Advanced Nonlinear Studies **8** (2008), 597–602

An Elementary Proof of a Characterization of Constant Functions

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Received 14 December 2007
Communicated by Shair Ahmad

Abstract

We give two short and elementary proofs of a characterization of constants function by Brezis. Whereas the original proof involves some refined arguments on Sobolev spaces and BV functions, ours are based either on convolutions or on a sort of nonsmooth Mean Value Theorem which is new to our knowledge.

1991 Mathematics Subject Classification. 46E35, 26A30.
Key words. Convolution, Mean value

1 Introduction

We deal with the following nice characterization of constant functions, formulated by H. Brezis in [1].

Proposition 1.1 (H. Brezis) *Let Ω be a connected open set in $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(y) - f(x)|}{|y - x|^{N+1}}\, dx\, dy < +\infty. \tag{1.1}$$

Then f is a.e. equal to a constant.

We refer to [1] for some connections of the result with the VMO functions and related results on the degree theory for classes of discontinuous maps and lifting of maps with values in the unit circle. We are mainly concerned here with the *proof* of Proposition 1.1. Indeed, in spite of the simplicity of the statement of Proposition 1.1, its original proof actually involves a refined characterization of BV functions (it is first shown that f is BV and then that the total mass of the measure ∇f is zero); actually it is pointed out in Remark 1 of [1] that the author does not know a direct, elementary proof of the result. The purpose of this short note is to present a couple of proofs with no connections to BV or Sobolev spaces.

We first present a self contained argument based on convolutions; in the final part of the paper we give a short proof of Theorem 4.1, a more general result again due to Brezis, as an application of a Mean Value Theorem under mild assumptions that is new to our knowledge. Actually, both methods could be used either for Proposition 1.1 or for Theorem 4.1.

We denote by $B_r(x)$ (or simply B_r if $x = 0$) the ball of radius r of center x in $\mathbb{R}^N$; Σ is the unit sphere and $\mathcal{H}^\alpha$ is the Hausdorff $\alpha-$dimensional measure.

2 A convolution argument

We give here a proof of Proposition 1.1 based on a convolution argument.

Proof of Proposition 1.1. It is not restrictive to assume that f is bounded (otherwise one considers $\arctan(f)$ instead of f). We first consider the case where $N = 1$ and prove that f is constant on any $[a, b] \subset \Omega$. Let $\varepsilon < \text{dist}([a, b], \partial\Omega)$; the change of variables $y = x + t$ in (1.1) yields that

$$\int_0^\varepsilon \frac{1}{t} \int_a^b \frac{|f(x + t) - f(x)|}{t}\, dx\, dt < +\infty. \tag{2.1}$$

Claim: Condition (2.1) implies that f is constant on $[a, b]$. Indeed, since $1/t$ is not summable around 0, there exists a sequence t_n converging to 0 such that

$$\lim_{n \to +\infty} \int_a^b \frac{|f(x + t_n) - f(x)|}{t_n}\, dx = 0;$$

for every $t > 0$, $(f(x + t) - f(x))/t$ is the derivative of the Lipschitz function $\varphi_t * f$, where φ_t is the mollifier $t^{-1}\chi_{[-t,0]}$. Moreover $\varphi_{t_n} * f$ converges to f a.e.: if y, z are two points of $[a, b]$ where the sequence converges we have

$$|f(z) - f(y)| = \lim_n |\varphi_{t_n} * f(z) - \varphi_{t_n} * f(y)| \le \lim_n \int_a^b |(\varphi_{t_n} * f)'(x)|\, dx = 0;$$

we deduce that f is constant a.e. in $[a, b]$.

Let $N > 1$ and $\overline{x} \in \Omega$; let $r > 0$ be such that $2r < \operatorname{dist}(\overline{x}, \partial\Omega)$. Fix a smooth mollifier φ_r with support in B_r. We shall show that $\nabla(\varphi_r * f)(\overline{x}) = 0$: since $\varphi_r * f$ converges to f a.e. for r tending to 0, the same argument will imply that f is constant. The change of variables $y = x + t\theta$ ($\theta \in \Sigma, t > 0$) and (1.1) yield

$$\int_0^r t^{N-1} \int_\Sigma \int_{B_r(\overline{x})} \frac{|f(x+t\theta) - f(x)|}{t^{N+1}} \, dx \, d\mathcal{H}^{N-1}(\theta) \, dt < +\infty.$$

Fubini Theorem implies that, for a.e. θ in Σ,

$$\int_0^r \int_{B_r(\overline{x})} \frac{|f(x+t\theta) - f(x)|}{t^2} \, dx \, dt < +\infty. \tag{2.2}$$

Fix θ such that (2.2) holds; let B' be the orthogonal projection of $B_r(\overline{x})$ onto the plane through $\overline{x}$ that is perpendicular to θ and let $S_{x'}$ be the chord in direction θ of $B_r(\overline{x})$ whose intersection with B' is x'. By Fubini Theorem and by (2.2)

$$\int_0^r \frac{1}{t} \int_{S_{x'}} \frac{|f(x+t\theta) - f(x)|}{t} \, d\mathcal{H}^1(x) \, dt < +\infty$$

for a.e. x' in B'. Therefore, by applying the arguments of the just proven one dimensional case of Proposition 1 we infer that, for a.e. x', f is constant on the chord $S_{x'}$ in direction θ. Since φ_r is null on ∂B_r then $\displaystyle\int_{S_{x'}} \frac{\partial \varphi_r}{\partial \theta}(\bar{x} - x) \, d\mathcal{H}^1(x) = 0$ and thus the integral $\displaystyle\int_{S_{x'}} \frac{\partial \varphi_r}{\partial \theta}(\bar{x} - x) f(x) \, d\mathcal{H}^1(x)$ is null for a.e. x'. Thus, by a further application of Fubini Theorem, the partial derivative $\dfrac{\partial}{\partial \theta}(\varphi_r * f)(\overline{x}) = \dfrac{\partial \varphi_r}{\partial \theta} * f(\bar{x})$ equals

$$\int_{B_r(\overline{x})} \frac{\partial \varphi_r}{\partial \theta}(\bar{x} - x) f(x) \, dx = \int_{B'} \int_{S'_x} \frac{\partial \varphi_r}{\partial \theta}(\bar{x} - x) f(x) \, d\mathcal{H}^1(x) \, d\mathcal{H}^{N-1}(x') = 0.$$

By choosing N linearly independent values of θ we get that $\nabla(\varphi_r * f)(\bar{x}) = 0$.

Remark 2.1 Our proof of Proposition 1.1 is self contained and relies on the properties of the smooth functions $\varphi_r * f$. However, the final part of the proof could be carried out in an alternative way once it is shown that f is constant on a.e. chord of $B_r(\overline{x})$ in direction θ: by taking a set of N linearly independent values of θ such that (2.2) holds, we obtain that f is constant on a.e. chord of $B_r(\overline{x})$ that is parallel to one of these N vectors; it follows (for instance from [2, Lemma 2]) that f is constant a.e.. Our convolution argument actually gives an alternative proof of the just quoted result.

3 A nonsmooth mean value theorem

The following sort of Mean Value Theorem provides an alternative way of proving Proposition 1.1. To avoid repetitions we will actually apply it in the next section to prove a more general result.

Theorem 3.1 (A Nonsmooth Mean Value Theorem) *Let $(Y, \|\cdot\|)$ be a normed linear space; let Ω be an open connected subset of $\mathbb{R}^N$, and let $\varphi : \Omega \to Y$ be continuous. Assume that there exist N independent vectors $u_1, \dots, u_N$ such that*

$$\forall y \in \Omega \qquad \liminf_{t\to 0^+} \frac{\|\varphi(y+tu_j)-\varphi(y)\|}{t} = 0 \quad (j=1,\dots,N). \tag{3.1}$$

Then φ is constant on Ω.

Proof. Assume first that $N = 1$, thus Ω is an interval I and, with no restriction, $u_1 = 1$. Given $a, b \in I$ with $a < b$, we prove that $\varphi(a) = \varphi(b)$. For $\varepsilon > 0$, consider the set $T = \{t \in [a,b] : \|\varphi(t) - \varphi(a)\| \le \varepsilon(t-a)\}$. Clearly T is non-empty, since $a \in T$, and is closed by continuity of φ; hence T has a maximum. Let $c = \max T$. We claim that $c = b$. If not, (3.1) implies the existence of $\bar{t} > c$ such that $\|\varphi(\bar{t}) - \varphi(c)\|/(\bar{t}-c) \le \varepsilon$; but then $\|\varphi(c)-\varphi(a)\| \le \varepsilon(c-a)$ and $\|\varphi(\bar{t})-\varphi(c)\| \le \varepsilon(\bar{t}-c)$ imply $\|\varphi(\bar{t})-\varphi(a)\| \le \varepsilon(\bar{t}-a)$, and hence $\bar{t} \in T$. But $\bar{t} > c = \max T$, a contradiction. Thus $\|\varphi(b)-\varphi(a)\| \le \varepsilon(b-a)$, for every $\varepsilon > 0$, and then $\varphi(a) = \varphi(b)$. Assume now that $N > 1$. The case $N = 1$ shows that φ is constant on every line segment contained in Ω and parallel to one of the u_j's. Then φ is constant on every parallelotope spanned by vectors parallel to the u_j's, and hence is locally constant on Ω; thus constant on Ω by connectedness.

4 A more general theorem

Theorem 4.1 below is an extension of Proposition 1.1; it is referred to as a theorem of H. Brezis in [3, Theorem 3.1] where it is proven via some fine properties of Sobolev spaces or BV functions. An alternative proof of Theorem 4.1 could be given by using again a convolution argument as in Section 2. However we present here a short alternative proof of it as an application of Theorem 3.1.

Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of radial mollifiers, i.e. ρ_ε are measurable on $[0, +\infty[$,

$$\rho_\varepsilon \ge 0; \qquad \int_{\mathbb{R}^N} \rho_\varepsilon(|x|)\,dx = 1; \qquad \forall \delta > 0 \ \ \lim_{\varepsilon\to 0} \int_{|x|>\delta} \rho_\varepsilon(|x|)\,dx = 0.$$

Theorem 4.1 (H. Brezis) *Let Ω be a connected open set in $\mathbb{R}^N$ and $\omega : [0,+\infty[\to [0,+\infty[$ be a convex function such that $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$. Let $f : \Omega \to \mathbb{R}$ be measurable and such that*

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \omega\left(\frac{|f(y)-f(x)|}{|y-x|}\right) \rho_\varepsilon(|y-x|)\,dx\,dy = 0 \tag{4.1}$$

for a family $(\rho_\varepsilon)_{\varepsilon>0}$ of radial mollifiers. Then f is a.e. equal to a constant.

Remark 4.1 As it is pointed out in [1], Theorem 4.1 implies Proposition 1.1. We mention that the conclusion of Theorem 4.1 holds true if $\omega : [0,+\infty[\to [0,+\infty[$ is a continuous function such that $\liminf_{t\to\infty} \omega(t)/t = \alpha > 0$ and $\omega(0) = 0$; indeed this latter condition is equivalent to the fact that ω is bounded below by a convex function $\tilde{\omega}$ such that $\tilde{\omega}(0) = 0$ and $\tilde{\omega}(t) > 0$ for $t > 0$. In the case where $\omega(t) = t^p$ the result is [1, Theorem 1]; the claim follows there from the characterization of Sobolev spaces ($p > 1$) or of BV functions ($p = 1$) formulated in [1]. In this general case, with $\omega(t)$ instead of t^p, the original proof of Theorem 4.1 involves the properties of Orlicz Sobolev spaces (see [4]).

Proof of Theorem 4.1. We may assume that f is bounded; otherwise, ω being increasing, we could replace f by $\arctan(f)$. Let $\overline{x} \in \Omega$ and $2r < \operatorname{dist}(\overline{x}, \partial\Omega)$. Set

$$\psi(y) = \int_{B_r(\overline{x})} \omega\left(\frac{|f(x+y) - f(x)|}{|y|}\right) dx, \quad y \in B_r,\ y \neq 0.$$

Notice that, by Jensen's inequality, for every y in $B_r \setminus \{0\}$ we have

$$\psi(y) \geq |B_r|\, \omega\left(\frac{1}{|B_r|} \int_{B_r(\overline{x})} \frac{|f(x+y) - f(x)|}{|y|}\, dx\right) \qquad |B_r| = \operatorname{Vol}(B_r). \tag{4.2}$$

The change of variables $(x, y) \longmapsto (x, x+y)$ together with (4.1) yields that

$$\lim_{\varepsilon \to 0} \int_{B_r} \rho_\varepsilon(|y|) \int_{B_r(\overline{x})} \omega\left(\frac{|f(x+y) - f(x)|}{|y|}\right) dx\, dy = \lim_{\varepsilon \to 0} \int_{B_r} \rho_\varepsilon(|y|) \psi(y)\, dy = 0.$$

By passing to polar coordinates $y = t\theta$ ($t > 0$, $\theta \in \Sigma$) then we get

$$\lim_{\varepsilon \to 0} \int_0^r t^{N-1} \rho_\varepsilon(t) \overline{\psi}(t)\, dt = 0, \qquad \overline{\psi}(t) = \int_\Sigma \psi(t\theta)\, d\mathcal{H}^{N-1}(\theta). \tag{4.3}$$

Let $\delta < r$. If $\overline{\psi}(t) \geq a > 0$ on $[0, \delta]$ then the l.h.s. of (4.3) is greater than

$$a \lim_{\varepsilon \to 0} \int_0^\delta t^{N-1} \rho_\varepsilon(t) \mathcal{H}^{N-1}(\Sigma)\, dt = a \lim_{\varepsilon \to 0} \int_{B_\delta} \rho_\varepsilon(|y|)\, dy = a,$$

a contradiction. Thus, there exists a positive sequence t_n converging to 0 such that

$$\lim_{n \to \infty} \overline{\psi}(t_n) = \lim_{n \to \infty} \int_\Sigma \psi(t_n\theta)\, d\mathcal{H}^{N-1}(\theta) = 0.$$

Since ψ is positive, passing to a subsequence (that we call again t_n) we obtain

$$\lim_{n \to \infty} \psi(t_n\theta) = 0 \quad \text{for a.e.} \quad \theta \in \Sigma. \tag{4.4}$$

Fix θ such that (4.4) holds. It follows from (4.2) and (4.4) that

$$\lim_{n \to +\infty} \omega\left(\frac{1}{|B_r|} \int_{B_r(\overline{x})} \frac{|f(x + t_n\theta) - f(x)|}{t_n}\, dx\right) = 0.$$

Now ω is strictly increasing and thus

$$\lim_{n \to +\infty} \int_{B_r(\overline{x})} \frac{|f(x + t_n\theta) - f(x)|}{t_n}\, dx = 0. \tag{4.5}$$

Choose N independent values $u_1, \dots, u_N$ of θ such that (4.5) holds; set $B' = B_{r/2}(\overline{x})$ and let $\varphi : B' \to L^1(B')$ be the translate of f defined by $\varphi(y) = f(\cdot + y)$. Now,

$$\frac{\|\varphi(y+t\theta) - \varphi(y)\|_{L^1(B')}}{t} = \int_{B'} \frac{|f(w+y+t\theta) - f(w+y)|}{t}\, dw = \; (x = w+y)$$

$$\int_{y+B'} \frac{|f(x+t\theta) - f(x)|}{t}\, dx \leq \int_{B_r(\overline{x})} \frac{|f(x+t\theta) - f(x)|}{t}\, dx \quad (0 < t < r/2).$$

The continuity of the translate and Theorem 3.1 implies that φ is constant on B'; thus f is constant a.e. on $B_{r/2}(\overline{x})$: the conclusion follows, Ω being connected.

Remark 4.2 It is pointed out in the proof of [3, Theorem 1.5] that, assuming $f \in W^{1,1}_{\ell oc}(\Omega)$, then (4.5) implies that f is constant; the *new* fact here is that we *just* assume that f is measurable. Notice that the conclusion of Theorem 4.1 could also be obtained from (4.5) by slightly modifying the arguments of Section 2.

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