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Comparison principles and Dirichlet problem for fully nonlinear degenerate equations of Monge–Ampère type

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Abstract. We study fully nonlinear partial differential equations of Monge–Ampère type involving the derivatives with respect to a family $\mathcal{X}$ of vector fields. The main result is a comparison principle among viscosity subsolutions, convex with respect to $\mathcal{X}$, and viscosity supersolutions (in a weaker sense than usual), which implies the uniqueness of solution to the Dirichlet problem. Its assumptions include the equation of prescribed horizontal Gauss curvature in Carnot groups. By the Perron method we also prove the existence of a solution either under a growth condition of the nonlinearity with respect to the gradient of the solution, or assuming the existence of a subsolution attaining continuously the boundary data, therefore generalizing some classical result for Euclidean Monge–Ampère equations.

Keywords. Monge–Ampère equation, subelliptic equations, fully nonlinear degenerate equations, viscosity solutions, Carnot group, horizontally convex functions.

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1 Introduction

For a given family of $C^{1,1}$ vector fields $\mathcal{X} = \{X_1, \dots, X_m\}$ in $\mathbb{R}^n$, $m \le n$, the $\mathcal{X}$-gradient and symmetrized $\mathcal{X}$-Hessian matrix of a function u are

$$D_{\mathcal{X}}u := (X_1u, \dots, X_mu), \quad (D^2_{\mathcal{X}}u)_{ij} := (X_iX_ju + X_jX_iu)/2.$$

The main examples we have in mind are the vector fields that generate a homogeneous Carnot group [12, 16], and in that case $D_{\mathcal{X}}u$ and $D^2_{\mathcal{X}}u$ are called, respectively, the horizontal gradient and the horizontal Hessian. We consider fully nonlinear partial differential equations of the form

$$-\det D^2_{\mathcal{X}}u + H(x, u, D_{\mathcal{X}}u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

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where $\Omega \subseteq \mathbb{R}^n$ is open and bounded and H is at least continuous and nonnegative. In the case when the vector fields are the canonical basis of $\mathbb{R}^n$, which we call the Euclidean case, this is a classical equation of Monge–Ampère type. We recall that in the Euclidean case the Monge–Ampère equations are elliptic on convex functions. These equations arise in several problems, mostly of differential geometry, and have a wide literature, especially on the regularity of solutions, see, e.g., the books [3, 4, 31, 32, 38, 47] and the papers [19, 21, 22, 28, 39–41, 51, 52]. For the recent applications to optimal transportation problems we refer to [1, 17, 53] and the references therein.

Partial differential equations with an elliptic structure relative to vector fields that do not span the whole space $\mathbb{R}^n$ are degenerate elliptic, often called subelliptic, see, e.g., the recent book of Bonfiglioli, Lanconelli and Uguzzoni [16] for a comprehensive survey of the linear theory. A theory of fully nonlinear subelliptic equations was started a few years ago by Bieske [13, 14] and Manfredi [11, 44] using viscosity methods, and the Monge–Ampère equation

$$-\det D^2_{\mathcal{X}} u + f(x) = 0 \quad \text{in } \Omega \tag{1.2}$$

was listed among the main examples, with $X_1, \dots, X_m$ generators of a given Carnot group. Moreover, a number of authors studied in the last five years several notions of convexity in Carnot groups [5, 26, 27, 30, 33, 34, 37, 42, 43, 49, 54], and one of their motivations was the connection with Monge–Ampère equations on such groups. However, little is known about them so far. We mention the comparison principle among smooth sub- and supersolutions of (1.2) in the Heisenberg group proved by Gutierrez and Montanari [33] (among other results).

The dependence on the gradient $D_{\mathcal{X}} u$ in H is motivated by various possible applications. A first interesting example is the subelliptic analogue of the prescribed Gauss curvature equation. Danielli, Garofalo and Nhieu [26] defined the horizontal Gauss curvature of the graph of a smooth function u on a Carnot group as

$$K_h(x) := \det(D^2_{\mathcal{X}} u)(1 + |D_{\mathcal{X}} u|^2)^{-\frac{m+2}{2}}.$$

In fact, this is the classical Gaussian curvature of the graph of the restriction of u to the horizontal plane passing through x. Moreover, Capogna, Pauls and Tyson [20] showed the connection of K_h with the second fundamental form of the graph of u. Then (1.1) becomes the prescribed horizontal Gauss curvature equation if

$$H(x, r, q) = k(x)(1 + |q|^2)^{\frac{m+2}{2}} \tag{1.3}$$

for a given continuous $k : \overline{\Omega} \to [0, +\infty[$. A different hypoelliptic Monge–Ampère-type equation was proposed in [50] for a financial problem. Finally, the extension of the theory of optimal transportation to the realm of sub-Riemannian manifolds was started recently by Ambrosio and Rigot [2] and Figalli and Rifford [29] and it might lead to equations of the form (1.1), or variants of it.

This paper is devoted to a study of the degenerate elliptic Monge–Ampère-type equations (1.1) within the theory of viscosity solutions, see [6, 18, 23, 24]. In particular, we establish the well-posedness of the Dirichlet problem under rather general conditions. The first part of the paper deals with comparison results among sub- and supersolutions, and the second part with the existence of solution by the Perron method.

The new difficulties we encounter for the comparison principles are three.

1. The PDE (1.1) is degenerate elliptic only on convex functions with respect to the vector fields $X_1, \dots, X_m$, briefly *$\mathcal{X}$-convex.* We say that an u.s.c. function u on $\overline{\Omega}$ is $\mathcal{X}$-convex if it satisfies $-D^2_{\mathcal{X}} u \le 0$ in Ω in viscosity sense, that is,

$$D^2_{\mathcal{X}}\varphi(x) \ge 0 \quad \text{for all } \varphi \in C^2(\Omega),\ x \in \operatorname{argmax}(u-\varphi). \tag{1.4}$$

This notion was introduced by Lu, Manfredi and Stroffolini [42] for the Heisenberg group under the name of v-convexity. It was extended recently to general C^2 vector fields by the first author and Dragoni [7], who proved the equivalence with the convexity along trajectories of the fields. In the case of Carnot groups, it coincides with the geometric notion of *horizontal convexity*, a fact proved under different assumptions by several authors [5, 37, 42, 43, 49, 54], see also [26] for connections with other notions. Our comparison results will concern an $\mathcal{X}$-convex viscosity subsolution of (1.1) and a viscosity supersolution defined with strictly $\mathcal{X}$-convex test functions. This is inspired by the treatment of the Euclidean case by Ishii and Lions [36] and is equivalent to comparing sub- and supersolutions of

$$\max\{-\lambda_{\min}(D^2_{\mathcal{X}}u), -\det D^2_{\mathcal{X}}u + H(x,u,D_{\mathcal{X}}u)\} = 0 \quad \text{in } \Omega,$$

where $\lambda_{\min}$ denotes the minimal eigenvalue.

In the classical case, convex functions are locally Lipschitz continuous with respect to the Euclidean norm, so there is an interior gradient bound for the subsolution. The corresponding property for $\mathcal{X}$-convex functions is the local Lipschitz continuity with respect to the Carnot–Carathéodory metric associated to the vector fields: this was proved in [26, 37, 42, 43, 49] for the generators of a Carnot group and in [7] for general fields.

2. The operator in (1.1) does not satisfy the standard structure conditions in viscosity theory, unless the vector fields are constant. To overcome this problem, for $H > 0$ we take the log of both terms in (1.1) and show that the new equation verifies the Lipschitz-type condition with respect to x of Crandall, Ishii and Lions [24] for *uniformly $\mathcal{X}$-convex* subsolutions, i.e., functions such that, for a $\gamma > 0$,

$-D^2_{\mathcal{X}}u + \gamma I \leq 0$ in Ω in viscosity sense. Our first main result states the comparison among semicontinuous sub- and supersolutions of equations of the form

$$-\log\det(D^2_{\mathcal{X}}u) + K(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{1.5}$$

provided that either K is strictly increasing in u or the subsolution is strict. Here K is not restricted to $\log H(x, u, D_{\mathcal{X}}u)$; it can be any degenerate elliptic operator involving the full gradient and Hessian Du, D^2u and satisfying the structure conditions of [24].

3. To cover the cases of H not strictly increasing in u, which is the most frequent in applications, and satisfying only $H \geq 0$, we need to perturb an $\mathcal{X}$-convex subsolution to a uniformly $\mathcal{X}$-convex strict subsolution. This was done in the Euclidean case in [36] and we adapted the method to several nonlinear subelliptic equations in [8]. We are able to perform this construction for equation (1.1) under an additional condition on the vector fields, namely

$$X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} \tau_{ij}(x)\frac{\partial}{\partial x_i}, \quad j = 1, \dots, m. \tag{1.6}$$

In this case, we say the vector fields are of *Carnot type*, because this property is satisfied by the generators of a Carnot group. However we do not need all the other rich properties of such generators, not even the Hörmander bracket generating condition.

We therefore get the following comparison principle, containing the Euclidean result of Ishii and Lions [36] as a special case.

Theorem 1.1. *Assume $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty[$ is continuous, nondecreasing in the second entry, and for all $R > 0$ there is L_R such that*

$$\left|H^{1/m}(x, r, q + q_1) - H^{1/m}(x, r, q)\right| \leq L_R|q_1| \tag{1.7}$$

for all $x \in \overline{\Omega}$, $|r| \leq R$, $|q| \leq R$ and $|q_1| \leq 1$. Suppose the vector fields $X_1, \dots, X_m \in C^2$ satisfy (1.6)*. Let $u : \overline{\Omega} \to \mathbb{R}$ be a bounded, $\mathcal{X}$-convex, u.s.c. subsolution of* (1.1) *and $v : \overline{\Omega} \to \mathbb{R}$ be a bounded l.s.c. supersolution of* (1.1)*. Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+. \tag{1.8}$$

Note that the result applies to the prescribed horizontal Gauss curvature equation (1.1), (1.3). We also get a comparison principle for (1.2) that extends to the viscosity context a result of Rauch and Taylor [48] in $W^{2,n}$, the first result for not necessarily convex supersolutions.

Theorem 1.1 implies that there is at most one $\mathcal{X}$-convex continuous viscosity solution of the equation (1.1) with prescribed boundary data

$$u = g \quad \text{on } \partial\Omega, \quad g \in C(\partial\Omega). \tag{1.9}$$

The existence of solutions to the Dirichlet problem can be studied by the Perron method, as adapted to viscosity solutions by Ishii [24]. It turns out to fit very well with our modified notions of sub- and supersolution. A byproduct is the following subelliptic version of a classical result by Caffarelli, Nirenberg and Spruck [19] and Lions [41] in the Euclidean setting.

Theorem 1.2. *Under the assumptions of Theorem* 1.1 *suppose that* $g \in C^2(\overline{\Omega})$ *is* $\mathcal{X}$*-convex and* Ω *is* $\mathcal{X}$*-convex, i.e., it is the sublevel set of a* C^2 $\mathcal{X}$*-convex function. Then the solvability of* (1.1), (1.9) *is equivalent to the existence of an* $\mathcal{X}$*-convex subsolution attaining continuously the boundary data.*

The construction of a subsolution with the desired properties requires further assumptions, as it is well known in the Euclidean case [31,41]. Our main existence result is the following extension of a theorem of Lions [41].

Theorem 1.3. *Besides the assumptions of Theorem* 1.1 *suppose*

$$H^{1/m}(x, \max_{\partial\Omega} g, p) \leq L|p| + M \quad \textit{for all } x \in \overline{\Omega},\ p \in \mathbb{R}^m. \tag{1.10}$$

Assume also that Ω *is uniformly* $\mathcal{X}$*-convex, i.e., the sublevel set of a* C^2 *uniformly* $\mathcal{X}$*-convex function. Then there is a unique* $\mathcal{X}$*-convex solution in* $C(\overline{\Omega})$ *of the Dirichlet problem* (1.1), (1.9).

The growth condition (1.10) rules out the prescribed Gauss curvature equation (1.3), where it is known that k must satisfy some compatibility conditions [31,41]. We have an existence result in this case for the Koranyi ball of the Heisenberg group if $k(x) \leq k_{\mathbb{H}}(x)$, where $k_{\mathbb{H}}$ is the horizontal Gauss curvature of the graph of the gauge w, i.e., $w(x) = |x|_{\mathbb{H}}^4$ and $|x|_{\mathbb{H}}$ is the homogeneous norm of the Heisenberg group, see formula (4.6) for the explicit expression of $k_{\mathbb{H}}$.

Some special cases of the comparison results proved here were announced in the note [10] and in the conference proceedings [9]. More precisely, [10] contains the statement of Theorem 1.1 in the case of Carnot groups and $H > 0$, with a few hints on the proof, and [9] gives a different proof of Theorem 2.18 below for H strictly increasing in u (which excludes (1.2) and the prescribed horizontal Gauss curvature equation) and in Carnot groups.

The Dirichlet problem for subelliptic fully nonlinear equations was studied by Bieske [13, 14], Bieske and Capogna [15], and Wang [55] for the Aronsson equations of the calculus of variations in L^∞, and by ourselves [8] and Cutri and Tchou [25] for Pucci-type and other Bellman–Isaacs equations. Almost nothing is known on the regularity of solutions of fully nonlinear subelliptic equations. This is a challenging subject for future research.

The paper is organized as follows. Section 2 is devoted to the definitions, the comparison principle for the equation (1.5) (and variants of it), and its applications to (1.1) if $H > 0$ and either the subsolution is strict or H is strictly increasing in r. In Section 3, we build strict subsolutions for vector fields of Carnot type and complete the proof of Theorem 1.1. Section 4 deals with the existence issue for the Dirichlet problem.

2 Comparison principles with strict subsolutions

2.1 Definitions

Let us consider equations of the form

$$\begin{cases} -G\big(\sigma^T(x)D^2u\,\sigma(x) + A(x, Du)\big) + K(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \\ G = \det \text{ or } G = \log\det, \end{cases} \tag{2.1}$$

where the set $\Omega \subseteq \mathbb{R}^n$ is open and bounded. We denote with S^n the set of the symmetric $n\times n$ matrices, with $\leq$ the usual partial order, with I the identity matrix, and with $\operatorname{tr} M$ the trace of a square matrix M. By $M > 0$ we denote any positive definite matrix. $\mathrm{USC}(\overline{\Omega})$ and $\mathrm{LSC}(\overline{\Omega})$ denote the sets of functions $\overline{\Omega} \to \mathbb{R}$ that are upper semicontinuous and lower semicontinuous, respectively. The assumptions on the data are the following:

$$\begin{cases} K : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R} \text{ is continuous,} \\ K(x, r, p, X) \leq K(x, s, p, Y) \quad \text{for all } r \leq s,\ Y \leq X,\ x \in \overline{\Omega}, \\ \qquad\qquad\qquad\qquad\qquad\qquad u \in \mathbb{R},\ p \in \mathbb{R}^n,\ X, Y \in S^n, \end{cases} \tag{2.2}$$

$$K\Big(y, r, \frac{x-y}{\varepsilon}, Y\Big) - K\Big(x, r, \frac{x-y}{\varepsilon}, X\Big) \leq \omega\Big(|x-y|\Big(1 + \frac{|x-y|}{\varepsilon}\Big)\Big) \tag{2.3}$$

for some modulus ω and all $\varepsilon > 0$, $x, y \in \overline{\Omega}$, $r \in \mathbb{R}$, $X, Y \in S^n$ satisfying

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{2.4}$$

Moreover, we assume that

$$\begin{cases} \sigma(x) \text{ is a Lipschitz continuous } n \times m \text{ matrix valued function on } \overline{\Omega} \\ \text{with } m \leq n, \end{cases} \tag{2.5}$$

and that

$$\begin{cases} A(x,p) \text{ is a continuous } m \times m \text{ matrix valued function on } \overline{\Omega} \times \mathbb{R}^n \\ \text{such that} \\ -C_1|x-y|(1+|p|)I \leq A(x,p) - A(y,p) \leq C_1|x-y|(1+|p|)I. \end{cases} \tag{2.6}$$

Definition 2.1. If $\Psi : \overline{\Omega} \times \mathbb{R}^n \times S^n \to S^m$ and $M \in S^m$, we say that u is a (viscosity) subsolution of the matrix inequality $\Psi(x, Du, D^2u) \leq M$ in Ω, if u is USC in Ω and $\Psi(x, D\phi(x), D^2\phi(x)) \leq M$ for all $\phi \in C^2(\Omega)$ and $x \in \operatorname{argmax}(u - \phi)$.

The definition of (viscosity) subsolution u of (2.1) is given in a standard way, as in [24] (see also the comments in Remark 2.4 below).

Definition 2.2. A function $u \in \mathrm{USC}(\overline{\Omega})$ is a (viscosity) subsolution of (2.1) with $G = \det$ or $G = \log\det$ if for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a maximum point at x_0 we have

$$\begin{aligned} &-G\big(\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + A(x_0, D\phi(x_0))\big) \\ &\qquad + K\big(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\big) \leq 0. \end{aligned} \tag{2.7}$$

The definition of (viscosity) supersolution v of (2.1) is modified by restricting the test functions to the C^2 functions ϕ with $\sigma^T D^2\phi\,\sigma + A > 0$ at points $x \in \arg\min(v - \phi)$. In the Euclidean case, this coincides with the definition given in [36]. (See also [51] for viscosity solutions of other prescribed curvature equations).

Definition 2.3. A function $v \in \mathrm{LSC}(\overline{\Omega})$ is a (viscosity) supersolution of (2.1) with $G = \det$ or $G = \log\det$ if for all $\phi \in C^2(\overline{\Omega})$ such that $v - \phi$ has a minimum point at x_0 and

$$\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + A(x_0, D\phi(x_0)) > 0, \tag{2.8}$$

we have

$$\begin{aligned} &-G\big(\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + A(x_0, D\phi(x_0))\big) \\ &\qquad + K\big(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)\big) \geq 0. \end{aligned} \tag{2.9}$$

The restriction of the test functions is motivated by the consistency with classical supersolutions. In fact, if v is smooth and $v - \phi$ has a minimum at x_0, then

$$\sigma^T D^2\phi\,\sigma + A(x_0, D\phi) \le \sigma^T D^2 v\,\sigma + A(x_0, Dv)$$

at x_0; if the left-hand side is not positive semidefinite, no inequality is ensured between the determinant of the two sides and therefore (2.9) may fail. For instance, suppose that m is even, $\sigma^T\sigma(x_0) > 0$, A is growing at most linearly in p, K is independent of the derivatives, and take $\phi(x) = v(x) - \alpha\frac{|x-x_0|^2}{2}$, with $\alpha > 0$. Then

$$\det\bigl(\sigma^T D^2\phi\,\sigma + A(x_0, D\phi)\bigr) \ge c_1(-\alpha)^m + c_2$$

at x_0 for suitable constants $c_1 > 0$ and c_2, the inequality (2.9) is violated for α large enough and so a classical supersolution v would not be a viscosity supersolution.

Remark 2.4. In the next sections, we will compare a supersolution of (2.1) in the sense of Definition 2.3 with a function u subsolution of (2.1) as in Definition 2.2 satisfying also the matrix inequality

$$-\bigl(\sigma^T(x)D^2u\,\sigma(x) + A(x, Du)\bigr) \le 0 \tag{2.10}$$

in the sense of Definition 2.1. This is equivalent to comparing sub- and supersolutions in the standard sense of [24] of the equation

$$\begin{aligned}\max\bigl\{&-\lambda_{\min}\bigl(\sigma^T D^2u\,\sigma + A(x, Du)\bigr),\\ &- G\bigl(\sigma^T D^2u\,\sigma + A(x, Du)\bigr) + K(x, u, Du, D^2u)\bigr\} = 0,\end{aligned}$$

where $\lambda_{\min}(Z)$ denotes the minimal eigenvalue of $Z \in S^m$. This is obvious for subsolutions, whereas for a supersolution v of the last equation and a standard test function ϕ, either (2.9) holds, or

$$\lambda_{\min}\bigl(\sigma^T(x_0)D^2\phi(x_0)\,\sigma(x_0) + A(x_0, D\phi(x_0))\bigr) \le 0$$

at $x_0 \in \arg\min(v - \phi)$, which is equivalent to Definition 2.3.

In the case $G = \log\det$, we will further restrict subsolutions to functions satisfying the matrix inequality

$$-\bigl(\sigma^T(x)D^2u\,\sigma(x) + A(x, Du)\bigr) \le -\gamma I$$

in the sense of Definition 2.1. Then the first term in (2.7) is well defined because the argument of G is a positive definite matrix.

The generality of the term A in (2.1) includes equations of the form

$$-\log\det(D^2u + A(x)) + f(x,u) = 0,$$

arising in problems of Riemannian geometry (see [3,19] and the references therein) and their counterparts involving non-commutative vector fields. However, in this paper we are mostly interested in subelliptic equations

$$-\det D^2_{\mathcal{X}}u + F(x,u,D_{\mathcal{X}}u,D^2_{\mathcal{X}}u) = 0 \quad \text{in } \Omega, \tag{2.11}$$

where $D_{\mathcal{X}}u = (X_1u,\dots,X_mu)$ is the intrinsic (or horizontal) gradient with respect to a given family of $C^{1,1}$ vector fields $X_1,\dots,X_m$, and

$$(D^2_{\mathcal{X}}u)_{ij} = \big(X_i(X_ju) + X_j(X_iu)\big)/2$$

is the symmetrized intrinsic Hessian. If we take the $n\times m$ $C^{1,1}$ matrix-valued function σ, defined in $\overline{\Omega}\subseteq\mathbb{R}^n$, whose columns σ^j are the coefficients of X_j, $j=1,\dots,m$, we see that, for any smooth u,

$$D_{\mathcal{X}}u = \sigma^T(x)Du, \quad D^2_{\mathcal{X}}u = \sigma^T(x)D^2u\,\sigma(x) + Q(x,Du), \tag{2.12}$$

where $Q(x,p)$ is an $m\times m$ matrix whose elements are

$$Q_{ij}(x,p) := \frac{1}{2}\big(D\sigma^j(x)\sigma^i(x) + D\sigma^i(x)\sigma^j(x)\big)\cdot p. \tag{2.13}$$

Therefore the PDE (2.11) can be written in the form (2.1) with

$$A = Q, \quad G = \det, \quad K(x,r,p,X) = F\big(x,r,\sigma^Tp,\sigma^TX\sigma + Q\big).$$

In this case, the functions satisfying the matrix inequality (2.10) are called $\mathcal{X}$-convex, consistently with the theory of convex functions in Carnot groups [37,42] and in general Carnot–Carathéodory metric spaces [7].

Definition 2.5. $u\in \mathrm{USC}(\overline{\Omega})$ is convex in Ω with respect to the fields $X_1,\dots,X_m$, briefly $\mathcal{X}$-convex (resp., uniformly $\mathcal{X}$-convex), if it is a subsolution of

$$-D^2_{\mathcal{X}}u = -\sigma^T(x)D^2u\,\sigma(x) - Q(x,Du) \le 0 \quad \text{in } \Omega \tag{2.14}$$

(resp., $\le -\gamma I$ for some $\gamma > 0$).

Remark 2.6. Note that for a uniformly $\mathcal{X}$-convex subsolution of (2.11) the test functions can be restricted to C^2 strictly $\mathcal{X}$-convex functions (i.e., satisfying (2.8)), as for supersolutions.

2.2 The basic comparison principle

The first result is a comparison principle between a supersolution and a strict subsolution such that $-\sigma^T D^2 u\, \sigma - A(x, Du) \le -\gamma I$ of the equation

$$-\log\det\big(\sigma^T(x)D^2u\,\sigma(x) + A(x, Du)\big) + K(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{2.15}$$

Note that K is not strictly increasing with respect to the entry u.

Theorem 2.7. *Assume that conditions* (2.2) *through* (2.6) *hold. Let* $u \in \mathrm{USC}(\overline{\Omega})$ *be a bounded subsolution, for some* $\gamma, \gamma_1 > 0$, *of*

$$-\sigma^T(x)D^2u\,\sigma(x) - A(x, Du) \le -\gamma I \quad \textit{in } \Omega, \tag{2.16}$$

and

$$-\log\det\big(\sigma^T(x)D^2u\,\sigma(x) + A(x, Du)\big) + K(x, u, Du, D^2u) \le -\gamma_1 \quad \textit{in } \Omega. \tag{2.17}$$

Let $v \in \mathrm{LSC}(\overline{\Omega})$ *be a bounded supersolution of* (2.15). *Then*

$$\sup_{\Omega}(u - v) \le \max_{\partial\Omega}(u - v)^+. \tag{2.18}$$

To prove the comparison principle we need the following two lemmata.

Lemma 2.8. *If* $\gamma > 0$, *for all* $A \in S^N$, $A \ge \gamma I$,

$$\log\det(A) = \min\big\{N\log a - N + \mathrm{tr}(AM) : a > 0,\ M \in S^N,\ 0 \le M \le \tfrac{1}{\gamma} I,\ \det M = a^{-N}\big\}. \tag{2.19}$$

Proof. It is well known that

$$(\det A)^{1/N} = \min\big\{\mathrm{tr}(AB) : B \in S^N,\ B \ge 0,\ \det B = N^{-N}\big\}, \tag{2.20}$$

and the minimum is attained at $B_m = \frac{(\det A)^{1/N}}{N} A^{-1}$. On the other hand

$$\log[(\det A)^{1/N}] = \min\Big\{\log a + \frac{(\det A)^{1/N} - a}{a} : a > 0\Big\},$$

and the minimum is attained at $a_m = (\det A)^{1/N}$. We combine the two formulas to get a minimum representation for $\log[(\det A)^{1/N}]$ and we can restrict the search for the minimum to matrices of the form $B = \frac{a}{N} M$ with $M^{-1} \ge \gamma I$. Then

$$\frac{1}{N}\log\det(A) = \min\Big\{\log a - 1 + \frac{\mathrm{tr}(A\frac{a}{N}M)}{a} : a > 0,\ M \in S^N,\ 0 \le M \le \tfrac{1}{\gamma} I,\ a^N \det M = 1\Big\},$$

which gives (2.19). ◻

Lemma 2.9. *Consider the operator*

$$F(x,p,X) := -\log\det\big(\sigma^T(x)X\,\sigma(x) + A(x,p)\big)$$

with $x \in \overline{\Omega}$, $p \in \mathbb{R}^n$, $X \in S^n$, σ *and* A *satisfying* (2.5), (2.6). *Then for all* $\gamma > 0$ *there is a constant* $C > 0$ *such that*

$$F\Big(y, \frac{x-y}{\varepsilon}, Y\Big) - F\Big(x, \frac{x-y}{\varepsilon}, X\Big) \le C\Big(|x-y| + \frac{|x-y|^2}{\varepsilon}\Big) \tag{2.21}$$

for all $X, Y \in S^n$ *satisfying* (2.4) *and*

$$\begin{cases} \sigma^T(x)X\,\sigma(x) + A\Big(x, \dfrac{x-y}{\varepsilon}\Big) \ge \gamma I, \\ \sigma^T(y)Y\,\sigma(y) + A\Big(y, \dfrac{x-y}{\varepsilon}\Big) \ge \gamma I. \end{cases} \tag{2.22}$$

Proof. By Lemma (2.8) and (2.22) we can write $F(x,p,X)$ as the maximum of

$$m - m\log a - \mathrm{tr}\big(\sigma^T(x)X\,\sigma(x)M\big) - \mathrm{tr}(A(x,p)M)$$

as a, M vary over $a > 0$, $M \in S^m$, $0 \le M \le \frac{1}{\gamma}I$, $\det M = a^{-m}$. Then there is a choice of a and M such that the left-hand side of (2.21) is bounded above by the sum of

$$\mathrm{tr}\big(\sigma^T(x)X\,\sigma(x)M\big) - \mathrm{tr}\big(\sigma^T(y)Y\,\sigma(y)M\big), \tag{2.23}$$

and

$$\mathrm{tr}(A(x,p)M) - \mathrm{tr}(A(y,p)M), \quad p = \frac{x-y}{\varepsilon}. \tag{2.24}$$

By diagonalization we see that $|M| \le \sqrt{m}/\gamma$, where $|\cdot|$ denotes the Euclidean norm. Moreover there is $R \in S^m$ such that $M = RR^T$, $|R| \le \sqrt{\sqrt{m}/\gamma}$. We call $\Sigma(x)$ the $n \times n$ matrix whose first lines are $R\sigma^T(x)$ and the last $m-n$ lines are 0. Then (2.23) can be rewritten as

$$\mathrm{tr}\big(\Sigma^T(x)\Sigma(x)X\big) - \mathrm{tr}\big(\Sigma^T(y)\Sigma(y)Y\big).$$

A standard calculation in the theory of viscosity solutions (see, e.g., [24, Example 3.6]) shows that this quantity is bounded above by $3L^2|x-y|^2/\varepsilon$ for matrices satisfying (2.4), where L is a Lipschitz constant of $\Sigma(\cdot)$. Therefore we can take $L = L_\sigma\sqrt{m/\gamma}$ where L_σ is a Lipschitz constant for $\sigma(\cdot)$. As for (2.24),

$$\begin{aligned} \big|\mathrm{tr}(A(x,p) - A(y,p))M\big| &\le |A(x,p) - A(y,p)||M| \\ &\le \frac{C_1\sqrt{m}}{\gamma}|x-y|(1+|p|). \end{aligned}$$

In conclusion, we get

$$F\Big(y, \frac{x-y}{\varepsilon}, Y\Big) - F\Big(x, \frac{x-y}{\varepsilon}, X\Big) \le \frac{C_1\sqrt{m}}{\gamma}|x-y| + \frac{3L_\sigma^2 m + C_1\sqrt{m}}{\gamma}\,\frac{|x-y|^2}{\varepsilon}. \qquad \square$$

We can prove now the comparison theorem.

Proof of Theorem 2.7. For $\varepsilon > 0$ the function $\Phi_\varepsilon(x, y) = u(x)-v(y)-\frac{1}{2\varepsilon}|x-y|^2$ has a maximum point $(x_\varepsilon, y_\varepsilon)$. A standard argument gives

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0, \quad \text{as } \varepsilon \to 0^+. \tag{2.25}$$

If there is a sequence $\varepsilon_k \to 0$ such that $x_{\varepsilon_k} \to \hat{x} \in \partial\Omega$, then $y_{\varepsilon_k} \to \hat{x}$, and by the upper semicontinuity of $u(x) - v(y)$, we get

$$\max_{\overline{\Omega}}(u-v) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \to \max_{\partial\Omega}(u-v), \quad \text{as } \varepsilon \to 0.$$

The case of $y_{\varepsilon_j} \to \hat{y} \in \partial\Omega$ for some $\varepsilon_j \to 0$ is analogous. Therefore we are left with the case $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for all small ε. We use the theorem on sums, as in [23] and get $X, Y \in S^n$ (depending on ε) such that, for $p_\varepsilon := |x_\varepsilon - y_\varepsilon|/\varepsilon$,

$$(u(x_\varepsilon), p_\varepsilon, X) \in \overline{J}^{2,+}u(x_\varepsilon), \quad (v(y_\varepsilon), p_\varepsilon, Y) \in \overline{J}^{2,-}v(y_\varepsilon),$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{2.26}$$

Then (2.16) implies

$$G(x_\varepsilon, X) := \sigma^T(x_\varepsilon)X\,\sigma(x_\varepsilon) + A(x_\varepsilon, p_\varepsilon) \ge \gamma I.$$

We seek a similar inequality for $G(y_\varepsilon, Y) := \sigma^T(y_\varepsilon)Y\,\sigma(y_\varepsilon) + A(y_\varepsilon, p_\varepsilon)$. To this end we multiply on the left the second inequality in (2.26) by the $m \times 2n$ matrix whose first n columns are $\sigma^T(x_\varepsilon)$ and the last n are $\sigma^T(y_\varepsilon)$, and then on the right by the transpose of such matrix. Since the operation preserves the inequality, we get

$$\begin{aligned}\sigma^T(x_\varepsilon)X\sigma(x_\varepsilon) - \sigma^T(y_\varepsilon)Y\sigma(y_\varepsilon) &\le \frac{3}{\varepsilon}(\sigma(x_\varepsilon) - \sigma(y_\varepsilon))^T(\sigma(x_\varepsilon) - \sigma(y_\varepsilon)) \\ &\le \frac{3}{\varepsilon}C_\sigma|x_\varepsilon - y_\varepsilon|^2 I, \end{aligned} \tag{2.27}$$

where C_σ is a suitable constant related to the Lipschitz constant of σ. Then, by (2.27) and assumptions (2.6),

$$\begin{aligned} G(y_\varepsilon, Y) &\geq G(x_\varepsilon, X) - \frac{3}{\varepsilon} C_\sigma |x_\varepsilon - y_\varepsilon|^2 I + A(y_\varepsilon, p_\varepsilon) - A(x_\varepsilon, p_\varepsilon) \\ &\geq \Big(\gamma - \frac{3}{\varepsilon} C_\sigma |x_\varepsilon - y_\varepsilon|^2 - C_1 |x_\varepsilon - y_\varepsilon| - C_1 \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}\Big) I \geq \frac{\gamma}{2} I \end{aligned}$$

for ε small enough, by (2.25). Now we use the facts that u satisfies (2.17) and that v is a supersolution to get

$$\begin{cases} -\log\det\big(\sigma^T(x_\varepsilon) X\, \sigma(x_\varepsilon) + A(x_\varepsilon, p_\varepsilon)\big) + K(x_\varepsilon, u(x_\varepsilon), p_\varepsilon, X) \leq -\gamma_1 < 0, \\ -\log\det\big(\sigma^T(y_\varepsilon) Y\, \sigma(y_\varepsilon) + A(y_\varepsilon, p_\varepsilon)\big) + K(y_\varepsilon, v(y_\varepsilon), p_\varepsilon, Y) \geq 0. \end{cases} \tag{2.28}$$

If $u(x_\varepsilon) \leq v(y_\varepsilon)$ for some ε, we conclude that

$$\max_{\overline{\Omega}} (u - v) \leq u(x_\varepsilon) - v(y_\varepsilon) \leq 0.$$

Otherwise, by the monotonicity of K with respect to the second entry r, we get

$$-\log\det\big(\sigma^T(y_\varepsilon) Y\, \sigma(y_\varepsilon) + A(y_\varepsilon, p_\varepsilon)\big) + K(y_\varepsilon, u(x_\varepsilon), p_\varepsilon, Y) \geq 0.$$

Now we subtract this inequality from the first of (2.28), we use Lemma 2.9 and the structure condition on K to obtain

$$C|x_\varepsilon - y_\varepsilon| \Big(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\Big) + \omega\Big(|x_\varepsilon - y_\varepsilon| \Big(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\Big)\Big) \leq -\gamma_1 < 0$$

which gives a contradiction as $\varepsilon \to 0^+$, by (2.25). □

Remark 2.10. If $K = K(x, p, X)$ is independent of r, the previous proof shows that

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} (u - v).$$

Remark 2.11. Theorem 2.7 remains true if we relax the strict subsolution condition (2.17) to the following: for any open Ω_1 such that $\overline{\Omega_1} \subseteq \Omega$ there exists $\gamma_1 > 0$ such that

$$-\log\det\big(\sigma^T(x) D^2 u\, \sigma(x) + A(x, Du)\big) + K(x, u, Du, D^2 u) \leq -\gamma_1 \quad \text{in } \Omega_1$$

holds. The proof is the same, because if no sequence x_{ε_k} or y_{ε_j} converges to a boundary point then $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega_1} \times \overline{\Omega_1}$ for some $\overline{\Omega_1} \subseteq \Omega$ and for all ε small enough.

Theorem 2.12. *The conclusion of Theorem* 2.7 *remains true if* u *is a subsolution of* (2.15) *instead of a strict subsolution* (2.17)*, provided that, for some* $C > 0$*,*

$$K(x, r, p, X) - K(x, s, p, X) \geq C(r - s), \quad -M \leq s \leq r \leq M,$$
$$M := \max\{\|u\|_\infty, \|v\|_\infty\},\ x \in \overline{\Omega},\ p \in \mathbb{R}^n,\ X \in S^n.$$

Under this condition there is at most one viscosity solution u *of* (2.15) *such that* $-\sigma^T(x)D^2u\,\sigma(x) - A(x, Du) \leq -\gamma I$ *with prescribed continuous boundary data.*

Proof. It is a standard variant of the preceding one. □

Remark 2.13. Note that, if we consider Monge–Ampère equations without log, the structure condition (2.21) can be not true. Take for example

$$\hat{F}(x, p, X) := -\det\bigl(\sigma^T(x)X\,\sigma(x) + A(x, p)\bigr)$$

with $A \equiv 0$ and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2y & -2x \end{pmatrix},$$

the matrix associated to the Heisenberg group. If X and Y satisfy the matrix inequality (2.4), then (2.27) holds. If we take Y diagonal, it is easy to see that

$$\det(M(y_\varepsilon, Y) + \lambda I) = \det M(y_\varepsilon, Y) + \lambda^2 + \lambda \operatorname{tr} M(y_\varepsilon, Y),$$

where $M(y_\varepsilon, Y) := \sigma^T(y_\varepsilon)Y\,\sigma(y_\varepsilon)$. Then from (2.27), taking $\lambda = 3C_\sigma \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}$,

$$\begin{aligned} \det M(y_\varepsilon, Y) - \det M(x_\varepsilon, X) &\geq \det M(y_\varepsilon, Y) - \det(M(y_\varepsilon, Y) + \lambda I) \\ &\geq -\lambda^2 - \lambda \operatorname{tr} M(y_\varepsilon, Y). \end{aligned}$$

The term $\lambda \operatorname{tr} M(y_\varepsilon, Y)$ does not necessarily tend to zero as $\varepsilon \to 0$. Taking, for example, $v(y)$ Lipschitz continuous, since the function $\Phi_\varepsilon(x, y) = u(x) - v(y) - \frac{1}{2\varepsilon}|x - y|^2$, introduced in the proof of Theorem 2.7, has a maximum point in $(x_\varepsilon, y_\varepsilon)$, we deduce $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \Phi_\varepsilon(x_\varepsilon, x_\varepsilon)$, i.e. $|v(x_\varepsilon) - v(y_\varepsilon)| \geq \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2$. From the Lipschitz continuity of v with constant L, we obtain $|x_\varepsilon - y_\varepsilon| \leq 2\varepsilon L$. In this case, since $\operatorname{tr} M(y_\varepsilon, Y) \leq \frac{C}{\varepsilon}$ for $C > 0$, the best one can say is that

$$\lambda \operatorname{tr} M(y_\varepsilon, Y) \leq K\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \leq 4KL^2$$

for a suitable $K > 0$.

2.3 Monge–Ampère equations with bounded right-hand side

In this section, we apply the basic comparison principle to equations of the form

$$-\det D_{\mathfrak{X}}^2 u + F(x, u, D_{\mathfrak{X}} u, D_{\mathfrak{X}}^2 u) = 0 \quad \text{in } \Omega, \tag{2.29}$$

introduced in Section 2.1, where $D_{\mathfrak{X}} u$ and $D_{\mathfrak{X}}^2 u$ are, respectively, the intrinsic gradient and the symmetrized Hessian with respect to the vector fields $X_1, \ldots, X_m \in C^{1,1}$.

Lemma 2.14. *Let* $0 < F(x, u, p, X) \leq C_1$, *for any* $x \in \overline{\Omega}$, $u \in \mathbb{R}$, $p \in \mathbb{R}^m$ *and* $X \in S^m$. *Let* $w \in \mathrm{USC}(\overline{\Omega})$ *be a uniformly* $\mathfrak{X}$*-convex bounded function satisfying*

$$-\det D_{\mathfrak{X}}^2 w + F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w) \leq -\alpha < 0 \quad \text{in } \Omega.$$

Then there is $\alpha_1 > 0$ *such that*

$$-\log\det(D_{\mathfrak{X}}^2 w) + \log F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w) \leq -\alpha_1 < 0 \quad \text{in } \Omega. \tag{2.30}$$

Proof. By the properties of log, we have

$$\begin{aligned} &\log\big(F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w) + \alpha\big) \\ &\quad = \log F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w) + \log\Big(1 + \frac{\alpha}{F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w)}\Big) \\ &\quad \geq \log F(x, w, D_{\mathfrak{X}} w, D_{\mathfrak{X}}^2 w) + \log\Big(1 + \frac{\alpha}{C_1}\Big), \end{aligned}$$

which gives (2.30) with $\alpha_1 = \log(1 + \frac{\alpha}{C_1})$. □

Corollary 2.15. *Suppose that* $0 < F(x, u, p, X) \leq C_1$, $K = \log F$ *satisfies* (2.2) *and* (2.3)*, and that the* $X_1, \ldots, X_m$ *are* $C^{1,1}$ *in* $\overline{\Omega}$. *Then the comparison principle holds between a uniformly* $\mathfrak{X}$*-convex strict subsolution* u *and a supersolution* v *of equation* (2.29).

Proof. We apply Theorem 2.7 with $K = \log F$ and $A = Q$ given by (2.13), noting that the strict subsolution of (2.29) is a strict subsolution of (2.15) by Lemma 2.14. □

From Theorem 2.12 we immediately get:

Corollary 2.16. *The comparison principle is true if* σ *is a* $C^{1,1}$ $n \times m$ *matrix valued function,* $F > 0$, $K = \log F$ *satisfies* (2.2) *and* (2.3)*, and* u *is a uniformly*

$\mathcal{X}$-convex subsolution of (2.29)*, not necessarily strict, provided that, for some* $C > 0$,

$$F(x, r, p, X) - F(x, s, p, X) \geq C(r - s),$$
$$-M \leq s \leq r \leq M, \quad M := \max\{\|u\|_\infty, \|v\|_\infty\},$$

for all $x \in \overline{\Omega}$*,* $p \in \mathbb{R}^m$*,* $X \in S^m$*.*

Note that here the upper bound for F is not required.

2.4 Equations with unbounded gradient terms

In this section, we prove a comparison result for the equation

$$-\log\det(D^2_{\mathcal{X}} u) + K_1(x, u, D_{\mathcal{X}} u) = 0 \tag{2.31}$$

with Hamiltonian K_1 unbounded but independent of the second derivatives. In this case, we can exploit the boundedness of $D_{\mathcal{X}} u$ for any $\mathcal{X}$-convex function u to decrease the assumptions on the Hamiltonian $K(x, r, p) = K_1(x, r, \sigma^T(x)p)$. The gradient estimate in viscosity sense of the next proposition was proved very recently by the first author and Dragoni for general vector fields [7]. In the special case of Carnot groups, various authors showed, under different assumptions, the Lipschitz continuity of $\mathcal{X}$-convex functions with respect to the intrinsic metric of the group and bounds on their horizontal gradient in the sense of distributions [26, 37, 42, 43, 49]. From those results one can obtain a short proof of the gradient bound in viscosity sense, which we give for the convenience of readers mostly interested in the Carnot group setting (see Section 3.1 for the definitions).

Proposition 2.17 ([7]). *Let the vector fields* $X_1, \ldots, X_m$ *be of class* C^2 *and* u *be* $\mathcal{X}$*-convex and bounded in* $\overline{\Omega}$*. Then, for every open* Ω_1 *with* $\overline{\Omega}_1 \subseteq \Omega$*, there exists a constant* C *such that*

$$|\sigma^T(x) Du| \leq C \quad \text{in } \Omega_1$$

in viscosity sense.

Proof in the case of Carnot groups. It is known that $\mathcal{X}$-convexity implies local Lipschitz continuity with respect to the Carnot–Carathéodory distance by a result of Magnani [43] and Rickly [49], see also [37]. In particular, u is continuous in Ω. We mollify u by convolution with kernels adapted to the group structure, as in [16, 26]. The approximating u_ε converge to u uniformly on compact subsets of

Ω, and they are smooth and $\mathcal{X}$-convex. Moreover, from the proof of [26, Theorem 9.1] we get, for R small enough,

$$\sup_{B_C(x_0,R)} \Big(\sum_{j=1}^{m} (X_j u_\varepsilon)^2 \Big)^{1/2} \le \frac{2}{R} \sup_{B_C(x_0,3R)} |u|,$$

where the balls B_C are taken with respect to the gauge pseudo-distance and $X_j u$ denotes the derivative of u along the trajectory of the vector field X_j. Since u_ε is C^∞, we have $X_j u_\varepsilon(x) = \sigma^j(x) D u_\varepsilon(x)$. Therefore there is a constant C depending only on $\sup_{\overline{\Omega}} |u|$ and the pseudo-distance of Ω_1 from $\partial\Omega$ such that

$$|\sigma^T(x) D u_\varepsilon| \le C \quad \text{in } \Omega_1.$$

By letting $\varepsilon \to 0$, we obtain that u is a viscosity subsolution of the same inequality. $\square$

Theorem 2.18. *Assume $K_1 : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is continuous and nondecreasing with respect to r, and $X_1, \dots, X_m$ are of class C^2. Suppose $u \in \mathrm{USC}(\overline{\Omega})$ is bounded, uniformly $\mathcal{X}$-convex and a subsolution of* (2.31)*, whereas $v \in \mathrm{LSC}(\overline{\Omega})$ is a bounded supersolution of* (2.31)*. Finally, assume that either K_1 is strictly increasing in r, $K_1(x, r, q) - K_1(x, s, q) \ge C(r - s)$ for some $C > 0$ and all $r, s \in [-M, M]$, $M = \max\{\|u\|_\infty, \|v\|_\infty\}$, or u is a strict subsolution of* (2.31) *in each $\overline{\Omega}_1$ with $\overline{\Omega}_1 \subseteq \Omega$. Then*

$$\sup_{\Omega}(u - v) \le \max_{\partial\Omega}(u - v)^+.$$

Proof. We only show how we can avoid the structure condition (2.3) on the Hamiltonian in the proof of Theorem 2.7. Since $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$ is in the superdifferential of u at $x_\varepsilon \in \Omega_1$, Proposition 2.17 gives

$$|\sigma^T(x_\varepsilon) p_\varepsilon| \le C.$$

Moreover

$$\left|\sigma^T(x_\varepsilon) p_\varepsilon - \sigma^T(y_\varepsilon) p_\varepsilon\right| \le L_\sigma \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0,$$

where L_σ is a Lipschitz constant of σ, and therefore, for ε small,

$$|\sigma^T(y_\varepsilon) p_\varepsilon| \le C + 1.$$

Let ω_1 be the modulus of continuity of K_1 on $\overline{\Omega} \times [-M, M] \times \overline{B}(0, C+1)$. Then

$$\begin{aligned}
&\left|K(x_\varepsilon, u(x_\varepsilon), p_\varepsilon) - K(y_\varepsilon, u(x_\varepsilon), p_\varepsilon)\right| \\
&\quad = \left|K_1\big(x_\varepsilon, u(x_\varepsilon), \sigma^T(x_\varepsilon)p_\varepsilon\big) - K_1\big(y_\varepsilon, u(x_\varepsilon), \sigma^T(y_\varepsilon)p_\varepsilon\big)\right| \\
&\quad \le \omega_1\Big(|x_\varepsilon - y_\varepsilon| + L_\sigma \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}\Big) \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

The rest of the proof is the same as that of Theorem 2.7, taking into account Remark 2.11 and Theorem 2.12. □

A different proof of this theorem under the strict monotonicity assumption on K_1 is given in our paper [9].

3 The comparison principle for vector fields of Carnot type

3.1 Carnot groups

We begin with recalling some well-known definitions. We adopt the terminology and notations of the recent book [16]. Consider a group operation $\circ$ on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$ with identity 0, such that

$$(x, y) \mapsto y^{-1} \circ x \qquad \text{is smooth},$$

and the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$,

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{n_i}.$$

If δ_λ is an automorphism of the group $(\mathbb{R}^n, \circ)$ for all $\lambda > 0$, $(\mathbb{R}^n, \circ, \delta_\lambda)$ is a homogeneous Lie group on $\mathbb{R}^n$. We say that $m = n_1$ smooth vector fields $X_1, \dots, X_m$ on $\mathbb{R}^n$ generate $(\mathbb{R}^n, \circ, \delta_\lambda)$, and that this is a (homogeneous) Carnot group, if

- $X_1, \dots, X_m$ are invariant with respect to the left translations on $\mathbb{R}^n$, $\tau_\alpha(x) := \alpha \circ x$ for all $\alpha \in \mathbb{R}^n$,
- $X_i(0) = \partial/\partial x_i$, $i = 1, \dots, m$,
- the rank of the Lie algebra generated by $X_1, \dots, X_m$ is n at every point $x \in \mathbb{R}^n$.

We refer, e.g., to [12, 16] for the connections of this definition with the classical one in the context of abstract Lie groups and for the properties of the generators. We will use only the following property, and refer to [16, p. 59, Remark 1.4.6] for more precise information.

Proposition 3.1. *If $X_1, \dots, X_m$ are generators of a Carnot group, then*

$$X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} \sigma_{ij}(x) \frac{\partial}{\partial x_i}$$

with $\sigma_{ij}(x) = \sigma_{ij}(x_1, \dots, x_{i-1})$ homogeneous polynomials of a degree $\le n - m$.

The previous proposition implies that $\sigma(x) = \binom{I}{\tau(x)}$ where I is the $m \times m$ identity matrix and $\tau(x)$ is an $(n-m) \times m$ matrix.

If $X_1, \dots, X_m$ are the generators of a Carnot group $\mathcal{G}$, the definition (2.5) of $\mathcal{X}$-convexity coincides with the definition of convexity in $\mathcal{G}$ in viscosity sense (v-convexity) of Lu, Manfredi and Stroffolini [42]. A more geometric notion of convexity in $\mathcal{G}$, called *horizontal convexity* (or weak H-convexity), was introduced and studied in the same seminal paper [42] and, independently, by Danielli, Garofalo and Nhieu [26]. The equivalence of the two notions was studied by several authors, first in the Heisenberg groups [5, 42], and then in general Carnot groups [37,43,54].

3.2 Construction of strict subsolutions

In this section, we construct a uniformly $\mathcal{X}$-convex strict subsolution of the subelliptic Monge–Ampère equation

$$-\det D^2_{\mathcal{X}} u + H(x, u, D_{\mathcal{X}} u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

from an $\mathcal{X}$-convex subsolution u (here the $\mathcal{X}$ derivatives are defined by (2.12) and (2.13)). We therefore get a comparison principle for usual viscosity subsolutions, without the strictness assumption.

The first assumption is on the vector fields and it is motivated by the properties of generators of Carnot groups recalled in the preceding section:

$$\begin{cases} \sigma(x) \text{ is an } n \times m \text{ matrix such that } \sigma(x) = \binom{I}{\tau(x)} \\ \text{where } I \text{ is the } m \times m \text{ identity matrix and} \\ \tau(x) \text{ is a } C^{1,1}\ (n-m) \times m \text{ matrix.} \end{cases} \tag{3.2}$$

When the matrix σ satisfies (3.2), we will say that the vector fields are of *Carnot type*, following the terminology of [45]. However, different from [45], we do not assume the Hörmander condition on the rank of the Lie algebra generated by the fields.

The second assumption is on H. In the Euclidean case, it coincides with the one made by Ishii and Lions for their comparison principle in that context [36].

$$\begin{cases} H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty) \text{ is continuous and} \\ \text{nondecreasing in } r\text{; for any } R > 0 \text{ there exists } L_R \text{ such that} \\ \left| H^{1/m}(x, r, q + q_1) - H^{1/m}(x, r, q) \right| \leq L_R |q_1| \\ \text{for all } x \in \overline{\Omega}, |r| \leq R, |q| \leq R, |q_1| \leq 1. \end{cases} \tag{3.3}$$

Theorem 3.2. *Assume* (3.2) *and* (3.3) *and let u be an $\mathcal{X}$-convex subsolution of equation* (3.1). *Then for any open set Ω_1 with $\overline{\Omega}_1 \subseteq \Omega$ there exist $\alpha, \varepsilon_0 > 0$ and a sequence $u_\varepsilon \in \mathrm{USC}(\overline{\Omega})$ of uniformly $\mathcal{X}$-convex functions such that $u_\varepsilon \leq u$, $u_\varepsilon \to u$ uniformly in Ω as $\varepsilon \to 0$, and, for all $\varepsilon \leq \varepsilon_0$,*

$$-\det D^2_{\mathcal{X}} u_\varepsilon + H(x, u_\varepsilon, D_{\mathcal{X}} u_\varepsilon) \leq -\alpha \quad \text{in } \Omega_1. \tag{3.4}$$

Proof. We consider

$$u_\varepsilon(x) := u(x) + \varepsilon\big(e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2} - \lambda\big),$$

and we want to show that it is a strict subsolution for λ and μ sufficiently large, independent of $\varepsilon > 0$. First we choose

$$\lambda := \max_{x \in \overline{\Omega}} e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2}$$

for all $x \in \Omega$, and this implies $u_\varepsilon(x) \leq u(x)$. We set

$$\nu := \varepsilon \mu e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2}, \quad \varepsilon_0 := \min_{x \in \overline{\Omega}} \frac{\exp(-\frac{\mu}{2}\sum_{i=1}^m x_i^2)}{\mu(\sum_{i=1}^m x_i^2)^{1/2}},$$

and compute

$$\begin{aligned} Du_\varepsilon &= Du + \nu(x_1, \ldots, x_m, 0, \ldots, 0), \\ D^2 u_\varepsilon &= D^2 u \\ &\quad + \nu \left(\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} + \mu(x_1, \ldots, x_m, 0, \ldots, 0) \otimes (x_1, \ldots, x_m, 0, \ldots, 0) \right), \end{aligned}$$

where $(q \otimes q)_{ij} = q_i q_j$. Note that $|\nu(x_1, \ldots, x_m)| \leq 1$ for $\varepsilon \leq \varepsilon_0$. Then

$$\sigma^T Du_\varepsilon = \sigma^T Du + \nu(x_1, \ldots, x_m)$$

and

$$\sigma^T(x)D^2u_\varepsilon\,\sigma(x) + Q(x, Du_\varepsilon) = \sigma^T(x)D^2u\,\sigma(x) + Q(x, Du) \\ + \nu(I_m + \mu(x_1, \dots, x_m) \otimes (x_1, \dots, x_m)) + \nu Q(x, (x_1, \dots, x_m, 0, \dots, 0)).$$

From the structure of the coefficients of the matrices Q and σ, (2.13) and (3.2), we have that

$$D\sigma^i = \begin{pmatrix} 0 \\ D\tau^i \end{pmatrix}, \quad D\sigma^i\,\sigma^j = \begin{pmatrix} 0 \\ D\tau^i\,\tau^j \end{pmatrix},$$

τ^i being the i-th column of the matrix τ. Then $Q_{ij}(x, (x_1, \dots, x_m, 0, \dots, 0)) \equiv 0$ for any $i, j = 1, \dots, m$. Hence, since u is $\mathcal{X}$-convex,

$$-\sigma^T(x)D^2u_\varepsilon\,\sigma(x) - Q(x, Du_\varepsilon) + \nu I_m \le 0,$$

i.e., u_ε is uniformly $\mathcal{X}$-convex.

Now we want to find a sufficiently large μ such that u_ε satisfies (3.4). Let us consider the auxiliary equation

$$G(x, u, Du, D^2u) := -\det{}^{1/m}\big(\sigma^T D^2u\,\sigma + Q(x, Du)\big) + H^{1/m}(x, u, \sigma^T Du) \\ = 0. \tag{3.5}$$

To prove that u_ε is a strict subsolution of (3.5) for large μ, we compute

$$G(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = -\det{}^{1/m}\big(\sigma^T D^2u\,\sigma + Q(x, Du) \\ + \nu(I_m + \mu(x_1, \dots, x_m) \otimes (x_1, \dots, x_m))\big) \\ + H^{1/m}\big(x, u_\varepsilon, \sigma^T Du + \nu(x_1, \dots, x_m)\big). \tag{3.6}$$

From Minkowski's inequality [35] we deduce

$$\det{}^{1/m}(A + B) \ge \det{}^{1/m}(A) + \det{}^{1/m}(B), \tag{3.7}$$

for all matrices of order m, $A > 0$, $B \ge 0$, and

$$G(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \le -\det{}^{1/m}\big(\sigma^T D^2u\,\sigma + Q(x, Du)\big) \\ - \nu\det{}^{1/m}(I_m + \mu(x_1, \dots, x_m) \otimes (x_1, \dots, x_m)) \\ + H^{1/m}\big(x, u_\varepsilon, \sigma^T Du + \nu(x_1, \dots, x_m)\big).$$

By Proposition 2.17, $|\sigma^T(x)Du| \le C$ in Ω_1, so we use the Lipschitz continuity of $H^{1/m}$ with $L = L_C$ and its monotonicity in u (see (3.3)), with the fact that u

is a subsolution of (3.1), to obtain

$$G(x,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon)\le \nu\big(L|(x_1,\dots,x_m)| \\ -\det{}^{1/m}(I_m+\mu(x_1,\dots,x_m)\otimes(x_1,\dots,x_m))\big).$$

We want to find a suitably large μ independent of ε such that

$$L^m|(x_1,\dots,x_m)|^m<\det(I_m+\mu(x_1,\dots,x_m)\otimes(x_1,\dots,x_m)) \tag{3.8}$$

for any $x\in\Omega_1$. We use the following equality [46]:

$$\det(I+q\otimes q)=1+|q|^2, \tag{3.9}$$

where q is an $m\times 1$ column vector. Then (3.8) becomes

$$1+\mu|(x_1,\dots,x_m)|^2-L^m|(x_1,\dots,x_m)|^m>0. \tag{3.10}$$

If $|(x_1,\dots,x_m)|<\frac{1}{L}$, then (3.10) is true for any $\mu>0$. If $|(x_1,\dots,x_m)|\ge\frac{1}{L}$, we take

$$\mu>\max_x\big(L^m|(x_1,\dots,x_m)|^m-1\big)L^2,$$

and (3.10) holds also in this case. With this choice of μ, for some $\alpha>0$ we have $G(x,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon)<-\alpha$ for any $x\in\Omega_1$. Then (3.4) holds and the proof is complete. □

Remark 3.3. We can obtain the same result also in the case

$$\sigma(x)=\begin{pmatrix}K\\ \tau(x)\end{pmatrix},$$

where K is a nonsingular constant $m\times m$ matrix. In this case, we use a generalization of (3.9):

$$\det\big(K^T(I+\mu vv^T)K\big)=\det(K^TK)(1+\mu|v|^2),\quad \text{if } |K^Tv|\ne 0. \tag{3.11}$$

Remark 3.4. Another case where it is possible to construct a uniformly $\mathcal{X}$-convex strict subsolution is when

$$\sigma^T(x)\sigma(x)+Q(x,x)\ge\eta I\quad \text{for all } x\in\overline{\Omega}, \text{ for some } \eta>0. \tag{3.12}$$

This condition is equivalent to saying that $|x|^2$ is uniformly $\mathcal{X}$-convex in Ω. In this case, we consider, for a given viscosity subsolution u,

$$u_\varepsilon(x):=u(x)+\varepsilon\big(e^{\mu|x|^2/2}-\lambda\big)$$

and we show that it is a strict subsolution for $\lambda \gg \mu \gg 1$, independent of $\varepsilon > 0$, following the procedure used in the proof of Theorem 3.2. To do this we apply the inequality [40]

$$\det(A + \mu q \otimes q) \geq \eta^N \left(1 + \frac{\mu}{N\eta}|q|^2\right), \tag{3.13}$$

where $A \in S^N$ such that $A \geq \eta I$, $\eta > 0$, $\mu \geq 0$, $q \in \mathbb{R}^N$.

3.3 The comparison principle with non-strict subsolutions

We are now ready to prove the main comparison principle for the equation

$$-\det(D^2_{\mathcal{X}} u) + H(x, u, D_{\mathcal{X}} u) = 0 \quad \text{in } \Omega. \tag{3.14}$$

Theorem 3.5. *Assume H satisfies* (3.3) *and the vector fields $X_1, \dots, X_m \in C^2$ are of Carnot type or satisfy* (3.12). *Let $u \in \mathrm{USC}(\overline{\Omega})$ be a bounded $\mathcal{X}$-convex subsolution of* (3.14) *and $v \in \mathrm{LSC}(\overline{\Omega})$ be a bounded supersolution of* (3.14). *Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Proof. We fix $\eta > 0$ and set $s := \max_{\partial\Omega}(u - v)^+$. By the upper semicontinuity of $u - v$ there is $\delta > 0$ such that

$$(u - v)(x) \leq s + \eta \quad \text{for all } x \text{ such that } \operatorname{dist}(x, \partial\Omega) \leq \delta.$$

Now we set

$$\Omega_\delta := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\right\}$$

and we must prove that $\sup_{\Omega_\delta}(u - v) \leq s + \eta$. To this goal we consider the uniformly $\mathcal{X}$-convex strict subsolutions u_ε constructed in Theorem 3.2. We claim that each u_ε is also a strict subsolution of

$$-\log\det(D^2_{\mathcal{X}} u_\varepsilon) + \log\left(H(x, u_\varepsilon, D_{\mathcal{X}} u_\varepsilon) + \frac{\alpha}{2}\right) \leq 0 \quad \text{in } \Omega_\delta,$$

where $\alpha > 0$ is the constant provided by Theorem 3.2. Since v is a supersolution of the same equation, we can then use Theorem 2.18 in Ω_δ with $K_1 = \log(H + \alpha/2)$ to get

$$\sup_{\Omega_\delta}(u_\varepsilon - v) \leq \max_{\partial\Omega_\delta}(u_\varepsilon - v)^+ \leq s + \eta \quad \text{for all } \varepsilon \leq \varepsilon_0,$$

where the last inequality follows from $u_\varepsilon \leq u$. Since $u_\varepsilon \to u$, we let $\varepsilon \to 0$ and obtain that $u - v \leq s + \eta$ in all Ω, which gives the conclusion by the arbitrariness of η.

To prove the claim we recall that for each ε there is C such that $|D_{\mathcal{X}}u_\varepsilon| \le C$ in Ω_δ, by Proposition 2.17. Then

$$H(x, u_\varepsilon, D_{\mathcal{X}}u_\varepsilon) \le \max_{x\in\overline{\Omega}, |p|\le C} H(x, \sup u, p) =: C_1.$$

From this we get the conclusion as in Lemma 2.14, because

$$\begin{aligned}&\log(H(x, u_\varepsilon, D_{\mathcal{X}}u_\varepsilon) + \alpha)\\ &\quad = \log\Big(H(x, u_\varepsilon, D_{\mathcal{X}}u_\varepsilon) + \frac{\alpha}{2}\Big) + \log\Big(1 + \frac{\alpha}{2H(x, u_\varepsilon, D_{\mathcal{X}}u_\varepsilon) + \alpha}\Big)\\ &\quad \ge \log\Big(H(x, u_\varepsilon, D_{\mathcal{X}}u_\varepsilon) + \frac{\alpha}{2}\Big) + \log\Big(1 + \frac{\alpha}{2C_1 + \alpha}\Big). \qquad \square\end{aligned}$$

Remark 3.6. If H is independent of u, then the conclusion of Theorem 3.5 can be strengthened to $\sup_\Omega(u - v) \le \max_{\partial\Omega}(u - v)$ by Remark 2.10.

Example 3.7. The assumptions of Theorem 3.5 cover equations of the form

$$-\det(D^2_{\mathcal{X}}u) + k(x, u)(1 + |D_{\mathcal{X}}u|^2)^\alpha = 0 \quad \text{in } \Omega,$$

for any $\alpha \ge 0$, $k \in C(\overline{\Omega} \times \mathbb{R})$, $k \ge 0$ and nondecreasing in the second entry. If the vector fields are the canonical basis of the Euclidean space $\mathbb{R}^n$ and $\alpha = (n + 2)/2$ this is the classical equation satisfied by a function u whose graph has Gauss curvature k. In Carnot groups and for $\alpha = (m + 2)/2$, $k = k(x)$, it is the equation of prescribed horizontal Gauss curvature as defined by Danielli, Garofalo and Nhieu [26]. As a corollary of Theorem 3.5 we obtain the uniqueness of a viscosity solution $u \in C(\overline{\Omega})$ of this PDE with prescribed boundary data.

Example 3.8. In problems of optimal transportation (see [17,53] for the Euclidean case and [2, 29]), H has the form $H(x, q) = f(x)/h(q)$ with $f, h \ge 0$ and $\int_\Omega f(x)\,dx = \int_{\mathbb{R}^m} h(q)\,dq < +\infty$. The assumption (3.3) of Theorem 3.5 is satisfied if $f \in C(\overline{\Omega})$, $h \in C(\mathbb{R}^m)$, $h > 0$, and $h^{-1/m}$ is locally Lipschitz. This is true, for instance, if $h(q) = 1/(c + |q|)^\alpha$ with $\alpha > m$ and $c > 0$.

In [48], Rauch and Taylor proved the following comparison principle for the classical Monge–Ampère equation:

If Ω is strictly convex, $u \in C(\overline{\Omega})$ is convex, $v \in W^{2,n}(\Omega)$, and $\det D^2u \ge \det D^2v$ *in Ω, then* $\max_\Omega(u - v) \le \max_{\partial\Omega}(u - v)$.

The last result of this section, which is a special case of Theorem 3.5 with Remark 3.6, gives a version of such statement in the context of viscosity solutions and non-commutative vector fields. It extends also a proposition of Gutierrez and Montanari [33] for $u, v \in C^2(\Omega)$, $n = 3$, $m = 2$, and X_1, X_2 generators of the Heisenberg group.

Corollary 3.9. *Assume the vector fields $X_1, \dots, X_m \in C^2$ are of Carnot type or satisfy* (3.12)*, $u \in \mathrm{USC}(\overline{\Omega})$ bounded and $\mathcal{X}$-convex, $v \in \mathrm{LSC}(\overline{\Omega})$ bounded, and*

$$-\det(D^2_{\mathcal{X}}u) + f(x) \le 0, \quad -\det(D^2_{\mathcal{X}}v) + f(x) \ge 0 \quad \textit{in } \Omega$$

in viscosity sense for some $f \in C(\overline{\Omega})$, $f \ge 0$. Then $\sup_\Omega(u-v) \le \max_{\partial\Omega}(u-v)$.

4 Solvability of the Dirichlet problem

In this section, we apply the results of Section 3 to solve the Dirichlet problem for the PDE (3.14):

$$\begin{cases} -\det D^2_{\mathcal{X}}u + H(x, u, D_{\mathcal{X}}u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $g \in C(\partial\Omega)$.

4.1 Some explicit solutions in the Heisenberg group

In $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, t)$ the generators of the Heisenberg group are the vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial t}. \tag{4.2}$$

The norm

$$|x|_{\mathbb{H}} := w(x)^{1/4}, \quad w(x_1, x_2, t) := (x_1^2 + x_2^2)^2 + t^2 \tag{4.3}$$

is positively 1-homogeneous with respect to the dilations

$$\delta_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda x_2, \lambda^2 t).$$

The Koranyi ball of radius $R > 0$ centered at the origin is

$$B_{\mathbb{H}}(R) := \left\{x = (x_1, x_2, t) \in \mathbb{R}^3 : |x|_{\mathbb{H}} < R\right\}.$$

Proposition 4.1. *Let $\Omega = B_{\mathbb{H}}(R)$ and X_1, X_2 be the generators of the Heisenberg group* (4.2)*. Then $w(x) = |x|_{\mathbb{H}}^4$ is the unique $\mathcal{X}$-convex viscosity solution of the Dirichlet problems*

$$\begin{cases} -\det D_{\mathcal{X}}^2 u + 144(x_1^2 + x_2^2)^2 = 0 & \text{in } \Omega, \\ u = R^4 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

and

$$\begin{cases} -\det D_{\mathcal{X}}^2 u + k_{\mathbb{H}}(x)(1 + |D_{\mathcal{X}} u|^2)^2 = 0 & \text{in } \Omega, \\ u = R^4 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where

$$k_{\mathbb{H}}(x) := \left(\frac{12(x_1^2 + x_2^2)}{1 + 16(x_1^2 + x_2^2)|x|_{\mathbb{H}}^4} \right)^2. \tag{4.6}$$

In particular, w is the unique $\mathcal{X}$-convex function on $B_{\mathbb{H}}(R)$ with horizontal Gauss curvature $k_{\mathbb{H}}$ and boundary value R^4.

Proof. The uniqueness follows from Theorem 3.5. A straightforward calculation gives

$$D_{\mathcal{X}}^2 w(x) = 12(x_1^2 + x_2^2) I, \tag{4.7}$$

so w is $\mathcal{X}$-convex and a classical solution of (4.4). Moreover

$$D_{\mathcal{X}} w = 4(x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 4t \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \tag{4.8}$$

so

$$|D_{\mathcal{X}} w|^2 = 16(x_1^2 + x_2^2)\big((x_1^2 + x_2^2)^2 + t^2\big) \tag{4.9}$$

and w is a classical solution of (4.5). □

The next result is the analogue in the Heisenberg group of the fact that the Euclidean norm $|x|$ in $\mathbb{R}^n$ is the unique convex function solving $\det D^2 u = 0$ in the punctured Euclidean ball $B(R) \setminus \{0\}$ and taking the values R on $\partial B(R)$ and 0 at 0.

Proposition 4.2. *Let X_1, X_2 be the generators of the Heisenberg group* (4.2)*. Then the homogeneous norm $|\cdot|_{\mathbb{H}}$ is the unique $\mathcal{X}$-convex viscosity solution of the Dirichlet problem*

$$\begin{cases} -\det D_{\mathcal{X}}^2 u = 0 & \text{in } B_{\mathbb{H}}(R) \setminus \{0\}, \\ u = R & \text{on } \partial B_{\mathbb{H}}(R), \\ u(0) = 0. \end{cases} \tag{4.10}$$

Proof. The uniqueness follows from Theorem 3.5. Next we compute, for $x \neq 0$,

$$D^2_{\mathcal{X}}|x|_{\mathbb{H}} = \frac{1}{4|x|^3_{\mathbb{H}}}\Big[D^2_{\mathcal{X}}w - \frac{3}{4w}D_{\mathcal{X}}w \otimes D_{\mathcal{X}}w\Big]. \tag{4.11}$$

If $(x_1, x_2) = 0$, then $D^2_{\mathcal{X}}|x|_{\mathbb{H}} = 0$. If $(x_1, x_2) \neq 0$, to show that the matrix in brackets $[\cdots]$ is positive semidefinite, we take a unit vector ζ and set $\psi := x_1^2 + x_2^2$. First observe that

$$\zeta^T D^2_{\mathcal{X}} w\zeta = 12\psi|\zeta|^2 = 12\psi$$

by (4.7). Next we compute, using (4.9),

$$\begin{aligned}\frac{3}{4w}\zeta^T(D_{\mathcal{X}}w \otimes D_{\mathcal{X}}w)\zeta &= \frac{3}{4w}|\zeta \cdot D_{\mathcal{X}}w|^2\\ &\leq \frac{3}{4w}|D_{\mathcal{X}}w|^2 = \frac{3}{4w}16\psi w = 12\psi.\end{aligned}$$

Then $D^2_{\mathcal{X}}|x|_{\mathbb{H}} \geq 0$ in $\mathbb{R}^3 \setminus \{0\}$ in the classical sense and $|\cdot|_{\mathbb{H}}$ is $\mathcal{X}$-convex.

To prove that $\det D^2_{\mathcal{X}}|x|_{\mathbb{H}} = 0$, it is enough to show, by (4.11), that

$$\Big[D^2_{\mathcal{X}}w - \frac{3}{4w}D_{\mathcal{X}}w \otimes D_{\mathcal{X}}w\Big]D_{\mathcal{X}}w = 0, \tag{4.12}$$

because $D_{\mathcal{X}}w \neq 0$ for $(x_1, x_2) \neq 0$. By (4.7) and (4.8) the first term is

$$D^2_{\mathcal{X}}wD_{\mathcal{X}}w = 48\psi^2\begin{pmatrix}x_1\\ x_2\end{pmatrix} + 48t\psi\begin{pmatrix}x_2\\ -x_1\end{pmatrix}.$$

For the second term we use $(D_{\mathcal{X}}w \otimes D_{\mathcal{X}}w)D_{\mathcal{X}}w = |D_{\mathcal{X}}w|^2 D_{\mathcal{X}}w$, (4.8), and finally (4.9) to compute

$$\frac{3}{4w}(D_{\mathcal{X}}w \otimes D_{\mathcal{X}}w)D_{\mathcal{X}}w = \frac{3}{4w}16\psi wD_{\mathcal{X}}w = 48\psi^2\begin{pmatrix}x_1\\ x_2\end{pmatrix} + 48t\psi\begin{pmatrix}x_2\\ -x_1\end{pmatrix},$$

which gives (4.12). □

Remark 4.3. The calculations of this section hold as well in $\mathbb{R}^{2j+1}$ with the generators of the j-th Heisenberg group and the corresponding homogeneous norm. The fact that such norm solves $\det D^2_{\mathcal{X}}u = 0$ off the origin was proved in [26] for general stratified groups of Heisenberg type.

4.2 Perron method

Here we describe the construction of solutions in the general case. We denote by $\mathcal{S}$ and $\mathcal{Z}$, respectively, the sets of sub- and supersolutions of (4.1):

$$\mathcal{S} := \big\{w \in \mathrm{USC}(\overline{\Omega}) : w \text{ bounded } \mathcal{X}\text{-convex subsolution of (3.14)}, \\ w \le g \text{ on } \partial\Omega\big\},$$

$$\mathcal{Z} := \big\{W \in \mathrm{LSC}(\overline{\Omega}) : W \text{ bounded supersolution of (3.14)}, \ W \ge g \text{ on } \partial\Omega\big\}.$$

The Perron method proposes

$$\underline{u}(x) := \sup_{w \in \mathcal{S}} w(x), \quad x \in \overline{\Omega}, \quad \text{if } \mathcal{S} \neq \emptyset$$

as a candidate solution of (4.1). Note that, if $W \in \mathcal{Z}$ and the comparison principle holds, then $\underline{u}(x) \le W(x) < +\infty$ for all x. Under no further assumptions $\underline{u}$ is a generalized, possibly discontinuous solution of the Dirichlet problem (4.1) in the following sense.

Theorem 4.4. *If $\mathcal{S} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$, and the comparison principle holds for* (4.1)*, then the u.s.c. envelope $\underline{u}^*$ is a $\mathcal{X}$-convex subsolution and the l.s.c. envelope $\underline{u}_*$ is a supersolution of* (3.14).

Proof. By a standard argument, if a function v is the sup of a set of subsolutions, then its u.s.c. envelope $v^*(x) := \inf\{V(x) : V \in \mathrm{USC}(\overline{\Omega}), v \le V\}$ is a subsolution, see [6, 24]. In particular, $\underline{u}^*(x)$ is $\mathcal{X}$-convex. The proof that $\underline{u}_*$ is a supersolution of (3.14) is achieved by contradiction: one assumes that $\underline{u}_*$ fails to be a supersolution at some point $y \in \Omega$ and constructs a subsolution that is larger than $\underline{u}$ near y, therefore contradicting the maximality of $\underline{u}$. For the Monge–Ampère equations we must show that the we can construct an $\mathcal{X}$-convex subsolution larger than $\underline{u}$.

If $\underline{u}_*$ fails to be a supersolution at some point, say $y = 0$, by Definition 2.3 there is a C^2 test function φ such that $\varphi(0) = \underline{u}_*(0)$, $\varphi(x) \le \underline{u}_*(x)$ for $|x|$ small,

$$-\det(D^2_{\mathcal{X}}\varphi(0)) + H(0, \underline{u}_*(0), D_{\mathcal{X}}\varphi(0)) < 0,$$

and

$$D^2_{\mathcal{X}}\varphi(0) > 0. \tag{4.13}$$

The usual "bump" construction [6, 24] considers $v(x) := \varphi(x) + \varepsilon - \gamma|x|^2$ and

$$U(x) := \begin{cases} \max\{\underline{u}(x), v(x)\} & \text{if } |x| < r, \\ \underline{u}(x) & \text{otherwise.} \end{cases}$$

Then one checks that for small ε, γ, r, the function U^* is a subsolution of the PDE (3.14) and $\sup(U^* - \underline{u}) > 0$. Thanks to (4.13), by further restricting γ, r if necessary, we also have that $-D^2_{\mathcal{X}} U^* \leq 0$ in viscosity sense, so U^* is $\mathcal{X}$-convex and we achieve the contradiction by the usual argument [6, 24]. □

To give examples of equations that satisfy the last theorem, as well as the next results on the existence of continuous solutions, we will use the assumption that for some $L, M, R > 0$,

$$H^{1/m}(x, R, p) \leq L|p| + M \quad \text{for all } x \in \overline{\Omega},\ p \in \mathbb{R}^m. \tag{4.14}$$

Since H is nondecreasing in the second entry, the same inequality holds for $H^{1/m}(x, r, p)$ and all $r \leq R$. In the next results, we will take $R = \min_{\partial\Omega} g$ or $\max_{\partial\Omega} g$, and L, M will depend on it. This is a slightly weaker version of the growth condition used by Lions [41] in the Euclidean case, see Examples 4.16 and 4.19 below.

Example 4.5. Assume H satisfies (3.3), (4.14) with $R = \min_{\partial\Omega} g$, and the vector fields $X_1, \ldots, X_m \in C^2$ are of Carnot type. Then $\underline{u}(x)$ is a generalized solution of problem (4.1) in the sense described by Theorem 4.4. In fact, by Theorem 3.5 we know that the comparison principle for (4.1) holds. Therefore, it is enough to prove that both sets $\mathcal{S}$ and $\mathcal{Z}$ are nonempty. First of all we note that $W \equiv \max_{\partial\Omega} g$ is an element of $\mathcal{Z}$. As far as the set $\mathcal{S}$ is concerned, we consider

$$w(x) = e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2} - \max_{x \in \overline{\Omega}} e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2} + \min_{\partial\Omega} g,$$

so that $w \leq \min_{\partial\Omega} g$. As in the proof of Theorem 3.2, we have that $D^2_{\mathcal{X}} w \geq \mu I$. Moreover, for

$$\nu := \mu e^{\frac{\mu}{2}\sum_{i=1}^m |x_i|^2},$$

by (3.9) we get

$$\begin{aligned}
&-\det{}^{1/m}(D^2_{\mathcal{X}} w) + H^{1/m}(x, w, D_{\mathcal{X}} w) \\
&\quad = -\nu \det{}^{1/m}(I_m + \mu(x_1, \ldots, x_m) \otimes (x_1, \ldots, x_m)) \\
&\qquad + H^{1/m}(x, w, \nu(x_1, \ldots, x_m)) \\
&\quad \leq \nu\big(-1 - \mu|(x_1, \ldots, x_m)|^{2/m} + L|(x_1, \ldots, x_m)|\big) + M
\end{aligned}$$

which becomes negative for μ large enough.

Example 4.6. Assume H satisfies (3.3), (4.14) with $R = \min_{\partial\Omega} g$, and the vector fields are of class C^2 and such that $|x|^2$ is $\mathcal{X}$-convex in Ω, that is, the inequality (3.12) holds. Then $\underline{u}(x)$ is a generalized solution of problem (4.1) as in the preceding example. The proof is the same except that now we use

$$w(x) = e^{\mu\frac{|x|^2}{2}} - \max_{x\in\overline{\Omega}} e^{\mu\frac{|x|^2}{2}} + \min_{\partial\Omega} g.$$

As in the classical potential theory, the continuity at the boundary of the Perron solution requires the existence of barriers.

Definition 4.7. We say that w is a lower (respectively, upper) barrier for problem (4.1) at a point $x \in \partial\Omega$ if $w \in \mathcal{S}$ (respectively, $w \in \mathcal{Z}$) and

$$\lim_{y\to x} w(y) = g(x).$$

Corollary 4.8. *Suppose that the comparison principle holds for* (4.1) *and that for all $x \in \partial\Omega$ there exist a lower and an upper barrier. Then $\underline{u} \in C(\overline{\Omega})$ is the solution of* (4.1)*, that is, the unique $\mathcal{X}$-convex viscosity solution of* (3.14) *attaining continuously the boundary data g.*

Proof. The existence of a lower and an upper barrier at $x \in \partial\Omega$ implies the continuity of $\underline{u}$ at x and $\underline{u}(x) = \underline{u}^*(x) = \underline{u}_*(x) = g(x)$. Then the comparison principle gives $\underline{u}^* = \underline{u}_*$ in Ω and therefore $\underline{u}$ is a continuous viscosity solution of (3.14). □

Remark 4.9 (Interior regularity of the solution). Since the solution $\underline{u}$ of the Dirichlet problem is $\mathcal{X}$-convex, it is locally Lipschitz continuous with respect to the Carnot–Carathéodory distance d associated to the vector fields $\mathcal{X}$ (see, e.g., [7,12] for the definition). If, in addition, the identity map $(\mathbb{R}^n, d) \to (\mathbb{R}^n, |\cdot|)$ is a homeomorphism (e.g., the vector fields $\mathcal{X}$ are smooth and satisfy the Hörmander condition), then the distributional derivatives $X_j\underline{u}$ exist a.e. and are locally bounded. All this is known in Carnot groups by [5, 26, 42, 43, 49] and was proved in [7] for general vector fields.

In Carnot groups of step 2, horizontally convex functions are also twice differentiable a.e. [27, 34, 43]. Therefore in this case the Perron solution $\underline{u}$ solves the PDE (3.14) also pointwise almost everywhere.

4.3 Construction of barriers

To find explicit examples where the Perron method works and the Dirichlet problem is solvable, we make some assumptions on the bounded open set Ω. We say it is smooth if

$$\begin{cases} \text{there exists } \Phi \in C^2 \text{ such that} \\ \Omega = \{x \in \mathbb{R}^n : \Phi(x) > 0\},\ D\Phi(x) \neq 0 \text{ for all } x \in \partial\Omega. \end{cases} \tag{4.15}$$

The main additional assumption is that the domain be uniformly convex with respect to the vector fields X_j. It is the natural extension for subelliptic Monge–Ampère equations of the standard uniform convexity in the Euclidean case [19, 31, 40, 41, 52].

Definition 4.10. A domain Ω smooth in the sense of (4.15) is called convex with respect to the fields $X_1, \dots, X_m$ (briefly $\mathcal{X}$-convex) if $D^2_{\mathcal{X}}\Phi(x) \leq 0$, and uniformly $\mathcal{X}$-convex if

$$D^2_{\mathcal{X}}\Phi(x) \leq -\gamma I \quad \text{for some } \gamma > 0, \text{ for any } x \in \Omega. \tag{4.16}$$

Example 4.11. Any Euclidean ball centered in x_0 is uniformly $\mathcal{X}$-convex if and only if $|x - x_0|^2$ is $\mathcal{X}$-convex in $\mathbb{R}^n$. It is well known that this is true in all Carnot groups of step 2, in particular the Heisenberg groups. For $x_0 = 0$ the $\mathcal{X}$-convexity of $|x|^2$ is equivalent to the inequality (3.12) in $\mathbb{R}^n$ that we already used in the comparison principles.

The Koranyi ball $B_{\mathbb{H}}(R)$ in $\mathbb{R}^3$ is $\mathcal{X}$-convex but not uniformly $\mathcal{X}$-convex with respect to the generators of the Heisenberg group (4.2).

The next result is an analogue in the context of Carnot-type vector fields of a classical result of Caffarelli, Nirenberg and Spruck [19] and Lions [41] in the Euclidean case, saying that the solvability of the Dirichlet problem is equivalent to the existence of a convex subsolution attaining continuously the boundary data (see also [51] for other curvature equations).

Corollary 4.12. *Assume H satisfies* (3.3) *and the vector fields $X_1, \dots, X_m \in C^2$ are either of Carnot type or they satisfy* (3.12). *Suppose also that either* (i) $g \equiv 0$, *or* (ii) Ω *and* $g \in C^2(\overline{\Omega})$ *are $\mathcal{X}$-convex, or* (iii) Ω *is uniformly $\mathcal{X}$-convex and* $g \in C^2(\overline{\Omega})$. *If for all $x \in \partial\Omega$ there exists a lower barrier, then $\underline{u} \in C(\overline{\Omega})$ is the unique solution of* (4.1).

Proof. We are going to apply Corollary 4.8. The comparison principle comes from Theorem 3.5. An upper barrier at all points of the boundary is $W \equiv 0$ in case (i).

In the other cases, we take $W(x) := \lambda\Phi(x) - g(x)$, $\lambda > 0$. Then in case (ii) $D^2_{\mathcal{X}} W(x) \leq 0$ for any λ and in case (iii) $D^2_{\mathcal{X}} W(x) \leq -\lambda\gamma I - D^2_{\mathcal{X}} g(x) \leq 0$ for λ large enough. Therefore any test function φ such that $W - \varphi$ attains a minimum at x has $D^2_{\mathcal{X}}\varphi(x) \leq 0$. Then W is a supersolution of (3.14) by Definition 2.3. □

Remark 4.13. In the papers [19] and [41], the vector fields are the canonical basis of $\mathbb{R}^n$ and the authors assume the existence of a convex function w attaining continuously the boundary data at all points and such that either w is a subsolution of class C^2 or w solves the PDE in the sense of Alexandrov. On the other hand the existence of a C^∞ solution to the Dirichlet problem is proved.

Next we give some explicit conditions on the data ensuring the existence of a lower barrier and therefore the solvability of the Dirichlet problem.

Proposition 4.14. *Suppose that Ω is uniformly $\mathcal{X}$-convex and $g \in C^2(\overline{\Omega})$. Assume H satisfies* (4.14) *with $R = \max_{\partial\Omega} g$ and it is nondecreasing with respect to the second entry r. Then there exists $w \in \mathcal{S} \cap C(\overline{\Omega})$ such that $w = g$ on $\partial\Omega$.*

Proof. We consider

$$w(x) = \lambda(e^{-\mu\Phi(x)} - 1) + g(x), \quad \mu, \lambda > 0, \tag{4.17}$$

where Φ is defined in (4.15). Clearly $w(x) = g(x)$ for any $x \in \partial\Omega$ and $w(x) < g(x)$ for any $x \in \Omega$. Moreover

$$\begin{aligned} Dw(x) &= -\mu\lambda\, e^{-\mu\Phi} D\Phi + Dg, \\ D^2 w(x) &= \mu\lambda\, e^{-\mu\Phi}(-D^2\Phi + \mu D\Phi \otimes D\Phi) + D^2 g, \\ D^2_{\mathcal{X}} w &= \mu\lambda\, e^{-\mu\Phi}\big(-D^2_{\mathcal{X}}\Phi + \mu\sigma^T D\Phi \otimes \sigma^T D\Phi\big) + D^2_{\mathcal{X}} g. \end{aligned}$$

From the regularity of the function g (there is c such that $-cI \leq D^2_{\mathcal{X}} g$) and the uniform $\mathcal{X}$-convexity of Ω with constant γ we have

$$D^2_{\mathcal{X}} w \geq (\mu\lambda\, e^{-\mu\Phi}\gamma - c)I + \mu^2\lambda\, e^{-\mu\Phi}\sigma^T D\Phi \otimes \sigma^T D\Phi.$$

First we choose λ such that $\mu\lambda\, e^{-\mu\Phi(x)}\gamma - c \geq \frac{1}{2}\mu\lambda\, e^{-\mu\Phi(x)}\gamma$, i.e.

$$\lambda \geq \frac{2c}{\mu\gamma} \max_{\overline{\Omega}} e^{\mu\Phi}. \tag{4.18}$$

With this λ, $D^2_{\mathcal{X}} w \geq 0$ and w is $\mathcal{X}$-convex. Moreover, from inequality (3.13),

$$\begin{aligned} \det(D^2_{\mathcal{X}} w) &\geq (\mu\lambda\, e^{-\mu\Phi})^m \det\Big(\frac{\gamma}{2} I + \mu\sigma^T D\Phi \otimes \sigma^T D\Phi\Big) \\ &\geq (\mu\lambda\, e^{-\mu\Phi})^m \Big(\frac{\gamma}{2}\Big)^m \Big(1 + \frac{2\mu}{m\gamma}|\sigma^T D\Phi|^2\Big). \end{aligned} \tag{4.19}$$

From the monotonicity of H, the growth assumption (4.14) with $R = \max_{\partial\Omega} g$, and the boundedness of Dg we get, for some $K > 0$,

$$\begin{aligned} H(x, w, D_{\mathcal{X}} w) &\le H(x, R, D_{\mathcal{X}} w) \le (M + L|D_{\mathcal{X}} w + D_{\mathcal{X}} g|)^m \\ &\le K + K(\mu\lambda\, e^{-\mu\Phi(x)})^m\, |\sigma^T D\Phi|^m. \end{aligned} \tag{4.20}$$

To prove that w is a subsolution we have to show that the right-hand side of (4.19) is larger than the right-hand side of (4.20) for μ and λ large enough. This is equivalent to

$$1 + \frac{2\mu}{m\gamma}|\sigma^T D\Phi|^2 \ge \frac{2^m K}{(\gamma\mu\lambda\, e^{-\mu\Phi})^m} + \left(\frac{2}{\gamma}\right)^m K|\sigma^T D\Phi|^m.$$

First we choose μ so large that

$$\frac{2\mu}{m\gamma}|\sigma^T D\Phi|^2 \ge \left(\frac{2}{\gamma}\right)^m K|\sigma^T D\Phi|^m$$

and then we choose λ such that $2^m K/(\gamma\mu\lambda\, e^{-\mu\Phi})^m \le 1$, i.e.,

$$\lambda \ge \frac{2K^{1/m}}{\mu\gamma} \max_{\overline{\Omega}} e^{\mu\Phi}.$$

From (4.18) we can conclude the proof by choosing

$$\lambda \ge \frac{2}{\mu\gamma} \max(c, K^{1/m}) \max_{\overline{\Omega}} e^{\mu\Phi}. \qquad \square$$

Theorem 4.15. *Suppose that Ω is uniformly $\mathcal{X}$-convex, $g \in C(\partial\Omega)$, and the vector fields $X_1, \dots, X_m \in C^2$ are either of Carnot type or they satisfy* (3.12). *Assume H satisfies* (3.3) *and* (4.14) *with $R = \max_{\partial\Omega} g$. Then $\underline{u} \in C(\overline{\Omega})$ is the unique solution of* (4.1).

Proof. We take a sequence $g_n \in C^2(\overline{\Omega})$ that converges uniformly to g. By Corollary 4.12 and Proposition 4.14 there is a solution $u_n \in C(\overline{\Omega})$ of the Dirichlet problem with boundary condition g_n. By the estimate of Theorem 3.5

$$\sup_{\Omega}|u_n - u_m| \le \max_{\partial\Omega}|g_n - g_m| \quad \text{for all } n, m.$$

Since g_n is a Cauchy sequence, also u_n is such and therefore it converges uniformly to $u \in C(\overline{\Omega})$. By the stability of viscosity solutions, u solves (4.1) and by the comparison principle Theorem 3.5 it coincides with $\underline{u}$. $\square$

Next we list several equations to which the last theorem on the well-posedness of the Dirichlet problem applies.

Example 4.16. Assume that the Hamiltonian H in (4.1) satisfies (3.3) and, for some $R \in \mathbb{R}$,

$$\limsup_{|p|\to+\infty} \frac{H(x,R,p)}{|p|^m} < +\infty \quad \text{uniformly in } x \in \overline{\Omega}.$$

Then the growth assumption (4.14) holds for some L, M and Theorem 4.15 applies for all data $g \in C(\partial\Omega)$ such that $\max_{\partial\Omega} g \le R$. In the Euclidean case ($m = n$ and the vector fields are the canonical basis of $\mathbb{R}^n$), we therefore recover one of the main results of Lions' paper [41].

Example 4.17. The previous example includes the basic subelliptic Monge–Ampère equation [26, 33, 44]

$$-\det(D^2_{\mathcal{X}}u) + f(x) = 0, \quad f \in C(\overline{\Omega}), \quad f \ge 0,$$

as well as the case $H(x,r,p) = (1+\lambda|p|^2)^{m/2}$, with $\lambda > 0$, that in the Euclidean case was studied in [40].

Example 4.18. Equations of the form

$$-\det(D^2_{\mathcal{X}}u) + f(x,u) = 0, \quad f \in C(\overline{\Omega} \times \mathbb{R}) \text{ nondecreasing in } u, \quad f \ge 0,$$

satisfy the assumptions of Theorem 4.15. The dependence of f on u appears in various problems of differential geometry, see [3, 21] and the references therein. An example that appears in several papers [3, 22, 28] is

$$f(x,u) = \phi(x)e^{\lambda(x)u}, \quad \lambda, \phi \in C(\overline{\Omega}), \quad \lambda, \phi \ge 0.$$

Example 4.19. Another special case of Example 4.16 is

$$-\det(D^2_{\mathcal{X}}u) + k(x,u)(1+|D_{\mathcal{X}}u|^2)^{\alpha} = 0, \quad k \ge 0,\ 0 \le \alpha \le \frac{m}{2},$$

with $k \in C(\overline{\Omega} \times \mathbb{R})$ nondecreasing in u, whose structure is reminiscent of the equation of prescribed Gauss curvature (where, however, $\alpha = \frac{m}{2} + 1$). We recall that the exponent $\frac{m}{2}$ is optimal without additional compatibility conditions on k. In the Euclidean case, it is well known that a necessary condition for the existence of a classical subsolution attaining the boundary data is

$$\int_{\Omega} k(x)\,dx \le \int_{\mathbb{R}^n} (1+|p|^2)^{-\alpha}\,dp,$$

and the right-hand side is finite if and only if $\alpha > n/2$.

Example 4.20. A simple example of nonexistence of the viscosity solution to the Dirichlet problem when the growth assumption (4.14) fails is the ODE

$$-u'' + H(x,u,u') = 0 \quad \text{in }]-1,1[\quad \text{with } H(x,u,p) \geq \frac{\pi}{2}(1+p^2)$$

and boundary conditions $u(-1) = u(1) = 0$. Indeed there are no viscosity subsolutions, i.e., $\mathcal{S} = \emptyset$. To prove this we solve $-u'' + k(1+(u')^2) = 0$ with $k < \frac{\pi}{2}$ and find $u_k(x) = \frac{1}{k}\log\frac{\cos k}{\cos kx}$. If $w \in \mathcal{S}$, the comparison principle gives $w \leq u_k$, but $u_k(x) \to -\infty$ as $k \to \frac{\pi}{2}-$, for all $|x| < 1$.

Remark 4.21. In the theory of linear PDEs, it is important to distinguish the characteristic and non-characteristic points of the boundary. For fully nonlinear, degenerate elliptic PDEs

$$F(x,u,Du,D^2u) = 0 \quad \text{in } \Omega,$$

we gave in [8] the following definition: a point $z \in \partial\Omega$ is called *characteristic* for the operator F if

$$F(z,0,-n(z),X + \mu n(z)\otimes n(z)) = F(z,0,-n(z),X) \tag{4.21}$$

for all X and all $\mu > 0$. For the subelliptic Monge–Ampère equations (3.14) the operator F is

$$F(x,r,p,X) := -\det\big(\sigma^T X\sigma + Q(x,p)\big) + H(x,r,\sigma^T p),$$

with Q given by (2.13). The characteristic points z are determined by the equation

$$\begin{aligned}\det\big(\sigma^T(z)X\sigma(z) + Q(z,p) + \mu\sigma^T(z)n(z)\otimes\sigma^T(z)n(z)\big)\\ = \det\big(\sigma^T(z)X\sigma(z) + Q(z,p)\big).\end{aligned} \tag{4.22}$$

We recall that if A, B are square matrices, $A > 0$ and $B \geq 0$, then

$$\det(A+B) = \det(A) \quad \text{if and only if} \quad B \equiv 0.$$

Then (4.22) for $\mu > 0$ yields $|\sigma^T(z)n(z)| = 0$. In the proof of Theorem 4.15, we did not need to treat these points differently from the non-characteristic ones because of the uniform $\mathcal{X}$-convexity of the domain. For more general domains we expect that the lower barrier must be constructed in different ways at characteristic and non-characteristic points, as for the linear and the Hamilton–Jacobi–Bellman subelliptic PDEs, see [8, 16] and the references therein.

We end the section with two results in the Koranyi ball of the Heisenberg group. Note that $B_{\mathbb{H}}(R)$ is a $\mathcal{X}$-convex set but it is not uniformly $\mathcal{X}$-convex, because the condition (4.16) fails at the characteristic points of the boundary, namely, the intersections with the t axis.

Corollary 4.22. *Let X_1, X_2 be the generators of the Heisenberg group* (4.2). *Then, for any $f \in C(\overline{B}_{\mathbb{H}}(R) \times \mathbb{R})$ nondecreasing in the second entry and such that $f(x, R^4) \leq 144(x_1^2 + x_2^2)^2$, $\underline{u} \in C(\overline{B}_{\mathbb{H}}(R))$ is the unique $\mathcal{X}$-convex viscosity solution of the Dirichlet problem*

$$\begin{cases} -\det D^2_{\mathcal{X}} u + f(x,u) = 0 & \text{in } B_{\mathbb{H}}(R), \\ u = R^4 & \text{on } \partial B_{\mathbb{H}}(R). \end{cases} \tag{4.23}$$

Proof. By Proposition 4.1, $w(x) = |x|^4_{\mathbb{H}}$ is a subsolution of (4.23) that attains continuously the boundary data. The conclusion follows from Corollary 4.12. □

The last result is about the prescribed horizontal Gauss curvature equation. Although it does not satisfy the growth condition (4.14), a lower barrier is given by $w(x) = |x|^4_{\mathbb{H}}$ if the prescribed curvature is lower than the horizontal curvature $k_{\mathbb{H}}$ of the graph of w.

Corollary 4.23. *Let X_1, X_2 be the generators of the Heisenberg group* (4.2). *Assume $k \in C(\overline{B}_{\mathbb{H}}(R) \times \mathbb{R})$ is nondecreasing in the second entry and satisfies $k(x, R^4) \leq k_{\mathbb{H}}(x)$, where $k_{\mathbb{H}}$ is given by* (4.6). *Then $\underline{u} \in C(\overline{B}_{\mathbb{H}}(R))$ is the unique $\mathcal{X}$-convex viscosity solution of the Dirichlet problem*

$$\begin{cases} -\det D^2_{\mathcal{X}} u + k(x,u)(1 + |D_{\mathcal{X}} u|^2)^2 = 0 & \text{in } B_{\mathbb{H}}(R), \\ u = R^4 & \text{on } \partial B_{\mathbb{H}}(R). \end{cases} \tag{4.24}$$

In particular, $\underline{u}$ is the unique $\mathcal{X}$-convex function on $B_{\mathbb{H}}(R)$ with horizontal Gauss curvature k and boundary value R^4.

Proof. By Proposition 4.1, $w(x) = |x|^4_{\mathbb{H}}$ is a subsolution of (4.24) that attains continuously the boundary data. The conclusion follows from Corollary 4.12. □

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