Forum Math. 16 (2004), 539–552

Forum Mathematicum

Tilting modules over valuation domains

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(Communicated by Rüdiger Göbel)

Abstract. The structure of tilting modules over valuation domains R is investigated. It is proved that the S-divisible modules δ_S introduced by Fuchs-Salce are canonical generators for the tilting torsion classes over valuation domains, assuming $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ when the tilting generator has uncountable rank, where $\hat{R}$ is the pure-injective hull of R.

2000 Mathematics Subject Classification: 13C05, 13F30; 13D07.

1 Introduction

Looking at the wide literature existing on tilting modules since their appearance in the early 80's, one could expect that their structure is well understood in most cases. This is not true if one considers infinitely generated tilting modules. For instance, restricting to modules over commutative integral domains, the structure of infinitely generated tilting modules is known only for Dedekind domains satisfying certain cardinal conditions—including the ring of the integers—and assuming Gödel's Axiom of Constructibility ($V = L$) (see the papers by Göbel-Trlifaj [GT] for abelian groups, and by Trlifaj-Wallutis [TW]).

The goal of this paper is to study tilting modules over valuation domains. A crucial role is played by certain modules which can be viewed as prototypes of tilting modules over general commutative domains, namely, the modules δ_S. They have been introduced by Fuchs and the author in 1992 [FS1], as a generalization of the Fuchs' divisible module δ (see [FS2]). Facchini first noted the tilting character of the module δ and proved a remarkable result of Brenner-Butler type using it (see [Fa] and [Fa1]).

In Section 2, after reviewing some preliminary facts, we recall the definition and the main properties of the modules δ_S over an arbitrary domain R, where S denotes a multiplicative subset of R. Section 3 is devoted to prove that a countable rank torsionfree module M of projective dimension 1 over a valuation domain R satisfying the condition $\mathrm{Ext}^1_R(M, N^{(\omega)}) = 0$, where N is an arbitrary torsionfree R-module, is necessarily a free module over the localization of R at the prime ideal $N^{\#}$ canonically

Research supported by MIUR in the program PRIN 2002.

associated with N, provided that $N^\# \geq M^\#$. The same result is proved for torsionfree modules of arbitrary rank, assuming $V = L$ and that the pure-injective hull $\hat{R}$ of R has cardinality not exceeding the continuum. A consequence of these results is that, among the torsionfree R-modules, only the free ones are tilting (under the convenient set-theoretic hypotheses in the uncountable rank case). Even if the above results closely remind the analogous results for modules over Dedekind domains in [TW], their proofs involve quite different techniques typical of the valuation domain setting.

In Section 4, using the structure results for divisible modules of projective dimension 1 over a valuation domain R obtained by Fuchs (see [FS1, VII.3]), we show that such a module is tilting if and only if it is mixed, equivalently, if it is a generator of the class of the divisible modules. This result and the crucial fact that the torsion part $t(T)$ of a tilting R-module T is S-torsion, where $S = S(T)$ is a multiplicative subset of R canonically associated with T, enable us to prove our main result (see Theorem 4.11): every tilting R-module T generates the same class of modules as δ_S, for $S = S(T)$, namely, the class of S-divisible modules (under convenient set-theoretic hypotheses in the uncountable rank case). An example shows that this result is not true for Dedekind domains. A more satisfactory structure theorem is obtained for the tilting modules T such that $T/t(T)$ has projective dimension ≤ 1 (see Theorem 4.13).

2 Preliminaries

For general notions and facts on modules over commutative integral domains R we refer to [FS2]. Given any R-module M, there is a saturated multiplicative subset $S(M)$ of R naturally associated with M, namely

$$S(M) = \{s \in R \mid sM = M\}.$$

Clearly, $S(M)$ is the largest multiplicative subset S of R such that M is S-divisible; it is called the *divisibility set* of M.

If R is a valuation domain, since unions of prime ideals of R are still prime ideals, every saturated multiplicative subset S of R is the complement of a prime ideal L. We prefer to indicate in the usual way by R_L the localization of R at S. If M is an R-module, then the complement in R of its divisibility set $S(M)$ coincides with the well-known prime ideal associated with $M : M^\# = \{r \in R \mid rM < M\}$ (see [FS2, p. 338]). Given a module M, $\mathrm{Add}(M)$ denotes the class of the direct summands of direct sums of copies of M. Given a cardinal κ, M is said a κ-splitter if $\mathrm{Ext}^1_R(M, M^{(\kappa)}) = 0$. We follow Colpi-Trlifaj [CT] defining a tilting module (over an arbitrary ring R) as a module T satisfying the following three conditions:

T1) $\mathrm{p.d.}T \leq 1$;

T2) T is κ-splitter for every cardinal κ;

T3) there exists an exact sequence $0 \to R \to T_1 \to T_2 \to 0$, where $T_1, T_2 \in \mathrm{Add}(T)$.

(The up-to-date name of modules satisfying conditions T1–T3 is 1-tilting modules). We will use the following characterization of a tilting module T due to Colpi-Trlifaj

[CT]: $\mathrm{Gen}(T) = T^{\perp}$, where $\mathrm{Gen}(T)$ is the class of modules which are quotients of direct sums of copies of T, and $T^{\perp} = \{M \mid \mathrm{Ext}^1_R(T, M) = 0\}$. Another useful result proved in [CT] and used in the following states that, given two tilting modules T_1 and T_2, $T_1 \in \mathrm{Add}(T_2)$ if and only if $T_2 \in \mathrm{Add}(T_1)$, if and only if $\mathrm{Gen}(T_1) = \mathrm{Gen}(T_2)$; here, as usual, given a module M, $\mathrm{Add}(M)$ denotes the class of the direct summands of arbitrary direct sums of copies of M.

Example. If R is a Matlis integral domain (i.e., the field of quotients Q of R has projective dimension 1), then, given any pair α, β of non-zero cardinals, the module $Q^{(\alpha)} \oplus (Q/R)^{(\beta)}$ is tilting. In fact, conditions T1) and T3) are trivially satisfied. Condition T2) easily follows from the fact that, for a module M, p.d.$M \leq 1$ if and only if $\mathrm{Ext}^1_R(M, D) = 0$ for all h-divisible modules, and since divisible modules over Matlis domains are h-divisible (see [FS2, VII.2]).

Recall that in [TW] a Dedekind domain R is said to be *small* if it has countable spectrum and $|R| \leq 2^{\aleph_0}$. The main result in [TW], proved assuming Gödel's Axiom of Constructibility ($V = L$), is the following.

Theorem (Trlifaj-Wallutis [TW]). ($V = L$) *Let R be a small Dedekind domain. A module T is tilting if and only if it is of the form*

$$T = \bigoplus_{P \in \Sigma} E(R/P)^{(\alpha_P)} \oplus N$$

where Σ is a subset (possibly empty) of the maximal spectrum, the α_P are non-zero cardinals, and N is a non-zero projective R_0-module, where $R_0 = \bigcap_{P \in \mathrm{Spec}(R) \setminus \Sigma} R_P$.

Notice that, for $\Sigma = \emptyset$, the preceding characterization furnishes a non-zero projective R-module. On the opposite side, for $\Sigma = \mathrm{Max}(R)$, the tilting R-module T is a mixed divisible module with all non-zero primary components.

A remarkable consequence of the above theorem is the next corollary; recall that, if $\Sigma \subseteq \mathrm{Max}(R)$, a module M is said to be Σ-divisible if $M = PM$ for all $P \in \Sigma$, equivalently, if $\mathrm{Ext}^1_R(R/P, M) = 0$ for all $P \in \Sigma$.

Corollary (Trlifaj-Wallutis [TW]). ($V = L$) *Let R be a small Dedekind domain. If T is a tilting R-module, then* $\mathrm{Gen}(T)$ *is the class of the Σ-divisible modules for a suitable subset Σ of* $\mathrm{Max}(R)$.

Let now R be an arbitrary domain and S a multiplicative subset of R. We recall the definition of the module δ_S introduced in [FS1]: δ_S is generated by all n-tuples

$$(s_1, \ldots, s_n) \quad \text{with } s_i \in S \text{ for all } i, n \geq 0.$$

If $n = 0$, we have as generator the empty set $\emptyset$, denoted for convenience by w.

The generators of δ_S are subject to the following relations

$$s_n(s_1,\dots,s_n)=(s_1,\dots,s_{n-1})\quad (n>1),\quad s(s)=w.$$

The submodule Rw is isomorphic to R and the quotient module δ_S/Rw is clearly S-torsion. Since all the generators of δ_S are S-divisible, δ_S itself is S-divisible.

We collect now the main properties of δ_S derived from [FS1].

I) p.d.$\delta_S = 1$;

II) there exists an exact sequence $0 \to t(\delta_S) \to \delta_S \to R_S \to 0$ where the torsion submodule $t(\delta_S)$ is S-torsion;

III) given a module M and an element $a \in d_S(M)$, there exists a homomorphism $\varphi : \delta_S \to M$ such that $\varphi(w) = a$; whence $\mathrm{Gen}(\delta_S)$ is the class of the S-divisible modules;

IV) the factor module δ_S/Rw is isomorphic to a direct summand of δ_S;

V) a module D is S-divisible if and only if $\mathrm{Ext}^1_R(\delta_S, D) = 0$; whence δ_S is a κ-splitter for every cardinal κ;

VI) δ_S is a tilting module; this fact follows from I), IV) and V);

VII) the exact sequence in II) splits $\Leftrightarrow$ δ_S is h_S-divisible $\Leftrightarrow$ all S-divisible modules are h_S-divisible $\Leftrightarrow$ p.d.$R_S \leq 1$.

Lemma 2.1. *Assume that* p.d.$R_S \leq 1$. *If D is an S-divisible module such that $D/t(D)$ is a free R_S-module, then $D \cong t(D) \oplus D/t(D)$.*

Proof. By VII), D is h_S-divisible, hence there exists an epimorphism $\varphi : \bigoplus R_S \to D$. The composition of φ with the canonical surjection $\pi : D \to D/t(D)$ is an R_S-morphism, hence it splits, whence π splits too. □

3 Torsionfree ω-splitters over valuation domains

From now on, we will consider only modules over valuation domains, so R will always denote such a domain; P will denote its maximal ideal and Q its field of quotients. The main goal of this section is to characterize the torsionfree R-modules which are ω-splitters (see next Corollary 3.7). As usual, given a module M, gen M denotes the minimal cardinality of a system of generators for M.

Lemma 3.1. *Let R be a valuation domain with maximal ideal P. If J is a submodule of Q such that* gen $J = \aleph_0$, *and N is an R-module such that $N^\# = P$, there exists an exact sequence*

$$0 \to \bigoplus_{n\in\omega} N_n \to M \to J \to 0$$

with $N_n = N$ for all $n \in \omega$, such that for every $k \geq 0$ the induced exact sequence

$$0 \to \bigoplus_{n \geq k} N_n \to M \Big/ \bigoplus_{n<k} N_n \to J \to 0$$

is not splitting.

Proof. Assume that J is the union of the chain $Rr_0 < Rr_1 < \cdots < Rr_n < \cdots$, with $r_n \in R$ for all n, and let $s_{n+1} = r_n r_{n+1}^{-1}$. Let M be the R-module defined by

$$M = \Big(\bigoplus_{n \in \omega} N_n \oplus \bigoplus_{n \in \omega} Rz_n \Big) \Big/ K$$

where the z_n's are free generators, and the submodule K of the relations is generated by the sequence of elements

$$s_{n+1} z_{n+1} - z_n - y_n \quad (n \geq 0)$$

with y_n a fixed element of $N_n \backslash s_{n+1} N_n$ for all n; the choice of y_n is possible in view of the equality $N^{\#} = P$. Notice that $(\bigoplus_{n \in \omega} N_n) \cap K = 0$, as it is easily checked, so we can think of $A = \bigoplus_{n \in \omega} N_n$ as embedded in M. Let us define a map

$$\varphi : \bigoplus_{n \in \omega} N_n \oplus \bigoplus_{n \in \omega} Rz_n \to J$$

by setting

$$\varphi\Big(\bigoplus_{n \in \omega} N_n \Big) = 0, \quad \varphi(z_n) = r_n \text{ for all } n.$$

Obviously $\varphi(K) = 0$ and φ is surjective, hence φ induces an epimorphism $\pi : M \to J$. We shall prove now that $\mathrm{Ker}(\pi) = A$. The inclusion $\mathrm{Ker}(\pi) \geq A$ is obvious. To prove that $\mathrm{Ker}(\pi)$ is not larger than A, it suffices to prove that the composite map from $M' = \bigoplus N_n \oplus \bigoplus Rz_n \to M \to J$ has kernel not larger than $A + K$. But $M'/(A + K)$ is torsionfree of rank 1 with an epimorphism onto J, so this must be an isomorphism.

In order to show that for every $k \geq 0$ the exact sequence

$$0 \to \bigoplus_{n \geq k} N_n \to M \Big/ \bigoplus_{n<k} N_n \to J \to 0$$

is not splitting, note that the factor module $M / \bigoplus_{n<k} N_n$ can be defined in a similar way as M, just eliminating the generators $z_0, \ldots, z_{k-1}$ and the first k relations in K. So it is enough to prove that the exact sequence $0 \to \bigoplus_{n \in \omega} N_n \to M \to J \to 0$ does not split.

Let us assume, by way of contradiction, that there exists a splitting map ψ for π. Then for each $k \geq 0$ there are elements $x_k \in A$ such that

$$\psi(r_k) = z_k + x_k + K.$$

Combining these equalities with the relations in K we obtain, for every $k \geq 0$, the following equality:

$$z_0 + x_0 + K$$
$$= z_0 + y_0 + s_1 y_1 + s_1 s_2 y_2 + \cdots + s_1 \cdot \ldots \cdot s_k y_k + s_1 \cdot \ldots \cdot s_{k+1} x_{k+1} + K.$$

In view of $(\bigoplus_{n \in \omega} N_n) \cap K = 0$, the above equality produces the next one:

$$x_0 = y_0 + s_1 y_1 + s_1 s_2 y_2 + \cdots + s_1 \cdot \ldots \cdot s_k y_k + s_1 \cdot \ldots \cdot s_{k+1} x_{k+1}.$$

Since for each $i \leq k$ we have that $s_1 \cdot \ldots \cdot s_i y_i \in N_i \backslash s_1 \cdot \ldots \cdot s_{i+1} N_i$ and $s_1 \cdot \ldots \cdot s_{k+1} x_{k+1} \in s_1 \cdot \ldots \cdot s_{k+1} A$, and since the above equality holds for all $k \geq 0$, we deduce that the element x_0 has infinitely many non-zero coordinates, thus obtaining the desired contradiction. □

Lemma 3.2. *Let R be a valuation domain and $0 \neq J$ a submodule of Q such that* p.d.$J \leq 1$. *Let N be a torsionfree R-module such that $N^\# \geq J^\#$. Then* $\mathrm{Ext}^1_R(J, N^{(\omega)}) = 0$ *if and only if $J \cong R_L$, with $L = N^\#$.*

Proof. If $J \cong R_L$, since $N^{(\omega)}$ is a torsionfree R_L-module, we get $\mathrm{Ext}^1_R(J, N^{(\omega)}) = \mathrm{Ext}^1_{R_L}(J, N^{(\omega)})$, whence the conclusion trivially follows.

Conversely, assume that $0 = \mathrm{Ext}^1_R(J, N^{(\omega)})$. Without loss of generality we can assume that $N^\# = P$ (notice that p.d.$_{R_L} J \leq 1$), and we must prove that J is a principal ideal. If this is not the case, since p.d.$J = 1$ is equivalent to gen $J = \aleph_0$, the preceding lemma yields a contradiction. □

The next lemma improves on Lemma 3.2, passing from the rank 1 case to the finite rank case; it makes use of the full force of Lemma 3.1.

Lemma 3.3. *Let R be a valuation domain and $0 \neq X$ a torsionfree R-module of finite rank such that* p.d.$X \leq 1$. *Let N be a torsionfree R-module such that $N^\# \geq X^\#$. Then* $\mathrm{Ext}^1_R(X, N^{(\omega)}) = 0$ *if and only if X is a free R_L-module, with $L = N^\#$.*

Proof. The sufficiency is proved as in Lemma 3.2. We shall prove the necessity by induction on $\mathrm{rk}(X) = n$. The case $n = 1$ is covered by Lemma 3.2. Assume $n > 1$ and the claim true for $n - 1$. There exists an exact sequence of torsionfree modules

$$0 \to Y \to X \to J \to 0$$

with $\mathrm{rk}(Y) = n - 1$ and $J \leq Q$. Notice that $Y^\# \geq X^\#$ and $J^\# \geq X^\#$. By [FS, VI.3.8] we have that X is countably generated, so J is countably generated; furthermore Y is also countable generated, by [FS, VI.3.5]. Hence both Y and J have projective dimension ≤ 1. Applying the functor $\mathrm{Hom}_R(-, N^{(\omega)})$ to the above exact sequence we obtain the long exact sequence

$$\operatorname{Hom}_R(Y, N^{(\omega)}) \xrightarrow{\psi} \operatorname{Ext}^1_R(J, N^{(\omega)}) \to \operatorname{Ext}^1_R(X, N^{(\omega)}) = 0 \to \operatorname{Ext}^1_R(Y, N^{(\omega)}) \to 0$$

where the last 0 is due to p.d.$J \leq 1$. There follows that $\operatorname{Ext}^1_R(Y, N^{(\omega)}) = 0$, so the inductive hypothesis ensures that $Y \cong R_L^{n-1}$. We must now prove that $J \cong R_L$. Without loss of generality, we can assume, as in the proof of the preceding lemma, that $L = P$, and we must prove that J is principal. Assume, by way of contradiction, that $\operatorname{gen} J = \aleph_0$. Consider the exact sequence provided by Lemma 3.1. Since the map $\psi : \operatorname{Hom}_R(Y, N^{(\omega)}) \to \operatorname{Ext}^1_R(J, N^{(\omega)})$ in the above long exact sequence is epic, there exists a homomorphism $\varphi : Y \to \bigoplus_{n\in\omega} N_n$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & J & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle id_J} & & \\ 0 & \longrightarrow & \bigoplus_{n\in\omega} N_n & \longrightarrow & M & \longrightarrow & J & \longrightarrow & 0. \end{array}$$

Since Y has finite rank, $\varphi(Y) \leq \bigoplus_{n<k} N_n$ for a certain $k > 0$. Now we have

$$M = \bigoplus_{n\in\omega} N_n + \gamma(X), \quad \varphi(Y) = \gamma(X) \cap \bigoplus_{n\in\omega} N_n \leq \bigoplus_{n<k} N_n$$

hence we get the splitting exact sequence

$$0 \to \bigoplus_{n\geq k} N_n \to M \Big/ \bigoplus_{n<k} N_n \cong \left(\bigoplus_{n\geq k} N_n\right) \oplus (\gamma(X)/\varphi(Y)) \to J \to 0$$

which contradicts the conclusion of Lemma 3.1. □

We can now easily prove the main result of this section.

Theorem 3.4. *Let R be a valuation domain, M a torsionfree R-module of countable rank such that* p.d.$M \leq 1$, *N a torsionfree R-module such that $N^\# \geq M^\#$. Then* $\operatorname{Ext}^1_R(M, N^{(\omega)}) = 0$ *if and only if M is a free R_L-module, with $L = N^\#$.*

Proof. The sufficiency is proved as in Lemma 3.2. Concerning the proof of the necessity, Lemma 3.3 takes care of the finite rank case. If M has countable rank, by [FS, VI.3.10] it is enough to show that every pure submodule A of M of finite rank is a free R_L-module. Since p.d.$M/A \leq 1$, we have the exact sequence

$$0 = \operatorname{Ext}^1_R(M, N^{(\omega)}) \to \operatorname{Ext}^1_R(A, N^{(\omega)}) \to \operatorname{Ext}^2_R(M/A, N^{(\omega)}) = 0$$

hence $\operatorname{Ext}^1_R(A, N^{(\omega)}) = 0$ and Lemma 3.3 enables us to conclude. □

The extension of Theorem 3.4 to modules M of arbitrary rank requires, as in the Dedekind case, additional hypotheses. First we prove a lemma which is the analogous of Proposition 8 in [TW]; the argument goes back to P. Schultz [S]. Recall that $\hat{M}$ denotes the pure-injective hull of the module M.

Lemma 3.5. *Let R be a valuation domain. Then every torsionfree R-module N has a torsionfree homomorphic image N' of cardinality not exceeding that of $\hat{R}$, and such that $N^{\#} = N'^{\#}$.*

Proof. The torsionfree module N contains a pure-essential submodule $B = \bigoplus_i J_i$, where each J_i is isomorphic to a submodule of Q (see [FS2, XI.5]). Clearly $B^{\#} = \bigcup_i J_i^{\#}$, and, because of purity in the following inclusions

$$B \le N \le \hat{N} \le \prod_i \hat{J}_i,$$

we get

$$B^{\#} \le N^{\#} \le \hat{N}^{\#} \le \bigcup_i \hat{J}_i^{\#} = \bigcup_i J_i^{\#}$$

(recall that $J^{\#} = \hat{J}^{\#}$ for every $0 \ne J \le Q$) whence all the preceding inequalities are actually equalities. If there exists a summand J_i of B, call it J_0, such that $J_0^{\#} = B^{\#}$, then extend the canonical projection $B \to J_0$ followed by the inclusion $J_0 \le \hat{J}_0$ to a map $\gamma : N \to \hat{J}_0$. Set $N' = \gamma(N)$; then $J_0^{\#} \le N'^{\#} \le N^{\#} \le J_0^{\#}$, so $N^{\#} = N'^{\#}$, and

$$|N'| \le |\hat{J}_0| = |J_0 \hat{R}_{N^{\#}}| = |\hat{R}_{N^{\#}}| \le |\hat{R}|.$$

On the other side, if no summand J_i of B satisfies $J_i^{\#} = B^{\#}$, setting $L = \bigcup_i J_i^{\#}$, it is easy to define an epimorphism $\varphi : B \to L$. Extend φ to a map $\psi : N \to \hat{L}$. Therefore we have:

$$L \le \psi(N)^{\#} \le N^{\#} = L.$$

Set now $N' = \psi(N)$ and conclude as before, replacing J_0 by L. □

Theorem 3.6 ($V = L$). *If $|\hat{R}| \le 2^{\aleph_0}$, then the same conclusion as in Theorem 3.4 holds for torsionfree R-modules M of arbitrary rank.*

Proof. The proof goes, as in [TW, Theorem 11], by transfinite induction on $\mathrm{rk}(M)$. Theorem 3.4 covers the cases up to rank $\aleph_0$. When $\mathrm{rk}(M)$ is a singular cardinal, apply Shelah's Singular Compactness Theorem. When $\mathrm{rk}(M)$ is an uncountable regular cardinal, use the preceding lemma and argue as in [TW, Theorem 11]; $V = L$ is used to prove that $\mathrm{Ext}_R^1(M, N'^{(\omega)}) = 0$ implies that M is a projective R_L-module, using Theorem 1.15 at page 353 of [EM], where the hypothesis $|N'| \le 2^{\aleph_0}$ is needed. □

From the preceding Theorems 3.4 and 3.6, we immediately deduce the following

Corollary 3.7. *A torsionfree module M of countable rank of projective dimension ≤ 1 over a valuation domain R is an ω-splitter if and only if M is a free R_L-module, with $L = M^{\#}$. If $(V = L)$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed, then the same conclusion holds for torsionfree R-modules M of arbitrary rank.*

As a consequence of the preceding corollary we obtain the structure of torsionfree tilting modules.

Corollary 3.8. *A torsionfree module M of countable rank over a valuation domain R is tilting if and only if it is free. If $(V = L)$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed, then the same conclusion holds for torsionfree R-modules M of arbitrary rank.*

Proof. By Corollary 3.7, M is a free R_L-module, with $L = M^{\#}$. Every module in $\mathrm{Add}(M)$ is also a free R_L-module, so, by the condition T3), R is isomorphic to a pure submodule of a free R_L-module; this is possible only if $L = P$. □

4 The structure of tilting modules over valuation domains

The goal of this section is to determine the structure of tilting modules over a valuation domain. So, also in this section, R always denotes a valuation domain, P its maximal ideal, and Q its field of quotients.

Proposition 4.1. *Let T be a tilting R-module, and let $S = S(T)$ be its divisibility set. Then the torsion part $t(T)$ is S-divisible and S-torsion.*

Proof. The S-divisibility derives trivially from the S-divisibility of T and the purity of $t(T)$ in T. Let us assume that $0 \neq a \in t(T)$. Then $\mathrm{p.d.}T/aR \leq 1$ and $\mathrm{p.d.}aR \leq 1$, by [FS2, VI.6.4], hence aR is finitely presented, by [FS2, VI.6.2], thus $aR \cong R/rR$ for a non-zero element $r \in R$. From the exact sequence $0 \to aR \to T \to T/aR \to 0$ we get the exact sequence

$$0 = \mathrm{Ext}^1_R(T, T) \to \mathrm{Ext}^1_R(aR, T) \to \mathrm{Ext}^2_R(T/aR, T) = 0$$

hence $\mathrm{Ext}^1_R(R/rR, T) = 0$. Since $\mathrm{Ext}^1_R(R/rR, T) \cong T/rT$, we deduce that $r \in S$, so a is an S-torsion element. □

From now on, given a module M, we shall denote by $\overline{M}$ the quotient module $M/t(M)$. The next proposition shows that, passing from a tilting module T to the factor module $\overline{T}$, the divisibility set does not increase. The following lemma first proves this fact for splitting tilting modules.

Lemma 4.2. *If T is a tilting R-module such that $t(T)$ is a summand in T, then $T^{\#} = \overline{T}^{\#}$.*

Proof. The inclusion $T^{\#} \geq \overline{T}^{\#}$ being obvious, we will show that the strict inclusion is impossible. Condition T3) defining tilting modules implies the existence of a short exact sequence

$$0 \to R \to T^{(\alpha)} \xrightarrow{\pi} B \to 0$$

where $B \in \mathrm{Add}(T)$. By hypothesis, $T^{(\alpha)}$ splits, so, setting $L = \overline{T}^{\#}$, for some non-zero cardinal β we have

$$T^{(\alpha)} = A \oplus F$$

where A is an S-torsion module ($S = S(T)$), by Proposition 4.1, and F is a torsion-free R_L-module. Looking at R as included in $T^{(\alpha)}$, we have that $R \cap A = 0$, since A is torsion, hence there is an embedding $\eta : R \to F$. Let J be the purification of $\eta(R)$ in F. Since J is a rank 1 R_L-module, if $T^{\#} > L$ the factor module $J/\eta(R)$ is not S-torsion (recall that $S = R \backslash T^{\#}$). But $J/\eta(R)$ is isomorphic to a submodule of $B/\pi(A)$, whose torsion part is S-torsion, a contradiction. So the strict inclusion $T^{\#} > \overline{T}^{\#}$ is impossible, as desired. □

Proposition 4.3. *If T is a tilting R-module, then $T^{\#} = \overline{T}^{\#}$.*

Proof. Assume, by way of contradiction, that $T^{\#} > \overline{T}^{\#}$. Pick an element $r \in T^{\#} \backslash \overline{T}^{\#}$. Since $t(T)$ is S-torsion ($S = R \backslash T^{\#}$) by Proposition 4.1, we have that $rt(T) = 0$, hence $rT \cap t(T) = 0$. On the other side, since $r \notin \overline{T}^{\#}$, $\overline{T} = r\overline{T}$, that is, $T = rT + t(T)$. We deduce that $T = rT \oplus t(T)$, with $rT \cong \overline{T}$. From Lemma 4.2 we deduce that $T^{\#} = \overline{T}^{\#}$, a contradiction. □

The next proposition holds for tilting modules over any domain.

Proposition 4.4. *If T is a tilting R-module, then $\mathrm{Ext}^1_R(\overline{T}, \overline{T}^{(\kappa)}) = 0$ for all cardinals κ.*

Proof. We have the exact sequence

$$0 = \mathrm{Ext}^1_R(T, T^{(\kappa)}) \to \mathrm{Ext}^1_R(T, \overline{T}^{(\kappa)}) \to \mathrm{Ext}^2_R(T, t(T)^{(\kappa)}) = 0$$

hence $\mathrm{Ext}^1_R(T, \overline{T}^{(\kappa)}) = 0$. We also have the exact sequence

$$0 = \mathrm{Hom}_R(t(T), \overline{T}^{(\kappa)}) \to \mathrm{Ext}^1_R(\overline{T}, \overline{T}^{(\kappa)}) \to \mathrm{Ext}^1_R(T, \overline{T}^{(\kappa)}) = 0$$

whence the middle term vanishes and $\overline{T}$ is a κ-splitter. □

Notice that, given a tilting R-module T, the quotient module $\overline{T}$ is canonically an R_L-module, where $L = T^{\#} = \overline{T}^{\#}$; whence $\mathrm{Ext}^1_R(\overline{T}, \overline{T}^{(\kappa)}) = \mathrm{Ext}^1_{R_L}(\overline{T}, \overline{T}^{(\kappa)})$. In order to apply the results obtained in Section 3 to $\overline{T}$, we need the next lemma, which shows that $\overline{T}$ has projective dimension ≤ 1 as an R_L-module. Note that, in general, p.d.$\overline{T}$ can be larger than 1 (think of δ_S when p.d.$R_S > 1$).

Lemma 4.5. *If T is a tilting R-module, then $\overline{T} \cong T \otimes_R R_L$, where $L = T^{\#}$. Consequently,* p.d.$_{R_L} \overline{T} \leq 1$.

Proof. Tensoring the exact sequence $0 \to t(T) \to T \to \overline{T} \to 0$ with R_L, since $t(T) \otimes_R R_L = 0$ by Lemma 4.1, we get that $T \otimes_R R_L \cong \overline{T} \otimes_R R_L$. But $\overline{T}$ is a torsionfree R_L-module, hence $\overline{T} \otimes_R R_L \cong \overline{T}$. The last statement follows from the isomorphism $\overline{T} \cong T \otimes_R R_L$ and a projective resolution of T tensorized with R_L. □

As an immediate consequence of Propositions 4.3 and 4.4, of Corollary 3.7 and of the preceding lemma, we get the following

Proposition 4.6. *Let T be a tilting R-module and $L = T^\#$. Then $\overline{T}$ is a free R_L-module, provided that T has countable rank, or $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed.*

Notice that, if the tilting module T in the preceding proposition is divisible, then $R_L = Q$, so $\overline{T}$ is certainly a free R_L-module, without any additional hypothesis. Actually, the structure results obtained by Fuchs for divisible R-modules of projective dimension 1 produce easily the next theorem; in its proof δ' denotes the "lean version" of the module δ, and $\bar{\delta}$ denotes the quotient module δ/Rw (see [FS2, VII.I]).

Theorem 4.7. *A divisible module D of projective dimension 1 over a valuation domain R is tilting if and only if it is mixed. If this happens, then* $\mathrm{Gen}(D) = \mathrm{Gen}(\delta)$.

Proof. An inspection to the structure of divisible R-modules of projective dimension ≤ 1 is enough: see [FS2, VII.3.5] for the case p.d.$Q = 1$; see [FS2, VII.3.9] for the case p.d.$Q > 1$. In the first case, $D \cong \bigoplus_\alpha Q \oplus \bigoplus_\beta Q/R$ $(\alpha \neq 0)$. In the latter case, $D \cong \bigoplus_\alpha \delta' \oplus \bigoplus_\beta \bar{\delta}$ $(\alpha \neq 0)$. The last statement is clear. □

Since for divisible tilting modules we have the favourable situation described in Theorem 4.7, our next goal is to try to reduce general tilting modules to the divisible case. This is accomplished by means of the next result.

Proposition 4.8. *Let T be a tilting R-module, and $L = T^\#$. Then*

1) $\mathrm{Tor}_1^R(T, R/L) = 0$;

2) $\mathrm{p.d.}_{R/L} T \otimes_R R/L \leq 1$;

3) *$T \otimes_R R/L$ is a divisible R/L-module whose torsion part is isomorphic to $t(T)$.*

Proof. 1) See the proof of Lemma 5.1 in [FS1].

2) From a projective resolution of T tensored with R/L we obtain

$$\mathrm{Tor}_1^R(T, R/L) \to \oplus R/L \to \oplus R/L \to T \otimes R/L \to 0$$

where the first term is zero, by point 1).

3) The first statement is obvious, since T is S-divisible and $S = R \backslash L$. Since $t(T)$ is S-torsion, by Proposition 4.1, it is naturally an R/L-module. Tensoring the exact sequence $0 \to L \to R \to R/L \to 0$ with $t(T)$, we get the exact sequence $t(T) \otimes_R L \to t(T) \to t(T) \otimes_R R/L \to 0$. But $t(T)$ is S-torsion and L is S-divisible, so $t(T) \otimes_R L = 0$ and $t(T) \cong t(T) \otimes_R R/L$. Finally, the isomorphism $t(T) \otimes R/L \cong t(T \otimes R/L)$

holds, since $\bar{T} \otimes_R R/L$ is a torsionfree R/L-module, and from the exact sequence $0 \to t(T) \otimes_R R/L \to T \otimes_R R/L \to \bar{T} \otimes_R R/L \to 0$. □

Now we can put together Theorem 4.7 and Proposition 4.8, obtaining immediately the following

Corollary 4.9. *Let T be a tilting R-module, $L = T^\#$ and $S = S(T)$. Then*

1) $T \otimes_R R/L$ *is a divisible tilting R/L-module, whence* $\mathrm{Gen}(T \otimes_R R/L) = \mathrm{Gen}(\delta_S \otimes_R R/L)$*;*

2) $t(T) \in \mathrm{Gen}(T)$*;*

3) $\mathrm{Add}(t(T)) = \mathrm{Add}(t(\delta_S))$.

Proof. 1) $T \otimes_R R/L$ is clearly a mixed divisible R/L-module, as well as $\delta_S \otimes_R R/L$, thus Theorem 4.7 applies.

2) By point 1), $\mathrm{Gen}(T \otimes_R R/L)$ is the all class of divisible R/L-modules, hence it is closed under pure submodules. Thus $t(T) \cong t(T \otimes_R R/L) \in \mathrm{Gen}(T \otimes_R R/L)$. But $T \otimes_R R/L$ is a quotient of T, hence $\mathrm{Gen}(T \otimes_R R/L) \subseteq \mathrm{Gen}(T)$, so the claim follows.

3) From point 1) we get that $\mathrm{Add}(T \otimes_R R/L) = \mathrm{Add}(\delta_S \otimes_R R/L)$. The same equality holds for the two respective torsion parts. Thus the conclusion follows by Proposition 4.8, 3). □

We are very close to prove our main result, stating that, for a tilting module T over a valuation domain, $\mathrm{Gen}(T) = \mathrm{Gen}(\delta_S)$, where $S = S(T)$. The next result characterizes the tilting modules for which this happens; notice that condition 2) just says that $\mathrm{Gen}(T)$ is a definable class (see [CB]).

Proposition 4.10. *Let T be a tilting R-module, $T^\# = L$ and $S = S(T)$. The following properties are equivalent:*

1) $\mathrm{Gen}(T) = \mathrm{Gen}(\delta_S)$*;*

2) $\mathrm{Gen}(T)$ *is closed under pure submodules;*

3) R_L *belongs to* $\mathrm{Gen}(T)$.

Proof. 1) $\Rightarrow$ 2) is obvious: $\mathrm{Gen}(\delta_S)$ is the class of S-divisible modules.

2) $\Rightarrow$ 3) We claim that R_L is isomorphic to a pure submodule of $\bar{T}$. In fact, by Lemma 4.5, $\mathrm{p.d.}_{R_L} \bar{T} \leq 1$; so, by [FS2, VI.6.6], $\bar{T}$ has a countably generated pure R_L-submodule J of rank 1 such that $\mathrm{p.d.}_{R_L} \bar{T}/J \leq 1$. Thus $\mathrm{Ext}^1_{R_L}(J, \bar{T}^{(\omega)}) = 0$ holds, hence $J \cong R_L$ by Lemma 3.2. Since $\bar{T} \in \mathrm{Gen}(T)$, we are done.

3) $\Rightarrow$ 1) Obviously, $\mathrm{Gen}(T) \subseteq \mathrm{Gen}(\delta_S)$. In order to prove the converse inclusion, it is enough to show that $\delta_S \in \mathrm{Gen}(T)$, and since $\mathrm{Gen}(T)$ is closed under extensions, this amounts to prove that both $t(\delta_S)$ and R_L belong to $\mathrm{Gen}(T)$. The latter fact holds by hypothesis. Furthermore, $t(\delta_S) \in \mathrm{Add}(t(T))$ by Corollary 4.9, so from $t(T) \in \mathrm{Gen}(T)$ we get the proof. □

A consequence of Proposition 4.10 is that, if R is a maximal valuation domain, then $\mathrm{Gen}(T) = \mathrm{Gen}(\delta_S)$ for any tilting module T. In fact, every localization of R at a prime ideal is a pure injective R-module, hence, being R_L pure in $\overline{T}$, it is a summand; thus $R_L \in \mathrm{Gen}(T)$ and the preceding proposition applies.

Recalling that $\overline{T}$ is a free R_L-module, provided that T has countable rank, or $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed, by Proposition 4.6, we get immediately from Proposition 4.10 our main result.

Theorem 4.11. *Let R be a valuation domain, T a tilting R-module, and $S = S(T)$. Then* $\mathrm{Gen}(T) = \mathrm{Gen}(\delta_S)$*, provided that* $\mathrm{rk}(T) \leq \aleph_0$*, or $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed.*

We can improve the preceding result, in case the projective dimension of the quotient module $\overline{T} = T/t(T)$ is still 1. The next lemma clarifies when this happens.

Note that Theorem 4.11 ensures that the tilting module T is a direct summand of a direct sum of copies of δ_S, and conversely.

Lemma 4.12. *Let T be a tilting R-module and let $L = T^{\#}$. Consider the following conditions:*

(1) *$\overline{T}$ is a free R_L-module and* $\mathrm{p.d.}R_L \leq 1$*;*

(2) $\mathrm{p.d.}R_L \leq 1$*;*

(3) *the torsion part $t(T)$ is a summand in T;*

(4) $\mathrm{p.d.}\overline{T} \leq 1$.

Then (1) ⇒ (2) ⇒ (4) *and* (1) ⇒ (3) ⇒ (4). *Furthermore,* (4) ⇒ (1) *provided that either T is of countable rank, or $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed.*

Proof. Trivially (1) ⇒ (2) and (3) ⇒ (4). (1) ⇒ (3) by Lemma 2.1.

(2) ⇒ (4) $\mathrm{p.d.}_{R_L}\overline{T} \leq 1$, by Lemma 4.5. By [FS2, VI.6.6], there exists a chain of pure R_L-submodules of $\overline{T}$: $0 = C_0 < C_1 < \cdots < C_\sigma < \cdots < C_\tau = \overline{T}$ such that $\mathrm{rk}_{R_L}(C_{\alpha+1}/C_\alpha) = 1$, $\mathrm{p.d.}_{R_L}(C_{\alpha+1}/C_\alpha) \leq 1$ for all α. The C_α's are pure R-submodules of $\overline{T}$, $C_{\alpha+1}/C_\alpha$ has rank 1 as R-module and it is $\aleph_0$-generated as R_L-module for all α. But $\mathrm{p.d.}R_L \leq 1$ implies that R_L is $\aleph_0$-generated as R-module, hence $C_{\alpha+1}/C_\alpha$ is $\aleph_0$-generated even as R-module, so $\mathrm{p.d.}(C_{\alpha+1}/C_\alpha) \leq 1$. Now apply again [FS2, VI.6.6] to obtain that $\mathrm{p.d.}\overline{T} \leq 1$.

(4) ⇒ (1) under the stated conditions, by Proposition 4.6. □

We can derive now the structure result for tilting modules T such that $\mathrm{p.d.}R_L \leq 1$ ($L = T^{\#}$), which closely reminds the main result in [TW].

Theorem 4.13. *Let T be a module over a valuation domain R such that* $\mathrm{p.d.}R_L \leq 1$*, where $L = T^{\#}$.*

1) *If $T \cong (R_L/R)^{(\alpha)} \oplus R_L^{(\beta)}$ for some non-zero cardinals α, β, then T is tilting;*

2) *The converse holds true, provided that either T is of countable rank, or $V = L$ and $|\hat{R}| \leq 2^{\aleph_0}$ are assumed.*

Proof. 1) Conditions T1) and T3) are trivially satisfied by T. The fact that T is a κ-splitter for any cardinal κ follows from the fact that $T \in \mathrm{Add}(\delta_S)$, where $S = R \setminus L$.

2) Assume T tilting. Theorem 5.2 in [FS1] shows that the S-divisible S-torsion module $t(T)$ of projective dimension ≤ 1 is isomorphic to $(R_L/R)^{(\alpha)}$, for some cardinal α. Thus the statement follows from Lemma 4.12. □

In view of Theorem 4.11, one could guess that also for tilting modules T of countable rank over Dedekind domains $\mathrm{Gen}(T) = \mathrm{Gen}(\delta_S)$, where $S = S(T)$. This is not the case, as the next example shows. The example also shows that the statement of Lemma 3 (i) of [TW] is not correct.

Example 4.14. Let R be a Dedekind domain with a maximal ideal P such that P^n is not principal for all $n > 0$. Then it is easy to see that $P \subseteq \bigcup\{P' \in \mathrm{Max}(R) \mid P' \neq P\}$. Setting $R^{[P]} = \bigcap_{P' \neq P} R_{P'}$, one has that $R^{[P]}/R \cong E(R/P)$. Then $T = E(R/P) \oplus R^{[P]}$ is a tilting module such that $S = S(T) = U(R)$, whence $\delta_S = R$, while $\mathrm{Gen}(T)$ coincides with the class of the P-divisible R-modules.

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Received January 23, 2003

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