SIAM J. CONTROL AND OPTIMIZATION
Vol. 34, No. 1, pp. 199–225, January 1996

DYNAMIC PROGRAMMING FOR NONLINEAR SYSTEMS DRIVEN BY ORDINARY AND IMPULSIVE CONTROLS*

MONICA MOTTA† AND FRANCO RAMPAZZO†

Abstract. A dynamic programming approach is considered for a class of minimum problems with impulses. The minimization domain consists of trajectories satisfying an ordinary differential equation whose right-hand side depends not only on a measurable control v but also on a second control u and on its time derivative $\dot{u}$. For this reason, the control u and the differential equation are called *impulsive*.

The value function of the considered minimum problem turns out to depend on the time, the state, the u variable, and the variation allowed to the impulsive control. It is shown that the value function satisfies, in a generalized sense, a dynamic programming equation (DPE), which is obtained from a dynamic programming principle involving space–time trajectories. Moreover the value function is the unique map-solving equation (DPE) satisfying either an inequality condition or a supersolution condition at each point of the boundary. Incidentally this extends a result by Barron, Jensen, and Menaldi [Nonlinear Anal., 21 (1993), pp. 241–268], where the impulsive control is scalar monotone and the corresponding vector field is independent of the state variable. Next, a maximum principle is proved, and the well-known relationship between adjoint variables and value function is suitably extended to impulsive control systems. A fully elaborated example concludes the paper.

Key words. impulsive control, minimum problem, dynamic programming

AMS subject classifications. 34A37, 49N25, 49L20, 49L25

1. Introduction.

The optimal control problem. This paper concerns the dynamic programming approach to minimum problems involving impulsive control systems of the form

$$\text{(E)}\qquad \begin{aligned} \dot{x} &= g_0(t,x,u,v) + \sum_{i=1}^{m} g_i(t,x,u,v)\dot{u}_i, \\ x(\bar{t}) &= \bar{x}, \end{aligned}$$

where the state x belongs to $\mathbb{R}^n$ and the controls u and v map a time interval $[\bar{t}, T]$ into a closed subset $U \subset \mathbb{R}^m$ and a compact subset $V \subset \mathbb{R}^q$, respectively. Moreover u is subject to the directional constraint $\dot{u} \in C$, where $C \subset \mathbb{R}^m$ is a closed cone. Optimum problems involving a dynamics of the form (E) arise in applications to rational mechanics [13]–[15], [35], economics [17], space navigation [25], [29], [33], and advertising strategy [20], [39].

Because of the presence of the derivative $\dot{u}$ on the right-hand side of (E) the state can jump in consequence of a discontinuity of the control u. However, the notion of solution to (E) is provided by the Carathéodory theory of ordinary differential equations only if the control u is absolutely continuous. Moreover, it is known—see, e.g., [9]–[12], [19], [21], [23], [28], [30], [32], [37], [41]—that whenever the fields $g_1, \ldots, g_m$ depend on x, u, and v, a mere measure-theoretic extension of this notion to the case of a discontinuous u does not agree with elementary requirements of continuity

* Received by the editors September 2, 1993; accepted for publication (in revised form) August 22, 1994.

† Dipartimento di Matematica Pura e Applicata, Università di Padova, via Belzoni 7, I-35131 Padova, Italy (motta@pdmath1.math.unipd.it), (rampazzo@pdmath1.math.unipd.it).

of the input-output map. In order to overcome this difficulty, in [10], [32], [34] one extends system (E) to the space-time system

$$\text{(STE)} \qquad x' = g_0(t,x,u,v)t' + \sum_{i=1}^{m} g_i(t,x,u,v)u_i',$$

where the *controls* $t(s), u(s)$ are Lipschitz continuous and the superscript denotes differentiation with respect to the pseudo-time parameter $s \in [0,1]$. In this space-time setting a discontinuous control $u(t)$ is regarded as the space projection of a space-time control $t(s), u(s)$ whose first component $t(s)$ is allowed to be nondecreasing. We just recall—see, e.g., [10], [30], [32], [37]—that, because of the noncommutativity of the vector fields $g_1, \dots, g_m$, the evolution of x depends on the particular space-time control $t(s), u(s)$ which *completes* the graph of $u(t)$. Incidentally we remark that in the standard impulse control theory there is no need of considering space-time controls. Indeed, in that case the fields $g_1, \dots, g_m$ $(m = n)$ coincide with the canonical basis; in particular they commute. As a consequence each completion of a control u produces the same trajectory which in turn coincides with the unique trajectory resulting from the measure-theoretic approach; see, e.g., [3].

As a prototype of a minimum problem initially formulated for the original system (E) we consider an unconstrained Mayer problem with finite horizon and a bound on the total variation of u.

More precisely, let $\Phi : \mathbb{R}^n \times U \to \mathbb{R}$ be a continuous map, C be a closed cone of $\mathbb{R}^m$, and $K > 0$ be an upper bound for the total variation of the control u. For every $(\bar t, \bar x, \bar u, \bar k) \in [0,T) \times \mathbb{R}^n \times U \times [0,K]$, we consider the following problem:

$$(\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}) \qquad \text{minimize}\,\{\Phi(x(T), u(T))\}$$

over all end points $(x(T), u(T))$ of (E) corresponding to control policies $(u(\cdot), v(\cdot))$, where $v : [\bar t, T] \to V$ is a Borel-measurable map and $u : [\bar t, T] \to U$ is an absolutely continuous map which satisfies

$$u(\bar t) = \bar u, \quad V_{\bar t}^T(u) \le K - \bar k, \quad \text{and } \dot u(t) \in C \text{ for a.e. } t \in [\bar t, T].$$

($V_{\bar t}^T(u)$ denotes the total variation of $u(\cdot)$ on the interval $[\bar t, T]$.) Since the unbounded control $\dot u$ appears linearly on the right-hand side of (E), problem $\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}$ does not display anyone of the standard coercivity assumptions which guarantee the existence of an optimal control. This justifies the introduction of the extended system (STE) and of the corresponding space-time reformulation of problem $\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}$. Actually this extension is *proper*, i.e., the infimum of the original problem turns out to coincide with the infimum of the extended problem. Hence the value functions determined by the two problems coincide. Moreover the set of original controls is dense in the set of space-time controls, and under some further assumptions, there exists an optimal control $(t(s), u(s), v(s))$ for the extended problem; see [32].

The dynamic programming approach. We call *value function* the map $\mathcal{V} : [0,T) \times \mathbb{R}^n \times U \times [0,K] \to \mathbb{R}$ which associates the infimum of problem $\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}$ to every $(\bar t, \bar x, \bar u, \bar k)$. Actually $\mathcal{V}$ can be identified with the value function corresponding to the extended problem, for the two maps turn out to coincide on $[0,T) \times \mathbb{R}^n \times U \times [0,K]$. Moreover, in the extended setting $\mathcal{V}$ can be defined also at $\bar t = T$.

In the particular case when the control u is a scalar nondecreasing map (i.e., $m = 1$, $\dot u \in C \doteq \mathbb{R}^+$, and $u \equiv k \in [0,K]$) and the vector field g_1 *does not depend on x and u*, the dynamic programming approach has been already pursued by E. N. Barron,

R. Jensen, and J. L. Menaldi [8]. Their main result consisted in proving that the value function $\mathcal{V}$ is the unique continuous map which satisfies (in the viscosity sense) a certain Hamilton–Jacobi–Bellman equation together with the following, quite natural, Dirichlet conditions:

$(BC)_1$ $\mathcal{V}$ coincides with the value function of the corresponding nonimpulsive problem ($\dot u = 0$) on the strip $[0,T] \times \mathbb{R}^n \times \{K\}$ (where all the available variation of u has run out);

$(BC)_2$ $\mathcal{V}$ coincides with the value function of the corresponding purely impulsive problem ($g_0 = 0$) on the region $\{T\} \times \mathbb{R}^n \times [0,K]$(where no more time is available).

Moreover, Barron, Jensen, and Menaldi left the following questions as open problems:

a) *Can the well-known relationship between the adjoint variables of the maximum principle and the value function be extended in some way to impulsive problems?*

b) *Can we state a rigorous result (i.e., a verification theorem) which relates the dynamic programming equation with the problem of testing the optimality of a given control?*

c) *What can be said when g_1 depends also on x and u?*

This paper is also a trial to give an answer to the above questions, not only in the scalar control case but also in the general situation where u is vector valued. More precisely, we begin by proving that, under suitable assumptions on the set U and the cone C, the value function $\mathcal{V}$ is continuous on $[0,T] \times \mathbb{R}^n \times U \times [0,K]$. Next, via a dynamic programming principle involving space-time trajectories, we prove that the value function $\mathcal{V}$ is a viscosity solution on $[0,T[\times \mathbb{R}^n \times \mathring{U} \times [0,K[$ of the dynamic programming equation

$$\text{(DPE)} \qquad -H\left(t,x,u,\frac{\partial \mathcal{V}}{\partial t},\frac{\partial \mathcal{V}}{\partial x},\frac{\partial \mathcal{V}}{\partial u},\frac{\partial \mathcal{V}}{\partial k}\right) = 0,$$

where, for every $(p_t, p_x, p_u, p_k) \in \mathbb{R}^{1+n+m+1}$, the Hamiltonian function H is defined by

$$H(t,x,u,p_t,p_x,p_u,p_k) \doteq \min\left\{ \begin{array}{c} (p_t + p_x \cdot g_0(t,x,u,v))w_0 + \sum_{i=1}^{m}(p_x \cdot g_i(t,x,u,k) + p_{u_i})w_i + p_k|w|, \\ |(w_0,\dots,w_m)| = 1,\ w_0 \ge 0,\ w = (w_1,\dots,w_m) \in C,\ v \in V \end{array}\right\}.$$

Furthermore, $\mathcal{V}$ turns out to be the unique solution of (DPE) satisfying the following boundary conditions:

$(BC)_1'$ $\mathcal{V}$ is a (viscosity) *supersolution* of (DPE) at all points of $[0,T[\times\mathbb{R}^n \times \partial U \times [0,K[\cup [0,T[\times \mathbb{R}^n \times U \times \{K\}$;

$(BC)_2'$ at each boundary point $(T,x,u,k)\mathcal{V} \le \Phi$ either $\mathcal{V}$ is a supersolution of (DPE) or it satisfies the relation $\mathcal{V}(T,x,u,k) = \Phi(x,u)$.

We remark that, unlike conditions $(BC)_1$ and $(BC)_2$ above, boundary conditions $(BC)_1'$ and $(BC)_2'$ do not involve any auxiliary minimum problem and refer only to the cost function Φ and to equation (DPE).

We also prove a verification theorem (Theorem 5.1), which incidentally provides a possible answer to the open question b) mentioned above.

Finally, by applying standard results to the space-time embedding, we are able to clarify the relationship occurring between the adjoint variables of the maximum principle and the value function $\mathcal{V}$. This provides a possible answer to the open question a) above, while the answer to question c) is inherent to the general setting of the problem, for the vector fields $g_1, \dots, g_m$ *do depend* on x and u.

The paper ends with a simple, elaborated example where the theoretical results proved throughout the paper are explicitly applied to test the optimality of a feedback control previously computed by means of the maximum principle.

2. The minimum problem. Let us consider the control system

$$\dot{x} = g_0(t, x, u, v) + \sum_{i=1}^{m} g_i(t, x, u, v)\dot{u}_i(t), \tag{2.1}$$

$$x(\bar{t}) = \bar{x}, \qquad u(\bar{t}) = \bar{u} \tag{2.2}$$

defined on a time interval $[\bar{t}, T]$, where the state x ranges in $\mathbb{R}^n$ while the controls u and v take values on a closed arcwise connected subset $U \subset \mathbb{R}^m$ and a compact subset $V \subset \mathbb{R}^q$, respectively. Moreover the control u is subject to the directional constraint $\dot{u} \in C$, where $C \subset \mathbb{R}^m$ denotes a given closed cone.

Let K be a positive constant, and for every $\bar{k} \in [0, K]$ let us define the set

$$W_{K-\bar{k}}(\bar{t}, \bar{u}) \doteq \left\{ \begin{array}{l} (u, v) \in AC([\bar{t}, T], U) \times B([\bar{t}, T], V) \ : \ u(\bar{t}) = \bar{u}, \\ \dot{u}(t) \in C \text{ for a.e. } t \in [\bar{t}, T] \text{ and } V_{\bar{t}}^{T}(u) \leq K - \bar{k} \end{array} \right\}, \tag{2.3}$$

where $AC([\bar{t}, T], U)$ denotes the set of absolutely continuous functions from $[\bar{t}, T]$ into U, $B([\bar{t}, T], V)$ is the set of Borel-measurable functions from $[\bar{t}, T]$ into V, and $V_{\bar{t}}^{T}(u)$ denotes the total variation of u on the interval $[\bar{t}, T]$. We call $W_{K-\bar{k}}(\bar{t}, \bar{u})$ the set of *admissible regular controls from* $(\bar{t}, \bar{u})$ *such that the variation of* u *is less than or equal to* $K - \bar{k}$.

Let Φ be a continuous function defined on $\mathbb{R}^n \times U$. For any $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0, T[\times \mathbb{R}^n \times U \times [0, K]$ we consider the following minimum problem of Mayer type:

$$(\mathcal{P}_{(\bar{t}, \bar{x}, \bar{u}, \bar{k})}) \qquad \underset{(u,v) \in W_{K-\bar{k}}(\bar{t}, \bar{u})}{\text{minimize}} \Phi\big(x[\bar{t}, \bar{x}, \bar{u}; u, v](T), u(T)\big),$$

where $x[\bar{t}, \bar{x}, \bar{u}; u, v](\cdot)$ denotes the solution of (2.1), (2.2) corresponding to the control (u, v).

Throughout this paper we assume the following hypothesis **(H1)** on the vector fields $g_0, \dots, g_m$ and the function Φ:

(H1) $g_0, \dots, g_m$ and Φ are continuous in all of its variables, and there is a positive constant M such that

$$|g_i(t, x, u, v)| \leq M\big(1 + |(x, u)|\big), \quad |\Phi(x, u)| \leq M$$
$$\forall (t, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V \quad (i = 0, \dots, m).$$

Moreover, for any compact subset $Q \subset \mathbb{R}^n \times U$ there is a constant L such that

$$|g_i(t, x, u, v) - g_i(t, \bar{x}, u, v)| \leq L|x - \bar{x}|$$
$$\forall (t, x, u, v), (t, \bar{x}, u, v) \in [0, T] \times Q \times V \quad (i = 0, \dots, m).$$

In the following discussion, whenever the compact set Q is specified, we will denote by $\omega_{g_0}, \dots, \omega_{g_m}$ and ω_{Φ} the modulus of uniform continuity of the restrictions of the functions $g_0, \dots, g_m$ and Φ to $[0, T] \times Q \times V$ and Q, respectively.

Remark 2.1. The condition $|\Phi(x, u)| \leq M$ implies that the value function is globally bounded, which turns out to be very convenient for applying the theory of viscosity solutions. On the other hand one can skip such a limitation by replacing the cost function Φ with the bounded cost function $\arctan \Phi$. It is obvious that this transformation will not affect the essential character of the problem.

Since the right-hand side of (2.1) depends linearly on the derivative $\dot{u}$, in general no optimal controls can be found within the class $W_{K-\bar{k}}(\bar{t}, \bar{u})$. Hence, denoting the triple (t, x, u) by y, on the basis of the results in [10], [32], we embed (2.1) into the space–time system

$$y' = \hat{g}_0(y, v)t'(s) + \sum_{i=1}^{m} \hat{g}_i(y, v)u_i'(s) \tag{2.4}$$

together with the initial condition

$$y(0) = (\bar{t}, \bar{x}, \bar{u}). \tag{2.5}$$

In (2.4) the superscript denotes differentiation with respect to the new parameter $s \in [0, 1]$ and for every $i = 0, \dots, m$ the vector field $\hat{g}_i$ coincides with the ith column of the $(1 + n + m) \times (1 + m)$ matrix

$$\hat{G}(y, v) \doteq \begin{pmatrix} 1 & 0 & \dots & 0 \\ g_{0_1} & g_{1_1} & \dots & g_{m_1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{0_n} & g_{1_n} & \dots & g_{m_n} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

DEFINITION 2.1. *The control system* (2.4) *is called the* space–time control system *relative to* (2.1), *and a map*

$$(t, u, v) : [0, 1] \to [\bar{t}, T] \times U \times V$$

is called a space–time control *for* (2.4), (2.5) *whenever the following hold:*

(i) $(t, u)(0) = (\bar{t}, \bar{u})$;

(ii) $(t, u) : [0, 1] \to [\bar{t}, T] \times U$ *is Lipschitz continuous and* $u'(s) \in C$ *for almost every* $s \in [0, 1]$;

(iii) $t : [0, 1] \to [\bar{t}, T]$ *is surjective and nondecreasing;*

(iv) $v : [0, 1] \to V$ *is Borel measurable.*

The set of space–time controls will be denoted by $\Gamma(\bar{t}, \bar{u})$. *A solution of the space–time control system* (2.4) *will be called a* space–time trajectory.

We remark again—see the introduction—that a mere interpretation of the original system (2.1) as an equation in measure would lead to an ill-posed problem, for the dependence of $g_1, \dots, g_m$ on x and u makes it impossible to define a concept of (univalued) trajectory as a map of the original parameter t.

We refer to the appendix for some basic facts concerning the concept of *canonical parametrization* and the related topology on the set of space–time controls. Briefly, the parametrization of a space–time control (t, u, v) is called canonical if the norm $|(t', u')|$ is constant almost everywhere in $[0, 1]$. Any space–time control can be reparametrized in such a way that the resulting space–time control turns out to be canonical. And, up to reparametrization, the corresponding trajectories coincide (see Proposition A.2).

Observe that after introducing new equations we regard t and u both as state variables and as control variables. This allows us to embed problem $\mathcal{P}_{(\bar{t}, \bar{x}, \bar{u}, \bar{k})}$ into the extended problem

$$(\mathcal{P}^e_{(\bar t,\bar x,\bar u,\bar k)}) \qquad \underset{(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}{\text{minimize}} \Phi(y[\bar t,\bar x,\bar u;t,u,v](1)),$$

where

$$\Gamma_{K-\bar k}(\bar t,\bar u) \doteq \{(t,u,v)\in\Gamma(\bar t,\bar u) \ : \ V_0^1(u)\le K-\bar k\} \tag{2.6}$$

is the set of admissible space–time controls and $y[\bar t,\bar x,\bar u;t,u,v](\cdot)$ denotes the solution of (2.4), (2.5) corresponding to the space–time control (t,u,v).

Remark 2.2. We point out that Φ is a function of the only variables (x,u). With abuse of notation we write $\Phi(y)$, where $y=(t,x,u)$, instead of $\Phi(x,u)$, just to remind the reader that we are now referring to the space–time extension (2.4).

It is clear that in the space–time setting the original set $W_{K-\bar k}(\bar t,\bar u)$ of admissible controls has to be identified with the subset $\Gamma^+_{K-\bar k}(\bar t,\bar u)\subset\Gamma_{K-\bar k}(\bar t,\bar u)$ formed by the Lipschitz continuous reparametrizations of the graphs of the elements belonging to $W_{K-\bar k}(\bar t,\bar u)$.

The subset $\Gamma^+_{K-\bar k}(\bar t,\bar u)$ turns out to be dense—see [32] and the Appendix—in the set $\Gamma_{K-\bar k}(\bar t,\bar u)$ of space–time controls.

We now prove that the infimum of the extended problem $\mathcal{P}^e_{(\bar t,\bar x,\bar u,\bar k)}$ coincides with the infimum of the original problem $\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}$.

THEOREM 2.1. *For every initial condition $(\bar t,\bar x,\bar u,\bar k)\in[0,T)\times\mathbb{R}^n\times U\times[0,K]$ one has*

$$\inf_{(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1)) = \inf_{(u,v)\in W_{K-\bar k}(\bar t,\bar u)}\Phi(x[\bar t,\bar x,\bar u;u,v](T),u(T)). \tag{2.7}$$

Proof. Let $(\bar t,\bar x,\bar u,\bar k)$ be a fixed initial datum and let us observe that Gronwall's lemma, together with the bound on the total variation of u, guarantees that there is some positive constant M' such that

$$\begin{aligned}&|y[\bar t,\bar x,\bar u;t,u,v](s)|\le M',\\ &|g_i(y[\bar t,\bar x,\bar u;t,u,v](s),v(s))|\le M' \quad (i=0,\dots,m) \quad \text{for a.e. } s\in[0,1]\end{aligned} \tag{2.8}$$

for all $(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)$. Hence, setting $Q\doteq B_{n+m}[0,M']\cap\mathbb{R}^n\times U$ (where $B_{n+m}[0,M']$ denotes the closed ball of center 0 and radius M' in $\mathbb{R}^{n+m}$), we can identify the vector fields $g_0,\dots,g_m$ and the function Φ with their restrictions to the compact sets $[0,T]\times Q\times V$ and Q, respectively.

By the definition of $\Gamma^+_{K-\bar k}(\bar t,\bar u)$, proving (2.7) is equivalent to checking that the identity

$$\inf_{(t,u,v)\in\Gamma^+_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1)) = \inf_{(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1))$$

holds true. Hence it suffices to show that

$$\inf_{(t,u,v)\in\Gamma^+_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1)) \le \inf_{(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1)). \tag{2.9}$$

Since these infima are bounded, for any $\varepsilon>0$ there is a space–time control $(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)$ verifying

$$\inf_{(\tilde t,\tilde u,\tilde v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;\tilde t,\tilde u,\tilde v](1)) \ge \Phi(y[\bar t,\bar x,\bar u;t,u,v](1))-\varepsilon/2. \tag{2.10}$$

Note that on the basis of Proposition A.2 it is not restrictive to assume that the norm $|(t',u')|$ is constant almost everywhere in $[0,1]$, this constant being less than $K+T$. Then, by setting

$$t_\varepsilon(s)\doteq\bar t+(T-\bar t)\frac{(t(s)-\bar t)+s\rho_\varepsilon}{(T-\bar t+\rho_\varepsilon)} \qquad \forall s\in[0,1]$$

for a $\rho_\varepsilon\in(0,(T-\bar t)/2]$ to be chosen, we obtain a space–time control $(t_\varepsilon,u,v)\in\Gamma^+_{K-\bar k}(\bar t,\bar u)$ such that

$$|t_\varepsilon(s)-t(s)|\le 2\rho_\varepsilon \qquad \forall s\in[0,1],$$

and the corresponding trajectory $x_\varepsilon\doteq x[\bar t,\bar x,\bar u;t_\varepsilon,u,v]$ satisfies

$$\begin{aligned}|x_\varepsilon(s)-x(s)| &\le \int_0^s \left|g_0(t_\varepsilon(\sigma),x_\varepsilon(\sigma),u(\sigma),v(\sigma))-g_0(t(\sigma),x(\sigma),u(\sigma),v(\sigma))\right|t'(\sigma)\,d\sigma\\ &\quad+\int_0^s\sum_{i=1}^m\left|g_i(t_\varepsilon(\sigma),x_\varepsilon(\sigma),u(\sigma),v(\sigma))-g_i(t(\sigma),x(\sigma),u(\sigma),v(\sigma))\right||u_i'(\sigma)|\,d\sigma\\ &\quad+\frac{\rho_\varepsilon}{T-\bar t+\rho_\varepsilon}\int_0^s\left|g_0(t_\varepsilon(\sigma),x_\varepsilon(\sigma),u(\sigma),v(\sigma))\right|[(T-\bar t)+t'(\sigma)]\,d\sigma\\ &\le (K+T)\sum_{i=0}^m\omega_{g_i}(2\rho_\varepsilon)+(m+1)(K+T)L\int_0^s|x_\varepsilon(\sigma)-x(\sigma)|\,d\sigma+2\frac{T-\bar t}{T-\bar t+\rho_\varepsilon}M'\rho_\varepsilon.\end{aligned}$$

By Gronwall's lemma it follows that

$$\left|\Phi(x_\varepsilon(1),u(1))-\Phi(x(1),u(1))\right|\le\omega_\Phi\left((2M'\rho_\varepsilon+(K+T)\sum_{i=0}^m\omega_{g_i}(2\rho_\varepsilon))e^{(m+1)(K+T)L}\right).$$

Hence for a ρ_ε small enough from (2.10) we have

$$\inf_{(\tilde t,\tilde u,\tilde v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;\tilde t,\tilde u,\tilde v](1))\ge\Phi(x_\varepsilon(1),u(1))-\varepsilon,$$

which by the arbitrariness of $\varepsilon>0$ yields (2.9). □

3. The value function. In this section we introduce the so-called value function for the problem $\mathcal{P}_{(\bar t,\bar x,\bar u,\bar k)}$ and study its regularity properties.

DEFINITION 3.1. *The map*

$$\mathcal{F}(\bar t,\bar x,\bar u,\bar k)\doteq\inf_{(u,v)\in W_{K-\bar k}(\bar t,\bar u)}\Phi(x[\bar t,\bar x,\bar u;u,v](T),u(T)) \tag{3.1}$$

from $[0,T)\times\mathbb{R}^n\times U\times[0,K]$ *into* $\mathbb{R}$ *is called the* value function of the original minimum problem.

DEFINITION 3.2. *The map*

$$\mathcal{V}(\bar t,\bar x,\bar u,\bar k)\doteq\inf_{(t,u,v)\in\Gamma_{K-\bar k}(\bar t,\bar u)}\Phi(y[\bar t,\bar x,\bar u;t,u,v](1)) \tag{3.2}$$

from $[0,T]\times\mathbb{R}^n\times U\times[0,K]$ *into* $\mathbb{R}$ *is called the* value function of the extended minimum problem.

The following result follows from Theorem 2.1.

COROLLARY 3.1. *The value function $\mathcal{F}$ of the original minimum problem is bounded and coincides with the value function $\mathcal{V}$ of the extended minimum problem.*

Let us observe that the value function $\mathcal{V}$ of the extended problem is defined even at time $\bar t=T$. Furthermore, in Theorem 3.1 below we show that $\mathcal{V}$ is continuous provided that one of the following two hypotheses on the cone C and the closed set U holds:

(H2)$_C$ The set U coincides with the whole $\mathbb{R}^m$.

(H2)$_U$ the cone C coincides with the whole $\mathbb{R}^m$; moreover for any $\varepsilon > 0$ and $u_1 \in U$ there exists a $\delta > 0$ such that for each $u_2 \in U \cap B(u_1, \delta)$, there is a path $\gamma_{12} \in AC([0,1], U)$ satisfying $\gamma_{12}(0) = u_1$, $\gamma_{12}(1) = u_2$, and

$$\int_0^1 |\gamma_{12}'(s)|\, ds \le \varepsilon.$$

THEOREM 3.1. *Let $Q_x \subset \mathbb{R}^n$, $Q_u \subset U$ be compact subsets. Then for every $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0,T] \times Q_x \times Q_u \times [0,K]$ one has the following:*

i) the functions $x \mapsto \mathcal{V}(\bar{t}, x, \bar{u}, \bar{k})$, $t \mapsto \mathcal{V}(t, \bar{x}, \bar{u}, \bar{k})$, and $k \mapsto \mathcal{V}(\bar{t}, \bar{x}, \bar{u}, k)$ are continuous on Q_x, $[0,T]$, and $[0,K]$, respectively, uniformly with respect to the remaining variables on $[0,T] \times Q_x \times Q_u \times [0,K]$; furthermore, $k \mapsto \mathcal{V}(\bar{t}, \bar{x}, \bar{u}, k)$ is nondecreasing;

ii) in addition, if either hypothesis **(H2)**$_C$ *or hypothesis* **(H2)**$_U$ *is assumed, then the function $u \mapsto \mathcal{V}(\bar{t}, \bar{x}, u, \bar{k})$ is continuous on Q_u, uniformly with respect to the remaining variables on $[0,T] \times Q_x \times Q_u \times [0,K]$. In particular the value function $\mathcal{V}$ is continuous on its domain.*

Proof. By (2.8) the trajectories starting from points of $[0,T] \times Q_x \times Q_u$ lie in the compact set $[0,T] \times B_{n+m}[Q_x \times Q_u; M'] \cap (\mathbb{R}^n \times U)$. Let ω_{g_i} denote the modulus of uniform continuity of g_i $(i = 0, \dots, m)$ on $[0,T] \times B_{n+m}[Q_x \times Q_u; M'] \cap (\mathbb{R}^n \times U) \times V$, and let ω_Φ be the modulus of continuity of Φ on $B_{n+m}[Q_x \times Q_u; M'] \cap (\mathbb{R}^n \times U)$.

Let $x_1, x_2 \in Q_x$, and consider the difference

$$\mathcal{V}(\bar{t}, x_2, \bar{u}, \bar{k}) - \mathcal{V}(\bar{t}, x_1, \bar{u}, \bar{k}),$$

which can be assumed nonnegative. For any $\varepsilon > 0$ let $(t, u, v) \in \Gamma_{K-\bar{k}}(\bar{t}, \bar{u})$ be a space–time control satisfying

$$\mathcal{V}(\bar{t}, x_1, \bar{u}, \bar{k}) \ge \Phi\big(y[\bar{t}, x_1, \bar{u}; t, u, v](1)\big) - \varepsilon. \tag{3.3}$$

Thus by the definition of $\mathcal{V}$ we have

(3.4)

$$\mathcal{V}(\bar{t}, x_2, \bar{u}, \bar{k}) - \mathcal{V}(\bar{t}, x_1, \bar{u}, \bar{k}) \le \Phi\big(y[\bar{t}, x_2, \bar{u}; t, u, v](1)\big) - \Phi\big(y[\bar{t}, x_1, \bar{u}; t, u, v](1)\big) + \varepsilon.$$

Furthermore standard estimates for the trajectories of (2.4) yield

$$\big|x[\bar{t}, x_2, \bar{u}; t, u, v](s) - x[\bar{t}, x_1, \bar{u}; t, u, v](s)\big| \le |x_2 - x_1| e^{L(1+m)(K+T)s} \tag{3.5}$$

for all $s \in [0,1]$. Hence (3.4) and (3.5) imply

$$\mathcal{V}(\bar{t}, x_2, \bar{u}, \bar{k}) - \mathcal{V}(\bar{t}, x_1, \bar{u}, \bar{k}) \le \omega_\Phi\big(e^{L(1+m)(K+T)}|x_2 - x_1|\big) + \varepsilon,$$

which, by the arbitrariness of $\varepsilon > 0$, proves that $x \mapsto \mathcal{V}(\bar{t}, x, \bar{u}, \bar{k})$ is continuous uniformly with respect to $(\bar{t}, \bar{u}, \bar{k})$.

Now let $t_1, t_2 \in [0,T]$, $t_1 \ne t_2$, and consider the difference

$$\mathcal{V}(t_2, \bar{x}, \bar{u}, \bar{k}) - \mathcal{V}(t_1, \bar{x}, \bar{u}, \bar{k}),$$

which can be assumed nonnegative. For any $\varepsilon > 0$, let (t, u, v) be a space–time control for $(t_1, \bar{x}, \bar{u}, \bar{k})$ satisfying

$$\mathcal{V}(t_1, \bar{x}, \bar{u}, \bar{k}) \ge \Phi\big(y[t_1, \bar{x}, \bar{u}; t, u, v](1)\big) - \varepsilon,$$

and consider the space–time control $(\tilde{t}, u, v) \in \Gamma_{K-\bar{k}}(t_2, \bar{u})$, where $\tilde{t}$ is defined as follows:

if $t_1 < t_2$, set

$$\tilde{t}(s) = \begin{cases} t_2, & s \in [0, \bar{s}], \\ t(s), & s \in [\bar{s}, 1], \end{cases}$$

where

$$\bar{s} \doteq \min\{s \in [0,1] \ : \ t(s) = t_2\};$$

if $t_1 \ge t_2$, set

$$\tilde{t}(s) = t(s) - (t_1 - t_2)(1 - s), \qquad s \in [0,1].$$

In both cases the definition of $\tilde{t}$ and Gronwall's lemma imply

(3.6)

$$|\tilde{t}(s) - t(s)| \le |t_2 - t_1|,$$

$$\big|\tilde{x}(s) - x(s)\big| \le \Big[M'|t_2 - t_1| + (K+T)\sum_{i=0}^{m} \omega_{g_i}(|t_2 - t_1|)\Big] e^{L(1+m)(K+T)} \quad \forall s \in [0,1],$$

where we have set $\tilde{x}(\cdot) \doteq x[t_2, \bar{x}, \bar{u}; \tilde{t}, u, v](\cdot)$, $x(\cdot) \doteq x[t_1, \bar{x}, \bar{u}; t, u, v](\cdot)$. It follows that

$$\begin{aligned} &\mathcal{V}(t_2, \bar{x}, \bar{u}, \bar{k}) - \mathcal{V}(t_1, \bar{x}, \bar{u}, \bar{k}) \\ &\qquad \le \omega_\Phi\Big(\Big[M'|t_2 - t_1| + (K+T)\sum_{i=0}^{m} \omega_{g_i}(|t_2 - t_1|)\Big] e^{L(1+m)(K+T)}\Big) + \varepsilon, \end{aligned}$$

which, by the arbitrariness of ε, implies that $t \mapsto \mathcal{V}(t, \bar{x}, \bar{u}, \bar{k})$ is continuous uniformly with respect to the remaining variables $(\bar{x}, \bar{u}, \bar{k})$.

Now let $k_1, k_2 \in [0,K]$, with $k_1 \ne k_2$. Since the map $k \mapsto \mathcal{V}(\bar{t}, \bar{x}, \bar{u}, k)$ is nondecreasing, it is not restrictive to consider only the case $k_2 > k_1$. Choose a space–time control for $(\bar{t}, \bar{x}, \bar{u}, k_1)$ satisfying

$$\mathcal{V}(\bar{t}, \bar{x}, \bar{u}, k_1) \ge \Phi\big(y[\bar{t}, \bar{x}, \bar{u}; t, u, v](1)\big) - \varepsilon,$$

and set

$$\tilde{u}(s) = \begin{cases} u(s), & s \in [0, \bar{s}], \\ u(\bar{s}), & s \in [\bar{s}, 1], \end{cases}$$

where

$$\bar{s} \doteq \max\{s \in [0,1] \ : \ V_0^s(u) \le K - k_2\}.$$

Observe that either $K - k_2 < V_0^1(u) \le K - k_1$ and $V_0^{\bar{s}}(u) = K - k_2$ with $\bar{s} < 1$, or $\bar{s} = 1$; furthermore, $V_0^1(\tilde{u}) = V_0^{\bar{s}}(u) \le K - k_2$ so that $(t, \tilde{u}, v) \in \Gamma_{K-k_2}(\bar{t}, \bar{u})$. For every $s \in [\bar{s}, 1)$ one has

$$V_{\bar{s}}^s(u) \le V_{\bar{s}}^1(u) = V_0^1(u) - V_0^{\bar{s}}(u) = V_0^1(u) - K + k_2 \le k_2 - k_1.$$

Hence, from the definition of $\tilde{u}$ and applying Gronwall's lemma one obtains

$$\begin{aligned} |\tilde{u}(s) - u(s)| &\le V_{\bar{s}}^1(u) \le k_2 - k_1, \\ \big|x[\bar{t}, \bar{x}, \bar{u}; t, \tilde{u}, v](s) - x[\bar{t}, \bar{x}, \bar{u}; t, u, v](s)\big| &\le \big[mM' V_{\bar{s}}^s(u) + \omega_{g_0}(T|k_2 - k_1|)\big] e^{L(K+T)} \\ &\le \big[mM'|k_2 - k_1| + T\omega_{g_0}(|k_2 - k_1|)\big] e^{L(K+T)} \quad \forall s \in [0,1]. \end{aligned}$$

This implies

(3.7)
$$\mathcal{V}(\bar t,\bar x,\bar u,k_2)-\mathcal{V}(\bar t,\bar x,\bar u,k_1)\le \omega_\Phi\Big(\big[(1+mM')|k_2-k_1|+T\omega_{g_0}(|k_2-k_1|)\big]e^{L(K+T)}\Big)+\varepsilon;$$

thence $k\mapsto \mathcal{V}(\bar t,\bar x,\bar u,k)$ is continuous uniformly with respect to the variables $(\bar t,\bar x,\bar u)$. Thus thesis i) of the theorem is proved.

In order to prove ii), let $\varepsilon>0$ and, for a $\delta>0$ to be determined later, let $u_1,u_2\in Q_u$ satisfy $|u_i-\bar u|\le\delta$, $i=1,2$. Let us consider the difference

$$\mathcal{V}(\bar t,\bar x,u_2,\bar k)-\mathcal{V}(\bar t,\bar x,u_1,\bar k),$$

which it is not restrictive to assume is nonnegative. Let $(t,u,v)\in\Gamma_{K-\bar k}(\bar t,u_1)$ be a control satisfying

$$\mathcal{V}(\bar t,\bar x,u_1,\bar k)\ge\Phi\big(y[\bar t,\bar x,u_1;t,u,v](1)\big)-\varepsilon/2.$$

If $\mathbf{(H2)}_C$ is assumed, then the control $(t,\tilde u,v)\doteq(t,u_2-u_1+u(s),v)$ is in $\Gamma_{K-\bar k}(\bar t,u_2)$. The definitions of $\omega_{g_0},\dots,\omega_{g_m}$ and ω_Φ together with standard estimates for the trajectories of (2.4) imply

$$\begin{aligned}
&\mathcal{V}(\bar t,\bar x,u_2,\bar k)-\mathcal{V}(\bar t,\bar x,u_1,\bar k)\\
&\qquad\le\Phi\big(y[\bar t,\bar x,u_2;t,\tilde u,v](1)\big)-\Phi\big(y[\bar t,\bar x,u_1;t,u,v](1)\big)+\varepsilon/2\\
&\qquad\le\omega_\Phi\Big(|u_2-u_1|+(K+T)\sum_{i=0}^{m}\omega_{g_i}(|u_2-u_1|)e^{L(1+m)(K+T)}\Big)+\varepsilon/2.
\end{aligned}$$

This yields the continuity of the map $u\mapsto\mathcal{V}(\bar t,\bar x,u,\bar k)$ uniformly with respect to the remaining variables.

We conclude by proving ii) under hypothesis $\mathbf{(H2)}_U$. Let $\rho_\varepsilon\in(0,1)$. If $K-\bar k\le\rho_\varepsilon$, by setting $\tilde u(s)=u_2\ \ \forall s\in[0,1]$ we obtain

(3.8)
$$\begin{aligned}
&\mathcal{V}(\bar t,\bar x,u_2,\bar k)-\mathcal{V}(\bar t,\bar x,u_1,\bar k)\\
&\qquad\le\Phi\big(y[\bar t,\bar x,u_2;t,u_2,v](1)\big)-\Phi\big(y[\bar t,\bar x,u_1;t,u,v](1)\big)+\varepsilon/2\\
&\qquad\le\omega_\Phi\Big(|u_2-u_1|+\rho_\varepsilon+(T\omega_{g_0}(|u_2-u_1|+\rho_\varepsilon)+mM'\rho_\varepsilon)e^{L(1+m)(K+T)}\Big)+\varepsilon/2.
\end{aligned}$$

Suppose on the contrary that $K-\bar k>\rho_\varepsilon$. Then by $\mathbf{(H2)}_U$ there exists a $\bar\delta>0$ such that if $|u_1-\bar u|<\bar\delta$, $|u_2-\bar u|<\bar\delta$ one has

$$V_0^1(\gamma_{21})\le\rho_\varepsilon/2\qquad(<1)$$

for some path $\gamma_{21}:[0,1]\to U$ such that $\gamma_{21}(0)=u_2$, $\gamma_{21}(1)=u_1$. We set

$$u_{\bar s}(s)=\begin{cases}u(s), & s\in[0,\bar s],\\ u(\bar s), & s\in[\bar s,1],\end{cases}$$

where

$$\bar s\doteq\max\{s\in[0,1]\ :\ V_0^s(u)\le K-\bar k-V_0^1(\gamma_{21})\}.$$

Hence for any $\nu\in V$ the control defined by

$$(\tilde t,\tilde u,\tilde v)=\begin{cases}(\bar t,\gamma_{21}(s/\sigma),\nu), & s\in[0,\sigma],\\ (t,u_{\bar s},v)((s-\sigma)/(1-\sigma)), & s\in[\sigma,1],\end{cases}$$

where $\sigma\doteq V_0^1(\gamma_{21})$ is in $\Gamma_{K-\bar k}(\bar t,u_2)$. Standard estimates yield

$$|\tilde u(s)-u(s)|\le|u_2-u_1|+(3+K+T)V_0^1(\gamma_{21}),$$

from which proceeding as in the previous case one obtains an inequality similar to (3.8). Hence, by choosing $\delta\doteq\min\{\rho_\varepsilon/2,\bar\delta\}$, there exists some $\rho_\varepsilon>0$ such that we have

$$\mathcal{V}(\bar t,\bar x,u_2,\bar k)-\mathcal{V}(\bar t,\bar x,u_1,\bar k)\le\varepsilon,$$

which implies the continuity of $u\to\mathcal{V}(\bar t,\bar x,u,\bar k)$ on Q_u uniformly with respect to the variables $(\bar t,\bar x,\bar k)$.

Thus the continuity of the value function $\mathcal{V}$ is proved. $\square$

4. Dynamic programming principle and dynamic programming equation. Let us define the Hamiltonian function $H:[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^{1+n+m+1}\to\mathbb{R}$ by setting

(4.1)
$$\begin{aligned}
&H(t,x,u,p_0,p_1,\dots,p_n,p_{n+1},\dots,p_{n+m},p_\infty)\\
&\qquad\doteq\min_{\substack{v\in V\\(w_0,w)\in S_+^m}}\mathcal{H}(t,x,u,p_0,p_1,\dots,p_n,p_{n+1},\dots,p_{n+m},p_\infty,w_0,w,v),
\end{aligned}$$

where $\mathcal{H}$ denotes the unminimized Hamiltonian

(4.2)
$$\begin{aligned}
&\mathcal{H}(t,x,u,p_0,p_1,\dots,p_n,p_{n+1},\dots,p_{n+m},p_\infty,w_0,w,v)\\
&\qquad\doteq\Big\{\Big(p_0+\sum_{i=1}^{n}p_ig_0^i(t,x,u,v)\Big)w_0+\sum_{\substack{i=1,\dots,n\\ j=1,\dots,m}}\big(p_ig_j^i(t,x,u,v)+p_{j+n}\big)w^j+p_\infty|w|\Big\},
\end{aligned}$$

while S_+^m is the intersection of $[0,+\infty[\times C$ and the unit sphere $S^m=\{(w_0,w)\in\mathbb{R}^{1+m}:\ |(w_0,w)|=1\}$.

We shall prove that $\mathcal{V}$ solves the dynamic programming equation

(DPE)
$$-H(t,x,u,\nabla\mathcal{V})=0,$$

where $\nabla\mathcal{V}$ stands for $(\nabla_t\mathcal{V},\nabla_x\mathcal{V},\nabla_u\mathcal{V},\nabla_k\mathcal{V})$, and $\nabla_t\mathcal{V}$, $\nabla_x\mathcal{V}$, $\nabla_u\mathcal{V}$, and $\nabla_k\mathcal{V}$ denote the gradients of $\mathcal{V}$ with respect to t, x, u, and k, respectively. The presence of the minus sign in (DPE) is motivated by the fact that we wish to be consistent with the terminology of the theory of viscosity solutions. In fact, like in the nonimpulsive case, the value function $\mathcal{V}$ fails in general to be continuously differentiable, so it can satisfy (DPE) only in a generalized sense. Aiming at self-consistency we recall the definition of viscosity solution of a first-order partial differential equation; see, e.g., [18].

DEFINITION 4.1. *Let E be a subset of $\mathbb{R}^N$. A function $\mathcal{V}\in C^0(E)$ is a* viscosity subsolution *of* (DPE) *at $(t,x,u,k)\in E$ if for any $\lambda\in C^\infty(\mathbb{R}^N)$ such that (t,x,u,k) is a local maximum point of $\mathcal{V}-\lambda$ on E one has*

$$-H\big(t,x,u,\nabla\lambda(t,x,u,k)\big)\le0.$$

$\mathcal{V}\in C^0(E)$ is a viscosity supersolution *of* (DPE) *at $(t,x,u,k)\in E$ if for any $\lambda\in C^\infty(\mathbb{R}^N)$ such that (t,x,u,k) is a local minimum point of $\mathcal{V}-\lambda$ on E one has*

$$-H\big(t,x,u,\nabla\lambda(t,x,u,k)\big)\ge0.$$

$\mathcal{V}\in C^0(E)$ is a viscosity solution *of* (DPE) *at (t,x,u,k) if it is both a viscosity subsolution and a viscosity supersolution.*

In order to state Theorem 4.1 below, let us introduce the domain

$$\Omega\doteq[0,T)\times\mathbb{R}^n\times\mathring{U}\times[0,K)$$

and the boundary's subsets

$$\partial_T\Omega \doteq \{T\} \times \mathbb{R}^n \times U \times [0, K], \qquad (4.3)$$
$$\partial'\Omega \doteq \partial\Omega \setminus \partial_T\Omega.$$

THEOREM 4.1 (dynamic programming equation and boundary conditions). *Assume either hypothesis* $(\mathbf{H2})_C$ *or hypothesis* $(\mathbf{H2})_U$. *Then*

a) $\mathcal{V}$ *is a viscosity solution on* Ω *of the dynamic programming equation* (DPE);

b) $\mathcal{V}$ *satisfies*

$$\mathcal{V}(T, x, u, k) \le \Phi(x, u) \qquad \forall (T, x, u, k) \in \partial_T\Omega; \qquad (4.4)$$

c) $\mathcal{V}$ *is a viscosity supersolution of* (DPE) *on* $\partial'\Omega$ *and at any point* $(T, x, u, k) \in \partial_T\Omega$ *such that* $\mathcal{V}(T, x, u, k) < \Phi(x, u)$.

Remark 4.1. Note that although the cone $[0, +\infty[\times(T_uU \cap C)$ (where T_uU denotes the contingent cone to U at u; see, e.g., [1]) could be considered the natural range of the control's derivative $(t', u'_1, \dots, u'_m)$, the minimum in (4.1) is searched over the *compact* set S^m_+. This is due essentially to the bound on the variation of u and to the possibility of replacing any space–time control with its canonical parametrization (see the appendix). On the other hand, the positive homogenity of $\mathcal{H}$ in the variable (w_0, w) allows us to use S^m_+ in the definition of H instead of $B^{1+m}_+ \doteq [0, +\infty) \times C \cap \{(w_0, w) : |(w_0, w)| \le 1\}$. Actually by allowing the elements $(w_0, w, v) \equiv (0, 0, v)$ in the domain of minimization of $\mathcal{H}$, we would obtain an equation lacking uniqueness properties; see §5. As a direct consequence of having replaced the unbounded set $[0, +\infty) \times (T_uU \cap C)$ with a compact set, we achieve the continuity of the Hamiltonian H. Incidentally we observe that this approach presents some analogies with the one adopted by G. Barles [5] in an infinite horizon problem.

Remark 4.2. The fact that the domain of minimization of (w_0, w) is independent of u is strictly related to the very definition of viscosity supersolution on a closed set. Indeed it is well known (see, e.g., [40]) that the supersolution condition together with the subsolution condition on the interior accounts for a constraint on the state variables. Actually, in our case the situation is slightly different, since at the boundary points we have an alternative between supersolution condition and an inequality condition; a similar situation is encountered, e.g., in [2], [16], [24].

The proof of Theorem 4.1 will be based on the following dynamic programming principle, whose proof is an obvious adaptation to the parameter-free extended problem $(\mathcal{P}^e_{(\bar t, \bar x, \bar u, \bar k)})$ of the standard reasonings which yield to the dynamic programming principle in the ordinary case.

PROPOSITION 4.1 (dynamic programming principle). *The value function* $\mathcal{V}$ *has the following properties:*

i) *For an initial condition* $(\bar t, \bar x, \bar u, \bar k) \in [0, T] \times \mathbb{R}^n \times U \times [0, K]$ *and an admissible control* $(t, u, v) \in \Gamma_{K-\bar k}(\bar t, \bar u)$, *let* $y \doteq y[\bar t, \bar x, \bar u; t, u, v]$ *be the corresponding trajectory of the extended system* (2.4), (2.5). *Then the map*

$$s \mapsto \mathcal{V}(y(s), \bar k + V^s_0(u)) \qquad (4.5)$$

is nondecreasing.

ii) *If in* i) *the control* (t, u, v) *is optimal, then the map* (4.5) *is constant.*

Proof. Assume by contradiction that there exist s_1, s_2, $0 \le s_1 < s_2 \le 1$, and $\varepsilon > 0$ such that

$$\mathcal{V}(y(s_2), \bar k + V^{s_2}_0(u)) = \mathcal{V}(y(s_1), \bar k + V^{s_1}_0(u)) - \varepsilon. \qquad (4.6)$$

By the definition of $\mathcal{V}$ there is a space–time control $(\tilde t, \tilde u, \tilde v) \in \Gamma_{K-(\bar k+V^{s_2}_0(u))}((t, u)(s_2))$ satisfying

$$\Phi(y[y(s_2); \tilde t, \tilde u, \tilde v](1)) \le \mathcal{V}(y(s_2), \bar k + V^{s_2}_0(u)) + \varepsilon/2. \qquad (4.7)$$

Define the space–time control $(\hat t, \hat u, \hat v)$ by

$$(\hat t, \hat u, \hat v)(s) \doteq \begin{cases} (t, u, v)(s_1 + 2s(s_2 - s_1)), & s \in [0, 1/2], \\ (\tilde t, \tilde u, \tilde v)(2(s - 1/2)), & s \in (1/2, 1], \end{cases}$$

and set $\hat y \doteq y[y(s_1); \hat t, \hat u, \hat v]$. Note that $(\hat t, \hat u, \hat v) \in \Gamma_{K-(\bar k+V^{s_1}_0(u))}((t, u)(s_1))$, for we have

$$V^1_0(\hat u) = V^{s_2}_{s_1}(u) + V^1_0(\tilde u) \le V^{s_2}_{s_1}(u) + K - \bar k - V^{s_2}_0(u) = K - (\bar k + V^{s_1}_0(u)).$$

Moreover, by the parameter-free character of the extended system (2.4)—see Proposition A.2—we have

$$\hat y(1/2) = y(s_2),$$
$$\hat y(1) = y[y(s_2); \tilde t, \tilde u, \tilde v](1).$$

Hence, by (4.6) and (4.7), we obtain

$$\mathcal{V}(y(s_1), \bar k + V^{s_1}_0(u)) \le \Phi(\hat y(1)) = \Phi(y[y(s_2); \tilde t, \tilde u, \tilde v](1))$$
$$\le \mathcal{V}(y(s_2), \bar k + V^{s_2}_0(u)) + \varepsilon/2 = \mathcal{V}(y(s_1), \bar k + V^{s_1}_0(u)) - \varepsilon/2.$$

Since $\varepsilon > 0$, this proves i).

To prove ii) it is enough to observe that whenever the control (t, u, v) is optimal, on the basis of i) one has

$$\mathcal{V}(\bar t, \bar x, \bar u, \bar k) \le \mathcal{V}(y(s), \bar k + V^s_0(u)) \le \Phi(y(1)) = \mathcal{V}(\bar t, \bar x, \bar u, \bar k)$$

for every $s \in [0, 1]$. $\square$

Proof of Theorem 4.1. We begin by proving that $\mathcal{V}$ is a viscosity subsolution of (DPE) on Ω. Fix a point $(\bar y, \bar k) = (\bar t, \bar x, \bar u, \bar k) \in \Omega$ and consider a map $\lambda \in C^\infty(\mathbb{R}^{1+n+m+1})$ such that $\mathcal{V}(\bar y, \bar k) = \lambda(\bar y, \bar k)$ and $\mathcal{V} - \lambda$ has a local maximum at $(\bar y, \bar k)$. Then

$$\mathcal{V}(t, x, u, k) \le \lambda(t, x, u, k) \qquad \forall (t, x, u, k) \in \Omega \cap B((\bar y, \bar k), r)$$

for a sufficiently small $r > 0$. Choose $v \in V$ and $w = (w_1, \dots, w_m) \in B^m[0, 1] \cap C$, where $B^m[0, 1] = \{w \in \mathbb{R}^m : |w| \le 1\}$, and set $w_0 \doteq \sqrt{1 - |w|^2}$. Since $\bar t < T$, $\bar u \in \mathring{U}$, and $\bar k < K$, there exists some $\varepsilon \in (0, 1)$ such that the control (t, u, v) defined by

$$(t, u, v)(s) \doteq \begin{cases} (\bar t + sw_0, \bar u + sw, v), & s \in [0, \varepsilon], \\ (\bar t + \varepsilon w_0 + (T - \bar t - \varepsilon w_0)(s - \varepsilon)/(1 - \varepsilon), \bar u + \varepsilon w, v), & s \in (\varepsilon, 1], \end{cases}$$

is in $\Gamma_{K-\bar k}(\bar t, \bar u)$. Then by the dynamic programming principle one has

$$\lambda(\bar y, \bar k) = \mathcal{V}(\bar y, \bar k) \le \mathcal{V}(y[\bar t, \bar x, \bar u; t, u, v](s), \bar k + V^s_0(u)) \le \lambda(y[\bar t, \bar x, \bar u; t, u, v](s), \bar k + V^s_0(u)),$$

provided $0 < s \le \rho$, with ρ small enough. Dividing the last inequality by s one has

$$\frac{\lambda(y[\bar t, \bar x, \bar u; t, u, v](s), \bar k + V^s_0(u)) - \lambda(\bar y, \bar k)}{s} \ge 0 \qquad (4.8)$$

for every $s \in (0, \rho]$. Passing to the limit in (4.8) as $s \to 0^+$, we obtain

$$\left(\nabla_t \lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}) + \nabla_x \lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}) g_0(\bar{t}, \bar{x}, \bar{u}, v)\right) w_0 + \nabla_x \lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \sum_{i=1}^{m} g_i(\bar{t}, \bar{x}, \bar{u}, v) w^i$$
$$+ \nabla_u \lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}) w + \nabla_k \lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}) |w| \geq 0.$$

Since w and v are arbitrary in $B^m[0,1] \cap C$ and V, respectively, it follows that

$$-H\left(\bar{t}, \bar{x}, \bar{u}, \nabla\lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}\right) \leq 0.$$

Hence $\mathcal{V}$ is a subsolution of (DPE) on Ω.

Let us prove that $\mathcal{V}$ is a supersolution of (DPE) on $\Omega \cup \partial'\Omega$ and at any point $(t, x, u, k) \in \partial_T \Omega$ where $\mathcal{V}(t, x, u, k) < \Phi(x, u)$. Let $(\bar{y}, \bar{k}) = (\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in \overline{\Omega}$ and consider a function $\lambda \in C^\infty(\mathbb{R}^{1+n+m+1})$ such that $\mathcal{V} - \lambda$ has a local minimum on $\overline{\Omega}$ at $(\bar{y}, \bar{k})$ and $\mathcal{V}(\bar{y}, \bar{k}) = \lambda(\bar{y}, \bar{k})$. Then

$$\mathcal{V}(t, x, u, k) \geq \lambda(t, x, u, k) \qquad \forall (t, x, u, k) \in \overline{\Omega} \cap B\left((\bar{y}, \bar{k}), r\right)$$

for a sufficiently small $r > 0$. For any $n \in \mathbb{N} \setminus \{0\}$ let $(t_n, u_n, v_n) \in \Gamma_{K-\bar{k}}(\bar{t}, \bar{u})$ be a space–time control such that the corresponding trajectory y_n satisfies

$$\Phi(y_n(1)) \leq \mathcal{V}(\bar{y}, \bar{k}) + 1/n^2. \tag{4.9}$$

The dynamic programming principle yields

$$\lambda(y_n(s), \bar{k} + V_0^s(u_n)) \leq \mathcal{V}(y_n(s), \bar{k} + V_0^s(u_n)) \leq \mathcal{V}(\bar{y}, \bar{k}) + 1/n^2 = \lambda(\bar{y}, \bar{k}) + 1/n^2,$$

provided $0 < s \leq \rho$, with ρ small enough. By choosing $s = 1/n$ and dividing by $1/n$ we obtain

$$n \int_0^{1/n} \mathcal{H}\left(y_n, \nabla\lambda\left(y_n, \bar{k} + V_0^s(u_n)\right), t_n{}', u_n{}', v_n\right) ds \leq 1/n \tag{4.10}$$

for every n sufficiently large. Since it is not restrictive to assume that the controls (t_n, u_n, v_n) coincide with their canonical parametrizations, we have $|(t_n{}', u_n{}')|(s) = V_0^1(t_n, u_n)$ for almost every $s \in [0,1]$. Now if $(\bar{y}, \bar{k}) \in \Omega \cup \partial'\Omega$, one has $V_0^1(t_n, u_n) \geq V_0^1(t_n) \geq T - \bar{t} > 0$. Hence by the continuity —on the bounded set $\overline{\Omega} \cap B\left((\bar{t}, \bar{x}, \bar{u}, \bar{k}), r\right)$— of all the considered functions, there exists a map $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{n \to \infty} \varepsilon(n) = 0$ and

$$\begin{aligned}
\varepsilon(n) &\geq n \int_0^{1/n} \mathcal{H}\left(\bar{t}, \bar{x}, \bar{u}, \bar{k}, \nabla\lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}), t_n{}', u_n{}', v_n\right) ds \\
&\geq n V_0^1(t_n, u_n) \int_0^{1/n} \min_{\substack{v \in V \\ (w_0, w) \in S_+^m}} \mathcal{H}\left(\bar{t}, \bar{x}, \bar{u}, \bar{k}, \nabla\lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k}), w_0, w, v\right) ds \\
&\geq (T - \bar{t}) H\left(\bar{t}, \bar{x}, \bar{u}, \nabla\lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k})\right).
\end{aligned} \tag{4.11}$$

Therefore, as n tends to infinity, one has

$$-H\left(\bar{t}, \bar{x}, \bar{u}, \nabla\lambda(\bar{t}, \bar{x}, \bar{u}, \bar{k})\right) \geq 0,$$

i.e., $\mathcal{V}$ is a viscosity supersolution of (DPE) at $(\bar{t}, \bar{x}, \bar{u}, \bar{k})$.

Now let $(\bar{y}, \bar{k}) = (T, \bar{x}, \bar{u}, \bar{k}) \in \partial_T \Omega$ and observe that any space–time control $(t, u, v) \in \Gamma(T, \bar{u})$ having components $t(s)$,$u(s)$ coinciding with T,$\bar{u}$, respectively, gives

rise to the constant trajectory $y \equiv (T, \bar{x}, \bar{u})$. Hence thesis b) holds true by the very definition of $\mathcal{V}$; in particular the condition

$$\mathcal{V}(T, \bar{x}, \bar{u}, \bar{k}) = \Phi(\bar{x}, \bar{u}) \tag{4.12}$$

is equivalent to the optimality of any space–time control (t, u, v) with $t(s) \equiv T$ and $u(s) \equiv \bar{u}$. On the contrary, if (4.4) is satisfied as a strict inequality, set

$$\eta = \Phi(\bar{x}, \bar{u}) - \mathcal{V}(T, \bar{x}, \bar{u}, \bar{k}). \tag{4.13}$$

In order to show that $\mathcal{V}$ is a viscosity supersolution of (DPE) at $(T, \bar{x}, \bar{u}, \bar{k})$ we claim the existence of a sequence of controls $(t_n, u_n, v_n) \equiv (T, u_n, v_n)$ enjoying the following properties: i) there exists two positive constants δ, $\bar{n}$ such that

$$V_0^1(u_n) \geq \delta \qquad \forall n \geq \bar{n}; \tag{4.14}$$

ii) the trajectories $y_n \doteq y[\bar{t}, \bar{x}, \bar{u}; t_n, u_n, v_n]$ satisfy (4.9).

In order to prove this claim, assume by contradiction that for any minimizing sequence $\left((T, u_n, v_n)\right)_{n \in N}$ whose corresponding solutions satisfy (4.9) and for any $\delta > 0$, $\bar{n} > 0$ there exists a $n > \bar{n}$ such that

$$V_0^1(u_n) < \delta.$$

Then one can determine a subsequence, still denoted by $\left((T, u_n, v_n)\right)_{n \in N}$, such that the corresponding trajectories y_n satisfy

$$|y_n(s) - (T, \bar{x}, \bar{u})| \leq \sum_{i=1}^{m} \int_0^s |\bar{g}_i(y_n(s), v_n(s))| |u_{i\,n}'(s)| \, ds \leq M' m V_0^s(u_n) < M' m \delta.$$

Then, choosing δ such that $\omega_\Phi(mM'\delta) \leq \eta/4$, for any $n \geq 2/\sqrt{\eta}$ we obtain

$$\left|\Phi(y_n(1)) - \Phi(\bar{x}, \bar{u})\right| \leq \eta/4, \qquad 1/n^2 \leq \eta/4.$$

These inequalities and (4.9) provide a contradiction, for

$$\Phi(\bar{x}, \bar{u}) - \eta/2 \leq \Phi(y_n(1)) - 1/n^2 \leq \mathcal{V}(T, \bar{x}, \bar{u}, \bar{k}) = \Phi(\bar{x}, \bar{u}) - \eta.$$

Hence a sequence of controls (T, u_n, v_n) satisfying (4.14) exists, and the proof is completed by replacing $T - \bar{t}$ with δ in (4.11). □

We conclude this section by showing that (DPE) can be replaced by a quasi-variational inequality. We point out that the latter can be regarded as a generalization of the dynamic programming equation which was obtained in [8] in the special case where m is equal to 1, g_1 is independent of x and u, and C coincides with $[0, +\infty)$. Set

$$\bar{H}(t, x, u, p) \doteq \min\left\{H_1(t, x, u, p), H_2(t, x, u, p)\right\}, \tag{4.15}$$

where H_1, H_2 are defined by

$$\begin{aligned}
H_1(t, x, u, p) &\doteq \min_{v \in V} \left\{p_0 + \sum_{i=1}^{n} p_i g_0^i(t, x, u, v)\right\}, \\
H_2(t, x, u, p) &\doteq \min_{\substack{v \in V \\ |w|=1, w \in C}} \left\{p_\infty + \sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} p_i g_j^i(t, x, u, v) w^j + \sum_{j=1}^{m} p_{n+j} w^j\right\}.
\end{aligned} \tag{4.16}$$

THEOREM 4.2 (dynamic programming equation in the form of quasi-variational inequality). *Assume either hypothesis* (H2)$_C$ *or hypothesis* (H2)$_U$. *Then the following hold:*

a) $\mathcal{V}$ *is a viscosity solution of*

$$(\text{DPE})_{(\text{QVI})} \qquad -\tilde{H}(t,x,u,\nabla\mathcal{V}) = 0,$$

on Ω;

b) $\mathcal{V}$ *satisfies*

$$\mathcal{V}(t,x,u,k) \le \Phi(x,u) \qquad \forall (t,x,u,k) \in \partial_T\Omega;$$

c) $\mathcal{V}$ *is a viscosity supersolution of* (DPE)$_{(\text{QVI})}$ *on* $\partial'\Omega$ *and at any point* $(t,x,u,k) \in \partial_T\Omega$ *such that* $\mathcal{V}(t,x,u,k) < \Phi(x,u)$.

Proof. Since $H(t,x,u,p) \le \tilde{H}(t,x,u,p)$ for all $(t,x,u,p) \in \Omega\times\mathbb{R}^{1+n+m+1}$, by the fact that $\mathcal{V}$ is a viscosity subsolution of (DPE) on Ω it follows straightforwardly that $\mathcal{V}$ is a viscosity subsolution of (DPE)$_{(\text{QVI})}$ on Ω.

Now suppose that either $(\bar{y},\bar{k}) \doteq (\bar{t},\bar{x},\bar{u},\bar{k})$ belongs to $\Omega\cup\partial'\Omega$ or it belongs to $\partial_T\Omega$, and assume that $\mathcal{V}(\bar{t},\bar{x},\bar{u},\bar{k}) < \Phi(\bar{x},\bar{u})$. By Theorem 4.1 it follows that for any $\lambda \in C^\infty(\mathbb{R}^{1+n+m+1})$ such that $\mathcal{V}-\lambda$ has a local minimum on $\bar{\Omega}$ at $(\bar{y},\bar{k})$ and $\mathcal{V}(\bar{y},\bar{k}) = \lambda(\bar{y},\bar{k})$, there is a pair $(v,w) \in V\times B^m[0,1]\cap C$ satisfying

$$(4.17)\quad \big(\nabla_t\lambda(\bar{y},\bar{k}) + \nabla_x\lambda(\bar{y},\bar{k})g_0(\bar{t},\bar{x},\bar{u},v)\big)w_0 + \nabla_x\lambda(\bar{y},\bar{k})\sum_{i=1}^m g_i(\bar{t},\bar{x},\bar{u},v)w^i$$
$$+ \nabla_u\lambda(\bar{y},\bar{k})w + \nabla_k\lambda(\bar{y},\bar{k})|w| \le 0,$$

where $w_0 \doteq \sqrt{1-|w|^2}$. If $w=0$ or $|w|=1$, then $\mathcal{V}$ is a supersolution of (DPE)$_{(\text{QVI})}$. Otherwise, i.e., if $0<|w|<1$, divide (4.17) by $|w|$ and observe that either the first term or the sum of the remaining terms must be nonpositive. Hence $\mathcal{V}$ is a supersolution of (DPE)$_{(\text{QVI})}$. □

Remark 4.3. Theorem 4.2 exhibits a certain analogy of the considered problem with standard impulse control problems; see, e.g., [3], [8]. Indeed the value functions of the latter satisfy certain quasi-variational inequalities, which replace the usual Bellman equation. Actually, the dynamics considered in standard impulse theory can be considered as the simplest case of the dynamics considered in this paper, namely, the case where the vector fields $g_1,\dots,g_m$ are constant. Yet the comparison between the two approaches cannot be pushed further, for the two corresponding minimum problems are not equivalent. Instead, a more strict relation can be recognized between the problems considered here and the questions adressed in E. N. Barron and R. Jensen's paper [6], where (nonimpulsive) controls $\eta(\cdot)$ with bounded variation are considered. Indeed by adding the trivial (impulsive) equation $\dot{z} = \dot{\eta}$ the control system studied in [6] will be reduced to the form considered in this paper.

5. Uniqueness of the solution of (DPE) and verification theorem. In this section we prove a comparison result for viscosity solutions of (DPE). As a consequence we obtain a uniqueness result and a verification theorem for the extended problem $\mathcal{P}^e_{(t,x,u,k)}$.

We assume hypothesis (H3) below on the boundary of U. Hypothesis (H3), which excludes the presence of zero-amplitude corners in ∂U, is quite standard in problems involving state constraints; see, e.g., [40].

(H3) There exist a map $\eta \in BUC(U,\mathbb{R}^m)$ and two positive numbers q, r such that

$$B\big(u+t\eta(u), rt\big) \subset \mathring{U} \qquad \text{for } u\in\partial U \text{ and } 0<t\le q.$$

THEOREM 5.1. *Assume hypothesis* (H3) *and either* (H2)$_C$ *or* (H2)$_U$. *Let* $\mathcal{V}_1$ *be a bounded continuous viscosity subsolution of* (DPE) *in* Ω *which satisfies*

$$(5.1)\qquad \mathcal{V}_1(t,x,u,k) \le \Phi(x,u) \qquad \forall (t,x,u,k)\in\partial_T\Omega.$$

Let $\mathcal{V}_2$ *be a bounded continuous viscosity supersolution of* (DPE) *in* $\Omega\cup\partial'\Omega$ *such that for any* $(t,x,u,k)\in\partial_T\Omega$ *either* $\mathcal{V}_2$ *satisfies the inequality*

$$(5.2)\qquad \mathcal{V}_2(t,x,u,k) \ge \Phi(x,u)$$

or it is a viscosity supersolution of (DPE).

Then

$$(5.3)\qquad \mathcal{V}_1 \le \mathcal{V}_2 \qquad \text{on } \bar{\Omega}.$$

Proof. For every $(t,k)\in[0,T]\times[0,K]$ let us define the map $\mathcal{T}_{t,k}:\mathbb{R}^+\to\mathbb{R}$ by setting

$$\mathcal{T}_{t,k}(r) \doteq \frac{\log r}{1+t+k}.$$

Let M be a lower bound for the maps $\Phi, \mathcal{V}_1$, and $\mathcal{V}_2$, and let us set

$$Z_i(t,x,u,k) \doteq \mathcal{T}_{t,k}\big(\mathcal{V}_i(T-t,x,u,K-k) - M + 1\big), \qquad i=1,2,$$
$$\psi(t,x,u,k) \doteq \mathcal{T}_{t,k}(\Phi(x,u) - M + 1).$$

Then, on the one hand, the map Z_1 turns out to be a bounded continuous subsolution of

(TDPE)

$$Z + \max_{(v,w_0,w)\in V\times S^m_+}\left\{\frac{1+t+k}{w_0+|w|}\mathcal{H}(T-t,x,u,\nabla_t Z,-\nabla_x Z,-\nabla_u Z,\nabla_k Z,v,w_0,w)\right\} = 0$$

in Ω, where $\mathcal{H}$ is the unminimized Hamiltonian defined in (4.2); moreover Z_1 satisfies

$$Z_1(t,x,u,k) \le \psi(t,x,u,k)$$

on $\partial_0\Omega \doteq \{0\}\times\mathbb{R}^n\times U\times[0,K]$.

On the other hand, Z_2 is a bounded continuous supersolution of (TDPE) on $\bar{\Omega}\backslash\partial_0\Omega$. Furthermore, at each point $(0,x,u,k)\in\partial_0\Omega$, Z_2 either satisfies the inequality $Z_2(0,x,u,k)\ge\psi(0,x,u,k)$ or is a viscosity supersolution of (TDPE). Hence, a straightforward application of Theorem 1.1 in [2] implies that

$$Z_1 \le Z_2$$

on $\bar{\Omega}$, which in turn yields the thesis. □

THEOREM 5.2 (uniqueness). *Assume hypothesis* (H3) *and either* (H2)$_C$ *or* (H2)$_U$. *Then the value function* $\mathcal{V}$ *is the unique bounded continuous viscosity solution of* (DPE) *on* Ω *which satisfies the following boundary conditions:*

$(BC)'_1$ $\mathcal{V}$ *is a viscosity supersolution of* (DPE) *at all points of* $[0,T[\times\mathbb{R}^n\times\partial U\times[0,K[\,\cup[0,T[\times\mathbb{R}^n\times U\times\{K\}$;

$(BC)'_2$ *at each boundary point* (T,x,u,k) *one has* $\mathcal{V}(T,x,u,k)\le\Phi(x,u)$ *and, moreover, either* $\mathcal{V}$ *is a supersolution of* (DPE) *or it satisfies the relation* $\mathcal{V}(T,x,u,k) = \Phi(x,u)$.

Remark 5.1. By using the same arguments as in the previous theorem a uniqueness result for $(DPE)_{(QVI)}$ can be proved as well. Hence (DPE) and $(DPE)_{(QVI)}$ turn out to be equivalent as soon as one assumes the boundary conditions $(BC)'_1$, $(BC)'_2$. It is worthwhile comparing the latter conditions with the boundary conditions of Dirichlet type assumed by Barron, Jensen, and Menaldi [8] in the particular case when $m = 1$, $u \equiv k \in [0, K]$, g_1 is independent of (x, u), and $C = [0, +\infty)$. Barron–Jensen–Menaldi's conditions can be stated as follows:

$(BC)_1$ the map $\mathcal{V}$ coincides with the value function

$$h_T(\bar{x}, \bar{k}) \doteq \inf_{(T,u,v)\in\Gamma_{K-\bar{k}}(T,\bar{k})} \Phi(z[\bar{x}, \bar{k}; u, v](1))$$

on the strip $\{T\}\times\mathbb{R}^n\times[0, K)$, where $z[\bar{x}, \bar{k}; u, v](\cdot)$ is the solution of the *purely impulsive* (integrable) Cauchy problem

$$\begin{cases} z' = \bar{g}_1(T, v(s))u'(s), \\ z(0) = (\bar{x}, \bar{k}); \end{cases}$$

$(BC)_2$ the map $\mathcal{V}$ coincides with the value function

$$h_K(\bar{t}, \bar{x}) \doteq \inf_{v\in\mathcal{B}([\bar{t},T],V)} \Phi(x[\bar{t}, \bar{x}; v](T), K)$$

on the strip $[0, T) \times \mathbb{R}^n \times \{K\}$, where $x[\bar{t}, \bar{x}; v](\cdot)$ is the solution of the *nonimpulsive* Cauchy problem

$$\begin{cases} \dot{x} = g_0(t, x(t), K, v(t)), \\ x(\bar{t}) = \bar{x}. \end{cases}$$

In particular, in order to construct the maps h_T and h_K one needs solving a class of auxiliary optimization problems whose difficulty is often comparable to the difficulty of the original problem. Instead, conditions $(BC)'_1$, $(BC)'_2$ of Theorem 5.2 refer only to equation (DPE) and to the known function Φ (see also the example in §7).

We conclude this section with a verification theorem, which incidentally provides an answer—in the present, more general, framework—to the question posed by Barron, Jensen, and Menaldi [8] (see the introduction, question b)) about the relationship between optimal controls and dynamic programming equation.

THEOREM 5.3 (verification theorem). *Let $Z \in C(\overline{\Omega})$ be a bounded viscosity subsolution of* (DPE) *in Ω which satisfies the condition $Z \le \Phi$ on $\partial_T\Omega$. Then*

$$Z \le \mathcal{V} \quad \text{on } \overline{\Omega}. \tag{5.5}$$

Moreover, if for a given $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in \overline{\Omega}$ there exists a space-time control $(t, u, v) \in \Gamma_{K-\bar{k}}(\bar{t}, \bar{u})$ such that

$$\Phi(y[\bar{t}, \bar{x}, \bar{u}; t, u, v](1)) \le Z(\bar{t}, \bar{x}, \bar{u}, \bar{k}),$$

then the control (t, u, v) is optimal and

$$Z(\bar{t}, \bar{x}, \bar{u}, \bar{k}) = \mathcal{V}(\bar{t}, \bar{x}, \bar{u}, \bar{k}).$$

6. Costate, maximum principle, and gradient of the value function.

In ordinary control theory it is well known that the costate involved in the Pontryagin's maximum principle coincides—when no endpoint constraints are imposed—with the gradient of the value function evaluated along an optimal trajectory. More generally, if the value function is not differentiable at some point, the costate belongs to the supergradient of the value function; see, e.g., [7], [22].

It is clear that in order to prove an analogous result for an impulsive system we need to understand the behaviour of the costate in the presence of spatial jumps of the trajectory.

In the special case where $m = 1$ and g_1 is independent of (x, u) (and $C = [0, +\infty)$) the question is posed as an open problem in [8] (see the introduction, question a)). Since the problem with impulses has been reduced to a standard nonimpulsive control problem, under hypothesis $(H2)_C$ and by simply applying standard arguments (see [7], [22]), it is now easy to provide an answer to the above question in the general case treated in the present paper.

Throughout this section we assume that the vector fields $g_0, \dots, g_m$ and the map Φ are continuously differentiable with respect to the variables t, x and u.

We recall that the space–time Hamiltonian equations in the variables (y, k) and (p, p_k) have the form

$$\begin{aligned} y' &= \nabla_p\mathcal{H}(y, p, p_k, t', u', v), \\ k' &= \nabla_{p_k}\mathcal{H}(y, p, p_k, t', u', v), \\ p' &= -\nabla_y\mathcal{H}(y, p, p_k, t', u', v), \\ p'_k &= -\nabla_k\mathcal{H}(y, p, p_k, t', u', v), \end{aligned} \tag{6.1}$$

where $\mathcal{H}$ is the unminimized Hamiltonian introduced in §4. In components we have

$$\begin{cases} t' = t', \\ x' = g_0(t, x, u, v)t' + \sum_{j=1}^m g_j(t, x, u, v)u'_j, \\ u' = u', \\ k' = |u'|, \end{cases} \tag{6.2}$$

$$\begin{cases} p'_0 = -\langle p_x, \nabla_t g_0(t, x, u, v)t'\rangle - \sum_{j=1}^m \langle p_x, \nabla_t g_j(t, x, u, v)u'_j\rangle, \\ p'_x = -\langle p_x, \nabla_x g_0(t, x, u, v)t'\rangle - \sum_{j=1}^m \langle p_x, \nabla_x g_j(t, x, u, v)u'_j\rangle, \\ p'_u = -\langle p_x, \nabla_u g_0(t, x, u, v)t'\rangle - \sum_{j=1}^m \langle p_x, \nabla_u g_j(t, x, u, v)u'_j\rangle, \\ p'_k = 0. \end{cases} \tag{6.3}$$

Note that (6.2) is nothing but the control system (2.4) supplemented with the equation $k' = |u'|$.

By saying that a control $(t(\cdot), u(\cdot), v(\cdot))$ evolves instantaneously at a time $\bar{t} \in [0, T]$ we mean that the preimage $t^{-}(\bar{t})$ is a nondegenerate interval $[s_1, s_2]$ on which the component $u(\cdot)$ is not constant. Accordingly, one can compute the *jumps* at time $\bar{t}$ of both the state (y, k) and the costate (p, p_k) by solving the Hamiltonian equations (6.2), (6.3) on the interval $[s_1, s_2]$.

In order to state a maximum principle for the extended problem $\mathcal{P}^e_{(\bar{t},\bar{x},\bar{u},\bar{k})}$ in the unconstrained case defined by hypothesis $(\mathbf{H2})_C$ we recall that it is not restrictive to assume that the norm of the derivative $(\hat{t}'(s), \hat{u}'(s))$ is equal to the constant value $V_0^1(\hat{t}, \hat{u}) = \int_0^1 |(\hat{t}'(s), \hat{u}'(s))| ds$ almost everywhere in $[0,1]$; see the appendix.

THEOREM 6.1 (maximum principle). *Let us assume $(\mathbf{H2})_C$, i.e., $U = \mathbb{R}^m$. Fix $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0,T] \times \mathbb{R}^{n+m} \times [0,K]$ and let $(\hat{t}, \hat{u}, \hat{v})$ be an optimal control for the extended problem $\mathcal{P}^e_{(\bar{t},\bar{x},\bar{u},\bar{k})}$, with $|(\hat{t}', \hat{u}')| = L$ almost everywhere in $[0,1]$ for some positive constant L. Moreover denote the corresponding optimal trajectory by $(\hat{y}, \hat{k}) = (\hat{t}, \hat{x}, \hat{u}, \hat{k})$.*

Then there exists a costate map $(\hat{p}, \hat{p}_k) = (\hat{p}_0, \hat{p}_x, \hat{p}_u, \hat{p}_k) : [0,1] \to \mathbb{R}^{1+n+m+1}$ such that

i) *$(\hat{y}, \hat{k}, \hat{p}, \hat{p}_k)$ is a solution of the Hamiltonian equations (6.1) corresponding to the control $(\hat{t}, \hat{u}, \hat{v})$ and satisfies the boundary conditions*

$$(6.4)\qquad \begin{gathered}(\hat{t}(0), \hat{x}(0), \hat{u}(0), \hat{k}(0)) = (\bar{t}, \bar{x}, \bar{u}, \bar{k}),\\ \hat{p}_x(1) = \nabla_x \Phi(\hat{x}(1), \hat{u}(1)), \qquad \hat{p}_u(1) = \nabla_u \Phi(\hat{x}(1), \hat{u}(1)),\\ H(T, \hat{x}(1), \hat{u}(1), \hat{p}_0(1), \hat{p}_x(1), \hat{p}_u(1), \hat{p}_k(1)) = 0,\end{gathered}$$

where the Hamiltonian H is defined as in §2;

ii) *the minimum relation*

$$(6.5)\qquad \begin{aligned}&\mathcal{H}\left(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{p}_0(s), \hat{p}_x(s), \hat{p}_u(s), \hat{p}_k(s), \frac{\hat{t}'(s)}{L}, \frac{\hat{u}'(s)}{L}, v(s)\right)\\ &= H\left(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{p}_0(s), \hat{p}_x(s), \hat{p}_u(s), \hat{p}_k(s)\right)\end{aligned}$$

holds for almost every $s \in [0,1]$, and the equality

$$(6.6)\qquad H\left(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{p}_0(s), \hat{p}_x(s), \hat{p}_u(s), \hat{p}_k(s)\right) = 0$$

holds for all $s \in [0,1]$;

iii) *if $\hat{u}$ verifies $V_0^1(\hat{u}) < K - \bar{k}$, one has $\hat{p}_k = 0$ identically on $[0,1]$.*

This theorem, whose proof will be given after the statement of Theorem 6.2, is a straightforward consequence of the Pontryagin maximum principle when the latter is applied to the extended (nonimpulsive) control problem.

We point out that in the case when the fields g_i are independent of (x,u), some versions of the maximum principle already exist in the literature; see, e.g., [33], [36], [38], [42]. What is more, up to some formal changes a maximum principle for the general case considered here can be already found in [31]. Yet since our main goal is to establish a relationship between the costate and the value function, we prefer to give here a statement and a proof of the maximum principle in the theoretical framework introduced in the previous sections.

In order to state a relationship between the costate and the value function let us introduce the family of maps defined by

$$\begin{aligned}&\mathcal{V}^{t,k} : \mathbb{R}^{n+m} \to \mathbb{R}, \qquad (t,k) \in [0,T] \times [0,K],\\ &\mathcal{V}^{t,k}(x,u) \doteq \mathcal{V}(t,x,u,k) \qquad \forall (x,u) \in \mathbb{R}^{n+m}.\end{aligned}$$

Moreover, let us recall the definition of superdifferential of a continuous map.

DEFINITION 6.1. *Let f be a function in $C(\mathbb{R}^N)$. Then for every $x \in \mathbb{R}^N$ the subset $\nabla^+ f(x) \subset \mathbb{R}^N$ defined by*

$$\nabla^+ f(x) \doteq \left\{ p \in \mathbb{R}^N : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p\ ,\ y - x\rangle}{|y - x|} \leq 0 \right\}$$

is called the superdifferential *of f at x.*

In the statement of the following theorem $(\hat{t}, \hat{x}, \hat{u}, \hat{k})$ and $(\hat{p}, \hat{p}_k) = (\hat{p}_0, \hat{p}_x, \hat{p}_u, \hat{p}_k)$ have the same meaning as in Theorem 6.1.

THEOREM 6.2 (costate and value function). *For every $s \in [0,1]$ one has*

$$(6.7)\qquad (\hat{p}_x(s), \hat{p}_u(s)) \in \nabla^+ \mathcal{V}^{\hat{t}(s), \hat{k}(s)}(\hat{x}(s), \hat{u}(s)).$$

Proof of Theorem 6.1. This theorem follows straightforwardly from the Pontryagin maximum principle for nonimpulsive control systems. Indeed, thanks to the fact that we can restrict the family of controls to the subfamily formed by canonically parametrized control strategies, a control $(\hat{t}, \hat{u}, \hat{v})$ is optimal for problem $\mathcal{P}^e_{(\bar{t},\bar{x},\bar{u},\bar{k})}$ if and only if the control $(\hat{w}_0, \hat{w}, \hat{v}) \doteq (\hat{t}', \hat{u}', \hat{v})$ is optimal for the following ordinary control problem with endpoint constraints:

$$\text{minimize } \Phi(z[\bar{t}, \bar{x}, \bar{u}, \bar{k}; w_0, w, v](1))$$

over the trajectories $(z[\bar{t}, \bar{x}, \bar{u}, \bar{k}; (w_0, w, v)](\cdot)$ of

$$\begin{aligned}&z' = \tilde{g}_0(z(s), v(s)) w_0 + \sum_{i=1}^m \tilde{g}_i(z(s), v(s)) w_i(s),\\ &z_k' = |w|,\\ &(z(0), z_k(0)) = (\bar{t}, \bar{x}, \bar{u}, \bar{k}),\end{aligned}$$

satisfying the endpoint constraints

$$z_0(1) = T, \qquad z_k(1) \leq K$$

and corresponding to measurable control maps (w_0, w, v) from $[0,1]$ into

$$\{(w_0, w, v) \in [0, +\infty) \times C \times V : \quad |(w_0, w)| \leq T + K\}.$$

By applying the Pontryagin's maximum principle (see [26]) to this problem we obtain a statement which is equivalent to Theorem 6.1 except that relation (6.5) must be replaced by

$$(6.8)\qquad \begin{aligned}&\mathcal{H}\left(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{p}_0(s), \hat{p}_x(s), \hat{p}_u(s), \hat{p}_k(s), \frac{\hat{t}'(s)}{L}, \frac{\hat{u}'(s)}{L}, v(s)\right)\\ &= \min_{(w_0,w,v) \in B_+^{m+1}\left[\frac{T+K}{L}\right] \times V} \left\{ \mathcal{H}\left(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{p}_0(s), \hat{p}_x(s), \hat{p}_u(s), \hat{p}_k(s), w_0, w, v\right) \right\},\end{aligned}$$

where

$$B_+^{m+1}\left[\frac{T+K}{L}\right] \doteq \left\{ (w_0, w) \in [0, +\infty) \times C : |(w_0, w)| \leq \frac{T+K}{L} \right\}.$$

On the other hand, since $|(\hat{t}'(s), \hat{u}'(s))| = L$ almost everywhere, in the minimum relation (6.8) one can replace $B_+^{m+1}\left[\frac{T+K}{L}\right]$ with the set S_+^m. Hence (6.8) reduces to (6.5), and the theorem is proved. □

We observe that the proof of Theorem 6.2 cannot be derived directly from analogous results concerning nonimpulsive systems. Indeed, to our knowledge these results concern problems without endpoint constraints, while the trajectories of system (6.2) are subject to

$$\hat{t}(1) = T, \quad \hat{k}(1) \leq K.$$

However, since the coordinates $(\hat{x}(1), \hat{u}(1))$ are not constrained, the arguments we use to prove Theorem 6.2 are substantially the same as the ones used in the nonimpulsive case without endpoint constraints; see, e.g., [7], [22].

Proof of Theorem 6.2. By the definition of optimal trajectory there exists a measurable map $\hat{v}$ such that the solution corresponding to the space–time control $(\hat{t}, \hat{u}, \hat{v})$ coincides with the optimal trajectory $(\hat{t}, \hat{x}, \hat{u}, \hat{k})$. Let $s^* \in [0,1]$ and for every initial point $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ define the control map

$$\hat{u}_{\bar{u}} : [s^*, 1] \to \mathbb{R}^m, \qquad \hat{u}_{\bar{u}}(s) \doteq \hat{u}(s) + \bar{u} - \hat{u}(s^*).$$

Next consider the cost functional

$$J(\bar{x}, \bar{u}) \doteq \Phi(\hat{x}[\bar{x}, \bar{u}](1), \hat{u}_{\bar{u}}(1)),$$

where $\hat{x}[\bar{x}, \bar{u}](\cdot)$ denotes the solution on the interval $[s^*, 1]$ of the Cauchy problem

$$\begin{aligned} x' &= g_0\left(\hat{t}(s), x(s), \hat{u}_{\bar{u}}(s), \hat{v}(s)\right) \hat{t}'(s) + \sum_{i=1}^{m} g_i\left(\hat{t}(s), x(s), \hat{u}_{\bar{u}}(s), \hat{v}(s)\right) \hat{u}_i'(s), \\ x(s^*) &= \bar{x}. \end{aligned}$$

Up to a reparametrization from the interval $[s^*, 1]$ into the standard interval $[0,1]$, the control $(\hat{t}, \hat{u}_{\bar{u}}, \hat{v}) : [s^*, 1] \to \mathbb{R}^{1+m} \times V$ is feasible for the initial point $(\hat{t}(s^*), \bar{x}, \bar{u}, \hat{k}(s^*))$. Hence one has

$$\mathcal{V}\left(\hat{t}(s^*), \bar{x}, \bar{u}, \hat{k}(s^*)\right) = \mathcal{V}^{\hat{t}(s^*), \hat{k}(s^*)}(\bar{x}, \bar{u}) \le J(\bar{x}, \bar{u})$$

for every $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$, and, by the optimality of $(\hat{t}, \hat{u}, \hat{v})$,

$$J(\hat{x}(s^*), \hat{u}(s^*)) = \mathcal{V}^{\hat{t}(s^*), \hat{k}(s^*)}(\hat{x}(s^*), \hat{u}(s^*)).$$

Therefore by the definition of superdifferential of $\mathcal{V}^{\hat{t}(s^*), \hat{k}(s^*)}$ it is sufficient to prove that $J(\bar{x}, \bar{u})$ is differentiable at $(\hat{x}(s^*), \hat{u}(s^*))$ and satisfies

$$(p_x(s^*), p_u(s^*)) = \nabla_{x,u} J(\hat{x}(s^*), \hat{u}(s^*)). \tag{6.9}$$

By standard computations involving the differentiability of the solutions of (6.2) with respect to the initial data we obtain

$$\nabla_{x,u} J(\hat{x}(s^*), \hat{u}(s^*)) = \langle \nabla_{x,u} \Phi(\hat{x}(1), \hat{u}(1)) \;,\; Z(1) \rangle, \tag{6.10}$$

where the $(n+m) \times (n+m)$ matrix $Z(\cdot)$ is the solution in $[s^*, 1]$ of the variational Cauchy problem

$$\begin{aligned} Z'(s) &= \left\langle \left[\nabla_{x,u} g_0(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{v}(s)) \hat{t}'(s) \right.\right. \\ &\qquad \left.\left. + \sum_{i=1}^{m} \nabla_{x,u} g_i(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{v}(s)) \hat{u}_i'(s) \right], Z(s) \right\rangle, \\ Z(s^*) &= Id. \end{aligned}$$

Since $(\hat{p}_x, \hat{p}_u)$ coincides with the unique solution to the adjoint Cauchy problem

$$\begin{aligned} (p_x'(s), p_u'(s)) &= -\left\langle (p_x(s), p_u(s)), \left[\nabla_{x,u} g_0(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{v}(s)) \hat{t}'(s) + \right.\right. \\ &\qquad \left.\left. + \sum_{i=1}^{m} \nabla_{x,u} g_i(\hat{t}(s), \hat{x}(s), \hat{u}(s), \hat{v}(s)) \hat{u}_i'(s) \right] \right\rangle, \\ (p_x(1), p_u(1)) &= \nabla_{x,u} \Phi(\hat{x}(1), \hat{u}(1)), \end{aligned}$$

one has

$$\frac{d}{ds} \langle (\hat{p}_x(s), \hat{p}_u(s)), Z(s) \rangle = 0$$

on the whole interval $[s^*, 1]$, from which it follows that

$$(\hat{p}_x(s^*), \hat{p}_u(s^*)) = \langle (\hat{p}_x(s^*), \hat{p}_u(s^*)), Id \rangle = \langle \nabla_{x,u} \Phi(\hat{x}(s^*), \hat{u}(s^*)), Z(1) \rangle.$$

The latter equality and (6.10) yield (6.9), and the theorem is proved. □

7. Example. We apply the results proved in the previous sections to a simple minimum problem. In particular we check the optimality of a certain feedback control by showing that the corresponding cost function satisfies equation (DPE) and the boundary conditions $(BC)_1'$, $(BC)_2'$.

Let T, K, and c be positive constants, and for any $(\bar{t}, \bar{x}, \bar{k}) \in [0, T) \times \mathbb{R} \times [0, K]$ consider the minimum problem:

$$(\mathcal{P}_{(\bar{t}, \bar{x}, \bar{k})}) \qquad \text{minimize } \arctan(x(T))$$

over all the endpoints of the trajectories of

$$\begin{aligned} \dot{x} &= c + \dot{u}_1(t) + x\dot{u}_2(t) \qquad \forall t \in (\bar{t}, T], \\ x(\bar{t}) &= \bar{x}, \end{aligned} \tag{7.1}$$

corresponding to the absolutely continuous controls (u_1, u_2) satisfying $V_{\bar{t}}^T(u_1, u_2) \le K - \bar{k}$. Moreover assume that the derivatives $(\dot{u}_1, \dot{u}_2)$ belong (for a.e. $t \in [\bar{t}, T]$) to the closed cone

$$C \doteq \{(w_1, w_2) \in \mathbb{R}^2 : \; w_1 \le 0, \; w_2 \ge 0\}. \tag{7.2}$$

Following §2, let us consider the extended system relative to (7.1), (7.2):

$$\begin{aligned} x' &= ct'(s) + u_1'(s) + xu_2'(s) \qquad \forall s \in [0,1], \\ (t, x)(0) &= (\bar{t}, \bar{x}), \quad (u_1', u_2')(s) \in C \; \text{ for a.e. } s \in [0,1]. \end{aligned} \tag{7.3}$$

The form of the equation in (7.3) implies that the optimal solution of problem $\mathcal{P}_{(\bar{t}, \bar{x}, \bar{k})}$ —and hence the value function $\mathcal{V}$— is independent of the initial values $u_1(\bar{t})$ and $u_2(\bar{t})$ of the controls. Moreover, by the definition of space–time control one has $t'(s) \ge 0$ for a.e. $s \in [0,1]$. Since both $\dot{u}_1$ and $x\dot{u}_2$ are negative whenever x is negative, heuristics suggests the following strategy: at the initial time let the state jump to the minimum x reachable by spending all the available variation $K - \bar{k}$. After the jump set $\dot{u}_1 \equiv 0 \equiv \dot{u}_2$ and let the state evolve in time (with constant derivative equal to c). The maximum principle (6.5) yields

$$(w_0, w_1, w_2)(x,s) = \begin{cases} -(K-\bar{k}+T-\bar{t})(1+x^2)^{-1/2}(0,1,x) & \text{if } x<0, \\ -(K-\bar{k}+T-\bar{t})(0,1,0) & \text{if } x\geq 0,\ s\in[0, \frac{K-\bar{k}}{K-\bar{k}+T-\bar{t}}], \\ (K-\bar{k}+T-\bar{t})(1,0,0) & \text{if } x\geq 0,\ s\in[\frac{K-\bar{k}}{K-\bar{k}+T-\bar{t}}, 1], \end{cases} \tag{7.4}$$

as a control candidate to be optimal. Accordingly, for each initial condition $(\bar{t}, \bar{x}, \bar{k})$ the corresponding terminal position $\hat{x}(T;\bar{t},\bar{x},\bar{k})$ is given by

$$\hat{x}(T;\bar{t},\bar{x},\bar{k}) = \begin{cases} \sinh\big(\operatorname{arcsinh}(\bar{x}) - (K-\bar{k})\big) + c(T-\bar{t}), & \bar{x}\leq 0,\ 0\leq \bar{k}\leq K, \\ \sinh\big(\bar{x} - (K-\bar{k})\big) + c(T-\bar{t}), & \bar{x}>0,\ \bar{x}+\bar{k}<K,\ 0\leq\bar{k}\leq K, \\ \bar{x} - (K-\bar{k}) + c(T-\bar{t}), & \bar{x}>0,\ \bar{x}+\bar{k}\geq K,\ 0\leq \bar{k}\leq K, \end{cases}$$

so that the resulting cost is given by $\mathcal{V}(\bar{t},\bar{x},\bar{k}) \doteq \arctan\big(\hat{x}(T;\bar{t},\bar{x},\bar{k})\big)$. We claim that $\mathcal{V}(t,x,k)$ is the optimal cost; i.e., it coincides with the value function of problem $\mathcal{P}_{(t,x,k)}$. Since $\mathcal{V}$ is continuously differentiable on $\Omega \doteq (0,T)\times\mathbb{R}\times(0,K)$, on the basis of the uniqueness of Theorem 5.2 it is sufficient to verify that i) $\mathcal{V}$ is a classical solution on Ω of (DPE) equation

$$\min\left\{(\nabla_t\mathcal{V} + \nabla_x\mathcal{V}c)w_0 + \nabla_x\mathcal{V}(w_1 + xw_2) + \mathcal{V}_k\sqrt{w_1^2+w_2^2} :\ (w_0,w_1,w_2)\in S_+^2\right\} = 0; \tag{7.5}$$

ii) $\mathcal{V}$ is a viscosity supersolution of (DPE) on $\partial\Omega \setminus \big(\{T\}\times\mathbb{R}\times\{K\}\big)$ and satisfies

$$\mathcal{V}(T,x,K) = \arctan(x) \qquad \forall x\in\mathbb{R}. \tag{7.6}$$

Relations (7.5) and (7.6) can be checked by means of straightforward calculations. Hence it only remains to check the supersolution inequality for every $(t,x,k)\in\partial\Omega\setminus \big(\{T\}\times\mathbb{R}\times\{K\}\big)$. If $\lambda\in C^\infty(\overline{\Omega})$ is a map such that $\mathcal{V}-\lambda$ has a local minimum at (t,x,k), then it satisfies the following relations:

$$\nabla_t\lambda(t,x,k) \geq \nabla_t\mathcal{V}(t,x,k), \quad (\nabla_x\lambda, \nabla_k\lambda)(t,x,k) = (\nabla_x\mathcal{V}, \nabla_k\mathcal{V})(t,x,k) \quad \text{if } t=T$$

and

$$\nabla_k\lambda(t,x,k) \geq \nabla_k\mathcal{V}(t,x,k), \quad (\nabla_t\lambda, \nabla_x\lambda)(t,x,k) = (\nabla_t\mathcal{V}, \nabla_x\mathcal{V})(t,x,k) \quad \text{if } k=K.$$

Moreover observe that relation (7.5) holds at any $(t,x,k)\in\partial\Omega\setminus\big(\{T\}\times\mathbb{R}\times\{K\}\big)$ and the minimum in the right-hand side of (7.5) is achieved by a vector $(\bar{w}_0, \bar{w}_1, \bar{w}_2)$ verifying $\bar{w}_0 = 0$ if $t=T$ and $(\bar{w}_1,\bar{w}_2)=0$ if $k=K$. It follows that

$$(\nabla_t\lambda + \nabla_x\lambda c)\bar{w}_0 + \nabla_x\lambda(\bar{w}_1 + x\bar{w}_2) + \nabla_k\lambda\sqrt{\bar{w}_1^2+\bar{w}_2^2} = 0 \qquad \text{on } \partial\Omega\setminus\big(\{T\}\times\mathbb{R}\times\{K\}\big).$$

Hence $\mathcal{V}$ is a viscosity supersolution on $\partial\Omega\setminus\big(\{T\}\times\mathbb{R}\times\{K\}\big)$, so we can conclude that $\mathcal{V}$ coincides with the value function of problem $\mathcal{P}_{(t,x,k)}$ $\forall (t,x,k)\in\overline{\Omega}$. In particular the controls (7.4) are optimal.

Appendix.

Canonical parametrizations. We recall the notion of *canonical parametrization* from [32]. For this purpose, if (t,u) is not identically constant let us introduce the

map σ from $[0,1]$ onto itself defined by

$$\sigma(s) \doteq \frac{V_0^s(t,u)}{V_0^1(t,u)} = \frac{\int_0^s |(t',u')|ds}{\int_0^1 |(t',u')|ds}.$$

If (t,u) is constant on the whole interval $[0,1]$, we set

$$(t^c, u^c, v^c) \doteq (t,u,v).$$

If (t,u) is not constant, we set

$$(t^c,u^c,v^c)(\sigma) \doteq (t,u,v)(s), \qquad \sigma = \sigma(s). \tag{D}$$

In principle (D) defines a multivalued map. Yet (t^c, u^c) turns out to be univalued, while v^c is uniquely determined almost everywhere. More precisely we have the following proposition.

PROPOSITION A.1. *The relation* (D) *defines a Lipschitz-continuous map* (t^c,u^c) *on* $[0,1]$, *and the derivative* $(t^{c\prime}, u^{c\prime})$, *which exists almost everywhere, has constant norm equal to* $V_0^1(t,u)$. *Moreover* (D) *defines a univalued Borel-measurable map* v^c *almost everywhere in* $[0,1]$.

Thanks to Proposition 1 we can give the notion of *canonical parametrization.*

DEFINITION A.1. *The space–time control* (t^c,u^c,v^c) *defined by relation* (D) *is called the* canonical parametrization *of* (t,u,v).

DEFINITION A.2. *Let* (t_1,u_1,v_1), (t_2,u_2,v_2) *be two space–time controls and let* (t_1^c,u_1^c,v_1^c), (t_2^c,u_2^c,v_2^c) *be the corresponding canonical parametrizations. The space–time control* (t_1,u_1,v_1) *is called* equivalent *to* (t_2,u_2,v_2) *if* $(t_1^c,u_1^c)(s) = (t_2^c,u_2^c)(s)$ $\forall s\in[0,1]$ *and* $v_1^c(s) = v_2^c(s)$ *for almost every s in* $[0,1]$.

Proposition A.2 below illustrates the relationship between the trajectory $y[t,u,v]$ corresponding to a space–time control (t,u,v) and the trajectory $y[t^c,u^c,v^c]$ corresponding to the canonical parametrization (t^c,u^c,v^c) of (t,u,v).

PROPOSITION A.2. *Fix the initial condition* $y(0) = (t_1,x_1,u_1)$. *Then the trajectories* $y[t,u,v]$, $y[t^c,u^c,v^c]$ *satisfy the relation*

$$y[t^c,u^c,v^c](\xi) = y[t,u,v]\left(\sigma^{-}(\{\xi\})\right)$$

for every $\xi\in[0,1]$.

A pseudometric for space–time controls. The notion of canonical parametrization allows us to introduce a pseudometric δ^c on the space $\Gamma(\bar{t},\bar{u})$ of space–time controls. For every two space–time controls (t_1,u_1,v_1), (t_2,u_2,v_2) let us set

$$\delta^c\big((t_1,u_1,v_1),(t_2,u_2,v_2)\big) \doteq \|(t^c{}_1, u^c{}_1) - (t^c{}_2,u^c{}_2)\| + \|v_1^c - v_2^c\|_1,$$

where $\|\cdot\|$ and $\|\cdot\|_1$ denote the C^0 norm and the L^1 norm, respectively. In particular two space–time controls have δ^c pseudodistance equal to zero if and only if they are equivalent, so δ^c induces a metric on the quotient space.

The following density result was proved in [32].

PROPOSITION A.3. *Any set* $\Gamma^+_{K-\bar{k}}(\bar{t},\bar{u})$ *of regular controls is dense, in the topology induced by* δ^c, *in the corresponding set* $\Gamma_{K-\bar{k}}(\bar{t},\bar{u})$ *of space–time controls.*

Acknowledgments. The authors wish to thank Martino Bardi and Pierpaolo Soravia for their very useful bibliographical suggestions.

REFERENCES

[1] J. P. AUBIN AND A. CELLINA, *Differential Inclusion*, Springer-Verlag, Berlin, 1984.
[2] M. BARDI AND P. SORAVIA, *A comparison result for Hamilton–Jacobi equations and applications to some differential games lacking controllability*, Funkcial. Ekvac., 37 (1994), pp. 19–43.
[3] G. BARLES, *Deterministic impulse control problems*, SIAM J. Control Optim., 23 (1985), pp. 419–432.
[4] ———, *Uniqueness and regularity results for first-order Hamilton–Jacobi equations*, Indiana Math. J., 39 (1990), pp. 443–466.
[5] ———, *An approach of deterministic control problems with unbounded data*, Ann. Inst. Henri Poincaré Anal Non Linéaire, 7 (1990), pp. 235–258.
[6] E. N. BARRON AND R. JENSEN, *Optimal control problems with no turning back*, J. Differential Equations, 36 (1980), pp. 223–248.
[7] ———, *The Pontrjagin maximum principle from dynamic programming and viscosity solutions to first order partial differential equations*, Trans. Amer. Math. Soc., 298 (1986), pp. 635–641.
[8] E. N. BARRON, R. JENSEN, AND J. L. MENALDI, *Optimal control and differential games with measures*, Nonlinear Anal., 21 (1993), pp. 241–268.
[9] A. BRESSAN, *On differential systems with impulsive controls*, Rend. Sem. Mat. Univ. Padova, 78 (1987), pp. 227–236.
[10] A. BRESSAN AND F. RAMPAZZO, *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital. B(7), 2 (1988), pp. 641–656.
[11] ———, *Impulsive control systems with commutative vector fields*, J. Optim. Theory Appl., 71 (1991), pp. 67–83.
[12] ———, *Impulsive control systems without commutativity assumption*, J. Optim. Theory Appl., to appear.
[13] A. BRESSAN, *Hyperimpulsive motions and controllizable coordinates for Lagrangean systems*, Atti Accad. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Natur., 19 (1991), pp. 195–246.
[14] ———, *On some control problem concerning the ski and the swing*, Atti Accad. Naz. Lincei Mem. Cl. Sc. Fis. Mat. Natur., Series IX, 1 (1991), pp. 149–196.
[15] A. BRESSAN AND M. MOTTA, *Some optimization problems with a monotone impulsive character. Approximation by means of structural discontinuities*, Mem. Mat. Acc. Lincei, Series IX, 2 (1994), pp. 31–52.
[16] I. CAPUZZO-DOLCETTA AND P. L. LIONS, *Hamilton–Jacobi equations and state constrained problems*, Trans. Amer. Math. Soc., 318 (1990), pp. 643–668.
[17] C. W. CLARK, F. H. CLARKE, AND G. R. MUNRO, *The optimal exploitation of renewable resource stocks*, Econometrica, 48 (1979), pp. 25–47.
[18] M. G. CRANDALL, L. C. EVANS, AND P. L. LIONS, *Some properties of viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc. (1984), p. 282.
[19] G. DAL MASO AND F. RAMPAZZO, *On systems of ordinary differential equations with measures as controls*, Differential Integral Equations, 4 (1991), pp. 739–765.
[20] J. R. DORROH AND G. FERREYRA, *A multi-state, multi-control problem with unbounded controls*, SIAM J. Control Optim., 32 (1994), pp. 1322–1331.
[21] V. A. DYKHTA, *Impulse trajectory extension of degenerated optimal control problems*, in The Liapunov Functions Methods and Applications, P. Borne and V. Matrosov, eds., J. C. Baltzer AG, 1990, pp. 103–109.
[22] W. H. FLEMING AND H. M. SONER, *Controlled Markov Processes and Viscosity Solutions*, Springer–Verlag, New York, 1993.
[23] O. HÁJEK, Book review, Bull. Amer. Math. Soc., 12 (1985), pp. 272–279.
[24] H. ISHII, *A boundary value problem of the Dirichlet type for Hamilton–Jacobi equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) 16 (1989), pp. 105–135.
[25] D. F. LAWDEN, *Optimal Trajectories for Space Navigation*, Butterworth, London, 1963.
[26] E. B. LEE AND L. MARKUS, *Foundations of Optimal Control Theory*, SIAM Series in Applied Mathematics, John Wiley, New York, London, Sydney, 1977.
[27] P. L. LIONS, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, London, 1982.
[28] W. LIU AND H. J. SUSSMANN, *Limit of hight oscillatory controls and the approximation of general paths by admissible trajectories*, Proc. C.D.C. IEEE, 1991.
[29] J. P. MAREC, *Optimal Space Trajectories*, Elsevier, Amsterdam Oxford, 1979.
[30] B. M. MILLER, *Optimization of dynamic systems with a generalized control*, translated from

Avtomat. i Telemekh., 6 (1989), pp. 23–34.
[31] B. M. MILLER, *Conditions for the optimality in problems of generalized control. I. Necessary conditions for optimality*, translated from Avtomat. i Telemekh., 3 (1992), pp. 50–58.
[32] M. MOTTA AND F. RAMPAZZO, *Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls*, Differential Integral Equations, 8 (1995), pp. 269–288.
[33] L. W. NEUSTADT, *A general theory of minimum-fuel space trajectories*, SIAM J. Control, 3 (1965), pp. 317–356.
[34] F. RAMPAZZO, *Optimal impulsive controls with a constraint on the total variation*, in New Trends in Systems Theory, Progress in Systems and Control Theory, G. Conte, A. M. Perdon, and B. F. Wyman, eds., Springer-Verlag, Boston, MA, 1990, pp. 606–613.
[35] ———, *On the Riemannian structure of a Lagrangian system and the problem of adding time-dependent constraints as controls*, European J. Mech. A Solids, 10 (1991), pp. 405–431.
[36] R. W. RISHEL, *An extended Pontryagin principle for control systems whose control laws contain measures*, SIAM J. Control, 3 (1965), pp. 191–205.
[37] A. V. SARYCHEV, *Nonlinear systems with impulsive and generalized function controls*, Proc. Conf. on Nonlinear Synthesis, Sopron, Hungary, 1989.
[38] W. W. SCHMAEDEKE, *Optimal control theory for nonlinear differential equations containing measures*, SIAM J. Control, 3 (1965), pp. 231–279.
[39] S. P. SETHI, *Dynamic optimal control problems in advertising: a survey*, SIAM Rev., 19 (1977), pp. 685–725.
[40] H. M. SONER, *Optimal control with state-space constraints*, SIAM J. Control Optim., 24 (1986), pp. 552–561, 1110–1122.
[41] H. J. SUSSMANN, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probab., 6 (1978), pp. 17–41.
[42] R. B. VINTER AND M. F. L. PEREIRA, *A maximum principle for optimal processes with discontinuous trajectories*, SIAM J. Control Optim., 26 (1988), pp. 205–229.