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Convergence by Viscosity Methods in Multiscale Financial Models with Stochastic Volatility*

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Abstract. We study singular perturbations of a class of stochastic control problems under assumptions motivated by models of financial markets with stochastic volatilities evolving on a fast time scale. We prove the convergence of the value function to the solution of a limit (effective) Cauchy problem for a parabolic equation of HJB type. We use methods of the theory of viscosity solutions and of the homogenization of fully nonlinear PDEs. We test the result on some financial examples, such as Merton portfolio optimization problem.

Key words. singular perturbations, viscosity solutions, stochastic volatility, asymptotic approximation, portfolio optimization

AMS subject classifications. 35B25, 91B28, 93C70, 49L25

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1. Introduction.

In this paper we consider stochastic control systems with a small parameter $\varepsilon > 0$ in the form

$$\begin{cases} dX_t = \tilde{\phi}(X_t, Y_t, u_t)dt + \sqrt{2}\tilde{\sigma}(X_t, Y_t, u_t)dW_t, \\ dY_t = \frac{1}{\varepsilon}b(Y_t)dt + \sqrt{\frac{2}{\varepsilon}}\tau(Y_t)dW_t, \end{cases} \tag{1}$$

where $X_t \in \mathbb{R}^n$, $Y_t \in \mathbb{R}^m$, u_t is the control taking values in a given compact set U, W_t is a multidimensional Brownian motion, and the components of drift and diffusion of the slow variables X_t have the form

$$\tilde{\phi}^i := x^i \phi^i(x, y, u), \qquad \tilde{\sigma}_{ij} := x^i \sigma_i^j(x, y, u),$$

with ϕ^i, σ_i^j bounded and Lipschitz continuous uniformly in u, so that $X_t^i \geq 0$ for $t > t_o$ if $X_{t_o}^i \geq 0$. On the fast process Y_t we will assume that the matrix $\tau\tau^T$ is positive definite and a condition implying the ergodicity (see (3)). We also take payoff functionals of the form

$$\mathbf{E}[e^{\lambda(t-T)} g(X_T, Y_T) \mid X_t = x,\, Y_t = y], \quad 0 \leq t \leq T, \quad \lambda \geq 0,$$

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with g continuous and growing at most quadratically at infinity, and call $V^\varepsilon(t,x,y)$ the value function of this optimal control problem, i.e.,

$$V^\varepsilon(t,x,y) := \sup_{u.} \mathbf{E}[e^{\lambda(t-T)}g(X_T,Y_T) \mid X_t = x,\, Y_t = y,\, (X.,Y.) \text{ satisfy (1) with } u.].$$

We are interested in the limit V as $\varepsilon \to 0$ of V^ε, in particular in understanding the PDE satisfied by V and interpreting it as the HJB equation for an *effective* limit control problem. This is a singular perturbation problem for the system (1) and for the HJB equation associated with it. We treat it by methods of the theory of viscosity solutions to such equations.

Our motivations are the models of pricing and trading derivative securities in financial markets with stochastic volatility. The book by Fleming and Soner [20] is a general presentation of viscosity solution methods in stochastic control, and in Chapter 10 it gives an excellent introduction to the applications of this theory to the mathematical models of financial markets. In such markets with stochastic volatility the asset prices are affected by correlated economic factors, modeled as diffusion processes. This is motivated by empirical studies of stock price returns in which the estimated volatility exhibits random behavior. So, typically, volatility is assumed to be a function of an Itô process Y_t driven by another Brownian motion, which is often negatively correlated with the one driving the stock prices (this is the empirically observed leverage effect; i.e., asset prices tend to go down as volatility goes up). This approach seems to have success in taking into account the so-called smile effect, due to the discrepancy between the predicted and market traded option prices, and in reproducing much more realistic returns distributions (i.e., with fatter and asymmetric tails).

An important extension of the stochastic volatility approach was introduced recently by Fouque, Papanicolaou, and Sircar in the book [23] (see, in particular, Chapter 3). The idea is trying to describe the *bursty* behavior of volatility: in empirical observations volatility often tends to fluctuate to a high level for a while, then to a low level for another small time period, then again at high level, and so on, for several times during the life of a derivative contract. These phenomena are also related to another feature of stochastic volatility, which is mean reversion. A mathematical framework which takes into account both bursting and mean reverting behavior of the volatility is that of multiple time scale systems and singular perturbations. In this setting volatility is modeled as a process which evolves on a faster time scale than the asset prices and which is ergodic, in the sense that it has a unique invariant distribution (the long-run distribution) and asymptotically decorellates (in the sense that it becomes independent of the initial distribution). We refer the reader to the book [23] and to the references therein for a detailed presentation of these models and for their empirical justification.

Several extensions and applications to a variety of financial problems appeared afterward; see [31, 24, 25, 22, 43, 30, 41, 29, 37] and the references therein.

According to the previous discussion, stochastic control systems of the form (1) are appropriate for studying financial problems in this setting. Indeed, here the slow variables represent prices of assets or the wealth of the investor, whereas Y_t is an ergodic process representing the volatility and evolving on a faster time scale for ε small. The main example for Y_t is the Ornstein–Uhlenbeck process. The asymptotic analysis of such systems as $\varepsilon \to 0$ then yields a simple pricing and hedging theory which provides a correction to classical Black–Scholes formulas, taking into account the effect of uncertain and changing volatility.

Most of the papers we cited on fast mean reverting stochastic volatility use formal asymptotic expansions of the value function in powers of ε and compute the first terms of the expansions by solving suitable auxiliary elliptic and parabolic PDEs. These methods are closely related to homogenization theory and can be found in earlier papers of Papanicolaou and coauthors and, e.g., in the book [9]. They are particularly fit to problems without control, such as the pricing of many options, so that the price function is smooth and satisfies a linear PDE. In these cases the accuracy of the expansion can often be proved.

There is a wide literature on singular perturbations of diffusion processes, with and without controls. For results based on probabilistic methods we refer the reader to the books [33, 32], the recent papers [38, 39, 40, 12], and the references therein. An approach based on PDE-viscosity methods for the HJB equations was developed by Alvarez and one of the authors in [1, 2, 3]; see also [4] for problems with an arbitrary number of scales. It allows one to identify the appropriate limit PDE governed by the *effective Hamiltonian* and gives general convergence theorems of the value function of the singularly perturbed system to the solution of the effective PDE, under assumptions that include deterministic control (i.e., $\sigma \equiv 0$ and/or $\tau \equiv 0$) as well as differential games, deterministic and stochastic. However, this theory originating in periodic homogenization problems [35, 18] was developed so far for fast variables restricted to a compact set, mostly the m-dimensional torus. As we already observed, though, an a priori assumption of boundedness does not appear natural to model volatility in financial markets, according to the empirical data and in the discussion presented in [23] and the references therein.

The goal of this paper is extending the methods based on viscosity solutions of [1, 2, 3] to singular perturbation problems of the form (1), including several models of mathematical finance. The main new difficulty is that the fast variables Y_t are unbounded.

We first check that the value function V^ε is the unique (viscosity) solution to a Cauchy problem for the HJB equation under very general assumptions on the data. In particular, the diffusion matrix of the slow variables $\sigma\sigma^T$ may degenerate and V^ε may be merely continuous. The possible degeneration of the diffusion matrix $\sigma\sigma^T$ can also have interesting financial applications, e.g., to path-dependent options and to interest rate models in the Heath–Jarrow–Morton framework (see section 6.5 for more comments on this).

Next we assume that the *fast subsystem*

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t \tag{2}$$

has a Lyapunov-like function w satisfying

$$-\mathcal{L}w(y) \geq k > 0 \text{ for } |y| > R_0, \qquad \lim_{|y|\to+\infty} w(y) = +\infty, \tag{3}$$

where $\mathcal{L}$ is the infinitesimal generator of the process (2). We prove a Liouville property for sub- and supersolution of $\mathcal{L}v = 0$, the existence of a unique invariant measure μ for (2) (by exploiting the theory of Hasminskii [28]), and some crucial properties of the effective Hamiltonian and terminal cost

$$\overline{H}(x, D_xV, D^2_{xx}V) := \int_{\mathbb{R}^m} H(x, y, D_xV, D^2_{xx}V, 0)\, d\mu(y), \qquad \overline{g}(x) := \int_{\mathbb{R}^m} g(x, y)\, d\mu(y),$$

where H is the Bellman Hamiltonian associated with the slow variables of (1) and its last entry is for the mixed derivatives D_{xy}. The condition (3) is easier to check and looks weaker than other known sufficient conditions for ergodicity [28, 36]. It appears also in a remark of [34], where the proof of the existence of μ is different from ours. Lions and Musiela [34] also state that (3) is indeed equivalent to the ergodicity of (2) and to the classical Lyapunov-type condition of Hasminskii [28].

Our main result is the convergence of $V^\varepsilon(t,x,y)$ to $V(t,x)$ as $\varepsilon \to 0$ uniformly on compact subsets of $[0,T) \times \mathbb{R}^n_+ \times \mathbb{R}^m$, where V is the unique (viscosity) solution to

$$-V_t + \overline{H}\left(x, D_x V, D^2_{xx} V\right) + \lambda V(x) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n_+, \tag{4}$$

with final data $V(T,x) = \overline{g}(x)$ in $\overline{\mathbb{R}^n_+}$. Note that there is a boundary layer at the terminal time T if the utility g depends on y.

We test this convergence theorem on two examples of financial models chosen from [23]. The first is the problem of pricing n assets with an m-dimensional vector of volatilities. The second is the Merton portfolio optimization problem with one riskless bond and n risky assets. The control system driving wealth and volatility is

$$\begin{cases} d\mathcal{W}_t = \mathcal{W}_t \left(r + \sum_{i=1}^n (\alpha^i - r)u^i_t\right) dt + \sqrt{2}\mathcal{W}_t \sum_{i=1}^n u^i_t f_i(Y_t) \cdot d\overline{W}_t, \\ dY_t = \frac{1}{\varepsilon} b(Y_t) dt + \sqrt{\frac{2}{\varepsilon}} \nu(Y_t) d\overline{Z}_t, \end{cases} \tag{5}$$

with $\mathcal{W}_{t_o} = w > 0$, where $\overline{W}_t, \overline{Z}_t$ are possibly correlated Brownian motions, and the value function is

$$V^\varepsilon(t,w,y) := \sup_{u.} \mathbf{E}[g(\mathcal{W}_T, Y_T) \mid \mathcal{W}_t = w,\, Y_t = y].$$

Our convergence result for this problem appears to be new, to the best of our knowledge, although the formula for the limit is derived in [23] (by a different method and for $n = 1$, g independent of y; another term of an asymptotic expansion in powers of ε is also computed in [23]). We also show that we can handle a periodic day effect, i.e., $f_i = f_i(\frac{t}{\varepsilon}, Y_t)$ periodic in the first entry, as in section 10.2 of [23], and the presence of a component of the volatility evolving on a very slow time scale (dependent or not on ε), as in [25, 37]. A similar result for the infinite horizon Merton problem of optimal consumption [19, 20] is under investigation.

Finally we observe that our methods work if an additional unknown disturbance $\tilde{u}_t$ affects the dynamics of X_t and we maximize the payoff under the worst possible behavior of $\tilde{u}_t$. This situation is modeled as a 0-sum differential game: its value function is characterized by a Hamilton–Jacobi–Isaacs PDE that can be analyzed in the framework of viscosity solutions [21, 3]. In [1, 2, 3] the disturbance $\tilde{u}_t$ and/or the controls u_t may also affect the fast variables Y_t (constrained to a compact set). Then there is no invariant measure and the definition of the effective Hamiltonian and terminal cost is less explicit, but the convergence theorem still holds.

Our conclusion is that the theory of viscosity solutions is the appropriate mathematical framework for fully nonlinear Bellman–Isaacs equations that provides general methods for treating singular perturbation problems (relaxed semilimits, perturbed test function method, comparison principles, etc.). These can be useful additional tools for the rigorous analysis of multiscale financial problems with stochastic volatility, in particular when some variables are controlled, the value function is not smooth, or the complexity of the model prevents more explicit calculations.

The paper is organized as follows. Section 2 presents the standing assumptions and the HJB equation. Section 3 studies the initial value problem satisfied by V^ε. Section 4 is devoted to the ergodicity of a diffusion process in the whole spaces and the properties of the effective Hamiltonian and terminal cost. In section 5 we prove our main result, Theorem 5.1, on the convergence of V^ε to the solution of the effective Cauchy problem. In section 6 we apply our results to a multidimensional option pricing model and to the Merton portfolio optimization problem and then illustrate some extensions. Section 7 is the conclusion.

2. The two-scale stochastic control problem.

2.1. The control system. We consider stochastic control problems that can be written in the form

$$\tag{6} \begin{cases} dX_t^i = X_t^i \phi^i(X_t, Y_t, u_t)dt + \sqrt{2} X_t^i \sigma_i(X_t, Y_t, u_t) \cdot dW_t, & i = 1, \dots, n, \\ dY_t^k = \frac{1}{\varepsilon} b^k(Y_t)dt + \sqrt{\frac{2}{\varepsilon}} \tau_k(Y_t) \cdot dW_t, & k = 1, \dots, m, \end{cases}$$

with $X_{t_o}^i = x^i \geq 0$, $Y_{t_o}^k = y^k$, where $\varepsilon > 0$, U is a given compact set, $\phi = (\phi^1, \dots, \phi^n) : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$, $\sigma^i : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^r$ are bounded continuous functions, Lipschitz continuous in (x, y) uniformly with respect to $u \in U$, $b = (b^1, \dots, b^m) : \mathbb{R}^m \to \mathbb{R}^m$, $\tau_k : \mathbb{R}^m \to \mathbb{R}^r$ are locally Lipschitz continuous functions with linear growth, i.e.,

$$\tag{7} \text{for some } K_c > 0 \qquad |b(y)|, \|\tau_k(y)\| \leq K_c(1 + |y|) \qquad \forall\, y \in \mathbb{R}^m, \quad k = 1, \dots, m,$$

and W_t is an r-dimensional standard Brownian motion. These assumptions will hold throughout the paper.

We will use the symbols $\mathbb{M}^{k,j}$ and $\mathbb{S}^k$ to denote, respectively, the set of $k \times j$ matrices and the set of $k \times k$ symmetric matrices, and we set

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \,:\, x^i > 0 \;\forall\, i = 1, \dots, n\}.$$

To shorten the notation we call $\tilde\phi : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ the drift of the slow variables X_t, $\tilde\sigma \in \mathbb{M}^{n,r}$ the matrix whose ith row is $x^i \sigma_i$, and $\tau \in \mathbb{M}^{m,r}$ the matrix whose kth row is τ_k, i.e.,

$$\tilde\phi^i := x^i \phi^i, \quad \tilde\sigma_{ij} := x^i \sigma_i^j, \quad \tau_{kj} := \tau_k^j, \quad j = 1, \dots, r.$$

Then the system (6) can be rewritten with vector notations in the form

$$\tag{8} \begin{cases} dX_t = \tilde\phi(X_t, Y_t, u_t)dt + \sqrt{2}\tilde\sigma(X_t, Y_t, u_t)dW_t, & X_{t_o} = x \in \overline{\mathbb{R}}^n_+, \\ dY_t = \frac{1}{\varepsilon} b(Y_t)dt + \sqrt{\frac{2}{\varepsilon}} \tau(Y_t)dW_t, & Y_{t_o} = y. \end{cases}$$

The set of admissible control functions is

$$\mathcal{U} := \{u_\cdot \text{ progressively measurable processes taking values in } U\}.$$

In the following we will assume the uniform nondegeneracy of the diffusion driving the fast variables Y_t, i.e.,

$$\exists\, e(y) > 0 \text{ such that } \xi\tau(y)\tau^T(y)\cdot\xi = |\xi\tau(y)|^2 \geq e(y)|\xi|^2 \qquad \text{for every } y, \xi \in \mathbb{R}^m. \tag{9}$$

We will not make any nondegeneracy assumption on the matrix σ and remark that, in any case, $\tilde\sigma$ degenerates near the boundary of $\mathbb{R}^n_+$.

2.2. The optimal control problem. We consider a payoff functional depending only on the position of the system at a fixed terminal time $T > 0$ (Mayer problem). The utility function $g : \overline{\mathbb{R}}^n_+ \times \mathbb{R}^m \to \mathbb{R}$ is continuous and satisfies

$$\exists\, K_g > 0 \quad \text{such that} \quad \sup_{y\in\mathbb{R}^d} |g(x,y)| \leq K_g(1+|x|^2) \qquad \forall\, x \in \mathbb{R}^n_+, \tag{10}$$

and the discount factor is

$$\lambda \geq 0.$$

Therefore the value function of the optimal control problem is

$$V^\varepsilon(t,x,y) := \sup_{u.\in\mathcal{U}} \mathbf{E}[e^{\lambda(t-T)}g(X_T,Y_T) \mid X_t = x,\, Y_t = y], \qquad 0 \leq t \leq T, \tag{11}$$

where $\mathbf{E}$ denotes the expectation. This choice of the payoff is sufficiently general for the application to finance models presented in this paper, but we could easily include in the payoff an integral term keeping track of some running costs or earnings.

2.3. The HJB equation. For a fixed control $u \in U$ the generator of the diffusion process is

$$\operatorname{trace}\left(\tilde\sigma\tilde\sigma^T D^2_{xx}\right) + \frac{2}{\sqrt{\varepsilon}}\operatorname{trace}\left(\tilde\sigma\tau^T (D^2_{xy})^T\right) + \tilde\phi\cdot D_x + \frac{1}{\varepsilon}\operatorname{trace}\left(\tau\tau^T D^2_{yy}\right) + \frac{1}{\varepsilon} b\cdot D_y,$$

where the last two terms give the generator of the fast process Y_t.

The HJB equation associated via dynamic programming with the value function of this control problem is

$$-V_t + H\left(x,y,D_xV,D^2_{xx}V,\frac{D^2_{xy}V}{\sqrt{\varepsilon}}\right) - \frac{1}{\varepsilon}\mathcal{L}(y,D_yV,D^2_{yy}V) + \lambda V = 0 \tag{12}$$

in $(0,T)\times\mathbb{R}^n_+\times\mathbb{R}^m$, where

$$H(x,y,p,X,Z) := \min_{u\in U}\left\{-\operatorname{trace}\left(\tilde\sigma\tilde\sigma^T X\right) - \tilde\phi\cdot p - 2\operatorname{trace}\left(\tilde\sigma\tau^T Z^T\right)\right\}, \tag{13}$$

with $\tilde\sigma$ and $\tilde\phi$ computed at (x,y,u), $\tau = \tau(y)$, and

$$\mathcal{L}(y,q,Y) := b(y)\cdot q + \operatorname{trace}(\tau(y)\tau^T(y)Y). \tag{14}$$

This is a fully nonlinear degenerate parabolic equation (strictly parabolic in the y variables by the assumption (9)).

The HJB equation is complemented with the obvious terminal condition

$$V(T,x,y) = g(x,y).$$

However, there is no natural boundary condition on the space boundary of the domain, i.e.,

$$(0,T) \times \partial\mathbb{R}^n_+ \times \mathbb{R}^m = \{(t,x,y) \,:\, 0 < t < T,\ x^i = 0 \text{ for some } i\}.$$

We will prove in the next section that the initial boundary value problem is well posed without prescribing any boundary condition because the PDE "holds up to boundary"; namely, the value function is a viscosity solution in the set $(0,T)\times\overline{\mathbb{R}}^n_+\times\mathbb{R}^m$, and there is at most one such solution. The irrelevance of the space boundary $(0,T)\times\partial\mathbb{R}^n_+\times\mathbb{R}^m$ is essentially due to the fact that $\overline{\mathbb{R}}^n_+\times\mathbb{R}^m$ is an invariant set for the system (6) for all admissible control functions (almost surely); that is, the state variables cannot exit this closed domain.

2.4. The main assumption. Consider the diffusion process in $\mathbb{R}^m$ obtained putting $\varepsilon = 1$ in (1),

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t, \tag{15}$$

called the *fast subsystem*, and observe that its infinitesimal generator is $\mathcal{L}w := \mathcal{L}(y, D_y w, D^2_{yy} w)$, with $\mathcal{L}$ defined by (14). We assume the following condition:

There exist $w \in \mathcal{C}(\mathbb{R}^d)$ and constants $k, R_0 > 0$ such that

$$-\mathcal{L}w \geq k \textit{ for } |y| > R_0 \textit{ in viscosity sense}, \quad \textit{and} \quad w(y) \to +\infty \quad \textit{as } |y| \to +\infty. \tag{16}$$

It is reminiscent of other similar conditions about ergodicity of diffusion processes in the whole space; see, for example, [28, 9, 34, 12, 36].

Remark 2.1. Condition (16) can be interpreted as a weak Lyapunov condition for the process (15) relative to the set $\{|y| \leq R_0\}$. Indeed, a Lyapunov function for the system (15) relative to a compact invariant set K is a continuous, positive definite function L such that $L(x) = 0$ if and only if $x \in K$, the sublevel sets $\{y \mid L(y) \leq k\}$ are compact, and $-\mathcal{L}L(x) = l(x)$ in $\mathbb{R}^m$, where l is a continuous function with $l = 0$ on K and $l > 0$ outside. For more details see [28].

Example 2.1. The motivating model problem studied in [23] is the Ornstein–Uhlenbeck process with equation

$$dY_t = (m - Y_t)dt + \sqrt{2}\tau dW_t,$$

where the vector m and matrix τ are constant. In this case it is immediate to check condition (16) by choosing $w(y) = |y|^2$ and R_0 sufficiently large.

Example 2.2. More generally, condition (16) is satisfied if

$$\limsup_{|y|\to+\infty} \left[b(y)\cdot y + \mathrm{trace}(\tau\tau^T(y))\right] < 0.$$

Indeed, also in this case it is sufficient to choose $w(y) = |y|^2$. Pardoux and Veretennikov [38, 39, 40] assume $\tau\tau^T$ bounded and $\lim_{|y|\to+\infty} b(y)\cdot y = -\infty$, and they call it the *recurrence condition*.

3. The Cauchy problem for the HJB equation. We characterize the value function V^ε as the unique continuous viscosity solution with quadratic growth to the parabolic problem with terminal data in the form

$$\begin{cases} -V_t + F\left(x, y, V, D_x V, \frac{D_y V}{\varepsilon}, D^2_{xx} V, \frac{D^2_{yy} V}{\varepsilon}, \frac{D^2_{xy} V}{\sqrt{\varepsilon}}\right) = 0 & \text{in } (0,T) \times \mathbb{R}^n_+ \times \mathbb{R}^m, \\ V(T,x,y) = g(x,y) & \text{in } \overline{\mathbb{R}^n_+} \times \mathbb{R}^m, \end{cases} \tag{17}$$

where the Hamiltonian $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^m \times \mathbb{M}^{n,m} \to \mathbb{R}$ is defined as

$$F(x,y,s,p,q,X,Y,Z) := H(x,y,p,X,Z) - \mathcal{L}(y,q,Y) + \lambda s. \tag{18}$$

This is a variant of a standard result (see [20] and the references therein) where we must take care of the lack of boundary condition on $\partial\mathbb{R}^n_+$ and the unboundedness of the solution.

Proposition 3.1. *For any $\varepsilon > 0$, the function V^ε defined in (11) is the unique continuous viscosity solution to the Cauchy problem (17) with at most quadratic growth in x and y. Moreover the functions V^ε are locally equibounded.*

Proof. The proof is divided into several steps.

Step 1 (bounds on V^ε). Observe that, using the definition of V^ε and (10),

$$|V^\varepsilon(t,x,y)| \le K_g \mathbf{E}(1 + |X_T(t,x,y)|^2).$$

So, using standard estimates on the second moment of the solution to (49) (see, for instance, [27, Thms. 1 and 4, Chap. 2] or [20, App. D]) and the boundedness of $\tilde{\phi}$ and $\tilde{\sigma}$ with respect to y, we get that there exist $C, c > 0$ such that

$$|V^\varepsilon(t,x,y)| \le Ce^{cT}(1 + |x|^2) = K_V(1 + |x|^2), \quad t \in [0,T], \quad x \in \mathbb{R}^n_+, \quad y \in \mathbb{R}^m. \tag{19}$$

This estimate in particular implies that the sequence V^ε is locally equibounded.

Step 2 (the semicontinuous envelopes are sub- and supersolutions). We define the lower and upper semicontinuous envelopes of V^ε as

$$V^\varepsilon_*(t,x,y) = \liminf_{(t',x',y') \to (t,x,y)} V^\varepsilon(t',x',y'),$$
$$(V^\varepsilon)^*(t,x,y) = \limsup_{(t',x',y') \to (t,x,y)} V^\varepsilon(t',x',y'),$$

where $(t',x',y') \in \left([0,T] \times \overline{\mathbb{R}^n_+} \times \mathbb{R}^m\right)$. By definition, $V^\varepsilon_*(t,x,y) \le V^\varepsilon(t,x,y) \le (V^\varepsilon)^*(t,x,y)$ and, moreover, both V^ε_* and $(V^\varepsilon)^*$ satisfy the growth condition (19). A standard argument in viscosity solution theory, based on the dynamic programming principle (see, e.g., [20, Chap. V, sect. 2]), gives that V^ε_* and $(V^\varepsilon)^*$ are, respectively, a viscosity supersolution and a viscosity subsolution to (17) at every point $(t,x,y) \in (0,T) \times \mathbb{R}^n_+ \times \mathbb{R}^m$.

Step 3 (behavior of V^ε_* and $(V^\varepsilon)^*$ at time T). We show that the value function V_ε attains continuously the final data (locally uniformly with respect to (x,y)). This means that $\lim_{t\to T} V^\varepsilon(t,x,y) = g(x,y)$ locally uniformly in $(x,y) \in \overline{\mathbb{R}^n_+} \times \mathbb{R}^m$. This result is well known and follows from (10), (19), and the continuity in the mean square of X_t, Y_t. Indeed for every

$K>0$ and $\delta>0$ there exists a constant $C(K,\delta)$ depending also on the Lipschitz constants of the coefficients of the equation (see [27, Thms. 1 and 4, Chap. 2] or [20, App. D]), such that

$$\mathbf{P}\left(|X_T-x|\geq\delta \mid X_t=x,\, Y_t=y\right), \qquad \mathbf{P}\left(|Y_T-y|\geq\delta \mid X_t=x,\, Y_t=y\right)\leq C(K,\delta)(T-t)$$

for all $x\in\mathbb{R}^n_+$, $y\in\mathbb{R}^m$ such that $|x|,|y|\leq K$. Define $A:=\{|X_T-x|\geq\delta\}\cup\{|Y_T-y|\geq\delta\}$ so that

$$\mathbf{P}(A \mid X_t=x,\, Y_t=y)\leq 2C(K,\delta)(T-t).$$

Then for every $\eta>0$ there exists an admissible control u such that

$$\begin{aligned} |V^\varepsilon(t,x,y)-V^\varepsilon(T,x,y)| &\leq \mathbf{E}\left(|g(X^u_T,Y_T)-g(x,y)| \mid X_t=x,\, Y_t=y\right)+\eta \\ &\leq \mathbf{E}\big(\chi_{\Omega\setminus A}\,|g(X^u_T,Y_T)-g(x,y)| \mid X_t=x,\, Y_t=y\big)+\eta \qquad (20)\\ &\quad + 2^{1/2}C(K,\delta)^{1/2}(T-t)^{1/2}\left(\mathbf{E}\left(|g(X^u_T,Y_T)-g(x,y)|^2 \mid X_t=x,\, Y_t=y\right)\right)^{1/2}. \qquad (21)\end{aligned}$$

Term (21) can be computed using (10) and the estimates on the mean square of X_T and Y_T in terms of the initial data:

$$\begin{aligned}(21) &\leq [2C(K,\delta)(T-t)(2K_g)]^{1/2}\left[(1+|x|^2)+\left(\mathbf{E}\left(1+|X_T|^2 \mid X_t=x,\, Y_t=y\right)\right)^{1/2}\right]\\ &\leq 2C(K,\delta)^{1/2}(T-t)^{1/2}K_g^{1/2}C(1+|x|^2)\leq H(K,\delta,g)(T-t)^{1/2}\to 0\end{aligned}$$

uniformly as $T\to t$. Term (20) can be estimated as follows:

$$(20)\leq \mathbf{E}\left(\omega_{g,K}(|X^u_T-x|,|Y_T-y|) \mid X_t=x,\, Y_t=y\right)+\eta\to\eta$$

uniformly as $T\to t$, where $\delta<K$ and $\omega_{g,K}$ is the continuity modulus of g restricted to $\{(x,y)\mid |x|\leq 2K,\, |y|\leq 2K\}$. We conclude by the arbitrariness of η.

Finally, using the definitions, it is easy to show that $V^\varepsilon_*(T,x,y)=(V^\varepsilon)^*(T,x,y)=g(x,y)$ for every $(x,y)\in\mathbb{R}^n_+\times\mathbb{R}^m$.

Step 4 (behavior of V^ε_* and $(V^\varepsilon)^*$ at the boundary of $\mathbb{R}^n_+$). We check that all the points of the boundary of $\mathbb{R}^n_+$ are irrelevant, according to Fichera-type classification of boundary points for elliptic problems. This means the following. Suppose that ϕ is smooth and $(V^\varepsilon)^*-\phi$ has a local maximum (resp., $V^\varepsilon_*-\phi$ has a local minimum) relative to $(0,T)\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m$ at $(\bar t,\bar x,\bar y)$ with the ith coordinate $\bar x^i=0$ for some $i\in\{1,\dots,n\}$ and $0<\bar t<T$. Then

$$-\phi_t+F\left(\bar x,\bar y,V,D_x\phi,\frac{D_y\phi}{\varepsilon},D^2_{xx}\phi,\frac{D^2_{yy}\phi}{\varepsilon},\frac{D^2_{xy}\phi}{\sqrt{\varepsilon}}\right)\leq 0 \quad (\text{resp.}, \geq 0) \qquad \text{at } (\bar t,\bar x,\bar y). \tag{22}$$

We give the proof of this claim only for the subsolution inequality and for the case that only two components, say $\bar x^1$ and $\bar x^2$, are null. All the other cases can be proved in the same way with obvious changes.

Therefore we fix $(\bar t,\bar x,\bar y)$ with $0<\bar t<T$, $\bar x\in\mathbb{R}^n$ with $\bar x^1=\bar x^2=0$ and $\bar x^i>0$, for $i\neq 1,2$, $\bar y\in\mathbb{R}^m$, and a smooth function ψ such that the maximum of $(V^\varepsilon)^*-\psi$ in $\overline{B}=B((\bar t,\bar x,\bar y),r)\cap([0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m)$ is attained at $(\bar t,\bar x,\bar y)$. Without loss of generality we can assume that the maximum is strict, $\overline{x}^i > r$ for every $i = 3, \dots, n$, and $0 < \overline{t} - r < \overline{t} + r < T$. For $\delta > 0$ we define

$$\psi_\delta(t, x, y) := \psi(t, x, y) + \frac{\delta}{x^1} + \frac{\delta}{x^2}$$

and $(t_\delta, x_\delta, y_\delta)$ a maximum point of $(V^\varepsilon)^* - \psi_\delta$ in $\overline{B}$. Note that $x_\delta \in \mathbb{R}^n_+$ and $0 < t_\delta < T$. By taking a subsequence we can assume that

$$(t_\delta, x_\delta, y_\delta) \to (\tilde{t}, \tilde{x}, \tilde{y}) \in \overline{B} \quad \text{and} \quad ((V^\varepsilon)^* - \psi_\delta)(t_\delta, x_\delta, y_\delta) \to s \qquad \text{as } \delta \to 0.$$

Observe that, since $(V^\varepsilon)^* - \psi_\delta \le (V^\varepsilon)^* - \psi$ by definition, we get

$$s \le ((V^\varepsilon)^* - \psi)(\tilde{t}, \tilde{x}, \tilde{y}) \le ((V^\varepsilon)^* - \psi)(\overline{t}, \overline{x}, \overline{y}).$$

Moreover, for $\delta < r^2$, we get

$$((V^\varepsilon)^* - \psi_\delta)(t_\delta, x_\delta, y_\delta) \ge ((V^\varepsilon)^* - \psi_\delta)(\overline{t}, \sqrt{\delta}, \sqrt{\delta}, \overline{x}^3, \dots, \overline{x}^n, \overline{y}).$$

By letting $\delta \to 0$ we obtain $s \ge ((V^\varepsilon)^* - \psi)(\overline{t}, \overline{x}, \overline{y})$. Therefore,

$$(\tilde{t}, \tilde{x}, \tilde{y}) = (\overline{t}, \overline{x}, \overline{y}), \qquad s = ((V^\varepsilon)^* - \psi)(\overline{t}, \overline{x}, \overline{y}) \quad \text{and} \quad \frac{\delta}{x^1_\delta}, \frac{\delta}{x^2_\delta} \to 0 \qquad \text{as } \delta \to 0.$$

Now we use the fact that $(V^\varepsilon)^*$ is a subsolution to (17), that $(V^\varepsilon)^* - \psi_\delta$ has a maximum at $(t_\delta, x_\delta, y_\delta)$, and that $x_\delta \in \mathbb{R}^n_+$ and $0 < t_\delta < T$, so the PDE holds at such a point. We get

$$(23)\quad -\psi_t + H\left(x_\delta, y_\delta, D_x\psi - \delta p_\delta, D^2_{xx}\psi + 2\delta X_\delta, \frac{D^2_{xy}\psi}{\sqrt{\varepsilon}}\right) - \frac{1}{\varepsilon}\mathcal{L}(y_\delta, D_y\psi, D^2_{yy}\psi) + \lambda(V^\varepsilon)^* \le 0,$$

where all the derivatives of ψ and $(V^\varepsilon)^*$ are computed at $(t_\delta, x_\delta, y_\delta)$,

$$p_\delta := \left(\frac{1}{(x^1_\delta)^2}, \frac{1}{(x^2_\delta)^2}, 0, \dots, 0\right),$$

and X_δ is the diagonal matrix with

$$(X_\delta)_{ii} = \frac{1}{(x^i_\delta)^3} \qquad \text{for } i = 1, 2; \qquad (X_\delta)_{ii} = 0 \qquad \text{for } i = 3, \dots, n.$$

By the definition of H, $\tilde{\phi}$, and $\tilde{\sigma}$, the second term on the left-hand side of (23) is

$$(24)\quad \min_{u \in U}\left\{-\tilde{\phi} D_x\psi + \frac{\delta}{x^1_\delta}\phi^1 + \frac{\delta}{x^2_\delta}\phi^2 - \operatorname{trace}\left(\tilde{\sigma}\tilde{\sigma}^T D^2_{xx}\psi\right) \right.$$
$$\left. - \frac{2\delta}{x^1_\delta}|\sigma_1|^2 - \frac{2\delta}{x^2_\delta}|\sigma_2|^2 - \frac{2}{\sqrt{\varepsilon}}\operatorname{trace}\left(\tau\tilde{\sigma}^T D^2_{x,y}\psi\right)\right\},$$

where $\tilde{\phi}, \phi^i, \tilde{\sigma}, \sigma_i$ are computed at (x_δ, y_δ, u), the derivatives of ψ at $(t_\delta, x_\delta, y_\delta)$, and τ at y_δ. Since $\delta / x^i_\delta \to 0$ as $\delta \to 0$ for $i = 1, 2$, the quantity in (24) tends to

$$H\left(\overline{x}, \overline{y}, D_x\psi, D^2_{xx}\psi, \frac{D^2_{xy}\psi}{\sqrt{\varepsilon}}\right),$$

where all the derivatives are computed at $(\bar{t},\bar{x},\bar{y})$. Therefore the limit of (23) as $\delta \to 0$ gives (22) at $(\bar{t},\bar{x},\bar{y})$, as desired.

Step 5 (comparison principle and conclusion). We now use a recent comparison result between sub- and supersolutions to parabolic problems satisfying the quadratic growth condition

$$|V(t,x,y)| \le C(1+|x|^2+|y|^2)$$

proved in [15, Thm. 2.1]. We already observed that the estimate (19) holds also for V^ε_* and $(V^\varepsilon)^*$, so they both satisfy the appropriate growth condition. Moreover we proved in Step 3 that $(V^\varepsilon)^*(T,x,y) = V^\varepsilon_*(T,x,y) = g(x,y)$. The comparison result is stated in [15] for parabolic problems in the whole spaces $[0,T]\times\mathbb{R}^k$. Nevertheless, because of the fact that our sub- and supersolutions $(V^\varepsilon)^*$ and V^ε_* satisfy the equation also on the boundary of $\mathbb{R}^n_+$ as proved in Step 4, their argument applies without relevant changes to our case. Therefore $(V^\varepsilon)^*(t,x,y) \le V^\varepsilon_*(t,x,y)$ for every $(t,x,y) \in ([0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m)$. Using the definition of upper and lower envelopes and the comparison result in Step 5, we get $(V^\varepsilon)^*(t,x,y) = V^\varepsilon_*(t,x,y) = V^\varepsilon(t,x,y)$ for every $(t,x,y) \in ([0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m)$. Then V^ε is the unique continuous viscosity solution to (17) satisfying a quadratic growth condition. ∎

4. Ergodicity of the fast variables and the effective Hamiltonian and initial data. In this section we consider an ergodic problem in $\mathbb{R}^m$ whose solution will be useful for defining the limit problem as $\varepsilon \to 0$ of the singularly perturbed HJB equation with terminal condition (17). We consider the diffusion process in $\mathbb{R}^m$

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t \tag{25}$$

and the infinitesimal generator $\mathcal{L}$ of the process Y_t. Our standing assumptions are those of section 2. It is well known that such conditions imply the existence of a unique global solution for (25) (see [27, Chap. 2, sect. 6, Thms. 3 and 4]).

The first result of this section is a Liouville property that replaces the standard strong maximum principle of the periodic case and is the key ingredient for extending some results of [3] to the nonperiodic setting.

Lemma 4.1. *Consider the problem*

$$-\mathcal{L}(y, DV(y), D^2V(y)) = 0, \qquad y \in \mathbb{R}^m, \tag{26}$$

under the assumption (16). *Then the following hold:*

(i) *every bounded viscosity subsolution to* (26) *is constant;*

(ii) *every bounded viscosity supersolution to* (26) *is constant.*

Remark 4.1. This result holds also under a weaker condition than (16), namely,

$$\begin{aligned} &\exists\, w \in \mathcal{C}(\mathbb{R}^m) \quad \text{and} \quad R_0 > 0 \\ &\text{such that } -\mathcal{L}w \ge 0 \text{ for } |y| > R_0 \quad \text{and} \quad |w(y)| \to +\infty \qquad \text{as } |y| \to +\infty. \end{aligned} \tag{27}$$

Proof. This proof uses an argument borrowed from [34]. We start by proving (i). Let V be a bounded subsolution to (26). We can assume, without loss of generality, that $V \ge 0$. Define, for every $\eta > 0$, $V_\eta(y) = V(y) - \eta w(y)$, where w is as in (16).

We fix $R > R_0$, and we claim that V_η is a viscosity subsolution to (26) in $|y| > R$ for every $\eta > 0$. Indeed consider $\overline{y} \in \mathbb{R}^m$, $|\overline{y}| > R$, and a smooth function ψ such that $V_\eta(\overline{y}) = \psi(\overline{y})$ and $V_\eta - \psi$ has a strict maximum at $\overline{y}$.

Assume by contradiction that $-\mathcal{L}(\overline{y}, D\psi(\overline{y}), D^2\psi(\overline{y})) > 0$. By the regularity of ψ and of $\mathcal{L}$, there exists $0 < k < R - R_0$ such that $-\mathcal{L}(y, D\psi(y), D^2\psi(y) > 0$ for every y with $|y - \overline{y}| \le k$. Now we prove that $\eta w + \psi$ is a supersolution to (26) in $B(\overline{y}, k)$. Take $\tilde{y} \in B(\overline{y}, k)$ and ξ smooth such that $\eta w + \psi - \xi$ has a minimum at $\tilde{y}$. Using the fact that w is a supersolution to (26) in $|y| > R_0$ and the linearity of the differential operator $\mathcal{L}$, we obtain

$$
\begin{aligned}
0 &\le -\mathcal{L}\left(\tilde{y}, \frac{1}{\eta} D(\xi - \psi)(\tilde{y}), \frac{1}{\eta} D^2(\xi - \psi)(\tilde{y})\right) \\
&= -\frac{1}{\eta}\mathcal{L}\left(\tilde{y}, D\xi(\tilde{y}), D^2\xi(\tilde{y})\right) + \frac{1}{\eta}\mathcal{L}\left(\tilde{y}, D\psi(\tilde{y}), D^2\psi(\tilde{y})\right) < -\mathcal{L}\left(\tilde{y}, D\xi(\tilde{y}), D^2\xi(\tilde{y})\right),
\end{aligned}
$$

where in the last inequality we used that ψ is a supersolution in $B(\overline{y}, k)$. Recall that by our assumption $V - (\eta w + \psi)$ has a strict maximum at $\overline{y}$ and $V(\overline{y}) = (\eta w + \psi)(\overline{y})$. Then there exists $\alpha > 0$ such that $V(y) - (\eta w + \psi)(y) < -\alpha$ on $\partial B(\overline{y}, k)$. A standard comparison principle gives that $V(y) \le \eta w(y) + \psi(y) - \alpha$ on $\overline{B}(\overline{y}, k)$, a contradiction with our assumptions. This proves the claim: V_η is a viscosity subsolution to (26) in $|y| > R$ for every $\eta > 0$.

Now, observing that $V_\eta(y) \to -\infty$ as $|y| \to +\infty$, for every η we fix $M_\eta > R$ such that $V_\eta(y) \le \sup_{|z|=R} V_\eta(z)$ for every y such that $|y| \ge M_\eta$. By the maximum principle applied in $\{y,\, R \le |y| \le M_\eta\}$,

$$
V_\eta(y) \le \sup_{|z|=R} V_\eta(z) \qquad \forall\, |y| \ge R, \quad \forall\, \eta > 0. \tag{28}
$$

Next we let $\eta \to 0$ in (28) and obtain $V(y) \le \sup_{|z|=R} V(z)$ for every y such that $|y| > R$. Therefore V attains its global maximum at some interior point, so it is a constant by the strong maximum principle (see [7] for its extension to viscosity subsolutions).

The proof of (ii) for bounded supersolutions U is analogous, with minor changes. It is sufficient to define $U_\eta(y)$ as $U(y) + \eta w(y)$ and to prove that $U_\eta \to +\infty$ as $|y| \to +\infty$ and that it is a viscosity supersolution to (26) in $|y| > R$. So, the same argument holds exchanging the role of super- and subsolutions and using the strong minimum principle [7]. ■

The second result is about the existence of an invariant measure.

Proposition 4.2. *Under the standing assumptions, there exists a unique invariant probability measure μ on $\mathbb{R}^m$ for the process Y_t.*

Proof. Hasminskii in [28, Chap. IV] proves that there exists an invariant probability measure for Y_t (see Theorem IV.4.1 in [28]) if, besides the standing assumptions of section 2, the following condition is satisfied: there exists a bounded set K with smooth boundary such that

$$
\mathbf{E}\tau_K(y) \text{ is locally bounded} \qquad \text{for } y \in \mathbb{R}^m \setminus K, \tag{29}
$$

where $\tau_K(y)$ is the first time at which the path of the process (25) issuing from y reaches the set K. We claim that condition (16) implies (29), with $K = B(0, R)$, with $R > R_0$. We fix w as in (16) and $R > R_0$ such that $w(y) \ge 0$ for $|y| > R$. A standard superoptimality principle for viscosity supersolutions to equation $-\mathcal{L}w \ge k$ (see, e.g., [20, sect. V.2]) implies that

$$
w(y) \ge k\mathbf{E}\tau_K(y) + \mathbf{E}w(Y_{\tau_K(y)}) \ge k\mathbf{E}\tau_K(y) \qquad \text{for every } y \in \mathbb{R}^m \setminus K.
$$

This gives our claim immediately, because w is locally bounded.

The uniqueness of the invariant measure is a standard result under the current assumptions, because the diffusion is nondegenerate; see, e.g., [28, Cor. IV.5.2] or [16]. ■

The previous two results—the Liouville property in Lemma 4.1 and the existence and uniqueness of the invariant measure in Proposition 4.2—are the main tools used to define the candidate limit Cauchy problem of the singularly perturbed problem (17) as $\varepsilon \to 0$. The underlying idea is that Proposition 4.2 provides the ergodicity of the process Y_t. This property allows us to construct the effective Hamiltonian and the effective terminal data. In the following we will perform such constructions in Theorem 4.3 and Proposition 4.4 using mainly PDE methods; nevertheless it must be noted that the same results could also be obtained using direct probabilistic arguments (see Remark 4.2).

We start by showing the existence of an effective Hamiltonian giving the limit PDE. In principle, for each $(\overline{x}, \overline{p}, \overline{X})$ one expects the effective Hamiltonian $\overline{H}(\overline{x}, \overline{p}, \overline{X})$ to be the unique constant $c \in \mathbb{R}$ such that the *cell problem*

$$-\mathcal{L}(y, D\chi, D^2\chi) + H(\overline{x}, y, \overline{p}, \overline{X}, 0) = c \quad \text{in } \mathbb{R}^m \tag{30}$$

has a viscosity solution χ, called corrector (see [35, 18, 1]). Actually, for our approach, it is sufficient to consider, as in [2], a *δ-cell problem*

$$\delta w_\delta - \mathcal{L}(y, Dw_\delta, D^2 w_\delta) + H(\overline{x}, y, \overline{p}, \overline{X}, 0) = 0 \quad \text{in } \mathbb{R}^m, \tag{31}$$

whose solution w_δ is called *approximate corrector*. The next result states that δw_δ converges to $-\overline{H}$ and it is smooth.

Theorem 4.3. *For any fixed $(\overline{x}, \overline{p}, \overline{X})$ and $\delta > 0$ there exists a solution $w_\delta = w_{\delta;\overline{x},\overline{p},\overline{X}}(y)$ in $\mathcal{C}^2(\mathbb{R}^m)$ of (31) such that*

$$-\lim_{\delta\to 0} \delta w_\delta = \overline{H}(\overline{x}, \overline{p}, \overline{X}) := \int_{\mathbb{R}^m} H(\overline{x}, y, \overline{p}, \overline{X}, 0) d\mu(y) \quad \textit{locally uniformly in } \mathbb{R}^m, \tag{32}$$

where μ is the invariant probability measure on $\mathbb{R}^m$ for the process Y_t.

Proof. We borrow some ideas from ergodic control theory in periodic environments; see [5].

The PDE (31) is linear with locally Lipschitz coefficients and forcing term

$$f(y) := H(\overline{x}, y, \overline{p}, \overline{X}, 0)$$

bounded and Lipschitz by the assumptions of section 2. The existence and uniqueness of a viscosity solution satisfying

$$|w_\delta(y)| \leq C(1 + |y|^2) \tag{33}$$

for some C follow from the Perron–Ishii method and the comparison principle in [15] (here we are using the growth assumption (7) on the coefficients). Moreover $w_\delta \in \mathcal{C}^2(\mathbb{R}^m)$ by standard elliptic regularity theory.

By comparison with constant sub- and supersolutions we get the uniform bound

$$|\delta w_\delta(y)| \leq \sup |f| =: C_f.$$

Then the functions $v_\delta := \delta w_\delta$ are uniformly bounded and satisfy

$$|\mathcal{L}(y, Dv_\delta, D^2 v_\delta)| \leq 2\delta C_f.$$

By the Krylov–Safonov estimates for elliptic equations, in any compact set the family $\{v_\delta\}$ with $\delta \leq 1$ is equi-Hölder continuous for some exponent and constants depending only on C_f and the coefficients of $\mathcal{L}$. Therefore by the Ascoli–Arzelà theorem there is a sequence $\delta_n \to 0$ such that $v_{\delta_n} \to v$ locally uniformly and

$$\mathcal{L}(y, Dv, D^2 v) = 0 \quad \text{in } \mathbb{R}^m$$

in the viscosity sense. By Lemma 4.1 v is constant.

To complete the proof we show that on any subsequence the limit of $v_\delta := \delta w_\delta$ is the same and it is given by the formula (32). We claim that

$$w_\delta(y) = \mathbf{E} \int_0^{+\infty} f(Y_t) e^{-\delta t}\, dt, \tag{34}$$

where Y_t is the process defined by the fast subsystem (25) with initial condition $Y_0 = y$. In fact, the right-hand side is a viscosity solution of (31) by Itô's rule and other standard arguments [20]. Moreover it is bounded by C_f/δ, and so the growth assumption (33) is satisfied. Therefore it is the viscosity solution of (31) by the comparison principle in [15], which proves the claim. Next we recall that by definition of invariant measure

$$\mathbf{E} \int_{\mathbb{R}^m} f(Y_t)\, d\mu(y) = \int_{\mathbb{R}^m} f(y)\, d\mu(y) \qquad \forall\, t > 0.$$

As a consequence, by integrating both sides of (34) with respect to μ and exchanging the order of integration we get

$$\int_{\mathbb{R}^m} w_\delta(y)\, d\mu(y) = \int_0^{+\infty} \int_{\mathbb{R}^m} f(y)\, d\mu(y) e^{-\delta t}\, dt = \frac{\int_{\mathbb{R}^m} f(y)\, d\mu(y)}{\delta}.$$

Therefore the constant limit v of δw_δ must be $\int_{\mathbb{R}^m} f(y)\, d\mu(y)$. ■

We end this section by defining the effective terminal value for the limit as $\varepsilon \to 0$ of the singular perturbation problem (17). We fix $\overline{x}$ and consider the following Cauchy initial problem:

$$\begin{cases} w_t - \mathcal{L}(y, Dw, D^2 w) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^m, \\ w(0, y) = g(\overline{x}, y), \end{cases} \tag{35}$$

where g satisfies assumption (10).

Proposition 4.4. *Under our standing assumptions, for every $\overline{x}$ there exists a unique bounded classical solution $w(\cdot, \cdot; \overline{x})$ to (35) and*

$$\lim_{t \to +\infty} w(t, y; \overline{x}) = \int_{\mathbb{R}^m} g(\overline{x}, y)\, d\mu(y) =: \overline{g}(\overline{x}) \quad \textit{locally uniformly in } y. \tag{36}$$

Proof. The PDE in (35) is parabolic with coefficients which are locally Lipschitz and grow at most linearly, whereas the initial data are bounded and continuous, by the assumptions of section 2. Classical results on these equations give the existence of a bounded classical solution to the Cauchy problem (35) (see, e.g., Theorem 1.2.1 in [36] and the references therein), whereas uniqueness among viscosity solutions is given by Theorem 2.1 in [15]. This solution can be represented as $w(t,y;\overline{x}) = \mathbf{E}g(\overline{x}, Y_t)$, where Y_t is the process starting at y and satisfying (25). Moreover the function $w(t,y;\overline{x})$ is uniformly continuous in every domain $[t_0,+\infty)\times K$, where $K\subseteq\mathbb{R}^m$ is a compact set; see [26, Thm. 3.5] or [28, Lem. 4.6.2].

To complete the proof it is enough to show that $\overline{w}(y) = \limsup_{s\to+\infty} w(s,y;\overline{x})$ and $\underline{w}(y) = \liminf_{s\to+\infty} w(s,y;\overline{x})$ are constants, i.e., $\overline{w}(y)=\overline{w}$ and $\underline{w}(y)=\underline{w}$ for every y, and that they both coincide with $\overline{g}(\overline{x})$, i.e., $\underline{w}=\overline{w}=\overline{g}(\overline{x})$.

The proof that $\overline{w}(y)$ and $\underline{w}(y)$ are constants is the same as in the periodic case, Theorem 4.2 in [3], once we replace the strong maximum (and minimum) principle with the Liouville property, Lemma 4.1.

To conclude we show that $\overline{w}=\overline{g}(\overline{x})=\underline{w}$. We detail the argument only for $\underline{w}$, since it is completely analogous for $\overline{w}$. We fix a subsequence such that $\underline{w}=\lim_n w(t_n,0;\overline{x})$ and define $w_n(t,y)=w(t+t_n,y;\overline{x})$. Since w_n is equibounded and equicontinuous, by taking a subsequence we can assume that $w_n(t,y)\to\tilde{w}(t,y)$ locally uniformly. Note that by construction $\tilde{w}(t,y)\geq\underline{w}$ for every (t,y) and $\tilde{w}(0,0)=\underline{w}$. By stability results of viscosity solutions, $\tilde{w}$ is a viscosity solution to $w_t-\mathcal{L}(y,Dw,D^2w)=0$ in $(-\infty,+\infty)\times\mathbb{R}^m$. Then, by the strong minimum principle, we get that $\tilde{w}(0,y)=\underline{w}$ for every y. This means that $w(t_n,y;\overline{x})$ converges to $\underline{w}$ locally uniformly in y, in particular $w(t_n,y;\overline{x})\to\underline{w}$ μ-almost surely, where μ is the invariant probability measure for Y_t (see Proposition 4.2). Moreover $|w(t_n,y)|\leq\|w\|_\infty\in L^1(\mathbb{R}^m,\mu)$ and then, by the Lebesgue theorem and the definition of invariant measure,

$$\underline{w}=\int_{\mathbb{R}^m}\underline{w}\,d\mu(y)=\lim_n\int_{\mathbb{R}^m}\mathbf{E}g(\overline{x},Y_{t_n})\,d\mu(y)=\int_{\mathbb{R}^m}g(\overline{x},y)\,d\mu(y). \qquad\blacksquare$$

Remark 4.2. The results in Theorem 4.3 and Proposition 4.4 could also be proved using direct probabilistic methods and semigroup theory.

We consider the infinitesimal generator L of the Markov semigroup in $\mathcal{C}_b(\mathbb{R}^m)$ associated with the diffusion process Y_t. In this abstract setting, the cell problem (30) can be seen as the Poisson equation $L\chi=c-c(y)$, where $c(y):=H(\overline{x},y,\overline{p},\overline{X})$, and the δ-cell problem (31) is the resolvent equation $(\delta-L)w_\delta=-c(y)$. Finally the initial layer problem (35) is the abstract Cauchy problem $w_t-Lw=0$, $w(0,y)=g(\overline{x},y)$ (for more details see the monograph [36]). In particular, thanks to the existence of a unique invariant probability measure μ (see Proposition 4.2), the solution of the Poisson equation $L\chi=c-c(y)$ is given by the representation formula

$$w(y)=\int_0^\infty\int_{\mathbb{R}^n}f(z)\,(P(t,y,dz)-\mu(dz))\,dt,$$

where $P(t,y,\cdot)$ are the transition probabilities associated with Y_t, provided the convergence of $P(t,y,\cdot)$ to μ is fast enough. Using the same approach and appropriate representation formulas, the convergence results (32) and (36) can be obtained as consequences of a sufficiently strong convergence result of the transition probabilities to the invariant measure.

Related results on the (exponential) convergence of the transition probabilities to the unique invariant measure were obtained in [17, Thm. 5.2] under a stronger condition than (16), namely, the existence of a positive function w and positive constants b, c such that $\lim_{|y|\to+\infty} w(y) = +\infty$ and $-\mathcal{L}w \geq cw - b$ in $\mathbb{R}^m$.

5. The convergence theorem. We state now the main result of the paper, namely, the convergence theorem for the singular perturbation problem. We will prove that the value function $V^\varepsilon(t,x,y)$, solution to (17), converges locally uniformly, as $\varepsilon \to 0$, to a function $V(t,x)$ which can be characterized as the unique solution of the limit problem

$$(37) \qquad \begin{cases} -V_t + \overline{H}\left(x, D_x V, D^2_{xx} V\right) + \lambda V(x) = 0 & \text{in } (0,T)\times\mathbb{R}^n_+, \\ V(T,x) = \overline{g}(x) & \text{in } \overline{\mathbb{R}^n_+}. \end{cases}$$

The Hamiltonian $\overline{H}$ and the terminal data $\overline{g}$ have been defined, respectively, in (32) and in (36) as the averages of H (see (13)) and g with respect to the unique invariant measure μ for the process Y_t, defined in (15).

Theorem 5.1. *The solution V^ε to (17) converges uniformly on compact subsets of $[0,T)\times \overline{\mathbb{R}^n_+}\times\mathbb{R}^m$ to the unique continuous viscosity solution to the limit problem (37) satisfying a quadratic growth condition in x, i.e.,*

$$(38) \qquad \exists\, K>0 \ \ s.t. \ \ \forall\,(t,x)\in[0,T]\times\overline{\mathbb{R}^n_+} \quad |V(t,x)| \leq K(1+|x|^2).$$

Moreover, if g is independent of y, then the convergence is uniform on compact subsets of $[0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m$ and $\overline{g}=g$.

Proof. The proof is divided into several steps.

Step 1 (relaxed semilimits). Recall that by (19) the functions V^ε are locally equibounded in $[0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m$, uniformly in ε. We define the half-relaxed semilimits in $[0,T]\times\overline{\mathbb{R}^n_+}\times\mathbb{R}^m$ (see [6, Chap. V]):

$$\underline{V}(t,x,y) = \liminf_{\substack{\varepsilon\to 0\\ t'\to t,\, x'\to x,\, y'\to y}} V^\varepsilon(t',x',y'), \qquad \overline{V}(t,x,y) = \limsup_{\substack{\varepsilon\to 0\\ t'\to t,\, x'\to x,\, y'\to y}} V^\varepsilon(t',x',y')$$

for $t<T$, $x\in\overline{\mathbb{R}^n_+}$ and $y\in\mathbb{R}^d$, and

$$\underline{V}(T,x,y) = \liminf_{t'\to T^-,\, x'\to x,\, y'\to y} \underline{V}(t',x',y'), \qquad \overline{V}(T,x,y) = \limsup_{t'\to T^-,\, x'\to x,\, y'\to y} \overline{V}(t',x',y').$$

It is immediate to get by definitions that the estimates (19) hold also for $\overline{V}$ and $\underline{V}$. This means that

$$(39) \qquad |\underline{V}(t,x,y)|, |\overline{V}(t,x,y)| \leq K_V(1+|x|^2) \qquad \forall\, t\in[0,T], \quad x\in\overline{\mathbb{R}^n_+}, \quad y\in\mathbb{R}^m.$$

Step 2 ($\overline{V}, \underline{V}$ do not depend on y). We check that $\overline{V}(t,x,y), \underline{V}(t,x,y)$ do not depend on y for every $t\in[0,T)$ and $x\in\mathbb{R}^n_+$. We claim that $\overline{V}(t,x,y)$ (resp., $\underline{V}(t,x,y)$) is, for every $t\in(0,T)$ and $x\in\mathbb{R}^n_+$, a viscosity subsolution (resp., supersolution) to

$$(40) \qquad -\mathcal{L}(y, D_y V, D^2_{yy} V) = 0 \quad \text{in } \mathbb{R}^d,$$

where $\mathcal{L}$ is the differential operator defined in (14). If the claim is true, we can use Lemma 4.1, since $\overline{V}, \underline{V}$ are bounded in y according to estimates (39), to conclude that the functions $y \to \overline{V}(t,x,y)$, $y \to \underline{V}(t,x,y)$ are constants for every $(t,x) \in (0,T) \times \mathbb{R}^n_+$. Finally, using the definition it is immediate to see that this implies that also $\overline{V}(T,x,y)$ and $\underline{V}(T,x,y)$ do not depend on y. We prove the claim only for $\overline{V}$, since the other case is completely analogous.

First of all we show that the function $\overline{V}(t,x,y)$ is a viscosity subsolution to (40). To do this, we fix a point $(\overline{t},\overline{x},\overline{y})$ and a smooth function ψ such that $\overline{V} - \psi$ has a maximum at $(\overline{t},\overline{x},\overline{y})$. Using the definition of weak relaxed semilimits it is possible to prove (see [6, Lem. V.1.6]) that there exist $\varepsilon_n \to 0$ and $\overline{B} \ni (t_n,x_n,y_n) \to (\overline{t},\overline{x},\overline{y})$ maxima for $V^{\varepsilon_n} - \psi$ in $\overline{B}$ such that $V^{\varepsilon_n}(t_n,x_n,y_n) \to \overline{V}(\overline{t},\overline{x},\overline{y})$. Therefore, recalling that V^ε is a subsolution to (17), we get

$$-\psi_t + H\left(x_n, y_n, D_x\psi, D^2_{xx}\psi, \frac{1}{\sqrt{\varepsilon_n}} D^2_{xy}\psi\right) - \frac{1}{\varepsilon_n}\mathcal{L}(y_n, D_y\psi, D^2_{yy}\psi) + \lambda V^{\varepsilon_n} \le 0,$$

where V^{ε_n} and all the derivatives of ψ are computed in (t_n,x_n,y_n). This implies

$$-\mathcal{L}(y_n, D_y\psi, D^2_{yy}\psi) \le \varepsilon_n \left[\psi_t - H\left(x_n, y_n, D_x\psi, D^2_{xx}\psi, \frac{1}{\sqrt{\varepsilon_n}} D^2_{xy}\psi\right) - \lambda V^{\varepsilon_n}\right]. \tag{41}$$

We observe that the term in square brackets is uniformly bounded with respect to n in $\overline{B}$, and using the regularity properties of ψ and of the coefficients in the equation we get the desired conclusion as $\varepsilon_n \to 0$.

We show now that if $\overline{V}(t,x,y)$ is a subsolution to (40), then for every fixed $(\overline{t},\overline{x})$ the function $y \mapsto \overline{V}(\overline{t},\overline{x},y)$ is a subsolution to (40), which was our claim. To do this, we fix $\overline{y}$ and a smooth function ϕ such that $\overline{V}(\overline{t},\overline{x},\cdot) - \phi$ has a strict local maximum at $\overline{y}$ in $B(\overline{y},\delta)$ and such that $\phi(y) \ge 1$ for all $y \in B(\overline{y},\delta)$. We define, for $\eta > 0$, $\phi_\eta(t,x,y) = \phi(y)\left(1 + \frac{|x-\overline{x}|^2 + |t-\overline{t}|^2}{\eta}\right)$, and we consider (t_η,x_η,y_η) a maximum point of $\overline{V} - \phi_\eta$ in $B((\overline{t},\overline{x},\overline{y}),\delta)$. Repeating the same argument as in [6, Lem. II.5.17], it is possible to prove, eventually passing to subsequences, that, as $\eta \to 0$, $(t_\eta,x_\eta,y_\eta) \to (\overline{t},\overline{x},\overline{y})$ and $K_\eta := \left(1 + \frac{|x_\eta-\overline{x}|^2 + |t_\eta-\overline{t}|^2}{\eta}\right) \to K > 0$. Moreover, using the fact that $\overline{V}$ is a subsolution to (40), we get $-\mathcal{L}(y_\eta, K_\eta D\phi(y_\eta), K_\eta D^2\phi(y_\eta)) \ge 0$, which gives, using the linearity of $\mathcal{L}$ and passing to the limit as $\eta \to 0$, $-\mathcal{L}(\overline{y}, D\phi(\overline{y}), D^2\phi(\overline{y})) \ge 0$.

Step 3 ($\overline{V}$ and $\underline{V}$ are sub- and supersolutions of the limit PDE). First we claim that $\overline{V}$ and $\underline{V}$ are sub- and supersolutions to the PDE in (37) in $(0,T) \times \mathbb{R}^n_+$. We prove the claim only for $\overline{V}$ since the other case is completely analogous. The proof adapts the perturbed test function method introduced in [18] for the periodic setting. We fix $(\overline{t},\overline{x}) \in ((0,T) \times \mathbb{R}^n_+)$, and we show that $\overline{V}$ is a viscosity subsolution at $(\overline{t},\overline{x})$ of the limit problem. This means that if ψ is a smooth function such that $\psi(\overline{t},\overline{x}) = \overline{V}(\overline{t},\overline{x})$ and $\overline{V} - \psi$ has a maximum at $(\overline{t},\overline{x})$, then

$$-\psi_t(\overline{t},\overline{x}) + \overline{H}(\overline{x}, D_x\psi(\overline{t},\overline{x}), D^2_{xx}\psi(\overline{t},\overline{x})) + \lambda\overline{V}(\overline{t},\overline{x}) \le 0. \tag{42}$$

Without loss of generality we assume that the maximum is strict in $B((\overline{t},\overline{x}),r) \cap ([0,T] \times \overline{\mathbb{R}^n_+})$ and that $\overline{x}^i > r$ for every i and $0 < \overline{t} - r < \overline{t} + r < T$. We fix $\overline{y} \in \mathbb{R}^m$, $\eta > 0$ and consider a solution $\chi = w_\delta \in \mathcal{C}^2$ of the δ-cell problem (31) at $(\overline{x}, D_x\psi(\overline{t},\overline{x}), D^2_{xx}\psi(\overline{t},\overline{x}))$ (see Theorem 4.3) such that

$$|\delta\chi(y) + \overline{H}(\overline{x}, D_x\psi(\overline{t},\overline{x}), D^2_{xx}\psi(\overline{t},\overline{x}))| \le \eta \qquad \forall\, y \in B(\overline{y}, r). \tag{43}$$

We define the perturbed test function as

$$\psi^\varepsilon(t,x,y) := \psi(t,x) + \varepsilon\chi(y).$$

Observe that

$$\limsup_{\varepsilon\to 0,\, t'\to\bar t,\, x'\to\bar x,\, y'\to\bar y} V^\varepsilon(t',x',y') - \psi^\varepsilon(t',x',y') = \overline{V}(t,x) - \psi(t,x).$$

By a standard argument in viscosity solution theory (see [6, Lem. V.1.6]) we get that there exist sequences $\varepsilon_n \to 0$ and $(t_n,x_n,y_n) \in \overline{B} := B((\bar t,\bar x,\bar y),r) \cap ([0,T] \times \overline{\mathbb{R}^n_+} \times \mathbb{R}^m)$ such that the following hold:
$(t_n,x_n,y_n) \to (\bar t,\bar x,y)$ for some $y \in B(\bar y,r)$;
$V^{\varepsilon_n}(t_n,x_n,y_n) - \psi^{\varepsilon_n}(t_n,x_n,y_n) \to \overline{V}(\bar t,\bar x) - \psi(\bar t,\bar x)$;
(t_n,x_n,y_n) is a strict maximum of $V^{\varepsilon_n} - \psi^{\varepsilon_n}$ in $\overline{B}$.
Then, using the fact that V^ε is a subsolution to (17), we get

$$-\psi_t + H\left(x_n,y_n,D_x\psi,D^2_{xx}\psi,0\right) + \lambda V^{\varepsilon_n}(t_n,x_n,y_n) - \mathcal{L}(y_n,D_y\chi,D^2_{yy}\chi) \le 0, \tag{44}$$

where the derivatives of ψ and χ are computed, respectively, in (t_n,x_n) and in y_n. Using the fact that χ solves the δ-cell problem (31), we obtain

$$\begin{aligned}&-\psi_t(t_n,x_n) + H(x_n,y_n,D_x\psi(t_n,x_n),D^2_{xx}\psi(t_n,x_n),0) - \delta\chi(y_n)\\ &\quad - H(\bar x,y_n,D_x\psi(\bar t,\bar x),D^2_{xx}\psi(\bar t,\bar x),0) + \lambda V^{\varepsilon_n}(t_n,x_n,y_n) \le 0.\end{aligned}$$

By taking the limit as $n \to +\infty$ the second and third terms of the left-hand side of this inequality cancel out. Next we use (43) to replace $-\delta\chi$ with $\overline{H} - \eta$ and get that the left-hand side of (42) is $\le \eta$. Finally, by letting $\eta \to 0$ we obtain (42).

Now we claim that $\underline{V}$ and $\overline{V}$ are, respectively, a super- and a subsolution to (37) also at the boundary of $\mathbb{R}^n_+$. In this case it is sufficient to repeat exactly the same argument of Step 4 in the proof of Proposition 3.1 to get the conclusion, recalling that the Hamiltonian $\overline{H}$ is defined as

$$\overline{H}(x,p,X) = \int_{\mathbb{R}^m} \min_{u\in U}\left\{-\operatorname{trace}\left(\tilde\sigma\tilde\sigma^T(x,y,u)X\right) - \tilde\phi(x,y,u)\cdot p\right\} d\mu(y).$$

Step 4 (behavior of $\overline{V}$ and $\underline{V}$ at time T). The arguments in this step are based on analogous results given in [2, Thm. 3] in the periodic setting, with minor corrections due to the unboundedness of our domain. We repeat briefly the proof for the convenience of the reader. We prove only the statement for subsolution, since the proof for the supersolution is completely analogous.

We fix $\bar x \in \overline{\mathbb{R}^n_+}$ and consider the unique bounded solution w^r to the Cauchy problem

$$\begin{cases} w_t - \mathcal{L}(y,Dw,D^2w) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^m,\\ w(0,y) = \sup_{\{|x-\bar x|\le r,\ x\ge 0\}} g(x,y). \end{cases} \tag{45}$$

Using stability properties of viscosity solutions it is not hard to see that w^r converges, as $r \to 0$, to $w_{\bar x}$, solution to (35), uniformly on compact sets.

We fix $k > 0$. Using the definition of $\overline{g}$ given in (36) and the uniform convergence of w^r to $w_{\overline{x}}$, it is easy to see that for every $\eta > 0$ there exist $t_0 > 0$ and r_0 such that $|w_r(t_0, y) - \overline{g}(\overline{x})| \leq \eta$ for every $r < r_0$ and $|y| \leq k$. Moreover, since $\mathcal{L}(y, 0, 0) = 0$, using a comparison principle, we get that

$$|w_r(t, y) - \overline{g}(\overline{x})| \leq \eta \qquad \text{for every } r < r_0,\ t \geq t_0,\ |y| \leq k. \tag{46}$$

We now fix $r < r_0$ and a constant M such that $V^\varepsilon(t, x, y) \leq M$ for every $\varepsilon > 0$ and $x \in \overline{B} := \overline{B(\overline{x}, r)} \cap \overline{\mathbb{R}^n_+}$. Observe that this is possible by estimates (19). Moreover we fix a smooth nonnegative function ψ such that $\psi(\overline{x}) = 0$ and $\psi(x) + \inf_y g(x, y) \geq M$ for every $x \in \partial B$ (using condition (10)). Let C be a positive constant such that

$$|H(y, x, D\psi(x), D^2\psi(x))| \leq C \qquad \text{for } x \in \overline{B} \quad \text{and} \quad y \in \mathbb{R}^m,$$

where H is defined in (13). We define the function

$$\psi^\varepsilon(t, x, y) = w_r\left(\frac{T - t}{\varepsilon}, y\right) + \psi(x) + C(T - t),$$

and we claim that it is a supersolution to the parabolic problem

$$\begin{cases} -V_t + F\left(x, y, V, D_x V, \frac{D_y V}{\varepsilon}, D^2_{xx} V, \frac{D^2_{yy} V}{\varepsilon}, \frac{D^2_{xy} V}{\sqrt{\varepsilon}}\right) = 0 & \text{in } (T - r, T) \times B \times \mathbb{R}^m, \\ V(t, x, y) = M & \text{in } (T - r, T) \times \partial B \times \mathbb{R}^m, \\ V(T, x, y) = g(x, y) & \text{in } \overline{B} \times \mathbb{R}^m, \end{cases} \tag{47}$$

where F is defined in (18). Indeed if w_r is smooth,

$$\begin{aligned} &-\psi^\varepsilon_t + F\left(x, y, D_x\psi^\varepsilon, \frac{D_y\psi^\varepsilon}{\varepsilon}, D^2_{xx}\psi^\varepsilon, \frac{D^2_{yy}V\psi^\varepsilon}{\varepsilon}, \frac{D^2_{xy}\psi^\varepsilon}{\sqrt{\varepsilon}}\right) \\ &= \frac{1}{\varepsilon}(w_r)_t + C + H(y, x, D\psi(x), D^2\psi(x)) - \frac{1}{\varepsilon}\mathcal{L}(y, Dw_r, D^2 w_r) \\ &\geq \frac{1}{\varepsilon}\left((w_r)_t - \mathcal{L}(y, Dw_r, D^2 w_r)\right) \geq 0. \end{aligned}$$

This computation is made in the case where w_r is smooth but can be easily generalized to w_r continuous using test functions (see [2, Thm. 3]). Moreover

$$\psi^\varepsilon(T, x, y) = \sup_{|x - \overline{x}| \leq r} g(x, y) + \psi(x) \geq g(x, y).$$

Finally, recalling that by the comparison principle, $w_r(t, y) \geq \inf_y \sup_{|x - \overline{x}| \leq r} g(x, y)$, we get

$$\psi^\varepsilon(t, x, y) \geq \inf_y \sup_{|x - \overline{x}| \leq r} g(x, y) + M - \inf_y g(x, y) + C(T - t) \geq M$$

for every $x \in \overline{B}$. For our choice of M, we get that V^ε is a subsolution to (47). Moreover note that both V^ε and ψ^ε are bounded in $[0, T] \times \overline{B} \times \mathbb{R}^m$, because of the estimate (19), the boundedness of w_r, and the regularity of ψ. So, a standard comparison principle for viscosity solutions gives

$$V^\varepsilon(t,x,y) \le \psi^\varepsilon(t,x,y) = w_r\left(\frac{T-t}{\varepsilon}, y\right) + \psi(x) + C(T-t) \tag{48}$$

for every $\varepsilon > 0$, $(t,x,y) \in ([0,T] \times \overline{B} \times \mathbb{R}^m)$. We compute the upper limit of both sides of (48) as $(\varepsilon, t', x', y') \to (0,t,x,y)$ for $t \in (t_0,T)$, $x \in B$, $|y| < k$ and get, recalling (46),

$$\overline{V}(t,x) \le \overline{g}(\overline{x}) + \eta + \psi_0(x) + C(T-t).$$

This permits us to conclude, taking the upper limit for $(t,x) \to (T,\overline{x})$ and recalling that η is arbitrary.

Step 5 (uniform convergence). Observe that by definition $\underline{V} \le \overline{V}$ and that both $\underline{V}$ and $\overline{V}$ satisfy the same quadratic growth condition (39). Moreover the Hamiltonian $\overline{H}$ defined in (32) and the terminal data $\overline{g}$ in (36) inherit all the regularity properties of H in (13), and g in (10), as is easily seen by their definitions. Therefore we can again use the comparison result between sub- and supersolutions to parabolic problems satisfying a quadratic growth condition, given in [15, Thm. 2.1], to deduce $\underline{V} \ge \overline{V}$. Therefore $\underline{V} = \overline{V} =: V$. In particular V is continuous, and by the definition of half-relaxed semilimits, this implies that V^ε converges locally uniformly to V (see [6, Lem. V.1.9]). ∎

Remark 5.1. The result in Theorem 5.1 still holds if the fast variables Y_t have an extra term such as $\Lambda(y)/\sqrt{\varepsilon}$ in the drift, with $\Lambda : \mathbb{R}^m \to \mathbb{R}^m$ bounded and Lipschitz continuous. This means that fast variables in the singularly perturbed system (6) satisfy

$$dY_t^k = \frac{1}{\varepsilon} b^k(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\Lambda^k(Y_t)dt + \sqrt{\frac{2}{\varepsilon}}\tau_k(Y_t)\cdot dW_t, \qquad Y_{t_o}^k = y^k,\ k = 1,\dots,m,$$

and the singularly perturbed HJB equation is

$$-V_t^\varepsilon + H\left(x, y, D_xV^\varepsilon, D_{xx}^2V^\varepsilon, \frac{D_{xy}^2V^\varepsilon}{\sqrt{\varepsilon}}\right) - \frac{1}{\varepsilon}\mathcal{L}(y, D_yV^\varepsilon, D_{yy}^2V^\varepsilon) - \frac{\Lambda\cdot D_yV^\varepsilon}{\sqrt{\varepsilon}} + \lambda V^\varepsilon = 0.$$

The new term $\frac{1}{\sqrt{\varepsilon}}\Lambda(y)\cdot D_yV^\varepsilon$ appearing in the equation is a lower order term with respect to $\frac{1}{\varepsilon}\mathcal{L}(y, D_yV^\varepsilon, D_{yy}^2V^\varepsilon)$ and does not affect the convergence argument. In particular it is sufficient to check the validity of Steps 2, 3, and 4 in the proof of Theorem 5.1.

In Step 2, we substitute formula (41) with

$$\begin{aligned}&-\mathcal{L}(y_n, D_y\psi, D_{yy}^2\psi)\\ &\le \varepsilon_n\left[\psi_t - H\left(x_n, y_n, D_x\psi, D_{xx}^2\psi, \frac{1}{\sqrt{\varepsilon_n}}D_{xy}^2\psi\right) - \lambda V^{\varepsilon_n}\right] + \sqrt{\varepsilon_n}\Lambda(y_n)\cdot D_y\psi\end{aligned}$$

and observe that the right-hand side is vanishing as $\varepsilon_n \to 0$ since $D_y\psi$ is locally bounded and Λ is bounded.

In Step 3, we replace formula (44) with

$$-\psi_t + H\left(x_n, y_n, D_x\psi, D_{xx}^2\psi, 0\right) + \lambda V^{\varepsilon_n} - \mathcal{L}(y_n, D_y\chi, D_{yy}^2\chi) \le \sqrt{\varepsilon_n}\Lambda(y_n)\cdot D_y\chi$$

and repeat the same argument since the right-hand side is vanishing as $\varepsilon_n \to 0$, due again to the boundedness of Λ and the smoothness of the approximate corrector χ.

Finally in Step 4, we substitute the Cauchy problem (45) with

$$\begin{cases} w_t - \mathcal{L}(y, Dw, D^2 w) - \sqrt{\varepsilon}\Lambda(y) \cdot Dw = 0 & \text{in } (0, +\infty) \times \mathbb{R}^m, \\ w(0, y) = \sup_{\{|x-\overline{x}| \le r,\ x \ge 0\}} g(x, y) \end{cases}$$

and denote with $w^{r,\varepsilon}$ its unique bounded solution. Stability properties of viscosity solutions imply that $w^{r,\varepsilon}$ converges, as $r \to 0$, $\varepsilon \to 0$, to $w_{\overline{x}}$, solution to (35), uniformly on compact sets.

6. Examples and extensions.

6.1. The model problem: Risky assets with stochastic volatility. We consider N underlying risky assets with price X^i evolving according to the standard lognormal model:

$$\begin{cases} dX^i_t = \alpha^i X^i_t dt + \sqrt{2} X^i_t f_i(Y_t) \cdot d\overline{W}_t, & X^i_{t_o} = x^i \ge 0, \quad i = 1, \dots, n, \\ dY^j_t = \frac{1}{\varepsilon} b^j(Y_t) dt + \sqrt{\frac{2}{\varepsilon}} \nu_j(Y_t) d\overline{Z}^j_t, & Y^j_{t_o} = y^j \in \mathbb{R}, \quad j = 1, \dots, m, \quad \varepsilon > 0, \end{cases} \tag{49}$$

where $f_i : \mathbb{R}^m \to \mathbb{R}^k$ is a bounded Lipschitz continuous function, with each component bounded away from 0, and $b^i : \mathbb{R}^m \to \mathbb{R}$ and $\nu_j : \mathbb{R}^m \to \mathbb{R}$ are locally Lipschitz continuous functions with linear growth (see (7)). We assume that

$$\nu_j^2(y) > 0 \qquad \forall\, y \in \mathbb{R}^m, \quad j = 1, \dots, m. \tag{50}$$

The processes $\overline{W}_t$ and $\overline{Z}_t$ are, respectively, standard k- and m-dimensional Brownian motions, and they are correlated. In particular we assume that there exists an m-dimensional standard Brownian motion Z_t such that $W_t = (\overline{W}_t, Z_t)$ is a $(k+m)$-dimensional standard Brownian motion and

$$\overline{Z}^j_t = \sum_{i=1}^k \rho_{ij} \overline{W}^i_t + \left(1 - \sum_{i=1}^k \rho_{ij}^2\right)^{\frac{1}{2}} Z^j_t \qquad \forall\, j = 1, \dots, m,\ \forall\, t \ge 0. \tag{51}$$

This model problem is essentially the one described in [23, sect. 10.6], where $k = n = m$.

We denote with ρ the correlation $(k \times m)$-matrix (ρ_{ij}) and with c^j the quantity

$$c^j := \left(1 - \sum_{i=1}^k \rho_{ij}^2\right)^{\frac{1}{2}}. \tag{52}$$

In the following proposition we describe the main properties of ρ.

Proposition 6.1.

(i) $-1 \le \rho_{ij} \le 1$ *for every* $i \in \{1, \dots, k\}$ *and* $j \in \{1, \dots, m\}$;

(ii) $\sum_{i=1}^k \rho_{ij}^2 \le 1$ *for every* $j \in \{1, \dots, m\}$;

(iii) $\sum_{i=1}^k \rho_{ij}\rho_{il} = 0$ *for every* $l \ne j \in \{1, \dots, m\}$.

Proof. Items (i) and (ii) can be easily proved by exploiting the definition of ρ_{ij}. To show (iii), we multiply $\sum_{i=1}^{k} \rho_{ij}\rho_{il}$ by t, for fixed $l \neq j \in \{1, \dots, m\}$, and use the properties of $\overline{W}$. to get

$$t\sum_{i=1}^{k} \rho_{ij}\rho_{il} = \mathbf{E}\sum_{i=1}^{k} \rho_{ij}\overline{W}^i_t \rho_{il}\overline{W}^i_t = \mathbf{E}\left(\sum_{i=1}^{k} \rho_{ij}\overline{W}^i_t \sum_{i=1}^{k} \rho_{il}\overline{W}^i_t\right) \tag{53}$$

since the components of $\overline{W}_t$ are independent. Substituting (51) in (53) we get

$$\begin{aligned} t\sum_{i=1}^{k} \rho_{ij}\rho_{il} &= \mathbf{E}\left[\left(\overline{Z}^j_t - c^j Z^j_t\right)\left(\overline{Z}^l_t - c^l Z^l_t\right)\right] \\ &= \mathbf{E}(\overline{Z}^j_t \overline{Z}^l_t) - c^j \mathbf{E}(Z^j_t \overline{Z}^l_t) - c^l \mathbf{E}(\overline{Z}^j_t Z^l_t) + c^j c^l \mathbf{E}(Z^j_t Z^l_t) = 0 \end{aligned}$$

for $j \neq l$, since the components of the Brownian motions Z_t and $\overline{Z}_t$ are independent and, moreover,

$$\mathbf{E}(Z^j_t \overline{Z}^l_t) = 0,$$

as can be easily obtained using (51) and the fact that Z_t and $\overline{W}_t$ are independent Brownian motions. ■

Substituting (51) in (49) we get

$$\begin{cases} dX_t = \tilde{\phi}(X_t)dt + \sqrt{2}\tilde{\sigma}(X_t, Y_t)dW_t, \\ dY_t = \frac{1}{\varepsilon} b(Y_t)dt + \sqrt{\frac{2}{\varepsilon}}\tau(Y_t)dW_t, \end{cases} \tag{54}$$

where $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{\sigma} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{M}^{n,k+m}$ are defined as $\tilde{\phi}^i(x) = \alpha^i x^i$ and $\tilde{\sigma}_{ij}(x, y) = x^i f^j_i(y)$ for $j = 1, \dots, k$ and $\tilde{\sigma}_{ij}(x, y) = 0$ for $j = k+1, \dots, k+m$, while $\tau : \mathbb{R}^m \to \mathbb{R}^{m\times(k+m)}$ is the $(m \times (k+m))$-matrix

$$\tau(y) = \begin{pmatrix} \rho_{11}\nu_1(y) & \cdots & \rho_{k1}\nu_1(y) & c^1\nu_1(y) & 0 & \cdots & 0 \\ \rho_{12}\nu_2(y) & \cdots & \rho_{k2}\nu_2(y) & 0 & c^2\nu_2(y) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{1m}\nu_m(y) & \cdots & \rho_{km}\nu_m(y) & 0 & 0 & 0 & c^m\nu_m(y) \end{pmatrix}. \tag{55}$$

We consider now the matrix $\tau(y)\tau^T(y)$. An easy computation shows that the diagonal terms of this matrix are

$$(\tau(y)\tau^T(y))_{jj} = \nu_j^2(y)\left(\sum_{i=1}^{k} \rho_{ij}^2 + (c^j)^2\right) = \nu_j^2(y)$$

by the definition of c^j in (52). The extra diagonal terms are given by

$$(\tau(y)\tau^T(y))_{jl} = \nu_j(y)\nu_l(y)\left(\sum_{i=1}^{k} \rho_{ij}\rho_{il}\right) = 0$$

by item (iii) in Proposition 6.1. Then the matrix $\tau\tau^T$ is the diagonal matrix

$$\tau(y)\tau^T(y) = \begin{pmatrix} \nu_1^2(y) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_m^2(y) \end{pmatrix}$$

and in particular satisfies (9) by (50).

Observe that the system (54) fits in our basic assumptions of section 2. It includes as a special case the multidimensional option pricing model of [23, sect. 10.6], where each Y_t^i is a standard one-dimensional Ornstein–Uhlenbeck processes. Here we are only assuming, besides standard regularity conditions on b and τ and nondegeneracy (50), that the infinitesimal generator of the process satisfies the Lyapunov-like condition (16).

The problem we consider here is the pricing of a European option given by a nonnegative payoff function g depending on the underlying X^i and by a maturity time T. According to risk-neutral theory, to define a no-arbitrage derivative price we have to use an equivalent martingale measure $\mathbf{P}^*$ under which the discounted stock prices $e^{-rt}X_t^i$ are martingales, where r is the instantaneous interest rate for lending or borrowing money. For a brief review of no-arbitrage price theory in the context of stochastic volatility we refer the reader to [23, sect. 2.5]. The system (54) can be written, under a risk-neutral probability $\mathbf{P}^*$, as

$$\begin{cases} dX_t = rX_t dt + \sqrt{2}\tilde{\sigma}(X_t, Y_t)dW_t^*, \\ dY_t = \frac{1}{\varepsilon}\left[b(Y_t) - \sqrt{\varepsilon}\Lambda(Y_t)\right] dt + \sqrt{\frac{2}{\varepsilon}}\tau(Y_t)dW_t^* \end{cases} \tag{56}$$

for some volatility risk premium $\Lambda(Y)$ chosen by the market and describing the relationship between the physical measure $\mathbf{P}$ under which the stock prices are observed and the risk-neutral measure $\mathbf{P}^*$ (see [23, sect. 10.6] and [24]). In (56) W^* is a $(k+m)$-dimensional standard Brownian motion obtained by an appropriate shift of W, and Λ can be assumed bounded and smooth. In this setting, an European contract has no-arbitrage price given by the formula

$$V^\varepsilon(t,x,y) := \mathbf{E}^*[e^{\lambda(t-T)}g(X_T) \mid X_t = x,\, Y_t = y], \qquad 0 \le t \le T, \tag{57}$$

where $\lambda > 0$ and the payoff function g satisfies (10). When there is only one asset X_t (say $n = 1$ in the system (56)), typically the payoff function g is defined as $g(x) = \max\{(x-K), 0\}$ for call options and $g(x) = \max\{(K-x), 0\}$ for put options, where K is the contracted strike price.

The (linear) HJB equation associated with the price function is

$$\begin{aligned} -V_t^\varepsilon + H_P\left(x, y, D_x V^\varepsilon, D_{xx}^2 V^\varepsilon, \frac{D_{xy}^2 V^\varepsilon}{\sqrt{\varepsilon}}\right) + \lambda V^\varepsilon \\ = \frac{1}{\varepsilon}\left[\mathcal{L}(y, D_y V^\varepsilon, D_{yy}^2 V^\varepsilon) - \sqrt{\varepsilon}\Lambda(y)\cdot D_y V^\varepsilon\right] \end{aligned}$$

in $(0,T)\times\mathbb{R}_+^n\times\mathbb{R}^m$ complemented with the obvious terminal condition

$$V^\varepsilon(T,x,y) = g(x),$$

where

$$H_P(x,y,p,X,Z) := -\operatorname{trace}\left(\tilde\sigma\tilde\sigma^T X\right) - \phi_r \cdot p - 2\operatorname{trace}\left(\tilde\sigma\tau^T Z^T\right)$$

and $\mathcal{L}$ is defined in (14). The prices $V^\varepsilon(t,x,y)$ converge locally uniformly, as $\varepsilon \to 0$, to the unique viscosity solution V of the limit equation (37), due to our convergence result, Theorem 5.1 (see also Remark 5.1 describing the slight modifications to the argument in the proof needed to treat this case). V can be represented as

$$V(t,x) := \mathbf{E}^*\left[e^{\lambda(t-T)} g(X_T) \mid X_t = x\right], \qquad 0 \le t \le T,$$

where μ is the unique invariant measure associated with the fast subsystem (see section 4) and X_t satisfies the averaged *effective* system

$$dX_t = rX_t dt + \sqrt{2}\overline{\sigma}(X_t) dW_t^* \tag{58}$$

whose volatility is the so-called mean historical volatility

$$\overline{\sigma}(x) := \sqrt{\int_{\mathbb{R}^m} \tilde\sigma(x,y)\tilde\sigma^T(x,y)\,d\mu(y)}.$$

Therefore the limit of the pricing problem as $\varepsilon \to 0$ is a new pricing problem for the effective system (58). This convergence result complements and extends a bit section 10.6 of [23] on multidimensional problems.

Let us recall also that $\mu(y)$ is explicitly known in some interesting cases, in particular when the fast variables are an Ornstein–Uhlenbeck process, as in [23]. For instance, if Y_t and $\overline{Z}_t$ are scalar processes, the measure μ has the Gaussian density

$$d\mu(y) = \frac{1}{\sqrt{2\pi\tau^2}} e^{-(y-m)^2/2\tau^2}\,dy,$$

with the notations of Example 2.1.

6.2. Merton portfolio optimization problem. We consider now another classical problem in finance, the Merton optimal portfolio allocation, under the assumption of fast oscillating stochastic volatility.

We consider a financial market consisting of a nonrisky asset X^0 evolving according to the deterministic equation $dX_t^0 = rX_t^0 dt$, with $r > 0$, and n risky assets X_t^i evolving according to the stochastic system (54). We denote by $\mathcal{W}$ the wealth of an investor. The investment policy—which will be the control input—is defined by a progressively measurable process u taking values in a compact set U, and u_t^i represents the proportion of wealth invested in the asset X_t^i at time t. Then the wealth process evolves according to the following system:

$$\begin{cases} d\mathcal{W}_t = \mathcal{W}_t\left(r + \sum_{i=1}^n (\alpha^i - r)u_t^i\right)dt + \sqrt{2}\mathcal{W}_t \sum_{i=1}^n u_t^i f_i(Y_t)\cdot d\overline{W}_t, \quad \mathcal{W}_{t_o} = w > 0, \\ dY_t = \frac{1}{\varepsilon} b(Y_t) dt + \sqrt{\frac{2}{\varepsilon}}\nu(Y_t) d\overline{Z}_t, \end{cases} \tag{59}$$

with the same notations and assumptions as in section 6.1. Also this system is a special case of (6), now with a one-dimensional slow state variable $\mathcal{W}_t$, and it satisfies the assumptions of section 2.

The Merton problem consists in choosing a strategy $u.$ which maximizes a given utility function g at some final time T. In particular the problem can be described in terms of the value function

$$V^\varepsilon(t,w,y) := \sup_{u.\in\mathcal{U}} \mathbf{E}[g(\mathcal{W}_T, Y_T) \mid \mathcal{W}_t = w,\, Y_t = y]. \tag{60}$$

Typically the utility functions in financial applications are chosen in the class of HARA (hyperbolic absolute risk aversion) functions $g(w,y) = a(bw+c)^\gamma$, where a, b, c are bounded and continuous given functions of y, and $\gamma \in (0,1)$ is a given coefficient called the relative risk premium coefficient. Observe that the function g satisfies assumption (10).

We remark also that in the classical HARA functions typically a, b, c are constants. We choose to consider y dependent coefficients since our method also permits us to manage this general case and, moreover, utilities of such a form are employed in the pricing of derivatives with nontraded assets (see [44]).

The HJB equation associated with the Merton value function is

$$-V_t^\varepsilon + H_M\left(w, y, V_w^\varepsilon, V_{ww}^\varepsilon, \frac{D_y V_w^\varepsilon}{\sqrt{\varepsilon}}\right) - \frac{1}{\varepsilon}\mathcal{L}(y, D_y V^\varepsilon, D_{yy}^2 V^\varepsilon) = 0 \tag{61}$$

in $(0,T)\times\mathbb{R}_+\times\mathbb{R}^m$ complemented with the terminal condition $V^\varepsilon(T,x,y) = g(x,y)$. In (61) $\mathcal{L}$ is as in (14) and $H_M(w,y,p,X,Z)$ is defined as

$$\inf_{u\in U}\left\{ -\left[r + \sum_{i=1}^n (\alpha^i - r)u^i\right] wp - \sum_{j=1}^k \left|\sum_{i=1}^n u^i f_i^j(y)\right|^2 w^2 X \right.$$
$$\left. - 2\sum_{h=1}^m \sum_{j=1}^k \sum_{i=1}^n u^i f_i^j(y)\tau_{hj}(y) w Z_h \right\},$$

with the matrix τ given by (55). Our main theorem (Theorem 5.1) applies also in this case and says that the value function V^ε converges locally uniformly to the unique solution of the limit problem

$$\begin{cases} -V_t + \int_{\mathbb{R}^m} H_M(w,y,V_w,V_{ww},0)\,d\mu(y) = 0 & \text{for } t\in(0,T),\quad w>0,\\ V(T,w) = \int_{\mathbb{R}^m} g(w,y)\,d\mu(y) & \text{for } w>0, \end{cases} \tag{62}$$

where $\mu(y)$ is the invariant measure associated with the fast subsystem (15).

This convergence result is new even in the case of a single risky asset and g independent of y that is studied in [23]. Next we interpret it in terms of stochastic control.

For simplicity we restrict ourselves to the case of a single risky asset and a scalar fast process Y_t, i.e., $n = m = 1$. The equation for the wealth becomes

$$d\mathcal{W}_t = \mathcal{W}_t\,(r + (\alpha - r)u_t)\,dt + \sqrt{2}\mathcal{W}_t u_t f(Y_t)\cdot d\overline{W}_t, \qquad \alpha > r, \tag{63}$$

and the HJB equation for V^ε is

$$-\frac{\partial V^\varepsilon}{\partial t} - \sup_{u\in U}\left\{[r+(\alpha-r)u]\,w\frac{\partial V^\varepsilon}{\partial w} + u^2|f|^2w^2\frac{\partial^2 V^\varepsilon}{\partial w^2} + \frac{2uw}{\sqrt{\varepsilon}}\sum_{j=1}^{k}\rho_j f^j\nu\frac{\partial^2 V^\varepsilon}{\partial w\partial y}\right\} = \frac{1}{\varepsilon}\mathcal{L}\left(y, \frac{\partial V^\varepsilon}{\partial y}, \frac{\partial^2 V^\varepsilon}{\partial y^2}\right), \tag{64}$$

where ρ_j is the correlation factor between $\overline{Z}_t^j$ and $\overline{W}_t$; see (51). The effective PDE is

$$-\frac{\partial V}{\partial t} - \int_{\mathbb{R}^m}\max_{u\in U}\left\{[r+(\alpha-r)u]\,w\frac{\partial V}{\partial w} + u^2|f(y)|^2w^2\frac{\partial^2 V}{\partial w^2}\right\}d\mu(y) = 0. \tag{65}$$

Effective utility. Note that since the utility depends also on y, we have an initial boundary layer. The effective utility $\overline{g}$ can be interpreted as an averaged utility which is robust with respect to fast mean reverting fluctuations and uncertainty in the market (depending also, e.g., on nontraded assets). If g is independent of y, then the convergence is uniform up to time T.

Solution of the effective Cauchy problem. In some cases the effective Cauchy problem (62) can be solved explicitly. As a constraint on the control u_t we take the interval

$$U := [R_1, R], \quad \text{with } -R \le R_1 \le 0 < R.$$

We also assume that the terminal cost is the HARA function

$$g(w,y) = a(y)\frac{w^\gamma}{\gamma}, \quad 0<\gamma<1, \quad a(y)\ge a_o > 0.$$

Then the terminal condition in (62) is

$$V(T,w) = \overline{a}\frac{w^\gamma}{\gamma}, \qquad \overline{a} := \int_{\mathbb{R}^m} a(y)\,d\mu(y),$$

and we look for solutions of (62) of the form $V(t,w) = \frac{w^\gamma}{\gamma}v(t)$ with $v(t)\ge 0$. By plugging it into the Cauchy problem we get

$$\dot v = -\gamma\overline{h}v, \quad v(T) = \overline{a}, \quad \overline{h} := r + \int_{\mathbb{R}^m}\max_{u\in U}\left[(\alpha-r)u + (\gamma-1)|f(y)|^2u^2\right]d\mu(y).$$

Therefore the uniqueness of solution to (62) gives

$$V(t,w) = \overline{a}e^{\gamma\overline{h}(T-t)}\frac{w^\gamma}{\gamma}, \qquad 0<t<T. \tag{66}$$

We compute the rate of exponential increase $\overline{h}$ and get

$$\overline{h} = r + \int_{\{y\,:\,2R(1-\gamma)|f(y)|^2<\alpha-r\}}\left[(\alpha-r)R + (\gamma-1)R^2|f(y)|^2\right]d\mu(y) + \int_{\{y\,:\,2R(1-\gamma)|f(y)|^2\ge\alpha-r\}}\frac{(\alpha-r)^2}{4(1-\gamma)|f(y)|^2}\,d\mu(y).$$

The limit is a Merton problem. It is interesting to compare this solution with the value function of the Merton problem with constant volatility $\sigma > 0$, where the wealth dynamics is

$$d\mathcal{W}_t = \mathcal{W}_t \left(r + (\alpha - r) u_t \right) dt + \sqrt{2} \mathcal{W}_t u_t \sigma d\overline{W}_t$$

and the utility function is aw^γ/γ.

In the case $2R(1-\gamma)\sigma \geq \alpha - r$ (in particular, for a large or no upper bound on the control) the value function is given by the classical Merton formula

$$a \exp\left[\gamma\left(r + \frac{(\alpha-r)^2}{4(1-\gamma)\sigma^2}\right)(T-t)\right]\frac{w^\gamma}{\gamma}. \tag{67}$$

It coincides with the solution (66) of the effective HJB equation (65) with terminal condition $\overline{g} = \overline{a}w^\gamma/\gamma$ if and only if $a = \overline{a}$ and

$$\sigma = \overline{\sigma} := \frac{\alpha - r}{2\sqrt{(1-\gamma)(\overline{h} - r)}}.$$

Therefore these are the correct parameters to use in a Merton model with constant volatility if we consider it as an approximation of a model with fast and ergodic stochastic volatility. We can call it the *effective Merton model*.

The effective volatility. The preceding formula for the effective volatility $\overline{\sigma}$ simplifies considerably if the μ-probability of the set $\{y : 2R(1-\gamma)|f(y)|^2 \geq \alpha - r\}$ is 1, e.g., for large upper bound R on the control. In fact we get

$$\overline{\sigma} = \left(\int_{\mathbb{R}^m} \frac{1}{|f(y)|^2}\, d\mu(y)\right)^{-\frac{1}{2}},$$

a formula derived in section 10.1.2 of [23] in the case of unconstrained controls ($R = +\infty$).

We remark that $\overline{\sigma}$ for the Merton problem is the harmonically averaged long-run volatility that is smaller than the mean historical volatility derived in section 6.1 for uncontrolled systems. Therefore using the correct parameter in the model leads to an increase of the value function, i.e., of the optimal expected utility.

The limit of the optimal control. Consider the effective Merton problem ($a = \overline{a}$, $\sigma = \overline{\sigma}$) and suppose the upper bound R on the control large enough to allow all the usual calculations of the case $R = +\infty$. The control where the Hamiltonian attains the maximum is

$$u^* := \frac{\alpha - r}{2(1-\gamma)\overline{\sigma}^2} = \frac{\alpha - r}{2(1-\gamma)} \int_{\mathbb{R}^m} \frac{1}{|f(y)|^2}\, d\mu(y),$$

which is then the optimal control. We want to compare it with the optimal control for the problem with $\varepsilon > 0$. For the terminal condition $V^\varepsilon(T, w, y) = a(y)w^\gamma/\gamma$ we expect a solution of (64) of the form $V^\varepsilon(t, w, y) = v^\varepsilon(t, y)w^\gamma/\gamma$. Then we can compute the maximum in the Hamiltonian of (64) and get

$$u^*_\varepsilon(t, y) = \frac{\alpha - r}{2(1-\gamma)|f(y)|^2} + \frac{\Phi(y)}{\sqrt{\varepsilon}v^\varepsilon(t, y)}\frac{\partial v^\varepsilon}{\partial y}(t, y), \qquad \Phi(y) := \frac{\sum_{j=1}^k \rho_j f^j(y)\nu(y)}{(1-\gamma)|f(y)|^2}. \tag{68}$$

By our main theorem $v^\varepsilon(t,y) \to v(t)$ locally uniformly in $[0,T) \times \mathbb{R}$ as $\varepsilon \to 0$, so $\frac{\partial v^\varepsilon}{\partial y}(t,y) \to 0$ in the sense of distributions with respect to y, locally uniformly in $t < T$. Then we wonder if the second term of u^*_ε vanishes in some sense, despite the $\sqrt{\varepsilon}$ at the denominator, therefore giving

$$\lim_{\varepsilon\to 0} u^*_\varepsilon(t,y) = \frac{\alpha - r}{2(1-\gamma)|f(y)|^2} =: u^*_0(y). \tag{69}$$

Note that the candidate limit u^*_0 is different from u^*, but $u^* = \int_{\mathbb{R}^m} u^*_0(y)\, d\mu(y)$.

Let us assume for simplicity that

$$\mu \text{ has a density } \varphi \in C^1 \quad \text{and} \quad \lim_{|y|\to\infty} \varphi(y) = 0. \tag{70}$$

The former assumption is satisfied, for instance, if the coefficients b, ν of $\mathcal{L}$ are smooth, because $\mathcal{L}^*\mu = 0$ in the sense of distributions and the regularity theory for elliptic equations applies ($\mathcal{L}^*$ being the formal adjoint of $\mathcal{L}$). The latter assumption is natural for an integrable φ, and it is satisfied, for instance, by the Ornstein–Uhlenbeck process (φ is a Gaussian function). Then, when we take the integral of (68) with respect to μ and integrate by parts the second term, we get

$$\int_{\mathbb{R}^m} u^*_\varepsilon(t,y)\, d\mu(y) = u^* + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad \text{as } \varepsilon \to 0,$$

which again is not very insightful. To get some convergence we write an asymptotic expansion for $v^*_\varepsilon(t,y)$ in powers of $\sqrt{\varepsilon}$, in the spirit of section 10.1.2 of the book by Fouque, Papanicolaou, and Sircar [23] but under weaker assumptions and using different arguments.

Proposition 6.2. *Besides the standing assumptions of the section and* (70) *suppose*

$$v^\varepsilon(t,y) = v(t) + \sqrt{\varepsilon} v^\varepsilon_1(t,y), \;\; v^\varepsilon_1(t,y) \to v_1(t,y) \quad \textit{locally uniformly}, \; v_1 \textit{ bounded}. \tag{71}$$

Then the following hold:

(i) $v_1 = v_1(t)$, *so* $\frac{1}{\sqrt{\varepsilon}}\frac{\partial v^\varepsilon}{\partial y} = \frac{\partial v^\varepsilon_1}{\partial y}(t,y) \to 0$ *in the sense of distributions with respect to* y*;*

(ii) *if, in addition,*

$$|v^\varepsilon_1| \le C, \qquad \sqrt{\varepsilon}\int_{\mathbb{R}^m} \left|\frac{\partial v^\varepsilon_1}{\partial y}\right| d\mu(y) \to 0 \qquad \forall\, t < T, \tag{72}$$

then

$$u^* = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^m} u^*_\varepsilon(t,y)\, d\mu(y) \qquad \forall\, t < T; \tag{73}$$

(iii) *if, in addition,*

$$v^\varepsilon_1(t,y) = v_1(t) + \omega(\varepsilon) v^\varepsilon_2(t,y), \quad \omega(\varepsilon) \to 0, \quad \left|\frac{\partial v^\varepsilon_2}{\partial y}(t,y)\right| \le C(t,y), \tag{74}$$

then $\frac{1}{\sqrt{\varepsilon}}\frac{\partial v^\varepsilon}{\partial y} \to 0$ *and* (69) *holds uniformly on every set where* $C(\cdot,\cdot)$ *is bounded.*

Proof. (i) By plugging the optimal control (68) into the HJB equation (64) we get

$$-\frac{\partial v^\varepsilon}{\partial t} - \gamma r v^\varepsilon - F_1(y)\left((\alpha - r)v^\varepsilon + \frac{F_2(y)}{\sqrt{\varepsilon}}\frac{\partial v^\varepsilon}{\partial y}\right)^2 = \frac{1}{\varepsilon}\mathcal{L}\left(y, \frac{\partial v^\varepsilon}{\partial y}, \frac{\partial^2 v^\varepsilon}{\partial y^2}\right)$$

for suitable continuous F_i, $i = 1, 2$. Using the expansion (71) the equation becomes

$$-\mathcal{L}\left(y, \frac{\partial v_1^\varepsilon}{\partial y}, \frac{\partial^2 v_1^\varepsilon}{\partial y^2}\right) = \sqrt{\varepsilon}\left[\frac{\partial v^\varepsilon}{\partial t} + \gamma r v^\varepsilon + F_1(y)\left((\alpha - r)v^\varepsilon + F_2(y)\frac{\partial v_1^\varepsilon}{\partial y}\right)^2\right].$$

Letting $\varepsilon \to 0$ we obtain, by standard properties of viscosity solutions,

$$-\mathcal{L}\left(y, \frac{\partial v_1}{\partial y}, \frac{\partial^2 v_1}{\partial y^2}\right) = 0 \quad \text{in } \mathbb{R},$$

so v_1 is constant with respect to y by the Liouville property, Lemma 4.1.

(ii) First observe that v^ε is uniformly bounded and bounded away from 0. The upper bound follows from (19). The lower bound is obtained by using the definition (60) of V^ε and computing the payoff of the control $u. \equiv 0$. We get

$$V^\varepsilon(t, w, y) \geq \mathbf{E}[a(Y_T) \mid Y_t = y]e^{\gamma r(T-t)}\frac{w^\gamma}{\gamma}$$

and therefore

$$v^\varepsilon(t, y) \geq a_o e^{\gamma r(T-t)} \geq a_o \qquad \forall\, t \leq T,\ \forall\, y.$$

From (68) and the expansion (71) we get

$$\int_{\mathbb{R}^m} u_\varepsilon^*(t, y)\, d\mu(y) = u^* + \int_{\mathbb{R}^m} \frac{\Phi(y)}{v^\varepsilon(t, y)}\frac{\partial v_1^\varepsilon}{\partial y}(t, y)\varphi(y)\, dy.$$

Integrating by parts, the integral on the right-hand side becomes

$$-\int_{\mathbb{R}^m} \frac{\partial}{\partial y}\left(\frac{\Phi\varphi}{v^\varepsilon}\right) v_1^\varepsilon\, dy + \left[\Phi\varphi\frac{v_1^\varepsilon}{v^\varepsilon}\right]_{y\to-\infty}^{y\to+\infty}$$

and the second term is null by (70) and the uniform boundedness of $\Phi v_1^\varepsilon / v^\varepsilon$. The first term can be written as

$$-\int_{\mathbb{R}^m} \frac{\partial\,(\Phi\varphi)}{\partial y}\frac{v_1^\varepsilon}{v^\varepsilon}\, dy + \int_{\mathbb{R}^m} \sqrt{\varepsilon}\frac{\partial v_1^\varepsilon}{\partial y}\frac{\Phi v_1^\varepsilon}{(v^\varepsilon)^2}\varphi\, dy,$$

and we let $\varepsilon \to 0$: the second integral vanishes by (72) and the uniform boundedness of $\Phi v_1^\varepsilon/(v^\varepsilon)^2$, whereas the first converges to

$$-\frac{v_1(t)}{v(t)}\int_{\mathbb{R}^m} \frac{\partial\,(\Phi\varphi)}{\partial y}(y)\, dy = 0$$

by (70). This completes the proof of (73).

(iii) By (74)

$$\frac{1}{\sqrt{\varepsilon}}\frac{\partial v^\varepsilon}{\partial y} = \omega(\varepsilon)\frac{\partial v_2^\varepsilon}{\partial y}(t,y) \to 0$$

uniformly on every set where $\partial v_2^\varepsilon/\partial y$ is uniformly bounded. By (68) u_ε^* converges uniformly on every such set to u_0^*. ■

We can roughly summarize the preceding proposition by saying that an asymptotic expansion of v^ε of the form

$$v^\varepsilon = v + \sqrt{\varepsilon}v_1 + o(\sqrt{\varepsilon})v_2^\varepsilon$$

implies that the optimal control u^* of the effective Merton model is the limit of the averages and the average of the limit of the optimal controls for the models with $\varepsilon > 0$, i.e.,

$$u^* = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^m} u_\varepsilon^*(t,y)\,d\mu(y) = \int_{\mathbb{R}^m}\lim_{\varepsilon\to 0} u_\varepsilon^*(t,y)\,d\mu(y).$$

The financial interpretation of this statement is clear: the optimal control for the Merton problem with constant volatility $\overline{\sigma}$ approximates the expectation of the optimal control for the same problem with stochastic volatility, provided the volatility evolves much faster than the assets.

6.3. Periodic day effects and volatility with a slow component. Section 10.2 of [23] discusses a refinement of the model in section 6.1, where the volatilities of the prices depend on time on a fast periodic scale, thus modeling the daily oscillations. This amounts to replacing $f_i(Y_t)$ in (49) and (59) with

$$f_i = f_i\left(\frac{t}{\varepsilon}, Y_t\right),$$

where f_i is 1-periodic in the first entry. We incorporate this in our setting by adding the new variable $s := t/\varepsilon$ whose dynamics is $\dot{s} := 1/\varepsilon$. The fast subsystem now has the additional variable s_t that is trivially ergodic on the unit circle with respect to the Lebesgue measure. Now the effective Hamiltonian of the limit PDE is

$$\overline{H} = \int_0^1\int_{\mathbb{R}^m} H(x,y,s,p,X,0)\,d\mu(y)\,ds.$$

Another possible extension of the model in sections 6.1 and 6.2 is the addition of another stochastic quantity Z_t affecting the volatilities of the prices and evolving on a slower time scale than the prices:

$$\begin{aligned} &f_i = f_i(Y_t, Z_t),\\ &dZ_t = \theta c(Z_t)dt + \sqrt{\theta}d(Z_t)dW_t, \qquad Z_0 = z, \end{aligned} \tag{75}$$

with θ small, and c, d Lipschitz and growing at most linearly at infinity. This is done, for instance, in [25, 37]. This modeling allows much more flexibility and is motivated by various empirical studies (see [25] and the references therein) which outline a volatility composed by one highly persistent factor and one quickly mean reverting factor. The slow volatility factor in particular is useful when considering options with longer maturities.

The value function now depends also on the initial position z of the new variable Z_t, and the HJB equation (12) becomes

$$\lambda V - V_t + H\left(x, y, z, D_x V, D^2_{xx} V, \frac{D^2_{xy} V}{\sqrt{\varepsilon}}, \sqrt{\theta} D^2_{xz} V\right) - \frac{1}{\varepsilon}\mathcal{L}(y, D_y V, D^2_{yy} V)$$
$$- \theta[c \cdot D_z V + \theta\, \mathrm{trace}(dd^T D^2_{zz} V)] - \sqrt{\frac{\theta}{\varepsilon}}\, \mathrm{trace}\left[\tau d^T D^2_{yz} V + D^2_{yz} V \tau d^T(z)\right] = 0.$$

In particular this can be seen as a *regular perturbation* of (12). If θ is independent of ε and we let it tend to 0, the basic properties of viscosity solutions give the convergence of the value function $V^{\varepsilon,\theta}(t, x, y, z)$ to the solution $V(t, x, z)$ of the same effective Cauchy problem as before, with the only difference that $\overline{H}$ now depends also on z (but z appears only as a fixed parameter in the limit PDE). It possible to check this result regardless of the order of taking the limits $\theta \to 0$ and $\varepsilon \to 0$. Indeed the term $-\sqrt{\frac{\theta}{\varepsilon}}\, \mathrm{trace}\left[\tau(y) d^T(z) D^2_{yz} V + D^2_{yz} V \tau(y) d^T(z)\right]$ is a lower order term with respect to $\frac{1}{\varepsilon}\mathcal{L}(y, D_y V, D^2_{yy} V)$, and then a similar argument as in Remark 5.1 holds. If, instead, $\theta = \theta(\varepsilon)$, the same conclusion follows with a much more delicate argument, following a theorem on regular perturbations of singular perturbation problems proved in [4].

Of course the periodic oscillations in time and the slow component of the volatility can also be treated simultaneously. As an example, we consider the scalar Merton problem (60), (63) with volatility and utility functions given by

$$f_i = f_i\left(\frac{t}{\varepsilon}, Y_t, Z_t\right), \qquad g = a(Y_T, Z_T)\frac{\mathcal{W}_T^{\gamma}}{\gamma},$$

with Z_t satisfying (75). Then the value function $V^{\varepsilon,\theta}(t, x, y, z)$ converges locally uniformly to the classical Merton formula (67) for the problem with constant volatility

$$\sigma = \overline{\sigma}(z) := \left(\int_0^1 \int_{\mathbb{R}^m} \frac{1}{|f(s, y, z)|^2}\, d\mu(y)\, ds\right)^{-\frac{1}{2}},$$

at least when the upper bound R on the controls is large enough, and

$$a = \overline{a}(z) := \int_{\mathbb{R}^m} a(y, z)\, d\mu(y).$$

6.4. Worst case optimization under unknown disturbances. Assume that the general stochastic control system (8) is affected by an additional disturbance $\tilde{u}_t$ taking values in a compact set $\tilde{U}$, and suppose you want to maximize the payoff under the worst possible behavior of $\tilde{u}_t$. There are several possible reasons for this choice, such as the lack of statistical information on the disturbance or the desire to avoid with probability one some catastrophic events caused by a particularly nasty behavior of $\tilde{u}_t$. The mathematical framework for modeling these problems is the theory of two-person 0-sum differential games, where the controller is the first player and the disturbance is considered as the control of a second player wishing to minimize the payoff.

For simplicity we suppose the following form of the drift and diffusion in (6):

$$\phi^i = \phi_1^i(x,y,u) + \phi_2^i(x,y,\tilde{u}), \qquad \sigma^i = \sigma_1^i(x,y,u) + \sigma_2^i(x,y,\tilde{u}),$$

with ϕ_j^i, σ_j^i bounded, continuous, and Lipschitz in (x,y) uniformly in $u, \tilde{u}$. For the system written in vector form (8) we then have $\tilde{\phi}^i = \tilde{\phi}_1^i(x,y,u) + \tilde{\phi}_2^i(x,y,\tilde{u})$ and $\tilde{\sigma}^i = \tilde{\sigma}_1^i(x,y,u) + \tilde{\sigma}_2^i(x,y,\tilde{u})$ with the obvious definitions. The Isaacs equation associated with the game is again of the form (12), but now the Hamiltonian is $H = H_1 + H_2$ with

$$H_1(x,y,p,X,Z) := \min_{u\in U}\left\{-\operatorname{trace}\left(\tilde{\sigma}_1\tilde{\sigma}_1^T X\right) - \tilde{\phi}_1\cdot p - 2\operatorname{trace}\left(\tilde{\sigma}_1\tau^T Z^T\right)\right\},$$
$$H_2(x,y,p,X,Z) := \max_{\tilde{u}\in\tilde{U}}\left\{-\operatorname{trace}\left(\tilde{\sigma}_2\tilde{\sigma}_2^T X\right) - \tilde{\phi}_2\cdot p - 2\operatorname{trace}\left(\tilde{\sigma}_2\tau^T Z^T\right)\right\}.$$

The precise definition of value function is more delicate for a stochastic differential game, as well as the proof that it is a viscosity solution of (12), and we refer the reader to [21]. We remark that the comparison principle of [15] still holds for the Cauchy problem (17) with the new convex-concave Hamiltonian, and therefore there is a unique viscosity solution V^ε. The convergence theorem (Theorem 5.1) of $V^\varepsilon(t,x,y)$ to $V(t,x)$ holds with no changes, because its proof never uses the convexity of H with respect to (p,X). The effective Hamiltonian now is

$$\overline{H} = \int_{\mathbb{R}^m} \min_{u\in U}\left\{-\operatorname{trace}\left(\tilde{\sigma}_1\tilde{\sigma}_1^T X\right) - \tilde{\phi}_1\cdot p\right\} + \max_{\tilde{u}\in\tilde{U}}\left\{-\operatorname{trace}\left(\tilde{\sigma}_2\tilde{\sigma}_2^T X\right) - \tilde{\phi}_2\cdot p\right\} d\mu(y).$$

6.5. Applications to problems with degenerate diffusion. We pointed out in the introduction that we do not make any nondegeneracy assumption on the diffusion matrix $\sigma\sigma^T$ for the slow variables X_t. This makes our methods applicable to a wide range of models, even in deterministic control, if one wants to study the sensitivity to random parameters evolving on a fast time scale. For instance, some differential games arising in marketing and advertising are under investigation.

Within mathematical finance, path-dependent models, such as Asian options, involve degenerate diffusion processes; see [42, 8] and the references therein. In these models one augments the state space by a new variable A_s that is the time integral of some functions of a price S_s. Therefore an ODE is added to the system, such as $dA_s = S_s\,ds$ for problems involving the arithmetic mean of the prices or $dA_s = \log(S_s)\,ds$ for the geometric mean. Therefore the process $X_s = (S_s, A_s)$ is a degenerate diffusion. Models of Asian options with fast stochastic volatility are studied in Chapter 8.3 of [25] and in [22, 43].

Interest rate models are another area where the uniform nondegeneracy of the diffusion matrix would not be a reasonable assumption. The LIBOR models with stochastic volatility reviewed in Chapter 11 of [13] all have a volatility function $\sigma(X_s, Y_s)$ vanishing at $Y_s = 0$. This event usually has null probability, by the choice of the dynamics for Y_s. So the associated PDE is parabolic but not uniformly parabolic. Some of these models with two time scales are studied in Chapter 11 of [25].

A stronger form of degeneracy occurs in the Heath–Jarrow–Morton (HJM) framework for forward rate models, where there are an infinite number of traded assets (one for each maturity) and a finite number of sources of randomness (components of the Brownian motion); see, e.g., Chapter 23 of [10]. The possibility of arbitrage is ruled out by the HJM drift condition. If one considers a large but finite number of maturities, the assets evolve as a degenerate diffusion and our methods can be used for the asymptotics of the fast stochastic volatility problem. HJM models with stochastic volatility (with the same time scale as the prices) were studied in [11].

7. Conclusion. In this paper we study stochastic control problems with random parameters driven by a fast ergodic process. Our methods are based on viscosity solutions theory and the Hamilton–Jacobi approach to singular perturbations. The assumptions are chosen to fit problems of pricing derivative securities and optimizing the portfolio allocation in financial markets with fast mean reverting stochastic volatility.

The main steps of our HJB approach to singular perturbations are the following:

- Write the HJB equation for the value function V^ε and characterize it as the unique viscosity solution of the Cauchy problem for such an equation (see section 3).
- Define a limit (effective) PDE and limit (effective) initial data resolving appropriate ergodic-type problems (see section 4).
- Prove the (locally) uniform convergence of V^ε to a function V, which can be characterized as the unique solution of the effective Cauchy problem (see section 5).
- Interpret the effective PDE as the HJB equation for a limit (effective) control problem. Such a problem approximates the one with $\varepsilon > 0$, and it has lower-dimensional state variables; therefore it is easier to solve. There is no general recipe for this step, and we do it in section 6 for a multidimensional option pricing model and for the Merton portfolio optimization problem.

The main contributions of the present paper are the following. On the mathematical side we extend the HJB approach from the setting of periodic fast variables (see [1, 2, 3] and the references therein) to the case of unbounded fast variables. The probabilistic literature on singular perturbations in stochastic control (see the monographs [32, 33] and the references therein) allows unbounded fast variables but makes other restrictive assumptions that rule out some financial models such as the Merton optimization problem (e.g., in [33] the diffusion matrix $\tilde{\sigma}$ is assumed uncontrolled).

On the side of financial models our approach complements the methods of Fouque, Papanicolaou, and Sircar [23]. They assume an asymptotic expansion for V^ε of the form

$$V^\varepsilon = V + \sqrt{\varepsilon} V_1 + \varepsilon V_2 + \cdots , \tag{76}$$

plug it into the HJB PDE for V^ε, set equal to 0 each term multiplying a power of ε, and solve iteratively such PDEs to compute the correctors V_i. This gives information not only on the limit but also for ε positive with various orders of magnitude. The validity of the expansion can be proved in some problems without control; this is done, for instance, in [24] for the option pricing of a single asset. Our result in section 6.1 complements it by treating the multiasset problem but only up to the first term of the expansion. Since the PDE is linear we believe that the arguments can be carried on to study further terms, but we do not try to do it here.

For problems with controls, however, the validity of the asymptotic expansion (76) is not known, even for particular problems like Merton, and presumably it is not true in general. Section 10.1 of [23] assumes (76) for the Merton problem and gets some interesting insight on the correction of the optimal control. Our contribution in section 6.2 is a rigorous proof of the locally uniform convergence of the value function with stochastic volatility to the value of the Merton problem with constant effective volatility $\overline{\sigma}$ (instead of the historical volatility)

$$\lim_{\varepsilon\to 0} V^\varepsilon(t,w,y) = V(t,w), \qquad \overline{\sigma}^2 = \int_{\mathbb{R}^m} \frac{1}{|f(y)|^2}\, d\mu(y)$$

and also for utility functions depending on the fast variable y. The problem of justifying further terms of the asymptotic expansion is wide open in stochastic control and fully nonlinear PDEs, even for the first corrector V_1. The only related result we know is in the very recent paper by Camilli and Marchi [14] and concerns the rate of convergence in periodic homogenization. We plan to study this issue for particular models arising in applications. As for the convergence of the optimal control, at the end of section 6.2 we assume the expansion

$$V^\varepsilon = V + \sqrt{\varepsilon} V_1 + o(\sqrt{\varepsilon}) V_2^\varepsilon$$

and prove that

$$u^* = \lim_{\varepsilon\to 0} \mathbf{E}\left[u_\varepsilon^*(t,Y)\right] = \mathbf{E}\left[\lim_{\varepsilon\to 0} u_\varepsilon^*(t,Y)\right], \qquad Y \sim \mu,$$

which has a clear financial interpretation.

Finally, we remark that our method is very general and can be used for a number of models, financial or not, including 0-sum differential games and degenerate diffusions. The case of controls appearing also in the fast variables was studied in [1, 2, 3] and the references therein when the fast variables are bounded; see also [12]. We plan to push the methods of the present paper further and treat problems with controlled and unbounded fast variables.

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