Gauge conservation laws and the momentum equation in nonholonomic mechanics

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Abstract

The gauge mechanism is a generalization of the momentum map which links conservation laws to symmetry groups of nonholonomic systems. This method has been so far employed to interpret conserved quantities as momenta of vector fields which are sections of the constraint distribution. In order to obtain the largest class of conserved quantities of this type, we extend this method to an over–distribution of the constraint distribution, the so–called reaction–annihilator distribution, which encodes the effects that the nonholonomic reaction force has on the conservation laws. We provide examples showing the effectiveness of this generalization. Furthermore, we discuss the Noetherian properties of these conserved quantities, that is, whether and to which extent they depend only on the group, and not on the system. In this context, we introduce a notion of 'weak Noetherianity'. Finally, we point out that the gauge mechanism is equivalent to the momentum equation (at least for locally free actions), we generalize the momentum equation to the reaction–annihilator distribution, and we introduce a 'gauge momentum map' which embodies both methods. For simplicity, we treat only the case of linear constraints, natural Lagrangians, and lifted actions.

Keywords: Nonholonomic systems, First integrals, Symmetries of nonholonomic systems, Nonholonomic momentum map, Momentum equation, Reaction–annhilator distribution, Nonholonomic Noether theorem, Gauge momenta, Gauge momentum map, Weakly Noetherian first integrals.

MSC:

1 Introduction

A. The link between symmetries and conservation laws of nonholonomic systems has been extensively studied, particularly for the case of natural Lagrangians and lifted actions, which is the case we consider in this article, see e.g. [2, 18, 19, 21, 3, 12, 7, 13, 26, 17, 22, 6, 14, 28, 15] and references therein. Due to the reaction forces exerted by the nonholonomic constraint, only certain components of the momentum map of a lifted action which leaves the Lagrangian invariant are conserved quantities.

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However, a 'gauge–like' mechanism which is more general than the momentum map has been devised to link symmetry groups to conservation laws of nonholonomic systems [5]. Within this method, first integrals are produced as momenta of certain vector fields which are tangent to the group orbits, but need not be infinitesimal generators of the group action. (Here, the 'momentum' of a vector field defined on the configuration manifold is the Hamiltonian of its (co)tangent lift; later on, we will however reserve this term for the components of the momentum map). So far, the study of this gauge mechanism has been restricted to the consideration of vector fields which are sections of the constraint distribution, that is, using a common terminology, which are *horizontal* vector fields. Accordingly, we will call *horizontal gauge momenta* the resulting conserved momenta.

More precisely, as proposed in [5], the gauge method consists of the idea of constructing horizontal vector fields which generate conserved momenta as pointwise linear combinations of infinitesimal generators of the group action, with coefficients which are smooth functions on Q. Even though this method is in principle constructive, to our knowledge it has not been employed so far to find new first integrals of specific nonholonomic systems. Instead, the effectiveness of this idea was demonstrated by showing that a number of known first integrals of sample nonholonomic systems, whose relation to a symmetry group was previously unknown, are in fact related to it in this gauge–like way [5, 22]. In our opinion, the gauge mechanism has a significant conceptual interest, in that it points out a precise way of extending, beyond the momentum map, the link symmetries–conservation laws for nonholonomic systems.

In view of this, in this article we will show that the gauge mechanism can be extended beyond the horizontal case, and can accordingly account for more conservation laws than what known so far. Moreover, we will study some of the properties of the conserved 'gauge momenta' it produces, and we will clarify its relation to the nonholonomic momentum map and momentum equation of [7, 9, 10], that we will extend as well beyond the horizontal case.

B. Our approach rests on a recent characterization, due to [15], of the effects that the reaction force exerted by the nonholonomic constraint has on the conservation of the momentum of a vector field on the configuration manifold. As shown in [15], these effects can be encoded in a certain distribution $\mathcal{R}^\circ$ on the configuration manifold Q, which is called the *reaction–annihilator distribution*. The fiber $\mathcal{R}^\circ_q$ of this distribution over a point $q \in Q$ is the annihilator of all reaction forces on constrained motions through q, and contains the fiber $\mathcal{D}_q$ of the constraint distribution.

Any first integral linear in the velocities is the momentum of infinitely many vector fields, only one of which is horizontal [19]. Therefore, considering non–horizontal vector fields does not produce new first integrals, but this freedom can be used to select vector fields with some properties—such as to be tangent to the orbits of a group action. From this perspective, the unique role of the distribution $\mathcal{R}^\circ$ emerges from the fact that it is the largest distribution for which the following is true: *any vector field which is a section of* $\mathcal{R}^\circ$*, and whose lift to* TQ *preserves the Lagrangian, generates a conserved momentum* [15]. As a particular case, *infinitesimal generators of the group action which are sections of* $\mathcal{R}^\circ$ *generate conserved components of the momentum map* [15]. (See Section 2 for precise statements).

Thus, we will extend the gauge mechanism to sections of $\mathcal{R}^\circ$ and obtain in this way the largest class of first integrals which can be interpreted as conserved 'gauge momenta' (Proposition 3). This study will require a formalization of the gauge mechanism itself, that in [5] was only illustrated on examples. In doing this, we will find convenient to formulate the gauge mechanism in terms of vector fields tangent to the group orbits, rather than in terms of pointwise linear combinations of infinitesimal generators (these two characterizations are equivalent if the action is, for instance, locally free, see Section 3C).

The conserved gauge momenta so obtained can be classified into four classes (components of the momentum map or 'gauge momenta'; generated by sections of $\mathcal{D}$ or by sections of $\mathcal{R}^\circ$). The interrelation among these four classes is somewhat subtle, and we will produce examples of gauge momenta which illustrate all possible cases. In particular, we will show on an *ad hoc* example (a five–dimensional version of the nonholonomic particle of [25]) that there are indeed conserved gauge momenta whose unique horizontal generator is not tangent to the group orbits, but nevertheless, they possess generators with this property which are sections of $\mathcal{R}^\circ$.

We will also investigate whether the conserved quantities provided by the gauge mechanism have the ('Noetherian') property of the ordinary momentum map, that is, whether they are conserved quantities for all systems with invariant Lagrangian (and given nonholonomic constraint). This property is shared, in particular, by the momenta of horizontal infinitesimal generators, while horizontal gauge momenta satisfy a weaker version of it, that we will call 'weak Noetherianity', that is, they are first integrals for any invariant Lagrangian with fixed kinetic energy (Proposition 5).

C. The gauge mechanism has a very close connection with the nonholonomic momentum map and momentum equation introduced in [7] and then variously generalized [9, 10, 27]. In its original formulation [7], the momentum equation is the balance equation of the momentum of a horizontal vector field which, at each point, coincides with the value at that point of an infinitesimal generator of the group action. This is equivalent to saying that such a vector field is a pointwise linear combination of infinitesimal generators of the action—just as in the gauge method as introduced in [5]. However, this connection between the two methods seems to have passed completely unnoticed so far, presumably because of the fact that the gauge method has not received much attention and of the fact that, in the studies of the momentum equation, the emphasis has not been primarily on producing conservation laws (but see [28]).

Like the gauge mechanism, also the momentum equation has been considered so far only in the horizontal case. We will show that it is valid for all sections of the reaction–annihilator distribution $\mathcal{R}^\circ$, and not only for the horizontal ones. This will in fact provide a distinct introduction of the distribution $\mathcal{R}^\circ$, as the largest distribution on the configuration manifold for which the momentum equation holds (Proposition 6). We will then compare this extension of the momentum equation to (our formulation of) the gauge mechanism and we will prove that they are equivalent for locally free action, in the precise sense that the conserved solutions of the momentum equation are exactly the conserved gauge momenta.

In doing this, we will introduce a *gauge momentum map* which unifies the gauge approach and the nonholonomic momentum map. The gauge momentum map is defined on the space of sections of the Lie algebra bundle over the configuration manifold Q. To each such section, the gauge momentum map associates a C^∞–function on the phase space TQ. This function is a conserved quantity for the nonholonomic system if and only if the section belongs to the intersection of the reaction–annihilator distribution $\mathcal{R}^\circ$ and the distribution of the tangent spaces to the group orbits.

D. The article is organized as follows. Section 2 is devoted to a review of known results on conserved momenta and on the distribution $\mathcal{R}^\circ$, and is the basis for the entire treatment. In Section 3 we formalize the gauge mechanism and extend it to sections of $\mathcal{R}^\circ$. The examples are given in Section 4. Section 5 is devoted to the momentum equation and to the gauge momentum map. A short Section of Conclusions follows.

We will use the Lagrangian description of nonholonomic systems, which is fully adequate to the treatment of lifted actions. Since the geometry underlying all constructions is elementary, for simplicity of exposition and greater clarity we will resort to a coordinate description wherever possible and adequate. For general references on nonholonomic systems see e.g. [23, 8, 11, 6].

2 The reaction–annihilator distribution and its role

A. Nonholonomic systems and the reaction–annihilator distribution. In this Section, which is based on [15], we review the role of the reaction–annihilator distribution in the conservation of momenta of vector fields.

As a starting point, consider a holonomic Lagrangian system (L, Q) with n-dimensional configuration manifold Q and smooth Lagrangian $L = T - V$, with kinetic energy $T(q,\dot q) = \frac{1}{2}\dot q \cdot A(q)\dot q$ and potential energy $V(q)$. Here and in the following $(q,\dot q)$ are bundle coordinates on TQ and the dot denotes the inner product in $\mathbb{R}^n$.

A *linear nonholonomic constraint of rank* r, $1 \le r < n$, is a non–integrable smooth distribution $\mathcal{D}$ on Q of constant rank r. This distribution, which is called the *constraint distribution*, can be locally defined by annihilation of $k = n - r$ linearly independent differential 1–forms on Q. Using local coordinates, the fibers $\mathcal{D}_q$ of the constraint distribution can be described as the kernel of a $k \times n$ matrix $S(q)$, which depends smoothly on q and has everywhere rank k:

$$\mathcal{D}_q \;=\; \ker S(q) \;=\; \{\dot q \in T_qQ \,:\, S(q)\dot q = 0\,\}\,. \tag{1}$$

The constraint distribution $\mathcal{D}$ can also be thought of as a submanifold D of TQ of dimension $2n - k$, which is called the *constraint manifold*:

$$D \;=\; \{(q,\dot q) \in TQ \,:\, \dot q \in \mathcal{D}_q\,\}\,.$$

D'Alembert's principle states that the reaction force annihilates (an appropriate jet extension of) the distribution $\mathcal{D}$, and leads to a dynamical system on the constraint submanifold D of TQ, that is, a vector field $X_{L,Q,D}$ on D, that we will call *nonholonomic (Lagrangian) system* (L, Q, D). This vector field is given by Lagrange equations with the reaction force, which in bundle coordinates are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = R\,. \tag{2}$$

For a given system, namely for given S, A and V, the reaction force R is a known function of $(q,\dot q) \in D$, that is

$$R(q,\dot q) \;=\; S(q)^T\big[S(q)A(q)^{-1}S(q)^T\big]^{-1}\Big[S(q)A(q)^{-1}\beta(q,\dot q) + S(q)A(q)^{-1}V'(q) - \gamma(q,\dot q)\Big] \tag{3}$$

where $\beta(q,\dot q) \in \mathbb{R}^n$, $V'(q) \in \mathbb{R}^n$ and $\gamma(q,\dot q) \in \mathbb{R}^k$ have components

$$\beta_i(q,\dot q) \;=\; \sum_{j,h}\left(\frac{\partial A_{ij}}{\partial q_h}(q) - \frac{1}{2}\frac{\partial A_{jh}}{\partial q_i}(q)\right)\dot q_j\dot q_h\,, \quad V_i'(q) \;=\; \frac{\partial V}{\partial q_i}(q)\,, \quad \gamma_a(q,\dot q) \;=\; \sum_{j,h}\frac{\partial S_{aj}}{\partial q_h}(q)\dot q_j\dot q_h$$

$(i, j, h = 1,\dots,n,\ a = 1,\dots,k)$ [2].

For each $q \in Q$, the range of $S(q)^T$ is the annihilator $\mathcal{D}_q^\circ$ of $\mathcal{D}_q$ and, in agreement with d'Alembert principle, the reaction force takes values in $\mathcal{D}_q^\circ$. Given that, at each point q, the matrix $S^T(SA^{-1}S^T)^{-1}SA^{-1}$ is the A^{-1}–orthogonal projector onto $\mathcal{D}_q^\circ$, the union over all potentials $V : Q \to \mathbb{R}$ of the vectors $R(q,\dot q)$ equals $\mathcal{D}_q^\circ$ for any $\dot q \in \mathcal{D}_q$. However, for a given system (namely, for given S, A and V), the image

$$\mathcal{R}_q \;:=\; \bigcup_{\dot q \in \mathcal{D}_q} R(q,\dot q)$$

of the fiber $\mathcal{D}_q$ under the map $\dot q \mapsto R(q,\dot q)$ may be only a proper subset of $\mathcal{D}_q^\circ$. The annihilators $\mathcal{R}_q^\circ \subset T_qQ$ of these sets, being linear spaces, are the fibers of a distribution $\mathcal{R}^\circ$ on Q, possibly of non–constant rank and non–smooth. Since $\mathcal{R}_q^\circ$ consists of the tangent vectors $\dot q \in T_qQ$ which annihilate all possible values of the reaction force on constrained motions through q, $\mathcal{R}^\circ$ was called the *reaction–annihilator distribution* [15]. Clearly $\mathcal{R}_q^\circ \supseteq \mathcal{D}_q$ for all $q \in Q$.

B. Notation and terminology. From now on, thinking of the Lagrangian $L = T - V$ as given, we denote by

$$p(q,\dot q) \;:=\; \frac{\partial L}{\partial \dot q}(q,\dot q) \;=\; A(q)\dot q$$

the conjugate momenta. If $\xi^Q = \sum_i \xi_i^Q \partial_{q_i}$ is a vector field on Q, then we denote by ξ^{TQ} its tangent lift, namely the vector field on TQ which, in bundle coordinates $(q, \dot{q})$, is given by

$$\xi^{TQ} \;=\; \sum_i \xi_i^Q \partial_{q_i} + \sum_{ij} \dot{q}_j \frac{\partial \xi_i^Q}{\partial q_j} \partial_{\dot{q}_i} \,. \tag{4}$$

If $\mathcal{E}$ is a distribution on Q, then we denote by $\Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$. Moreover, we denote by $\mathcal{E}^\perp$ the distribution whose fibers are the A–orthogonal complements of the fibers of $\mathcal{E}$, where A is the kinetic matrix.

Given two distributions $\mathcal{E}$ and $\mathcal{F}$ on Q, we denote by $\mathcal{E} \cap \mathcal{G}$ the distribution on Q with fibers $\mathcal{E}_q \cap \mathcal{G}_q$. Moreover, we say that $\mathcal{E}$ is an *over–distribution* of $\mathcal{F}$, and that $\mathcal{F}$ is an *under–distribution* of $\mathcal{E}$, if their fibers satisfy $\mathcal{E}_q \supseteq \mathcal{F}_q$ for all $q \in Q$. By a regular distribution we mean a smooth constant rank distribution.

C. First integrals linear in the velocities. A first integral of a nonholonomic system (L, Q, D) is a smooth function $F : D \to \mathbb{R}$ which is constant along the solutions of (2), that is, $X_{L,Q,D}(F) = 0$. A first integral which is linear in the velocities can be written as

$$F(q, \dot{q}) = \xi^Q(q) \cdot p(q, \dot{q})|_D$$

for some smooth vector field ξ^Q on Q, that we call a *generator* of F; we also say that F *is generated* by ξ^Q. It is well known that, because of the restriction to the constraint manifold D, the generator of a first integral F which is linear in the velocities is never unique. Specifically, there is a unique generator $\xi^Q_{\mathcal{D}}$ of F which is a section of $\mathcal{D}$, and a vector field on Q is a generator of F if and only if its A–orthogonal projection onto $\mathcal{D}$ equals $\xi^Q_{\mathcal{D}}$ [19]. Thus, a first integral which is linear in the velocities has infinitely many generators which are sections of any over–distribution of $\mathcal{D}$. The unique role of $\mathcal{R}^\circ$ emerges from the following

Proposition 1 [15] *Given a nonholonomic system (L, Q, D) and a smooth vector field ξ^Q on Q, any two of the following three conditions imply the third:*
C1. $\xi^Q \in \Gamma(\mathcal{R}^\circ)$
C2. $\xi^{TQ}(L)|_D = 0$
C3. $\xi^Q \cdot p\,|_D$ *is a first integral of* (L, Q, D).

The proof of Proposition 1 can be found in [15], but it is essentially the same as that of Proposition 6 below.

Since $\mathcal{R}^\circ$ is an over–distribution of $\mathcal{D}$, Proposition 1 implies the well known result that any smooth section ξ^Q of $\mathcal{D}$ which satisfies the invariance condition C2 is the generator of a linear first integral of (L, Q, D), see particularly [2, 19, 14]. On this regard, it should be remarked that, if $\xi^Q \cdot p$ is a first integral, then the A–orthogonal projection of ξ^Q onto $\mathcal{D}$ satisfies the three conditions C1, C2, C3 [15]. Therefore, Proposition 1 does not prove the existence of any new first integral with respect to the case $\xi^Q \in \Gamma(\mathcal{D})$. However, the consideration of sections of the larger distribution $\mathcal{R}^\circ$ gives significant advantages when a group action is considered, see next Subsection.

Furthermore, the consideration of $\mathcal{R}^\circ$ gives insight on various questions. For instance, among the consequences of Proposition 1 is the following fact, that we will need in the sequel: if $\xi^Q \cdot p$ is a first integral of the holonomic system (L, Q), then $\xi^Q \cdot p|_D$ is a first integral of the nonholonomic system (L, Q, D) if and only if $\xi^Q \in \Gamma(\mathcal{R}^\circ)$ [15].

Remark: To our knowledge, the result just quoted, that a function which is linear in the velocity is a first integral if and only if the tangent lift of the A–orthogonal projection onto $\mathcal{D}$ of its generator preserves the Lagrangian on the points of D is due to Iliev [19]. A characterization of *all* first integrals, not only those which are linear in the velocities, has been given more recently in [12] for linear constraints and then in [13, 26] for nonlinear constraints.

D. Conserved momenta of lifted actions. Assume now that a Lie group G acts smoothly on Q. We denote by ξ_η^Q the infinitesimal generator corresponding to a Lie algebra vector $\eta \in \mathfrak{g}$, and by ξ_η^{TQ} the tangent lift of ξ_η^Q. In bundle coordinates $(q, \dot q)$, the momentum map $J : TQ \to \mathfrak{g}^*$ of the lifted action on TQ is defined by $\langle J(q, \dot q), \eta\rangle = J_\eta(q, \dot q)$, where

$$J_\eta := \xi_\eta^Q \cdot p$$

is the momentum associated to $\eta \in \mathfrak{g}$. See e.g. [4, 1] for details. If the Lagrangian L is G–invariant, namely $\xi_\eta^{TQ}(L) = 0$ for all $\eta \in \mathfrak{g}$, then the momentum map J is a conserved quantity of the holonomic system (L, Q). With regard to the nonholonomic system, Proposition 1 implies the following

Proposition 2 [15] *Assume that a Lie group G acts smoothly on Q and that the Lagrangian L is G–invariant. Given $\eta \in \mathfrak{g}$, $J_\eta|_D$ is a first integral of (L, Q, D) if and only if $\xi_\eta^Q \in \Gamma(\mathcal{R}^\circ)$.*

We shall denote by $\mathcal{G}$ the (possibly non–regular) distribution on Q whose fibers $\mathcal{G}_q$ are the tangent spaces to the orbits of G in Q, that is $\mathcal{G}_q = T_q(G.q)$. Then, it follows from Proposition 2 that *the restriction to D of the momentum map J is a conserved quantity of (L, Q, D) if and only if $\mathcal{G} \subseteq \mathcal{R}^\circ$.*

Let $\mathcal{E}$ be a distribution on Q. Under the hypotheses of Proposition 2, assume that a Lie algebra element $\eta \in \mathfrak{g}$ is such that the infinitesimal generator ξ_η^Q is a section of $\mathcal{E}$. Then, following [15], we will say that

- η and ξ_η^Q are *$\mathcal{E}$–symmetries*
- $J_\eta|_D$ is a *$\mathcal{E}$–momentum.*

Of course, the interesting cases are $\mathcal{E} = \mathcal{D}$ and $\mathcal{E} = \mathcal{R}^\circ$. In fact, Proposition 2 states that *the set of conserved components of the momentum map coincides with that of $\mathcal{R}^\circ$–momenta.*

In the literature, $\mathcal{D}$–symmetries are known as *horizontal symmetries*. Since $\mathcal{D} \subseteq \mathcal{R}^\circ$, Proposition 2 implies the very well known fact that horizontal symmetries generate conserved momenta [21, 3, 7, 10, 22]. The advantage of considering the distribution $\mathcal{R}^\circ$ is that it can host more infinitesimal generators of the group actions than $\mathcal{D}$, and the set of conserved components of the momentum map may thus be larger than that of $\mathcal{D}$–symmetries. A number of examples of first integrals of nonholonomic systems which are $\mathcal{R}^\circ$–momenta but not $\mathcal{D}$–momenta have been given in [15], see also Section 4.

Remark: Even though the terminology that we adopt does not stress this fact, 'symmetries' and 'momenta' (and later on 'gauge symmetries' and 'gauge momenta') are relative to a given group action.

3 Gauge symmetries and gauge momenta

A. The gauge mechanism. In this Section we consider a nonholonomic Lagrangian system (L, Q, D) with Lagrangian invariant under the (lift of) a smooth action of a Lie group G on Q.

Motivated by the fact that horizontal symmetries are rather rare and account only for some of the known first integrals linear in the velocities of sample nonholonomic systems with symmetry, Bates, Graumann and MacDonnell pointed out in [5] that there is another, more general mechanism to produce integrals of motion of nonholonomic systems out of a group action. (See also [22, 28] for related considerations). Specifically, given that any section of $\mathcal{D}$ which infinitesimally preserves the Lagrangian on the constraint manifold generates a linear first integral [2, 21, 3, 12, 14], they constructed such sections in a number of examples by taking linear combinations of generators of the group actions, but with nonconstant coefficients. By Proposition 1, the same remains true when the sections are $\mathcal{R}^\circ$–valued. Thus, we formalize here the gauge method in this more general case. We freely use the notation introduced in the previous Section. First, we introduce some terminology to be used throughout:

Definition 1 *Consider a nonholonomic system (L,Q,D) and a smooth action of a Lie group G on Q such that L is G–invariant.*

(i) A gauge symmetry *is a smooth section of $\mathcal{G}$ which satisfies the invariance condition C2. The* gauge momentum $\widehat{J}_{\xi^Q} : D \to \mathbb{R}$ of a gauge symmetry ξ^Q *is the function*

$$\widehat{J}_{\xi^Q} \;:=\; p\cdot \xi^Q|_D\,.$$

(ii) If $\mathcal{E}$ is a distribution on Q, then a gauge symmetry which is also a section of $\mathcal{E}$ is called $\mathcal{E}$–gauge symmetry. *The gauge momentum of an $\mathcal{E}$–gauge symmetry is called* $\mathcal{E}$–gauge momentum.

(iii) A gauge momentum *is a function on D which is the gauge momentum of some gauge symmetry.*

The reason why in the definition of gauge symmetry we require the invariance condition C2 is that the lift of a section of $\mathcal{G}$ need not be tangent to the orbits of the lifted action. In the sequel, the important cases are those with either $\mathcal{E}=\mathcal{D}$ or $\mathcal{E}=\mathcal{R}^\circ$. The connection between gauge momenta and first integrals descends immediately from Proposition 1:

Proposition 3 *Consider a smooth action of a Lie group G on Q such that L is G–invariant.*

i. A gauge momentum is a first integral of (L,Q,D) if and only if it is a $\mathcal{R}^\circ$–gauge momentum.

ii. A first integral linear in the velocities of (L,Q,D) is a gauge momentum if and only if it has a generator which is a section of $\mathcal{G}\cap\mathcal{R}^\circ$.

Proof. (i) A gauge symmetry ξ^Q satisfies the invariance condition C2. Hence, by Proposition 1, $\xi^Q\cdot p|_D$ is a first integral if and only if ξ^Q is a section of $\mathcal{R}^\circ$. (ii) Let $\xi^Q\cdot p|_D$ be a first integral. Then, by Proposition 1, ξ^Q satisfies C2 and is a section of $\mathcal{G}$ if and only if it is a section of $\mathcal{R}^\circ$ and of $\mathcal{G}$. ■

Thus, $\mathcal{R}^\circ$–gauge symmetries form the class of generators of first integrals which satisfy the invariance condition C2 and are sections of $\mathcal{G}$. And among all gauge momenta, the $\mathcal{R}^\circ$–gauge momenta are the conserved ones.

The term 'gauge' was first used in [5], which considered only the horizontal case with $\mathcal{E}=\mathcal{D}$. This term refers to the fact that at each point $q\in Q$, $\xi^Q(q)$ coincides with the value at that point of the infinitesimal generator of an element of $\mathfrak{g}$, but this element changes from point to point. Since the infinitesimal generators of the vectors of $\mathfrak{g}$ span the fibers of $\mathcal{G}$, a practical (though possibly not exhaustive, see subsection 3.C) way of looking for $\mathcal{E}$–gauge symmetries, $\mathcal{E}=\mathcal{D}$ or $\mathcal{R}^\circ$, is that of choosing a basis $\eta_1,\dots,\eta_m$ of $\mathfrak{g}$ and then looking for smooth functions $f_1,\dots,f_m : Q\to\mathbb{R}$ such that the vector field

$$\xi^Q := \sum_{b=1}^m f_b\eta_b^Q \tag{5}$$

is a section of $\mathcal{E}$ and satisfies the invariance condition C2. In such a case, the $\mathcal{E}$–gauge momentum $\widehat{J}_{\xi^Q}$ is the restriction of $\sum_{b=1}^m f_bJ_{\eta_b}$ to D. Note that $\mathcal{E}$–symmetries are given by constant f_b's.

Several instances of first integrals of sample nonholonomic systems which are not $\mathcal{D}$–momenta but are $\mathcal{D}$–gauge momenta have been obtained in [5, 22] using a variant of this procedure. (Instead of verifying that a section (5) of $\mathcal{D}$ satisfies condition C1, it was verified that it satisfies the equivalent condition C3). As shown in [15], some of these $\mathcal{D}$–gauge momenta are $\mathcal{R}^\circ$–momenta, and are thus linked to the group action without the need of a gauge–like construction. Nevertheless, some others among the $\mathcal{D}$–gauge momenta found in [5, 22] are not momenta. We shall provide elementary examples of all possible cases in the next Section.

The fact that the class of $\mathcal{R}^\circ$–gauge momenta may be larger than that of $\mathcal{D}$–gauge momenta is important when $\mathcal{R}^\circ \cap \mathcal{G}$ is an over–distribution of $\mathcal{D} \cap \mathcal{G}$. In particular, if the constraint distribution $\mathcal{D}$ and the distribution $\mathcal{G}$ of the tangent spaces to the group orbits have trivial intersection, then there cannot be $\mathcal{D}$–gauge momenta while there may be $\mathcal{R}^\circ$–gauge momenta and even $\mathcal{R}^\circ$–momenta. Systems with this property form the so–called 'purely kinematic case' and include the (generalized) Chaplygin systems [20, 7, 10, 16]. An example of this situation will be given in Section 4.E.

In certain cases it is possible to make some statements about the (non) existence of $\mathcal{D}$– and $\mathcal{R}^\circ$–gauge momenta:

Proposition 4 *Consider a smooth action of a Lie group G on Q such that L is G–invariant.*

i. If $\mathcal{D} \subseteq \mathcal{G}$, then every first integral which is linear in the velocities is a $\mathcal{D}$–gauge momentum.

ii. If $\mathcal{G} \subseteq \mathcal{D}^\perp$, then there are no nonzero conserved gauge momenta for the considered action.

Proof. (i) Let $F = \xi^Q \cdot p|_D$ be a first integral. As already remarked in Section 2, the A–orthogonal projection $\xi^Q_{\mathcal{D}}$ of ξ^Q onto $\mathcal{D}$ is a generator of F and, by Proposition 1, it satisfies the invariance condition C2. Since $\mathcal{D} = \mathcal{D} \cap \mathcal{G}$, it follows that $\xi^Q_{\mathcal{D}}$ is a $\mathcal{D}$–gauge symmetry and hence F is a $\mathcal{D}$–gauge momentum.

(ii) Recall that a gauge momentum is conserved if and only if it is a $\mathcal{R}^\circ$–gauge momentum. If $F = \xi^Q \cdot p|_D$ is a $\mathcal{R}^\circ$–gauge momentum, then $\xi^Q \in \Gamma(\mathcal{R}^\circ \cap \mathcal{G})$. Since $\mathcal{R}^\circ \cap \mathcal{G} \subseteq \mathcal{D}^\perp$, $\xi^Q \in \Gamma(\mathcal{D}^\perp)$ and $\xi^Q \cdot p|_D = 0$. ■

Case i. of Proposition 4 is verified, for instance, whenever the action is transitive on Q. For an example, see Section 4.A.

B. Noetherianity and weak Noetherianity. In the Hamiltonian formulation of holonomic systems, the momentum map of a group action on Q has the property that it is a conserved quantity for any system on T^*Q whose Hamiltonian is invariant under that action. This fundamental property of the momentum map is sometimes called "Noetherian condition" [24]. In the Lagrangian formulation, where the conjugate momenta change with the kinetic energy, this property is of course owned by the infinitesimal generators, not by the components of the momentum map. We now discuss whether momenta and gauge momenta share this property—or some weaker version of it.

Definition 2 *Consider a manifold Q, a distribution $\mathcal{D}$ on Q, and a Lie group G which acts smoothly on Q.*

i. A section ξ^Q of $\mathcal{G}$ is Noetherian *if $\xi^Q \cdot p|_D$ is a first integral of (L, Q, D) for any G–invariant Lagrangian $L = T - V$ on TQ.*

ii. Fix a G–invariant kinetic energy $T(q, \dot q) = \frac{1}{2}\dot q \cdot A(q)\dot q$. A section ξ^Q of $\mathcal{G}$ is weakly Noetherian *if $\xi^Q \cdot p|_D$ is a first integral of $(T - V, Q, D)$ for any G–invariant potential V on Q.*

Proposition 5 *Under the hypotheses of Definition 2:*

i. $\mathcal{D}$–symmetries are Noetherian.

ii. $\mathcal{D}$–gauge symmetries are weakly Noetherian.

Proof. Statement i. is obvious, given that the momentum map fullfills the Noetherian condition and the distribution $\mathcal{D}$ is kept fixed. In order to prove statement ii., consider a $\mathcal{D}$–gauge symmetry $\xi^Q \in \Gamma(\mathcal{G} \cap \mathcal{D})$ of (T, Q, D). If V is a G–invariant function of Q, then $\xi^{TQ}(T - V)|_D = \xi^{TQ}(T)|_D - \xi^Q(V) = 0$. Hence ξ^Q is a first integral of $(T - V, Q, D)$. ■

A particular instance of statement ii. was noticed in [5], where it was remarked that a certain $\mathcal{D}$–gauge momentum of the so called 'nonholonomic particle' is a first integral for any invariant choice of the potential.

Thus, from the point of view of the Noetherian conditions there is an important difference between $\mathcal{D}$–(gauge) momenta and $\mathcal{R}^\circ$–(gauge) momenta. $\mathcal{R}^\circ$–symmetries need not be Noetherian because the reaction–annihilator distribution depends on A and V. For the same reason, $\mathcal{R}^\circ$–gauge symmetries need not be weakly Noetherian. The reason why $\mathcal{D}$–gauge symmetries need not be Noetherian is that, if $\xi^Q \in \Gamma(\mathcal{D})$ is a section of $\mathcal{G}$ but not an infinitesimal generator of the G–action, then $\xi^{TQ}(T)$ need not vanish for all G–invariant kinetic energies T. (Consider for instance $\xi^Q = \sum_b f_b \xi^Q_{\eta_b}$, as in (5). It follows from (4) that $\xi^{TQ} = \sum_b f_b \xi^{TQ}_{\eta_b} + \sum_b (\dot q \cdot \partial_q f_b)(\xi^Q_{\eta_b} \cdot \partial_{\dot q})$ and hence, if T is G–invariant, $\xi^{TQ}(T) = \sum_b (\dot q \cdot \partial_q f_b) J_{\eta_b}$, which need not be zero even when restricted to D). Nevertheless, there may be $\mathcal{D}$–gauge symmetries which are not $\mathcal{D}$–symmetries and are Noetherian. An elementary example is given by the angular momentum considered in the Conclusions.

C. Some remarks. We add now some remarks on the gauge method. First, we note that, without some hypotheses on the group action, it is in general not true that every section of $\mathcal{G}$ can be written as a linear combination (5) with *smooth* coefficients f_b. This is true if, for instance, the action is locally free. (Under this hypothesis $\mathcal{G}$ is a regular distribution and can hence be thought of as a submanifold $\hat{\mathcal{G}}$ of TQ. The map $\Phi : Q \times \mathfrak{g} \to \hat{\mathcal{G}}$, $(q,\eta) \mapsto \xi^Q_\eta(q)$ is a bijection. Since its Jacobian matrix is invertible, Φ is a diffeomorphism. It follows that to each $\mathcal{G}$-valued vector field ξ^Q there corresponds a unique smooth $\mathfrak{g}$–valued vector field defined by $q \mapsto \pi_{\mathfrak{g}}(\Phi^{-1}(\xi^Q(q)))$, where $\pi_{\mathfrak{g}}$ is the projector onto the second factor).

Second, we note that, even though the constraint distribution $\mathcal{D}$ is not integrable, it may happen that the reaction–annihilator distribution $\mathcal{R}^\circ$ is integrable. In this case, if ξ^Q_1 and ξ^Q_2 are two sections of $\mathcal{R}^\circ$, then their Lie bracket $[\xi^Q_1, \xi^Q_2]$ is a section of $\mathcal{R}^\circ$. Moreover, since $[\xi^Q_1, \xi^Q_2]^{TQ} = [\xi^{TQ}_1, \xi^{TQ}_2]$, if ξ^Q_1 and ξ^Q_2 satisfy the invariance condition C2, then also $[\xi^Q_1, \xi^Q_2]$ satisfies C2. Therefore, the set of all generators of first integrals linear in the velocities which satisfy C2 form a Lie algebra. Since $[\xi^Q_1, \xi^Q_2] \cdot p = \{\xi^Q_1 \cdot p, \xi^Q_2 \cdot p\}$, this in turn implies that these first integrals form a Lie algebra with respect to the Poisson brackets $\{\ ,\ \}$ on T^*Q.

Third, as observed in [5], the notion of gauge momenta has an obvious extension to non–lifted actions. Since the study of this extension is probably more appropriately done in the Hamiltonian setting, we do not consider it here. We note however that the consideration of momenta of vector fields on TQ which are not lifts of sections of $\mathcal{G}$ might open up new possibilities even in the case of lifted actions. In fact, a (non–equivalent) alternative to the gauge method we have described here would use, instead of (5), linear combinations of the lifts η^{TQ}_b of the infinitesimal generators, with coefficients which are functions on TQ. A comparison of the two methods requires a more general setting than the one we adopted here, and therefore lies outside the scope of the present work.

4 Examples

The relations among the four classes of $\mathcal{D}$– and $\mathcal{R}^\circ$–momenta and $\mathcal{D}$– and $\mathcal{R}^\circ$–gauge momenta are depicted in Figure 1. We provide here a few simple examples, chosen to illustrate the various situations: $\mathcal{R}^\circ$–momenta which are or which are not $\mathcal{D}$–gauge momenta, $\mathcal{D}$–gauge momenta which are not $\mathcal{R}^\circ$–momenta, and $\mathcal{R}^\circ$–gauge momenta which are neither $\mathcal{R}^\circ$–momenta nor $\mathcal{D}$–gauge momenta, see Figure 1. Even though our emphasis is not on $\mathcal{D}$–momenta, several examples of which are well known, see e.g. [7, 10, 6, 11], we will incidentally find here one instance of them. For other examples of $\mathcal{D}$–gauge momenta see [5, 22] and for other examples of $\mathcal{R}^\circ$–momenta see [15].

A. The vertical coin. Consider a disk which is constrained to roll without slipping on a horizontal plane while standing vertically, under the action of no active forces (except for gravity, which however plays no role), see e.g. [12, 7, 10, 15]. For a natural choice of the symmetry group, this system has two $\mathcal{R}^\circ$–momenta which are not $\mathcal{D}$–momenta [15]. We shall show here that they are $\mathcal{D}$–gauge momenta as well. (For another natural choice of the symmetry group, however, one of the

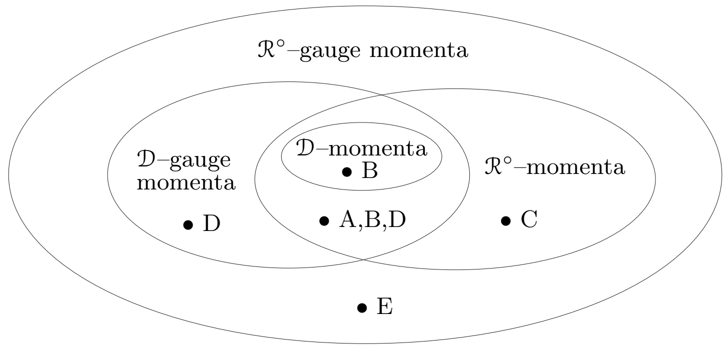

Figure 1: Classification of conserved gauge–momenta. The points locate the first integrals exhibited by Examples A, B, C, D and E of Section 4.

two is a $\mathcal{D}$–momentum, see Subsection 4.B). This is an elementary computation, which is however useful as the basis for the next example.

The holonomic system consists of a disk of unit radius which stands vertically and touches the plane. Its configuration manifold is $Q = \mathbb{R}^2 \times S^1 \times S^1 \ni (x, y, \varphi, \theta)$, where $(x, y) \in \mathbb{R}^2$ are cartesian coordinates of the point of contact, φ is the angle between the x–axis and the projection of the disk on the plane, and θ is the angle between a fixed radius of the disk and the vertical. Assuming that the disk has unit mass, the kinetic energy is $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2 + \frac{1}{2}I\dot{\theta}^2$, where J and I are the pertinent moments of inertia. All four conjugate momenta $p_x = \dot{x}$, $p_y = \dot{y}$, $p_\varphi = J\dot{\varphi}$, and $p_\theta = I\dot{\theta}$ are first integrals of the holonomic system.

The nonholonomic constraint of rolling without slipping leads to a constraint distribution $\mathcal{D}$ with fibers

$$\mathcal{D}_{(x,y,\varphi,\theta)} = \mathrm{span}_{\mathbb{R}}\big\{\partial_\varphi\,,\ \cos\varphi\,\partial_x + \sin\varphi\,\partial_y + \partial_\theta\big\}$$

and, by (3), to a reaction–annihilator distribution $\mathcal{R}^\circ$ with fibers

$$\mathcal{R}^\circ_{(x,y,\varphi,\theta)} = \mathrm{span}_{\mathbb{R}}\{\partial_\varphi\,,\ \cos\varphi\,\partial_x + \sin\varphi\,\partial_y\,,\ \partial_\theta\,\}\,.$$

Note that $\mathcal{D}$ has rank two while $\mathcal{R}^\circ$ has rank three. The constraint manifold D is six–dimensional and can be globally parameterized with the coordinates $(x, y, \varphi, \theta, \dot{\varphi}, \dot{\theta})$. In these coordinates, the equations of motion are

$$\dot{x} = \dot{\theta}\cos\varphi\,, \quad \dot{y} = \dot{\theta}\sin\varphi\,, \quad \ddot{\varphi} = 0\,, \quad \ddot{\theta} = 0\,.$$

Thus, $p_\varphi|_D$ and $p_\theta|_D$ are first integrals. Note that their generators ∂_φ and ∂_θ are sections of $\mathcal{R}^\circ$ (consistently with the fact that they are restrictions to D of first integrals of the holonomic system).

Following [7], we now introduce a symmetry group. The Lagrangian is invariant under the action of $\mathrm{SE}(2) \times S^1$ given by

$$\big((a, b, \alpha), \beta\big).(x, y, \varphi, \theta) \;=\; (a + x\cos\alpha - y\sin\alpha\,,\ b + x\sin\alpha + y\cos\alpha\,,\ \varphi + \alpha\,,\ \theta + \beta)$$

for $(a, b, \alpha) \in \mathrm{SE}(2)$ and $\beta \in S^1$, namely roto–translations of the disk on the plane and rotations of the disk around its axis. Since this action is transitive on the configuration manifold, Proposition 4 ensures that both $p_\varphi|_D$ and $p_\theta|_D$ are $\mathcal{D}$–gauge momenta.

Since ∂_θ is a section of $\mathcal{R}^\circ$, $p_\theta|_D$ is a $\mathcal{R}^\circ$–momentum. That $p_\varphi|_D$ is a $\mathcal{R}^\circ$–momentum is seen by observing that, since $p_x\sin\varphi + p_y\cos\varphi = -p_\theta$ on D, another generator of $p_\varphi|_D$ is $\partial_\varphi + \cos\varphi\,\partial_x + \sin\varphi\,\partial_y + \partial_\theta$, which is a section of $\mathcal{R}^\circ$ and an infinitesimal generator of the action.

It remains to show that neither $p_\varphi|_D$ nor $p_\theta|_D$ are $\mathcal{D}$–momenta. The generator ∂_φ of $p_\varphi|_D$ is a section of $\mathcal{D}$ but is not an infinitesimal generator of the action. The generator of $p_\theta|_D$ which is a section of $\mathcal{D}$ is seen to be $\cos\varphi\partial_x + \sin\varphi\partial_y + \partial_\theta$, which is not an infinitesimal generator of the action.

B. The vertical coin with a different group. Even though, in the previous example, ∂_φ is not a $\mathcal{D}$–symmetry for the action of $\mathrm{SE}(2)\times S^1$, it is a $\mathcal{D}$–symmetry for the abelian action of $\mathbb{R}^2\times S^1\times S^1 \ni (\lambda,\mu,\alpha,\beta)$ given by $(x,y,\varphi,\theta)\mapsto(x+\lambda, y+\mu,\varphi+\alpha,\theta+\beta)$. Note that the two actions have the same orbits. This example indicates that the distinction between the various categories of first integrals may be rather subtle.

C. A torqued vertical coin. As we have already remarked, a case in which there cannot be $\mathcal{D}$–gauge momenta, while there may be $\mathcal{R}^\circ$–momenta and $\mathcal{R}^\circ$–gauge momenta, is that for which the intersection between the constraint distribution $\mathcal{D}$ and the distribution $\mathcal{G}$ of the tangent spaces to the group orbits is trivial in an open subset of Q. In order to construct a system with one $\mathcal{R}^\circ$–momentum which is not a $\mathcal{D}$–gauge momentum, we thus add to the disk of Example A an external potential which is invariant under the action of a subgroup $G=\mathbb{T}^2$ of $\mathrm{SE}(2)\times S^1$, such that $\mathcal{D}_q\cap\mathcal{G}_q=\{0\}$ (in an open subset of Q).

Specifically, we assume that the disk is acted upon by external forces with potential energy $V(x,y,\varphi) = x\sin\varphi - y\cos\varphi$. The equations of motion of the nonholonomically constrained system $(L=T-V,Q,D)$ are

$$\dot x = \dot\theta\cos\varphi\,,\qquad \dot y = \dot\theta\sin\varphi\,,\qquad J\ddot\varphi = x\cos\varphi + y\sin\varphi\,,\qquad \ddot\theta = 0\,.$$

Thus, $p_\theta|_D$ is still a first integral.

The Lagrangian L is invariant under the action of $\mathbb{T}^2$ given by

$$(\alpha,\beta).(x,y,\varphi,\theta) = (x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, \varphi+\beta, \theta+\alpha)\,.$$

The distribution $\mathcal{G}$ tangent to the orbits of this action has fibers

$$\mathcal{G}_{(x,y,\varphi,\theta)} = \mathrm{span}_{\mathbb{R}}\{\partial_\theta, \partial_\varphi - y\partial_x + x\partial_y\}\,.$$

The intersection $\mathcal{D}_q\cap\mathcal{G}_q$ is trivial in all of Q, except where $x=\cos\varphi$ and $y=-\sin\varphi$. Therefore, there are no (nonconstant) $\mathcal{D}$–gauge momenta.

The reaction annihilator distribution $\mathcal{R}^\circ$ turns out to be the same as in Example A. Thus, ∂_θ is a $\mathcal{R}^\circ$-symmetry and $p_\theta|_D$ is a $\mathcal{R}^\circ$–momentum. Note that, since $\mathcal{G}_q\cap\mathcal{R}^\circ_q = \mathrm{span}_{\mathbb{R}}\{\partial_\theta\}$, there are no other $\mathcal{R}^\circ$–gauge momenta.

D. The heavy ball in a cylinder. We consider now a system which is known to possess a $\mathcal{D}$–momentum and a $\mathcal{D}$–gauge momentum and we show that the latter is not a $\mathcal{R}^\circ$–momentum.

The system is the classical system of a heavy homogeneous ball constrained to roll without slipping inside a vertical cylinder, see particularly [5, 22]. Let m be the mass of the ball, r its radius, I its moment of inertia, and $R+r$ the radius of the cylinder. The configuration space is $Q=\mathbb{R}\times S^1\times\mathrm{SO}(3)$. Let $(z,\alpha)\in\mathbb{R}\times S^1$ be the coordinates of the center of mass of the ball and $(\varphi,\psi,\theta)\in S^1\times S^1\times(0,\pi)$ the Euler angles relative to a body frame with the origin in the center of the ball and to a spatial frame with the same origin and axes x,y,z with z aligned to the axis of the cylinder (as for the definition of the angles, we adopt the convention of [4]). Then, the Lagrangian is $L=\frac{m}{2}\left(R^2\dot\alpha^2+\dot z^2\right)+\frac{I}{2}(\dot\theta^2+\dot\varphi^2+\dot\psi^2+2\dot\varphi\dot\psi\cos\theta) - mgz$.

The no–slipping constraint $R\dot\alpha + r[\dot\varphi+\dot\psi\cos\theta]=0$, $\dot z + r[\dot\psi\sin\theta\cos(\alpha-\varphi)+\dot\theta\sin(\alpha-\varphi)]=0$ produces the constraint distribution with fibers

$$\mathcal{D}_{(z,\alpha,\varphi,\psi,\theta)} = \mathrm{span}_{\mathbb{R}}\left\{r\partial_\alpha - R\partial_\varphi\,,\ \partial_\theta - r\sin(\alpha-\varphi)\,\partial_z\,,\ \partial_\psi - r\cos(\alpha-\varphi)\sin\theta\,\partial_z - \cos\theta\,\partial_\varphi\right\}\,.$$

The reaction force is $R(\alpha,z,\varphi,\psi,\theta,\dot\alpha,\dot z,\dot\varphi,\dot\psi,\dot\theta) = \big(0,K,0,Kr\cos(\alpha-\varphi)\sin\theta,Kr\sin(\alpha-\varphi)\big)$ with $K = mJ(J+mr^2)^{-1}[g - r\cos(\alpha-\varphi)\dot\alpha\dot\theta + r\sin(\alpha-\varphi)\sin\theta\dot\alpha\dot\psi]$, so that

$$\mathcal{R}^\circ_{(\alpha,z,\varphi,\psi,\theta)} = \mathrm{span}_{\mathbb{R}}\,\{\partial_\alpha\,,\ \partial_\varphi\,,\ \partial_\theta - r\sin(\alpha-\varphi)\,\partial_z\,,\ \partial_\psi - r\cos(\alpha-\varphi)\sin\theta\,\partial_z\}\ .$$

(Note that $\mathcal{R}^\circ$ has rank four while $\mathcal{D}$ has rank three).

The holonomic system of Lagrangian L has the linear first integrals $p_\alpha = mR^2\dot\alpha$, $p_\varphi = I(\dot\varphi + \dot\psi\cos\theta)$ and $p_\psi = I(\dot\psi+\dot\varphi\cos\theta)$. Since the generators ∂_α of p_α and ∂_φ of p_φ are sections of $\mathcal{R}^\circ$, $p_\alpha|_D$ and $p_\varphi|_D$ are first integrals of the nonholonomic system, even though they are not independent given that $Ip_\alpha|_D = -mrRp_\varphi|_D$.

The Lagrangian L is invariant under the right action of $S^1\times\mathrm{SO}(3)$ on the $S^1\times\mathrm{SO}(3)$ factor of Q. The tangent spaces to the group orbits are spanned by ∂_α and by three infinitesimal generators of the SO(3)–action, which are easily computed from the three components of the momentum map (namely, the angular momentum vector in space) as

$$\sin\varphi(\partial_\psi - \cos\theta\,\partial_\varphi) + \cos\varphi\sin\theta\,\partial_\theta\,, \qquad \cos\varphi(\partial_\psi - \cos\theta\,\partial_\varphi) - \sin\theta\,\sin\varphi\,\partial_\theta\,, \qquad \partial_\varphi\,.$$

Thus, ∂_α and ∂_φ are $\mathcal{R}^\circ$–symmetries and $p_\alpha|_D$ is therefore a $\mathcal{R}^\circ$–momentum. However, as remarked in [5, 22], $p_\alpha|_D$ is also a $\mathcal{D}$–momentum because it is also generated by the $\mathcal{D}$–symmetry $\frac{1}{2}(\partial_\alpha - \frac{R}{r}\partial_\varphi)$.

The nonholonomic system possesses yet another first integral linear in the velocities and independent of $p_\alpha|_D$, which has been shown to be a $\mathcal{D}$–gauge momentum for the considered group action [5, 22]. Such integral is not a $\mathcal{R}^\circ$–momentum because the only linear combinations with constant coefficients of infinitesimal generators which are sections of $\mathcal{R}^\circ$ are of the form $a\partial_\alpha + b\partial_\varphi$ with $a,b\in\mathbb{R}$, and generate first integrals that have already been considered.

E. A five–dimensional nonholonomic particle. The 'nonholonomic particle' of ref. [25] has been used as an example of a system having $\mathcal{D}$–gauge momenta which are not $\mathcal{D}$–momenta [5]. Here we use a higher–dimensional analogue of this system to show that the class of $\mathcal{R}^\circ$–gauge momenta is actually larger than the union of $\mathcal{D}$–gauge momenta and $\mathcal{R}^\circ$–momenta. This example can be viewed either as a nonholonomically constrained particle in 5–dimensional space, or as a system of two nonholonomically constrained particles with equal masses, one of which is holonomically constrained to a plane.

Consider the Lagrangian $L(q,\dot q) = \frac{1}{2}\,|\dot q|^2$ on $T\mathbb{R}^n \ni (q,\dot q)$, $n>3$, and the linear nonholonomic constraint given by the non–integrable rank–two distribution with fibers

$$\mathcal{D}_q \;=\; \mathrm{span}_{\mathbb{R}}\{\partial_{q_1}\,,\ q_1\partial_{q_2} + \cdots + q_{n-2}\partial_{q_{n-1}} + \partial_{q_n}\}\,.$$

The matrix $S(q)$ such that $\mathcal{D}_q = \ker S(q)$ is the $(n-2)\times n$ matrix with block structure

$$S(q) \;=\; \begin{pmatrix} 0_{n-2} & \mathbb{I}_{n-2} & -\hat q \end{pmatrix},$$

where $0_{n-2}\in\mathbb{R}^{n-2}$ is the zero vector, $\mathbb{I}_{n-2}$ is the $(n-2)\times(n-2)$ identity matrix, and $\hat q \in \mathbb{R}^{n-2}$ is the vector whose components are the first $n-2$ coordinates of q. In the sequel we write vectors indifferently as rows or columns, depending on typographical convenience.

Since the kinetic matrix A equals the identity and there is no potential energy, equation (3) gives the reaction force

$$R(q,\dot q) = -S(q)^T\big[S(q)S(q)^T\big]^{-1}\gamma(q,\dot q)\,.$$

In order to determine $\mathcal{R}_q = \bigcup_{\dot q\in\mathcal{D}_q} R(q,\dot q)$ we first compute the span of the vectors $\gamma(q,\dot q)$ for $\dot q\in\mathcal{D}_q$ and then we apply to such a vector space the linear transformation $S^T[SS^T]^{-1}$. Note that $\gamma(q,\dot q) = -\dot q_n\dot{\hat q}$ and that the vectors of $\mathcal{D}_q$ are the vectors $(a, bq_1,\ldots,bq_{n-2},b)$ for $a,b\in\mathbb{R}$. Thus

$$\gamma\big(q,(a,bq_1,\ldots,bq_{n-2},b)\big) \;=\; \big(-ba, -b^2q_1,\ldots,-b^2q_{n-3}\big)\,.$$

Varying a, b and taking the linear closure generates the vector space

$$E \;=\; \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q_1 \\ \vdots \\ q_{n-3} \end{pmatrix} \right\}.$$

Since $SS^T = \mathbb{I}_{n-2} + \hat{q}\hat{q}^T$ has inverse $\mathbb{I}_{n-2} - (1+|\hat{q}|^2)^{-1}\hat{q}\hat{q}^T$, we have $S^T[SS^T]^{-1} = S^T - (1+|\hat{q}|^2)^{-1}S^T\hat{q}\hat{q}^T$. Hence, the image of E under the matrix $S^T(SS^T)^{-1}$ is the vector space

$$\mathcal{R}_q \;=\; \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 \\ -q_1q_2-\ldots-q_{n-3}q_{n-2} \\ q_1^2 \\ q_1q_2 \\ \vdots \\ q_1q_{n-3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1-|\hat{q}|^2 \\ q_1q_2 \\ \vdots \\ q_1q_{n-2} \\ q_1 \end{pmatrix} \right\}.$$

Since $\mathcal{R}_q$ has dimension 2 everywhere except on the hyperplane $q_1 = 0$, where it is one–dimensional, the distribution $\mathcal{R}^\circ$ has rank $n-2$ in the complement of such hyperplane, and has rank $n-1$ on it. Thus, for $n \geq 5$ the fibers of $\mathcal{R}^\circ$ strictly contain everywhere those of $\mathcal{D}$. We restrict ourselves to $n = 5$. In this case

$$\mathcal{R}_q^\circ \;=\; \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q_1 \\ q_2 \\ q_3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -q_2 \\ q_1 \\ q_2^2-q_1q_3 \end{pmatrix} \right\}$$

except on the points of the hyperplane $q_1 = 0$, where the fibers of $\mathcal{R}^\circ$ are $\{\dot{q} \in \mathbb{R}^5 \,:\, \dot{q}_2 = 0\}$.

Consider now the action of $\mathbb{R}^3$ on the configuration space $\mathbb{R}^5$ by translation on the coordinates q_2, q_4, q_5. All smooth sections of the distribution $\mathcal{G} \cap \mathcal{R}^\circ$ must be multiples of the vector field

$$v(q) \;=\; \begin{pmatrix} 0 \\ q_1 \\ 0 \\ q_1+q_3 \\ 1+q_2^2-q_1q_3 \end{pmatrix}$$

while the only smooth section of the distribution $\mathcal{G} \cap \mathcal{D}$ is the zero section. This last fact implies that there cannot be $\mathcal{D}$–momenta nor $\mathcal{D}$–gauge momenta. Moreover, since no function $f : \mathbb{R}^5 \to \mathbb{R}$ can make $f(q)v(q)$ an infinitesimal generator of the group action, there cannot be $\mathcal{R}^\circ$–momenta either.

On the other hand, a computation shows that the tangent lift of the vector field

$$\xi^Q(q) := (1+|\hat{q}|^2)^{-1/2} v(q)$$

preserves the Lagrangian $\frac{1}{2}\,|\dot{q}|^2$ on the constraint manifold D. In fact, on the points of D,

$$\xi^{TQ}(q) = \left(q_1^2+q_2^2+q_3^2+1\right)^{3/2} \begin{pmatrix} 0 \\ q_1\left(q_1^2+q_2^2+q_3^2+1\right) \\ 0 \\ (q_1+q_3)\left(q_1^2+q_2^2+q_3^2+1\right) \\ \left(q_2^2-q_1q_3+1\right)\left(q_1^2+q_2^2+q_3^2+1\right) \\ 0 \\ -\dot{q}_5q_2q_1^2-\dot{q}_5q_2q_3q_1+\dot{q}_1\left(q_2^2+q_3^2+1\right) \\ 0 \\ \dot{q}_1q_3^2-q_1(\dot{q}_1+2\dot{q}_5q_2)q_3+(\dot{q}_1+\dot{q}_5q_2)\left(q_2^2+1\right) \\ \dot{q}_5q_2q_1^3+\dot{q}_5q_2q_3q_1^2-\left(\dot{q}_1q_2^2-2\dot{q}_5q_3^2q_2+\dot{q}_1\right)q_1-\dot{q}_1q_3^3-(\dot{q}_1+\dot{q}_5q_2)\left(q_2^2+1\right)q_3 \end{pmatrix}$$

and $\frac{\partial T}{\partial(q,\dot{q})}(q,\dot{q}) = (0,0,0,0,0,\dot{q}_1,\dot{q}_5q_1,\dot{q}_5q_2,\dot{q}_5q_3,\dot{q}_5)$, so that the scalar product of the two vectors is zero in D. Therefore, ξ^Q is a $\mathcal{R}^\circ$–gauge symmetry for the considered action of $\mathbb{R}^3$. Its gauge momentum, which is obtained by restricting $\xi^Q \cdot p = \xi^Q \cdot (\dot q_1, \dot q_5 q_1, \dot q_5 q_2, \dot q_5 q_3, \dot q_5)$ to the fibers of $\mathcal{D}$, equals

$$\dot q_5 \sqrt{1+q_1^2+q_2^2+q_3^2}\,, \qquad (q,\dot q)\in D\,.$$

This is an example of a $\mathcal{R}^\circ$–gauge momentum which is neither a $\mathcal{R}^\circ$–momentum nor a $\mathcal{D}$–gauge momentum.

Note that the existence of $\mathcal{R}^\circ$–gauge momenta which are not $\mathcal{D}$–gauge momenta does not contradict the fact that, as we have remarked in Section 2, any first integral which is linear in the velocities has a generator which is a section of $\mathcal{D}$ and satisfies condition C2, because the A–orthogonal projection onto $\mathcal{D}$ of a section of $\mathcal{G}\cap\mathcal{R}^\circ$ need not be in $\mathcal{G}\cap\mathcal{D}$.

5 The momentum equation

A. The nonholonomic momentum map and the momentum equation. We investigate now the 'nonholonomic momentum map' and 'momentum equation' studied in [7, 9, 10, 27]. Our aim is to extend this construction from the constraint distribution to the reaction–annihilator distribution, and to clarify its link with the gauge method of Section 3. As above, for clarity, we privilege coordinate expressions.

Consider a nonholonomic Lagrangian system (L,Q,D) and a Lie group G which acts smoothly on Q, with Lie algebra $\mathfrak{g}$. Consider the vector bundle over Q defined as the disjoint union

$$\mathfrak{g}_Q \;:=\; \coprod_{q\in Q} \mathfrak{g}\,,$$

that is, $Q\times\mathfrak{g}$ with projection map given by the projection onto the first factor. Fix now a (not necessarily regular) distribution $\mathcal{E}$ on Q. Then, for each $q\in Q$ the set

$$\mathfrak{g}_q^{\mathcal{E}} \;:=\; \big\{\eta\in\mathfrak{g} \,|\, \xi_\eta^Q(q)\in\mathcal{E}_q\big\}$$

is a vector subspace of $\mathfrak{g}$ and the disjoint union

$$\mathfrak{g}_Q^{\mathcal{E}} \;:=\; \coprod_{q\in Q} \mathfrak{g}_q^{\mathcal{E}}$$

is a subset of $\mathfrak{g}_Q$. In general, $\mathfrak{g}_Q^{\mathcal{E}}$ is not a subbundle of $\mathfrak{g}_Q$, but this is not relevant for the argument. (It can be regarded as a generalized subbundle of $\mathfrak{g}_Q$; see e.g. [8] for this notion). Note that $\mathfrak{g}_Q^{\mathcal{E}} = \mathfrak{g}_Q^{\mathcal{G}\cap\mathcal{E}}$ and that $\mathfrak{g}_Q^{\mathcal{G}} = \mathfrak{g}_Q$.

Any smooth section $\widetilde\eta$ of $\mathfrak{g}_Q^{\mathcal{E}}$ induces a smooth section $\widetilde\xi_{\widetilde\eta}^Q$ of $\mathcal{G}\cap\mathcal{E}$, which is defined as

$$\widetilde\xi_{\widetilde\eta}^Q(q) \;:=\; \xi_{\widetilde\eta(q)}^Q(q)\,, \qquad q\in Q\,,$$

that is, the value at q of $\widetilde\xi_{\widetilde\eta}^Q$ equals the value at q of the infinitesimal generator corresponding to $\widetilde\eta(q)\in\mathfrak{g}$. The converse is not true in general. However, if the action of G on Q is (for instance) locally free, then the above correspondence gives a bijection between smooth sections of $\mathcal{E}\cap\mathcal{G}$ and smooth sections of $\mathfrak{g}_Q^{\mathcal{E}}$. (In fact, as shown in Section 3.C, there are smooth functions f_b such that $\xi^Q = \sum_b f_b \xi_{\eta_b}^Q$ for a basis $\eta_1,\dots,\eta_m$ of $\mathfrak{g}$; hence, $\tilde\eta = \sum_b f_b\eta_b$).

Based on [7, 9], we introduce now the following

Definition 3 *Consider a nonholonomic system (L,Q,D) and a Lie group G which acts smoothly on Q. Then, the* nonholonomic momentum map *is the map*

$$\widetilde J \;:=\; \Gamma(\mathfrak{g}_Q)\times D \;\to\; \mathbb{R}\,, \qquad \big(\widetilde\eta,(q,\dot q)\big)\mapsto \widetilde\xi_{\widetilde\eta}^Q(q)\cdot p(q,\dot q)\,. \tag{6}$$

The nonholonomic momentum *of a section* $\widetilde{\eta}$ *of* $\mathfrak{g}_Q$ *is the function*

$$\widetilde{J}_{\widetilde{\eta}} \,:\, D \to \mathbb{R}\,, \qquad \widetilde{J}_{\widetilde{\eta}}(q,\dot{q}) \;=\; \widetilde{J}\big(\widetilde{\eta}(q),(q,\dot{q})\big)\,.$$

The nonholonomic momentum map was defined in [7] as the bundle map $J^{\rm nh} : TQ \to (\mathfrak{g}_Q^{\mathcal{D}})^*$ such that $\langle J^{\rm nh}(q,\dot{q}), \widetilde{\eta}\rangle = \widetilde{J}_{\widetilde{\eta}}(q,\dot{q})$ for all $(q,\dot{q}) \in TQ$ and $\widetilde{\eta} \in \Gamma(\mathfrak{g}_Q^{\mathcal{D}})$, where $(\mathfrak{g}_Q^{\mathcal{D}})^*$ is the bundle over Q whose fibers are the duals of those of $\mathfrak{g}_Q^{\mathcal{D}}$. This definition formally reproduces the structure of the ordinary momentum map $J : TQ \to \mathfrak{g}^*$, while the restriction to $\Gamma(\mathfrak{g}_Q^{\mathcal{D}})$ reflects the idea that the nonholonomic momentum map gives "just some of the components of the standard momentum map, namely, along those symmetry directions that are consistent with the constraints" [7]. Definition 3 extends this definition to all sections of $\mathfrak{g}_Q$ and, in doing so, we have also preferred to define the nonholonomic momentum map in a way which emphasizes its role as a means to construct conserved quantities out of sections of $\mathfrak{g}_Q$. (In this respect, observe that the ordinary momentum map as well can be regarded as a map $J : \Gamma(\mathfrak{g}) \times TQ \to \mathbb{R}$).

In the quoted references, no name is assigned to the functions $\widetilde{J}_{\widetilde{\eta}}$, which however explicitely appear in [9]. The name 'nonholonomic momentum' we chose here is only meant to avoid confusion in the forthcoming comparison of this method to the gauge method. A minor difference between our definition of the nonholonomic momenta and those of [7, 9] is that, in order to simplify some of the forthcoming statements, we restrict the domain from TQ to the constraint manifold D.

The relevance of the extension of the nonholonomic momentum map to over–distributions of $\mathcal{D}$ emerges from the following

Proposition 6 *Consider a nonholonomic system* (L,Q,D)*, a Lie group* G *which acts smoothly on* Q*, and a distribution* $\mathcal{E}$ *on* Q*. Then, the nonholonomic momentum map* $\widetilde{J}$ *satisfies*

$$X_{L,Q,D}\big(\widetilde{J}_{\widetilde{\eta}}\big) \;=\; \widetilde{\xi}^{TQ}_{\widetilde{\eta}}(L)|_D \qquad \forall\, \widetilde{\eta} \in \Gamma(\mathfrak{g}_Q^{\mathcal{E}}) \tag{7}$$

if and only if $\mathcal{E} \subseteq \mathcal{R}^\circ$.

Proof. Denote by $\frac{d}{dt}$ the derivative along the flow of the vector field $X_{L,Q,D}$, namely, the flow of Lagrange equations (2). Let $\widetilde{\xi}^{TQ}_{\widetilde{\eta}}$ be the tangent lift of $\widetilde{\xi}^{Q}_{\widetilde{\eta}}$. Thus, on the points of D we have $X_{L,Q,D}(\widetilde{J}_{\widetilde{\eta}}) = \frac{d}{dt}\big(\frac{\partial L}{\partial \dot{q}} \cdot \widetilde{\xi}^Q_{\widetilde{\eta}}\big) = \big(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\big) \cdot \widetilde{\xi}^Q_{\widetilde{\eta}} + \frac{\partial L}{\partial \dot{q}} \cdot \big(\frac{d}{dt}\widetilde{\xi}^Q_{\widetilde{\eta}}\big) = \big(\frac{\partial L}{\partial q} + R\big) \cdot \widetilde{\xi}^Q_{\widetilde{\eta}} + \frac{\partial L}{\partial \dot{q}} \cdot \widetilde{\xi}^{TQ}_{\widetilde{\eta}}$, that is, $\frac{d}{dt}\widetilde{J}^{\mathcal{E}}_{\widetilde{\xi}} = \widetilde{\xi}^{TQ}(L) + R\cdot\widetilde{\xi}^Q$. The last term vanishes in all of D if and only if $\widetilde{\xi}^Q_{\widetilde{\eta}}$ is a section of $\mathcal{R}^\circ$. ■

Equation (7) is called 'momentum equation' and was derived in [7, 9] for the case $\mathcal{E} = \mathcal{D}$. Proposition 6 states that this equation is valid for any distribution $\mathcal{E} \subseteq \mathcal{R}^\circ$, and only for them. Hence, the reaction–annihilator distribution emerges as the largest distribution for which the momentum equation is valid.

Remarks: (i) In view of Proposition 6, one might as well restrict the definition of the nonholonomic momentum map $\widetilde{J}$ to $\Gamma(\mathfrak{g}_Q^{\mathcal{R}^\circ}) \times TQ$.

(ii) The coordinate free expression of the definition (6) of the gauge momentum map is manifestly $\widetilde{J} \,:=\, \Gamma(\mathfrak{g}_Q) \times TQ \,\to\, \mathbb{R}$, $\big(\widetilde{\eta}, v_q\big) \mapsto \langle \mathbb{F}L(v_q), \widetilde{\xi}^Q_{\widetilde{\eta}}(q)\rangle$ where $\mathbb{F}L$ is the fiber derivative.

(iii) The core of the proof of Proposition 6, and in fact of Proposition 1 too, is the fact that, if ξ^Q is a section of a distribution $\mathcal{E}$ on Q, then its momentum $P_{\xi^Q} := \xi^Q \cdot A\dot{q}$ satisfies the equation $X_{L,Q,D}(P_{\xi^Q})|_D = \xi^{TQ}(L)|_D$ if and only if $\mathcal{E} \subseteq \mathcal{R}^\circ$. The validity of this equation in the horizontal case $\mathcal{E} = \mathcal{D}$ has been noticed in [27], where it is called 'momentum equation'. We prefer using this name to denote the case in which a preassigned group action is given.

B. Relationship between the momentum equation and the gauge method. The relationship between the gauge method and the momentum equation rests on the fact that the nonholonomic momentum $\widetilde{J}_{\widetilde{\eta}}$ of a section $\widetilde{\eta}$ of $\mathfrak{g}_Q$ is nothing but the gauge momentum of the vector field $\widetilde{\xi}^Q_{\widetilde{\eta}}$. Thus:

- *The nonholonomic momentum $\widetilde{J}_{\widetilde{\eta}}$ of a smooth section $\widetilde{\eta}$ of $\mathfrak{g}_Q^{\mathcal{R}^\circ}$ is a first integral of (L, Q, D) if and only if $\widetilde{\xi}^Q_{\widetilde{\eta}}$ is a gauge symmetry.*

As for the converse, in presence of isotropy it might happen that not every smooth section of a distribution $\mathcal{E}$ on Q corresponds to a smooth section of $\mathfrak{g}_Q^{\mathcal{E}}$. As noticed above, however, this happens if (for instance) the action has no isotropy:

- *If the action of G on Q is locally free, then any gauge momentum is the nonholonomic momentum of a smooth section of $\mathfrak{g}_Q$.*

Thus, for locally free actions the difference between the two approaches is only in the parameterization of the conserved quantities: the gauge method uses sections of $\mathcal{G}\cap\mathcal{E}$ while the nonholonomic momentum map uses sections of $\mathfrak{g}_Q^{\mathcal{E}}$, where $\mathcal{E}$ is typically either $\mathcal{D}$ or $\mathcal{R}^\circ$. This relationship between the two methods seems to have been passed unnoticed so far, even in the case $\mathcal{E} = \mathcal{D}$.

In presence of isotropy, however, the situation is not as a clear. In principle, there might be gauge momenta which are not parameterizable with sections of $\mathfrak{g}_Q$.

C. The gauge momentum map. We conclude this section by remarking that there is an analogue of the nonholonomic momentum map on the gauge side, that we call *gauge momentum map*:

Definition 4 *Given a nonholonomic system (L, Q, D) and a smooth action of a Lie group G on Q, the* gauge momentum map *is the map*

$$\widehat{J} \,:\, \Gamma(\mathcal{G}) \times D \,\to\, \mathbb{R}, \qquad (\xi^Q, (q, \dot{q})) \mapsto \widehat{J}_{\xi^Q}(q, \dot{q})$$

where $\widehat{J}_{\xi^Q} \;=\; \xi^Q \cdot p|_D$, see Definition 1.

The relation to the nonholonomic momentum map is obvious: if $\xi^Q = \widetilde{\xi}^Q_{\widetilde{\eta}}$ for some smooth section of $\mathfrak{g}_Q$, then $\widehat{J}^Q_{\xi} = \widetilde{J}(\widetilde{\eta}, \cdot)$. Equivalently, if $\Xi : \Gamma(\mathfrak{g}_Q) \to \Gamma(\mathcal{G} \cap \mathcal{E})$ is the injection $\widetilde{\eta} \mapsto \widetilde{\xi}^Q_{\widetilde{\eta}}$, then

$$\widetilde{J} \;=\; \widehat{J} \circ (\Xi \times \mathrm{id}_D)\,.$$

Since Ξ need not be surjective, the gauge momentum map is a generalization of the nonholonomic momentum map. Nevertheless, there is a 'momentum equation' for the gauge momentum map as well: as follows from the proof of Proposition 6, given a distribution $\mathcal{E}$ on Q, the gauge momentum map $\widehat{J}$ satisfies

$$X_{L,Q,D}(\widehat{J}_{\xi^Q}) = \xi^{TQ}(L)|_D \qquad \forall \xi^Q \in \Gamma(\mathcal{E} \cap \mathcal{G})$$

if and only if $\mathcal{E} \subseteq \mathcal{R}^\circ$.

6 Conclusions

The goal of this article was to produce a comprehensive picture of the relationship between the invariance under a lifted action of the Lagrangian of a linear nonholonomic system, and the existence of first integrals linear in the velocities. The two starting points of our analysis were (a) the characterization of the conserved components of the momentum map as those generated by infinitesimal generators of the group action which are sections of the reaction–annihilator distribution $\mathcal{R}^\circ$ [15] and (b) the gauge mechanism [5]. We have shown that the gauge mechanism is equivalent to the momentum equation of [7, 9, 10, 27] if the action is locally free, and that they both extend to sections of $\mathcal{R}^\circ$, providing in this way the most general framework to link conservation laws to symmetry groups. All these objects and points of view can be formulated in terms of a single object, the gauge momentum map, which is a reformulation of the nonholonomic momentum map of [7, 9, 10].

The analysis of simple examples has demonstrated that both extensions, from the constraint distribution $\mathcal{D}$ to the reaction–annihilator distribution $\mathcal{R}^\circ$, and from the momentum map to the gauge mechanism, are necessary in order to obtain a complete picture of the relationship symmetries–conservation laws for nonholonomic systems.

The extension from $\mathcal{D}$ to $\mathcal{R}^\circ$ encodes the extent to which the conserved momenta of a nonholonomic system depend on the system, rather than just on the group. As remarked, there is a deep difference between $\mathcal{D}$–gauge momenta and $\mathcal{R}^\circ$–gauge momenta, in that the formers have a 'weakly Noetherian' character, that is, they are shared by all nonholonomic systems with fixed constraints and G–invariant kinetic energy, and any G–invariant potential. On the other hand, the class of $\mathcal{R}^\circ$–gauge momenta is the class of all conserved gauge momenta.

Overall, our impression is that the gauge mechanism—that is, the idea that the link symmetry group–conserved quantities should be relaxed so as to include also momenta of vector fields which are tangent to the group orbits but are not infinitesimal generators of the action—should be taken into account for a full comprehension of the relationship between symmetries and conservation laws in nonholonomic mechanics.

Whether the gauge method should be considered a 'fundamental' mechanism to link symmetries and conservation laws, is a difficult question to answer. Consider, for instance, the holonomic system constituted by a free particle in $\mathbb{R}^2 \ni (x, y)$, and the group $G = \mathbb{R}^2$ of translations in $\mathbb{R}^2$. The momentum map leads to the conservation of the linear momentum $(\dot{x}, \dot{y})$, and the angular momentum $x\dot{y} - y\dot{x}$ is a gauge momentum relative to this action. However, $x\dot{y} - y\dot{x}$ is the momentum of the rotation group. So, one might suspect that the class of gauge momenta for a given group might be 'too' large. However, one may also note that the invariance under translations along the x and y axes implies in this case the invariance under rotations, so one might even claim that it is unnecessary to consider the rotation group in this context. In fact, whenever the orbits of a group action are subsets of the orbits of another group action, the components of the momentum map of the latter action can be regarded as gauge momenta of the former action. Perhaps, the comprehension of more cases, and a better appreciation of the role of the gauge method in nonholonomic mechanics, will bring some light on the question.

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