Forum Math. **21** (2009), 579–599
DOI 10.1515/FORUM.2009.029

Forum
Mathematicum
© de Gruyter 2009

A general notion of algebraic entropy and the rank-entropy

Luigi Salce and Paolo Zanardo

(Communicated by Rüdiger Göbel)

Abstract. We give a general definition of a subadditive invariant i of Mod(R), where R is any ring, and the related notion of algebraic entropy of endomorphisms of R-modules, with respect to i. We examine the properties of the various entropies that arise in different circumstances. Then we focus on the rank-entropy, namely the entropy arising from the invariant 'rank' for Abelian groups. We show that the rank-entropy satisfies the Addition Theorem. We also provide a uniqueness theorem for the rank-entropy.

2000 Mathematics Subject Classification: 20K30, 20K20, 13C99.

Introduction

The notion of entropy is ubiquitous. After its first appearance in thermodynamics in the first half of the XIX$^{\text{th}}$ century, it received a rigorous mathematical definition in statistical mechanics in the second half of the XIX$^{\text{th}}$ century. It was defined inside the mathematical formalism of quantum mechanics developed by John von Neumann in the 1930's, and then by Shannon in the setting of information theory, starting from the middle of the XX$^{\text{th}}$ century. Entropy has arisen recently in mathematical statistics, social sciences and life sciences.

Passing to mathematical subjects, entropy played a fundamental role in the ergodic theory of dynamical systems, developed by the Russian school since the beginning of the 1950's. Entropy appeared also in topology, first in a paper by Adler-Konheim-McAndrew [AKM] in 1965 for continuous self-maps of compact spaces. Later on, topological entropy was extended to more general spaces and deeply investigated by many authors. We refer to the notes by T. Ward [W] for a comprehensive presentation of these subjects.

Passing now to the algebraic setting, the algebraic entropy was introduced in the paper [AKM] quoted above, where one can find just a sketch of its definition for endomorphisms of Abelian groups. It was developed further in a paper by Weiss [W] in 1975, where some basic properties have been proved. But Weiss was mainly interested in comparing the algebraic entropy of an endomorphism ϕ of an Abelian group G, with the topological and the measure-theoretic (with respect to the Haar measure) entropies of the adjoint map of ϕ restricted to the Pontryagin dual of tG, the torsion subgroup of G. The definition of algebraic entropy given in [AKM] has the intrinsic limitation of being useful only for torsion groups, since

Research supported by MIUR, PRIN 2005

it trivializes for torsionfree groups. In order to overcome this limitation, Peters slightly modified in 1979, [P], the definition of algebraic entropy, and proved a result similar to Weiss's result, comparing the new entropy of ϕ with the Haar-measure-theoretic entropy of the adjoint map of ϕ itself, but only for automorphisms and under additional hypotheses. Recently, the two authors thoroughly investigated, with D. Dikranjan and B. Goldsmith in [DGSZ] and in [SZ], the algebraic entropy as defined in [AKM], and this has proved itself to be a very useful tool in the study of endomorphism rings of Abelian p-groups.

In all these different contexts, the entropy is always viewed as a measure of "expansiveness" of a transformation and a tool to measure the expansion "at infinity". So the technical definition of entropy, which obviously changes depending on the physical or mathematical setting one is dealing with, always involves the limit of a certain quantity, measured at the n-th iteration of the transformation and divided by n.

The first goal of this paper is to extend the notion of algebraic entropy to the setting of endomorphisms of modules over general unital rings (for a notion of algebraic entropy in a completely different setting see [BV] and [Br]). Our first goal is to start a formal development of the algebraic entropy associated with module theoretic invariants; this is done in Section 1.

The measure-theoretic entropy of a measure-preserving transformation of a space X with a measure μ, is defined in terms of finite measurable partitions of X. Passing to modules over an arbitrary unitary ring R, in order to define the algebraic entropy of an endomorphism of a module M, one needs, in analogy with the measure μ, a tool to "measure" the size of the submodules of M. More precisely, the n-th iteration of the endomorphism, applied to a submodule F, gives rise to a submodule called the n-th partial trajectory of F, whose size is measured by an invariant.

In module theory there are many different invariants that measure various finiteness properties of the modules. There are two minimal requirements that an invariant of Mod(R) must satisfy in order to be able to associate an algebraic entropy to it. These two conditions identify what we call a *subadditive invariant* (see Definition 1). Additional conditions on the invariant ensure that the associated algebraic entropy is easily computable and more manageable; for instance, if the invariant is *additive* (see Definition 2), then the asymptotic growth of the n-th partial trajectories, measured by the invariant, becomes constant at large iterations (see Proposition 1.10). Therefore, dealing with additive invariants, one can avoid the calculation of the limit in order to compute the algebraic entropy of endomorphisms. We will consider some typical examples of invariants of modules, such as the logarithm of the cardinality, the rank, the minimal cardinality of a generating set, etc. We will see that some invariants fit nicely in our general definition of algebraic entropy, and some other invariants are not suitable for this purpose.

The properties satisfied by the different kind of invariants are the subject of Section 2.

The second goal of this paper is to investigate more carefully, in Section 3, the algebraic entropy obtained by using the rank of an Abelian group as an invariant of Mod($\mathbb{Z}$). We will call it the *rank-entropy*. We will show that it is a useful tool in the investigation of torsionfree groups, and that it trivializes for torsion groups. So the rank-entropy can be viewed as a counterpart of the algebraic entropy defined in [AKM] and investigated in [DGSZ] and [SZ]. We will give characterizations of endomorphisms with zero and finite rank-entropy. A version of the Addition Theorem will also be proved. Finally, we provide an axiomatic characterization of the rank-entropy, similar to that given for algebraic entropy in [DGSZ]. In fact, we prove that the rank-entropy is the unique collection of functions $h_G : \mathrm{End}(G) \to \mathbb{R}_{\geq 0} \cup \infty$ (where G ranges in the class of Abelian groups) which satisfy five characterizing conditions, including the Addition Theorem.

General references are the monographs [FS] for notions on modules, and [F] for notions on Abelian groups.

1 Algebraic entropy related to invariants

Let us start with the following

Definition 1. Given a ring R, a *subadditive invariant* of Mod(R) is a map $i : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is invariant under isomorphism and satisfies the following conditions:

(i) if $M \leq N$ are R-modules, then $i(N/M) \leq i(N)$;
(ii) if M and M' are R-modules, then $i(M+M') \leq i(M) + i(M')$.

The usual conventions for ∞ are assumed. The invariant i is said to be *faithful* if $i(M) = 0$ implies $M = 0$.

Remark 1. We could substitute condition (i) in Definition 1 by the weaker condition: if $\phi : N \to N$ is an endomorphism of an R-module N and $M \leq N$, then $i(M) \geq i(\phi M)$. Some points in the sequel should then be modified accordingly.

The notion of subadditive invariant defined above differs from the notion of invariant given by Vámos [V], who required that the map i takes cardinal values, and that, for any pair of modules M and N, the following equality holds:

(iii) $i(M \oplus N) = i(M) + i(N)$.

Note that (i) and (iii) imply (ii), but (i) and (ii) do not imply (iii), as Example 1.5 shows.

Definition 2. An *additive invariant* of Mod(R) is a map $i : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies the following condition:

(iv) if $0 \to M \to N \to N/M \to 0$ is an exact sequence in Mod(R), then $i(N) = i(M) + i(N/M)$.

Note that an additive invariant is subadditive and satisfies condition (iii), as well as the following one:

(v) if $M \leq N$ are R-modules, then $i(M) \leq i(N)$.

Let i be a subadditive invariant of Mod(R) and let Fin_i denote the subclass of Mod(R) consisting of the modules F such that $i(F) < \infty$. In view of the conditions (i) and (ii), the class Fin_i is evidently closed under quotients and under finite sums. If i is an additive invariant, then Fin_i is closed also under submodules and extensions. Fixed a module M, let $\mathrm{Fin}_i(M)$ denote the family of the submodules of M contained in Fin_i.

Example 1.1. If R is a field, for every vector space M over R let $i(M) = \dim(M)$ if M is finite-dimensional, otherwise let $i(M) = \infty$. Then i is a faithful additive invariant and Fin_i is the class of finite-dimensional vector spaces.

Example 1.2. If $R = \mathbb{Z}$, for every Abelian groups M set $i(M) = \log|M|$ if M is finite, otherwise set $i(M) = \infty$. Then i is a faithful additive invariant, in view of Lagrange's theorem, and Fin_i is the class of all finite Abelian groups.

Example 1.3. Let R be a Dedekind domain and F a finitely generated torsion R-module. Then F is isomorphic to a module of the form

$$\bigoplus_{1\le j\le k} R/P_j^{e_j} \cong \bigoplus_{1\le j\le k} R_{P_j}/P_j^{e_j}R_{P_j} \cong \bigoplus_{1\le j\le k} R_{P_j}/p_j^{e_j}R_{P_j},$$

where $e_j \geq 1$, $P_j \in \mathrm{Max}(R)$ for all j, and p_j is an element of R generating the ideal $P_jR_{P_j}$ of the localization R_{P_j}, which is a DVR. Using the above notation, let us consider the map $v : \mathrm{Mod}(R) \to \mathbb{N} \cup \{\infty\}$ defined by setting: $v(F) = \sum_{1\le j\le k} e_j$ if F is finitely generated torsion, and $v(M) = \infty$ if M is not finitely generated torsion. When R is local, i.e., a DVR, and F is a cyclic torsion module, then v coincides with the value of the generators of the annihilator ideal of F. From the theory of torsion modules over Dedekind domains it follows that the map v is a faithful additive invariant. Fin_v is the class of the finitely generated torsion R-modules.

Example 1.4. When $R = \mathbb{Z}$, for every Abelian group M set $\mathrm{rk}(M) = \dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$ if this dimension is finite, otherwise set $\mathrm{rk}(M) = \infty$. Then rk is an additive invariant, but not faithful, since $\mathrm{rk}(T) = 0$ for every torsion group T. $\mathrm{Fin}_{\mathrm{rk}}$ is the class of all Abelian groups of finite rank.

Example 1.5. If R is an arbitrary ring and M is a finitely generated R-module, let $\mathrm{gen}(M)$ be the minimal cardinal number of a generating set for M; set $\mathrm{gen}(M) = \infty$ if M is not finitely generated. Then gen is a faithful subadditive invariant. It satisfies condition (iii) under the assumption that R is commutative and local. In general it is not an additive invariant and it does not satisfy condition (v). Obviously, $\mathrm{Fin}_{\mathrm{gen}}$ is the class of all finitely generated modules. It is worth noting that, for a suitable choice of the ring R, the invariant gen may satisfy (iii) but not (v). For instance, take R to be a local commutative domain which is Noetherian but not a DVR. Then R satisfies (iii), but, if I is a non-principal ideal of R, we have $\mathrm{gen}(R) < \mathrm{gen}(I) < \infty$, hence (v) does not hold.

Example 1.6. If R is an arbitrary ring, let $\mathrm{Gd}(M)$ denote the Goldie dimension of M (see [FS], Ch. I). Then Gd is a faithful invariant which satisfies conditions (ii) and (iii), but, in general, it does not satisfy (i). So the Goldie dimension is not a subadditive invariant, and we shall not deal with it.

Example 1.7. Let R be a valuation domain and let M be an R-module. The Malcev rank $\mathrm{Mr}(M)$ of M is defined as the supremum of $\mathrm{gen}(N)$, where N ranges over the set of the finitely generated submodules of M. The invariant Mr is subadditive: in fact, it obviously satisfies condition (i), and, since R is a valuation domain, also condition (ii) holds (see [FS], Ch. XII). The invariant Mr obviously satisfies condition (v). It also satisfies (iii), since R is local.

Let now R denote a unitary ring and i a subadditive invariant of Mod(R).

Given an R-module M, an endomorphism $\phi : M \to M$ and a submodule F of M belonging to $\mathrm{Fin}_i(M)$, we define, for every $n > 0$,

$$T_n(\phi, F) = F + \phi F + \cdots + \phi^{n-1}F,$$

and

$$T(\phi, F) = \sum_{n>0} T_n(\phi, F) = \sum_{n\geq 0} \phi^n F.$$

The submodule $T(\phi, F)$ is called the *ϕ-trajectory* of F, and $T_n(\phi, F)$ is called the *n-th partial trajectory*.

Note that, since $F \in \mathrm{Fin}_i$, conditions (i) and (ii) imply that $T_n(\phi, F) \in \mathrm{Fin}_i$, as well.

Since $T_n(\phi, F) \leq T_{n+1}(\phi, F) = T_n(\phi, F) + \phi^n F$, using (i) and (ii) we get, for all n, m,

$$i(T_{n+m}(\phi, F)) \leq i(T_n(\phi, F)) + i(\phi^n(T_m(\phi, F))) \leq i(T_n(\phi, F)) + i(T_m(\phi, F)).$$

By a standard exercise in analysis (see, e.g., [Wa], Exercise 6.5, p. 42), the limit

$$H_i(\phi, F) = \lim_{n\to\infty} i(T_n(\phi, F))/n$$

exists and coincides with the infimum of the sequence $\{i(T_n(\phi, F))/n : n > 0\}$.

Definition 3. Let i be a subadditive invariant of Mod(R) and $\phi : M \to M$ an endomorphism of $M \in$ Mod(R). The *i-entropy* of ϕ is defined as

$$\mathrm{ent}_i(\phi) = \sup\{H_i(\phi, F) : F \in \mathrm{Fin}_i(M)\}$$

and the *i-entropy* of M as

$$\mathrm{ent}_i(M) = \sup\{\mathrm{ent}_i(\phi) : \phi \in \mathrm{End}_R(M)\}\,.$$

We have a first property for subadditive invariants which satisfy property (v).

Proposition 1.8. *Let R be a ring and i a subadditive invariant of* Mod(R) *satisfying (v). If $M \in \mathrm{Fin}_i$, then* $\mathrm{ent}_i(M) = 0$.

Proof. Pick arbitrary $\phi \in \mathrm{End}_R(M)$ and $F \leq M$. Since $T_n(\phi, F) \leq M$, for all $n > 0$, and property (v) holds, we readily get $H_i(\phi, F) \leq \lim_{n\to\infty} i(M)/n = 0$. Since ϕ and F were arbitrary, we get $\mathrm{ent}(M) = 0$. □

If i is just a subadditive invariant, it is difficult in general to compute the i-entropy of an endomorphism. But if i is an additive invariant, then this computation is made easier, since one can avoid computing the limit in the definition. To see this, we need the next technical lemma, which holds for any subadditive invariant, but which is particularly useful for the additive ones. Given an endomorphism $\phi : M \to M$, for each $F \in \mathrm{Fin}_i(M)$ and $n \geq 1$ we set

$$\alpha_{n+1} = i\left(\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)}\right) = i\left(\frac{\phi^n F}{T_n(\phi, F) \cap \phi^n F}\right).$$

In the present notation we have the following

Lemma 1.9. *If i is a subadditive invariant of* Mod(R)*, then $\alpha_{n+1} \le \alpha_n$ for each $n \ge 1$.*

Proof. Since ϕ and F remain unchanged during the proof, we write T_n in place of $T_n(\phi, F)$. The module $\phi^n F/(T_n \cap \phi^n F)$ is a quotient of the module $B_n = \phi^n F/(\phi T_{n-1} \cap \phi^n F)$, since $\phi T_{n-1} \cap \phi^n F$ is contained in $T_n \cap \phi^n F$. So $\alpha_{n+1} \le i(B_n)$ by the property (i) of the subadditive invariant i. From $\phi T_n = \phi T_{n-1} + \phi^n F$ we conclude that

$$B_n \cong \frac{\phi T_n}{\phi T_{n-1}} \cong \frac{T_n}{T_{n-1} + (T_n \cap \mathrm{Ker}\phi)}.$$

Since the latter group is a quotient of T_n/T_{n-1}, we conclude by the same argument that $i(B_n) \le \alpha_n$. Therefore $\alpha_{n+1} \le \alpha_n$. □

The preceding lemma allows us to control the growth of $i(T_n(\phi, F))$, as n increases, provided that the invariant i is additive. In fact in such case we get

$$i\left(\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)}\right) = i(T_{n+1}(\phi, F)) - i(T_n(\phi, F)).$$

If, furthermore, the invariant i takes values in a subset of $\mathbb{R}$ with the minimum condition, we have

Proposition 1.10. *If i is an additive invariant with values in a subset of $\mathbb{R}_{\ge 0}$ order-isomorphic to $\mathbb{N}$, then*

(i) *$H_i(\phi, F) = 0$ if and only if $i(T(\phi, F)) = i(T_n(\phi, F))$ for some n;*

(ii) *If $i(T(\phi, F)) > i(T_n(\phi, F))$ for all n, then $H_i(\phi, F) = \alpha$, where $\alpha = i(T_{n+1}(\phi, F)) - i(T_n(\phi, F))$ for all n large enough.*

Proof. Since i is additive, $\alpha_{n+1} = i(T_{n+1}(\phi, F)) - i(T_n(\phi, F)))$ for each n. From Lemma 1.9 it follows that the decreasing sequence of the α_n is stationary, hence, for n large enough, we have that $\alpha_n = \alpha$ for a fixed non-negative real number α. Clearly $\alpha = 0$ exactly when $i(T_{n+1}(\phi, F)) = i(T_n(\phi, F)))$ for each n large enough, in which case $i(T(\phi, F)) = i(T_n(\phi, F)))$. If $\alpha > 0$, then

$$H_i(\phi, F) = \lim_{k\to\infty} \frac{i(T_{n+k}(\phi, F))}{n+k} = \lim_{k\to\infty} \frac{i(T_n(\phi, F)) + k\alpha}{n+k} = \alpha.$$

□

The preceding proposition motivates the following

Definition 4. A subadditive invariant i is said to be *discrete* if its finite values form a subset of $\mathbb{R}_{\ge 0}$ order-isomorphic to $\mathbb{N}$.

Corollary 1.11. *If i is a discrete additive invariant, then the i-entropy of every endomorphism and of every module is either a maximum or ∞.* □

In the next example we show that Proposition 1.10 is not applicable to discrete subadditive invariants which are not additive. Recall the notion of "right shift endomorphism" of a countable direct sum $M = \bigoplus_{n\ge 0} H_n$ of copies H_n of the same module H (see also [DGSZ]).

The right shift endomorphism $\sigma : M \to M$ is defined extending the following assignments: for all $n \geq 0$, if $a \in H_n$, then $\sigma(a) = a \in H_{n+1}$.

The next example will show that, in general, the i-entropy of the right shift of a countable direct sums of copies of the same module H, with $i(H) < \infty$, does not coincide with $i(H)$. However, when either $i = \mathrm{rk}$ or $i = \log|-|$ (see Examples 1.4 and 1.2), the entropy of the right shift actually coincides with $i(H)$. Note that in the example we make use of the invariant gen, which does not satisfy condition (iii).

Example 1.12. Let R be a Dedekind domain admitting a non-principal ideal I. Let $M = \bigoplus_{j\in\mathbb{N}} I_j$, where $I_j = I$ for all j. Let $\sigma : M \to M$ be the right shift. As the only torsion submodule of M is $\{0\}$, the v-entropy of σ is 0, where v is the additive invariant defined in Example 1.3. If $i = \mathrm{rk}$, as in Example 1.4, for every $n \geq 1$ and every non-zero submodule F of M of finite rank we have that $\mathrm{rk}(T_{n+1}(\sigma,F)/T_n(\sigma,F)) = 1$, hence Proposition 1.10 yields $\mathrm{ent}_{\mathrm{rk}}(\sigma) = 1 = \mathrm{rk}(I)$. Let now $i = \mathrm{gen}$, and let $F = I_1$. For every $n \geq 1$ we have $\mathrm{gen}(T_{n+1}(\sigma,F)/T_n(\sigma,F)) = 2$, as $\mathrm{gen}(I) = 2$, but we cannot apply Proposition 1.10, since the invariant gen is only subadditive. Indeed, we have

$$\mathrm{gen}(T_n(\sigma,F)) = \mathrm{gen}(I_1 \oplus \cdots \oplus I_n) = \mathrm{gen}(R^{(n-1)} \oplus J),$$

where $J = I_1 \cdots I_n$, in view of the Steinitz property. Note that $\mathrm{gen}(R^{(n-1)} \oplus J)$ equals either n or $n+1$, according as J is principal or not. It follows that

$$H_i(\sigma,F) = \lim_{n\to\infty} \mathrm{gen}(T_n(\sigma,F))/n = 1,$$

while $\mathrm{gen}(T_{n+1}(\sigma,F)/T_n(\sigma,F)) = 2$.

The next lemma provides a useful inequality for additive invariants. It will be applied to obtain a similar inequality for i-entropies in Proposition 2.1, but an additional assumption on i will be needed there.

Lemma 1.13. *If i is an additive invariant of* Mod(R)*, $\phi : M \to M$ is an endomorphism of $M \in$* Mod(R) *and N is a ϕ-invariant submodule of M, then, for every $F \in \mathrm{Fin}_i(M)$ we have*

$$H_i(\phi,F) \geq H_i(\bar{\phi},(F+N)/N) + H_i(\phi|_N, F\cap N),$$

where $\bar{\phi} : M/N \to M/N$ is the induced endomorphism.

Proof. For each n consider the exact sequence

$$0 \to T_n(\phi,F)\cap N \to T_n(\phi,F) \to (T_n(\phi,F)+N)/N \to 0.$$

Since i is additive and $(T_n(\phi,F)+N)/N = T_n(\bar{\phi},(F+N)/N)$, we get

$$i(T_n(\phi,F)) = i(T_n(\phi,F)\cap N) + i(T_n(\bar{\phi},(F+N)/N)).$$

Since $T_n(\phi|N, F\cap N)$ is a submodule of $T_n(\phi,F)\cap N$ and the invariant i satisfies condition (v), we deduce that

$$i(T_n(\phi,F)) \geq i(T_n(\bar{\phi},(F+N)/N)) + i(T_n(\phi|_N, F\cap N)).$$

Dividing by n and passing to the limit we get the desired inequality. □

2 Properties of the i-entropy

In what follows, R will be an arbitrary unital ring, i a subadditive invariant of Mod(R), and $\phi : M \to M$ an endomorphism of an R-module M. We will prove some basic properties of the i-entropy of ϕ; only the first three properties hold without additional hypotheses on the invariant i.

(1) Let $\theta : M \to M'$ be an isomorphism of R-modules. Then $\mathrm{ent}_i(\phi) = \mathrm{ent}_i(\theta\phi\theta^{-1})$.

If $F' \in \mathrm{Fin}_i(M')$, for each n we have $T_n(\theta\phi\theta^{-1}, F') = \theta T_n(\phi, \theta^{-1}F')$. Thus, applying the invariant i, dividing by n and passing to the limit, we get $H_i(\theta\phi\theta^{-1}, F') = H_i(\phi, \theta^{-1}F')$. From this equality the conclusion follows easily, since $F' \in \mathrm{Fin}_i(M')$ if and only if $\theta^{-1}F' \in \mathrm{Fin}_i(M)$. □

(2) if $\phi : M \to M$ is an automorphism, then $\mathrm{ent}_i(\phi) = \mathrm{ent}_i(\phi^{-1})$.

It is immediate to check that $T_n(\phi, F) = \phi^{n-1} T_n(\phi^{-1}, F)$ for all n and for each $F \in \mathrm{Fin}_i(M)$, from which we derive $H_i(\phi, F) = H_i(\phi^{-1}, F)$, hence the desired equality follows. □

(3) $\mathrm{ent}_i(\phi) \geq \mathrm{ent}_i(\phi|_H)$ for every ϕ-invariant submodule H of M.

This fact follows from the inclusion $\mathrm{Fin}_i(H) \leq \mathrm{Fin}_i(M)$. □

(4) For each $k \geq 1$, we have $\mathrm{ent}_i(\phi^k) \geq k \cdot \mathrm{ent}_i(\phi)$, and the equality holds whenever i satisfies condition (v), in particular if it is an additive invariant.

Let us pick $F \in \mathrm{Fin}_i(M)$. For each $k, n \geq 1$ we have

$$T_{nk}(\phi, F) = T_n(\phi^k, T_k(\phi, F)).$$

Setting $F' = T_k(\phi, F)$, we deduce that

$$\begin{aligned} k \cdot H_i(\phi, F) &= k \cdot \lim_{n\to\infty} i(T_{nk}(\phi, F))/nk \\ &= \lim_{n\to\infty} i(T_n(\phi^k, F'))/n = H_i(\phi^k, F') \leq \mathrm{ent}_i(\phi^k), \end{aligned}$$

hence $k \cdot \mathrm{ent}_i(\phi) \leq \mathrm{ent}_i(\phi^k)$. Conversely, we have

$$\begin{aligned} \mathrm{ent}_i(\phi) \geq H_i(\phi, F) &= \lim_{n\to\infty} i(T_{nk}(\phi, F))/nk \\ &= \lim_{n\to\infty} i(T_n(\phi^k, T_k(\phi, F)))/nk = H_i(\phi^k, F')/k. \end{aligned}$$

If i satisfies condition (v), then $T_n(\phi^k, F) \leq T_n(\phi^k, F')$ implies that $i(T_n(\phi^k, F)) \leq i(T_n(\phi^k, F'))$. Therefore we get $\mathrm{ent}_i(\phi) \geq H_i(\phi^k, F)/k$, hence we can conclude that $k \cdot \mathrm{ent}_i(\phi) \geq \mathrm{ent}_i(\phi^k)$. □

(5) If $\phi_j : M_j \to M_j$ are endomorphisms ($j = 1, 2$) and i is a subadditive invariant satisfying condition (iii), then $\mathrm{ent}_i(\phi_1 \oplus \phi_2) \geq \mathrm{ent}_i(\phi_1) + \mathrm{ent}_i(\phi_2)$. Equality holds when i satisfies also property (v), in particular, if it is an additive invariant.

If $F_j \in \mathrm{Fin}_i(M_j)$ ($j = 1, 2$), using (iii) it is easily seen that

$$H_i(\phi_1, F_1) + H_i(\phi_2, F_2) = H_i(\phi_1 \oplus \phi_2, F_1 \oplus F_2),$$

hence $\mathrm{ent}_i(\phi_1 \oplus \phi_2) \geq \mathrm{ent}_i(\phi_1) + \mathrm{ent}_i(\phi_2)$. Assume now that property (v) is satisfied by i. For any given $F \in \mathrm{Fin}_i(M_1 \oplus M_2)$, let F_j be the projection of F onto M_j $(j = 1,2)$. As $F \leq F_1 \oplus F_2$, we get

$$H_i(\phi_1 \oplus \phi_2, F) \leq H_i(\phi_1 \oplus \phi_2, F_1 \oplus F_2) = H_i(\phi_1, F_1) + H_i(\phi_2, F_2),$$

hence the opposite inequality also holds. □

Property (5) does not hold for invariants that do not satisfy condition (iii). For instance, consider the entropy associated with the invariant gen on $\mathrm{Mod}(\mathbb{Z})$, which we denote by $\mathrm{ent}_{\mathrm{gen}}$. Consider the right shift σ of the Abelian group $\bigoplus_{n\geq 0} G_n$, where $G_n = \mathbb{Z}/6\mathbb{Z}$ for all $n \geq 0$. It is easy to check that $\mathrm{ent}_{\mathrm{gen}}(\sigma) = 1$ and that $\sigma = \sigma_2 \oplus \sigma_3$, where σ_2 is the right shift of the 2-component of G and σ_3 is the right shift of the 3-component of G. Clearly $\mathrm{ent}_{\mathrm{gen}}(\sigma_2) = 1 = \mathrm{ent}_{\mathrm{gen}}(\sigma_3)$, hence property (5) does not hold for $\mathrm{ent}_{\mathrm{gen}}$.

For the next property we need a new notion.

Definition 5. We say that a subadditive invariant i of $\mathrm{Mod}(R)$ is *liftable* if, given $H \leq K$ in $\mathrm{Mod}(R)$ such that $i(K/H) < \infty$, there exists $F \leq K$ such that $K/H = (F+H)/H$ and $i(F) < \infty$.

Note that, if i is a liftable subadditive invariant, then the class Fin_i is closed under extensions.

(6) If i is a liftable subadditive invariant, and H is a ϕ-invariant submodule of M, then $\mathrm{ent}(\bar{\phi}) \leq \mathrm{ent}(\phi)$, where $\bar{\phi}$ is the endomorphism of M/H induced by ϕ.

Let K/H be a submodule of M/H such that $i(K/H) < \infty$. Take $F \leq M$ as in the definition above. Then for each n we have

$$T_n(\bar{\phi}, K/H) = (T_n(\phi, F) + H)/H \cong T_n(\phi, F)/(T_n(\phi, F) \cap H),$$

hence $H_i(\bar{\phi}, K/H) \leq H_i(\phi, F)$. The desired inequality follows. □

Note that the invariants of the Examples 1.1, 1.4 and 1.6 above are all liftable. On the other hand, the invariants of Examples 1.2, 1.3 and 1.7 are not liftable. For the invariant $\log|-|$, in the notation of Definition 5, consider $H = n\mathbb{Z}$ and $K = \mathbb{Z}$. In a similar way one sees that the invariant in Example 1.3 is not liftable. For the invariant Mr, consider the natural epimorphism of $\mathbb{Z}_p$-modules $\bigoplus_{\aleph_0} \mathbb{Z}_p \to \mathbb{Z}(p^\infty)$; note that $\mathrm{Mr}(\mathbb{Z}(p^\infty)) = 1$. As a matter of fact, Example 1.11 in [DGSZ] provides an endomorphism which does not satisfy property (6) with respect to the invariant $\log|-|$.

We will give now a result which is the starting point to prove the Addition Theorem. It is worth noting that it holds for additive liftable invariants.

Proposition 2.1. *Let i be a discrete additive liftable invariant of* $\mathrm{Mod}(R)$*, $\phi : M \to M$ an endomorphism of $M \in$* $\mathrm{Mod}(R)$ *and N a ϕ-invariant submodule of M. Then*

$$\mathrm{ent}_i(\phi) \geq \mathrm{ent}_i(\bar{\phi}) + \mathrm{ent}_i(\phi|_N)$$

where $\bar{\phi} : M/N \to M/N$ is the induced endomorphism.

Proof. The properties (3) and (6) above allow us to assume that both $\mathrm{ent}_i(\bar{\phi})$ and $\mathrm{ent}_i(\phi|_N)$ are finite. By Corollary 1.11, both these entropies are reached as a maximum. Since i is liftable, there exists a submodule F_1 of M such that $i(F_1) < \infty$ and $\mathrm{ent}_i(\bar{\phi}) = H_i(\bar{\phi},(F_1+N)/N)$. There is also a submodule F_2 of N such that $i(F_2) < \infty$ and $\mathrm{ent}_i(\phi|_N) = H_i(\phi,F_2)$. Set $F = F_1+F_2$; then obviously $\mathrm{ent}_i(\bar{\phi}) = H_i(\bar{\phi},(F+N)/N)$. Furthermore $H_i(\phi,F_2) \geq H_i(\phi,F\cap N) \geq H_i(\phi,F_2)$, where the first inequality is due to the fact that $H_i(\phi,F_2)$ is a maximum, and the latter since i satisfies condition (v). Thus from Lemma 1.13 we deduce that

$$H_i(\phi,F) \geq H_i(\bar{\phi},(F+N)/N) + H_i(\phi,F\cap N).$$

Since $\mathrm{ent}_i(\phi) \geq H_i(\phi,F)$, the conclusion follows. □

In order to get the next property (7) of the right shifting map, we need another notion.

Definition 6. We say that a subadditive invariant i of $\mathrm{Mod}(R)$ is *small* if, given $F \in \mathrm{Fin}_i(\bigoplus_{j\in J} M_j)$, where J is an arbitrary index set, there exists a finite subset J' of J such that $F \leq \bigoplus_{j\in J'} M_j$.

It is immediate to see that the invariant i is small exactly if all modules in Fin_i are small (for the notion of small object of an additive category see [FS], p. 51). The invariants in the Examples 1.1–1.5 are small, but the invariant Mr in Example 1.7 is not (to get a counterexample for $\mathbb{Z}_p$-modules, we may construct an embedding $\beta : \mathbb{Z}(p^\infty) \to \bigoplus_{\aleph_0} \mathbb{Z}(p^\infty)$ such that the composite of β with any projection is nonzero).

As the final part of Example 1.12 shows, the next property is not satisfied by the invariant gen.

(7) Let i be a small discrete additive invariant. Let $\sigma : \bigoplus_{\aleph_0} K \to \bigoplus_{\aleph_0} K$ be the right shift. Then $\mathrm{ent}_i(\sigma) = i(K)$.

Let $F \leq \bigoplus_{\aleph_0} K$ be such that $i(F) < \infty$. The smallness of i ensures that $F \leq F' = \bigoplus_{j\leq n} K$. By property (v), $H_i(\sigma,F) \leq H_i(\sigma,F')$ and clearly $T_{n+1}(\sigma,F')/T_n(\sigma,F') \cong K$, hence $H_i(\sigma,F') = i(K)$, by Proposition 1.10. Thus the conclusion easily follows. □

Our last property deals with endomorphisms with infinite i-entropy. Its statement and its proof are similar to those of [DGSZ, Theorem 1.12]; we sketch the proof for the sake of completeness.

(8) Let i be a small discrete additive invariant. Let $M = \bigoplus_{n\geq 1} M_n$ be a countable direct sum of modules M_n such that $i(M_1) > 0$ and there is an embedding $\phi_n : M_n \to M_{n+1}$ for every n. Then there exists an endomorphism ϕ of M such that $\mathrm{ent}_i(\phi) = \infty$.

Let $\sigma : M \to M$ be the right shift relative to the embeddings ϕ_n,

$$\sigma(x_1,x_2,\ldots,x_n,\ldots) = (0,\phi_1(x_1),\phi_2(x_2),\ldots,\phi_n(x_n),\ldots),$$

where the $x_n \in M_n$ are almost all zero. Let $\dot{\bigcup}_{k\geq 1} I_k = \mathbb{N}$ be a partition of $\mathbb{N}$, where, for each $k \geq 1, I_k = \{i_{k1} < i_{k2} < \cdots < i_{kn} < \cdots\}$ is an infinite increasing sequence of positive integers. For each $k \geq 1$, set $A_k = \bigoplus_{n\in I_k} M_n$, so that $M = \bigoplus_{k\geq 1} A_k$; each A_k has a right shifting endomorphism $\sigma_k : A_k \to A_k$ induced by the embeddings $\psi_{kn} : M_{i_{kn}} \to M_{i_{k,n+1}}$ obtained by composing the maps ϕ_n. Let $\phi = \oplus_k \sigma_k : M \to M$ be the endomorphism which is the direct sum of the endomorphisms σ_k. For every $k \geq 1$, we have $H_i(\sigma_k, M_{i_{k1}}) = i(M_{i_{k1}})$, by property

(7), and, since i is additive, we have $i(M_{i_{k1}}) \geq i(M_1) > 0$. Let us define now, for each $n \geq 1$, the submodule of M:

$$F_n = \bigoplus_{k \leq n} M_{i_{k1}}.$$

Obviously $H_i(\phi, F_n) = \sum_{k \leq n} H_i(\sigma_k, M_{i_{k1}}) \geq \sum_{k \leq n} i(M_{i_{k1}}) \geq n \cdot i(M_1)$. We derive that

$$\mathrm{ent}_i(\phi) = \sup_{F \in \mathrm{Fin}_i(M)} H_i(\phi, F) \geq \sup_{n \geq 1} H_i(\phi, F_n) = \infty. \qquad \square$$

All the subadditive invariants considered so far were discrete. Let us give an example of a subadditive invariant which is not discrete.

Example 2.2. We refer to [FS, V] for definitions and results on finitely generated modules over valuation domains which we need in the present example.

Let R be a valuation domain whose value group is a dense subgroup of $\mathbb{R}$. For every ideal I of R, let $v(I) = \inf\{v(r) : r \in I\}$. Note that for the zero ideal we have $v(0) = \infty$, and for the maximal ideal P of R we have $v(P) = 0$, since the value group of R is dense in $\mathbb{R}$. For the sake of simplicity, we assume that R is almost maximal (although analogous results hold in a general situation). Then every finitely generated R-module F is a direct sum of cyclic submodules

$$F = \bigoplus_{i=1}^{m} R/I_i,$$

where the I_i are ideals of R (possibly equal to zero). The above decomposition is unique, up to isomorphism. Then we set

$$\delta(F) = \sum_{i=1}^{m} v(I_i).$$

Note that, if some ideal is zero, say $I_j = 0$, then $\delta(F) = \infty$. We define the invariant δ of $M \in \mathrm{Mod}(R)$ as

$$\delta(M) = \sup\{\delta(F) : F \leq M, F \in \mathscr{F}(R)\}$$

(recall that $\mathscr{F}(R)$ denote the class of finitely generated R-modules).

From [FS, V], we know that $\delta(F) \leq \delta(F')$ if $F \leq F'$ are finitely generated. Hence the above definition is consistent. Moreover, from the definition of δ, we at once get $\delta(F_1 \oplus F_2) = \delta(F_1) + \delta(F_2)$. We easily derive that the invariant δ satisfies condition (i). Moreover, for $M, N \in \mathrm{Mod}(R)$, we have $\delta(M \oplus N) = \delta(M) + \delta(N)$. In fact, the definition readily yields $\delta(M \oplus N) \geq \delta(M) + \delta(N)$. Let now $F \leq M \oplus N$ be finitely generated. Then $F \leq F_1 \oplus F_2$, where F_1, F_2 are the projections of F onto M and N, respectively. It follows that $\delta(F) \leq \delta(F_1 \oplus F_2) = \delta(F_1) + \delta(F_2) \leq \delta(M) + \delta(N)$. Since F was arbitrary, we conclude that $\delta(M \oplus N) \leq \delta(M) + \delta(N)$, and the desired equality follows.

Thus δ satisfies (i) and (iii), hence it also satisfies (ii), and so it is a subadditive invariant. Of course, δ is not discrete, since it takes values in the value group of R.

The invariant δ is not faithful. In fact, $\delta(R/P) = 0$, since $v(P) = 0$. More generally, $\delta(V) = 0$ for every R/P-vector space V.

If $M = \bigoplus_{n=1}^{\infty} R/I_n$, we have $\delta(M) = \sum_{n=1}^{\infty} v(I_n)$. Thus $M \in \mathrm{Fin}_\delta$ if the series of the $v(I_n)$ converges. In such case we know that $\mathrm{ent}_\delta(M) = 0$, in view of Proposition 1.8.

However, it is interesting to note that we may have $\mathrm{ent}_\delta(M) = 0$ even for a direct sum of cyclic modules $M \notin \mathrm{Fin}_\delta$. For instance, we may construct $M = \bigoplus_{n=1}^{\infty} R/I_n$, where $I_1 < I_2 < \cdots$, such that the partial sums s_n of the series defined by $\delta(M)$ are asymptotic to $\sqrt{n}$. Let us take any $\phi \in \mathrm{End}_R(M)$ and compute its entropy. Pick any $0 \neq x \in M$. Using results on finitely generated R-modules, one can prove that, for all $n > 0$, $\delta(T_n(\phi, x)) \leq v(I_1) + \cdots + v(I_n) = s_n \sim \sqrt{n}$. It follows that, for any $F \leq M$ finitely generated by, say, k elements, we have $\delta(T_n(\phi, F) \leq k\sqrt{n}$, for all $n > 0$. Then

$$H_\delta(\phi, F) \leq \lim_{n\to\infty} k\sqrt{n}/n = 0.$$

We conclude that $\mathrm{ent}_\delta(\phi) = 0$, hence also $\mathrm{ent}_\delta(M) = 0$, since ϕ was arbitrary.

3 The rank entropy

In this section we focus on the entropy obtained from the additive invariant $i = \mathrm{rk}$ defined by the rank (see Example 1.4); we will call it the *rank-entropy* and will denote it by $\mathrm{ent}_{\mathrm{rk}}$. Accordingly, the set of the subgroups of finite rank of the Abelian group M will be denoted by $\mathrm{Fin}_{\mathrm{rk}}(M)$, and $H_i(\phi, F)$ by $H_{\mathrm{rk}}(\phi, F)$.

Recall that, given an endomorphism $\phi : M \to M$, by definition we have $\mathrm{ent}_{\mathrm{rk}}(\phi) = \sup\{H_{rk}(\phi, F) : F \in \mathrm{Fin}_{\mathrm{rk}}(M)\}$. As in the preceding sections, we denote by $\mathscr{F}(M)$ the set of the finitely generated subgroups of M.

We start with a simple observation.

Lemma 3.1. *Let M be an Abelian group and $\phi \in \mathrm{End}(M)$. Then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = \sup_{F\in\mathscr{F}(M)} H_i(\phi, F)$.

Proof. A finitely generated subgroup has finite rank, hence $\sup_{F\in\mathscr{F}(M)} H_{\mathrm{rk}}(\phi, F) \leq \mathrm{ent}_{\mathrm{rk}}(\phi)$. Conversely, let F be a subgroup of finite rank of M. Then F contains a free essential submodule H of the same rank and for all $n > 0$ we have:

$$\mathrm{rk}(T_n(\phi, F)) = \mathrm{rk}(T_n(\phi, H)).$$

In fact, the tensor product commutes with finite sums and $\mathrm{rk}(\phi^n F) = \mathrm{rk}(\phi^n H)$ for all n, since F/H torsion implies that $\phi^n F/\phi^n H$ is torsion for all n. Consequently, $\mathbb{Q} \otimes T_n(\phi, F) = \mathbb{Q} \otimes T_n(\phi, H)$. Hence the conclusion follows. □

The proof of the preceding lemma shows that the invariant rk is liftable; it is obviously small, too, hence the rank-entropy satisfies all the properties (1)–(8) considered in the preceding section, since rk is additive, as noted before.

In the following lemma, point a) reduces the investigation of the rank-entropy to torsionfree groups, and point b) shows that it trivializes for torsion groups. Hence, as noted in the introduction, the rank-entropy is a counterpart of the entropy obtained from the invariant $\log|-|$, introduced by Adler-Konheim-McAndrew in [AKM], which trivializes for torsionfree groups.

Lemma 3.2. *Let $\phi : M \to M$ be an endomorphism of the Abelian group M. Then*
a) $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathit{ent}_{\mathrm{rk}}(\bar{\phi})$*, where $\bar{\phi} : M/tM \to M/tM$ is the induced endomorphism, tM being the torsion subgroup of M;*
b) *if M is a torsion group, then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = 0$.
c) *If N is a ϕ-invariant subgroup of M such that M/N is a torsion group, then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi|_N)$.

Proof. a) The inequality $\mathrm{ent}_{\mathrm{rk}}(\phi) \geq \mathrm{ent}_{\mathrm{rk}}(\bar{\phi})$ holds by property (6). Conversely, let F be a finitely generated subgroup of M, and let $\bar{F} = (F + tM)/tM$. Then for each n we have: $T_n(\bar{\phi}, \bar{F}) = (T_n(\phi, F) + tM)/tM$, hence $\mathrm{rk}(T_n(\bar{\phi}, \bar{F}) = \mathrm{rk}(T_n(\phi, F))$. Dividing by n and passing to the limit, we deduce that $H_{\mathrm{rk}}(\phi, F) = H_{\mathrm{rk}}(\bar{\phi}, \bar{F})$, hence the inequality $\mathrm{ent}_{\mathrm{rk}}(\phi) \leq \mathrm{ent}_{\mathrm{rk}}(\bar{\phi})$ also holds.
b) is a trivial consequence of a).
c) The inequality $\mathrm{ent}_{\mathrm{rk}}(\phi) \geq \mathrm{ent}_{\mathrm{rk}}(\phi|_N)$ holds by property (3). Conversely, let F be a finitely generated subgroup of M; then there exists a non-zero $r \in \mathbb{Z}$ such that $rF \leq N$. Since $rT_n(\phi, F) = T_n(\phi|_N, rF)$ for all $n > 0$, it follows that $H_{\mathrm{rk}}(\phi, F) = H_{\mathrm{rk}}(\phi|N, rF)$, so it is easy to conclude that the inequality $\mathrm{ent}_{\mathrm{rk}}(\phi) \leq \mathrm{ent}_{\mathrm{rk}}(\phi|_N)$ also holds. □

In view of Lemma 3.2. a), from now on we will consider only endomorphisms of torsionfree groups.

If N is a subgroup of a torsionfree group M, we denote as usual by N_* the purification of N in M, i.e., the subgroup of M satisfying the condition $N_*/N = t(M/N)$.

Lemma 3.3. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M, N a ϕ-invariant subgroup of M, $\bar{\phi} : M/N \to M/N$ the induced endomorphism. Then*
a) *N_* is a ϕ-invariant subgroup of M;*
b) $\mathrm{ent}_{\mathrm{rk}}(\phi|_N) = \mathrm{ent}_{\mathrm{rk}}(\phi|_{N_*})$*;*
c) $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) = \mathrm{ent}_{\mathrm{rk}}(\bar{\phi}_*)$*, where $\bar{\phi}_* : M/N_* \to M/N_*$ is the induced endomorphism.*

Proof. a) Let $x \in N_*$. Then $rx \in N$ for some $0 \neq r \in \mathbb{N}$, so $r\phi(x) = \phi(rx) \in N$, which shows that $\phi(x) \in N_*$.
b) Immediate consequence of Lemma 3.2, c), since N_*/N is a torsion group.
c) Immediate consequence of Lemma 3.2, a), since N_*/N is the torsion subgroup of M/N. □

The next lemma gives two particular versions of the so-called "Addition Theorem", that will be proved in the general form in the subsequent Theorem 3.11.

Lemma 3.4. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M, N a ϕ-invariant pure subgroup of M, $\bar{\phi} : M/N \to M/N$ the induced endomorphism.*
a) *If* $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) = 0$*, then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi|_N)$.
b) *If N has finite rank, then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\bar{\phi})$.

Proof. a) By property (3), it is enough to prove that $H_{\mathrm{rk}}(\phi, F) \leq \mathrm{ent}_{\mathrm{rk}}(\phi|_N)$, for any finitely generated subgroup F of M. In view of Proposition 1.10, the hypothesis ensures that there exists an integer m such that

$$\mathrm{rk}(T_m(\bar{\phi}, \bar{F})) = \mathrm{rk}(T(\bar{\phi}, \bar{F}))$$

where $\bar{F} = (F + N)/N$. It follows that

$$(\phi^m F + N)/N \le (T_m(\phi, F) + N)_*/N,$$

and therefore $\phi^m F \le (T_m(\phi, F) + N)_*$. Since F is finitely generated, there exists a finitely generated subgroup F_1 of N such that $\phi^m F \le (T_m(\phi, F) + F_1)_*$, hence for every positive integer k we derive

$$\phi^{m+k} F \le \phi^k (T_m(\phi, F) + F_1)_* = (\phi^k T_m(\phi, F) + \phi^k F_1)_*.$$

Thus we get $\phi^k T_m(\phi, F) \le (T_m(\phi, F) + T_k(\phi, F_1))_*$, so that

$$\mathrm{rk}(T_{m+k}(\phi, F)) \le \mathrm{rk}(T_m(\phi, F) + T_k(\phi, F_1)).$$

Dividing by $m + k$ and passing to the limit with respect to k we derive

$$H_{\mathrm{rk}}(\phi, F) \le H_{\mathrm{rk}}(\phi|_N, F_1) \le \mathrm{ent}_{\mathrm{rk}}(\phi|_N),$$

as desired.

b) By property (6), it is enough to prove that $H_{\mathrm{rk}}(\phi, F) = H_{\mathrm{rk}}(\bar{\phi}, \bar{F})$, where F is any assigned finitely generated subgroup of M, and $\bar{F} = (F + N)/N$. For each integer m we have $T_m(\bar{\phi}, \bar{F}) = (T_m(\phi, F) + N)/N \cong T_m(\phi, F)/(T_m(\phi, F) \cap N)$, hence $\mathrm{rk}(T_m(\phi, F) \le \mathrm{rk}(T_m(\bar{\phi}, F)) + \mathrm{rk} N$. Since N has finite rank, dividing by m and passing to the limit we get the desired equality. ☐

We will need the following

Lemma 3.5. *Let $\phi : M \to M$ be an endomorphism of a group M, which is the direct limit of a directed family of ϕ-invariant subgroups M_σ. Then* $\mathrm{ent}_{\mathrm{rk}}(\phi) = \sup_\sigma \mathrm{ent}_{\mathrm{rk}}(\phi|_{M_\sigma})$.

Proof. The inequality $\mathrm{ent}_{\mathrm{rk}}(\phi) \ge \sup_\sigma \mathrm{ent}_{\mathrm{rk}}(\phi|_{M_\sigma})$ follows from property (3). For the converse inequality, let F be a finitely generated subgroup of M; it is enough to prove that $H_{\mathrm{rk}}(\phi, F) \le \mathrm{ent}_{\mathrm{rk}}(\phi|_{M_\sigma})$ for some σ. But this is obvious, since F embeds into M_σ for some σ, and property (1) holds. ☐

We give now some characterizations of the endomorphisms of zero rank-entropy. In what follows, if ϕ is an endomorphism of a torsionfree group M and $x \in M$, we write $T(\phi, x)$ for the ϕ-trajectory of the subgroup generated by x.

Theorem 3.6. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M. The following conditions are equivalent:*

a) $\mathrm{ent}_{\mathrm{rk}}(\phi) = 0$*;*
b) *for every $x \in M$,* $\mathrm{rk}(T(\phi, x))$ *is finite;*
c) *M is the union of a well-ordered smooth ascending chain of pure ϕ-invariant subgroups M_σ ($\sigma < \lambda$), such that $M_0 = 0$ and* $\mathrm{rk}(M_{\sigma+1}/M_\sigma)$ *is finite for all σ.*
d) *ϕ is point-wise algebraic, i.e., for every $x \in M$ there exists a polynomial f, depending on x, with rational (or, equivalently, integral) coefficients, such that* $f(\phi)(x) = 0$.

Proof. a) $\Rightarrow$ b). Since $\mathrm{ent}_{\mathrm{rk}}(\phi) = 0$, it follows, in particular, that $H_{\mathrm{rk}}(\phi,x) = 0$ for every $x \in M$. Then Proposition 1.10 (i) shows that $\mathrm{rk}(T(\phi,x)) = \mathrm{rk}(T_n(\phi,x))$, for some $n > 0$. Then we are done, as $T_n(\phi,x)$ is a free finitely generated group. (Note that we cannot conclude that $T(\phi,x)$ is finitely generated, since it could happen that $T_n(\phi,x) < T(\phi,x)$ for all n.)

b) $\Rightarrow$ c). We construct the submodules M_σ by transfinite induction on σ, starting with $M_0 = 0$. Assume that M_σ has been already constructed, and that $M \neq M_\sigma$. Pick a non-zero element $x + M_\sigma \in M/M_\sigma$. Let $M_{\sigma+1}$ be the purification of $T(\phi,x) + M_\sigma$. Clearly $M_{\sigma+1}$ is ϕ-invariant in M and $\mathrm{rk}(M_{\sigma+1}/M_\sigma)$ is finite, since, by hypothesis $\mathrm{rk}(T(\phi,x))$ is finite, hence $(T(\phi,x)+M_\sigma)/M_\sigma \cong T(\phi,x)/T(\phi,x) \cap M_\sigma$ also has finite rank. At limit ordinals σ define M_σ as the union of the M_ρ for $\rho < \sigma$.

c) $\Rightarrow$ a). Let $\phi_\sigma : M_\sigma \to M_\sigma$ be the map induced by ϕ for each σ. If we show that $\mathrm{ent}_{\mathrm{rk}}(\phi_\sigma) = 0$ for all $\sigma < \lambda$, then we get $\mathrm{ent}_{\mathrm{rk}}(\phi) = 0$, applying Lemma 3.5. In fact, at non limit ordinals $\sigma + 1$, from Proposition 1.8 we get $\mathrm{ent}_{\mathrm{rk}}(\phi_{\sigma+1}) = 0$, since $\mathrm{rk}(M_{\sigma+1}/M_\sigma)$ is finite. At limit ordinals, we apply Lemma 3.5.

b) $\Leftrightarrow$ d). Let n denote the degree of the polynomial f; then $f(\phi)(x) = 0$ amounts to say that $\phi^n(x) \in T_n(\phi,x)_*$, which is equivalent to b). $\square$

Remark 2. The equivalence a) $\Leftrightarrow$ d) in Theorem 3.6 has its analog for entropies of endomorphisms with respect to the invariant $\log|-|$, see Proposition 2.4 of [DGSZ]. However, it is worth remarking that this equivalence fails, in general.

On the one hand, we consider the invariant δ for modules over a valuation domain R with value group dense in $\mathbb{R}$, as defined in Example 2.2. Up to a factor $\log p$, this invariant actually extends the invariant $\log|-|$, acting on $\mathbb{Z}_p$-modules (compare also with the invariant v of Example 1.12). Let us show that, for suitable $M \in \mathrm{Mod}(R)$ and $\phi \in \mathrm{End}_R(M)$, we may have $\mathrm{ent}(\phi) = 0$, even if ϕ is not point-wise algebraic over the field of quotients Q of R.

We get a first obvious example if we take a R/P-vector space V of countable dimension, and the right shift $\sigma \in \mathrm{End}_R(V)$. It is readily proved that the shift cannot be point-wise algebraic, but $\mathrm{ent}_\delta(\sigma) = 0$, since $\delta(V) = 0$.

A second, more significant, example is given by a module $M = \bigoplus_{n=1}^\infty R/I_n$, where the partial sum s_n of the series defined by $\delta(M)$ is asymptotic to $\sqrt{n}$, and $I_1 < I_2 < \cdots$. Then the shift σ is an endomorphism of M, that cannot be point-wise algebraic. However, at the end of Example 2.2 we have proved that $\mathrm{ent}_\delta(M) = 0$, hence, in particular, $\mathrm{ent}_\delta(\sigma) = 0$.

On the other hand, we consider the invariant gen for modules over the Noetherian local ring $R = K[X,Y]_{(X,Y)}$, where K is a field and X, Y are indeterminates. Let us regard the field of quotients Q of R as an R-module, and the multiplication by $\phi = X/Y$ as an element of $\mathrm{End}_R(Q)$. Of course, $\phi = X/Y$ is algebraic over Q. However, $\mathrm{ent}_{\mathrm{gen}}(\phi) > 0$. In fact, consider the cyclic R-module $F = R$. In the present circumstances, it is an easy exercise to check that $T_n(\phi,F) \cong (X,Y)^{n-1}$, for all $n > 0$. Since $\mathrm{gen}((X,Y)^n) = n+1$, we at once get $H_{\mathrm{gen}}(\phi,F) = 1$, and therefore $\mathrm{ent}_{\mathrm{gen}}(\phi) \geq 1$, as required.

Given an endomorphism $\phi : M \to M$ of the torsionfree group M, we define the subset $t_\phi M$ of M as

$$t_\phi M = \{x \in M : \mathrm{rk}(T(\phi,x)) < \aleph_0\},$$

which is clearly the maximum ϕ-invariant subgroup N of M such that $\mathrm{ent}_{\mathrm{rk}}(\phi|_N) = 0$. The subgroup $t_\phi M$ is called the *ϕ-torsion part* of M, and is obviously pure in M. We say that M is *ϕ-torsion* (respectively, *ϕ-torsionfree*), if $M = t_\phi M$ (respectively, $t_\phi M = 0$). A condition equivalent to those of Theorem 3.6 is that M is ϕ-torsion. A straightforward computation shows that the quotient group $M/t_\phi M$ is $\bar{\phi}$-torsionfree, where $\bar{\phi} \in \mathrm{End}(M/t_\phi M)$ is the induced map.

In order to simplify the notation, from now on we will make a little abuse of language. Namely, if ϕ is an assigned endomorphism of a torsionfree group M and H is ϕ-invariant, we will just say that M/H is ϕ-torsion, when M/H is torsion with respect to the map induced by ϕ.

We examine now the basic examples of endomorphisms of torsionfree groups of rank-entropy 1, namely, the right shifts.

Example 3.7. 1) Consider the free group of countable rank $M = \bigoplus_{n\geq 1} x_n\mathbb{Z}$. Let σ be the endomorphism right shift of M defined by the assignments $\sigma(x_n) = x_{n+1}$ for all $n \geq 1$. Then easy calculations show that $\mathrm{ent}_{\mathrm{rk}}(\sigma) = 1$, $M = T(\sigma, x_1)$ and $t_\phi M = 0$, i.e., M is a ϕ-torsionfree group.

2) Consider the free group of countable rank $M = \bigoplus_{n\in\mathbb{Z}} x_n\mathbb{Z}$. Let ψ be the endomorphism right shift of M defined by the assignments $\psi(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Also in this case $\mathrm{ent}_{\mathrm{rk}}(\psi) = 1$ and $t_\psi M = 0$, but M strictly contains $T(\psi, x)_*$ for every $x \in M$; furthermore, $M/T(\psi, x)_*$ is a ψ-torsion group.

The situation of Example 3.7 is typical of endomorphisms of rank-entropy 1, as shown by Lemma 3.8.

In the next results, we will make use of the following simple fact. If ϕ is an endomorphism of a torsionfree group M and $x \in M$, the rank of the ϕ-trajectory $T(\phi, x)$ is infinite if and only if the elements $x, \phi(x), \phi^2(x), \ldots$ are linearly independent.

Lemma 3.8. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M. The following are equivalent:*
a) $\mathrm{ent}_{\mathrm{rk}}(\phi) = 1$*;*
b) *there exists $0 \neq x \in M$ such that $T(\phi, x) = \bigoplus_{n\geq 0} \phi^n(x)\mathbb{Z}$ is a free group of countable rank and both $M/T(\phi, x)$ and $M/T(\phi, x)_*$ are ϕ-torsion.*

Proof. a) $\Rightarrow$ b) Since $\mathrm{ent}_{\mathrm{rk}}(\phi) = 1$, there exists an element $x \in M$ such that $\mathrm{rk}(T(\phi, x)) = \aleph_0$, in view of Theorem 3.6. Then $x, \phi(x), \phi^2(x), \ldots$ are linearly independent, hence $T(\phi, x) = \bigoplus_{n\geq 0} \phi^n(x)\mathbb{Z}$ is a free group of countable rank. Since ϕ acts as the right shift on $T = T(\phi, x)$, from Example 3.7, 1) we get $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) = 1$. Then, from Proposition 2.1 we get $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) = 0$, where $\bar{\phi} : M/T \to M/T$ is the induced endomorphism. Moreover, Lemma 3.3, c) shows that $0 = \mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) = \mathrm{ent}_{\mathrm{rk}}(\bar{\phi}_*)$, where $\bar{\phi}_* : M/T_* \to M/T_*$ is the induced map, hence we are done.

b) $\Rightarrow$ a) Apply Lemma 3.4, a) to the submodule $N = T(\phi, x)_*$ and recall that $\mathrm{ent}_{\mathrm{rk}}(\phi|_N) = 1$, by Example 3.7, 1) and Lemma 3.3, b). □

The next lemma is the main technical ingredient in the proof of Theorem 3.10.

Lemma 3.9. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M, N a pure ϕ-invariant subgroup of M, and $\bar{\phi} : M/N \to M/N$ the induced endomorphism. If* $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) > 0$, *there exists an element $x \in M$ such that $M \geq N \oplus (\bigoplus_{n\geq 0} \phi^n(x)\mathbb{Z})$, where $\phi^n(x) \neq 0$ for all n.*

Proof. By Theorem 3.6, there exists $\bar{x} = x + N$ such that the trajectory $T(\bar{\phi}, \bar{x})$ has countable rank. This implies that the partial trajectory $T_n(\bar{\phi}, \bar{x})$ is free of rank n, for all $n > 0$, hence

$$T(\bar{\phi}, \bar{x}) = (T(\phi, x) + N)/N = \bigoplus_{n\geq 0} \bar{\phi}^n(\bar{x})\mathbb{Z},$$

where $\bar{\phi}^n \bar{x} \neq \bar{0}$ for all $n \geq 0$. An easy computation shows that the elements $\{\phi^n(x)\}_n$ are independent and that $N \cap \bigoplus_{n\geq 0} \phi^n(x)\mathbb{Z} = 0$. □

We can now prove the main result of this section.

Theorem 3.10. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M. The following conditions are equivalent:*

a) $\mathrm{ent}_{\mathrm{rk}}(\phi) = k$ *for a positive integer k;*

b) *there exists an ascending chain of ϕ-invariant pure subgroups of M*

$$t_\phi M = N_0 < N_1 < N_2 < \cdots < N_k = M$$

such that $\mathrm{ent}_{\mathrm{rk}}(\phi_i) = 1$ *for all $1 \leq i \leq k$, where $\phi_i : N_i/N_{i-1} \to N_i/N_{i-1}$ is the map induced by ϕ, and* $\mathrm{ent}_{\mathrm{rk}}(\phi|_{N_i}) = i$ *for all $i \leq k$;*

c) *there exist k elements $x_1, x_2, \ldots, x_k \in M$ such that their trajectories $T(\phi, x_i) = \bigoplus_{n\geq 0} \phi^n(x_i)\mathbb{Z}$, $1 \leq i \leq k$, are free of countable rank and independent: $\sum_{i\leq k} T(\phi, x_i) = \bigoplus_{i\leq k} T(\phi, x_i)$. Furthermore, setting $S = (\bigoplus_{i\leq k} T(\phi, x_i))_*$, the quotient group M/S is ϕ-torsion.*

Proof. a) ⇒ b) We construct the subgroups N_i by induction on i. For $i = 0$ we have nothing to prove. Assume that $i > 0$ and N_{i-1} has been already constructed. Then the endomorphism $\bar{\phi}_{i-1}$ of M/N_{i-1} induced by ϕ cannot have zero rank-entropy, otherwise Lemma 3.4, a) would imply $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi|_{N_{i-1}}) = i - 1 < k$, a contradiction. Hence, in view of Lemma 3.9, there exists $x_i \in M$ such that its trajectory $T_i = T(\phi, x_i)$ is free of countable rank, and $T_i + N_{i-1} = T_i \oplus N_{i-1}$ is the direct sum of two ϕ-invariant subgroups. Recall also that $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i}) = 1$, since ϕ acts as the right shift on T_i. Then property (5) ensures that

$$\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i \oplus N_{i-1}}) = \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i}) + \mathrm{ent}_{\mathrm{rk}}(N_{i-1}) = 1 + i - 1 = i.$$

Let us define N_i as the subgroup of M containing $(T_i \oplus N_{i-1})_*$ such that

$$N_i/(T_i \oplus N_{i-1})_* = t_\phi(M/(T_i \oplus N_{i-1})_*).$$

Since $(T_i \oplus N_{i-1})_*$ is a pure ϕ-invariant subgroup of M, so also is N_i. Now we show that $\mathrm{ent}_{\mathrm{rk}}(\phi_i) = 1$. Consider the exact sequence

$$0 \to (T_i \oplus N_{i-1})_*/N_{i-1} \to N_i/N_{i-1} \to N_i/(T_i \oplus N_{i-1})_* \to 0.$$

Since the third term is ϕ-torsion, the entropy of its endomorphism induced by ϕ has zero rank-entropy. Therefore, by Lemma 3.4, a), $\mathrm{ent}_{\mathrm{rk}}(\phi_i)$ equals the rank-entropy of the endomorphism induced by ϕ on $(T_i \oplus N_{i-1})_*/N_{i-1}$. By Lemma 3.3, b) we can disregard the purification, hence

$$\mathrm{ent}_{\mathrm{rk}}(\phi_i) = \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i}) = 1,$$

as desired. Finally, Lemma 3.4, a) and the exact sequence

$$0 \to (T_i \oplus N_{i-1})_* \to N_i \to N_i/(T_i \oplus N_{i-1})_* \to 0$$

show that $\mathrm{ent}_{\mathrm{rk}}(\phi|_{N_i}) = \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i \oplus N_{i-1}}) = i$, as proved above.

It remains to check that we actually have $M = N_k$. In fact, by construction we have $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_k \oplus N_{k-1}}) = k = \mathrm{ent}_{\mathrm{rk}}(\phi|_{(T_k \oplus N_{k-1})_*})$, and so $M/(T_k \oplus N_{k-1})_*$ is ϕ-torsion, in view of Proposition 2.1; hence, by definition, $N_k = M$.

b) $\Rightarrow$ a) is trivial.

a) $\Rightarrow$ c) Let m be a positive integer such that there exist elements $x_1, x_2, \ldots, x_m \in M$ such that their trajectories $T_i = T(\phi, x_i) = \bigoplus_{n \geq 0} \phi^n(x_i)\mathbb{Z}$ are free of countable rank and independent: $\sum_{i \leq m} T_i = \bigoplus_{i \leq m} T_i$. Note that Lemma 3.8 ensures that at least the case $m = 1$ is possible. Moreover $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_i}) = 1$, for $1 \leq i \leq m$, as observed above. Since $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_1 \oplus \cdots \oplus T_m}) = m$, by property (5), and $\mathrm{ent}_{\mathrm{rk}}(\phi) \geq \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_1 \oplus \cdots \oplus T_m})$, we must have $m \leq k$. Let us now assume m to be maximum with respect to the above requirement; let us verify that $m = k$. Assume for a contradiction that $m < k$. Let $N = (T_1 \oplus \cdots \oplus T_m)_*$; then $\mathrm{ent}_{\mathrm{rk}}(\phi|_N) = m < k$, by Lemma 3.3, b), and therefore the rank-entropy of the induced endomorphism on M/N cannot be zero, in view of Lemma 3.4, a). Thus we are in the position to apply Lemma 3.9, and we can find $x_{m+1} \in M$ such that $T(\phi, x_{m+1})$ is free of countable rank, and $M \geq N \oplus T(\phi, x_{m+1})$. But this contradicts the maximality of m, impossible. Thus $m = k$ and the preceding argument shows that the quotient group $M/(T_1 \oplus \cdots \oplus T_k)_*$ must be ϕ-torsion. The desired conclusion follows.

c) $\Rightarrow$ a) From Lemma 3.4, a) we deduce that $\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi|_S)$, which equals k, in view of Lemma 3.3, b), property (5) and Example 3.7, 1). □

The above theorem has some remarkable consequences, that have their analogues for the algebraic entropy of endomorphisms of Abelian p-groups, defined by the invariant $\log|-|$, which was discussed in [DGSZ]. The first one is the so-called "Addition Theorem".

Theorem 3.11 (Addition Theorem). *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M, let H be a ϕ-invariant subgroup of M, and let $\bar{\phi} : M/H \to M/H$ be the induced endomorphism. Then we have*

$$\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) + \mathrm{ent}_{\mathrm{rk}}(\phi|_H).$$

Proof. By Lemma 3.3 we may assume that H is pure in M. If either $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi})$ or $\mathrm{ent}_{\mathrm{rk}}(\phi_H)$ is infinite, the conclusion follows from Proposition 2.1. Thus let us assume that $\mathrm{ent}_{\mathrm{rk}}(\phi|_H) = h$ and $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}) = k$ $(h, k \in \mathbb{N})$. By Theorem 3.10, H contains a subgroup $A = (\bigoplus_{1 \leq i \leq k} T(\phi, x_i))_*$ $(x_i \in H)$, such that H/A is ϕ-torsion. In a similar way, M/H contains a subgroup $B/H = (\bigoplus_{1 \leq j \leq k} T(\bar{\phi}, \bar{y}_j))_*$ $(y_j \in M)$ such that M/B is ϕ-torsion. Note that, by Lemma 3.9, $B/H = (\bigoplus_{1 \leq j \leq k} T(\phi, y_j) \oplus H)_*/H$. Let us set

$$K = (\bigoplus_{1 \leq i \leq h} T(\phi, x_i)) \oplus (\bigoplus_{1 \leq j \leq k} T(\phi, y_j))$$

and note that $K_* \leq B$. Then $\mathrm{ent}_{\mathrm{rk}}(\phi|_{K_*}) = h+k$, by property (5) and Lemma 3.3, b). In view of Lemma 3.4, a), it suffices to prove now that M/K_* is a ϕ-torsion group. Consider the exact sequence

$$0 \to B/K_* \to M/K_* \to M/B \to 0.$$

Since M/B is ϕ-torsion, by Lemma 3.4, a), it is enough to check that B/K_* is ϕ-torsion, namely, that $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}|_B) = 0$, where $\bar{\phi}|_B : B/K_* \to B/K_*$ is the map induced by ϕ. Indeed, by Lemma 3.3, $\mathrm{ent}_{\mathrm{rk}}(\bar{\phi}|_B)$ coincides with the rank-entropy of the endomorphism induced on

$$\Big(\bigoplus_{1\leq j\leq k} T(\phi, y_j) \oplus H\Big)/K,$$

which clearly vanishes, since H/A ϕ-torsion implies that $H/(\bigoplus_{1\leq i\leq h} T(\phi, x_i))$ is ϕ-torsion. □

Remark 3. Let us observe that for the invariant gen the Addition Theorem does not hold. For instance, consider the $\mathbb{Z}_p$-module $A = \bigoplus_{\aleph_0} \mathbb{Z}_p$ and its shift endomorphism σ. Then the restriction $\sigma|_{pA}$ and the induced endomorphism $\bar{\sigma} : A/pA \to A/pA$ are the shifts of pA and A/pA, respectively. It is readily seen that $1 = \mathrm{ent}_{\mathrm{gen}}(\sigma) = \mathrm{ent}_{\mathrm{gen}}(\sigma|_{pA}) = \mathrm{ent}_{\mathrm{gen}}(\bar{\sigma})$, hence

$$\mathrm{ent}_{\mathrm{gen}}(\sigma) < \mathrm{ent}_{\mathrm{gen}}(\sigma|_{pA}) + \mathrm{ent}_{\mathrm{gen}}(\bar{\sigma}),$$

and the addition theorem does not hold.

Indeed, we have seen that the formula in Proposition 2.1 is not valid for the invariant gen. This depends on the fact that gen is not additive.

Corollary 3.12. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M and let H and K be two ϕ-invariant subgroups of M such that $M = H + K$. Then*

$$\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi|_H) + \mathrm{ent}_{\mathrm{rk}}(\phi|_K) - \mathrm{ent}_{\mathrm{rk}}(\phi|_{H\cap K}).$$

Proof. Consider the exact sequence

$$0 \to H\cap K \to H\oplus K \to H+K = M \to 0$$

where $x \in H\cap K$ maps to $(x, -x) \in H\oplus K$ and $(x,y) \in H\oplus K$ maps to $x+y$. The endomorphism $\phi|_H \oplus \phi|_K : H\oplus K \to H\oplus K$ sends the image of $H\cap K$ in $H\oplus K$ into itself, hence it induces the endomorphism ϕ on M. Apply now the Addition Theorem. □

Corollary 3.13. *Let $\phi : M \to M$ be an endomorphism of the torsionfree group M, let $F \leq M$ be a finitely generated subgroup of rank m, and let $T = T(\phi, F)$. Then $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) \leq m$.*

Proof. We argue by induction on $\mathrm{rk}(F) = m$. If $\mathrm{rk}(F) = 1$, say $F = x\mathbb{Z}$, then $T = T(\phi, x)$ has either finite rank, in which case $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) = 0$, or $T = \bigoplus_{n\geq 0} \phi^n(x)\mathbb{Z}$. In the latter case, ϕ acts on T as the right shift, and therefore $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) = 1$. Assume now that $m > 1$; then $F = x\mathbb{Z} \oplus F_1$, where $\mathrm{rk}(F_1) = m-1$, and $T = T(\phi, x) + T(\phi, F_1) = T_0 + T_1$. Applying Corollary 3.12 to T, T_0, T_1 we get $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) \leq \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_0}) + \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_1})$. Since $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_0}) \leq 1$ and, by induction, $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_1}) \leq m-1$, the desired inequality follows. □

It is worth noting that there exist indecomposable torsionfree groups G of infinite rank which admit non-trivial endomorphisms (i.e., not multiplications by rational numbers) of zero rank-entropy. For instance, take any reduced subring A of a number field. By Corner's theorem [C] there exists a torsionfree group G of countable rank such that $\mathrm{End}(G) \cong A$; more generally, one can construct G such that $\mathrm{rk}(G)$ is an arbitrarily large cardinal, satisfying suitable conditions (see [GT], Theorem 12.3.4, p. 428). Since A is an integral domain, G is indecomposable, and every endomorphism of G has zero rank-entropy by Theorem 3.6, since A is algebraic over $\mathbb{Q}$, hence pointwise algebraic.

Our last result shows the uniqueness of the rank-entropy, once some "minimal" conditions are satisfied.

Theorem 3.14. *The rank-entropy is the unique collection of functions $h_G : \mathrm{End}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where G ranges over the class of Abelian groups, which satisfy the following conditions:*
(i) *the Addition Theorem holds, i.e., given an endomorphism $\phi : G \to G$ and a ϕ-invariant subgroup H of G, then $h_G(\phi) = h_H(\phi|_H) + h_{G/H}(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the induced endomorphism;*
(ii) *h_G is invariant under conjugation, i.e., if $\theta : G \to G'$ is an isomorphism, then $h_G(\phi) = h_{G'}(\theta\phi\theta^{-1})$ for all $\phi \in \mathrm{End}(G)$;*
(iii) *$h_G(\phi) = 0$ for all $\phi \in \mathrm{End}(G)$, whenever $\mathrm{rk}(G)$ is finite;*
(iv) *given $\phi \in \mathrm{End}(G)$, if G is a direct limit of ϕ-invariant subgroups G_α, then $h_G(\phi) = \sup_\alpha h_{G_\alpha}(\phi|_{G_\alpha})$;*
(v) *if $G = \bigoplus_{n\geq 1} K_n$, with $K_n = K$ for all n, and $\sigma : G \to G$ is the right shift, then $h_G(\sigma) = \mathrm{rk}(K)$.*

Proof. The fact that the rank-entropy satisfies conditions (i)–(v) is proved, respectively, in Theorem 3.11, property (1), Theorem 3.6, Lemma 3.5 and property (7).

Conversely, we want to prove that $h_G(\phi) = \mathrm{ent}_{\mathrm{rk}}(\phi)$ for every endomorphism $\phi : G \to G$. By (i) and (iii) from one side, and by Theorem 3.11 and Proposition 1.8 on the other side, we can assume that G is torsionfree of infinite rank.

Let us first assume that $\mathrm{ent}_{\mathrm{rk}}(\phi) = k < \infty$. If $\mathrm{ent}_{\mathrm{rk}}(\phi) = 0$, then Theorem 3.6 ensures that G is the union of a well-ordered smooth ascending chain of pure ϕ-invariant subgroups G_σ ($\sigma < \lambda$), such that $G_0 = 0$ and $\mathrm{rk}(G_{\sigma+1}/G_\sigma)$ is finite for all σ. By condition (iii) we have $h_{G_{\sigma+1}/G_\sigma}(\phi_\sigma) = 0$ for all σ, where $\phi_\sigma \in \mathrm{End}(G_{\sigma+1}/G_\sigma)$ is the map induced by ϕ. Using (i) and (iv) we immediately deduce that $h_G(\phi) = 0$, as desired. Assume now that $\mathrm{ent}_{\mathrm{rk}}(\phi) = k > 0$. By Theorem 3.10, there exists an ascending chain of ϕ-invariant pure subgroups of G

$$t_\phi G = N_0 < N_1 < N_2 < \cdots < N_k = G$$

such that $\mathrm{ent}_{\mathrm{rk}}(\phi_i) = 1$ for all $1 \leq i \leq k$, where $\phi_i : N_i/N_{i-1} \to N_i/N_{i-1}$ is the map induced by ϕ, and $\mathrm{ent}_{\mathrm{rk}}(\phi|_{N_i}) = i$ for all $i \leq k$. It is enough to prove that $h_{N_0}(\phi|_{N_0}) = 0$ and $h_{N_i/N_{i-1}}(\phi_i) = 1$ for $i > 0$, and then to iterate applications of condition (i). Since $\mathrm{ent}_{\mathrm{rk}}(\phi|_{N_0}) = 0$, the preceding argument shows that $h_{N_0}(\phi|_{N_0}) = 0$. To simplify the notation, for $i > 0$ we set $M = N_i/N_{i-1}$ and $\psi = \phi_i$. Then, applying Lemma 3.8 to M and ψ, we can find $x \in M$ such that $T(\psi, x) = T$ is free of countable rank, and $\mathrm{ent}(\bar{\psi}) = 0$, where $\bar{\psi} \in \mathrm{End}(M/T)$ is the induced endomorphism. Then $h_{M/T}(\bar{\psi}) = 0$, as seen above, and $h_T(\psi|_T) = 1$, since ψ acts on T as the right shift, and condition (v) holds. Using condition (i) we get $h_M(\psi) = h_{N_i/N_{i-1}}(\phi_i) = 1$, as desired.

Let us observe that, if $T = T(\phi, F)$ is the trajectory of a finitely generated subgroup F of G, from the above argument we get $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) = h_T(\phi|_T)$, since Corollary 3.13 shows that $\mathrm{ent}_{\mathrm{rk}}(\phi|_T) < \infty$.

Let us now assume that $\mathrm{ent}_{\mathrm{rk}}(\phi) = \infty$. Since G is the direct limit of the ϕ-trajectories T_α of finitely generated subgroups F_α $(\alpha < \lambda)$, and $\mathrm{ent}_{\mathrm{rk}}(\phi|_{T_\alpha}) = h_{T_\alpha}(\phi|_{T_\alpha})$ for all α, using Lemma 3.5 and condition (iv) we get $\infty = \mathrm{ent}_{\mathrm{rk}}(\phi) = \sup_\alpha \mathrm{ent}_{\mathrm{rk}}(\phi|_{T_\alpha}) = \sup_\alpha h_{T_\alpha}(\phi|_{T_\alpha}) = h_G(\phi)$, as desired. $\square$

References

[AKM] Adler R. L., Konheim A. G., McAndrew M. H.: Topological entropy. Trans. Amer. Math. Soc. **114** (1965), 309–319

[BV] Bellon M. P., Viallet C.-M.: Algebraic entropy. Comm. Math. Pys. **204** (1999), 425–437

[Br] Brown R. J.: The algebraic entropy of the special linear character automorphisms of a free group on two generators. Trans. Amer. Math. Soc. **359** (2007), 1445–1470

[C] Corner A. L. S.: Every countable reduced torsion-free ring is an endomorphism ring. Proc. London Math. Soc. (3) **13** (1963), 687–710

[DGSZ] Dikranjan D., Goldsmith B., Salce L., Zanardo P.: Algebraic entropy for Abelian groups, to appear in Trans. Amer. Math. Soc.

[F] Fuchs L.: Infinite Abelian Groups, Vol. I and II. Academic Press, 1970 and 1973

[FS] Fuchs L., Salce L.: Modules over Non-Noetherian Domains. AMS Math. Surveys and Monographs 84, 2001

[GT] Göbel R., Trlifaj J.: Approximations and Endomorphism Algebras of Modules. de Gruyter Expositions in Mathematics 41, Berlin 2006

[P] Peters J.: Entropy on discrete Abelian groups. Adv. Math. **33** (1979), 1–13

[SZ] Salce L., Zanardo P.: Commutativity modulo small endomorphisms and endomorphisms of zero algebraic entropy, to appear in Contribution to Module Theory – Corner Memorial Volume, de Gruyter, 2008

[V] Vámos P.: The Holy Grail of algebra: seeking complete sets of invariants. Abelian groups and modules (Padova, 1994), pp. 475–483. Kluwer Acad. Publ., Dordrecht 1995

[Wa] Ward T.: Entropy of compact group automorphisms. Notes for the Graduate Course, 1994, Ohio State University

[W] Weiss M. D.: Algebraic and other entropies of group endomorphisms. Math. System Theory **8** (1975), 243–248

Received 25 June 2007

Dipartimento di Matematica Pura e Applicata, Via Trieste 63, 35121 Padova, Italy
salce@math.unipd.it; pzanardo@math.unipd.it