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014

A NONCONVEX VARIATIONAL PROBLEM WITH CONSTRAINTS*

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Abstract. A multidimensional version of Liapunov-type theorems is proven. As an application, it is proven that, under proper hypothesis on the possibly nonconvex function f, the problem $\min \int_0^T f(u'(t))\,dt$ on the subset of $W^{1,p}([0,T],\mathbb{R}^n)$ of those functions u satisfying the prescribed boundary conditions and whose trajectory lies out of a prescribed open subset of $\mathbb{R}^n$ admits at least one solution.

Key words. relaxed problem, bipolar, Liapunov, simplex, convex, extremal point

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1. Introduction. The most general scalar problem that has been investigated without the classical Tonelli convexity condition on the function $\xi \to h(t,s,\xi)$ is that of minimizing

$$\text{(P}'\text{)}\qquad \min I(u) = \min \left\{ \int_0^T h(t,u(t),u'(t))\,dt \right\}, \\ u \in W^{1,p}([0,T],\mathbb{R}^n), \quad u(0)=a, \quad u(T)=b.$$

Under differentiability assumptions on the integrand, this problem was studied by Aubert and Tahraoui in [2] and [3], Raymond in [11], and Tahraoui in [13].

In the case $h(t,s,\xi) = g(t,s) + f(t,\xi)$, this problem was studied by Olech (see [10]), Marcellini (see [8]), Cellina and Colombo (see [4]), and Raymond (see [12]), under weaker assumptions on the regularity of g and f.

In particular, in [4] the main tool is a Liapunov-type theorem, which allows the modification of a solution to the convexified problem in order to obtain a solution of the original one. The same technique has also been used in [12] and [9].

For $n = 1$, i.e., for functions with values in $\mathbb{R}$, a more precise version of Liapunov's theorem has recently been given in [1].

THEOREM 1.1. *Let $\Phi : [0,T] \to 2^{\mathbb{R}}$ be a measurable multifunction with values in the closed intervals of $\mathbb{R}$. Then for each integrable selection $\tilde{u}'$ of $\Phi(t)$, there exists a measurable selection $\bar{u}'$ with values in the extreme points of $\Phi(t)$ such that $\int_0^T \bar{u}'(t)\,dt = \int_0^T \tilde{u}'(t)\,dt$ and for each $t \in [0,T]$, $\bar{u}(t) \le \tilde{u}(t)$.*

This result has been successfully applied in [5] in order to prove that there exists a dense subset D of $C([0,T],\mathbb{R})$ such that, for g in it, the problem of minimizing $\int_0^T g(u(t))\,dt + \int_0^T f(u'(t))\,dt$ does always admit at least one solution for each f satisfying growth conditions.

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A more than one-dimensional version of the above Liapunov type lemma does not hold, in general.

Example. Let $n = 2$, $T = 1$, $\Phi(t) = \{\lambda(1,t) \ : \lambda \in [0,1]\}$, $\tilde{u}'(t) = (\tilde{u}_1'(t), \tilde{u}_2'(t)) = (\frac{1}{2}, (\frac{1}{2})t) \in \Phi(t)$ a.e. in $[0,1]$. Assume, by contradiction, that there exists $\bar{u}'(t) = (\bar{u_1}'(t), \bar{u_2}'(t)) \in \{(0,0),(1,t)\}$ a.e. such that

$$\int_0^1 \bar{u}'(t)\,dt = \int_0^1 \tilde{u}'(t)\,dt, \tag{1.1}$$

$$\bar{u}_1(t) \geq \tilde{u}_1(t) \quad \text{for a.e. } t \in [0,1], \tag{1.2}$$

$$\bar{u}(0) = \tilde{u}(0). \tag{1.3}$$

Then there exists a measurable subset E of $[0,1]$ such that

$$\bar{u}'(t) = (0,0)\chi_{[0,1]\setminus E} + (1,t)\chi_E;$$

whence, $\bar{u}_2'(t) = t\bar{u}_1'(t)$. Conditions (1.1) and (1.3) and integration by parts of the second component give

$$\int_0^1 \bar{u}_1(t)\,dt = \int_0^1 \tilde{u}_1(t)\,dt$$

so that, by (1.2), $\bar{u}_1(t) = \tilde{u}_1(t)$, i.e., $\chi_E = \frac{1}{2}$. This is a contradiction.

Neverthless, we prove here that a multidimensional version of the above theorem holds if the measurable function Φ is identically equal to a convex bounded subset of $\mathbb{R}^n$. As an application, we study the problem of minimizing

$$\int_0^T f(u'(t))\,dt$$

on the subset of $W^{1,p}([0,T],\mathbb{R}^n)$ of those functions u satisfying prescribed boundary conditions and whose trajectory lies out of a prescribed open subset of $\mathbb{R}^n$.

2. Notation and preliminary results. In the following, Γ will denote an open convex poligone contained in $\mathbb{R}^n$ and, given $a, b \in \mathbb{R}^n \setminus \Gamma$, K will be the set of those functions $u : [0,T] \to \mathbb{R}^n$ that are in the Sobolev space $W^{1,p}((0,T),\mathbb{R}^n)$ ($p \geq 1$) and such that $u(0) = a, u(T) = b$.

Given a set A, we denote by ∂A the boundary of A, by $\text{extr}A$ the extremal points of A, and by meas(A) the Lebesgue measure of A. Finally, given two vectors v_1 and v_2 of $\mathbb{R}^n$, we denote by $v_1 \cdot v_2$ the usual scalar product in $\mathbb{R}^n$ and by $|v_1|$ the euclidean norm of v_1 in $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnecessarily convex and lower semicontinuous function that satisfies the following growth conditions:

$$\text{(F)} \qquad \begin{aligned} c_1|\xi|^p - c_2 &\leq f(\xi) \quad \forall \xi \in \mathbb{R}^n \quad \text{if } p > 1, \\ \psi(|\xi|) - c_2 &\leq f(\xi) \quad \forall \xi \in \mathbb{R}^n \quad \text{if } p = 1, \end{aligned}$$

where c_1 and c_2 are real positive constants and $\psi : [0,+\infty) \to [0,+\infty)$ is a convex and lower semicontinuous function such that $\lim_{r\to+\infty} \frac{\psi(r)}{r} = +\infty$.

Given a function f, we denote by f^{**} its bipolar function.

LEMMA 2.1 (see, for instance, [6, Prop. I.4.1]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Then f^{**} is the greatest lower semicontinuous and convex function not greater than f.*

Let us consider the set

$$E = \{\xi \in \mathbb{R}^n \ : \ f^{**}(\xi) < f(\xi)\}. \tag{2.1}$$

In the following, we shall assume that $E = \bigcup_{i\in\mathbb{N}} A_i$ and f^{**} is affine on every A_i, where A_i is a convex open and bounded subset of $\mathbb{R}^n$.

Notice that, while the hypothesis on the structure of E is quite natural, the hypothesis on the form of f^{**} on every A_i is a technical hypothesis, which is automatically satisfied only in the scalar case (i.e., when $n = 1$). On the contrary, when n is strictly greater than one, this hypothesis is not always fulfilled.

LEMMA 2.2. *Let A be a simplex in $\mathbb{R}^n$, $I \subseteq [0,T]$ be a measurable set, $u' : [0,T] \to A$ be a measurable function, and η be a fixed vector in $\mathbb{R}^n$. Then there exists a measurable function, $\omega : [0,T] \to \operatorname{extr} A$ depending on u', A, and η, such that*

$$\int_I \omega(s)\, ds = \int_I u'(s)\, ds, \tag{2.2}$$

$$\forall t \in [0,T] \quad \int_0^t [\omega(s) \cdot \eta\chi_I(s)]\, ds \geq \int_0^t [u'(s) \cdot \eta\chi_I(s)]\, ds. \tag{2.3}$$

Proof. Let $v_0, \dots, v_n$ be the $n+1$ vertices of the simplex A; then $u'(s) = p_0(s)v_0 + \cdots + p_n(s)v_n$ for a proper choice of $p_0, \dots, p_n : [0,T] \to [0,1]$ with $p_0(s) + \cdots + p_n(s) \equiv 1$. Moreover, for every $i = 0, \dots, n$, let us define $a_i := v_i \cdot \eta$. Without loss of generality, we may assume that $a_0 > \cdots > a_n$.

We shall prove that there exists a measurable partition $E_0, \dots, E_n$ of I such that

$$\operatorname{meas}(E_i) = \int_I p_i(s)\, ds \quad \forall\, i = 0, \dots, n, \tag{2.4}$$

$$\int_0^t \sum_{i=0}^n a_i p_i(s)\chi_I(s)\, ds \leq \int_0^t \sum_{i=0}^n a_i \chi_{E_i}(s)\, ds \quad \forall t \in [0,T]. \tag{2.5}$$

It is clear that, setting $\omega(s) = \sum_{i=0}^n v_i \chi_{E_i}(s)$, (2.2) and (2.3) follow from (2.4) and (2.5). In order to prove (2.4) and (2.5), we proceed by induction. When $n = 0$, we have that $p_0(s) \equiv 1$; and if we set $E_0 = I$, the thesis is trivially satisfied. Let us assume now that $n > 0$. Let $0 = t_0 < t_1 < \cdots < t_{n+1} = T$ be a partition of $[0,T]$ such that

$$\int_{t_i}^{t_{i+1}} \chi_I(s)\, ds = \int_I p_i(s)\, ds \quad \forall\, i = 0, \dots, n.$$

Such partition exists since $p_0(s) + \cdots + p_n(s) \equiv 1$. Let us define

$$E_0 = [t_0, t_1] \cap I, \quad \text{and} \quad E_i = (t_i, t_{i+1}] \cap I \quad \forall\, i = 0, \dots, n.$$

First, by the very definition of E_i, (2.4) trivially holds. In order to prove (2.5), we proceed as follows.

Let us define

$$\tilde{E}_i = E_i, \quad \tilde{p}_i = p_i \quad \forall i = 0, \dots, n-2,$$
$$\tilde{E}_{n-1} = E_{n-1} \cup E_n, \qquad \tilde{p}_{n-1} = p_{n-1} + p_n.$$

Clearly, $\bigcup_{i=0}^{n-1} \tilde{E}_i = I$, $\sum_{i=0}^{n-1} \tilde{p}_i = 1$, and (2.4) is satisfied by $\tilde{E}_i$ and $\tilde{p}_i$ for $i =$

$0,\dots,n-1$. Moreover, the hypothesis of induction assures that

$$\int_0^t \sum_{i=0}^{n-1} a_i \chi_{\tilde{E}_i}(s)\,ds \geq \int_0^t \sum_{i=0}^{n-1} a_i \tilde{p}_i(s)\chi_I(s)\,ds. \tag{2.6}$$

Assume that $t \leq t_n$. We observe that, in this case, $E_n \cap [0,t] = \emptyset$; hence for every $i = 0,\dots,n-1$ we have that $E_i \cap [0,t] = \tilde{E}_i \cap [0,t]$. Then, by (2.6), it follows that

$$\begin{aligned}\int_0^t \sum_{i=0}^{n} a_i \chi_{E_i}(s)\,ds &= \int_0^t \sum_{i=0}^{n-1} a_i \chi_{E_i}(s)\,ds \\ &= \int_0^t \sum_{i=0}^{n-1} a_i \chi_{\tilde{E}_i}(s)\,ds \geq \int_0^t \sum_{i=0}^{n-1} a_i \tilde{p}_i(s)\chi_I(s)\,ds \\ &\geq \int_0^t \sum_{i=0}^{n} a_i p_i(s)\chi_I(s)\,ds.\end{aligned}$$

Assume now that $t_n < t \leq T$. Then

$$\begin{aligned}\int_0^t \sum_{i=0}^{n} a_i \chi_{E_i}(s)\,ds &= \sum_{i=0}^{n-1} a_i \operatorname{meas}(E_i) + \int_{t_n}^t a_n \chi_{E_n}(s)\,ds \\ &= \sum_{i=0}^{n-1} a_i \int_0^T p_i(s)\chi_I(s)\,ds + \int_{t_n}^t a_n \chi_{E_n}(s)\,ds \\ &\geq \int_0^t \sum_{i=0}^{n-1} a_i p_i(s)\chi_I(s)\,ds + a_n\left[\int_t^T \sum_{i=0}^{n-1} p_i(s)\chi_I(s)\,ds + \int_0^t \chi_{E_n}(s)\,ds\right] \\ &= \int_0^t \sum_{i=0}^{n-1} a_i p_i(s)\chi_I(s)\,ds + a_n\left[\int_{t_n}^T \sum_{i=0}^{n-1} p_i(s)\chi_I(s)\,ds + \int_{t_n}^t p_n(s)\chi_I(s)\,ds\right] \\ &= \int_0^t \sum_{i=0}^{n-1} a_i p_i(s)\chi_I(s)\,ds + a_n \operatorname{meas}(E_n) - a_n \int_t^T p_n(s)\chi_I(s)\,ds \\ &= \int_0^t \sum_{i=0}^{n-1} a_i p_i(s)\chi_I(s)\,ds + \int_0^t a_n p_n(s)\chi_I(s)\,ds = \int_0^t \sum_{i=0}^{n} a_i p_i(s)\chi_I(s)\,ds.\end{aligned}$$

Hence, also (2.5) holds and the lemma is proved. □

LEMMA 2.3. *Let A be an open convex bounded subset of $\mathbb{R}^n$. Then A can be covered by a countable family of simplexes whose vertices are contained in the boundary of A.*

Proof. Let $x_1,\dots,x_{n+1}$ be $n+1$ points of the boundary ∂A of A, such that they generate a closed $(n+1)$-dimensional simplex denoted by S_1. We denote by F_i (for $i = 1,\dots,n+1$) the face generated by

$$\{x_1,\dots,\check{x}_i,\dots,x_{n+1}\}$$

and let q_i be a point of F_i such that

$$d(q_i,\partial A) = \max\{d(x,\partial A) : x \in F_i\}.$$

Moreover, let ν_i be the external half-line normal to F_i at the point q_i and let $x_{1,i}$ be its intersection with ∂A. Let $T_{1,i}$ $(i = 1,\dots,n+1)$ be the closed $(n+1)$-dimensional simplex generated by

$$\{x_1, \ldots, x_{i-1}, x_{1,i}, x_{i+1}, \ldots, x_{n+1}\},$$

and set

$$S_2 = S_1 \cup \bigcup_{i=1}^{n+1} T_{1,i}.$$

Recursevely, one obtains an increasing family of convex polygons whose vertices lie in ∂A; moreover, each S_{j+1} is obtained by adding to S_j a finite union of $(n+1)$-simplexes $T_{j,k}$ $(1 \leq k \leq k(j) < +\infty)$ with vertices in ∂A. We claim that

$$A \subset \bigcup_j S_j.$$

Clearly, it is enough to prove that

$$\lim_j \max_{x \in \partial S_j} d(x, \partial A) = 0. \tag{2.7}$$

In order to prove (2.7), let us remark that, if it does not hold, there exists $d_0 > 0$ such that

$$\max_{x \in \partial S_j} d(x, \partial A) \geq d_0$$

for each $j \in \mathbb{N}$. It follows that, by construction,

$$\max_{x \in \partial T_{j,k}} d(x, \partial A) \geq d_0$$

for each $j \in \mathbb{N}$ and $k \leq k(j)$, so that the "heights" of the simplexes $T_{j,k}$ (and hence their volumes) are bounded below by a positive constant, a contradiction, the set A being bounded. □

3. Main results.

THEOREM 3.1. *Let A be an open convex and bounded subset of $\mathbb{R}^n$, I a measurable subset of $[0,T]$, $u' : I \to A$ a measurable function, and η an arbitrary vector in $\mathbb{R}^n$. Then there exists a function $\omega : I \to \partial A$, depending on u', A, and η, such that*

$$\int_I \omega(s)\, ds = \int_I u'(s)\, ds, \tag{3.1}$$

$$\forall t \in [0,T] \quad \int_0^t [\omega(s) \cdot \eta\chi_I(s)]\, ds \geq \int_0^t [u'(s) \cdot \eta\chi_I(s)]\, ds. \tag{3.2}$$

COROLLARY 3.2. *Assume that A, I, u', ω, and η are as in the previous theorem. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous, f^{**} is affine on A, and $f(\xi) = f^{**}(\xi)$ when $\xi \in \partial A$. Then*

$$\int_I f^{**}(u'(s))\, ds = \int_I f(\omega(s))\, ds. \tag{3.3}$$

Proof. Since f^{**} is affine on A, there exist two vectors v_1 and v_2 such that

$$f^{**}(\xi) = v_1 \cdot \xi + v_2 \quad \forall \xi \in A;$$

hence, by (2.2) of Lemma 2.2, it easily follows that

$$\int_I f^{**}(u'(s))\, ds = \int_I f^{**}(\omega(s))\, ds.$$

Finally, recalling that ω takes values in ∂A and f^{**} coincides with f on ∂A, (2.4) follows. □

Proof of Theorem 3.1. By Lemma 2.3 $A = \bigcup_{j\in\mathbb{N}} S_j$, where S_j is a simplex contained in $\mathbb{R}^n$ whose vertices belongs to ∂A. Let us set, for every $j \in \mathbb{N}$, $I_j := (u')^{-1}(S_j) \cap I$ and $u'_j := u'\chi_{I_j} : I_j \to S_j$. Applying Lemma 2.2 to u'_j in I_j, we obtain a function $\omega_j : I_j \to \text{extr } S_j \subset \partial A$, which satisfies (2.2) and (2.3). Hence, defining $\omega(s) := \sum_{j\in\mathbb{N}} \omega_j(s)$, it is clear that ω takes values in ∂A and satisfies (3.1) and (3.2). □

4. Applications. We consider the following minimum problem with obstacle

$$\min_{\substack{u\in K\\ u(t)\notin\Gamma}} \int_0^T f(u'(t))\,dt.$$

As we have already announced in the introduction, our main goal is to prove the existence of a solution for this minimum problem.

THEOREM 4.1. *Let $\Gamma \subset \mathbb{R}^n$, $K \subset W^{1,p}([0,T],\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$ be as in §2. Assume further that $f^{**}(0) = f(0)$. Then the problem*

$$\text{(P)} \qquad \min\left\{\int_0^T f(u'(s))\,ds \;:\; u \in K,\; u(t) \notin \Gamma\right\}$$

admits at least one solution.

Proof. Assume, for the sake of simplicity, that the set defined by

$$\{\xi \in \mathbb{R}^n : f^{**}(\xi) < f(\xi)\}$$

coincides with a simplex E on which f^{**} is affine; by the remark following Lemma 2.1 and by Lemmas 2.3 and 2.2, this is not restrictive. Assume further that $n = 2$, the general case being similar. Let p_i $(i = 1,\dots,m)$ be the vertices of Γ; by G_i we denote the relative interior of the side $\overline{p_ip_{i+1}}$ and by ν_i their external normal vector. The set Γ being open, there exists a solution $\tilde u$ to the associated relaxed problem

$$\min\left\{\int_0^T f^{**}(u'(s))\,ds \quad : \quad u \in K, u(t) \notin \Gamma\right\}.$$

Since the measure of the interval $[0,T]$ is finite, for each vertex p_i there exists an external half-line L_i containing p_i such that, setting $N = \{t : \tilde u(t) \in L_i \setminus \{p_i\}\}$, we have $\text{meas}(N) = 0$.

Fix G_i and consider the "external" unbounded set O_i defined by the interior of the region delimited by the half-lines L_i, L_{i+1} and the side G_i, jointly with the side G_i itself.

Clearly, each O_i is open in the relative topology of $\mathbb{R}^2 \setminus \Gamma$; moreover, the solution $\tilde u$ does not belong to Γ. Hence, the inverse image of $\mathcal{O} = \bigcup_i O_i$ under $\tilde u$ is a countable union of relative open intervals (α_j, β_j) of $[0,T]$: for every $j \in \mathbb{N}$ let $i(j)$ be such that

$$\tilde u(\alpha_j, \beta_j) \subset O_{i(j)}.$$

Let us define by $\mathcal{K}$ the subset of $[0,T]$ where $f \circ \tilde u'$ does not coincide with $f^{**} \circ \tilde u'$, i.e.,

$$\mathcal{K} = (\tilde u')^{-1}(E) = \{t : f^{**}(\tilde u'(t)) \neq f(\tilde u'(t))\};$$

and set, for each $j \in \mathbb{N}$,

$$S_j = (\alpha_j, \beta_j) \cap \mathcal{K}.$$

Now, for each j, $\tilde{u}'(\mathcal{S}_j) \subset E$, on which f^{**} is affine; by Corollary 3.2 there exists a measurable function

$$\sigma_j : \mathcal{S}_j \to \mathbb{R}^2$$

with values in ∂E (on which $f^{**} = f$) satisfying

$$\int_{\mathcal{S}_j} \sigma_j(t)\, dt = \int_{\mathcal{S}_j} \tilde{u}'(t)\, dt, \tag{4.1}$$

$$\forall t \in [\alpha_j, \beta_j] \qquad \int_0^t \sigma_j(s) \cdot \nu_{i(j)} \chi_{\mathcal{S}_j}(s)\, ds \geq \int_0^t \tilde{u}'(s) \cdot \nu_{i(j)} \chi_{\mathcal{S}_j}(s)\, ds, \tag{4.2}$$

$$\int_{\mathcal{S}_j} f(\sigma_j(t))\, dt = \int_{\mathcal{S}_j} f^{**}(\tilde{u}'(t))\, dt. \tag{4.3}$$

Let $\bar{u}' : [0, T] \to \mathbb{R}^2$ be the measurable function defined by

$$\bar{u}' = \tilde{u}' \chi_{[0,T] \setminus \bigcup_j \mathcal{S}_j} + \sum_{j \in \mathbb{N}} \sigma_j \chi_{\mathcal{S}_j}.$$

The growth conditions on f and relation (4.1) show, together with Vitali's convergence theorem, that $\bar{u}' \in L^p$. Let $\bar{u}$ be the function defined by

$$\bar{u}(t) = a + \int_0^t \bar{u}'(s)\, ds.$$

We claim that $\bar{u}$ is a solution to (P).

Clearly, by (4.1) and the definition of $\bar{u}$ we have

$$\bar{u}(0) = \tilde{u}(0), \qquad \bar{u}(T) = \tilde{u}(T).$$

In order to prove that

$$\int_0^T f(\bar{u}'(t))\, dt = \int_0^T f^{**}(\tilde{u}'(t))\, dt = \min_{\substack{u \in K \\ u(t) \notin \Gamma}} \int_0^T f^{**}(u'(t))\, dt \tag{4.4}$$

we first remark that $[0, T]$ can be partitioned as a disjoint union of four measurable subsets N, D_1, D_2, D_3 where

$$D_1 = \bigcup_j \mathcal{S}_j = \tilde{u}^{-1}(\mathcal{O}) \cap \mathcal{K}, \qquad D_2 = \tilde{u}^{-1}(\mathcal{O}) \cap ([0, T] \setminus \mathcal{K}), \qquad D_3 = \tilde{u}^{-1}(\{p_1, \ldots, p_m\}).$$

By (4.3) we have

$$\int_{D_1} f(\bar{u}'(t))\, dt = \int_{D_1} f^{**}(\tilde{u}'(t))\, dt; \tag{4.5}$$

by the very definitions of $\bar{u}$ and $\mathcal{K}$ we have

$$\text{for a.e. } t \in [0, T] \setminus \mathcal{K} \qquad f(\bar{u}'(t)) = f(\tilde{u}'(t)) = f^{**}(\tilde{u}'(t))$$

so that in particular

$$\int_{D_2} f(\bar{u}'(t))\, dt = \int_{D_2} f^{**}(\tilde{u}'(t))\, dt. \tag{4.6}$$

Finally, by [7, Lemma 7.7], on $\tilde{u}^{-1}(\{p_1,\ldots,p_m\}) = D_3$ we have $\tilde{u}' = 0$ a.e.; since by the very definition $\bar{u}' = 0$ on D_3 and by our assumption $f^{**}(0) = f(0)$

$$\text{for a.e. } t \in D_3 \qquad f(\bar{u}'(t)) = f(0) = f^{**}(0) = f^{**}(\tilde{u}'(t))$$

so that

$$\int_{D_3} f(\bar{u}'(t))\,dt = \int_{D_3} f^{**}(\tilde{u}'(t))\,dt. \tag{4.7}$$

Taking into account that N has measure zero, equalities (4.5), (4.6), and (4.7) together give (4.4).

At this stage we only need to show that

$$\forall t \in [0,T]: \quad \bar{u}(t) \notin \Gamma.$$

Fix t in $[0,T]$: either there exists $j_0 \in \mathbb{N}$ such that $t \in (\alpha_{j_0},\beta_{j_0})$ or t does not belong to $\tilde{u}^{-1}(\mathcal{O})$. In the first case let $i \in \mathbb{N}$ be such that $\tilde{u}(\alpha_{j_0},\beta_{j_0}) \subset O_i$; in order to prove that $\bar{u}(t) \notin \Gamma$ it is enough to show that

$$\bar{u}(t)\cdot\nu_i \geq \tilde{u}(t)\cdot\nu_i. \tag{4.8}$$

Since $\tilde{u}' = \bar{u}'$ on $[0,T]\setminus\bigcup_j \mathcal{S}_j$ then by (4.1) and (4.2) we have

$$\begin{aligned}
(\bar{u}(t)-\tilde{u}(t))\cdot\nu_i &= \int_0^t (\bar{u}'(s)-\tilde{u}'(s))\cdot\nu_i\,ds \\
&= \int_0^t (\bar{u}'(s)-\tilde{u}'(s))\cdot\nu_i\chi_{\cup_j\mathcal{S}_j}(s)\,ds \\
&= \sum_{\{j:\mathcal{S}_j\subset[0,t]\}} \int_{\mathcal{S}_j} (\sigma_j(s)-\tilde{u}'(s))\cdot\nu_i\,ds \\
&\quad + \int_0^t (\sigma_{j_0}(s)-\tilde{u}'(s))\cdot\nu_i\chi_{\mathcal{S}_{j_0}}(s)\,ds \\
&= \int_0^t (\sigma_{j_0}(s)-\tilde{u}'(s))\cdot\nu_i\chi_{\mathcal{S}_{j_0}}(s)\,ds \geq 0,
\end{aligned}$$

which proves (4.8).

In the second case ($t \notin \tilde{u}^{-1}(\mathcal{O})$) there is no interval (α_j,β_j) containing t. It follows that for each j in $\mathbb{N}$ either $\mathcal{S}_j \subset [0,t]$ or $\mathcal{S}_j \cap [0,t] = \emptyset$. As a consequence we have

$$\begin{aligned}
\bar{u}(t)-\tilde{u}(t) &= \int_0^t \bar{u}'(s)-\tilde{u}'(s)\,ds \\
&= \int_0^t (\bar{u}'(s)-\tilde{u}'(s))\chi_{\cup_j\mathcal{S}_j}(s)\,ds \\
&= \sum_{\{j:\mathcal{S}_j\subset[0,t]\}} \int_{\mathcal{S}_j} \sigma_j(s)-\tilde{u}'(s)\,ds.
\end{aligned}$$

Equality (4.1) yields $\bar{u}(t) = \tilde{u}(t)$; in particular $\bar{u}(t) \notin \Gamma$, the conclusion follows. □

As a consequence of the proof of the above theorem, we have the following result, with no assumption on the bipolar of f in 0.

THEOREM 4.2. *Let $\Gamma \subset \mathbb{R}^n$ be an open half-space, $K \subset W^{1,p}([0,T],\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$ be as in §2. Then the problem*

$$\min \left\{ \int_0^T f(u'(s))\, ds \,:\, u \in K,\, u(t) \notin \Gamma \right\}$$

admits at least one solution.

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