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3-charge geometries and their CFT duals

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Abstract

We consider two families of D1–D5–P states and find their gravity duals. In each case the geometries are found to 'cap off' smoothly near $r = 0$; thus there are no horizons or closed timelike curves. These constructions support the general conjecture that the interior of black holes is nontrivial all the way up to the horizon.

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1. Introduction

The traditional picture of a black hole has a horizon, a central singularity, and essentially 'empty space' in between. This picture leads to contradictions with quantum mechanics—Hawking radiation leads to a loss of unitarity [1]. More recently a different picture of the black hole interior has been suggested, where the information of the state of the hole is distributed throughout the interior of the horizon, creating a 'fuzzball' [2]. While the general state of a Schwarzschild hole is expected to be very nonclassical inside the horizon, we expect that for extremal holes we can find appropriately selected states that will be represented by classical solutions. In [3] it was found that the generic state of the 2-charge extremal D1–D5 system could be understood by studying classical solutions of supergravity, and in [4–6] classical solutions were constructed for specific families of 3-charge extremal D1–D5–P states. The traditional picture of the 3-charge extremal hole is

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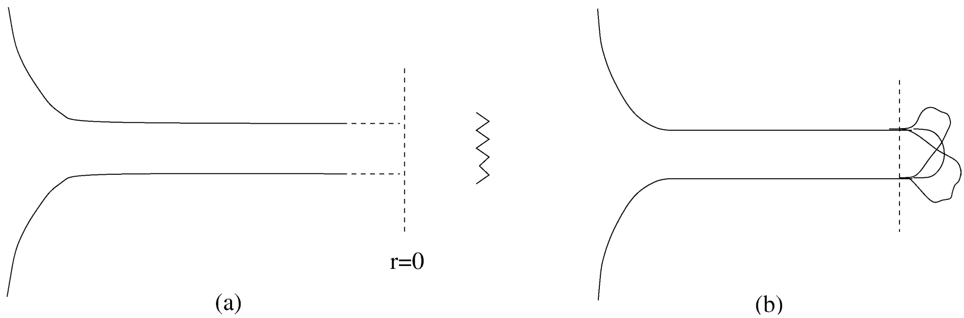

Fig. 1. (a) Naive geometry of 3-charge D1–D5–P; there is a horizon at $r = 0$ and a singularity past the horizon. (b) Expected geometries for D1–D5–P; the area at the dashed line will give $\frac{A}{4G} = 2\pi\sqrt{n_1 n_5 n_p}$.

pictured in Fig. 1(a). But the 3-charge geometries constructed in [5,6] were of the form Fig. 1(b); the throat 'caps off' without any horizon or singularity. (All 2-charge extremal states have a geometry that caps off like Fig. 1(b); the 'naive geometry' in this case has a zero area horizon coinciding with the singularity at $r = 0$.)

In this paper we pursue this program further, by finding further sets of 3-charge extremal CFT states and their dual geometries. One can write down a large class of 3-charge extremal solutions of classical supergravity, and these will in general have pathologies. But a basic tenet of our conjecture is that the geometries that are dual to actual microstates of the 3-charge CFT will be regular solutions with no horizons, singularities or closed timelike curves. The solutions in [5,6] were smooth, and we will find that the solutions we now construct will also be free of horizons and closed timelike curves (though most will have an orbifold singularity along certain curves). Thus our solutions will be like Fig. 1(b) rather than 1(a), and will lend support to the general 'fuzzball' picture of the black hole interior.[1]

In more detail, we do the following.

(a) In [11,12] a family of D1–D5 geometries was obtained, by taking extremal limits in the general family of rotating 3-charge solutions constructed in [13]. This family is labelled by a parameter $0 < \gamma \leqslant 1$. The geometries have no horizons and the only singularity is an orbifold singularity along an S^1 in the noncompact directions. The corresponding CFT duals can be identified [3,11,12]. The orbifold singularity vanishes for the special case $\gamma = 1$.

If we perform a spectral flow on the left sector of the CFT then from a 2-charge D1–D5 state we get a 3-charge D1–D5–P state. In [6] we found the geometries for the 3-charge states that are related by spectral flow to the 2-charge state with $\gamma = 1$. Here we extend this computation to find the geometries for 3-charge states starting from 2-charge states for arbitrary γ. It is straightforward to identify the corresponding CFT duals.

(b) Given a D1–D5–P state we can use dualities to interchange any two of its charges. This leads to a new geometry which must also represent a true state of the 3-charge system, since these dualities are exact symmetries of the theory. We construct these geometries

[1] Additional evidence for this picture comes from a study of the nonzero size of supertubes [7–10].

obtained by S, T dualities. These geometries turn out to have orbifold singularities along two nonintersecting S^1 curves. These geometries and the ones obtained in (a) all fall into a general class that we identify; they are rotating extremal solutions with parameters such that there are no horizons and no closed timelike curves.

(c) It is not immediately obvious what the CFT states corresponding to the geometries in (b) are. To find information about the state, we study the infall of a quantum down the 'throat' of this geometry, and study the travel time $\Delta t_{\mathrm{SUGRA}}$ for a complete 'bounce'. It was found in [3] that this bounce time exactly equalled the time Δt_{CFT} for excitations to travel around the corresponding 'effective string' in the CFT. By computing $\Delta t_{\mathrm{SUGRA}}$ for the geometries we find the length of each component of the effective string, and thus identify the CFT state.

(d) We observe that the result found in (c) supports the picture of 'spacetime bits' arrived at in [3]: Under duality the number of components of the effective string remains the same, though the total winding number of the effective string changes. We also observe that the travel time in the geometry and in the CFT are related by a redshift factor η which relates the time coordinate at infinity to the time coordinate in the AdS region (η becomes unity if the momentum charge P vanishes).

2. D1–D5–P states from spectral flow of D1–D5 states

2.1. The D1–D5 CFT

We take IIB string theory compactified to $M_{4,1} \times S^1 \times T^4$. Let y be the coordinate along S^1 with

$$0 \leqslant y < 2\pi R. \tag{2.1}$$

The T^4 is described by 4 coordinates z_1, z_2, z_3, z_4, and the noncompact space is spanned by t, x_1, x_2, x_3, x_4. We wrap n_1 D1 branes on S^1, and n_5 D5 branes on $S^1 \times T^4$. Let $N = n_1 n_5$. The bound state of these branes is described by a $(1+1)$-dimensional sigma model, with base space (y, t) and target space a deformation of the orbifold $(T^4)^N/S_N$ (the symmetric product of N copies of T^4). The CFT has $\mathcal{N} = 4$ supersymmetry, and a moduli space which preserves this supersymmetry. It is conjectured that in this moduli space we have an 'orbifold point' where the target space is just the orbifold $(T^4)^N/S_N$ [14].

The rotational symmetry of the noncompact directions $x_1, \ldots, x_4$ gives a symmetry $so(4) \approx su(2)_L \times su(2)_R$, which is the R symmetry group of the CFT.

The CFT with target space just one copy of T^4 is described by 4 real bosons X^1, X^2, X^3, X^4 (which arise from the 4 directions z_1, z_2, z_3, z_4), 4 real left moving fermions ψ^1, ψ^2, ψ^3, ψ^4 and 4 right moving fermions $\bar\psi^1$, $\bar\psi^2$, $\bar\psi^3$, $\bar\psi^4$. The central charge is $c = 6$. The complete theory with target space $(T^4)^N/S_N$ has N copies of this $c = 6$ CFT, with states that are symmetrized between the N copies. The orbifolding also generates 'twist' sectors, which are created by twist operators σ_k. A detailed construction of the twist operators is given in [15,16], but we summarise here the properties that will be relevant to us.

The twist operator of order k links together k copies of the $c = 6$ CFT so that the X^i, ψ^i act as free fields living on a circle of length kL (L is the length of the spatial circle of the CFT). Let us first discuss the NS sector. The left fermions ψ^i carry spin $\frac{1}{2}$ under the $su(2)_L$ and the right fermions $\bar{\psi}^i$ carry spin $\frac{1}{2}$ under the $su(2)_R$. The 'charge' of a state is given by the quantum numbers $(j, \bar{j}) = (j_L^3, j_R^3)$. Adding a suitable charge to the twist operator we get a chiral primary

$$\sigma_k^{--}:\quad h = j = \frac{k-1}{2},\qquad \bar{h} = \bar{j} = \frac{k-1}{2}. \tag{2.2}$$

We can act on this chiral primary with J_{-1}^{+} to get another chiral primary

$$\sigma_k^{+-} \equiv J_{-1}^{+}\sigma_k^{--}:\quad h = j = \frac{k+1}{2},\qquad \bar{h} = \bar{j} = \frac{k-1}{2}. \tag{2.3}$$

Similarly we also get

$$\sigma_k^{-+} \equiv \bar{J}_{-1}^{+}\sigma_k^{--}:\quad h = j = \frac{k-1}{2},\qquad \bar{h} = \bar{j} = \frac{k+1}{2}, \tag{2.4}$$

$$\sigma_k^{++} \equiv J_{-1}^{+}\bar{J}_{-1}^{+}\sigma_k^{--}:\quad h = j = \frac{k+1}{2},\qquad \bar{h} = \bar{j} = \frac{k+1}{2}. \tag{2.5}$$

(We can get additional chiral primaries by applying for example $\psi^{+}_{-\frac{1}{2}}$ (which increase h and j by $\frac{1}{2}$, but we will not need such states in this paper).)

2.1.1. A subclass of states

In the NS sector we can start with the NS vacuum

$$|0\rangle_{\mathrm{NS}}:\quad h = j = 0,\qquad \bar{h} = \bar{j} = 0 \tag{2.6}$$

and act with $\sigma_k^{\pm\pm}$ to generate chiral primaries. Consider the subclass of states

$$\left(\sigma_k^{--}\right)^{\frac{N}{k}}|0\rangle_{\mathrm{NS}}:\quad h = j = \frac{N}{k}\frac{(k-1)}{2},\qquad \bar{h} = \bar{j} = \frac{N}{k}\frac{(k-1)}{2}. \tag{2.7}$$

All copies of the CFT are linked into 'long circles' which are all of the same length kL, and the spin orientation (given by the choice $(--)$ for σ) is also the same for each circle. We therefore expect that the corresponding states exhibit some symmetry; it will turn out that their gravity duals have axial symmetry around two circles ψ, ϕ. Similarly we have the states

$$\left(\sigma_k^{+-}\right)^{\frac{N}{k}}|0\rangle_{\mathrm{NS}}:\quad h = j = \frac{N}{k}\frac{(k+1)}{2},\qquad \bar{h} = \bar{j} = \frac{N}{k}\frac{(k-1)}{2}, \tag{2.8}$$

$$\left(\sigma_k^{-+}\right)^{\frac{N}{k}}|0\rangle_{\mathrm{NS}}:\quad h = j = \frac{N}{k}\frac{(k-1)}{2},\qquad \bar{h} = \bar{j} = \frac{N}{k}\frac{(k+1)}{2}, \tag{2.9}$$

$$\left(\sigma_k^{++}\right)^{\frac{N}{k}}|0\rangle_{\mathrm{NS}}:\quad h = j = \frac{N}{k}\frac{(k+1)}{2},\qquad \bar{h} = \bar{j} = \frac{N}{k}\frac{(k+1)}{2}. \tag{2.10}$$

2.1.2. Spectral flow

The NS sector states can be mapped to R sector states by 'spectral flow' [17], under which the conformal dimensions and charges change as

$$h' = h - \alpha q + \alpha^2 \frac{c}{24}, \tag{2.11}$$

$$q' = q - \alpha \frac{c}{12}. \tag{2.12}$$

Setting $\alpha = 1$ gives the flow from the NS sector to the R sector, and we can see that under this flow chiral primaries of the NS sector (which have $h = q$) map to Ramond ground states with $h = \frac{c}{24}$.

The field theory on the D1–D5 branes system is in the R sector. This follows from the fact that the branes are solitons of the gravity theory, and the fermions on the branes are induced from fermions on the bulk. The latter are periodic around the S^1; choosing antiperiodic boundary conditions would give a nonvanishing vacuum energy and disallow the flat space solution that we have assumed at infinity.

If we set $\alpha = 2$ in (2.12) then we return to the NS sector, and setting $\alpha = 3$ brings us again to the R sector. More generally, the choice

$$\alpha = 2n + 1, \quad n \in \mathbb{Z} \tag{2.13}$$

brings us to the R sector.

2.1.3. The states we consider

Suppose we start with a chiral primary in the NS sector. Perform a spectral flow (2.13) on the right movers with $\alpha = 1$; this brings us to an R ground state for the right movers. Perform a spectral flow with $\alpha = 2n + 1$ on the left movers. This brings us to the R sector but not in general to an R ground state. The state thus has a momentum charge

$$n_p = h - \bar{h}. \tag{2.14}$$

Applying this procedure to the state (2.7) we get an R sector state

$$\left[\left(\sigma_k^{--}\right)^{\frac{N}{k}} |0\rangle_{\mathrm{NS}}\right]_{\alpha_L = 2n+1, \alpha_R = 1} \equiv \left|\Psi^{--}(k, n)\right\rangle \tag{2.15}$$

with

$$h = N\left(n^2 + \frac{n}{k} + \frac{1}{4}\right), \qquad j = -\frac{N}{2}\left(2n + \frac{1}{k}\right), \qquad \bar{h} = \frac{N}{4}, \qquad \bar{j} = -\frac{N}{2k} \tag{2.16}$$

and

$$n_p = h - \bar{h} = Nn\left(n + \frac{1}{k}\right). \tag{2.17}$$

2.1.4. Explicit representations of the states

Let us construct explicitly the above CFT states. Consider one copy of the $c = 6$ CFT, in the R sector. The fermions have modes ψ^i_m. The 4 real fermions can be grouped into 2 complex fermions ψ^+, ψ^- which form a representation of $su(2)$. (ψ^+ has $j = \frac{1}{2}$ and ψ^-

has $j=-\frac{1}{2}$.) The anti-commutation relations are

$$\{(\psi^+)^*_m, \psi^+_p\} = \delta_{m+p,0}, \qquad \{(\psi^-)^*_m, \psi^-_p\} = \delta_{m+p,0}. \tag{2.18}$$

The $su(2)$ currents are

$$J^+_m = (\psi^-)^*_{m-p}\psi^+_p,\ J^-_m = (\psi^+)^*_{m-p}\psi^-_p,$$
$$J^3_m = \frac{1}{2}\big[(\psi^-)^*_{m-p}\psi^-_p - (\psi^+)^*_{m-p}\psi^+_p\big]. \tag{2.19}$$

In the full theory with $n_1 n_5$ copies of the $c=6$ CFT the currents are the sum of the currents in the individual copies

$$J^{a,\text{total}}_n = (J^a_n)_1 + \cdots + (J^a_n)_{n_1 n_5}. \tag{2.20}$$

First consider the state $|\Psi^{--}(k,0)\rangle$ ((2.15) for $n=0$). This gives a D1–D5 state with momentum charge zero

$$\big[(\sigma_k^{--})^{\frac{N}{k}}|0\rangle_{\text{NS}}\big]_{\alpha_L=1,\alpha_R=1}: \quad h=\frac{N}{4}, \qquad j=-\frac{N}{2k},$$
$$\bar h=\frac{N}{4}, \qquad \bar j=-\frac{N}{2k}, \qquad n_p=0. \tag{2.21}$$

Each set of k copies of the $c=6$ CFT which are joined together by σ_k^{--} behave like one copy of the $c=6$ CFT but on a circle of length kL.

Note. We will call each such set of linked copies a *component string*.

Thus in the presence of a twist operator of order k we can apply fractional modes of currents

$$J^a_{-\frac{m}{k}}, \qquad \bar J^a_{-\frac{m}{k}}. \tag{2.22}$$

Since we are in the R sector the fermions have fractional modes $\psi^{\pm}_{-\frac{m}{k}}$. Apart from this fractionation the situation is identical to the one studied in [6] where we had no twist ($k=1$) and we applied currents to find the states arising after spectral flow. In the present case the lowest dimension current operator that we can apply to lower charge is $J^-_{-\frac{2}{k}}$ to (2.21); this is equivalent to applying $(\psi^-)^*_{-\frac{1}{k}}\psi^-_{-\frac{1}{k}}$. The next operator we can apply is $J^-_{-\frac{4}{k}}$, and so on. The orbifold CFT requires that the total momentum on each component string be an integer—we will discuss this issue (and possible exceptions) in more detail in Section 6. Thus we can apply

$$J^-_{-\frac{2(k-1)}{k}} \cdots J^-_{-\frac{4}{k}} J^-_{-\frac{2}{k}} \tag{2.23}$$

which adds dimension and charge

$$\Delta h = k-1, \qquad \Delta j = -(k-1) \tag{2.24}$$

or we can apply

$$J^-_{-2} J^-_{-\frac{2(k-1)}{k}} \cdots J^-_{-\frac{4}{k}} J^-_{-\frac{2}{k}} \tag{2.25}$$

which adds dimension and charge

$$\Delta h = k+1, \qquad \Delta j = -k. \tag{2.26}$$

To get states with a high symmetry we apply the same set of current operators to all the $\frac{N}{k}$ twist operators in the state. Applying (2.23) we get a state

$$\Bigl(\prod \bigl[J^-_{-\frac{2(k-1)}{k}} \cdots J^-_{-\frac{4}{k}} J^-_{-\frac{2}{k}}\bigr]\Bigr)\bigl|\Psi^{--}(k,0)\bigr\rangle \tag{2.27}$$

where the product runs over the $\frac{N}{k}$ connected components of the CFT created by the twists. This state has dimensions and charges

$$\begin{aligned} &h = \frac{N}{4} + \frac{N}{k}(k-1), \qquad j = -\frac{N}{2k} - \frac{N}{k}(k-1), \\ &\bar{h} = \frac{N}{4}, \qquad \bar{j} = -\frac{N}{2k} \end{aligned} \tag{2.28}$$

and

$$n_p = \frac{N}{k}(k-1). \tag{2.29}$$

Applying (2.25) instead to each of the component strings we get the state

$$\Bigl(\prod \bigl[J^-_{-2} J^-_{-\frac{2(k-1)}{k}} \cdots J^-_{-\frac{4}{k}} J^-_{-\frac{2}{k}}\bigr]\Bigr)\bigl|\Psi^{--}(k,0)\bigr\rangle \tag{2.30}$$

with dimensions and charges

$$h = \frac{N}{4} + \frac{N}{k}(k+1), \qquad j = -\frac{N}{2k} - N, \qquad \bar{h} = \frac{N}{4}, \qquad \bar{j} = -\frac{N}{2k}, \tag{2.31}$$

$$n_p = \frac{N}{k}(k+1). \tag{2.32}$$

We now observe that the state (2.30) has the correct dimensions and charges to be the member of the spectral flow family (2.15) with $n = 1$. Similarly, the state (2.27) can be identified with the state obtained by spectral flow, with $n = 1$, from the state $(\sigma_k^{+-})^{\frac{N}{k}}|0\rangle_{\rm NS}$ in Eq. (2.8).

We can apply further sets of currents to get states with spectral flow by $n > 1$ units

$$\Bigl(\prod \bigl[J^-_{-2n} \cdots J^-_{-\frac{2}{k}}\bigr]\Bigr)\bigl|\Psi^{--}(k,0)\bigr\rangle. \tag{2.33}$$

These states have

$$h = \frac{N}{4} + \frac{N}{k}n(nk+1), \qquad j = -\frac{N}{2k} - nN, \qquad \bar{h} = \frac{N}{4}, \qquad \bar{j} = -\frac{N}{2k}, \tag{2.34}$$

$$n_p = \frac{N}{k}n(nk+1). \tag{2.35}$$

We can similarly get those with $n < 0$; latter are obtained by applying modes of J^+ instead of J^-.

2.2. Gravity duals

2.2.1. Duals of 2-charge states

In [13] a set of D1–D5–P solutions was given. The solutions had axial symmetry along two circles ψ, ϕ, and angular momenta J_ψ, J_ϕ. In [11,12] an extremal limit was obtained for solutions with $P = 0$, getting the geometries

$$\begin{aligned} ds^2 = {}& -\frac{1}{h}\left(dt^2 - dy^2\right) + hf\left(\frac{dr^2}{r^2 + a^2\gamma^2} + d\theta^2\right) \\ & + h\left(r^2 + \frac{a^2\gamma^2 Q_1 Q_5 \cos^2\theta}{h^2 f^2}\right)\cos^2\theta\, d\psi^2 \\ & + h\left(r^2 + a^2\gamma^2 - \frac{a^2\gamma^2 Q_1 Q_5 \sin^2\theta}{h^2 f^2}\right)\sin^2\theta\, d\phi^2 \\ & - \frac{2a\gamma\sqrt{Q_1 Q_5}}{hf}\left(\cos^2\theta\, dy\, d\psi + \sin^2\theta\, dt\, d\phi\right) + \sqrt{\frac{H_1}{H_5}}\sum_{i=1}^{4} dx_i^2, \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} & a = \frac{\sqrt{Q_1 Q_5}}{R}, \qquad f = r^2 + a^2\gamma^2\cos^2\theta, \\ & H_1 = 1 + \frac{Q_1}{f}, \qquad H_5 = 1 + \frac{Q_5}{f}, \qquad h = \sqrt{H_1 H_5}. \end{aligned} \tag{2.37}$$

These metrics have angular momenta

$$J_\psi = -\bar{j} + j = 0, \qquad J_\phi = -\bar{j} - j = \gamma n_1 n_5 \tag{2.38}$$

with n_1 and n_5 the numbers of D1 and D5 branes. $j, \bar{j}$ are the angular momenta in the L, R factors of the $so(4) \approx su(2)_L \times su(2)_R$ describing the angular directions. For

$$\gamma = \frac{1}{k}, \quad k = 1, 2 \ldots \tag{2.39}$$

we obtain geometries that are the duals of the states $|\Psi^{--}(k, 0)\rangle$ (Eq. (2.21)) [3,11,12].

2.2.2. Duals of 3-charge states obtained by spectral flow of 2-charge states

We would now like to find the duals of the 3-charge states obtained by spectral flow of the above 2-charge states. We again start from the 3-charge nonextremal solutions and take an extremal limit, keeping the charges and angular momenta at the values given by the CFT state. For the case where all twists were trivial ($k = 1$ for all twist operators) this procedure was carried out in [6]. The starting nonextremal solution was derived in [6] by starting with the neutral rotating hole in $4 + 1$ dimensions, and applying a sequence of boosts and dualities. This solution is reproduced in Appendix A. Taking the limit for $P \neq 0$ needs some care, but the procedure was described in detail in [6] and needs no changes for the more general case here.

We fix the values of the angular momenta and the momentum charge n_p to the values we desire, and then take the extremal limit. For general values of these parameters, we obtain the solution given in Appendix A. These solutions have pathologies in general; for example they can have closed timelike curves. But we must select only those that correspond to microstates of the 3-charge system, and these we expect to be free of pathologies.

We set the charges and momentum to equal those of the states (2.33)

$$J_\psi = -n n_1 n_5, \qquad J_\phi = (n+\gamma) n_1 n_5, \qquad n_p = n(n+\gamma) n_1 n_5. \tag{2.40}$$

The metric is written naturally in terms of the dimensionful quantities

$$\tilde\gamma_1 = \frac{4G^{(5)}}{\pi\sqrt{Q_1Q_5}} J_\psi, \qquad \tilde\gamma_2 = \frac{4G^{(5)}}{\pi\sqrt{Q_1Q_5}} J_\phi,$$
$$Q_p = \frac{4G^{(5)}}{\pi R} n_p, \qquad Q_1 = \frac{(2\pi)^4 g\alpha'^3}{V} n_1, \qquad Q_5 = g\alpha' n_5, \tag{2.41}$$

where $G^{(5)}$ is the 5D Newton's constant

$$G^{(5)} = \frac{G^{(10)}}{V(2\pi R)} = \frac{4\pi^5 g^2 \alpha'^4}{VR}, \tag{2.42}$$

V is the volume of T^4, R the radius of the S^1 and g the string coupling. We find

$$\tilde\gamma_1 = \frac{\sqrt{Q_1Q_5}}{R}\frac{J_\psi}{n_1n_5} \equiv \frac{\sqrt{Q_1Q_5}}{R}\gamma_1, \qquad \tilde\gamma_2 = \frac{\sqrt{Q_1Q_5}}{R}\frac{J_\phi}{n_1n_5} \equiv \frac{\sqrt{Q_1Q_5}}{R}\gamma_2 \tag{2.43}$$

and thus, for the duals of the states (2.33),

$$\tilde\gamma_1 = -\frac{\sqrt{Q_1Q_5}}{R} n, \qquad \tilde\gamma_2 = \frac{\sqrt{Q_1Q_5}}{R}(n+\gamma), \qquad Q_p = \frac{Q_1Q_5}{R^2} n(n+\gamma). \tag{2.44}$$

We observe that

$$Q_p = -\tilde\gamma_1\tilde\gamma_2. \tag{2.45}$$

We will see that it is this relation that selects, from the class of all axisymmetric solutions, the geometries that are free of pathologies.[2] Using (2.45) to simplify the extremal solution, we get

$$\begin{aligned} ds^2 = {} & -\frac{1}{h}\left(dt^2 - dy^2\right) + \frac{Q_p}{hf}(dt-dy)^2 + hf\left(\frac{dr^2}{r^2 + (\tilde\gamma_1+\tilde\gamma_2)^2\eta} + d\theta^2\right) \\ & + h\left(r^2 + \tilde\gamma_1(\tilde\gamma_1+\tilde\gamma_2)\eta - \frac{Q_1Q_5(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta\cos^2\theta}{h^2f^2}\right)\cos^2\theta\, d\psi^2 \\ & + h\left(r^2 + \tilde\gamma_2(\tilde\gamma_1+\tilde\gamma_2)\eta + \frac{Q_1Q_5(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta\sin^2\theta}{h^2f^2}\right)\sin^2\theta\, d\phi^2 \\ & + \frac{Q_p(\tilde\gamma_1+\tilde\gamma_2)^2\eta^2}{hf}\left(\cos^2\theta d\psi + \sin^2\theta\, d\phi\right)^2 \end{aligned}$$

[2] We certainly have states of the system with $Q_p \neq 0$ and small or vanishing angular momenta, but as seen in the 2-charge case [3] such states will break axial symmetry, and thus not be in the class that we are considering at present.

$$
\begin{aligned}
&- \frac{2\sqrt{Q_1 Q_5}}{hf}\big(\tilde{\gamma}_1 \cos^2\theta\, d\psi + \tilde{\gamma}_2 \sin^2\theta\, d\phi\big)(dt - dy) \\
&- \frac{2\sqrt{Q_1 Q_5}(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta}{hf}\big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big)\, dy + \sqrt{\frac{H_1}{H_5}} \sum_{i=1}^{4} dx_i^2, \qquad (2.46)
\end{aligned}
$$

$$
\begin{aligned}
C_2 = &-\frac{\sqrt{Q_1 Q_5}\cos^2\theta}{H_1 f}(\tilde{\gamma}_2\, dt + \tilde{\gamma}_1\, dy) \wedge d\psi \\
&- \frac{\sqrt{Q_1 Q_5}\sin^2\theta}{H_1 f}(\tilde{\gamma}_1\, dt + \tilde{\gamma}_2\, dy) \wedge d\phi \\
&+ \frac{(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta Q_p}{\sqrt{Q_1 Q_5} H_1 f}(Q_1\, dt + Q_5\, dy) \wedge \big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big) \\
&- \frac{Q_1}{H_1 f}\, dt \wedge dy - \frac{Q_5 \cos^2\theta}{H_1 f}\big(r^2 + \tilde{\gamma}_2(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta + Q_1\big)\, d\psi \wedge d\phi, \qquad (2.47)
\end{aligned}
$$

$$
e^{2\Phi} = \frac{H_1}{H_5}, \qquad (2.48)
$$

where

$$
\begin{aligned}
&\eta = \frac{Q_1 Q_5}{Q_1 Q_5 + Q_1 Q_p + Q_5 Q_p}, \\
&f = r^2 + (\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta\big(\tilde{\gamma}_1 \sin^2\theta + \tilde{\gamma}_2 \cos^2\theta\big), \\
&H_1 = 1 + \frac{Q_1}{f}, \qquad H_5 = 1 + \frac{Q_5}{f}, \qquad h = \sqrt{H_1 H_5}. \qquad (2.49)
\end{aligned}
$$

3. Obtaining new solutions by *S*, *T* dualities

As mentioned in the introduction, we are interested in making geometries that are dual to actual bound states of the D1–D5–P system, and not just formal solutions of supergravity carrying D1, D5, P charges. In the previous section we started with known states in the CFT and found their gravity duals by looking for solutions with the same symmetries and quantum numbers. In this section we make D1–D5–P solutions by a different method: we start with a D1–D5–P geometry that we have already constructed and perform S, T dualities to permute the charges. Since these dualities are exact symmetries of the theory, we know that the resulting geometry represents a true microstate. But it will not be immediately obvious what the dual CFT state is. We will identify the CFT state later, by analyzing the properties of the supergravity solution, and find that the change of CFT state under these S, T dualities provides insight into the AdS/CFT duality map.

The metric (2.46) is invariant under the interchange of Q_1, Q_5, so the only nontrivial duality is the one which interchanges the momentum charge with, say, the D1 charge

$$
\begin{pmatrix} \mathrm{P} \\ \mathrm{D1} \\ \mathrm{D5} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \mathrm{P} \\ \mathrm{F1} \\ \mathrm{NS5} \end{pmatrix} \xrightarrow{T_y, T_{z_1}} \begin{pmatrix} \mathrm{F1} \\ \mathrm{P} \\ \mathrm{NS5} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \mathrm{D1} \\ \mathrm{P} \\ \mathrm{D5} \end{pmatrix}. \qquad (3.1)
$$

The metric which results from this chain of dualities, and the corresponding dilaton are

$$
\begin{aligned}
ds^2 = {} & -\frac{1}{\tilde h}\bigl(dt^2 - dy^2\bigr) + \frac{Q_1}{\tilde h f}(dt-dy)^2 + \tilde h f\left(\frac{dr^2}{r^2 + (\tilde\gamma_1+\tilde\gamma_2)^2\eta} + d\theta^2\right) \\
& + \tilde h\left(r^2 + \tilde\gamma_1(\tilde\gamma_1+\tilde\gamma_2)\eta - \frac{Q_p Q_5(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta\cos^2\theta}{\tilde h^2 f^2}\right)\cos^2\theta\, d\psi^2 \\
& + \tilde h\left(r^2 + \tilde\gamma_2(\tilde\gamma_1+\tilde\gamma_2)\eta + \frac{Q_p Q_5(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta\sin^2\theta}{\tilde h^2 f^2}\right)\sin^2\theta\, d\phi^2 \\
& + \frac{Q_p(\tilde\gamma_1+\tilde\gamma_2)^2\eta^2}{\tilde h f}\bigl(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\bigr)^2 \\
& - \frac{2\sqrt{Q_1 Q_5}}{\tilde h f}\bigl(\tilde\gamma_1\cos^2\theta\, d\psi + \tilde\gamma_2\sin^2\theta\, d\phi\bigr)(dt-dy) \\
& - \frac{2Q_p(\tilde\gamma_1+\tilde\gamma_2)\eta}{\tilde h f}\sqrt{\frac{Q_5}{Q_1}}\bigl(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\bigr)dy + \sqrt{\frac{H_p}{H_5}}\sum_{i=1}^{4} dx_i^2, \\
C_2 = {} & -\frac{\sqrt{Q_1 Q_5}\cos^2\theta}{H_p f}(\tilde\gamma_2\, dt + \tilde\gamma_1\, dy)\wedge d\psi - \frac{\sqrt{Q_1 Q_5}\sin^2\theta}{H_p f}(\tilde\gamma_1\, dt + \tilde\gamma_2\, dy)\wedge d\phi \\
& + \frac{(\tilde\gamma_1+\tilde\gamma_2)\eta}{H_p f}\sqrt{\frac{Q_1}{Q_5}}(Q_p\, dt + Q_5\, dy)\wedge\bigl(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\bigr) \\
& - \frac{Q_p}{H_p f}\, dt\wedge dy - \frac{Q_5\cos^2\theta}{H_p f}\bigl(r^2 + \tilde\gamma_2(\tilde\gamma_1+\tilde\gamma_2)\eta + Q_p\bigr)\, d\psi\wedge d\phi, \\
e^{2\Phi} = {} & \frac{H_p}{H_5},
\end{aligned}
\tag{3.2}
$$

where

$$
H_p = 1 + \frac{Q_p}{f}, \qquad \tilde h = \sqrt{H_p H_5}. \tag{3.3}
$$

The solution above is again of the general form (2.46) but with different parameters[3]

$$
\begin{aligned}
& Q_1' = Q_p, \qquad Q_5' = Q_5, \qquad Q_p' = Q_1, \\
& \tilde\gamma_1' = \sqrt{\frac{Q_1}{Q_p}}\tilde\gamma_1, \qquad \tilde\gamma_2' = \sqrt{\frac{Q_1}{Q_p}}\tilde\gamma_2.
\end{aligned}
\tag{3.4}
$$

[3] The quantities Q_i have units of (length)^2, so we have to be careful about the meaning of (3.4). If we start with the F–NS5–P system and do T_y to get P–NS5–F then we get $\frac{Q_1'}{l_s'^2} = \frac{Q_p}{l_s^2}$, $\frac{Q_p'}{l_s'^2} = \frac{Q_1}{l_s^2}$, etc., where l_s is the string length. Starting with D1–D5–P and applying $ST_yT_{z_1}S$ to get P–D5–D1 we get $\frac{Q_1'}{l_d'^2} = \frac{Q_p}{l_d^2}$, $\frac{Q_p'}{l_d'^2} = \frac{Q_1}{l_d^2}$ where $l_d = g^{\frac{1}{2}} l_s$ is the D-string length. Since the classical geometry is unchanged by an overall rescaling, we can write (3.4).

4. Conical defect angles

The geometries (2.46) possess, generically, an orbifold singularity along a circle in the noncompact space directions, just like the subset of 2-charge metrics (2.36). Let us recall the physics of these singularities, and then study the 'conical defect' angle (created in the AdS part of the geometry by the orbifolding) for the metrics constructed in the above sections.

4.1. The physics of conical defects

Let us recall the reason why we get conical defects in 2-charge geometries. In [3] a method was developed to compute the gravity duals for *all* states of the 2-charge system (not just the subclass giving the geometries (2.36)). By S, T dualities we map the D1–D5 system to the FP system, which has a fundamental string (F) wrapped on S^1 carrying momentum (P) along S^1. The bound state of the FP system has the strands of the F string all joined up into one 'multiwound' string, and all the momentum is carried as travelling waves on the string. Metrics for the vibrating string were constructed, and dualized back to get D1–D5 geometries. The general geometry was thus parametrized by the vibration profile $\vec{F}(v)$ of the F string.

The detailed map between D1–D5 states and D1–D5 geometries is found in the following way. The vibration on the F string is written in terms of harmonics. If we have a quantum of the kth harmonic on the string then we get a twist operator σ_k acting on the NS vacuum in the D1–D5 CFT. The polarization of the vibration is given by an index $i = 1, \ldots, 4$ labeling the four noncompact directions. The $so(4)$ symmetry group of the angular directions is $\approx su(2)_L \times su(2)_R$, and writing the vector index i in terms of the representation $(\frac{1}{2}, \frac{1}{2})$ of $su(2)_L \times su(2)_R$ we get the choice of superscripts of the twist operator $\sigma_k^{\pm,\pm}$. The collection of all twist operators (arising from all quanta of vibration on the F string) give an NS state, which is spectral flowed to get an R sector state in the D1–D5 CFT. The geometry for this state is known, since it is obtained by S, T dualities from the geometry created by the vibrating F string.

The strands of the F string wrap the y direction, but under the vibration they carry they separate out from each other, and spread out over a simple closed curve in the transverse space $x^1, \ldots, x^4$. It appears at first that there would be a singularity in the FP and D1–D5 geometries at this curve, but it was found in [3] that all waves reflect trivially off this singularity. The reason for this was explained in [18] where it was found that for the D1–D5 geometries the singularity was just a *coordinate* singularity similar to that at the origin of a KK monopole; we have a 'KK monopole tube' (KK monopole $\times S^1$) centered at the above curve. As long as this curve does not self-intersect (it generically does not self-intersect since it is just an S^1 in $\mathbb{R}^4$) the 2-charge D1–D5 geometry is completely smooth, with no horizon or singularity.

The generic solution has no particular symmetry. If we look at solutions that have axial symmetry then there are very few possibilities. We must let the F string swing in a uniform helix in the covering space of the y circle; let this helix have k turns. The vibrations are now all in the kth harmonic, so the D1–D5 CFT state is created by $(\sigma_k^{--})^{\frac{N}{k}}$ (the choice $(--)$ says that we let the F string swing in the x_1–x_2 plane; changing this plane changes the superscripts). The KK monopole tube now 'runs over itself' k times, so all points on the S^1 exhibit the geometry of k KK monopoles coming together. But it is known that this generates an ALE singularity, which has a conical defect angle of $2\pi(1-\frac{1}{k})$.

4.2. Singularity structure of 3-charge metrics

We thus see that a conical defect is a 'harmless singularity'—it arises only if we make nongeneric states by letting the S^1 run over itself, and in the full quantum theory one can imagine that quantum fluctuations separate the intersecting strands and smooth out the singularity. It is important though that the conical defect angle be of the form $2\pi(1-\frac{1}{k})$; if we find an irrational angle for instance then we would not be able to understand how the given geometry sits in a family of geometries that are generically smooth. We would like to understand more precisely the nature of these defects. In particular we would also like to know the values of the parameters γ_1 and γ_2 for which such defects might arise. The arguments we use below are similar to the ones given in [6], where more details can be found.

Around $r=0$ the 6-dimensional part of the 3-charge metric (2.46) have the following form:

$$ds^2 \approx \frac{hf}{(\tilde\gamma_1+\tilde\gamma_2)^2\eta}\left(dr^2+r^2\frac{d\tilde y^2}{R^2}\right) + hf\big(d\theta^2+\tilde g_{\psi\psi}\cos^2\theta\, d\tilde\psi^2+\tilde g_{\phi\phi}\sin^2\theta\, d\tilde\phi^2+2\tilde g_{\psi\phi}\cos^2\theta\sin^2\theta\, d\tilde\psi\, d\tilde\phi\big) + g_{tt}\,dt^2+2\tilde g_{t\psi}\cos^2\theta\, dt\, d\tilde\psi+2\tilde g_{t\phi}\sin^2\theta\, dt\, d\tilde\phi, \tag{4.1}$$

where

$$\gamma = |\gamma_1+\gamma_2|, \tag{4.2}$$

$$\tilde y=\gamma y, \qquad \tilde\psi=\psi-\gamma_2\frac{y}{R}, \qquad \tilde\phi=\phi-\gamma_1\frac{y}{R}, \tag{4.3}$$

$$f\approx(\tilde\gamma_1+\tilde\gamma_2)\eta\big(\tilde\gamma_1\sin^2\theta+\tilde\gamma_2\cos^2\theta\big) \tag{4.4}$$

and $\tilde g_{\psi\psi}$, $\tilde g_{\phi\phi}$, $\tilde g_{\psi\phi}$, g_{tt}, $\tilde g_{t\psi}$, $\tilde g_{t\phi}$ are differentiable functions of θ with

$$\tilde g_{\psi\psi}(\pi/2)=1, \qquad \tilde g_{\phi\phi}(0)=1. \tag{4.5}$$

The above form of the metric shows that at $r=0$ the $\tilde y$ cycle shrinks. It is important to know if, for some particular value of θ, some other cycle shrinks at the same time. This can be understood by looking at the determinant of the metric restricted to the t, $\tilde\psi$ and $\tilde\phi$ coordinates

$$\det g|_{t,\tilde\psi,\tilde\phi} = -\frac{(Q_1Q_5)^2}{R^2}\frac{\eta^2\gamma^2}{hf}\sin^2\theta\cos^2\theta. \tag{4.6}$$

This determinant only vanishes at $\theta=0,\pi/2$: At $\theta=0$ the $\tilde\phi$ coordinate decouples from $\tilde\psi$ and t and the coefficient of $d\tilde\phi^2$ vanishes, i.e., the $\tilde\phi$ cycle shrinks at this point. At $\theta=\pi/2$ the same happens for the $\tilde\psi$ cycle. No other combination of the $\tilde\phi$ and $\tilde\psi$ cycles vanishes at any other value of θ.

To proceed further, we need to look at the actual values of γ, γ_1 and γ_2.

4.2.1. Geometries obtained by spectral flow: orbifold singularities

Let us consider first the case of the 3-charge metrics obtained by spectral flow from the 2-charge metrics in (2.36). For them

$$\gamma_1 = -n, \qquad \gamma_2 = \left(n + \frac{1}{k}\right), \qquad \gamma = \frac{1}{k}, \quad k \in \mathbb{N},\ n \in \mathbb{Z}. \tag{4.7}$$

In this case the coordinates $\tilde{y}$, $\tilde{\psi}$ and $\tilde{\phi}$ are

$$\tilde{y} = \frac{y}{k}, \qquad \tilde{\psi} = \psi - \left(n + \frac{1}{k}\right)\frac{y}{R}, \qquad \tilde{\phi} = \phi + n\frac{y}{R}. \tag{4.8}$$

If $\tilde{y}/R$, $\tilde{\psi}$ and $\tilde{\phi}$ were periodic coordinates with period 2π, i.e., if

$$\left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) \sim \left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) + 2\pi(l_1, l_2, l_3) \tag{4.9}$$

with l_1, l_2, l_3 integers, then the metric (4.1) would be smooth as can be seen from the coefficients of $dy, d\tilde{\psi}$ and $d\tilde{\phi}$ in Eqs. (4.1)–(4.5). However, it follows from the definition (4.8) and from the periodicity of the asymptotic coordinates y/R, ψ, ϕ, that $\tilde{y}/R$, $\tilde{\psi}$ and $\tilde{\phi}$ are subject to the further identifications:

$$\left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) \sim \left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) + 2\pi l\left(\frac{1}{k}, -\frac{1}{k}, 0\right) \tag{4.10}$$

with $l = 0, \ldots, k-1$. The identifications above generate a group isomorphic to $\mathbb{Z}_k$ and the space characterized by the metric (4.1) is topologically equivalent to an orbifold $(\mathbb{R}^3 \times S^3)/\mathbb{Z}_k$. The orbifold action $\mathbb{Z}_k$ has fixed points where both the $\tilde{y}$ and $\tilde{\psi}$ cycles shrink to zero size, which happens at $r = 0$ and $\theta = \pi/2$. Thus the spectral flow metrics with $k > 1$ have $\mathbb{Z}_k$ orbifold singularities of the same kind as the original 2-charge metrics (2.36).

4.2.2. Metrics obtained after $ST_yT_{z_1}S$ duality: orbifold singularities

Let us now turn to the case of the metrics obtained from the spectral flow metrics by S, T dualities. Since these dualities interchange n_1 and n_p and do not change the angular momenta J_ψ and J_ϕ, we find, using (2.43), (2.44)

$$\gamma_1' = \frac{R'}{\sqrt{Q_1'Q_5'}}\tilde{\gamma}_1' = \frac{J_\psi}{n_p n_5} = -\frac{k}{n_5(kn+1)},$$
$$\gamma_2' = \frac{R'}{\sqrt{Q_1'Q_5'}}\tilde{\gamma}_2' = \frac{J_\phi}{n_p n_5} = \frac{1}{n_5 n}. \tag{4.11}$$

We thus see that the parameter γ' after duality

$$\gamma' = \gamma_1' + \gamma_2' = \frac{1}{n_5 n(kn+1)} \tag{4.12}$$

is again of the form (4.7) with the integer k replaced by the integer

$$k' = n_5 n(kn+1). \tag{4.13}$$

In this case, then, the $\tilde{y}$, $\tilde{\psi}$ and $\tilde{\phi}$ coordinates can be written as

$$\tilde{y} = \frac{y}{k'}, \qquad \tilde{\psi} = \psi - \frac{1}{n_5 n}\frac{y}{R}, \qquad \tilde{\phi} = \phi + \frac{k}{n_5(kn+1)}\frac{y}{R} \tag{4.14}$$

and the space (4.1) is topologically equivalent to $(\mathbb{R}^3 \times S^3)/\mathbb{Z}_{k'}$, where now the $\mathbb{Z}_{k'}$ group acts as

$$\left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) \sim \left(\frac{\tilde{y}}{R}, \tilde{\psi}, \tilde{\phi}\right) + 2\pi l \left(\frac{1}{k'}, -\frac{1}{n_5 n}, \frac{k}{n_5(kn+1)}\right) \tag{4.15}$$

with $l = 0, \ldots, k'-1$. Denote by ω the generator of this $\mathbb{Z}_{k'}$ group. We notice that, in contrast with the orbifold action (4.10), ω acts nontrivially on all the three cycles $\tilde{y}$, $\tilde{\psi}$ and $\tilde{\phi}$. However, the group element $\omega^{n_5 n}$ only acts by a -2π rotation on the cycle $\tilde{\psi}$. Thus at $r=0$ and $\theta=0$, where the other two cycles $\tilde{y}$ and $\tilde{\phi}$ shrink, $\omega^{n_5 n}$ has a fixed point and the 6-dimensional space (4.1) has an orbifold singularity. The order of this orbifold singularity is $k'/(n_5 n) = kn+1$. Similarly, if k and n_5 have no common factors, the group element $\omega^{n_5(kn+1)}$ acts trivially on $\tilde{\phi}$ and has a fixed point when $\tilde{y}$ and $\tilde{\psi}$ shrink, which happens at $r=0$ and $\theta=\pi/2$. Thus at $r=0$ and $\theta=\pi/2$ there is an orbifold singularity of order $k'/(n_5(kn+1)) = n$. If n_5 and k have a common factor m, i.e., $n_5 = m\tilde{n}_5$ and $k = m\tilde{k}$, then $\omega^{\tilde{n}_5(kn+1)}$ has a fixed point at $r=0$ and $\theta=\pi/2$ and the order of the orbifold singularity increases to $k'/(\tilde{n}_5(kn+1)) = mn$.

In conclusion, the metrics obtained from the spectral flow metrics by S and T dualities have orbifold singularities at both $(r=0, \theta=0)$, and at $(r=0, \theta=\pi/2)$. The order of the orbifold singularity is $kn+1$ at $(r=0,\theta=0)$ and mn at $(r=0,\theta=\pi/2)$, where m is the highest common factor shared by n_5 and k.

Note: In the classical limit of the D1–D5 system we take $n_1, n_5 \to \infty$, though BPS states exist of course for all n_1, n_5. If we start with a large n_5 then the orbifold shifts involving $\frac{1}{n_5}$ in (4.15) are very close together, and cannot be seen in the classical limit $n_5 \to \infty$. But since microstates exist for all n_1, n_5 the geometries studied in this section can be considered for n_5 of order unity, and then they give well-defined classical metrics with the orbifold group (4.15).

4.2.3. Absence of horizons and closed timelike curves

If we write down a generic 3-charge solution with rotation, we find closed timelike curves (see for example [19]). But we expect that geometries that actually arise as duals to 3-charge states will be free of pathologies. In [6] computations were developed to show that the geometries constructed there had no horizons and no closed timelike curves. A similar result holds for the geometries found in this paper; the computations are given in Appendix B.

5. Wave equation for a scalar

In [3,20] the wave equation for a massless minimally coupled scalar was studied in the 2-charge geometry. Such a scalar arises for instance from fluctuations of the metric on T^4, for example the component $h_{z_1 z_2}$. It was found that the wavepacket spent a time $\Delta t_{\mathrm{SUGRA}}$

traveling down and back up the 'throat' of the supergravity solution. The time $\Delta t_{\mathrm{SUGRA}}$ exactly equalled the time taken for excitations to travel around the 'effective string' in the dual CFT. We now compute $\Delta t_{\mathrm{SUGRA}}$ for the 3-charge solutions we have found, and then use the result to find the dual state in the CFT.

The wave equation for a massless minimally coupled scalar in the 6D geometry is[4]

$$\Box\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi\right) = 0. \tag{5.1}$$

We give $g^{\mu\nu}$, $\det g$ in Appendix C. Writing

$$\Phi(t,y,r,\theta,\psi,\phi) = \exp\left(-i\omega\frac{t}{R} + i\lambda\frac{y}{R} + im_1\psi + im_2\phi\right)\tilde{\Phi}(r,\theta) \tag{5.2}$$

we get a wave equation that is separable in r,θ [21]. We write

$$\tilde{\Phi}(r,\theta) = H(r)\Theta(\theta). \tag{5.3}$$

We introduce the dimensionless radial coordinate

$$x = r^2\frac{R^2}{Q_1Q_5} \tag{5.4}$$

and the following convenient quantities

$$\begin{aligned}
&\delta = \sqrt{\eta}|\gamma_1+\gamma_2| = \sqrt{\eta}\gamma, \qquad \sigma^2 = \left[\left(\omega^2-\lambda^2\right)\frac{Q_1Q_5}{R^4}\right]^{-1},\\
&\nu = \left(1+\Lambda-\left(\omega^2-\lambda^2\right)\frac{Q_1+Q_5}{R^2}-(\omega-\lambda)^2\frac{Q_p}{R^2}\right)^{1/2},\\
&\xi = \sqrt{\eta}\left(\frac{\omega}{\eta}-\lambda\frac{Q_p(Q_1+Q_5)}{Q_1Q_5}-m_1\gamma_1-m_2\gamma_2\right),\\
&\zeta = \sqrt{\eta}(\lambda+m_1\gamma_2+m_2\gamma_1).
\end{aligned} \tag{5.5}$$

Then the radial and angular part of the wave equation become

$$4\frac{d}{dx}\left(x\left(x+\delta^2\right)\frac{d}{dx}\right)H + \left[\sigma^{-2}x+1-\nu^2+\frac{\xi^2}{x+\delta^2}-\frac{\zeta^2}{x}\right]H = 0, \tag{5.6}$$

$$\begin{aligned}
&\frac{1}{\sin 2\theta}\frac{d}{d\theta}\left(\sin 2\theta\frac{d}{d\theta}\right)\Theta + \Lambda\Theta\\
&\quad + \left[-\frac{m_1^2}{\cos^2\theta}-\frac{m_2^2}{\sin^2\theta}+\frac{\tilde{\gamma}_1+\tilde{\gamma}_2}{\sigma^2}\eta\left(\tilde{\gamma}_1\sin^2\theta+\tilde{\gamma}_2\cos^2\theta\right)\right]\Theta = 0.
\end{aligned} \tag{5.7}$$

Reality of the metric implies that the wave equation is real; thus the complex conjugate of a solution gives another solution. We can thus take

$$\xi \geqslant 0. \tag{5.8}$$

The solutions with $\xi < 0$ are obtained by complex conjugation.

---

[4] The 6D string metric is obtained by ignoring the T^4, and the 6D Einstein metric turns out to be the same as the 6D string metric.

5.1. Solving the wave equation

The radial wave equation can be solved in the two regions

$$\begin{aligned} &\text{outer region:} \quad x \gg 1, \\ &\text{inner region:} \quad x \ll \sigma^2. \end{aligned} \tag{5.9}$$

If one chooses the frequency of the scattering wave to be very low

$$\sigma^2 \gg 1 \tag{5.10}$$

the inner and outer regions have a wide overlap, where the two limiting solutions can be matched. Due to this large overlapping region, we expect that a reliable solution can be found, in the low frequency limit (5.10), without the need to introduce a further region or to make any further assumption on the parameters. In particular the quantity

$$\left(\omega^2 - \lambda^2\right)\frac{Q_1 + Q_5}{R^2} + (\omega - \lambda)^2 \frac{Q_p}{R^2} \tag{5.11}$$

is *not* assumed to be small. For large ω it can be seen from (5.5) that ν becomes imaginary; this corresponds to energies where the quantum travels over the potential barrier in the 'neck' region instead of tunneling through it. Since the CFT is known to describe the low energy dynamics of the system, we will restrict ourselves to real ν; we can choose the sign of ν to be positive.

Note also that in the low frequency limit (5.10) the angular part of the wave equation (5.7) reduces to the angular equation in flat space and thus the eigenvalue Λ is

$$\Lambda = l(l+2). \tag{5.12}$$

The technique of matching solutions across the two regions is well known, and details of the computation are given in Appendix D. Here we outline the main steps and results. The solution in the outer region is a linear combination of Bessel's functions

$$H_{\text{out}} = \frac{1}{\sqrt{x}}\left[C_1 J_\nu\left(\sigma^{-1}\sqrt{x}\right) + C_2 J_{-\nu}\left(\sigma^{-1}\sqrt{x}\right)\right]. \tag{5.13}$$

The coefficients C_1 and C_2 are fixed by demanding continuity with the inner solution. The latter is uniquely determined by the requirement of regularity at $x = 0$ and is given in terms of the hypergeometric function

$$H_{\text{in}} = x^\alpha\left(x + \delta^2\right)^\beta F\left(p, q; 1 + 2\alpha; -\frac{x}{\delta^2}\right) \tag{5.14}$$

with

$$\begin{aligned} \alpha &= \frac{|\zeta|}{2\delta} = \frac{\sqrt{\eta}}{2\delta}|\lambda + m_1\tilde{\gamma}_2 + m_2\tilde{\gamma}_1|, \\ \beta &= \frac{\xi}{2\delta} = \frac{\sqrt{\eta}}{2\delta}\left(\frac{\omega}{\eta} - \lambda\frac{Q_p(Q_1+Q_5)}{Q_1Q_5} - m_1\tilde{\gamma}_1 - m_2\tilde{\gamma}_2\right), \\ p &= \frac{1}{2} + \alpha + \beta + \frac{\nu}{2}, \qquad q = \frac{1}{2} + \alpha + \beta - \frac{\nu}{2}. \end{aligned} \tag{5.15}$$

As a result of the matching we get

$$\frac{C_2}{C_1} = \frac{\Gamma(-\nu+1)\Gamma(-\nu)}{\Gamma(\nu+1)\Gamma(\nu)} \frac{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu}{2})\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu}{2})\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu}{2})}\left(\frac{\delta^2}{4\sigma^2}\right)^{\nu}. \tag{5.16}$$

Note that since $\delta^2 < 1 \ll \sigma^2$ we have $C_2 \ll C_1$.

5.2. *Time of travel and absorption probability*

For very large x we get from (5.13)

$$\begin{aligned} H_{\text{out}} = \sqrt{\frac{2\sigma}{\pi}}\frac{1}{x^{3/4}}\Big[&e^{i(\sigma^{-1}\sqrt{x}-\frac{\pi}{4})}\big(C_1 e^{-i\frac{\pi\nu}{2}} + C_2 e^{i\frac{\pi\nu}{2}}\big) \\ &+ e^{-i(\sigma^{-1}\sqrt{x}-\frac{\pi}{4})}\big(C_1 e^{i\frac{\pi\nu}{2}} + C_2 e^{-i\frac{\pi\nu}{2}}\big)\Big]\big(1+O\big(x^{-\frac{1}{2}}\big)\big). \end{aligned} \tag{5.17}$$

The ratio between the outgoing and the ingoing wave amplitude is (ignoring the constant phase shift caused the by factors $-\frac{\pi}{4}$)

$$\begin{aligned} \mathcal{R} &= \frac{C_1 e^{-i\frac{\pi\nu}{2}} + C_2 e^{i\frac{\pi\nu}{2}}}{C_1 e^{i\frac{\pi\nu}{2}} + C_2 e^{-i\frac{\pi\nu}{2}}} \\ &= e^{-i\pi\nu} + \big(1-e^{-2i\pi\nu}\big)\left(\frac{C_2}{C_1} + O\left(\frac{C_2}{C_1}\right)^2\right). \end{aligned} \tag{5.18}$$

In [22] a procedure was given to compute the travel time in the throat from $\mathcal{R}$; we summarize the method here. The quantum coming in from infinity tunnels through the 'neck' region with some probability $p \ll 1$ and enters the 'throat'. Here it travels freely down to the 'cap' and bounces back up. We again have the same probability p that it emerges to infinity, while with probability $1-p$ it turns back for another trip in the throat. We thus get emergent waves at times separated by a fixed interval Δt_{SUGRA}.

To find p and Δt_{SUGRA} we note that $\mathcal{R}$ can be written in the form

$$\mathcal{R} = a + b\sum_{n=1}^{\infty} e^{2\pi i n\frac{\omega}{R}\Delta t}, \tag{5.19}$$

where a and b are some real functions of ω. Let us send in from infinity a wave packet

$$\int dk_r\, f(k_r) e^{-i\frac{k_r}{R}r - i\frac{\omega}{R}t}, \tag{5.20}$$

where k_r/R is the radial wave number $k_r = \sqrt{\omega^2-\lambda^2}$. After scattering from the geometry the wavepacket will be

$$\int dk_r\, f(k_r)\big[e^{-i\frac{k_r}{R}r - i\frac{\omega}{R}t} + \mathcal{R}e^{i\frac{k_r}{R}r - i\frac{\omega}{R}t}\big]. \tag{5.21}$$

From the form (5.19) of $\mathcal{R}$ we see that the wave packet will have peaks at

$$k_r r = \omega(t - 2\pi n \Delta t), \quad n = 0, 1, \ldots. \tag{5.22}$$

The $n \geqslant 1$ peaks represent waves that have travelled n times down the throat and back, and we identify $\Delta t = \Delta t_{\mathrm{SUGRA}}$. From (5.19) we also see that the probability to enter the throat and reemerge is

$$P = p^2 = |b|^2. \tag{5.23}$$

Note. In order to be able to distinguish between successive peaks of the wavepacket, the separation between the peaks (seen at infinity) should be larger than the width of the wavepackets. This implies

$$\Delta t \gg \left(\frac{\Delta k_r}{R}\right)^{-1} \sim \frac{R}{k_r}. \tag{5.24}$$

Our approximate solution (5.16) is only valid in the low frequency limit $k_r \ll R^2/\sqrt{Q_1 Q_5}$, so we should have $\Delta t \gg \sqrt{Q_1 Q_5}/R$. We will find that $\Delta t \sim R/\delta$ so the requirement (5.24) is

$$\frac{R^2}{\delta} \gg \sqrt{Q_1 Q_5}. \tag{5.25}$$

The above condition can be satisfied either by taking R large or δ small.

We now list the results found in Appendix E.

5.2.1. Energy threshold for absorption

If the wave frequency ω is low enough so that

$$\beta < \alpha + \frac{\nu + 1}{2} \tag{5.26}$$

then the wave is reflected back at the neck region and the absorption probability vanishes.

We can interpret this threshold in the following way. If the energy of the quantum is low enough then we cannot fit a complete wavelength in the 'throat'. In the limit of large R the throat is a large asymptotically AdS region. The spectrum of a scalar in such an asymptotically AdS geometry was computed in [23]. Adapting those results to the notation of our paper, we find that the energy levels are given by solutions to the equation

$$\beta = \alpha + \frac{l + 2}{2} + k, \quad k = 0, 1, 2, \ldots. \tag{5.27}$$

If we take the limit $R \to \infty$ then we get a large AdS region, and we can compare quantities to those computed in asymptotically AdS space. Taking this limit we see that the frequencies (5.26) which are not absorbed are those which lie below the AdS_3 excitation spectrum.

5.2.2. Travel time in the geometry

If ω is high enough so that

$$\beta > \alpha + \frac{\nu+1}{2} \tag{5.28}$$

then the reflection amplitude computed in (E.6) has a form similar to (5.19) with the sum over phase factors

$$\sum_{n=1}^{\infty} e^{2\pi i n(\beta-\alpha-\frac{1+\nu}{2})} \equiv \sum_{n=1}^{\infty} e^{2\pi i n \frac{\omega}{R}\Delta t}. \tag{5.29}$$

The factor $\beta - \alpha$ depends linearly on ω; the parameter ν, on the other hand has a nonlinear dependence on ω that will distort the wave packet. Note however that

$$\beta - \alpha - \frac{1+\nu}{2} = \frac{\xi - |\zeta| - \delta(1+\nu)}{2\delta} = \left(\beta - \alpha - \frac{l+2}{2}\right)\bigl(1 + O(\delta\epsilon)\bigr), \tag{5.30}$$

where

$$\epsilon \equiv (l+1) - \sqrt{(l+1)^2 - \bigl(\omega^2 - \lambda^2\bigr)\frac{Q_1+Q_5}{R^2} - (\omega-\lambda)^2\frac{Q_p}{R^2}}. \tag{5.31}$$

We take all the Q_i to be of the same order. We see from (5.25) that travel time makes good sense only if $\delta\frac{Q_i}{R^2} \ll 1$. Thus either $\delta \ll 1$ or $\frac{Q_i}{R^2} \ll 1$ (or both). In either case we find that $\delta\epsilon \ll 1$. We then find

$$\beta - \alpha - \frac{1+\nu}{2} \approx \beta - \alpha - \frac{l+2}{2} = \frac{\omega}{2\sqrt{\eta}\delta} + (\omega \text{ independent terms}). \tag{5.32}$$

In this limit the wave packet is not distorted and it travels up and down the throat in the time

$$\Delta t = \frac{\pi R}{\sqrt{\eta}\delta} = \frac{\pi R}{\eta\gamma}. \tag{5.33}$$

5.2.3. Absorption probability

The probability for the wave to be absorbed and reemitted in the throat is found to be

$$P = \left(\frac{4\pi^2}{\Gamma^2(\nu)\Gamma^2(\nu+1)}\right)^2 \left(\frac{\delta}{2\sigma}\right)^{4\nu} \times \left(\frac{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha+\frac{\nu}{2})}{r\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha-\frac{\nu}{2})}\right)^2. \tag{5.34}$$

The probability for just absorption or just emission is $p = \sqrt{P}$.

5.2.4. The factor η as a redshift

In [3] the travel time was computed for 2-charge D1–D5 geometries, and was found to be

$$\Delta t_{\mathrm{SUGRA}} = \frac{\pi R}{\gamma}. \tag{5.35}$$

This is seen to differ by a factor η from the 3-charge (5.33). We offer a simple physical explanation for this factor.

Consider first the D1–D5 system. In the dual CFT the absorption of the quantum is described by the creation of a set of left and a set of right moving excitations, which travel at the speed of light around the 'component string' in a time $\Delta t_{\rm CFT} = \Delta t_{\rm SUGRA}$. Now suppose we have a P charge as well. This corresponds to the presence of left movers on the component strings. But the left and right excitations in the CFT travel around the component without interacting with each other, so one may think that one again gets the same Δt as in the 2-charge case and thus the value of the P charge does not enter Δt.

But this cannot be right, since the D1, D5, P charges can all be permuted by duality. Indeed the factor η makes Δt in (5.33) invariant under such permutations. To understand the role of η consider the limit of the metric (2.46) for small r and small conical defect δ

$$r \ll \sqrt{Q_i} \quad (i = 1, 5), \qquad \delta \ll \frac{R^2}{\sqrt{Q_1 Q_5}}. \tag{5.36}$$

In this case we have a large AdS type region which would possess a CFT dual description. In this limit $f \ll Q_i$ and thus one can replace H_i by Q_i/f, obtaining an asymptotically $AdS_3 \times S^3$ geometry. As is clear from our computation above, the time of travel is dominated by the time spent by the wave in this part of the geometry. Let us look at the form of the metric (2.46) in the 'near horizon' limit (5.36). We get

$$\begin{aligned}\frac{ds^2_{\rm n.h.}}{\sqrt{Q_1 Q_5}} &= -\left(\rho^2 + \gamma^2\right)(\eta\, d\tilde{t})^2 + \frac{d\rho^2}{\rho^2 + \gamma^2} + \rho^2\, d\tilde{y}^2 + d\theta^2 \\ &\quad + \cos^2\theta\, d\tilde{\psi}^2 + \sin^2\theta\, d\tilde{\phi}^2,\end{aligned} \tag{5.37}$$

where we have made the following coordinate redefinitions

$$\begin{aligned}&\rho^2 = \frac{r^2}{\eta}\frac{R^2}{Q_1 Q_5}, \qquad \tilde{t} = \frac{t}{R}, \qquad \tilde{y} = \frac{1}{R}\left[y - \eta\frac{Q_p(Q_1 + Q_5)}{Q_1 Q_5}t\right], \\ &\tilde{\psi} = \psi - \eta\left[\gamma_1 - \gamma_2\frac{Q_p(Q_1 + Q_5)}{Q_1 Q_5}\right]\frac{t}{R} - \gamma_2\frac{y}{R}, \\ &\tilde{\phi} = \phi - \eta\left[\gamma_2 - \gamma_1\frac{Q_p(Q_1 + Q_5)}{Q_1 Q_5}\right]\frac{t}{R} - \gamma_1\frac{y}{R}.\end{aligned} \tag{5.38}$$

As anticipated, the near horizon metric $ds^2_{\rm n.h.}$ is locally $AdS_3 \times S^3$ with curvature radius $(Q_1 Q_5)^{1/4}$, but the time is rescaled by η with respect to the time t at asymptotically flat infinity. Thus the time computed in the CFT will be a factor of η times the time between wavepackets measured at infinity. If we take a limit $R \to \infty$ keeping other parameters fixed then $Q_p \to 0$, $\eta \to 1$ and we can directly compare the CFT time to the gravity time. We will take such a large R limit in the more detailed analysis of the CFT state below.

6. Finding the CFT duals

We started with CFT states (2.15) and found their gravity duals (2.46). We then made new solutions by applying $ST_yT_{z_1}S$ duality to (2.46) and obtained the solutions (3.2). What are the CFT states dual to the geometries (3.2)? We use two closely related tools to identify the CFT duals: the time of travel $\Delta t_{\mathrm{SUGRA}}$ down the throat of the supergravity solution, and the threshold of absorption into this throat. Longer travel times map to longer components of the 'effective string' in the CFT, and lower absorption thresholds also reflect longer effective strings.

6.1. *Time of travel*

In [3] it was found that for the 2-charge D1–D5 geometries (2.36) a quantum falling into the throat emerges after a time

$$\Delta t_{\mathrm{SUGRA}} = \frac{\pi R}{\gamma} \tag{6.1}$$

or its integer multiples. In the dual CFT the corresponding state is (2.21), where twist operators have joined together

$$k = \frac{1}{\gamma} \tag{6.2}$$

copies of the CFT together to create 'effective strings' of length $2\pi Rk = \frac{2\pi R}{\gamma}$. In the gravity picture a quantum can fall down the throat of the geometry; in the dual CFT description the energy of the quantum gets converted to a set of left and a set of right moving vibrations on the 'effective string' [24,25]. These vibrations travel at the speed of light and meet halfway around the effective string, so that the energy can leave the effective string after a time

$$\Delta t_{\mathrm{CFT}} = \frac{1}{2} 2\pi Rk = \pi Rk \tag{6.3}$$

in exact agreement with (6.1). (If the vibrations fail to collide and leave the string, then they encounter each other again after a time (6.3), etc., this corresponds to the successive waves emerging at separations $\Delta t_{\mathrm{SUGRA}}$ in the gravity picture.)

Note that the number of effective strings created by the twists is

$$m = \frac{n_1 n_5}{k} = \frac{N}{k}. \tag{6.4}$$

Recall that each connected piece of the effective string is termed a 'component string'.

6.1.1. *The number of component strings m' for the 3-charge states (3.2)*

The 3-charge geometries (2.46) had D5, D1, P charges (n_5, n_1, n_p). The orbifold CFT had $N = n_1 n_5$, the winding number of each component string was k, and thus the number of component strings was

$$m = \frac{N}{k} = \frac{n_1 n_5}{k}. \tag{6.5}$$

By the duality $ST_yT_{z_1}S$ we obtained the geometries (3.2) which had charges[5]

$$n'_5 = n_5, \qquad n'_1 = n_p, \qquad n'_p = n_1. \tag{6.6}$$

The orbifold CFT now has

$$N' = n'_1 n'_5 = n_5 n_p. \tag{6.7}$$

The travel time for such geometries is derived in (5.33). Take the limit $R' \to \infty$; this gives a large AdS region which will be the dual of the CFT. We get $Q'_p \to 0$, $\eta' \to 1$ and the travel time is

$$\Delta t_{\mathrm{SUGRA}} = \frac{\pi R'}{\gamma'} = \pi R' n_5 n(nk+1) \tag{6.8}$$

where we used (4.12). Thus we expect the winding number of each component string to be

$$k' = k\frac{n_p}{n_1} = n_5 n(nk+1). \tag{6.9}$$

The number of components of the effective string are then

$$m' = \frac{N'}{k'} = \frac{n_5 n_p}{k}\frac{n_1}{n_p} = \frac{n_1 n_5}{k}. \tag{6.10}$$

We thus see that while N, k change to N', k', *the number of components of the effective string remains unchanged*

$$m' = m. \tag{6.11}$$

To interpret this fact we recall the physical significance of m found in [3]. When we put a particle in the throat of the gravity solution then we excite left and right movers on one component string in the dual CFT. Putting another particle in the throat excites another component string, and so on. Let each particle have the longest possible wavelength which can still fit in the throat of the geometry. When we have enough particles in the throat so that we use up all the m component strings in the CFT then we find that we have enough energy in the supergravity solution to give a backreaction of order unity in the geometry; the geometry distorts and we can no longer study the particles as independent excitations. We now see that under S, T dualities this critical number of particles for the geometry stays unchanged.

[5] S, T dualities certainly map a state of the 3-charge system to another state of the system, but after duality we may not be in a range of parameters where the state is well described by the *conformal* field theory. The conformal limit is the low energy limit, and is attained for small $\frac{\sqrt{Q_1 Q_5 \gamma}}{R}$. If we start with a large circle radius R then after dualities we get small R'. But since we are dealing with BPS states we can follow the state as we increase R' and get back to a CFT domain. It is this latter CFT state that we will mean when we look at the states after $ST_yT_{z_1}S$ duality.

6.1.2. Level of excitation of the component strings

The CFT state for the geometries (2.46) had n_p units of momentum distributed equally over m component strings; thus each component had

$$T = \frac{n_p}{m} \tag{6.12}$$

units of momentum. It will be helpful to write this in terms of the basic units of momentum on the component string. Since the string has winding number k the excitations come in units of $\frac{1}{k}$. Thus the number of these basic units of momentum on each component string is

$$\hat{T} = kT = \frac{n_p k}{m} = \frac{n_p}{m}\frac{N}{m} = \frac{n_1 n_5 n_p}{m^2}. \tag{6.13}$$

For the state after $ST_yT_{z_1}S$ duality we have $n'_p = n_1$ units of momentum, distributed over $m' = m$ component strings, and

$$T' = \frac{n_1}{m}, \qquad \hat{T}' = k'T = \frac{n_1}{m}k\frac{n_p}{n_1} = \frac{n_1 n_5 n_p}{m^2} \tag{6.14}$$

so we see that

$$\hat{T}' = \hat{T} = \frac{n_1 n_5 n_p}{m^2}. \tag{6.15}$$

Thus $\hat{T}$ is a duality invariant; the charges permute under dualities and we have seen that m stays unchanged. We note that the number of possible states on each component string depends on the number $\hat{T}$: we have to just excite free bosons and fermions on the component string so that the total level (as measured in units of the basic excitation on the component string) is $\hat{T}$. Thus we see that under $ST_yT_{z_1}S$ duality the number of allowed states on the component strings remains unchanged.

6.1.3. The state after $ST_yT_{z_1}S$ duality

The angular momentum of the state does not change under the $ST_yT_{z_1}S$ duality (the T dualities are all along compact directions, while angular momentum reflects the properties of the state in the noncompact directions). For the entire state we have from (2.16)

$$j = -\frac{n_1 n_5}{2}\left(2n + \frac{1}{k}\right), \qquad \bar{j} = -\frac{n_1 n_5}{2k}. \tag{6.16}$$

Both before and after the $ST_yT_{z_1}S$ duality we have m component strings, so the angular momentum on each component string is

$$\hat{j} = -\frac{1}{2} - nk, \qquad \hat{\bar{j}} = -\frac{1}{2}. \tag{6.17}$$

Both before and after the duality we have the same 'level' of excitation $\hat{T}$ on each component string. The state before the duality was given by (2.33)—the charge $(j, \bar{j})$ was attained by the lowest energy possible by having all fermion spins aligned and all fermions in the lowest levels allowed by the Pauli exclusion principle. We see that the only way to make the state after $ST_yT_{z_1}S$ duality is to make a construction similar to (2.33)

$$\Big(\prod\big[J^-_{-2n}J^-_{-2n\frac{k'-1}{k'}}\cdots J^-_{-\frac{2}{k'}}\big]\Big)\big|\Psi^{--}(k',0)\big\rangle. \tag{6.18}$$

The component strings now have winding k' each instead of k, but the rest of the construction is the same.

6.2. *Absorption threshold*

In Section 5 it was found that an incoming wave was absorbed only if

$$\beta-\alpha>\frac{\nu+1}{2}. \tag{6.19}$$

Take the CFT limit $R\to\infty$ which gives $Q_p\to 0, \eta\to 1$. The angular momentum components m_1, m_2 imply for the two $su(2)$ factors the eigenvalues

$$m=\frac{m_1-m_2}{2},\qquad \bar m=-\frac{m_1+m_2}{2}. \tag{6.20}$$

The condition (6.19) then gives a pair of conditions from the two possible signs of the absolute value in α

$$\omega-2\left(m\frac{j}{N}+\bar m\frac{\bar j}{N}\right)-\lambda+2\left(m\frac{j}{N}-\bar m\frac{\bar j}{N}\right)>\gamma(l+2),$$
$$\omega-2\left(m\frac{j}{N}+\bar m\frac{\bar j}{N}\right)+\lambda-2\left(m\frac{j}{N}-\bar m\frac{\bar j}{N}\right)>\gamma(l+2). \tag{6.21}$$

Here $(j,\bar j)$ are the angular momenta of the geometry into which the quantum is falling

$$\frac{j}{N}=-\left(n+\frac{\gamma}{2}\right),\qquad \frac{\bar j}{N}=-\frac{\gamma}{2}. \tag{6.22}$$

Note that $\frac{\omega\pm\lambda}{2}$ are the increments of $L_0, \bar L_0$ caused by the incoming quantum, so we can write (6.21) as

$$\Delta h>\gamma+\left(\frac{l}{2}-m\right)\gamma-2mn, \tag{6.23}$$

$$\Delta\bar h>\gamma+\left(\frac{l}{2}-\bar m\right)\gamma. \tag{6.24}$$

6.2.1. *Absorption of quanta in the CFT description*

In [24–26] the CFT description of absorption was studied. We have supposed that the quantum being absorbed is a scalar arising from the component h_{ij} of the metric on the T^4. The quantum has angular momentum l, which means that it is in the representation $(\frac{l}{2},\frac{l}{2})$ of $su(2)_L\times su(2)_R$. In the CFT state this quantum creates an excitation that has

$$\partial X,\psi\cdots\psi,\qquad \bar\partial X,\bar\psi\cdots\bar\psi, \tag{6.25}$$

where the X variables carry the indices i, j (we must symmetrize over the two permutations) and there are l fermions on each of the left and right sectors. The fermions are in the Ramond sector, and thus both fermionic and bosonic excitations come in multiples of the basic harmonic on the component strings

$$\Delta h = \Delta\bar{h} = \frac{1}{k} = \gamma. \tag{6.26}$$

The $(J^3, \bar{J}^3)$ quantum numbers in $su(2)_L \times su(2)_R$ are $(m, \bar{m})$. There are two species of left moving fermions with $m = \frac{1}{2}$ and two with $m = -\frac{1}{2}$; similarly for the right movers. We will simply write $\psi^{\pm}$, $\bar{\psi}^{\pm}$ without distinguishing the two species since the difference is irrelevant for the discussion below. Thus on the left sector we must have $\frac{l}{2} + m$ fermions ψ^+ and $\frac{l}{2} - m$ fermions ψ^-. On the right sector we have $\frac{l}{2} + \bar{m}$ fermions $\bar{\psi}^+$ and $\frac{l}{2} - \bar{m}$ fermions $\bar{\psi}^-$.

6.2.2. *Identifying the excitations*

Recall the structure of the CFT state (2.33) into which we are absorbing the quantum. On each component string we have fermion zero modes. For the left movers we started by choosing the state which is killed by the ψ_0^-, and then applied operators J^- which resulted in the application of modes $\psi^-_{-\frac{1}{k}} \cdots \psi^-_{-n}$ (for both species of ψ^-). For the right movers we also have the vacuum killed by the $\bar{\psi}_0^-$, but applied no other excitations.

Consider first the right movers. The excitation $\bar{\partial} X$ needs a minimum $\Delta\bar{h}$ of $\frac{1}{k} = \gamma$; this is the first term on the RHS of (6.24). Now consider the fermions. Suppose that $\bar{m} = \frac{l}{2}$. Then we have l operators $\bar{\psi}^+$ acting on the state of the CFT. But we can choose each of these operators to be a zero mode $\bar{\psi}_0^+$ which changes the vacuum on a component string to one that is killed by $\bar{\psi}_0^+$. The fermions then do not contribute to $\Delta\bar{h}$ and we find exact agreement with (6.24). Note that we could find at most two such zero modes on any given component string, so we will have to apply the $\bar{\psi}_0^+$ in general to many different component strings.[6]

Now suppose that $\bar{m} = \frac{l}{2} - 1$. We have $l - 1$ operators $\bar{\psi}^+$ and one $\bar{\psi}^-$. The $\bar{\psi}^+$ again give no contribution to $\Delta\bar{h}$, but the lowest allowed mode for $\bar{\psi}^-$ is $\bar{\psi}^-_{-\frac{1}{k}}$ and we again get agreement with (6.24). Proceeding this way, we find agreement for all $\bar{m}$ for the right movers.

Now consider the left movers, and let $m = \frac{l}{2}$. We have to apply l operators ψ^+. The lowest excitation results if we use the operators ψ^+ to *annihilate* l modes ψ^-_{-n}, which gives a total $\Delta h = \frac{1}{k} - nl$ (including the boson ∂X) which agrees with (6.23). Next consider $m = \frac{l}{2} - 1$. Now we have $l - 1$ operators ψ^+ which again annihilate modes ψ^-_{-n} and one operator ψ^- which creates the lowest allowed mode $\psi^-_{-(n+\frac{1}{k})}$, again in exact agree-

[6] We can read off the value of $\gamma = \frac{1}{k}$ from the classical geometry, for example by the conical defect angles of the metrics (2.46). But n_1, n_5 are infinite in the classical limit, and so $m = \frac{n_1 n_5}{k}$ is also infinite in this limit. Thus the absorption thresholds computed from the classical geometry have an essentially infinite number of component strings, and we will not 'run out of component strings' for any value of l that we choose.

ment with (6.23). (The operator ψ^- cannot just fill in one of the empty levels created by annihilation of modes ψ^-_{-n} since the overall operator describing the absorption is taken to be normal ordered.)

We thus see that the absorption threshold seen in the wave equation analysis can be understood in detail in terms of the occupied levels on the effective strings. This computation supports the conjecture that the state after duality has the form (6.18) analogous to the states before the $ST_yT_{z_1}S$ duality; the absorption computation applies equally to both sets of geometries.

6.3. *A puzzle about the orbifold theory*

Consider the CFT state (6.18) dual to the geometries obtained after $ST_yT_{z_1}S$ duality. The momentum T' per component string (given in (6.14)) is *fractional* in general, not an integer. Interestingly, when this momentum is measured in units of the basic excitation $\frac{1}{k}$ on the component string then we get an *integer* $\hat{T}'$. (If $\hat{T}'$ were nonintegral, we would have a severe contradiction, since we could not carry the excitation on the component string.)

But here we face a puzzle, since in the orbifold CFT the quantity T' is *also* required to be integral. The reason for this is as follows. We have to orbifold by the symmetric group $S_{N'}$ which permutes the N' copies of the $c=6$ CFT. In a given state of the CFT we can label the copies by how they make up the different component strings, and even inside each component string the copies can be ordered by the sequence in which they link up to make the 'long cycle'. But one part of the symmetry group still survives: we can cyclically permute the copies inside a component string

$$c_1 \to c_2 \to \cdots \to c_k \to c_1. \tag{6.27}$$

This symmetry forces the momenta on the component string to be integral.

Faced with this problem, we first review the steps that led us to our solutions. The fundamental string (F) is an elementary excitation of the theory on $M_{4,1}\times S^1\times T^4$, so the 2-charge FP solutions we started with certainly correspond to BPS states of the full string theory. Since S, T dualities are exact symmetries of the theory the 2-charge D1–D5 states are also states of the theory. Spectral flow was just a coordinate change, and so the 3-charge solutions that were obtained by spectral flow must also be valid. Finally, the exact symmetry $ST_yT_{z_1}S$ was used to get the solutions which we are now discussing, so we conclude that they must be allowed states of the string theory.

Even though one may accept the gravity solutions one may question the identification of the CFT state. We have made each component string have the same winding number k', but if we allowed the winding numbers to be different then we could have carried the momentum $n'_p=n_1$ on the component strings in such a way that each component string had an integral number T' of units of momentum. But let us recall the evidence we have that the component strings should all be equal: (a) We have learn from the explicit construction of 2-charge systems that states with axial symmetry have all component strings identical; if the component strings are different then all symmetries are broken in general. (b) The return time from the throat gave a precise value that could be put in correspondence with the length of the component strings; if component strings had different lengths then we would get distortion of the wavepacket since different parts would be returned after different times. (c) The threshold of absorption worked out exactly—if we change the structure of the filled levels on the component strings then the allowed levels that could be excited by the incoming quantum would change. For all these reasons it seems hard to have any other construction of the CFT state.

It may be that we need changes in our understanding of the CFT dual to the gravity theory. It is not completely clear where the orbifold point sits in the moduli space, though there are some leading candidates [27,28]. It is also not clear if the orbifold $(T^4)^N/S_N$ does describe some point in the moduli space, or if we need to consider other related orbifold theories like the iterated orbifolds [29].

Before concluding we note a point about the winding number of the component strings. The geometries (2.46) had an integral k, since this number can be traced back to the number of turns of the helix of the F string in the starting FP solution that led to (2.46). For the geometries obtained after $ST_yT_{z_1}S$ duality we still found in Eq. (6.9) that the winding number of each component string k' was integral. This was important, since if k' turned out fractional we could make no sense of the CFT state. But if we assume that the quantities T do not need to be integral and only the $\hat{T}$ need to be integral in general, then after $ST_yT_{z_1}S$ duality we would get fractional k' in general. What then are the rules for the allowed CFT states?

For the D1–D5 CFT we can have m component strings with equal winding if

$$\frac{n_1 n_5}{m} \in \mathbb{Z}. \tag{6.28}$$

Given that m was found to be a duality invariant, and that n_1, n_5, n_p permute under duality, we conjecture that the state can have m equal length component strings if all the following conditions are true

$$\frac{n_1 n_5}{m} \in \mathbb{Z}, \qquad \frac{n_5 n_p}{m} \in \mathbb{Z}, \qquad \frac{n_p n_1}{m} \in \mathbb{Z}, \qquad \hat{T} = \frac{n_1 n_5 n_p}{m^2} \in \mathbb{Z}, \tag{6.29}$$

where we have included the requirement that $\hat{T}$ be integral. We hope to return to these issues elsewhere.

7. Discussion

Our basic conjecture states that the black hole interior is not ‘empty space with a central singularity’ but a ‘fuzzball’ of horizon size. If we consider extremal holes, and look at states where in the dual CFT we have many component strings in the same state, then we can have a good description of the geometry in classical supergravity. All 2-charge states could be approached through such classical geometries, and in [5,6] some classes of 3-charge extremal states were considered and their dual geometries identified; the geometries were smoothly capped as in Fig. 1(b). In the present paper we have looked at two families of states and their dual geometries. The geometries were again found to be ‘capped’; there were no horizons or closed timelike curves.

The first family arose from spectral flow of a subfamily of 2-charge states. These 2-charge states generically had an orbifold singularity along a curve; this was understood as a ‘trivial’ singularity in the sense that it arose from the coincidence of two or more

'KK monopoles tubes' and was thus arose only as a limit in a family of regular solutions. Since spectral flow is given by a coordinate transformation in $AdS_3 \times S^3$ [11,12] it is not surprising that a similar orbifold singularity arose also for the 3-charge states obtained by spectral flow of the 2-charge states. What was interesting to note was that the conical defect angle did not get corrected when the asymptotically AdS solution is modified to become an asymptotically flat solution. If the conical defect parameter had changed away from the form $\frac{1}{k}$ to say an irrational value then we would not be able to understand the orbifold singularity in any simple way as a limit of nonsingular solutions.

The second family we considered was found by applying S, T dualities to the first family of geometries so that the D1 and P charges got interchanged. We identified the CFT states dual to this second family of geometries by computing the time of travel Δt_{SUGRA} for a quantum to fall down the throat and bounce back out. The winding number of the 'component strings' in the CFT, computed by this method, gave a minimum threshold energy for excitations of the CFT state. This minimum energy agreed with the threshold of energy below which the incident quantum was unable to enter the throat of the geometry, thus confirming the identification of the CFT state. We noted that the CFT states found this way had a fractional momentum on each 'component string' (the total momentum was of course an integer). This suggested that we need to go beyond the simple orbifold CFT to understand all 3-charge bound states; we may need to understand deformations away from the orbifold point [30] or perhaps we may have to consider iterated orbifolds [29].

The conical defect in the 2-charge D1–D5 geometries could be directly linked to the winding number k of each component string. For the metrics obtained by spectral flow from 2-charge geometries we have a similar relation since spectral flow is just a coordinate transformation on the geometry [11,12]. The conical defect is caused by an orbifold singularity of order k along the S^1 given by $(r = 0, \theta = \pi/2)$ and the component strings in the dual CFT state have winding number k. For the geometries obtained after S, T dualities the situation is more complex—we have orbifold singularities along two different S^1 curves, with order $kn + 1$ at $(r = 0, \theta = 0)$ and order mn at $(r = 0, \theta = \pi/2)$ (m is the highest common factor shared by n_5, k). There is a suggestive pattern though that we observe. It was argued in [30] that the D1–D5 CFT with charges (n_1, n_5) could be mapped to the theory with charges $(n_1 n_5, 1)$, and that the orbifold point occurs at this latter value of the charges. If we set $n_5 = 1$ to enable a comparison to the orbifold theory, then we observe that the product of the orders of the two orbifold groups is $(kn + 1)n$ which in this case equals k', the winding number of the effective string in the dual CFT state.

We have noted that the travel time is symmetric between the three charges of the solution, but the factor η enforcing this symmetry is responsible for providing a 'redshift' which relates the time at infinity to the time in the AdS region. The AdS region is the one that is dual to the CFT description, so understanding such effects in more detail may help us to identify the deformations in the CFT that correspond to deforming AdS space to flat space at infinity.

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Appendix A. Non-extremal rotating 3-charge metrics and their extremal limits

The metric for general rotating 3-charge solutions was given in [13], and the metric, 2-form field and dilaton were given in [6] by generating the solution by a different technique. We have

$$
\begin{aligned}
ds^2 = {} & -\left(1-\frac{M\cosh^2\delta_p}{f}\right)\frac{dt^2}{\sqrt{H_1H_5}} + \left(1+\frac{M\sinh^2\delta_p}{f}\right)\frac{dy^2}{\sqrt{H_1H_5}} \\
& - \frac{M\sinh 2\delta_p}{f\sqrt{H_1H_5}}\,dt\,dy + f\sqrt{H_1H_5}\left(\frac{r^2\,dr^2}{(r^2+a_1^2)(r^2+a_2^2)-Mr^2}+d\theta^2\right) \\
& + \left[(r^2+a_1^2)\sqrt{H_1H_5} + \frac{(a_2^2-a_1^2)K_1K_5\cos^2\theta}{\sqrt{H_1H_5}}\right]\cos^2\theta\,d\psi^2 \\
& + \left[(r^2+a_2^2)\sqrt{H_1H_5} + \frac{(a_1^2-a_2^2)K_1K_5\sin^2\theta}{\sqrt{H_1H_5}}\right]\sin^2\theta\,d\phi^2 \\
& + \frac{M}{f\sqrt{H_1H_5}}\left(a_1\cos^2\theta\,d\psi + a_2\sin^2\theta\,d\phi\right)^2 \\
& + \frac{2M\cos^2\theta}{f\sqrt{H_1H_5}}\big[(a_1\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_2\sinh\delta_1\sinh\delta_5\sinh\delta_p)\,dt \\
& + (a_2\sinh\delta_1\sinh\delta_5\cosh\delta_p - a_1\cosh\delta_1\cosh\delta_5\sinh\delta_p)\,dy\,d\psi\big] \\
& + \frac{2M\sin^2\theta}{f\sqrt{H_1H_5}}\big[(a_2\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_1\sinh\delta_1\sinh\delta_5\sinh\delta_p)\,dt \\
& + (a_1\sinh\delta_1\sinh\delta_5\cosh\delta_p - a_2\cosh\delta_1\cosh\delta_5\sinh\delta_p)\,dy\big]\,d\phi \\
& + +\sqrt{\frac{H_1}{H_5}}\sum_{i=1}^{4}dx_i^2, \\
C_2 = {} & \frac{M\cos^2\theta}{fH_1}\big[(a_2\cosh\delta_1\sinh\delta_5\cosh\delta_p - a_1\sinh\delta_1\cosh\delta_5\sinh\delta_p)\,dt \\
& + (a_1\sinh\delta_1\cosh\delta_5\cosh\delta_p - a_2\cosh\delta_1\sinh\delta_5\sinh\delta_p)\,dy\big]\wedge d\psi \\
& + \frac{M\sin^2\theta}{fH_1}\big[(a_1\cosh\delta_1\sinh\delta_5\cosh\delta_p - a_2\sinh\delta_1\cosh\delta_5\sinh\delta_p)\,dt \\
& + (a_2\sinh\delta_1\cosh\delta_5\cosh\delta_p - a_1\cosh\delta_1\sinh\delta_5\sinh\delta_p)\,dy\big]\wedge d\phi \\
& - \frac{M\sinh 2\delta_1}{2fH_1}\,dt\wedge dy
\end{aligned}
$$

$$- \frac{M \sinh 2\delta_5}{2 f H_1} \big(r^2 + a_2^2 + M \sinh^2 \delta_1\big) \cos^2\theta \, d\psi \wedge d\phi,$$

$$e^{2\Phi} = \frac{H_1}{H_5}. \tag{A.1}$$

Here

$$f = r^2 + a_1^2 \sin^2\theta + a_2^2 \cos^2\theta,$$

$$H_i \equiv 1 + K_i = 1 + \frac{M \sinh^2 \delta_i}{f}, \quad i = 1, 5. \tag{A.2}$$

In terms of the parameters appearing in the metric, the D1, D5 and momentum charges and the two angular momenta are

$$Q_1 = \frac{M}{2} \sinh 2\delta_1, \qquad Q_5 = \frac{M}{2} \sinh 2\delta_p, \qquad Q_p = \frac{M}{2} \sinh 2\delta_p, \tag{A.3}$$

$$J_\psi = -M (a_1 \cosh\delta_1 \cosh\delta_5 \cosh\delta_p - a_2 \sinh\delta_1 \sinh\delta_5 \sinh\delta_p) \frac{\pi}{4G^{(5)}}$$

$$= \tilde\gamma_1 \frac{\pi \sqrt{Q_1 Q_5}}{4G^{(5)}},$$

$$J_\phi = -M (a_2 \cosh\delta_1 \cosh\delta_5 \cosh\delta_p - a_1 \sinh\delta_1 \sinh\delta_5 \sinh\delta_p) \frac{\pi}{4G^{(5)}}$$

$$= \tilde\gamma_2 \frac{\pi \sqrt{Q_1 Q_5}}{4G^{(5)}} \tag{A.4}$$

with $G^{(5)}$ the 5D Newton's constant. The extremal limit is the limit in which $M \to 0$ while Q_1, Q_5, Q_p, $\tilde\gamma_1$, $\tilde\gamma_2$ are kept finite. We will give the extremal metric for generic values of the charges and of the angular momenta satisfying

$$(\tilde\gamma_1 - \tilde\gamma_2)^2 - 4Q_p \geqslant 0. \tag{A.5}$$

In this case we find it convenient to parametrize Q_p as

$$Q_p = \left(\frac{\tilde\gamma_1 - \tilde\gamma_2}{2}\right)^2 - \left(\frac{\tilde\gamma_1 + \tilde\gamma_2}{2\mu}\right)^2, \quad \mu > 0. \tag{A.6}$$

The extremal metric, Ramond field and dilaton are

$$ds^2 = -\frac{1}{h}\big(dt^2 - dy^2\big) + \frac{Q_p}{hf}(dt - dy)^2 + hf\left(\frac{dr^2}{r^2 + \mu^{-1}(\tilde\gamma_1 + \tilde\gamma_2)^2 \eta} + d\theta^2\right)$$

$$+ h\left(r^2 + (\tilde\gamma_1 + \tilde\gamma_2)\eta \frac{(1+\mu)\tilde\gamma_1 + (1-\mu)\tilde\gamma_2}{2\mu}\right.$$

$$\left. - \frac{(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta Q_1 Q_5 \cos^2\theta}{h^2 f^2}\right) \cos^2\theta \, d\psi^2$$

$$+ h\left(r^2 + (\tilde\gamma_1 + \tilde\gamma_2)\eta \frac{(1-\mu)\tilde\gamma_1 + (1+\mu)\tilde\gamma_2}{2\mu}\right.$$

$$\left. + \frac{(\tilde\gamma_1^2 - \tilde\gamma_2^2)\eta Q_1 Q_5 \sin^2\theta}{h^2 f^2}\right) \sin^2\theta \, d\phi^2$$

$$+\frac{Q_p(\tilde\gamma_1+\tilde\gamma_2)^2\eta^2}{hf}\big(\cos^2\theta\,d\psi+\sin^2\theta\,d\phi\big)^2$$
$$-\frac{2\sqrt{Q_1Q_5}}{hf}\big[\tilde\gamma_1\cos^2\theta\,d\psi+\tilde\gamma_2\sin^2\theta\,d\phi\big](dt-dy)$$
$$-\frac{2(\tilde\gamma_1+\tilde\gamma_2)\eta\sqrt{Q_1Q_5}}{hf}\big[\cos^2\theta\,d\psi+\sin^2\theta\,d\phi\big]dy+\sqrt{\frac{H_1}{H_5}}\sum_{i=1}^4 dx_i^2, \tag{A.7}$$

$$C_2=-\frac{\sqrt{Q_1Q_5}\cos^2\theta}{H_1f}(\tilde\gamma_2\,dt+\tilde\gamma_1\,dy)\wedge d\psi$$
$$-\frac{\sqrt{Q_1Q_5}\sin^2\theta}{H_1f}(\tilde\gamma_1\,dt+\tilde\gamma_2\,dy)\wedge d\phi$$
$$+\frac{(\tilde\gamma_1+\tilde\gamma_2)\eta Q_p}{\sqrt{Q_1Q_5}H_1f}(Q_1\,dt+Q_5\,dy)\wedge\big(\cos^2\theta\,d\psi+\sin^2\theta\,d\phi\big)$$
$$-\frac{Q_1}{H_1f}dt\wedge dy$$
$$-\frac{Q_5\cos^2\theta}{H_1f}\left(r^2+(\tilde\gamma_1+\tilde\gamma_2)\eta\frac{(1-\mu)\tilde\gamma_1+(1+\mu)\tilde\gamma_2}{2\mu}+Q_1\right)d\psi\wedge d\phi, \tag{A.8}$$

$$e^{2\Phi}=\frac{H_1}{H_5}, \tag{A.9}$$
$$f=r^2+(\tilde\gamma_1+\tilde\gamma_2)\eta\left[\frac{(1+\mu)\tilde\gamma_1+(1-\mu)\tilde\gamma_2}{2\mu}\sin^2\theta\right.$$
$$\left.+\frac{(1-\mu)\tilde\gamma_1+(1+\mu)\tilde\gamma_2}{2\mu}\cos^2\theta\right],$$
$$H_1=1+\frac{Q_1}{f},\qquad H_5=1+\frac{Q_5}{f},\qquad h=\sqrt{H_1H_5}. \tag{A.10}$$

The metric (A.7) with $\mu\neq1$ can be obtained from the metric (2.46), which corresponds to the case $\mu=1$, via a boost in the y direction:

$$t\to t\cosh\delta+y\sinh\delta,\qquad y\to y\cosh\delta+t\sinh\delta \tag{A.11}$$

with

$$e^{2\delta}=\frac{\mu-1}{\eta}+1. \tag{A.12}$$

Appendix B. Singularities, closed timelike curves and horizons

The determinant of the metric (A.7)

$$\sqrt{-g}=hf\sin\theta\cos\theta r \tag{B.1}$$

only vanishes at $r=0$ and $\theta=0,\pi/2$. Note that $\theta=0,\pi/2$ are the points where spherical coordinates degenerate, so singularities of (A.7) can only occur at $r=0$. In a neighborhood of $r=0$ y can be decoupled from all the other coordinates by the coordinate change

$$\begin{aligned}\tilde{t} &= t - c_t y,\\ \tilde{\psi} &= \psi - c_\psi y,\\ \tilde{\phi} &= \phi - c_\phi y\end{aligned} \tag{B.2}$$

with

$$c_t = -\frac{(\mu-1)[\mu^2(\tilde{\gamma}_1-\tilde{\gamma}_2)^2 - (\tilde{\gamma}_1+\tilde{\gamma}_2)^2 + 2\mu^2\sqrt{Q_1Q_5}]}{(\mu-1)[\tilde{\gamma}_1(1+\mu)+\tilde{\gamma}_2(1-\mu)][\tilde{\gamma}_1(1-\mu)+\tilde{\gamma}_2(1+\mu)] + 2\mu^2(1+\mu)\sqrt{Q_1Q_5}},$$

$$c_\psi = 2\frac{\mu^2[\tilde{\gamma}_1(1-\mu)+\tilde{\gamma}_2(1+\mu)]}{(\mu-1)[\tilde{\gamma}_1(1+\mu)+\tilde{\gamma}_2(1-\mu)][\tilde{\gamma}_1(1-\mu)+\tilde{\gamma}_2(1+\mu)] + 2\mu^2(1+\mu)\sqrt{Q_1Q_5}},$$

$$c_\phi = 2\frac{\mu^2[\tilde{\gamma}_1(1+\mu)+\tilde{\gamma}_2(1-\mu)]}{(\mu-1)[\tilde{\gamma}_1(1+\mu)+\tilde{\gamma}_2(1-\mu)][\tilde{\gamma}_1(1-\mu)+\tilde{\gamma}_2(1+\mu)] + 2\mu^2(1+\mu)\sqrt{Q_1Q_5}}.$$

In the new coordinates, the part of the metric that involves y and r is

$$ds^2_{r-y} = \frac{hf}{(\tilde{\gamma}_1+\tilde{\gamma}_2)^2\eta}\left(dr^2 + c_y^2 r^2\,dy^2\right) + O\left(r^4\right) \tag{B.3}$$

with

$$c_y = \frac{4(\tilde{\gamma}_1+\tilde{\gamma}_2)\mu^2}{(\mu-1)[\tilde{\gamma}_1(1+\mu)+\tilde{\gamma}_2(1-\mu)][\tilde{\gamma}_1(1-\mu)+\tilde{\gamma}_2(1+\mu)] + 2\mu^2(1+\mu)\sqrt{Q_1Q_5}}. \tag{B.4}$$

From the expressions above we see that, unless $\mu=1$, the transformation (B.3) involves a shift of t by the periodic coordinate y. At $r=0$ the y circle shrinks. Thus, as explained in [6], metrics with $\mu\neq 1$ have closed timelike curves.

The geometries (2.46) dual to CFT states have $\mu=1$. (The condition $\mu=1$ is seen to give, using (A.6), the relation $Q_p=\tilde{\gamma}_1\tilde{\gamma}_2$.) For these metrics one can show that there are no closed timelike curves by looking at the determinant $\tilde{g}$ of the metric restricted to the three periodic coordinates y, ψ and ϕ:

$$\tilde{g} = \frac{r^2\sin^2\theta\cos^2\theta}{\sqrt{(Q_1+f)(Q_5+f)}}\left[\left(r^2+(\tilde{\gamma}_1+\tilde{\gamma}_2)^2\eta\right)(f+Q_1+Q_5+Q_p) + \frac{Q_1Q_5}{\eta}\right]. \tag{B.5}$$

Using an argument given in [6], it is enough to show that the determinant above never vanishes to prove that the metric is free of closed timelike curves. By explicit inspection, we know that the zeros of $\tilde{g}$ at $r=0$ or $\theta=0,\pi/2$ do not signal the presence of closed timelike curves. So we need to show that $\tilde{g}$ has no other zeros. This follows from the fact that $f+Q_i>0$ for $i=1,5,p$. In order to prove this last statement, consider first the geometries obtained by spectral flow, which have

$$\tilde{\gamma}_1 = -an, \qquad \tilde{\gamma}_2 = a(n+\gamma), \qquad Q_p = a^2 n(n+\gamma). \tag{B.6}$$

Since $n\in\mathbb{Z}$ and $\gamma<1$ we have $Q_p\geqslant 0$ and thus

$$\eta \equiv \frac{Q_1Q_5}{Q_1Q_5+(Q_1+Q_5)Q_p} \leqslant \frac{Q_1}{Q_p} \tag{B.7}$$

and

$$\eta \leqslant 1. \tag{B.8}$$

Let us look at $f + Q_1$ and distinguish the two cases $n > 0$ and $n < 0$. (It will be very important here that n is an integer and that $0 < \gamma < 1$.) If $n > 0$

$$f + Q_1 \geqslant Q_1 - a^2 n\eta\gamma \geqslant Q_1\left(1 - \frac{a^2 n\gamma}{Q_p}\right) = Q_1\left(1 - \frac{\gamma}{n+\gamma}\right) > 0. \tag{B.9}$$

If $n < 0$ (and thus $n + \gamma < 0$)

$$f + Q_1 \geqslant Q_1 + a^2(n+\gamma)\eta\gamma \geqslant Q_1\left(1 + \frac{a^2(n+\gamma)\gamma}{Q_p}\right) = Q_1\left(1 + \frac{\gamma}{n}\right) > 0. \tag{B.10}$$

In both cases we used (B.7). The symmetry of the metric under interchange of Q_1 and Q_5 then also implies $f + Q_5 > 0$. Look now at $f + Q_p$. If $n > 0$

$$f + Q_p \geqslant Q_p - a^2 n\eta\gamma = a^2 n\big(n + (1-\eta)\gamma\big) > 0 \tag{B.11}$$

as $1 - \eta > 0$. If $n < 0$ (and thus $n + \gamma < 0$)

$$f + Q_p \geqslant Q_p + a^2(n+\gamma)\eta\gamma = a^2(n+\gamma)(n+\eta\gamma) > 0 \tag{B.12}$$

as $n + \eta\gamma \leqslant n + \gamma < 0$. In order to prove that the same results hold for the metrics after S, T dualities it is enough to notice that f is duality invariant, since the transformation of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ (3.4) cancel that of η:

$$\eta' = \frac{Q_p Q_5}{Q_1 Q_5 + (Q_1 + Q_5)Q_p} = \frac{Q_p}{Q_1}\eta. \tag{B.13}$$

One can also verify that the metrics (2.46) do not have any horizon. For this purpose we again refer to an argument given in [6]: there is no horizon if one can find a timelike vector in the forward light cone which has a nonzero positive component along the radial direction. In turn, the existence of such a vector follows from the fact that the determinant of the metric restricted to the t, y, ψ, ϕ coordinates

$$\hat{g} = -r^2\big(r^2 + \eta(\tilde{\gamma}_1 + \tilde{\gamma}_2)^2\big)\sin^2\theta\cos^2\theta \tag{B.14}$$

is always negative (apart from the points $r = 0$ and $\theta = 0, \pi/2$, where we know there is no horizon by direct analysis[7]).

The form of the metric around $r = 0$ given in (B.3) shows that the metrics (A.7) generically have orbifold singularities. If R is the radius of the y circle, the conical defect parameter is given by

$$\gamma = |c_y|R. \tag{B.15}$$

[7] The naive geometry with no rotation [31] has $\tilde{\gamma}_1 + \tilde{\gamma}_2 = 0$, and this case has a horizon at $r = 0$; for the geometries we constructed as duals of actual microstates $\tilde{\gamma}_1 + \tilde{\gamma}_2 \neq 0$ and the analysis around $r = 0$ done in Section 4.2 shows that there is no horizon at $r = 0$.

Appendix C. Inverse metric

The determinant of the extremal metric (2.46) is

$$\sqrt{-g} = hf \sin\theta \cos\theta r. \tag{C.1}$$

The inverse of the metric (2.46) is

$$\begin{aligned}
g^{tt} &= -\frac{1}{hf}\left(f + Q_1 + Q_5 + Q_p + \frac{Q_1Q_5 + Q_1Q_p + Q_5Q_p}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\right),\\
g^{yy} &= \frac{1}{hf}\left(f + Q_1 + Q_5 - Q_p + \frac{Q_1Q_5\eta}{r^2} - \frac{Q_p^2\eta}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\frac{(Q_1+Q_5)^2}{Q_1Q_5}\right),\\
g^{ty} &= -\frac{Q_p}{hf}\left(1 + \frac{Q_1 + Q_5}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\right),\\
g^{\psi\psi} &= \frac{1}{hf}\left(\frac{1}{\cos^2\theta} + \frac{\tilde\gamma_2^2\eta}{r^2} - \frac{\tilde\gamma_1^2\eta}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\right),\\
g^{\phi\phi} &= \frac{1}{hf}\left(\frac{1}{\sin^2\theta} + \frac{\tilde\gamma_1^2\eta}{r^2} - \frac{\tilde\gamma_2^2\eta}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\right),\\
g^{\psi\phi} &= -\frac{Q_p\eta}{hf}\left(\frac{1}{r^2} - \frac{1}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\right),\\
g^{t\psi} &= -\frac{\sqrt{Q_1Q_5}}{hf}\frac{\tilde\gamma_1}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta},\\
g^{t\phi} &= -\frac{\sqrt{Q_1Q_5}}{hf}\frac{\tilde\gamma_2}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta},\\
g^{y\psi} &= \frac{\sqrt{Q_1Q_5}\tilde\gamma_2\eta}{hf}\left(\frac{1}{r^2} + \frac{\tilde\gamma_1^2}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\frac{Q_1+Q_5}{Q_1Q_5}\right),\\
g^{y\phi} &= \frac{\sqrt{Q_1Q_5}\tilde\gamma_1\eta}{hf}\left(\frac{1}{r^2} + \frac{\tilde\gamma_2^2}{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}\frac{Q_1+Q_5}{Q_1Q_5}\right),\\
g^{rr} &= \frac{r^2 + (\tilde\gamma_1 + \tilde\gamma_2)^2\eta}{hf}, \qquad g^{\theta\theta} = \frac{1}{hf}, \qquad g^{x_ix_j} = \sqrt{\frac{H_5}{H_1}}\delta^{ij}.
\end{aligned} \tag{C.2}$$

Appendix D. Solution of the wave equation

D.1. Matching region

In the matching region $1 \ll x \ll \sigma^2$ the wave equation becomes

$$4\frac{d}{dx}\left(x^2\frac{d}{dx}\right)H_{\text{match}} + (1-\nu^2)H_{\text{match}} = 0. \tag{D.1}$$

A basis of independent solutions is

$$H^{(1)}_{\text{match}} = x^{\frac{\nu-1}{2}}, \qquad H^{(2)}_{\text{match}} = x^{-\frac{\nu+1}{2}}. \tag{D.2}$$

D.2. Outer region

In the outer region $x \gg 1$ the radial equation is

$$4\frac{d}{dx}\left(x^2\frac{dH_{\rm out}}{dx}\right) + \left[\sigma^{-2}x + 1 - \nu^2\right]H_{\rm out} = 0 \tag{D.3}$$

with solution

$$H_{\rm out} = \frac{1}{\sqrt{x}}\left[C_1 J_\nu\left(\sigma^{-1}\sqrt{x}\right) + C_2 J_{-\nu}\left(\sigma^{-1}\sqrt{x}\right)\right]. \tag{D.4}$$

In the matching region this gives

$$H_{\rm out} = C_1\frac{1}{\sqrt{x}}\left(\frac{\sigma^{-2}x}{4}\right)^{\frac{\nu}{r2}}\frac{1}{\Gamma(\nu+1)} + C_2\frac{1}{\sqrt{x}}\left(\frac{\sigma^{-2}x}{4}\right)^{-\frac{\nu}{2}}\frac{1}{\Gamma(-\nu+1)}. \tag{D.5}$$

D.3. Inner region

In the inner region, $\sigma^{-2}x \ll 1$, the wave equation simplifies to

$$4\frac{d}{dx}\left(x\left(x+\delta^2\right)\frac{d}{dx}\right)H_{\rm in} + \left[1 - \nu^2 + \frac{\xi^2}{x+\delta^2} - \frac{\zeta^2}{x}\right]H_{\rm in} = 0. \tag{D.6}$$

By defining

$$H_{\rm in} = x^\alpha\left(x+\delta^2\right)^\beta F \tag{D.7}$$

with

$$\alpha = \frac{|\zeta|}{2\delta}, \qquad \beta = \frac{\xi}{2\delta} \tag{D.8}$$

the equation above reduces to an hypergeometric equation for F. Of the two independent solutions, only one is regular at $x = 0$:

$$H_{\rm in} = x^\alpha\left(x+\delta^2\right)^\beta F\left(p, q; 1+2\alpha; -\frac{x}{\delta^2}\right), \tag{D.9}$$

where

$$p = \frac{1}{2} + \alpha + \beta + \frac{\nu}{2}, \qquad q = \frac{1}{2} + \alpha + \beta - \frac{\nu}{2}. \tag{D.10}$$

To get the large x behavior we write

$$\begin{aligned} &F\left(p, q; 1+2\alpha, -\frac{x}{\delta^2}\right) \\ &\quad = \frac{\Gamma(1+2\alpha)\Gamma(-\nu)}{\Gamma(q)\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu}{2})}\left(\frac{x}{\delta^2}\right)^{-p} F\left(p, p-2\alpha; \nu+1; -\frac{\delta^2}{x}\right) \\ &\qquad + \frac{\Gamma(1+2\alpha)\Gamma(\nu)}{\Gamma(p)\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu}{2})}\left(\frac{x}{\delta^2}\right)^{-q} F\left(q, q-2\alpha; -\nu+1; -\frac{\delta^2}{x}\right) \end{aligned} \tag{D.11}$$

and note that

$$F(p,q;\nu;z)=\sum_{n=0}^{\infty}\frac{\Gamma(p+n)\Gamma(q+n)}{\Gamma(p)\Gamma(q)}\frac{\Gamma(\nu)}{\Gamma(\nu+n)}\frac{z^n}{n!}. \tag{D.12}$$

One thus gets (dropping an overall constant)

$$H_{\text{in}}=\frac{1}{\sqrt{x}}\left[\frac{\Gamma(\nu)}{\Gamma(p)\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu}{2})}\left(\frac{x}{\delta^2}\right)^{\frac{\nu}{2}}\left(1+O\left(\delta^2x^{-1}\right)\right)\right.$$
$$\left.+\frac{\Gamma(-\nu)}{\Gamma(q)\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu}{2})}\left(\frac{x}{\delta^2}\right)^{-\frac{\nu}{2}}\left(1+O\left(\delta^2x^{-1}\right)\right)\right]. \tag{D.13}$$

D.4. Matching the solutions

Matching the coefficients of $x^{(-1\pm\nu)/2}$ between H_{out} and H_{in} we find the ratio C_2/C_1 given in (5.16).

Appendix E. Computation of travel time and absorption probability

In order to write the reflection amplitude $\mathcal{R}$ given in (5.16)–(5.18) as in (5.19) it is important to know the sign of the arguments

$$\alpha\pm\beta+\frac{\nu+1}{2} \tag{E.1}$$

of the gamma functions appearing in (5.16). As we chose $\beta\geqslant 0$, $\alpha+\beta+\frac{\nu+1}{2}$ is always positive and thus we are left with only two possibilities.

Case 1

$$\alpha-\beta+\frac{\nu+1}{2}>0. \tag{E.2}$$

In this case one can use the series expansion

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\sum_{k=0}^{\infty}\frac{(-1)^k(b-a)_kB(k,a-b+1,a)z^{-k}}{k!} \tag{E.3}$$

($(a)_k$ is the Pochhammer symbol and $B(k,a,b)$ the Bernoulli polynomial) which is valid for $z+a>0$, to show that $\mathcal{R}$ has no oscillating factor like $\exp(2\pi in\omega/R\Delta t)$, and thus there is no absorption.

Case 2

$$\alpha-\beta+\frac{\nu+1}{2}<0. \tag{E.4}$$

One can rewrite the ratio of gamma functions in (5.16) with negative arguments as

$$\frac{\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu}{2})}$$

$$
\begin{aligned}
&= \frac{\Gamma(\frac{1}{2}+\beta-\alpha+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\beta-\alpha-\frac{\nu}{2})} \frac{\sin(\pi(\alpha-\beta+\frac{1-\nu}{2}))}{\sin(\pi(\alpha-\beta+\frac{1+\nu}{2}))} \\
&= \frac{\Gamma(\frac{1}{2}+\beta-\alpha+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\beta-\alpha-\frac{\nu}{2})} \left[e^{-i\pi\nu} + \left(e^{-i\pi\nu} - e^{i\pi\nu}\right) \sum_{n=1}^{\infty} e^{2\pi i n(\beta-\alpha-\frac{1+\nu}{2})} \right] \qquad \text{(E.5)}
\end{aligned}
$$

so that $\mathcal{R}$ is brought into the form (5.19)

$$
\begin{aligned}
\mathcal{R} &= e^{-i\pi\nu} + \frac{4\pi^2}{\Gamma^2(\nu)\Gamma^2(\nu+1)} \frac{e^{-i\pi\nu}}{e^{2\pi i\nu}-1} \left(\frac{\delta}{2\sigma}\right)^{2\nu} \\
&\quad \times \frac{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha-\frac{\nu}{2})} \\
&\quad - \frac{4\pi^2 e^{-i\pi\nu}}{\Gamma^2(\nu)\Gamma^2(\nu+1)} \left(\frac{\delta}{2\sigma}\right)^{2\nu} \frac{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu}{2})\Gamma(\frac{1}{2}+\beta-\alpha-\frac{\nu}{2})} \\
&\quad \times \sum_{n=1}^{\infty} e^{2\pi i n(\beta-\alpha-\frac{1+\nu}{2})}. \qquad \text{(E.6)}
\end{aligned}
$$

From the equation above we can read off the probability of absorption/emission (5.34) and the time of travel (5.33).

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