_R(z)$ is given by (3.45), then from (3.44) in Theorem 3.16 we conclude that the $H$-mean curvature of $E_o$ is given by

$$\mathscr{H} = \frac{Q-2}{R}.$$ □

This completes proof of Theorem 1.1.

*Proof of Theorem 1.4.* We have already established the restricted isoperimetric inequality. Furthermore, the invariance of the isoperimetric quotient with respect to the group translations and dilations is a consequence of Propositions 2.11 and 2.12. We are left with the computation of the constant $C_Q$. To this end, we use the set $E_R$ described by $u_o$. We note that the integrals (3.47) and (3.50) give $|E_R|/2$ and $P_H(E;\mathbb{H}^n_+)$ respectively, and therefore after some elementary simplifications we obtain

$$C_Q = \frac{|E_R|^{(Q-1)/Q}}{P_H(E_R;\mathbb{H}^n)} = \frac{(Q-1)\Gamma\left(\frac{Q}{2}\right)^{2/Q}}{Q^{(Q-1)/Q}(Q-2)\Gamma\left(\frac{Q+1}{2}\right)^{1/Q}\pi^{(Q-1)/2Q}}.$$

This completes the proof. □

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Received 16 January 2006; revised 5 June 2006

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