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A singularly perturbed nonlinear traction boundary value problem for linearized elastostatics. A functional analytic approach

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Dedicated to Professor Heinrich Begehr on the occasion of his 70th birthday.

Summary: In this paper, we consider two bounded open subsets of Ω^i and Ω^o of $\mathbb{R}^n$ containing 0 and a (nonlinear) function G^o of $\partial\Omega^o \times \mathbb{R}^n$ to $\mathbb{R}^n$, and a map T of $]1-(2/n), +\infty[$ times the set $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries to $M_n(\mathbb{R})$, and we consider the problem

$$\begin{cases} \mathrm{div}\,(T(\omega, Du)) = 0 & \text{in } \Omega^o \setminus \epsilon\mathrm{cl}\Omega^i, \\ -T(\omega, Du)\nu_{\epsilon\Omega^i} = 0 & \text{on } \epsilon\partial\Omega^i, \\ T(\omega, Du(x))\nu^o(x) = G^o(x, u(x)) & \forall x \in \partial\Omega^o, \end{cases}$$

where $\nu_{\epsilon\Omega^i}$ and ν^o denote the outward unit normal to $\epsilon\partial\Omega^i$ and $\partial\Omega^o$, respectively, and where $\epsilon > 0$ is a small parameter. Here $(\omega - 1)$ plays the role of ratio between the first and second Lamé constants, and $T(\omega, \cdot)$ plays the role of (a constant multiple of) the linearized Piola Kirchhoff stress tensor, and G^o plays the role of (a constant multiple of) a traction applied on the points of $\partial\Omega^o$. Then we prove that under suitable assumptions the above problem has a family of solutions $\{u(\epsilon, \cdot)\}_{\epsilon \in]0,\epsilon'[}$ for ϵ' sufficiently small and we show that in a certain sense $\{u(\epsilon, \cdot)\}_{\epsilon \in]0,\epsilon'[}$ can be continued real analytically for negative values of ϵ.

1 Introduction

In this paper, we consider a linearly elastic homogeneous isotropic body with a small hole subject to a traction free boundary condition on the boundary of the hole and to an external traction depending nonlinearly on the deformation on the outer boundary of the body.

We assume that the constitutive relations of our body are expressed by means of the linearized tensor $T(\omega, \cdot)$ defined by

$$T(\omega, A) \equiv (\omega - 1)(\mathrm{tr}\, A)I + (A + A^t) \qquad \forall A \in M_n(\mathbb{R}),$$

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where $\omega \in]1-(2/n), +\infty[$ is a parameter such that $(\omega-1)$ plays the role of ratio between the first and second Lamé constants, $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries, I denotes the identity matrix, tr A and A^t denote the trace and the transpose matrix of the matrix A, respectively. We also note that the classical linearization of the Piola Kirchhoff tensor equals the second Lamé constant times $T(\omega, \cdot)$.

First we introduce a problem in the case in which the body has no hole, and then we shall consider the problem with the hole, which is the goal of this paper.

We assume that the body with no hole occupies an open bounded connected subset Ω^o of $\mathbb{R}^n$ of class $C^{m,\alpha}$ for some $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0, 1[$ such that $0 \in \Omega^o$ and such that the exterior of Ω^o is also connected.

Then we assign a function G^o of $\partial\Omega^o \times \mathbb{R}^n$ to $\mathbb{R}^n$ which plays the role of the reciprocal of the second Lamé constant times a field of forces applied to the boundary of the body and depending both on the point on $\partial\Omega^o$ and on the deformation of the body, and we assume that the nonlinear traction boundary value problem

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \Omega^o, \\ T(\omega, Du(x))\nu^o(x) = G^o(x, u(x)) & \forall x \in \partial\Omega^o, \end{cases} \tag{1.1}$$

where ν^o denotes the outward unit normal to $\partial\Omega^o$, admits at least a solution $\tilde{u}$ in the space $C^{m,\alpha}(\mathrm{cl}\Omega^o, \mathbb{R}^n)$, and that the gradient matrix $D_u G^o(x, \tilde{u}(x))$ with respect to the second variable of G^o satisfies certain nondegeneracy conditions (see (3.13)). A classical argument based on the use of topological degree shows that solutions as $\tilde{u}$ exist under reasonable conditions on G^o.

Next we make a hole in the body Ω^o. Namely, we consider another bounded open connected subset Ω^i of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $0 \in \Omega^i$ and such that the exterior of Ω^i is also connected, and we take $\epsilon_0 > 0$ such that $\epsilon\mathrm{cl}\Omega^i \subseteq \Omega^o$ for $|\epsilon| < \epsilon_0$, and we consider the perforated domain

$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\mathrm{cl}\Omega^i.$$

Obviously, $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$. For each $\epsilon \in]0, \epsilon_0[$, we consider the nonlinear traction boundary value problem

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \Omega(\epsilon), \\ -T(\omega, Du)\nu_{\epsilon\Omega^i} = 0 & \text{on } \epsilon\partial\Omega^i, \\ T(\omega, Du(x))\nu^o(x) = G^o(x, u(x)) & \forall x \in \partial\Omega^o, \end{cases} \tag{1.2}$$

where $\nu_{\epsilon\Omega^i}$ denotes the outward unit normal to $\epsilon\partial\Omega^i$. Then we prove that possibly shrinking ϵ_0, the boundary value problem (1.2) has a solution $u(\epsilon, \cdot) \in C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon), \mathbb{R}^n)$ for all $\epsilon \in]0, \epsilon_0[$, which converges to the unperturbed solution $\tilde{u}$ as ϵ tends to 0, and which is unique in a sense which we clarify in Theorem 5.1, and we pose the following two questions.

(j) Let x be a fixed point in $\mathrm{cl}\Omega^o \setminus \{0\}$. What can be said on the map $\epsilon \mapsto u(\epsilon, x)$ when ϵ is close to 0 and positive?

(jj) What can be said on the energy integral

$$\mathcal{E}(\omega, u(\epsilon,\cdot)) \equiv \frac{1}{2}\int_{\Omega(\epsilon)} \operatorname{tr}\Big(T(\omega, D_x u(\epsilon,x)) D_x^t u(\epsilon,x)\Big)\, dx \tag{1.3}$$

when ϵ is close to 0 and positive?

(We note that the classical energy integral is $\mathcal{E}(\omega, u(\epsilon,\cdot))$ times the second Lamé constant.) Questions of this type have long been investigated for linear problems with the methods of Asymptotic Analysis, which aims at giving complete asymptotic expansions of the solutions in terms of the parameter ϵ. It is perhaps difficult to provide a complete list of the contributions. Here, we mention the work of Kozlov, Maz'ya, and Movchan [10], Maz'ya, Nazarov, and Plamenewskii [23], Movchan [26], Ozawa [28], Ward and Keller [35].

For nonlinear problems far less seems to be known. We mention the seminal paper of Ball [2] for problems as (1.2) for nonlinear hyperelasticity, but with linear boundary conditions and with geometric assumptions on the symmetry of the domain. For related problems, we refer to Horgan and Polignone [9] and to Sivaloganathan, Spector, and Tilakraj [31]. We also mention here the vast literature on homogenization theory (cf. Dal Maso and Murat [4]) and the computation of the expansions in the case of quasilinear equations of Titcombe and Ward [32], Ward, Henshaw and Keller [33], Ward and Keller [34].

Here we wish to characterize the behavior of $u(\epsilon,\cdot)$ at $\epsilon = 0$ by a different approach. Thus for example, if we consider a certain functional, say $f(\epsilon)$, relative to the solution such as for example one of those considered in questions (j)–(jj) above, one could resort to Asymptotic Analysis and may succeed (depending of course on the functional f under consideration) to write out an expansion of the type

$$f(\epsilon) = \sum_{j=0}^{r} a_j \epsilon^j + o(\epsilon^r) \qquad \text{as } \epsilon \to 0^+, \tag{1.4}$$

for suitable coefficients a_j. Instead, in the same circumstance we would try to prove that $f(\cdot)$ can be continued real analytically around $\epsilon = 0$. More generally, we would try to represent $f(\epsilon)$ for $\epsilon > 0$ in terms of real analytic maps and in terms of possibly singular at $\epsilon = 0$, but known functions of ϵ (such as ϵ^{-1}, $\log\epsilon$, etc.). We observe that our approach does have its advantages. Indeed, if for example we know that $f(\epsilon)$ equals for $\epsilon > 0$ a real analytic function of ϵ defined in a whole neighborhood of $\epsilon = 0$, then we know that an asymptotic expansion as (1.4) for all r would necessarily generate a convergent series $\sum_{j=0}^{\infty} a_j\epsilon^j$, and that the sum of such a series would be $f(\epsilon)$ for $\epsilon > 0$.

Such a project has been carried out for suitable nonlinear operators associated to the dependence of the conformal representation of $\Omega(\epsilon)$ in the planar case (see [14, 15]), and for the linear Dirichlet problem for the Laplace and for the Poisson equation (see [17, 18, 19]), where the dependence has been considered upon the complex of variables determined by ϵ, and by global charts of $\partial\Omega^i$, $\partial\Omega^o$, and for other nonlinear problems (see [16, 21, 13]).

In particular, in [16], a nonlinear Robin problem for the Laplace operator on a domain as $\Omega(\epsilon)$ has been considered. Here we generalize the techniques of [16] to the case of the elliptic *system* of linearized elasticity.

2 Preliminaries and notation

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}$ $\forall(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the normed space of the continuous and linear maps of $\mathcal{X}$ and $\mathcal{Y}$. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [29]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N} \setminus \{0, 1\}$. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices with real entries. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{R})$, or more simply by $C^m(\Omega)$. Let $f \in (C^m(\Omega))^n$. The s-th component of f is denoted f_s, and Df (or ∇f) denotes the gradient matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{s,l=1,\dots,n}$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0, 1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, (cf. e.g. Gilbarg and Trudinger [7]). Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{D})$ denotes $\left\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : \ f(\mathrm{cl}\,\Omega) \subseteq \mathbb{D}\right\}$. A similar notation holds if $\mathbb{D}$ is replaced by $M_n(\mathbb{R})$. Now let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\,\Omega)$ endowed with the norm $\|f\|_{C^m(\mathrm{cl}\,\Omega)} \equiv \sum_{|\eta|\leq m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. If $f \in C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its Hölder constant $|f : \Omega|_\alpha$ is defined as $\sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \mathrm{cl}\,\Omega, x \neq y\right\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, equipped with its usual norm $\|f\|_{C^{m,\alpha}(\mathrm{cl}\,\Omega)} = \|f\|_{C^m(\mathrm{cl}\,\Omega)} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha$, is well-known to be a Banach space. We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. e.g., Gilbarg and Trudinger [7, §6.2]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to Gilbarg and Trudinger [7] (see also [20, §2, Lemmas 3.1 and 4.26, Theorem 4.28], Lanza and Rossi [22, §2]). We retain the standard notation of L^p spaces and of corresponding norms. We note that throughout the paper 'analytic' means 'real analytic'. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [29, p. 89].

We denote by S_n the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(\xi) \equiv \begin{cases} \frac{1}{s_n} \log|\xi| & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n}|\xi|^{2-n} & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2, \end{cases} \tag{2.1}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0, 1)$. S_n is well-known to be the fundamental solution of the Laplace operator.

We denote by $\Gamma_n(\cdot,\cdot)$ the matrix valued function of $(\mathbb{R}\setminus\{-1\})\times(\mathbb{R}^n\setminus\{0\})$ to $M_n(\mathbb{R})$ which takes a pair (ω,ξ) to the matrix $\Gamma_n(\omega,\xi)$ defined by

$$\Gamma^j_{n,i}(\omega,\xi)\equiv\frac{\omega+2}{2(\omega+1)}\delta_{i,j}S_n(\xi)-\frac{\omega}{2(\omega+1)}\frac{1}{s_n}\frac{\xi_i\xi_j}{|\xi|^n},$$

where $\delta_{i,j}=1$ if $i=j$, $\delta_{i,j}=0$ if $i\neq j$. As is well known, $\Gamma_n(\omega,\xi)$ is the fundamental solution of the operator

$$L[\omega]\equiv\Delta+\omega\nabla\mathrm{div}.$$

We note that the classical operator of linearized homogeneous isotropic elastostatics equals $L[\omega]$ times the second constant of Lamé, and that $L[\omega]u=\mathrm{div}\,T(\omega,Du)$ for all regular vector valued functions u, and that the classical fundamental solution of the operator of linearized homogeneous and isotropic elastostatics equals $\Gamma_n(\omega,\xi)$ times the reciprocal of the second constant of Lamé. We find also convenient to set

$$\Gamma^j_n(\cdot,\cdot)\equiv(\Gamma^j_{n,i}(\cdot,\cdot))_{i=1,\dots,n},$$

which we think of as a column vector for all $j=1,\dots,n$. Let $\alpha\in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We shall denote by ν_Ω the outward unit normal to $\partial\Omega$. We also set

$$\Omega^-\equiv\mathbb{R}^n\setminus\mathrm{cl}\Omega.$$

Let $\omega\in]1-(2/n),+\infty[$. Then we set

$$v[\omega,\mu](x)\equiv\int_{\partial\Omega}\Gamma_n(\omega,x-y)\mu(y)\,d\sigma_y,$$
$$w[\omega,\mu](x)\equiv-\left(\int_{\partial\Omega}\mu^t(y)T(\omega,D_\xi\Gamma^i_n(\omega,x-y))\nu_\Omega(y)\,d\sigma_y\right)_{i=1,\dots,n},$$

for all $x\in\mathbb{R}^n$ and for all $\mu\equiv(\mu_j)_{j=1,\dots,n}\in L^2(\partial\Omega,\mathbb{R}^n)$. As is well known, if $\mu\in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $v[\omega,\mu]$ is continuous in the whole of $\mathbb{R}^n$. We set

$$v^+[\omega,\mu]\equiv v[\omega,\mu]_{|\mathrm{cl}\Omega}\qquad v^-[\omega,\mu]\equiv v[\omega,\mu]_{|\mathrm{cl}\Omega^-}.$$

Also if $\mu\in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $w[\omega,\mu]_{|\Omega}$ admits a unique continuous extension to $\mathrm{cl}\Omega$, which we denote by $w^+[\omega,\mu]$, and $w[\omega,\mu]_{|\Omega^-}$ admits a unique continuous extension to $\mathrm{cl}\Omega^-$, which we denote by $w^-[\omega,\mu]$.

We now shortly review some facts on the linear traction problem, which we need in the sequel. Let a be a continuous map of $\partial\Omega$ to $M_n(\mathbb{R})$ satisfying the following assumptions.

$$\text{The determinant } \det a(\cdot) \text{ does not vanish identically in } \partial\Omega, \tag{2.2}$$
$$\xi^t a(x)\xi\geq 0\qquad\forall x\in\partial\Omega,\ \forall\xi\in\mathbb{R}^n. \tag{2.3}$$

Then we consider the following linear boundary value problem

$$\begin{cases}\mathrm{div}\,(T(\omega,Du))=0 & \text{in }\Omega,\\ T(\omega,Du)\nu_\Omega+au=g & \text{on }\partial\Omega,\end{cases} \tag{2.4}$$

for a given boundary data g. Then we have the following known result.

Proposition 2.1 *Let $\omega \in]1-(2/n), +\infty[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class C^1. Let $a \in C^0(\partial\Omega, M_n(\mathbb{R}))$ satisfy conditions (2.2) and (2.3). Let $g \in C^0(\partial\Omega, \mathbb{R}^n)$. Then problem (2.4) has at most one solution $u \in C^1(\mathrm{cl}\Omega, \mathbb{R}^n)$.*

Proof: Let u belong to $C^1(\mathrm{cl}\Omega, \mathbb{R}^n)$ solve problem (2.4) with $g = 0$. Then by interior elliptic regularity theory, we have $u \in C^2(\Omega, \mathbb{R}^n)$ and the Divergence Theorem implies that

$$\int_\Omega \mathrm{tr}\Big(T(\omega, Du)D^t u\Big)\,dx = \int_{\partial\Omega} u^t T(\omega, Du)\nu_\Omega\,d\sigma = -\int_{\partial\Omega} u^t a u\,d\sigma \le 0. \tag{2.5}$$

Then by an elementary argument (cf. e.g., Lemma A.1 of the Appendix applied to $A = Du$), we deduce that $|Du + D^t u|$ must equal zero almost everywhere in Ω. Then as is well known, there exist a skew-symmetric element $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ such that $u(x) = Ax + b$ for all $x \in \Omega$. We now prove that $A = 0$. To do so, we assume by contradiction that $A \neq 0$. Since $A \neq 0$, the set $\{x \in \mathbb{R}^n : Ax+b = 0\}$ is an affine subspace of $\mathbb{R}^n$ of codimension at least 2. Instead, the boundary condition in (2.4) with $g = 0$ and the obvious identity $T(\omega, A) = 0$ ensure that the set $\{x \in \partial\Omega : u(x) = Ax + b = 0\}$ contains $\{x \in \partial\Omega : \det a(x) \neq 0\}$ and thus at least a manifold of codimension 1 of $\mathbb{R}^n$, a contradiction. Hence, $A = 0$. Then again by condition (2.2) and by the boundary condition in (2.4), we conclude that $u(x) = b$ must vanish identically. □

As customary, we associate to problem (2.4) an integral equation. For each $a \in C^0(\partial\Omega, M_n(\mathbb{R}))$, $\omega \in]1-(2/n), +\infty[$, $\mu \in L^2(\partial\Omega, \mathbb{R}^n)$, we set

$$\mathbf{J}_a[\omega, \mu] \equiv -\frac{1}{2}\mu + v_*[\omega, \mu] + av[\omega, \mu] \qquad \text{on } \partial\Omega,$$

where

$$v_*[\omega, \mu](x) \equiv \int_{\partial\Omega} \sum_{l=1}^{n} \mu_l(y) T(\omega, D_\xi \Gamma_n^l(\omega, x-y))\nu_\Omega(x)\,d\sigma_y \quad \forall x \in \partial\Omega.$$

We shall denote by I the identity operator in a function space. Also, if $\mathcal{X}$ is a vector subspace of $L^1(\partial\Omega, \mathbb{R}^n)$, we find convenient to set

$$\mathcal{X}_0 \equiv \left\{ f \in \mathcal{X} : \int_{\partial\Omega} f\,d\sigma = 0 \right\}. \tag{2.6}$$

Then we have the following certainly known result.

Theorem 2.2 *Let $\alpha \in]0, 1[$, $\omega \in]1-(2/n), +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $a \in C^{m-1,\alpha}(\partial\Omega, M_n(\mathbb{R}))$ satisfy conditions (2.2), (2.3). Then the following statements hold.*

(i) $\mathbf{J}_a[\omega, \cdot]$ *is a Fredholm operator of index zero of $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself.*

(ii) The map $\tilde{\mathbf{J}}_a[\omega,\cdot,\cdot]$ *of* $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ *defined by*

$$\tilde{\mathbf{J}}_a[\omega,c,\mu] \equiv \mathbf{J}_a[\omega,\mu] + ac \qquad \forall (c,\mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0,$$

is a homeomorphism of $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ *onto* $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$*. Furthermore, if* (d,g) *belongs to* $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$*, then there exists a unique pair* (c,μ) *in* $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ *such that* $\mathbf{J}_a[\omega,\mu] + ac = g$, $\int_{\partial\Omega} \mu \, d\sigma = d$.

(iii) Let $g \in C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$*. Then the problem (2.4) admits a unique solution* $u \in C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n)$*, and* $u = v^+[\omega,\mu] + c$*, where* (c,μ) *in* $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ *is the unique solution of equation*

$$\tilde{\mathbf{J}}_a[\omega,c,\mu] = g \qquad \text{on } \partial\Omega.$$

Proof: As is well known, the operator $-\frac{1}{2}I + v_*[\omega,\cdot]$ is a Fredholm operator of index 0 in $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ (cf. Theorem A.9 of the Appendix). Since $v[\omega,\cdot]_{|\partial\Omega}$ maps $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ into $C^{m,\alpha}(\partial\Omega,\mathbb{R}^n)$, which is compactly imbedded into the space $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$, and the product in $C^{m-1,\alpha}(\partial\Omega)$ is bilinear and continuous, we conclude that $\mathbf{J}_a[\omega,\cdot]$ is a compact perturbation of an operator of Fredholm of index 0 in $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$, and thus statement (i) holds (cf. e.g., Deimling [5, Theorem 9.8, p. 79]).

We now prove (ii). We write $\tilde{\mathbf{J}}_a[\omega,\cdot,\cdot]$ in the form $\tilde{\mathbf{J}}_a[\omega,\cdot,\cdot] = \mathbf{J}_1 \circ \mathbf{J}_2 \circ \mathbf{J}_3$, where $\mathbf{J}_1$ is the operator of $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ which takes a pair (c,f) to the function $f + ac$, and $\mathbf{J}_2$ is the operator of $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to itself which takes a pair (c,μ) to the pair $(c,\mathbf{J}_a[\omega,\mu])$, and $\mathbf{J}_3$ is the inclusion of $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ into $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$. Then we easily verify that $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$ are Fredholm operators of indexes n, and 0, and $-n$, respectively. Thus the composite operator $\tilde{\mathbf{J}}_a[\omega,\cdot,\cdot]$ is of index 0. Hence, it suffices to prove that $\tilde{\mathbf{J}}_a[\omega,\cdot,\cdot]$ is injective. Thus we now assume that $(c,\mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ and that $\tilde{\mathbf{J}}_a[\omega,c,\mu] = 0$. Standard jump properties of elastic single layer potentials and equality $\tilde{\mathbf{J}}_a[\omega,c,\mu] = 0$ imply that $v^+[\omega,\mu] + c$ solves problem (2.4) with $g = 0$. Accordingly, Proposition 2.1 implies that $v^+[\omega,\mu] + c = 0$ in $\mathrm{cl}\Omega$ and thus $-\frac{1}{2}\mu + v_*[\omega,\mu] = 0$. Then Theorem A.5 (iv) of the Appendix implies that $v^+[\omega,\mu]_{|\partial\Omega} = 0$ and that $c = 0$. Then by uniqueness of the Dirichlet problem for $L[\omega]$ in Ω, we also have $v^+[\omega,\mu] = 0$ in $\mathrm{cl}\Omega$. If $n = 2$, condition $\int_{\partial\Omega} \mu \, d\sigma = 0$ implies that $|x| v[\omega,\mu](x)$ and $|x|^2 Dv[\omega,\mu](x)$ are bounded in a neighborhood of infinity. If $n \geq 3$, we know that $|x|^{n-2} v[\omega,\mu](x)$ and $|x|^{n-1} Dv[\omega,\mu](x)$ are bounded in a neighborhood of infinity. Both in case $n = 2$ and $n \geq 3$, condition $v[\omega,\mu] = 0$ on $\partial\Omega$ implies that $v[\omega,\mu] = 0$ on $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ (cf. e.g., Kupradze et al. [12, Chapter III, §1]). Hence, the classical jump properties for $T(\omega, Dv[\omega,\mu])$ imply that $\mu = 0$. The last part of statement (ii) follows by taking $\mu = d|\partial\Omega|^{-1} + \mu_1$, with $\int_{\partial\Omega} \mu_1 \, d\sigma = 0$ and $\tilde{\mathbf{J}}_a[\omega,c,\mu_1] = g - \mathbf{J}_a[\omega, d|\partial\Omega|^{-1}]$.

We now prove statement (iii). By statement (ii), there exists a unique pair $(c,\mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ such that $\tilde{\mathbf{J}}_a[\omega,c,\mu] = g$. Then we set $u \equiv v^+[\omega,\mu] + c$. By Theorem A.2 (i) of the Appendix, we have $u \in C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n)$. By classical jump properties of elastic layer potentials, equation $\tilde{\mathbf{J}}_a[\omega,c,\mu] = g$ implies that the boundary condition of problem (2.4) holds. Since $v^+[\omega,\mu] + c$ must satisfy equation $L[\omega]u = 0$ in Ω, u solves problem (2.4). Then Proposition 2.1 completes the proof of statement (iii). □

Now let $G \in C^0(\partial\Omega\times\mathbb{R}^n, \mathbb{R}^n)$. We denote by F_G the (nonlinear) composition operator of $C^0(\partial\Omega, \mathbb{R}^n)$ to itself which maps $v \in C^0(\partial\Omega, \mathbb{R}^n)$ to the function $F_G[v]$ defined by

$$F_G[v](t) \equiv G(t, v(t)) \qquad \forall t \in \partial\Omega,$$

and we now transform our nonlinear traction boundary value problem into a problem for integral equations by means of the following.

Proposition 2.3 *Let $\alpha \in]0, 1[$, $\omega \in]1 - (2/n), +\infty[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $G \in C^0(\partial\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ be such that F_G maps $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself. Then the map of the set of pairs $(c, \mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0$ which satisfy the problem*

$$\tilde{\mathbf{J}}_0[\omega, c, \mu] = F_G[v[\omega, \mu]_{|\partial\Omega} + c] \tag{2.7}$$

to the set of $u \in C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n)$ which solve the problem

$$\begin{cases} \mathrm{div}\,(T(\omega, Du)) = 0 & \text{in } \Omega, \\ T(\omega, Du)\nu_\Omega = F_G[u_{|\partial\Omega}] & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

which takes (c, μ) to the function $v^+[\omega, \mu] + c$ is a bijection.

Proof: If (c, μ) satisfies (2.7), then $v^+[\omega, \mu]$ belongs to $C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n)$ (see Theorem A.2 (i)), and we have

$$T(\omega, D(v^+[\omega, \mu] + c))\nu_\Omega = \tilde{\mathbf{J}}_0[\omega, c, \mu] = F_G[v[\omega, \mu]_{|\partial\Omega} + c]$$

and $v^+[\omega, \mu] + c$ satisfies problem (2.8). Conversely, if $u \in C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n)$ satisfies problem (2.8), then by Theorem 2.2 with $a = I$ (the identity matrix), there exists a unique $(c, \mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0$ such that

$$\tilde{\mathbf{J}}_I[\omega, c, \mu] = F_G[u_{|\partial\Omega}] + u_{|\partial\Omega}, \tag{2.9}$$

and the function $V = v^+[\omega, \mu] + c$ is the only solution of the boundary value problem

$$\begin{cases} \mathrm{div}\,(T(\omega, DV)) = 0 & \text{in } \Omega, \\ T(\omega, DV)\nu_\Omega + V = F_G[u_{|\partial\Omega}] + u_{|\partial\Omega} & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

Since u satisfies (2.10), Proposition 2.1 implies that $u = V = v^+[\omega, \mu] + c$. Hence, (2.9) implies that

$$\tilde{\mathbf{J}}_I[\omega, c, \mu] = F_G[v[\omega, \mu]_{|\partial\Omega} + c] + v[\omega, \mu]_{|\partial\Omega} + c, \tag{2.11}$$

which implies the validity of equality (2.7).

We now show the uniqueness of (c, μ) such that $u = v^+[\omega, \mu] + c$ and for which (2.7) holds. If (c_1, μ_1) and (c_2, μ_2) belong to $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0$ and solve equation (2.7) and

$$u = v^+[\omega, \mu_1] + c_1 = v^+[\omega, \mu_2] + c_2,$$

then (2.9) and (2.11) must hold for both (c_1, μ_1) and (c_2, μ_2). Hence, Theorem 2.2 (ii) with $a = I$ implies that $(c_1, \mu_1) = (c_2, \mu_2)$. □

3 Formulation of the problem in terms of integral equations, and existence of the solution $u(\epsilon, \cdot)$

We now provide a formulation of problem (1.2) in terms of integral equations. We shall consider the following assumptions for some $\alpha \in]0, 1[$ and for some natural $m \geq 1$.

$$\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha}. \quad (3.1)$$
$$\text{Let } \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ be connected. Let } 0 \in \Omega.$$

Now let Ω^i, Ω^o be as in (3.1). Then we set

$$\epsilon_0 \equiv \sup\{\theta \in]0, +\infty[: \epsilon \mathrm{cl}\Omega^i \subseteq \Omega^o, \ \forall \epsilon \in]-\theta, \theta[\}. \quad (3.2)$$

Clearly, $\epsilon_0 > 0$. Moreover, a simple topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon \mathrm{cl}\Omega^i$ is connected, and that $\mathbb{R}^n \setminus \mathrm{cl}\Omega(\epsilon)$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \mathrm{cl}\Omega^o$, and that $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$, for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$. For brevity, we set

$$\nu^i \equiv \nu_{\Omega^i} \qquad \nu^o \equiv \nu_{\Omega^o} \qquad \nu_\epsilon \equiv \nu_{\Omega(\epsilon)}.$$

Obviously,

$$\nu_\epsilon(x) = -\nu^i(x/\epsilon)\,\mathrm{sgn}(\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i, \quad (3.3)$$
$$\nu_\epsilon(x) = \nu^o(x) \qquad \forall x \in \partial\Omega^o, \quad (3.4)$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. Now let $\epsilon \in]0, \epsilon_0[$. If $a^o \in C^0(\partial\Omega^o, M_n(\mathbb{R}))$, we denote by a the function of $\partial\Omega(\epsilon)$ to $M_n(\mathbb{R})$ defined by

$$a(x) \equiv a^o(x) \quad \text{if } x \in \partial\Omega^o, \qquad a(x) \equiv 0 \quad \text{if } x \in \epsilon\partial\Omega^i.$$

Then we shall consider the following assumptions.

$$G^o \in C^0(\partial\Omega^o \times \mathbb{R}^n, \mathbb{R}^n), \quad (3.5)$$
$$F_{G^o} \text{ maps } C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n) \text{ to itself}. \quad (3.6)$$

Furthermore, we denote by G the function of $\partial\Omega(\epsilon) \times \mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$G(x, c) \equiv G^o(x, c) \quad \text{if } (x, c) \in \partial\Omega^o \times \mathbb{R}^n,$$
$$G(x, c) \equiv 0 \quad \text{if } (x, c) \in \epsilon\partial\Omega^i \times \mathbb{R}^n.$$

We now convert our boundary value problems (1.1) and (1.2) into integral equations. We could exploit Proposition 2.3. However, we note that the corresponding representation formulas include integration on $\partial\Omega(\epsilon)$ and thus on $\epsilon\partial\Omega^i$, which depends on ϵ. In order to get rid of such a dependence, we shall introduce the following Theorem, in which we properly rescale the restriction of the unknown function μ to $\epsilon\partial\Omega^i$.

We find convenient to introduce the following abbreviation. We set

$$X_{m,\alpha} \equiv C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n) \times C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n).$$

Theorem 3.1 *Let $\alpha \in]0,1[$, $\omega \in]1-(2/n),+\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω^i, Ω^o be as in (3.1). Let ϵ_0 be as in (3.2). Let G^o be as in (3.5), (3.6). Let $M=(M_1,M_2,M_3)$ be the map of $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^n\times X_{m,\alpha}$ to $\mathbb{R}^n\times X_{m,\alpha}$ defined by*

$$
\begin{aligned}
M_1[\epsilon,c,\eta,\rho] &\equiv \int_{\partial\Omega^i}\eta\,d\sigma+\int_{\partial\Omega^o}\rho\,d\sigma, \qquad (3.7)\\
M_2[\epsilon,c,\eta,\rho](t) &\equiv \frac{1}{2}\eta(t)+v_*[\omega,\eta](t)\\
&\quad+\epsilon^{n-1}\int_{\partial\Omega^o}\sum_{l=1}^{n}\rho_l(s)T(\omega,D_\xi\Gamma_n^l(\omega,\epsilon t-s))\nu^i(t)\;d\sigma_s \quad \forall t\in\partial\Omega^i,\\
M_3[\epsilon,c,\eta,\rho](t) &\equiv -\frac{1}{2}\rho(t)+v_*[\omega,\rho](t)\\
&\quad+\int_{\partial\Omega^i}\sum_{l=1}^{n}\eta_l(s)T(\omega,D_\xi\Gamma_n^l(\omega,t-\epsilon s))\nu^o(t)\;d\sigma_s\\
&\quad-G^o\left(t,\int_{\partial\Omega^i}\Gamma_n(\omega,t-\epsilon s)\eta(s)\,d\sigma_s+v[\omega,\rho](t)+c\right)\ \forall t\in\partial\Omega^o,
\end{aligned}
$$

for all $(\epsilon,c,\eta,\rho)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}^n\times X_{m,\alpha}$. Then the following two statements hold.

(i) Let $\epsilon\in]0,\epsilon_0[$. The map $u[\epsilon,\cdot,\cdot,\cdot]$ of the set of solutions $(c,\eta,\rho)\in\mathbb{R}^n\times X_{m,\alpha}$ of equation

$$M[\epsilon,c,\eta,\rho]=0, \qquad (3.8)$$

to the set of solutions $u\in C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon),\mathbb{R}^n)$ of (1.2) which takes (c,η,ρ) to $v^+[\omega,\mu]+c$, where

$$\mu(x)\equiv\rho(x)\quad \text{if } x\in\partial\Omega^o,\qquad \mu(x)\equiv\epsilon^{1-n}\eta(x/\epsilon)\quad\text{if } x\in\epsilon\partial\Omega^i, \qquad (3.9)$$

is a bijection.

(ii) The triple $(c,\eta,\rho)\in\mathbb{R}^n\times X_{m,\alpha}$ satisfies the equation

$$M[0,c,\eta,\rho]=0 \qquad (3.10)$$

if and only if both the following conditions are satisfied

(j) $\eta=0$.

(jj) The pair (c,ρ) satisfies both the equations

$$M_1[0,c,0,\rho]=0,\qquad M_3[0,c,0,\rho]=0. \qquad (3.11)$$

The map $u[0,\cdot,0,\cdot]$ of the set of solutions (c,ρ) of (3.11) in $\mathbb{R}^n\times C^{m-1,\alpha}(\partial\Omega^o,\mathbb{R}^n)$ to the set of solutions $u\in C^{m,\alpha}(\mathrm{cl}\Omega^o,\mathbb{R}^n)$ of (1.1) which takes (c,ρ) to $u[0,c,0,\rho]\equiv v^+[\omega,\rho]+c$ is a bijection.

Proof: Let $\epsilon > 0$. A simple computation based on the rule of change of variables in the integrals over $\partial\Omega^i$ shows that (c, η, ρ) solves equation (3.8) if and only if the pair (c, μ) solves the integral equation (2.7) with $\Omega = \Omega(\epsilon)$ and $\int_{\partial\Omega(\epsilon)} \mu \, d\sigma = 0$. Thus statement (i) follows by Proposition 2.3.

We now prove statement (ii). To prove that (c, η, ρ) solves (3.10) if and only if both (j) and (jj) hold, it suffices to note that if $M[0, c, \eta, \rho] = 0$, then

$$\frac{1}{2}\eta + v_*[\omega, \eta] = 0 \qquad \text{on } \partial\Omega^i.$$

Since $\mathbb{R}^n \setminus \mathrm{cl}\Omega^i$ is connected, a classical result in potential theory implies that $\eta = 0$ (see Remark A.8 of the Appendix). The second part of the statement is an immediate consequence of Proposition 2.3 with $\Omega = \Omega^o$. □

Theorem 3.1 reduces the analysis of problem (1.1) or of problem (1.2) to that of equation $M = 0$. We shall now show that under reasonable assumptions on the data of (1.1), if problem (1.1) admits a solution $\tilde{u}$ satisfying certain nondegeneracy conditions, then for ϵ sufficiently small, problem (1.2) has a solution which is unique in a local sense which we clarify in Section 5. To show existence for (1.2), we shall apply the Implicit Function Theorem to the equation $M = 0$ around a zero $(0, \tilde{c}, 0, \tilde{\rho})$ of M such that $\tilde{u} = u[0, \tilde{c}, 0, \tilde{\rho}]$. Thus we now prove the following.

Theorem 3.2 *Let $\alpha \in]0, 1[$, $\omega \in]1 - (2/n), +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω^i, Ω^o be as in (3.1). Let ϵ_0 be as in (3.2). Let (3.5), (3.6) hold. Let*

$$F_{G^o} \text{ be real analytic in } C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n). \tag{3.12}$$

Assume that there exists a solution $\tilde{u} \in C^{m,\alpha}(\mathrm{cl}\Omega^o, \mathbb{R}^n)$ of (1.1) such that

$$\begin{aligned} &\text{there exists } t \in \partial\Omega^o \text{ such that } \det D_u G^o(t, \tilde{u}(t)) \neq 0, \\ &\xi^t D_u G^o(x, \tilde{u}(x))\xi \leq 0 \ \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in \partial\Omega^o. \end{aligned} \tag{3.13}$$

Let $(\tilde{c}, \tilde{\rho})$ be the unique solution of (3.11) in $\mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$ such that $\tilde{u}$ equals $u[0, \tilde{c}, 0, \tilde{\rho}]$ (cf. Theorem 3.1). Then there exist $\epsilon' \in]0, \epsilon_0[$, and an open neighborhood $\mathcal{V}$ of $(\tilde{c}, 0, \tilde{\rho})$ in $\mathbb{R}^n \times X_{m,\alpha}$, and a real analytic operator (C, E, R) of $]-\epsilon', \epsilon'[$ to $\mathcal{V}$ such that the set of zeros of M in $]-\epsilon', \epsilon'[\times\mathcal{V}$ coincides with the graph of (C, E, R). In particular, $(C[0], E[0], R[0]) = (\tilde{c}, 0, \tilde{\rho})$.

Proof: We plan to apply the Implicit Function Theorem to equation $M = 0$ around the point $(0, \tilde{c}, 0, \tilde{\rho})$. By assumption (3.12) and by standard properties of elastic layer potentials (cf. Theorem A.2 of the Appendix), and by known properties of (nonsingular) integral operators (cf. e.g., Theorem 6.2 of Appendix B of [16]), we conclude that the map M is real analytic. By definition of $(\tilde{c}, \tilde{\rho})$, we have $M[0, \tilde{c}, 0, \tilde{\rho}] = 0$. By standard calculus in Banach space (see also Proposition 6.3 of Appendix B of [16]), $D_u G^o(\cdot, \cdot)$

exists and the differential of M at $(0, \tilde{c}, 0, \tilde{\rho})$ with respect to (c, η, ρ) is delivered by the formula

$$\partial_{(c,\eta,\rho)} M_1[0, \tilde{c}, 0, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = \int_{\partial\Omega^i} \overline{\eta}\, d\sigma + \int_{\partial\Omega^o} \overline{\rho}\, d\sigma, \tag{3.14}$$

$$\partial_{(c,\eta,\rho)} M_2[0, \tilde{c}, 0, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = \frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}] \quad \text{on } \partial\Omega^i,$$

$$\begin{aligned}
&\partial_{(c,\eta,\rho)} M_3[0, \tilde{c}, 0, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho})(t) \\
&= -\frac{1}{2}\overline{\rho}(t) + v_*[\omega, \overline{\rho}](t) + \int_{\partial\Omega^i} \sum_{l=1}^{n} \overline{\eta}_l(s) T(\omega, D_\xi \Gamma_n^l(\omega, t))\nu^o(t)\, d\sigma_s \\
&\quad - D_u G^o\,(t, v[\omega, \tilde{\rho}](t) + \tilde{c}) \\
&\quad \cdot \left\{ \int_{\partial\Omega^i} \Gamma_n(\omega, t)\overline{\eta}(s)\, d\sigma_s + v[\omega, \overline{\rho}](t) + \overline{c} \right\} \qquad \forall t \in \partial\Omega^o,
\end{aligned}$$

for all $(\overline{c}, \overline{\eta}, \overline{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$. We now prove that $\partial_{(c,\eta,\rho)} M[0, \tilde{c}, 0, \tilde{\rho}]$ is a linear homeomorphism of $\mathbb{R}^n \times X_{m,\alpha}$ onto itself. By the Open Mapping Theorem, it suffices to show that $\partial_{(c,\eta,\rho)} M[0, \tilde{c}, 0, \tilde{\rho}]$ is a bijection of $\mathbb{R}^n \times X_{m,\alpha}$ onto itself. Let $(d, f^i, f^o) \in \mathbb{R}^n \times X_{m,\alpha}$. We must show that there exists a unique $(\overline{c}, \overline{\eta}, \overline{\rho})$ in $\mathbb{R}^n \times X_{m,\alpha}$ such that

$$\partial_{(c,\eta,\rho)} M[0, \tilde{c}, 0, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = (d, f^i, f^o)\,. \tag{3.15}$$

By Remark A.8 of the Appendix, we conclude that the second component of equation (3.15) has a unique solution $\overline{\eta} \in C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$. We now rewrite the first and third components of (3.15) in the form

$$\int_{\partial\Omega^o} \overline{\rho}\, d\sigma = d - \int_{\partial\Omega^i} \overline{\eta}\, d\sigma, \tag{3.16}$$

$$\begin{aligned}
&-\frac{1}{2}\overline{\rho}(t) + v_*[\omega, \overline{\rho}](t) - D_u G^o(t, v[\omega, \tilde{\rho}](t) + \tilde{c}) \cdot (v[\omega, \overline{\rho}](t) + \overline{c}) \\
&\qquad = f^o(t) - \int_{\partial\Omega^i} \sum_{l=1}^{n} \overline{\eta}_l(s) T(\omega, D_\xi \Gamma_n^l(\omega, t))\nu^o(t)\, d\sigma_s \\
&\qquad\quad + D_u G^o(t, v[\omega, \tilde{\rho}](t) + \tilde{c}) \int_{\partial\Omega^i} \Gamma_n(\omega, t)\overline{\eta}(s)\, d\sigma_s \qquad \forall t \in \partial\Omega^o.
\end{aligned}$$

By Theorem A.2 (i) and by the membership of $\tilde{\rho}$ in $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$, we have $v[\omega, \tilde{\rho}]_{|\partial\Omega} \in C^{m,\alpha}(\partial\Omega^o, \mathbb{R}^n)$. By assumption (3.12) and by standard properties of superposition operators (cf. [16, Proposition 6.3]), the superposition operator $F_{D_u G^o}$ must map the space $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$ to $C^{m-1,\alpha}(\partial\Omega^o, M_n(\mathbb{R}))$. Hence, we can conclude that the function $D_u G^o\ (t, v[\omega, \tilde{\rho}](t) + \tilde{c})$ of the variable $t \in \partial\Omega^o$ belongs to $C^{m-1,\alpha}(\partial\Omega^o, M_n(\mathbb{R}))$. Since the functions f^o and $T(\omega, D_\xi \Gamma_n(\omega, t))\nu^o(t)$ are also of class $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$, we conclude that the right-hand side of (3.16) belongs to $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$ and that

$$A(t) \equiv -D_u G^o(t, v[\omega, \tilde{\rho}](t) + \tilde{c}) \qquad \forall t \in \partial\Omega^o$$

defines a matrix valued function which satisfies assumptions (2.2) and (2.3) (see also (3.13)). Then we can invoke Theorem 2.2 (ii) and conclude that the system (3.16) admits one and only one solution $(\overline{c}, \overline{\rho}) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$. □

We are now ready to define our family of solutions.

Definition 3.3 *Let $\alpha \in]0,1[$, $\omega \in]1-(2/n), +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω^i, Ω^o be as in (3.1). Let ϵ_0 be as in (3.2). Let (3.5), (3.6), (3.12) hold. Assume that there exists a solution $\tilde{u} \in C^{m,\alpha}(\mathrm{cl}\Omega^o, \mathbb{R}^n)$ of (1.1) such that (3.13) holds. Let $\epsilon' \in]0, \epsilon_0[$ and $(C[\cdot], E[\cdot], R[\cdot])$ be as in in Theorem 3.2. Let $\epsilon \in]0, \epsilon'[$. Let $u[\epsilon, \cdot, \cdot, \cdot]$ be as in Theorem 3.1 (i). Then we set*

$$u(\epsilon, t) \equiv u[\epsilon, C[\epsilon], E[\epsilon], R[\epsilon]](t) \qquad \forall t \in \mathrm{cl}\Omega(\epsilon).$$

4 A functional analytic representation for the family $\{u(\epsilon,\cdot)\}_{\epsilon \in]0,\epsilon'[}$ and for its energy integral

Theorem 4.1 *Let the assumptions of Definition 3.3 hold. Let $\tilde{\Omega}$ be a bounded open subset of $\Omega^o \setminus \{0\}$ such that $0 \notin \mathrm{cl}\tilde{\Omega}$. Then there exist $\epsilon_{\tilde{\Omega}} \in]0, \epsilon'[$ and a real analytic operator $U_{\tilde{\Omega}}$ of $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega}, \mathbb{R}^n)$ such that $\tilde{\Omega} \subseteq \Omega(\epsilon)$ for all $\epsilon \in]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ and such that*

$$U_{\tilde{\Omega}}[\epsilon](\cdot) = u(\epsilon, \cdot)_{|\mathrm{cl}\tilde{\Omega}} \qquad \forall \epsilon \in]0, \epsilon_{\tilde{\Omega}}[. \tag{4.1}$$

Moreover, $U_{\tilde{\Omega}}[0] = \tilde{u}_{|\mathrm{cl}\tilde{\Omega}}$. In particular, we have $\lim_{\epsilon\to 0^+} u(\epsilon,\cdot)_{|\mathrm{cl}\tilde{\Omega}} = \tilde{u}_{|\mathrm{cl}\tilde{\Omega}}(\cdot)$ in $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega}, \mathbb{R}^n)$.

Proof: Let $\epsilon'_{\tilde{\Omega}} \in]0, \epsilon'[$ be such that $\tilde{\Omega} \subseteq \Omega(\epsilon)$ for all $\epsilon \in [-\epsilon'_{\tilde{\Omega}}, \epsilon'_{\tilde{\Omega}}]$. By definition of $u(\epsilon, \cdot)$, we have

$$u(\epsilon, t) = u[\epsilon, C[\epsilon], E[\epsilon], R[\epsilon]](t) = \int_{\partial\Omega(\epsilon)} \Gamma_n(\omega, t-s)\mu_\epsilon(s)\, d\sigma_s + C[\epsilon]$$

for all $t \in \mathrm{cl}\Omega(\epsilon)$ and for all $\epsilon \in]0, \epsilon'_{\tilde{\Omega}}[$, where

$$\mu_\epsilon(s) \equiv R[\epsilon](s) \quad \text{if } s \in \partial\Omega^o, \qquad \mu_\epsilon(s) \equiv \epsilon^{1-n} E[\epsilon](s/\epsilon) \quad \text{if } s \in \epsilon\partial\Omega^i.$$

Hence,

$$u(\epsilon, t) = \int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s) E[\epsilon](s)\, d\sigma_s + v[\omega, R[\epsilon]](t) + C[\epsilon] \quad \forall t \in \mathrm{cl}\Omega(\epsilon).$$

Now let $\epsilon_{\tilde{\Omega}} \in]0, \epsilon'_{\tilde{\Omega}}]$ be such that $\epsilon \mathrm{cl}\Omega^i \subseteq \epsilon'_{\tilde{\Omega}}\Omega^i$ for all $\epsilon \in [-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}]$. Thus it is natural to define

$$U_{\Omega(\epsilon'_{\tilde{\Omega}})}[\epsilon](t) \equiv \int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s) E[\epsilon](s)\, d\sigma_s + v[\omega, R[\epsilon]](t) + C[\epsilon], \tag{4.2}$$

for all $t \in \mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}})$ and for all $\epsilon \in]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$. Thus we are reduced to show that the right-hand side of (4.2) defines a real analytic operator of $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ to $C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}), \mathbb{R}^n)$.

Indeed, $\tilde{\Omega} \subseteq \Omega(\epsilon'_{\tilde{\Omega}})$, and thus we can take $U_{\tilde{\Omega}}$ equal to the restriction to $\mathrm{cl}\tilde{\Omega}$ of $U_{\Omega(\epsilon'_{\tilde{\Omega}})}[\epsilon]$. Since $\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}) \subseteq \mathrm{cl}\Omega^o$, Theorem A.2 (i) of the Appendix and the real analyticity of $R[\cdot]$ imply that the map of $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ to $C^{m,\alpha}(\mathrm{cl}\Omega^o, \mathbb{R}^n)$ which takes ϵ to $v^+[\omega, R[\epsilon]]_{|\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}})}$ is real analytic. By standard properties of integral operators depending on a parameter (see also [16, Proposition 6.1]), the map of $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times L^1(\partial\Omega^i, \mathbb{R}^n)$ to $C^{m+1}(\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}), \mathbb{R}^n)$ which takes (ϵ, f) to the function $\int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s) f(s)\, d\sigma_s$ of $t \in \mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}})$ is real analytic. Since $E[\cdot]$ is real analytic from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ to $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ and since $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ is continuously imbedded in $L^1(\partial\Omega^i, \mathbb{R}^n)$ and $C^{m+1}(\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}), \mathbb{R}^n)$ is continuously imbedded into the space $C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}), \mathbb{R}^n)$, we conclude that the map of $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$ to $C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}}), \mathbb{R}^n)$ which takes ϵ to the map $\int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s) E[\epsilon](s) d\sigma_s$ of $t \in \mathrm{cl}\Omega(\epsilon'_{\tilde{\Omega}})$ is real analytic. □

We now consider the energy integral of the family $\{u(\epsilon,\cdot)\}_{\epsilon\in\,]0,\epsilon'[}$, and we prove the following.

Theorem 4.2 *Let the assumptions of Definition 3.3 hold. Then there exist* $\tilde{\epsilon} \in\,]0, \epsilon'[$ *and a real analytic operator* $\mathcal{F}$ *of* $]-\tilde{\epsilon}, \tilde{\epsilon}[$ *to* $\mathbb{R}$ *such that*

$$\mathcal{F}[\epsilon] = \mathcal{E}(\omega, u(\epsilon,\cdot)) \qquad \forall \epsilon \in\,]0, \tilde{\epsilon}[.$$

Moreover, $\mathcal{F}[0] = \mathcal{E}(\omega, \tilde{u}) \equiv \frac{1}{2}\int_{\Omega^o} \mathrm{tr}\left(T(\omega, D_x\tilde{u}) D_x^t \tilde{u}\right) dx.$

Proof: By the Divergence Theorem, we have

$$\begin{aligned}\int_{\Omega(\epsilon)} &\mathrm{tr}\left(T(\omega, D_x u(\epsilon,x)) D_x^t u(\epsilon,x)\right) dx \\ &= -\int_{\epsilon\partial\Omega^i} u^t(\epsilon,s) T(\omega, D_x u(\epsilon,s)) \nu_{\epsilon\Omega^i}(s)\, d\sigma_s \\ &\quad + \int_{\partial\Omega^o} u^t(\epsilon,s) T(\omega, D_x u(\epsilon,s)) \nu_{\Omega^o}(s)\, d\sigma_s \\ &= \int_{\partial\Omega^o} u^t(\epsilon,s) T(\omega, D_x u(\epsilon,s)) \nu_{\Omega^o}(s)\, d\sigma_s \\ &= \int_{\partial\Omega^o} u^t(\epsilon,s) G^o(s, u(\epsilon,s))\, d\sigma_s \qquad \forall \epsilon \in\,]0, \epsilon'[.\end{aligned}$$

Then it suffices to take $\tilde{\epsilon} = \epsilon_{\Omega(\epsilon')}$ (see Theorem 4.1) and to set

$$\mathcal{F}[\epsilon] \equiv \frac{1}{2}\int_{\partial\Omega^o} U_{\Omega(\epsilon')}[\epsilon]^t(s) G^o(s, U_{\Omega(\epsilon')}[\epsilon](s))\, d\sigma_s \qquad \forall \epsilon \in\,]0, \tilde{\epsilon}[.$$

By assumption (3.12), and by the real analyticity of the map which takes $\epsilon \in\,]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $U_{\Omega(\epsilon')}[\epsilon]$ in $C^{m,\alpha}(\mathrm{cl}\Omega(\epsilon'), \mathbb{R}^n)$, we easily deduce that $\mathcal{F}$ is real analytic. Finally, equality $U_{\Omega(\epsilon')}[0] = \tilde{u}_{|\mathrm{cl}\Omega(\epsilon')}$ and the definition of $\mathcal{F}$ ensure that the last part of the statement holds. □

5 A property of local uniqueness for the family $\{u(\epsilon,\cdot)\}_{\epsilon\in\,]0,\epsilon'[}$

We now show by means of the following theorem, the local uniqueness of the family $\{u(\epsilon,\cdot)\}_{\epsilon\in\,]0,\epsilon'[}$.

Theorem 5.1 *Let the assumptions of Definition 3.3 hold. If* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *is a sequence of* $]0,\epsilon_0[$ *converging to* 0 *and if* $\{u_j\}_{j\in\mathbb{N}}$ *is a sequence of functions such that*

$$\begin{gathered} u_j \in C^{m,\alpha}(\mathrm{cl}\Omega(\varepsilon_j),\mathbb{R}^n), \\ u_j \text{ solves (1.2) for } \epsilon=\varepsilon_j, \\ \lim_{j\to\infty} u_{j\,|\partial\Omega^o} = \tilde{u}_{|\partial\Omega^o} \text{ in } C^{m-1,\alpha}(\partial\Omega^o,\mathbb{R}^n), \end{gathered}$$

then there exists $j_0\in\mathbb{N}$ *such that* $u_j(\cdot)=u(\varepsilon_j,\cdot)$ *for all* $j\geq j_0$.

Proof: Since u_j solves problem (1.2), Theorem 3.1 ensures that there exist $(c_j,\eta_j,\rho_j)\in\mathbb{R}^n\times X_{m,\alpha}$ and $(\tilde{c},\tilde{\rho})\in\mathbb{R}^n\times C^{m-1,\alpha}(\partial\Omega^o,\mathbb{R}^n)$ such that

$$\begin{aligned} u_j &= u[\varepsilon_j,c_j,\eta_j,\rho_j], \qquad & M[\varepsilon_j,c_j,\eta_j,\rho_j] &= 0,\\ \tilde{u} &= u[0,\tilde{c},0,\tilde{\rho}], & M[0,\tilde{c},0,\tilde{\rho}] &= 0, \end{aligned}$$

and that

$$u_j = v^+[\omega,\mu_j]+c_j, \qquad \tilde{u}=v^+[\omega,\tilde{\rho}]+\tilde{c},$$

where

$$\mu_j(y)=\rho_j(y) \quad \text{if } y\in\partial\Omega^o, \qquad \mu_j(y)=\varepsilon_j^{1-n}\eta_j(y/\varepsilon_j) \quad \text{if } y\in\varepsilon_j\partial\Omega^i.$$

Now we rewrite equation $M[\epsilon,c,\eta,\rho]=0$ in the following form

$$\begin{aligned} &M_1[\epsilon,c,\eta,\rho]=0, \\ &M_2[\epsilon,c,\eta,\rho]=0 \qquad \text{on } \partial\Omega^i, \\ &-\frac{1}{2}\rho(t)+v_*[\omega,\rho](t)+\int_{\partial\Omega^i}\sum_{l=1}^n \eta_l(s)T(\omega,D_\xi\Gamma_n^l(\omega,t-\epsilon s))\nu^o(t)\,d\sigma_s \\ &\qquad\qquad -D_uG^o(t,\tilde{u}(t))\left\{\int_{\partial\Omega^i}\Gamma_n(\omega,t-\epsilon s)\eta(s)\,d\sigma_s+v[\omega,\rho](t)+c\right\} \\ &\quad = G^o\left(t,\int_{\partial\Omega^i}\Gamma_n(\omega,t-\epsilon s)\eta(s)\,d\sigma_s+v[\omega,\rho](t)+c\right) \\ &\qquad\qquad -D_uG^o(t,\tilde{u}(t))\left\{\int_{\partial\Omega^i}\Gamma_n(\omega,t-\epsilon s)\eta(s)\,d\sigma_s+v[\omega,\rho](t)+c\right\}, \end{aligned} \tag{5.1}$$

for all $t\in\partial\Omega^o$. Next we denote by $N[\cdot,\cdot,\cdot,\cdot]\equiv(N_l[\cdot,\cdot,\cdot,\cdot])_{l=1,2,3}$ the function of $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^n\times X_{m,\alpha}$ to $\mathbb{R}^n\times X_{m,\alpha}$ defined by $N_l\equiv M_l$ for $l=1,2$, and such that

N_3 equals the left hand side of the third equation in (5.1). Thus equation (5.1) can be rewritten as

$$
\begin{aligned}
N_1[\epsilon, c, \eta, \rho] &= 0 \\
N_2[\epsilon, c, \eta, \rho] &= 0 \qquad \text{on } \partial\Omega^i \\
N_3[\epsilon, c, \eta, \rho](t) &= G^o\left(t, \int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s)\eta(s)\, d\sigma_s + v[\omega,\rho](t) + c\right) \\
&\quad - D_u G^o(t, \tilde{u}(t)) \left\{ \int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s)\eta(s)\, d\sigma_s + v[\omega,\rho](t) + c \right\}
\end{aligned}
\tag{5.2}
$$

for all $t \in \partial\Omega^o$. By our assumption on F_{G^o}, and by the known form of the differential of a composition operator, we have that $D_u G^o(t, \tilde{u}(t))$ must be an element of $C^{m-1,\alpha}(\partial\Omega^o, M_n(\mathbb{R}))$ (see [16, Proposition 6.3], where the scalar case has been worked out, but the proof is the same for matrix-valued functions). Then by standard properties of integrals depending on a parameter (see [16, Theorem 6.2]), and by Theorem A.2, the map N is real analytic. Next we note that $N[\epsilon, \cdot, \cdot, \cdot]$ is linear for all fixed $\epsilon \in]-\epsilon_0, \epsilon_0[$. Accordingly, the map of $]-\epsilon_0, \epsilon_0[$ to $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha})$ which takes ϵ to $N[\epsilon, \cdot, \cdot, \cdot]$ is real analytic. We also note that

$$
N[0, \cdot, \cdot, \cdot] = \partial_{(c,\eta,\rho)} M[0, \tilde{c}, 0, \tilde{\rho}](\cdot, \cdot, \cdot),
$$

and thus that $N[0, \cdot, \cdot, \cdot]$ is a linear homeomorphism (see the proof of Theorem 3.2). Since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map which takes a linear invertible operator to its inverse is real analytic (cf. e.g., Hille and Phillips [8, Theorems 4.3.2 and 4.3.4]), there exists $\epsilon'' \in]0, \epsilon_0[$ such that the map $\epsilon \mapsto N[\epsilon, \cdot, \cdot, \cdot]^{(-1)}$ is real analytic from $]-\epsilon'', \epsilon''[$ to $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha})$. Since $M[\varepsilon_j, c_j, \eta_j, \rho_j] = 0$, the invertibility of $N[\varepsilon_j, \cdot, \cdot, \cdot]$ and equality (5.2) guarantee that

$$
(c_j, \eta_j, \rho_j) = N[\varepsilon_j, \cdot, \cdot, \cdot]^{(-1)}[0, 0, F_{G^o}[u_{j|\partial\Omega^o}] - F_{D_u G^o}[\tilde{u}_{|\partial\Omega^o}] u_{j|\partial\Omega^o}]
$$

if $\varepsilon_j \in]0, \epsilon''[$. By (3.12), $F_{G^o}[\cdot]$ is continuous in $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$. Hence,

$$
\lim_{j\to\infty} F_{G^o}[u_{j|\partial\Omega^o}] - F_{D_u G^o}[\tilde{u}_{|\partial\Omega^o}] u_{j|\partial\Omega^o} = F_{G^o}[\tilde{u}_{|\partial\Omega^o}] - F_{D_u G^o}[\tilde{u}_{|\partial\Omega^o}] \tilde{u}_{|\partial\Omega^o}, \tag{5.3}
$$

in $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$. The analyticity of $\epsilon \mapsto N[\epsilon, \cdot, \cdot, \cdot]^{(-1)}$ guarantees that

$$
\lim_{j\to\infty} N[\varepsilon_j, \cdot, \cdot, \cdot]^{(-1)} = N[0, \cdot, \cdot, \cdot]^{(-1)}, \tag{5.4}
$$

in $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha})$. Since the evaluation map of $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha}) \times (\mathbb{R}^n \times X_{m,\alpha})$ to $\mathbb{R}^n \times X_{m,\alpha}$, which takes a pair (A, v) to $A[v]$ is bilinear and continuous, the limiting relations of (5.3) and (5.4) imply that

$$
\begin{aligned}
&\lim_{j\to\infty} (c_j, \eta_j, \rho_j) \\
&\qquad = \lim_{j\to\infty} N[\varepsilon_j, \cdot, \cdot, \cdot]^{(-1)}\left[0, 0, F_{G^o}[u_{j|\partial\Omega^o}] - F_{D_u G^o}[\tilde{u}_{|\partial\Omega^o}] u_{j|\partial\Omega^o}\right] \\
&\qquad = N[0, \cdot, \cdot, \cdot]^{(-1)}\left[0, 0, F_{G^o}[\tilde{u}_{|\partial\Omega^o}] - F_{D_u G^o}[\tilde{u}_{|\partial\Omega^o}] \tilde{u}_{|\partial\Omega^o}\right]
\end{aligned}
\tag{5.5}
$$

in $\mathbb{R}^n \times X_{m,\alpha}$. Since $M[0, \tilde{c}, 0, \tilde{\rho}] = 0$, the right-hand side of (5.5) equals $(\tilde{c}, 0, \tilde{\rho})$. Hence,

$$\lim_{j\to\infty}(\varepsilon_j, c_j, \eta_j, \rho_j) = (0, \tilde{c}, 0, \tilde{\rho})$$

in $]-\epsilon'', \epsilon''[\times\mathbb{R}^n \times X_{m,\alpha}$. Thus Theorem 3.2 implies that there exists $j_0 \in \mathbb{N}$ such that

$$c_j = C[\varepsilon_j], \quad \eta_j = E[\varepsilon_j], \quad \rho_j = R[\varepsilon_j] \qquad \text{for } j \geq j_0.$$

Accordingly, $u_j(\cdot) = u(\varepsilon_j, \cdot)$ for $j \geq j_0$ (see Definition 3.3). □

A Classical results of potential theory for linearized elasticity

As is well known, the following Lemma holds.

Lemma A.1 *Let* $\omega \in]1-(2/n), +\infty[$. *Then there exists a constant* $c > 0$ *such that*

$$\mathrm{tr}\Big(T(\omega, A)A^t\Big) \geq c|A + A^t|^2 \qquad \forall A \in M_n(\mathbb{R}). \tag{A.1}$$

Proof: By a simple computation, we have

$$\mathrm{tr}\Big(T(\omega, A)A^t\Big) = (\omega - 1)(\mathrm{tr}\, A)^2 + \frac{1}{2}|A + A^t|^2 \qquad \forall A \in M_n(\mathbb{R}). \tag{A.2}$$

If $\omega \geq 1$, we can take $c = 1/2$. If instead $\omega < 1$, we note that

$$\mathrm{tr}\Big(T(\omega, A)A^t\Big) = (\omega + 1)\sum_{i=1}^{n} A_{ii}^2 + (\omega - 1)\sum_{i,j=1,i\neq j}^{n} A_{ii}A_{jj} \tag{A.3}$$

$$+ \frac{1}{2}\sum_{i,j=1,i\neq j}^{n}(A_{ij} + A_{ji})^2 \qquad \forall A \in M_n(\mathbb{R}),$$

and that

$$\sum_{i,j=1,i\neq j}^{n} A_{ii}A_{jj} \leq \frac{1}{2}\sum_{i,j=1,i\neq j}^{n}\left[A_{ii}^2 + A_{jj}^2\right] = (n-1)\sum_{i=1}^{n} A_{ii}^2.$$

Hence, the right-hand side of (A.3) is greater or equal than

$$n(\omega - 1 + (2/n))\sum_{i=1}^{n} A_{ii}^2 + \frac{1}{2}\sum_{i,j=1,i\neq j}^{n}(A_{ij} + A_{ji})^2,$$

which in turn is greater or equal to $\frac{n}{4}(\omega - 1 + (2/n))|A + A^t|^2$. Hence, we can take $c \equiv \min\{(1/2), n(\omega - 1 + (2/n))/4\}$. □

Next we introduce the following known result concerning the regularity of simple and double layer potentials.

Theorem A.2 *Let $\alpha \in]0,1[$, $\omega \in]1-(2/n),+\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\Omega \subseteq \mathbb{B}_n(0,R)$.*

(i) If $\mu \in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $v[\omega,\mu] \in C^0(\mathbb{R}^n,\mathbb{R}^n)$. The map which takes μ to $v^+[\omega,\mu]$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n)$. The map which takes μ to $v^-[\omega,\mu]_{|\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega}$ is linear and continuous from the space $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega,\mathbb{R}^n)$.

(ii) The map which takes μ to $w^+[\omega,\mu]$ is linear and continuous from $C^{m,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n)$. The map which takes μ to $w^-[\omega,\mu]_{|\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega}$ is linear and continuous from $C^{m,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega,\mathbb{R}^n)$.

(iii) The map which takes μ to $v_*[\omega,\mu]$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$.

Proof: We first prove statement (i). Let B_n be the function of $\mathbb{R}^n\setminus\{0\}$ to $\mathbb{R}$ defined by

$$B_n(\xi) \equiv \begin{cases} (-1)^{(n-2)/2}(4s_n)^{-1}|\xi|^{4-n}\log|\xi| & \text{if } n \in \{2,4\},\\ [2(n-2)(n-4)s_n]^{-1}|\xi|^{4-n} & \text{if } n \in \mathbb{N}\setminus\{0,1,2,4\}, \end{cases}$$

for all $\xi \in \mathbb{R}^n\setminus\{0\}$. As is well known, B_n is the fundamental solution of the biharmonic operator Δ^2. Let $v_{B_n}[\mu]$ denote the single layer potential corresponding to the kernel B_n and density μ. Then a straightforward computation shows that

$$v[\omega,\mu] = \left(\Delta - \frac{\omega}{\omega+1}\nabla\mathrm{div}\right)\left[(v_{B_n}[\mu_i])_{i=1,\dots,n}\right],$$

and statement (i) follows by Miranda [25, Theorem 5.I].

We now prove statement (ii). Let $\mathcal{M}_{ij}(\nu_\Omega)$ denote the tangential differential operator defined by

$$\mathcal{M}_{ij}(\nu_\Omega(x)) \equiv \nu_{\Omega,i}(x)\frac{\partial}{\partial\xi_j} - \nu_{\Omega,j}(x)\frac{\partial}{\partial\xi_i}$$

for all $x \in \partial\Omega$ and $i,j = 1,\dots,n$. We denote by $\mathcal{M}(\nu_\Omega)$ the matrix $(\mathcal{M}_{ij}(\nu_\Omega))_{ij=1,\dots,n}$. By an elementary computation, we can verify that

$$\begin{aligned}\left(T(\omega, D\Gamma_n^i(\omega,\xi))\nu_\Omega(x)\right)_j &= \delta_{ij}\nu_\Omega(x)\cdot DS_n(\xi)\\ &\quad - \mathcal{M}_{ij}(\nu_\Omega(x))S_n(\xi) - 2\left(\mathcal{M}(\nu_\Omega(x))\Gamma_n(\omega,\xi)\right)_{ji}\end{aligned}$$

for all $x \in \partial\Omega$, $\xi \in \mathbb{R}^n\setminus\{0\}$ and $i,j = 1,\dots,n$. By the Divergence Theorem, we have

$$\begin{aligned}&\int_{\partial\Omega}(\mathcal{M}(\nu_\Omega(y))S_n(x-y))\mu(y)\,d\sigma_y\\ &\qquad = -\int_{\partial\Omega} S_n(x-y)(\mathcal{M}(\nu_\Omega(y))\mu(y))\,d\sigma_y,\\ &\int_{\partial\Omega}(\mathcal{M}(\nu_\Omega(y))\Gamma_n(\omega,x-y))^t\mu(y)\,d\sigma_y\\ &\qquad = \int_{\partial\Omega}\Gamma_n(\omega,x-y)(\mathcal{M}(\nu_\Omega(y))\mu(y))\,d\sigma_y,\end{aligned}$$

for all $x \in \mathbb{R}^n$ (cf. e.g., Kupradze et al. [12, Chapter V, §1]). Hence, we deduce that

$$w[\omega, \mu](x) = \int_{\partial\Omega} \left(\frac{\partial}{\partial\nu(y)} S_n(x-y) \right) \mu(y)\, d\sigma_y + \int_{\partial\Omega} S_n(x-y)(\mathcal{M}(\nu_\Omega(y))\mu(y))\, d\sigma_y - 2\int_{\partial\Omega} \Gamma_n(\omega, x-y)(\mathcal{M}(\nu_\Omega(y))\mu(y))\, d\sigma_y \qquad \forall x \in \mathbb{R}^n.$$

Then statement (ii) follows by statement (i) and by known properties of regularity of simple and double layer potentials corresponding to the fundamental solution S_n of the Laplace operator (cf. e.g., Miranda [25, Theorem 5.I], [22, Theorem 3.1]). Statement (iii) is an immediate consequence of statement (i) and of standard jump properties for $T(\omega, Dv[\omega, \mu])\nu_\Omega$. □

We also note that if α, ω, m and Ω are as in Theorem A.2, then we can write the Green formula in the form

$$w[\omega, u_{|\partial\Omega}](x) - v[\omega, T(\omega, (Du)_{|\partial\Omega})\nu_\Omega](x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^-, \end{cases} \tag{A.4}$$

for all $u \in C^{1,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n)$ which solve $L[\omega]u = 0$ in Ω (cf. Kupradze et al. [12, Chapter III, §2.1]). Then we have the following classical result. For a proof in case $n = 2$, we refer to the book of Muskhelishvili [27, Chapter 19] (see also Kupradze [11, Chapter VIII, §§5–6]). For a proof in case $n \geq 3$, we refer to the book of Mikhlin and Prössdorf [24, Chapter XIV, §6], who actually worked out the proof for the case $n = 3$. However, the proof is the same for $n \geq 3$.

Theorem A.3 *Let $\alpha \in]0, 1[$, $\omega \in]1 - (2/n), +\infty[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let W denote the map of $L^2(\partial\Omega, \mathbb{R}^n)$ to itself defined by*

$$W[\mu] \equiv w[\omega, \mu]_{|\partial\Omega} \qquad \forall \mu \in L^2(\partial\Omega, \mathbb{R}^n).$$

Then the adjoint W^* to W is delivered by the following equality

$$W^*[\mu] = v_*[\omega, \mu]_{|\partial\Omega} \qquad \forall \mu \in L^2(\partial\Omega, \mathbb{R}^n).$$

Moreover, the operators $\pm\frac{1}{2}I + W$ and $\pm\frac{1}{2}I + W^*$ are Fredholm of index zero in $L^2(\partial\Omega, \mathbb{R}^n)$.

Next we note that by Ševčenko [30, p. 929 of Engl. transl.] and Mikhlin and Prössdorf [24, Chapter XIII, Thm. 7.1], one can prove the following classical result (which can probably be considered as ‘folklore’).

Theorem A.4 *Let $\alpha \in]0, 1[$, $\omega \in]1 - (2/n), +\infty[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in L^2(\partial\Omega, \mathbb{R}^n)$. If at least one of the four functions $\pm\frac{1}{2}\mu + W[\mu]$ and $\pm\frac{1}{2}\mu + W^*[\mu]$ belongs to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$, then $\mu \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$.*

We now turn to describe the kernels of $\pm\frac{1}{2}I + W$ and of $\pm\frac{1}{2}I + W^*$ in a fashion which generalizes that of Folland [6, Chapter 3] for the potentials associated to the fundamental solution of the Laplace operator. To do so, we find convenient to denote by $\mathbb{R}^n_\Omega$ the set of functions of Ω to $\mathbb{R}^n$ which are constant, and by $\mathbb{R}^n_{\Omega,\mathrm{loc}}$ the set of functions of Ω to $\mathbb{R}^n$ which are constant on each connected component of Ω, and by $(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$ the set of functions on $\partial\Omega$ which are trace on $\partial\Omega$ of functions of $\mathbb{R}^n_{\Omega,\mathrm{loc}}$. Then we denote by $\mathcal{R}$ the set of functions ρ of $\mathbb{R}^n$ to $\mathbb{R}^n$ such that there exists a skew symmetric matrix $A \in M_n(\mathbb{R})$ and a constant $b \in \mathbb{R}^n$ such that $\rho(x) = Ax + b$ for all $x \in \mathbb{R}^n$, and we denote by $\mathcal{R}_\Omega$ the set of restrictions to Ω of the functions of $\mathcal{R}$, and we denote by $\mathcal{R}_{\Omega,\mathrm{loc}}$ the set of functions of Ω to $\mathbb{R}^n$ which equal an element of $\mathcal{R}$ on each connected component of Ω, and we denote by $(\mathcal{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$ the set of functions on $\partial\Omega$ which are trace on $\partial\Omega$ of functions of $\mathcal{R}_{\Omega,\mathrm{loc}}$. Also, if $\mathcal{X}$ is a vector subspace of $L^1(\partial\Omega, \mathbb{R}^n)$ with Ω of class C^1, we set

$$\mathcal{X}_{0'} \equiv \Big\{ f \in \mathcal{X} : \int_{\partial\Omega'} f\, d\sigma = 0, \quad \text{for all connected components } \Omega' \text{ of } \Omega \Big\}. \tag{A.5}$$

Then we have the following.

Theorem A.5 *Let $\alpha \in]0,1[$, $\omega \in]1-(2/n), +\infty[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) $v[\omega, \mu]_{|\partial\Omega} \in \mathrm{Ker}(-\frac{1}{2}I + W)$ *for all* $\mu \in \mathrm{Ker}(-\frac{1}{2}I + W^*)$.

(ii) The map of $\left(\mathrm{Ker}(-\frac{1}{2}I + W^*)\right)_0$ *to* $\mathrm{Ker}(-\frac{1}{2}I + W)$ *which takes μ to $v[\omega, \mu]_{|\partial\Omega}$ is injective (see (2.6)).*

(iii) Let $n \geq 3$. The map of $\mathrm{Ker}(-\frac{1}{2}I + W^*)$ *to* $\mathrm{Ker}(-\frac{1}{2}I + W)$ *which takes μ to $v[\omega, \mu]_{|\partial\Omega}$ is an isomorphism.*

(iv) $\mathrm{Ker}(-\frac{1}{2}I + W)$ *is the direct sum of* $v[\omega, \left(\mathrm{Ker}(-\frac{1}{2}I + W^*)\right)_{0'}]_{|\partial\Omega}$ *and of* $(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$. *Such a sum however, is not necessarily orthogonal.*

(v) $\mathrm{Ker}(-\frac{1}{2}I + W) = (\mathcal{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$.

Proof: Let $\mu \in \mathrm{Ker}(-\frac{1}{2}I + W^*)$. By Theorem A.4, we know that $\mu \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. Hence, Theorem A.2 (i) implies that $v^+[\omega, \mu] \in C^{1,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n)$. Now by the Green formula applied to the function $v[\omega, \mu]$, we have

$$w[\omega, v[\omega, \mu]_{|\partial\Omega}](x) - v[\omega, T(\omega, Dv^+[\omega, \mu]_{|\partial\Omega})\nu_\Omega](x) = 0,$$

for all $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$ (cf. (A.4)). Then by standard jump properties for simple elastic layer potentials, we have

$$w^-[\omega, v[\omega, \mu]_{|\partial\Omega}](x) = v^-\Big[\omega, -\frac{1}{2}\mu + W^*[\mu]\Big](x) = v^-[\omega, 0](x) = 0,$$

for all $x \in \partial\Omega$, *i.e.*, $w^-[\omega, v[\omega,\mu]_{|\partial\Omega}]$ vanishes on the boundary of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then standard jump properties of elastic double layer potentials imply that statement (i) holds.

We now prove statement (ii). Let $\mu \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ be such that $\frac{1}{2}\mu = W^*[\mu]$ on $\partial\Omega$ and $v[\omega,\mu]_{|\partial\Omega} = 0$. Since $v^+[\omega,\mu]$ solves the homogeneous Dirichlet problem for $L[\omega]$ in Ω, we deduce that $v[\omega,\mu] = 0$ in $\mathrm{cl}\Omega$. Since $v^-[\omega,\mu]$ solves the homogeneous Dirichlet problem for $L[\omega]$ in $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, the known properties of decay at infinity for the simple layer and condition $\int_{\partial\Omega} \mu \, d\sigma = 0$ in case $n = 2$, imply that $v[\omega,\mu] = 0$ in $\mathbb{R}^n \setminus \Omega$ and thus in $\mathbb{R}^n$. Then by standard jump properties of simple elastic layer potentials, we deduce that

$$\mu = T(\omega, Dv^-[\omega,\mu])\nu_\Omega - T(\omega, Dv^+[\omega,\mu])\nu_\Omega = 0 \qquad \text{on } \partial\Omega,$$

which implies the validity of statement (ii). Note that here we exploit condition $\int_{\partial\Omega} \mu \, d\sigma = 0$ only in case $n = 2$.

We now prove statement (iii). By Theorems A.3, A.4, we know that the kernels in statement (iii) are of equal finite dimension. Thus it suffices to show that the map which takes μ to $v[\omega,\mu]_{|\partial\Omega}$ induces an injection, a fact which follows by the proof of statement (ii).

We now prove statement (iv). We first prove that if $\mu \in (\mathrm{Ker}(-\frac{1}{2}I + W^*))_{0'}$ and $v[\omega,\mu]_{|\partial\Omega} = \rho \in (\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$, then $\mu = 0$. By standard jump relations for elastic simple layer potentials, we have

$$\mu = T(\omega, Dv^-[\omega,\mu]_{|\partial\Omega})\nu_\Omega. \tag{A.6}$$

Now let $\Omega_1, \dots, \Omega_N$ be the connected components of Ω. By the behavior at infinity of $v[\omega,\mu]$ and by the Divergence Theorem applied to the exterior of Ω, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} &\mathrm{tr}\Big(T(\omega, Dv^-[\omega,\mu])D^t v^-[\omega,\mu]\Big)\,dx \\ &= -\int_{\partial\Omega} v^-[\omega,\mu]^t T(\omega, Dv^-[\omega,\mu])\nu_\Omega \, d\sigma = -\int_{\partial\Omega} \rho^t_{|\partial\Omega}\mu \, d\sigma \\ &= -\sum_{j=1}^N \rho^t_{|\partial\Omega_j} \int_{\partial\Omega_j} \mu \, d\sigma = 0.\end{aligned}$$

Since $\omega \in]1-(2/n), +\infty[$, inequality (A.1) implies that $v^-[\omega,\mu]$ belongs to $\mathcal{R}_{\mathbb{R}^n\setminus\mathrm{cl}\Omega,\mathrm{loc}}$. Since $Dv^-[\omega,\mu]$ equals a skew-symmetric matrix on every connected component of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, we conclude that $T(\omega, Dv^-[\omega,\mu]) = 0$ in $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, and that accordingly $\mu = 0$ by (A.6). Next we note that $(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$ is contained in $\mathrm{Ker}(-\frac{1}{2}I + W)$. Indeed, if $\rho \in (\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$, then the Green representation formula implies that $w[\omega,\rho](x) = 0$ for all $x \in \mathbb{R}^n \setminus \mathrm{cl}\Omega$, and thus that $w^-[\omega,\rho](x) = 0$ for all $x \in \partial\Omega$, and accordingly $\rho \in \mathrm{Ker}(-\frac{1}{2}I + W)$. By Theorem A.3, we have

$$\dim \mathrm{Ker}\Big(-\frac{1}{2}I + W^*\Big) = \dim \mathrm{Ker}\Big(-\frac{1}{2}I + W\Big).$$

Clearly, we have

$$\dim \left\{ \Big(\mathrm{Ker}\Big(-\frac{1}{2}I + W^*\Big)\Big) \Big/ \Big(\mathrm{Ker}\Big(-\frac{1}{2}I + W^*\Big)\Big)_{0'} \right\} \le nN,$$

(cf. (A.5)). Hence, statement (ii) implies that

$$\dim\left\{\left(\mathrm{Ker}\Big(-\frac{1}{2}I+W\Big)\right)\Big/ v\Big[\omega,\Big(\mathrm{Ker}\Big(-\frac{1}{2}I+W^*\Big)\Big)_{0'}\Big]\right\}\le nN.$$

Since the dimension of $(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$ is nN, and

$$(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}\cap v\Big[\omega,\Big(\mathrm{Ker}\Big(-\frac{1}{2}I+W^*\Big)\Big)_{0'}\Big]=\{0\},$$

we conclude that statement (iv) holds.

We now prove statement (v). If $\rho\in\mathrm{Ker}(-\frac{1}{2}I+W)$, then (iv) implies that ρ is the sum of an element b of $(\mathbb{R}^n_{\Omega,\mathrm{loc}})_{|\partial\Omega}$ and of a simple layer $v[\omega,\mu]_{|\partial\Omega}$ with μ in $(\mathrm{Ker}(-\frac{1}{2}I+W^*))_{0'}$. Clearly, $T(\omega,Dv^+[\omega,\mu])\nu_\Omega=0$ on $\partial\Omega$, and thus the Divergence Theorem together with inequality (A.1) imply that $v^+[\omega,\mu]_{|\partial\Omega}\in(\mathcal{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$. Hence, $\rho=b+v^+[\omega,\mu]_{|\partial\Omega}\in(\mathcal{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$. Conversely, let $\psi\in\mathcal{R}_{\Omega,\mathrm{loc}}$. Let ρ be the trace of ψ on $\partial\Omega$. Then we clearly have $L[\omega](\psi)=0$ in Ω and $T(\omega,D\psi)\nu_\Omega=0$ on $\partial\Omega$. Then by the Green representation formula, we have $\psi(x)=w[\omega,\rho](x)$ for all $x\in\Omega$, and thus $\rho=w^+[\omega,\rho]$ on $\partial\Omega$. Hence, $w^-[\omega,\rho]=-\rho+w^+[\omega,\rho]=0$, and thus the proof of statement (v) is complete. □

Next we observe that if $\mu\in L^2(\partial\Omega,\mathbb{R}^n)$, then $-\frac{1}{2}\mu+W^*[\mu]$ must be orthogonal to the kernel of $-\frac{1}{2}I+W$, which we have just seen to coincide with $(\mathcal{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$. Hence, $\int_{\partial\Omega}-\frac{1}{2}\mu+W^*[\mu]\,d\sigma=0$ and

$$\int_{\partial\Omega}\mu\,d\sigma=\int_{\partial\Omega}\frac{1}{2}\mu+W^*[\mu]\,d\sigma. \tag{A.7}$$

In particular, if $\int_{\partial\Omega}\frac{1}{2}\mu+W^*[\mu]\,d\sigma=0$, then we have $\int_{\partial\Omega}\mu\,d\sigma=0$ no matter whether $n=2$ or $n=3$. Hence, by arguing so as to prove Theorem A.5, we can prove the following.

Theorem A.6 *Let $\alpha\in]0,1[$, $\omega\in]1-(2/n),+\infty[$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) The operator of $\mathrm{Ker}(\frac{1}{2}I+W^*)$ to $\mathrm{Ker}(\frac{1}{2}I+W)$ which takes μ to $v[\omega,\mu]_{|\partial\Omega}$ is an isomorphism.

(ii) $\mathrm{Ker}(\frac{1}{2}I+W)$ coincides with the set of $\rho\in(\mathcal{R}_{\mathbb{R}^n\setminus\mathrm{cl}\Omega,\mathrm{loc}})_{|\partial\Omega}$ which vanish on the boundary of the unbounded connected component of $\mathbb{R}^n\setminus\mathrm{cl}\Omega$.

Finally, we have the following.

Theorem A.7 *Let $\alpha\in]0,1[$, $\omega\in]1-(2/n),+\infty[$, $m\in\mathbb{N}\setminus\{0\}$. Let Ω be an open and bounded subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\mu\in L^2(\partial\Omega,\mathbb{R}^n)$. If either $\frac{1}{2}\mu+W^*[\mu]$ or $-\frac{1}{2}\mu+W^*[\mu]$ belongs to $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $\mu\in C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$.*

Proof: Case $m = 1$ follows by Theorem A.4. Thus we now consider case $m \geq 2$. We first assume that $-\frac{1}{2}\mu + W^*[\mu] \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$. By Theorem A.4, we have $\mu \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. Then by standard jump properties of simple elastic layer potentials, we have

$$T(\omega, Dv^+[\omega, \mu])\nu_\Omega = -\frac{1}{2}\mu + W^*[\mu] \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n).$$

Hence, standard results in elliptic regularity theory imply that $v^+[\omega, \mu] \in C^{m,\alpha}(\text{cl}\Omega, \mathbb{R}^n)$ (cf. Agmon, Douglis and Nirenberg [1, Theorem 9.3]). Now let $R > 0$ be such that $\text{cl}\Omega \subseteq \mathbb{B}_n(0, R)$. Since $v^-[\omega, \mu]_{|\partial\Omega} = v^+[\omega, \mu]_{|\partial\Omega} \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$ and $v^-[\omega, \mu]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0, R), \mathbb{R}^n)$, again a classical result in elliptic regularity theory implies that $v^-[\omega, \mu] \in C^{m,\alpha}(\text{cl}\mathbb{B}_n(0, R) \setminus \Omega, \mathbb{R}^n)$ (cf. Agmon, Douglis, and Nirenberg [1, Theorem 9.3]). Then by standard jump properties of elastic simple layer potentials, we deduce that

$$\mu = T(\omega, Dv^-[\omega, \mu])\nu_\Omega - T(\omega, Dv^+[\omega, \mu])\nu_\Omega \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n).$$

Case $\frac{1}{2}\mu + W^*[\mu] \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ can be treated similarly. □

Remark A.8 Under the assumptions of Theorem A.6, if $\mathbb{R}^n \setminus \text{cl}\Omega$ is connected, then Theorem A.6 (ii) together with Theorem A.3 and Theorem A.7 imply that $\frac{1}{2}I + W^*$ is a linear homeomorphism of $L^2(\partial\Omega, \mathbb{R}^n)$ onto itself and of $C^{r,\alpha}(\partial\Omega, \mathbb{R}^n)$ onto itself, for all $r \in \{0, \dots, m-1\}$.

Finally, we note that the following holds.

Theorem A.9 *Let* $\alpha \in]0, 1[$, $\omega \in]1 - (2/n), +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. *Let* Ω *be an open and bounded subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Then* $\pm\frac{1}{2}I + W^*$ *are Fredholm operators of index* 0 *in* $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$.

Proof: By Theorem A.7, the kernels of the operators $\pm\frac{1}{2}I + W^*$ acting in $L^2(\partial\Omega, \mathbb{R}^n)$ are actually contained in the space $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$. As is well known, $\pm\frac{1}{2}I + W^*$ is a Fredholm operator of index 0 in $L^2(\partial\Omega, \mathbb{R}^n)$ (cf. Theorem A.3). Then by exploiting again Theorem A.7, one can easily show that the image of $\pm\frac{1}{2}I + W^*$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ coincides with the subset of $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ consisting of those functions f such that $\int_{\partial\Omega} f\phi\, d\sigma = 0$ for all $\phi \in \text{Ker}\,(\pm\frac{1}{2}I + W)$. Since $\dim \text{Ker}\,(\pm\frac{1}{2}I + W) = \dim \text{Ker}\,(\pm\frac{1}{2}I + W^*)$ is finite, the operators $\pm\frac{1}{2}I + W^*$ are easily seen to be Fredholm of index 0 in $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$. □

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