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ON A VARIATIONAL PROBLEM OF SLOW GROWTH WITHOUT CONVEXITY ASSUMPTIONS

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1. INTRODUCTION

THE CLASSICAL approach to the problem of the calculus of variations of

$$\text{minimizing } I(x) = \int_0^T f(t, x(t), x'(t))\,\mathrm{d}t\colon x(0) = a,\ x(T) = b,\ x \in W^{1,p}([0, T], \mathbb{R}^n) \qquad \text{(P)}$$

is the following: by imposing superlinear growth conditions at infinity and Tonelli's assumption of convexity with respect to x', the integral functional I is weakly lower semicontinuous and each minimizing sequence is contained in a weakly compact set so that each of its limit points is a solution to (P). However, the above conditions are not necessary in order to obtain existence.

The first problem that has been investigated outside the realm of l.s.c. is that of minimizing

$$I(x) = \int_0^T f(t, x'(t))\,\mathrm{d}t\colon x(0) = a,\ x(T) = b,\ x \in W^{1,p}([0, T], \mathbb{R}^n)$$

under superlinear growth conditions but without Tonelli's assumption on f w.r. to x': Olech proved in [1] that a solution does always exist. When the variable x is involved, some other existence results have been given in [2–5]. The main difference with respect to the classical approach is that in their proofs, these authors build a solution. In the case where the independent variable is not a scalar, the same reasonings do not hold. The reason is as follows: an extension of Lyapunov's theorem on the range of a vector measure [6, Section 16] allows [3–5] substitution of another function to a solution to the relaxed problem, a candidate for being a solution to the original problem. This new function is not defined directly; rather, for problems seeking a solution $\bar{x}(t)$, one defines a measurable function $u(t)$ and $\bar{x}(t)$ is the primitive of u, i.e. its integral; instead for problems involving the gradient, a measurable function u is not, in general, the gradient of some function $\bar{x}$. Nevertheless, for some nonconvex problems involving the gradient, a necessary and sufficient condition for the existence of a solution has been given in [7, 8].

If a weaker growth condition is assumed,

$$f(t, x') \geq \alpha(t) + \gamma|x'| \qquad (\gamma > 0,\ \alpha \in L^1)$$

then Tonelli's convexity assumption is no longer sufficient for the existence of a solution (see [6, Section 14]), so that, opposite to the superlinear case, the associated relaxed problem does not admit, in general, a solution. The "convex cases" for which a solution exists are treated, for instance, in [6, Section 14].

In [9], we proved that if f is positive homogeneous of degree one in x' (i.e. $I(x) = \int_0^T f(x'(t))\,dt$ is a parametric integral), then Cesari–Tonelli's convexity assumption [6, 14.1.iv] can be omitted for the existence of a solution.

Here, we consider the problem of the minimum for the integral functional $I(x)$

$$I(x) = \int_{t_1}^{t_2} h(t, x'(t))\,dt$$

on the set of the absolutely continuous (A.C.) functions $x(t) = (x^1, \ldots, x^n)$, $0 \leq t_1 \leq t_2 \leq T$ such that $(t_1, x(t_1), t_2, x(t_2))$ belongs to a prescribed compact subset of $\mathbb{R}^{2n+2}$. Under slow growth conditions and Tonelli's assumption on h, it has been considered in [6, theorem 14.3.i–14.3.ii]. The purpose of this paper is to give conditions not involving convexity w.r. to x' in order to have a solution.

The idea underlying the proof of the main theorems is the following: under the assumption that the convex hull of the epigraph of the nonconvex integrand is closed, the relaxed associated problem can be formulated in the version of [10]. Then each of its minimizers (whether they exist) can be modified as in [3, 9] in order to obtain a minimizer to the original problem.

ASSUMPTIONS AND PRELIMINARY RESULTS

Let us denote by coepi(h) the convex hull of the epigraph of a function h. We shall assume the following hypothesis.

Hypothesis Φ. The function h: $[0, T] \times \mathbb{R}^n \to \mathbb{R}$ is such that
(h_1) $t \mapsto h(t, x')$ is measurable for each x';
(h_2) $x' \mapsto h(t, x')$ is continuous for a.e. t.
Moreover, there exist a positive constant a and a function $\alpha(\cdot)$ in L^1 such that
(h_3) $h(t, x') \geq a|x'| + \alpha(t)$.

Our main tool in this approach to nonconvex noncoercive problems is the following proposition, a consequence of the proof of Cellina–Colombo's theorem [3] and proposition IX.3.1 in [10].

PROPOSITION 1. Let h satisfy Φ such that the relaxed problem (PR) associated to (P) admits at least one solution. Assume further that coepi($h(t, \cdot)$) is closed for a.e. t. Then (P) admits at least one solution.

Sketch of the proof. Let $\tilde{x}$ be a solution to (PR). Then by the proof of lemma IX.3.1 in [10], there exist measurable

$$p_1, \ldots, p_{n+1}: [0, T] \to [0, 1]\left(\sum_i p_i \equiv 1\right), \qquad v_1, \ldots, v_{n+1}: [0, T] \to \mathbb{R}^n$$

such that

$$\tilde{x}' = \sum_i p_i v_i; \qquad h^{**}(t, \tilde{x}(t)) = \sum_i p_i(t) h(t, v_i(t)).$$

Now, the same argument of the proof of [3, theorem 1] enables us to find a measurable partition $E_1, \dots, E_{n+1}$ of $[0, T]$ such that

$$\sum_{i=1}^{n+1} v_i \chi_{E_i} \in L^1, \qquad \int_0^T \sum_{i=1}^{n+1} v_i(t)\chi_{E_i}(t)\,\mathrm{d}t = \tilde{x}(T) - \tilde{x}(0),$$

and

$$\int_0^T h\left(t, \sum_{i=1}^{n+1} v_i(t)\chi_{E_i}(t)\right)\mathrm{d}t = \int_0^T h^{**}(t, \tilde{x}'(t))\,\mathrm{d}t.$$

Clearly,

$$\bar{x}(t) = \tilde{x}(0) + \int_0^t \sum_{i=1}^{n+1} v_i(s)\chi_{E_i}(s)\,\mathrm{d}s$$

is a solution to (P).

Remark 1. Ekeland and Temam [10, lemma IX.3.3] proved that, for any function h satisfying (h_1), (h_2) and bounded below by $c|x'|^p + \gamma(t)$ $(c > 0, p > 1, \gamma \in L^1)$, then coepi$(h(t, \cdot))$ is closed a.e. for t.

Instead, it can be easily shown that, under slow growth conditions, the same conclusion does not hold, in general.

For $n = 1$, coepi(h) is closed if, for instance, h is continuous and $h = h^{**}$ in the complement of an interval $I = [a, b]$ in which the graph of h^{**} is a line joining the points $[a, h(a)]$ and $[b, h(b)]$. In fact, in this situation coepi(h) coincides with the epigraph of h^{**}, a closed set (the function h being continuous).

Remark 2. In [9, 11], we proved that coepi(h) is closed if h is positive homogeneous of degree one, so that, by [6, theorem 14.1.iv] the relaxed associated problem admits at least one solution. Then, by proposition 1, so does the original problem.

Let us denote by $H(t, p, u)$ the parametric integrand associated to the bipolar $h^{**}(t, x')$ of h [6, Chapter 14] defined by

$$H: [0, T] \times]0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$$

$$(t, p, u) \mapsto ph^{**}\left(t, \frac{u}{p}\right).$$

As described in [6], the function H is convex in (p, u). As a consequence, if h^{**} is supposed to be continuous, if (t, u) is fixed and we allow $p > 0$ to approach zero, then $H(t, p, u)$ must approach a finite limit or $+\infty$. This limit is taken as the definition of $H(t, 0, u)$. Since $H(t, kp, ku) = kH(t, p, u)$ for all $k > 0$, we define $H(t, 0, 0)$ to be zero, so that the homogeneity property holds for $k \geq 0$. It can be shown [6] that if h^{**} is continuous in its domain and $H(t, 0, u)$ is finite everywhere then $H(t, p, u)$ is continuous in $[0, T] \times [0, +\infty[\times \mathbb{R}^n$.

MAIN RESULTS

Let K_1, K_2 be two compact subsets of $\mathbb{R}^{n+1}$ such that, for every $(t_1, x_1, t_2, x_2) \in K_1 \times K_2$, we have $t_1 < t_2$ and set $K = K_1 \times K_2$. We consider the problem of the minimum of the integral

$$I(x) = \int_{t_1}^{t_2} h(t, x'(t))\, \mathrm{d}t$$

in the class Ω of all A.C. functions $x(t) = (x^1, \ldots, x^n)$, $(t_1, x(t_1), t_2, x(t_2)) \in K$ (we say that these are the admissible trajectories). The following existence theorems are the nonconvex analogues of [6, theorems 14.3.i and 14.3.ii].

THEOREM 1. Let h satisfy hypothesis Φ and be such that coepi$(h(t, \cdot))$ is closed a.e. for t. Assume further that the bipolar of h is of class $\mathcal{C}^1$ in its domain and that its associated parametric integrand H is continuous in $[0, T] \times [0, +\infty[\times \mathbb{R}^n$. Moreover, assume that

$$\forall\, t \in [0, T], \qquad \forall\, u \in \mathbb{R}^n, \qquad |u| = 1: \qquad \frac{\partial H}{\partial p}(t, 0, u) = -\infty$$

and there are constants M_1, M_2, $\delta > 0$ such that for all $t \in [0, T]$, $(p, u) \in [0, +\infty[\times \mathbb{R}^n$, $|p| + |u| = 1$ and t^* with $|t^* - t| < \delta$ we have

$$\left|\frac{\partial H}{\partial t}(t^{**}, p, u)\right| \leq M_1 H(t, p, u) + M_2.$$

Then, $I(x)$ has an absolute minimum in the class of all admissible trajectories.

THEOREM 2. Let h be independent of t, satisfy hypothesis Φ and be such that coepi$(h(\cdot))$ is closed for a.e. t. Assume that the parametric integrand H associated to the bipolar of h is continuous in $[0, +\infty[\times \mathbb{R}^n$ with continuous partial derivative $\partial H/\partial p$ in $[0, +\infty[\times \mathbb{R}^n/\{0, 0\}$. Assume further that for every $u \neq 0$ we have

$$\frac{\partial H}{\partial p}(t, p, u) = 0 \qquad \text{if and only if } p = 0.$$

Then, $I(x)$ has an absolute minimum in the class Ω of all admissible trajectories.

Proof of theorem 1. We claim that the relaxed problem

$$\text{minimize } I^{**}(x) = \int_{t_1}^{t_2} h^{**}(t, x'(t))\, \mathrm{d}t \tag{1}$$

admits a solution in the class of all admissible trajectories. For this purpose, we show that the assumptions of [6, theorem 14.3.i] are satisfied.

Let $a > 0$ and $\alpha \in L^1$ be such that

$$h(t, x') \geq a|x'| + \alpha(t).$$

Since the map $x' \mapsto a|x'| + \alpha(t)$ is convex and continuous, then the above inequality holds for h^{**} instead of h. As a consequence, if $x(t)$, $t_1 \leq t \leq t_2$ is an admissible trajectory, then we have

$$I^{**}(x) \geq \int_{t_1}^{t_2} a|x'(t)|\, \mathrm{d}t - \int_{t_1}^{t_2} |\alpha(t)|\, \mathrm{d}t.$$

Hence, if $t \in [t_1, t_2]$, the following inequality holds

$$|x(t) - x(t_1)| \leq AI^{**}(x) + D, \tag{2}$$

where $A = 1/a$ and $D = (\int_0^T |\alpha(t)|\, dt)/a$.

Let C_1 be the projection of the compact subset K_1 of $[0, T] \times \mathbb{R}^n$ onto $\mathbb{R}^n$. Then, if we fix x in Ω and if y is any admissible trajectory such that $I^{**}(y) \leq I^{**}(x)$, by (2) we have

$$y(t) \in C_1 + B(AI^{**}(x) + D) = C,$$

$B(\cdot)$ denoting the closed ball in $\mathbb{R}^n$ of radius $(\cdot)$.

It follows that the relaxed problem (1) is equivalent to minimizing I^{**} in the class of all A.C. functions x with $(t, x(t)) \in [0, T] \times C$ and $(t_1, x(t_1), t_2, x(t_2)) \in K$. Hence, the sets C and K being compact, in order to prove the above assumption, it is enough to show that condition (λ) of [6, theorem 14.3.i] holds if it is applied to the parametric integral associated to H.

Let $\mathcal{C}: t = t(s),\ X = X(s),\ 0 \leq s \leq L$ be a rectifiable parametric curve with graph in $[0, T] \times C$ such that $(t(0), X(0), t(L), X(L)) \in K$, $t(s)$ being monotone nondecreasing and s being the arc length parameter.

Let

$$\mathcal{I}^{**}(\mathcal{C}) = \int_0^L H(t(s), t'(s), X'(t(s)))\, ds$$

be the parametric integral associated to the nonparametric integral I^{**}. By the above arguments, it turns out that there exist $A > 0$ and D such that

$$L(\mathcal{C}) \leq A\mathcal{I}^{**}(\mathcal{C}) + D.$$

Hence, condition (λ) holds if we set $\Phi(\xi) = A\xi + D$. Then, proposition 1 yields the conclusion.

The proof of theorem 2 is based on the same arguments and on [6, theorem 14.3.ii].

Remark 3. It does not seem reasonable, in order to satisfy the assumptions of theorems 1 and 2, to require conditions only on h instead of h^{**}. For instance, the function defined by

$$h(\xi) = |\xi|(1 + \sin(2\pi\xi))$$

is such that the limit as p approaches zero of its associated parametric integrand $\tilde{H}(p, u) = |u|(1 + \sin(2\pi(u/p)))$ does not exist, whereas the parametric integrand H associated to the bipolar of h, given by $H(p, u) = |u|$, is continuously differentiable in $[0, +\infty[\times \mathbb{R}$.

Example. Let us consider the following nonconvex continuously differentiable function h defined by

$$h(\xi) = \begin{cases} \sqrt{(1 + \xi^2)} & \text{if } |\xi| \geq \pi, \\ \sqrt{(1 + \xi^2)} + \cos(\xi) + 1 & \text{otherwise.} \end{cases}$$

Let x_0 be the point in $]0, \pi[$ such that $h'(x_0) = 0$. It can be shown that the bipolar of h is given by

$$h^{**}(\xi) = \begin{cases} h(\xi) & \text{if } |\xi| \geq x_0; \\ h(x_0) & \text{if } |\xi| \leq x_0. \end{cases}$$

We claim that the parametric integrand H associated to the bipolar of h satisfies the conditions of theorem 2.

In fact, $h(\xi) \geq |\xi|$ and remark 1 shows that coepi(h) is closed. Furthermore, H is clearly of class $\mathcal{C}^1$, and, if we let $u \neq 0$, its partial derivative with respect to p is given by

$$\frac{\partial H}{\partial p}(p, u) = \begin{cases} \dfrac{p}{\sqrt{(p^2 + u^2)}} & \text{if } \dfrac{|u|}{p} \geq \pi; \\ h\left(\dfrac{u}{p}\right) - \dfrac{u}{p} h'\left(\dfrac{u}{p}\right) & \text{if } \pi \geq \dfrac{|u|}{p} \geq x_0; \\ h(x_0) & \text{if } \dfrac{|u|}{p} \leq x_0. \end{cases}$$

Since $h(x_0) \neq 0$ and for each $\xi \in [x_0, \pi)$, we have $h'(\xi) \neq h(\xi)/\xi$ then $(\partial H/\partial p)(u, p) = 0$ if and only if $p = 0$.

The problem of

$$\text{minimizing} \int_0^1 h(x'(t))\, \mathrm{d}t: \qquad x(0) = a, \qquad x(1) = b$$

admits, by theorem 3, at least one solution.

Clearly, if $|b - a| \leq x_0$ then each measurable selection $u(t)$ of $\{-x_0, x_0\}$ satisfying

$$\int_0^1 u(s)\, \mathrm{d}s = b - a$$

is the derivative of an admissible minimizer. Otherwise, if $|b - a| > x_0$, a minimizer cannot be easily found.

Let us remark further that, in general, there are solutions to the relaxed problem that are not solutions to the original one. This is the case, for instance, if $a = b = 0$: in this situation $\bar{x}(t) \equiv 0$ is a solution to (PR) but $I(0) > I^{**}(0)$.

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