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Mathematical programming techniques to solve biharmonic problems by a recursive projection algorithm

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Abstract: To solve a classical ill-conditioned problem in the sense of Hadamard as the initial Cauchy problem for a biharmonic operator after some a priori estimates, a posteriori estimates are evaluated using three different methods of minimization such as: linear programming, least squares and a recursive projection algorithm for least squares. Numerical comparisons will be made on these three methods.

Keywords: Biharmonic operator, extrapolation methods, least squares, mathematical programming.

1. Introduction

The Cauchy problem for the biharmonic operator is considered here and this problem is, as it is well known, an ill-posed problem in the sense of Hadamard. In the case that the initial data are not analytical but at most differentiable a certain number of times, it may seem convenient to proceed as follows: first of all, approximate the function expressing the initial data through polynomials and solve afterwards the problem by the use of the approximated initial data, believing that the solution obtained in such a way may be not very much different from the solution of the original problem. Unfortunately, such belief is false as it is completely explained by the famous Hadamard example [7]. Though, the problem considered has a physical relevance and the difficulty underlined by the famous Hadamard example for the Laplace operator may be overcome in the following manner: if the data prescribed are not really the Cauchy data, we still may solve the problem imposing to the solution to be uniformly bounded. It follows that the Cauchy problem for the biharmonic operator has a stable solution in the sense that it is possible to give some a priori estimates for the coefficients of the solution. The solution of the problem is therefore reduced to a solution of a problem obtained by minimizing a certain functional subject to some linear inequalities. In this paper, after having recalled all necessary a priori estimates for the biharmonic Cauchy problem, the solution may be only obtained by a numerical procedure.

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Three different numerical procedures are possible: linear programming, least squares with normal equations and least squares with a recursive projection algorithm. The first and second procedure have been presented in [5], the third one is presented here. A comparison is made among numerical results of the three procedures from the point of view of accuracy and computing time.

2. The Cauchy problem for a biharmonic operator

In the xy-plane let C denote a smooth simple closed rectifiable Jordan curve whose interior G is assumed to be star-shaped with respect to the origin $z = x + \mathrm{i}y = 0$. Let $r = R(\theta)$ denote the equation of C in polar coordinates. The problem of approximating the solution of $u = u(r, \theta)$ of

$$\Delta^2 u = 0, \quad z \in G, \tag{1}$$

$$u(R(\theta), \theta) = f(\theta), \tag{2}$$

$$\frac{\partial u}{\partial n}(R(\theta), \theta) = g(\theta), \tag{3}$$

where n is the direction of the interior normal to C and where the real-valued functions $f(\theta)$, $f'(\theta)$ and $g(\theta)$ are known only approximately as $F(\theta)$, $F_1(\theta)$ and $G(\theta)$ such that

$$\max\{\| F(\theta) - f(\theta) \|_{[0,2\pi]}, \| F_1(\theta) - f'(\theta) \|_{[0,2\pi]},$$
$$\| G(\theta) - g(\theta) \|_{[0,2\pi]}\} \leqslant \mu, \quad \mu > 0, \tag{4}$$

where for any real-valued function $f(x)$ on a set E

$$\| f(x) \|_E = \sup_{x \in E} | f(x) |. \tag{5}$$

In [4,5] it has been shown that a sequence of biharmonic functions $\{u_k\}$ can be determined numerically such that the u_k and their first derivatives converge uniformly to u and its first derivatives on $\bar{G} = G \cup C$ as k tends to infinity and μ tends to zero. Then an a priori estimate is given using a Miranda [8] maximum principle, that can be paraphrased for the restriction to simply connected regions.

Theorem 1. *If u is a biharmonic function in G which is continuous along with its first derivatives in $\bar{G} = G \cup C$, then for each $\epsilon > 0$ there exists a biharmonic polynomial*

$$u_N(r, \theta) = a_N + b_N r^2 + \sum_{k=1}^{N} c_{N,k} r^k \cos k\theta + \sum_{k=1}^{N} d_{N,k} r^k \sin k\theta$$
$$+ \sum_{k=1}^{N} e_{N,k} r^{k+2} \cos k\theta + \sum_{k=1}^{N} f_{N,k} r^{k+2} \sin k\theta, \tag{6}$$

where N depends upon ϵ and G, such that

$$\max\left\{\| u(r, \theta) - u_N(r, \theta) \|_G, \left\| \frac{\partial u}{\partial r}(r, \theta) - \frac{\partial u_N}{\partial r}(r, \theta) \right\|_G,\right.$$
$$\left.\left\| \frac{\partial u}{\partial \theta}(r, \theta) - \frac{\partial u_N}{\partial \theta}(r, \theta) \right\|_G\right\} \leqslant \epsilon. \tag{7}$$

Theorem 2 (Maximum principle). *There exist two positive constants K_1 and K_2, which only depend upon G, such that if u is a solution of* (1)–(3) *which is C^1 in $\bar{G}$, then for (r, θ) in G,*

$$\sqrt{(u_x)^2+(u_y)^2} \leqslant K_1\{\| g \|_{[0,2\pi]} + \| f' \|_{[0,2\pi]}\} + K_2 \| f \|_{[0,2\pi]} \tag{8}$$

and

$$| u(r, \theta) | \leqslant K_1\delta\{\| g \|_{[0,2\pi]} + \| f' \|_{[0,2\pi]}\} + (1+K_2\delta) \| f \|_{[0,2\pi]}, \tag{9}$$

where $\delta = \delta(r, \theta, C)$ is the distance of (r, θ) from C.

The use of such theorems allows to reduce the solution of the problem consideration to a minimization problem on the boundary of the region C. Using a minimization technique on a functional that is introduced as a posteriori estimation of the error depending on the Miranda maximum principle it is therefore possible to solve the problem.

3. Numerical solution of the initial-value Cauchy problem

Theorem 3. *There exists a constant K which only depends upon G and N such that*

$$\| a_N \| \leqslant K(H+\mu+\epsilon), \tag{10}$$

$$\| b_N \| \leqslant r_0^{-1}K(H+\mu+\epsilon), \tag{11}$$

$$\| c_{N,k} \| \leqslant r_0^{-k}K(H+\mu+\epsilon), \qquad k=1,\dots,N, \tag{12}$$

$$\| d_{N,k} \| \leqslant r_0^{-k}K(H+\mu+\epsilon), \qquad k=1,\dots,N, \tag{13}$$

$$\| e_{N,k} \| \leqslant r_0^{-k-2}K(H+\mu+\epsilon), \quad k=1,\dots,N, \tag{14}$$

$$\| f_{N,k} \| \leqslant r_0^{-k-2}K(H+\mu+\epsilon), \quad k=1,\dots,N, \tag{15}$$

where

$$H = \max\{\| F(\theta) \|_{[0,2\pi]}, \| F_1(\theta) \|_{[0,2\pi]}, \| G(\theta) \|_{[0,2\pi]}\} \tag{16}$$

and

$$r_0 = \inf_{0\leqslant\theta\leqslant 2\pi} R(\theta). \tag{17}$$

The proof of this theorem is given in [4]. By the maximum principle of Miranda [8], it follows that

$$\max\left\{\| u(r, \theta) \|_G, \left\| \frac{\partial u}{\partial r}(r, \theta) \right\|_G, \left\| \frac{\partial u}{\partial \theta}(r, \theta) \right\|_G\right\} \leqslant K(H+\mu), \tag{18}$$

where K is a positive constant that depends only upon G. In the remainder of the paper, K shall denote a sufficiently large positive constant whose dependence on various parameters will be emphasized only if the dependence is crucial to the discussion.

Let Φ_m denote the finite set $0=\theta_0<\theta_1< \cdots <\theta_{m-1}<\theta_m=2\pi$. Replacing a_N by a, b_N by b, $c_{N,k}$ by c_k, etc. in (6), denote the resulting function as $u_N(r, \theta; P)$, where

$$P = (a, b, c_1,\dots, c_N, d_1,\dots, d_N, e_1,\dots, e_N, f_1,\dots, f_N)$$

is a point in Euclidean $(4N+2)$-space. Consider the continuous functions

$$E_{N,m}(P) = \max\left\{ \| F(\theta_j) - u_N\left(R(\theta_j), \theta_j; P\right) \|_{\Phi_m}, \right.$$
$$\left\| F_1(\theta_j) - \left[\frac{\mathrm{d}u_N}{\mathrm{d}\theta}(R(\theta), \theta; P)\right]_{\theta=\theta_j} \right\|_{\Phi_m},$$
$$\left.\left\| G(\theta_j) - \frac{\partial u_N}{\partial n}\left(R(\theta_j), \theta_j; P\right) \right\|_{\Phi_m} \right\} \tag{19}$$

on the compact sets

$$D_N = \{ P \mid a, \ldots, f_N \text{ satisfy the inequalities (10)–(15) with } H + \mu + \epsilon \text{ replaced by } H + \mu + 1 \}. \tag{20}$$

Let

$$\mathscr{I}_{N,m} = \inf_{D_N} E_{N,m}(P). \tag{21}$$

Since $E_{N,m}$ is a continuous function on the compact set D_N, there exists a point $P_{N,m}$ in D_N such that

$$\mathscr{I}_{N,m} = E_{N,m}(P_{N,m}). \tag{22}$$

Consider the biharmonic functions

$$u_{N,m}(r, \theta) = u_N(r, \theta; P_{N,m}), \quad N = 1,2,\ldots, \quad m = 0, 1, \ldots . \tag{23}$$

For these functions, it follows from the maximum principle of Miranda [8] and the triangle inequality that for any point (r, θ) in $\bar{G}$,

$$\max\left\{ | u(r, \theta) - u_{N,m}(r, \theta) |, \left| \frac{\partial u}{\partial r}(r, \theta) - \frac{\partial u_{N,m}}{\partial r}(r, \theta) \right|, \left| \frac{\partial u}{\partial \theta}(r, \theta) - \frac{\partial u_{N,m}}{\partial \theta}(r, \theta) \right| \right\}$$
$$\leqslant K\Big\{ \max[\omega(f, \eta_m, [0, 2\pi]), \omega(f', \eta_m, [0, 2\pi]), \omega(g, \eta_m, [0, 2\pi])]$$
$$+ \mu + \mathscr{I}_{N,m} + \max\Big[\omega(u_{N,m}, K_1\eta_m, \{ |z| \leqslant d \}),$$
$$\omega\left(\frac{\partial u_{N,m}}{\partial r}, K_1\eta_m, \{ |z| \leqslant d \}\right),$$
$$\omega\left(\frac{\partial u_{N,m}}{\partial \theta}, K_1\eta_m, \{ |z| \leqslant d \}\right)\Big]\Big\}, \tag{24}$$

where for any continuous function $g(\xi)$ defined on a connected set S,

$$\omega(g, \eta, S) = \sup_{\xi_1, \xi_2 \in S, |\xi_1 - \xi_2| \leqslant \eta} | g(\xi_1) - g(\xi_2) |, \tag{25}$$

$$\eta_m = \max_{i \leqslant j < m} | \theta_{j+1} - \theta_j |, \tag{26}$$

K_1 is a positive constant depending upon the smoothness of $R(\theta)$, and d is the diameter of G.

This result comes from the following theorem.

Theorem 4. *There exist two numbers K_1 and K_2 depending only on the domain $\bar{G}$ such that for every biharmonic function $u(x, y)$, continuous in G with its first derivatives and verifying the boundary conditions* (2) *and* (3), *and with $f'(\theta)$ and $g(\theta)$ continuous, it is true that*

$$\sqrt{u_x^2+u_y^2} \leqslant K_1[\max|g|+\max|f'|]+K_2 \max|f|, \tag{27}$$

$$|u(P)| \leqslant K_1\delta[\max|f|+\max|f'|]+(1+K_2\delta)\max|f|, \tag{28}$$

where δ is the distance of a point $P(r, \theta)$ from the boundary of G.

Relation (28) comes immediately from (27). To show the relation (27) we take a biharmonic function $v(x, y)$ continuous with the first derivatives in G verifying (2) and such that

$$\sqrt{v_x^2+v_y^2} \leqslant H_1 \max|f'|+H_2 \max|f|, \tag{29}$$

where H_1 and H_2 are convenient constants depending only on G.

It is in fact sufficient to observe that the function $u-v$ is zero on the boundary and such that the normal derivative satisfies

$$\left|\frac{\partial(u-v)}{\partial n}\right| \leqslant \max|g|+H_1 \max|f'|+H_2 \max|f| \tag{30}$$

to obtain

$$\sqrt{(u_x-v_x)^2+(u_y-v_y)^2} \leqslant K_1[\max|g|+H_1 \max|f'|+H_2 \max|f|] \tag{31}$$

and from all these relations, (27) follows.

From the result in (24), we can state an a posteriori estimate, that may be solved only numerically, in the following form:

$$\sigma(P, \mathscr{I}) \equiv \sigma(a, b, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})=\mathscr{I}, \tag{32}$$

where $\boldsymbol{c}=(c_1,\dots,c_N)$, $\boldsymbol{d}=(d_1,\dots,d_N)$, $\boldsymbol{e}=(e_1,\dots,e_N)$ and $\boldsymbol{f}=(f_1,\dots,f_N)$, subject to the linear inequalities

$$|F(\theta_j)-u_N(R(\theta_j), \theta_j; P)| \leqslant \mathscr{I}, \qquad j=0,\dots,m-1, \tag{33}$$

$$\left|F_1(\theta_j)-\left[\frac{\mathrm{d}u_N}{\mathrm{d}\theta}(R(\theta), \theta; P)\right]_{\theta=\theta_j}\right| \leqslant \mathscr{I}, \qquad j=0,\dots,m-1, \tag{34}$$

$$\left|G(\theta_j)-\frac{\partial u_N}{\partial n}(R(\theta_j), \theta_j; P)\right| \leqslant \mathscr{I}, \qquad j=0,\dots,m-1, \tag{35}$$

$$\mathscr{I} \geqslant 0, \tag{36}$$

and the linear inequalities that state the assertion that P is in D_N (20) is equal to $\mathscr{I}_{N,m}$.

4. The numerical algorithms

Since the minimization problem of the functional (36) subject to (33)–(35) is a feasible linear programming problem, it follows that $u_{N,m}$ can be determined by its solution [6]. We shall now consider an alternative method of approximating the solution of (1)–(3) subject to (4) by the method of least squares subject to a quadratic constraint. From the fact that $u(r, \theta)$ can be represented in the form

$$u(r, \theta)=\int_0^{2\pi}\{\gamma_1(r, \theta, \tau)f(\tau)+\gamma_2(r, \theta, \tau)f'(\tau)+\gamma_3(r, \theta, \tau)g(\tau)\}\ d\tau, \tag{37}$$

where γ_i, $i=1, 2, 3$, are the particular parts of the biharmonic kernel function for G which are associated with f, f' and g, it follows that for any compact subset D of G there exists a constant K depending on D and the γ_i, $i=1, 2, 3$, such that for any point (r, θ) in D,

$$|u(r, \theta)|+\left|\frac{\partial u}{\partial r}(r, \theta)\right|+\left|\frac{\partial u}{\partial \theta}(r, \theta)\right|\leqslant K\{\|f\|_2+\|f'\|_2+\|g\|_2\}, \tag{38}$$

where for any function f defined on $[0, 2\pi]$

$$\|f\|_2=\left[\int_0^{2\pi}|f(\theta)|^2\ d\theta\right]^{1/2}. \tag{39}$$

In all methods, the values of u_N, $(d/d\theta)u_N(r(\theta), \theta)$, $(\partial u_N/\partial n)$ are taken at the points P_j as those of $3m$ linear functions φ_j, χ_j, ψ_j of the variables a_N, $b_N,\dots, f_{NN}$, where

$$\begin{aligned}\varphi_j(a_N, b_N,\dots, f_{NN})&=a_N+b_Nr_j^2+\sum_{k=1}^N c_{Nk}r_j^k\cos k\theta_j+\sum_{k=1}^N d_{Nk}r_j^k\sin k\theta_j\\&\quad+\sum_{k=1}^N e_{Nk}r_j^{k+2}\cos k\theta_j+\sum_{k=1}^N f_{Nk}r_j^{k+2}\sin k\theta_j,\end{aligned}$$

$$\begin{aligned}&\chi_j(a_N, b_N,\dots, f_{NN})\\&\quad=2b_Nr_j^4(\alpha-\beta)\sin\theta_j\cos\theta_j\\&\qquad+\sum_{k=1}^N c_{Nk}kr_j^k\big(-\sin k\theta_j+r^2(\alpha-\beta)\sin\theta_j\cos\theta_j\cos k\theta_j\big)\\&\qquad+\sum_{k=1}^N d_{Nk}kr_j^k\big(\cos k\theta_j+r^2(\alpha-\beta)\sin\theta_j\cos\theta_j\sin k\theta_j\big)\\&\qquad+\sum_{k=1}^N e_{Nk}\big((k+2)r_j^{k+4}(\alpha-\beta)\sin\theta_j\cos\theta_j\cos k\theta_j-kr_j^{k+2}\sin k\theta_j\big)\\&\qquad+\sum_{k=1}^N f_{Nk}\big((k+2)r_j^{k+4}(\alpha-\beta)\sin\theta_j\cos\theta_j\sin k\theta_j+kr_j^{k+2}\cos k\theta_j\big),\end{aligned} \tag{40}$$

$$\begin{aligned}\psi_j(a_N, b_N,\dots, f_{NN})&=2b_Nr_j(\alpha-\beta)\cos(\theta_j-\gamma_j)+\sum_{k=1}^N c_{Nk}kr_j^{k-1}\cos(k\theta_j-\theta_j+\gamma_j)\\&\quad+\sum_{k=1}^N d_{Nk}kr_j^{k-1}\sin(k\theta_j-\theta_j+\gamma_j)\\&\quad+\sum_{k=1}^N e_{Nk}r_j^{k+1}\big(2\cos k\theta_j\cos(\theta_j-\gamma_j)+k\cos(k\theta_j-\theta_j+\gamma_j)\big)\\&\quad+\sum_{k=1}^N f_{Nk}r_j^{k+1}\big(2\sin k\theta_j\cos(\theta_j-\gamma_j)+k\sin(k\theta_j-\theta_j+\gamma_j)\big),\end{aligned}$$

where

$$\gamma_j = \tan^{-1}\left(-\frac{\alpha}{\beta}\frac{1}{\tan\theta_j}\right), \quad r_j = r(\theta_j).$$

The points P_j are uniformly distributed points over the boundary, here an ellipse ε is considered described by the equation

$$r(\theta) = \left(\alpha\cos^2\theta + \beta\sin^2\theta\right)^{-1/2}, \tag{41}$$

where α and β are the squares of the reciprocal of the semiaxes of ε.

To consider the problem as a "linear programming" problem let us introduce a new variable ζ such that

$$|\varphi_j - F(\theta_j)| \leqslant \zeta, \qquad |\chi_j - F_1(\theta_j)| \leqslant \zeta, \qquad |\psi_j - G(\theta_j)| \leqslant \zeta, \tag{42}$$

which may be written as

$$\begin{aligned} &\varphi_j - \zeta \leqslant F(\theta_j), && -\varphi_j - \zeta \leqslant -F(\theta_j),\\ &\chi_j - \zeta \leqslant F_1(\theta_j), && -\chi_j - \zeta \leqslant -F_1(\theta_j),\\ &\psi_j - \zeta \leqslant G(\theta_j), && -\psi_j - \zeta \leqslant -G(\theta_j). \end{aligned} \tag{43}$$

The linear function σ to be minimized is simply

$$\sigma(\zeta, a_N, b_N, \dots, f_{NN}) \equiv \zeta, \tag{44}$$

subject to simultaneous linear inequalities (43).

To consider our problem as a "least squares" problem let us rename the vector $(a_N, b_N, \dots, f_{NN})$ as $x = (x_0, x_1, \dots, x_{4N+1})$. The functions (40) will then be written as follows:

$$\varphi_j = \sum_{\mu=0}^{4N+1} y_{j\mu}x_\mu, \qquad \chi_j = \sum_{\mu=0}^{4N+1} w_{j\mu}x_\mu, \qquad \psi_j = \sum_{\mu=0}^{4N+1} z_{j\mu}x_\mu. \tag{45}$$

Now, let us consider the polynomial of second degree with respect to the x_μ:

$$L_{N,m} = \sum_{j=0}^{m-1}\left\{\left[F(\theta_j) - \varphi_j\right]^2 + \left[F_1(\theta_j) - \chi_j\right]^2 + \left[G(\theta_j) - \psi_j\right]^2\right\}. \tag{46}$$

We want to determine the point in $(4N+2)$-dimensional Euclidean space which makes $L_{N,m}$ a minimum. We consider the solution of the following simultaneous equations:

$$\sum_{\mu=0}^{4N+1} A_{\delta\mu}x_\mu = B_\delta, \quad \delta = 0, \dots, 4N+1, \tag{47}$$

where

$$B_\delta = \sum_{j=0}^{m-1}\left(F(\theta_j)y_{j\delta} + F_1(\theta_j)w_{j\delta} + G(\theta_j)z_{j\delta}\right),$$

$$A_{\delta\mu} = \sum_{j=0}^{m-1}\left(y_{j\mu}y_{j\delta} + w_{j\mu}w_{j\delta} + z_{j\mu}z_{j\delta}\right).$$

A new method is considered in the following to solve the biharmonic problem by a recursive projection algorithm [1–3]. We want to approximate a function $f(\theta)$ using a function $f^*(\theta)$ that can be expressed as a linear combination

$$f^*(\theta) = c_0\phi_0(\theta) + c_1\phi_1(\theta) + \cdots + c_k\phi_k(\theta) \tag{48}$$

of $k+1$ functions $\phi_0, \phi_1, \ldots, \phi_k$, which are chosen in advance and $c_0, c_1, \ldots, c_k$ are coefficients whose value is to be determined. The function f is known at the m distinct points P_j, $j = 0, \ldots, m-1$. We want to determine the $k+1$ parameters $c_0, c_1, \ldots, c_k$, such that $f^*(\theta_j) = f(\theta_j)$ holds exactly or as close as possible at all the m points. In the case of least-squares approximation we determine the coefficients such that the Euclidean norm of the error function $f^* - f$ becomes as small as possible, that is, such that

$$\| f^* - f \|_2^2 = \sum_{j=0}^{m-1} | f^*(\theta_j) - f(\theta_j) |^2$$

becomes as small as possible.

In our case the $k+1$ functions are the $4N+2$ functions of the biharmonic polynomial $\phi_0 = 1$, $\phi_1 = r^2(\theta), \ldots, \phi_k$, and the undetermined coefficients are $(a_N, b_N, \ldots, f_{NN})$. The chosen points P_j are such that the number of the equations is less than the number of the unknowns and then we have an undetermined least squares problem that becomes an overdetermined least squares problem adding the tangential and normal derivatives known at the P_j points ($3m > 4N+2$). It is possible to obtain the solution to this linear system, in the least squares sense, through the recursive projection algorithm with the assumption that we consider the real vector space of functions E in which the Euclidean norm becomes the inner product of two real-valued continuous functions:

$$\| \cdot \|_2^2 = (f, g) = \sum_{j=0}^{m-1} f(\theta_j) g(\theta_j).$$

Then we define $3m$ linear functionals

$$\begin{aligned} &L_{1j}(\phi_\mu) = \phi_\mu(\theta_j), \\ &L_{2j}(\phi_\mu) = \frac{d\phi_\mu}{d\theta}(\theta_j), \quad j = 0, \ldots, m-1, \\ &L_{3j}(\phi_\mu) = \frac{\partial\phi_\mu}{\partial n}(\theta_j), \quad \mu = 0, \ldots, 4N+1, \end{aligned} \tag{49}$$

with

$$L_{1j}(f) = F(\theta_j), \qquad L_{2j}(f) = F_1(\theta_j), \qquad L_{3j}(f) = G(\theta_j). \tag{50}$$

Then the overdetermined linear system represented by $Ax = b$ with A a real $(3m \times (4N+2))$-matrix and b a $(3m)$-vector assume the form:

$$\begin{pmatrix} \phi_0(\theta_j) & \phi_1(\theta_j) & \cdots & \phi_{4N+1}(\theta_j) \\ \frac{d\phi_0}{d\theta}(\theta_j) & \frac{d\phi_1}{d\theta}(\theta_j) & \cdots & \frac{d\phi_{4N+1}}{d\theta}(\theta_j) \\ \frac{\partial\phi_0}{\partial n}(\theta_j) & \frac{\partial\phi_1}{\partial n}(\theta_j) & \cdots & \frac{\partial\phi_{4N+1}}{\partial n}(\theta_j) \end{pmatrix} (x) = \begin{pmatrix} F(\theta_j) \\ F_1(\theta_j) \\ G(\theta_j) \end{pmatrix}, \quad j = 0, \ldots, m-1, \tag{51}$$

or, in the functional form:

$$\begin{pmatrix} L_{1j}(\phi_0) & L_{1j}(\phi_1) & \cdots & L_{1j}(\phi_{4N+1}) \\ L_{2j}(\phi_0) & L_{2j}(\phi_1) & \cdots & L_{2j}(\phi_{4N+1}) \\ L_{3j}(\phi_0) & L_{3j}(\phi_1) & \cdots & L_{3j}(\phi_{4N+1}) \end{pmatrix}(x) = \begin{pmatrix} L_{1j}(f) \\ L_{2j}(f) \\ L_{3j}(f) \end{pmatrix}. \tag{52}$$

If we consider the $(4N+2)$-dimensional linear subspace spanned by the functions ϕ_μ the least-squares problem may be proposed as a geometric problem in a function space, because we look for the direction of the vector having the shortest distance to the vector represented by f. Then, if we are interested only in computing the approximate value, it is not necessary to compute the coefficients x_μ and we can use the E-algorithm [1] after modifying the initializations according to the Mühlbach–Neville–Aitken algorithm [9].

Having

$$L(f) = f(\theta), \qquad (f, g) = \sum_{j=0}^{m-1} \left(L_{1j}(f)L_{1j}(g) + L_{2j}(f)L_{2j}(g) + L_{3j}(f)L_{3j}(g) \right), \tag{53}$$

where θ is the angle whose values must be extrapolated, we utilize the recursive formulas

$$\begin{aligned} f_k^{*(n)} &= f_{k-1}^{*(n)} + \phi_{k-1,k}^{(n)} \frac{f_{k-1}^{*(n)} - f_{k-1}^{*(n+1)}}{\phi_{k-1,k}^{(n+1)} - \phi_{k-1,k}^{(n)}}, \\ \phi_{k,i}^{(n)} &= \phi_{k-1,i}^{(n)} + \phi_{k-1,k}^{(n)} \frac{\phi_{k-1,i}^{(n)} - \phi_{k-1,i}^{(n+1)}}{\phi_{k-1,k}^{(n+1)} - \phi_{k-1,k}^{(n)}}, \quad i = k+1,\ k+2, \ldots, \end{aligned} \tag{54}$$

with the initializations

$$f_0^{*(n)} = (\phi_n, f)\frac{L(\phi_0)}{(\phi_n, \phi_0)}, \qquad \phi_{0,i}^{(n)} = (\phi_n, \phi_i)\frac{L(\phi_0)}{(\phi_n, \phi_0)} - L(\phi_i), \tag{55}$$

and we shall obtain

$$f_{4N+1}^{*(0)} = L(f^*).$$

5. Some numerical results

Considering the comparison among the different methods some tables have been prepared where the input parameters are the integers N and m, and the two parameters α and β of the ellipse, with $\alpha = 0.5$ and $\beta = 1.0$. The results are presented for linear programming, least squares and the E-algorithm applied to three functions: $f = r \cos \theta$, $f = 1 + x^2 y$ and $f = 1 + \sinh y \sin x$. The case of the third function is the most interesting because this function cannot be expressed as a sum of a finite number of terms of a biharmonic function.

The accuracy of the approximation obtained in the first method is expressed by the value of ζ. The approximation obtained over the entire ellipse was good only where a small value of ζ was obtained.

Table 1
Results concerning the function $f = r\cos\theta$

m	$4N+2$	ζ (LP-90)	l (least squares)	E_1 (E-alg.)
4	6	0.19	0.175	$<10^{-14}$
8	6	$<10^{-6}$	$<10^{-6}$	$<10^{-13}$
8	10	$<10^{-6}$	$<10^{-6}$	$<10^{-12}$

Table 2
Results concerning the function $f = 1 + x^2y$

m	$4N+2$	ζ (LP-90)	l (least squares)	E_1 (E-alg.)
6	14	0.36	0.58	$<10^{-10}$
8	14	$<10^{-6}$	$<10^{-6}$	$<10^{-11}$
10	14	0.93	0.75	$<10^{-11}$
12	14	$<10^{-6}$	$<10^{-6}$	$<10^{-10}$

Table 3
Results concerning the function $f = 1 + \sinh y \sin x$ (linear programming method)

m	$4N+2$	ζ	R	$E_{1/4}$	$E_{1/2}$	$E_{3/4}$	E_1
8	14	0.28	0.03	0.19	0.11	0.06	0.28
8	22	$<10^{-3}$	$<10^{-4}$	$<10^{-3}$	$<10^{-2}$	$<10^{-3}$	$<10^{-3}$
16	14	0.38	0.14	0.29	0.23	0.19	0.38
16	22	0.24	0.08	0.23	0.19	0.15	0.31
32	22	0.02	0.01	0.02	0.02	0.01	0.02

In the second method the approximation is measured by the quantity

$$l = \sqrt{\frac{L_{N,m}}{3m - (4N+2)}},$$

and the solutions obtained for the coefficients of the biharmonic functions have been applied on as many as 1000 points.

In the third method E_1 represents the value $|f(\theta) - f^*(\theta)|$ over the boundary of ε.

In Tables 3–5, R means the error at the origin and $E_{1/4}$, $E_{1/2}$, $E_{3/4}$ are respectively the computed suprema of $|u - u_N| \equiv |f - f^*|$ on the three ellipses ε_μ which belong to the region G, concentrical and homothetical with ε, with ratio μ with respect to ε.

Table 4
Results concerning the function $f = 1 + \sinh y \sin x$ (least squares method)

m	$4N+2$	l	R	$E_{1/4}$	$E_{1/2}$	$E_{3/4}$	E_1
8	14	0.28	$<10^{-7}$	$<10^{-2}$	$<10^{-2}$	$<10^{-2}$	0.04
8	22	$<10^{-3}$	$<10^{-7}$	$<10^{-3}$	$<10^{-3}$	$<10^{-3}$	$<10^{-3}$
16	14	0.20	0.04	0.09	0.09	0.07	0.08
16	22	0.14	0.04	0.08	0.09	0.06	0.02
16	42	$<10^{-4}$	0.03	0.10	0.05	0.06	$<10^{-5}$
32	14	0.175	$<10^{-8}$	0.06	0.03	$<10^{-2}$	0.08
32	22	$<10^{-2}$	$<10^{-6}$	$<10^{-2}$	$<10^{-2}$	$<10^{-3}$	$<10^{-2}$
32	42	$<10^{-5}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$
32	62	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$	$<10^{-6}$
64	42	0.175	$<10^{-2}$	0.04	0.01	$<10^{-2}$	0.41
64	62	0.175	$<10^{-2}$	0.02	0.01	$<10^{-2}$	0.04
64	82	$<10^{-1}$	$<10^{-2}$	0.02	0.01	$<10^{-2}$	0.04

Table 5
Results concerning the function $f = 1 + \sinh y \sin x$ (E-algorithm)

m	$4N+2$	R	$E_{1/4}$	$E_{1/2}$	$E_{3/4}$	E_1
8	14	$<10^{-2}$	1.0	0.14	0.04	0.01
8	22	$<10^{-2}$	0.55	0.03	$<10^{-2}$	$<10^{-2}$
16	14	$<10^{-2}$	0.74	0.01	0.03	0.01
16	22	$<10^{-3}$	0.12	$<10^{-2}$	$<10^{-2}$	$<10^{-3}$
16	42	$<10^{-8}$	$<10^{-4}$	$<10^{-6}$	$<10^{-7}$	$<10^{-7}$
32	14	$<10^{-2}$	0.46	0.06	0.02	$<10^{-2}$
32	22	$<10^{-4}$	0.07	$<10^{-2}$	$<10^{-2}$	$<10^{-3}$
32	42	$<10^{-8}$	$<10^{-4}$	$<10^{-5}$	$<10^{-5}$	$<10^{-7}$
32	62	$<10^{-6}$	$<10^{-5}$	$<10^{-5}$	$<10^{-5}$	$<10^{-5}$
64	42	$<10^{-7}$	0.03	$<10^{-2}$	$<10^{-2}$	$<10^{-2}$
64	62	$<10^{-4}$	0.06	$<10^{-2}$	0.02	$<10^{-2}$
64	82	$<10^{-3}$	0.88	0.08	0.01	$<10^{-2}$

Since the experiments have been made on some known function f, the order of approximation is evaluated in points that are not taken as nodes.

The E-algorithm compares very well in accuracy with the other methods, however, the algorithm is totally reapplied on each test point. If only a few points are needed, the algorithm is very convenient from the point of view of the computing time; furthermore, the accuracy appears to be greater than for the other methods. In this case the RPA [3] can be used instead of the E-algorithm for implementing the procedure.

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