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Recognising dualities in finite simple groups

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Introduction

Given a group G, a subgroup K of G is called *cocyclic* (in G) if the interval $[G/K]$ is anti-isomorphic to the lattice of subgroups of a cyclic group. Let us denote by CG the partially ordered set of all cyclic subgroups of G and by $\mathrm{Co}\,G$ the partially ordered set of all cocyclic subgroups of G. We shall consider groups G, $\bar{G}$ for which the following properties hold:

D_1: there exists an anti-isomorphism τ of $\mathrm{Co}\,G$ onto $C\bar{G}$,
D_2: every subgroup of G is the intersection of cocyclic subgroups,
D_3: if X, Y, Z are cocyclic subgroups of G, then

$$X \geqslant Y \cap Z \quad \text{if and only if} \quad X^\tau \leqslant \langle Y^\tau, Z^\tau \rangle.$$

Two groups $G, \bar{G}$ will be said to be in *D-situation* (relative to the map τ) if the properties D_1, D_2, D_3 hold. Also a group G will be said to *admit* a D-situation if there exist a group $\bar{G}$ and a map τ such that G and $\bar{G}$ are in D-situation. In [12] it was shown that a finite soluble group admits a D-situation if and only if G has an auto-duality. The aim of this paper is to prove the following result.

Theorem. *A finite simple group G admits a D-situation if and only if G is abelian.*

Section 1 contains several preliminary results and technicalities. Section 2 deals with the alternating groups and the simple Chevalley groups, Section 3 with the twisted simple groups of Lie type, and finally in Section 4 we consider the sporadic simple groups. Our terminology and notation are quite standard (see for example [10] and [11]). When discussing simple groups, we follow mainly [1], [2] and [6]. We denote by C_m the cyclic group of order m and by S_n the symmetric group of degree n. For each prime p, G_p denotes a Sylow p-subgroup of the group G, $Z(G)$ is the center of G, and

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$\omega(G)$ is the set of prime divisors of the order of G. The quaternion group of order 8 is denoted by Q_8 and the four-group by V. Thus $V \cong C_2 \times C_2$. We denote by $(CG)_2$ the set of all subgroups of G that can be generated by at most two cyclic subgroups. Dually, $(\mathrm{Co}\,G)_2$ denotes the set of all subgroups of G that are the intersection of at most two cocyclic subgroups. We write $X <\cdot\, G$ when X is a maximal subgroup of G. Also $A < B$ means that A is a proper subgroup of B, and $A \lhd B$ means that $A \trianglelefteq B$, but $A \neq B$. A *Zassenhaus group* is a group with all Sylow subgroups cyclic. If P is a parabolic subgroup of a group of Lie type of characteristic p, then the *unipotent radical* of P is its maximal normal p-subgroup, denoted by U_P. All groups considered here are finite.

1 Preliminary results

Let G and $\bar{G}$ be two groups in D-situation via the map τ. Then the maps

$$\begin{cases} \delta : L(G) \to L(\bar{G}), & H \mapsto \langle X^{\tau} \mid X \in \mathrm{Co}[G/H] \rangle \\ \bar{\delta} : L(\bar{G}) \to L(G), & \bar{H} \mapsto \bigcap\limits_{\bar{X} \in C\bar{H}} \bar{X}^{\tau^{-1}} \end{cases} \tag{1}$$

are inclusion-reversing, with $\delta\bar{\delta} = 1$. Moreover, by [12],

$$\begin{array}{l} \delta|(\mathrm{Co}\,G)_2 \text{ is an anti-isomorphism of } (\mathrm{Co}\,G)_2 \text{ onto } (C\bar{G})_2 \\ \text{whose inverse is } \bar{\delta}|(C\bar{G})_2. \end{array} \tag{2}$$

The map δ is a duality if and only if G is projective to $\bar{G}$, in which case G is soluble and projective to an abelian group (see [15] and [17]).

Assume now that G is a finite simple group of Lie type and consider a maximal unipotent subgroup U of G. Let $B = \mathcal{N}(U)$ be the Borel subgroup above U, let H be a complement of U in B, as defined in [1], and let $N \leqslant \mathcal{N}(H)$ be such that N/H is the Weyl group. Then it is known that

$$\text{if } U \leqslant M <\cdot\, G \text{ then } B \leqslant M. \tag{3}$$

(see for example [3]). If one considers the map $\delta : G \to \bar{G}$, then since $[G/B]$ is a Boolean lattice, $\bar{B} = B^{\delta}$ is a cyclic group of square-free order and, by (3), it is generated by the set of minimal subgroups of $\bar{U} = U^{\delta}$. It follows that the minimal subgroups of $\bar{U}$ are all normal in $\bar{U}$, and the Sylow subgroups are either cyclic or generalized quaternion.

Assume that the Sylow 2-subgroup of $\bar{U}$ is a generalized quaternion group $\bar{Q}$ and set $\bar{T} = \bar{Q}\bar{B} \leqslant \bar{U}$. By [8], $L(\bar{T}) = (C\bar{T})_2$, so there exists $T \in [B/U] \cong [H/1]$ such that $\bar{T} = T^{\delta}$ and, by (2), $\delta|[B/T]$ is an anti-isomorphism of $[B/T]$ onto $\bar{T}/\bar{B} \cong \bar{Q}/Z(\bar{Q})$. So $\bar{Q}/Z(\bar{Q})$ is a modular group, since H is abelian, and therefore it is the four-group. Hence $\bar{Q} = \langle \bar{x}, \bar{y} \rangle$ is the quaternion group.

Now assume that $\bar{U}$ non-soluble. By [13, Theorem A], we have a Hall factorization

$$\bar{U}/\langle \bar{x}^2 \rangle = (\bar{L}/\langle \bar{x}^2 \rangle) \times (\langle \bar{Z}, \bar{x}^2 \rangle / \langle \bar{x}^2 \rangle)$$

with $\bar{L}/\langle\bar{x}^2\rangle \cong \mathrm{PSL}_2(p)$ and $\bar{Z}$ a Zassenhaus group of odd order. Since

$$\langle\bar{x}^2\rangle \leqslant \bar{L}\cap\bar{B} \trianglelefteq \bar{L} \leqslant \overline{U},$$

we conclude that $\bar{L}\cap\bar{B} = \langle\bar{x}^2\rangle$, a contradiction. We have therefore proved

> $\overline{U}$ is a soluble group with all Sylow subgroups cyclic or quaternion. Moreover $L(\overline{U}) = (C\overline{U})_2$ by [8], and the minimal subgroups of $\overline{U}$ are normal. (4)

Proposition 1.1. *Let G be a simple group of Lie type, with B a Borel subgroup of G, U its unipotent radical and H a maximal torus of B. Assume that G is in D-situation with $\bar{G}$ and $\delta : G \to \bar{G}$ is the associated anti-monomorphism of $L(G)$ into $L(\bar{G})$. Then the following are true.*

(i) *$\bar{B}$ is a cyclic group of square-free order generated by the minimal subgroups of $\overline{U}$, and $\delta|[G/U]$ is an anti-isomorphism of $[G/U]$ onto $[\overline{U}/1]$.*

(ii) *If $\overline{U} \neq \bar{B}$, then there is a Hall factorization*

$$\overline{U}/\bar{B} = P_1 \times P_2 \times \cdots \times P_t \times C.$$

Here $t \geqslant 0$, C is cyclic, and, for $i > 1$, P_i is a P-group of order $p_i q_i$ with $p_i > q_i$ while either

(a) *P_1 is a P-group of order $p_1 q_1$ with $p_1 > q_1$, or*

(b) *P_1 is the four-group, and this is the case if and only if $\overline{U}$ contains a quaternion subgroup.*

(iii) *The Sylow subgroups of H are elementary abelian of order p_i^2 or cyclic.*

(iv) *$\overline{U} = \overline{U}_1 \ltimes \overline{U}_2$, where $\overline{U}_2$ is a Hall subgroup and a Zassenhaus group, and there is a Hall decomposition*

$$\overline{U}_1 = \bar{R}_1 \times \bar{R}_2 \times \cdots \times \bar{R}_t.$$

Here, for $i > 1$, $\bar{R}_i$ a non-cyclic Zassenhaus group of order $p_i^2 q_i^2$ with center of order q_i. In case (a), *$\bar{R}_1$ is a non-cyclic Zassenhaus group of order $p_1^2 q_1^2$ with center of order q_1, and in case* (b), *$\bar{R}_1$ is the quaternion group. Moreover*

$$\bar{B} = (\bar{B}\cap(\bar{R}_1 \times \cdots \times \bar{R}_t)) \times (\bar{B}\cap\overline{U}_2).$$

Proof. As already pointed out, $\delta|[G/U]$ is an anti-isomorphism of $[G/U]$ onto $[\overline{U}/1]$. In particular we have a duality of $B/U \cong H$ onto $\overline{U}/\bar{B}$, and so, by (4) and [11, Theorem 8.2.2], (ii) must hold. Then (iii) follows from (ii).

Write $\bar{B} = \bar{B}_1 \times \bar{B}_2$, with $\omega(\bar{B}_2) = \omega(\bar{B})\backslash\omega(P_1 \times \cdots \times P_t)$. We have

$$\overline{U}/\bar{B} \cong \overline{U}_1/\bar{B}_1 \times \overline{U}_2/\bar{B}_2,$$

for certain subgroups $\overline{U}_1$, $\overline{U}_2$, with

$$\overline{U}_1/\bar{B}_1 \cong P_1 \times \cdots \times P_t, \quad \overline{U}_2/\bar{B}_2 \cong C.$$

Then $\overline{U}_1 \cap \overline{U}_2 = \bar{B}_1 \cap \bar{B}_2 = 1$, and hence $\overline{U} = \overline{U}_1\overline{U}_2$. Moreover, since $\Phi(\overline{U}_1) = \bar{B}_1 \trianglelefteq \overline{U}$, we have $\overline{U}_2 \trianglelefteq \overline{U}$ and $\overline{U}_1 = \bar{R}_1 \times \cdots \times \bar{R}_t$, where $\bar{R}_i$ has the required structure, while $\overline{U}_2$ is a Zassenhaus group with $\overline{U}_2' \leqslant \bar{B}_2 = \bar{B} \cap \overline{U}_2$.

It will be useful to observe that, in the above situation, the number of maximal subgroups of G containing U is equal to the number of distinct prime divisors of $|\overline{U}|$.

Lemma 1.2. *Let G be a group in D-situation with $\bar{G}$. If $H <\cdot X \in \operatorname{Co} G$, then $X^\delta <\cdot H^\delta$.*

Proof. This is clear if $H \in \operatorname{Co} G$. Otherwise we have $H = X \cap Y$ for some $Y \in \operatorname{Co} G$ and, by (2), $H^\delta = \langle X^\delta, Y^\delta \rangle$. Let $X^\delta <\cdot \overline{T} \leqslant H^\delta$; then $\overline{T} \in (C\bar{G})_2$, so by (2) there exists a (unique) subgroup $T \leqslant G$ such that $\overline{T} = T^\delta$. It follows that $H \leqslant T < X$. Hence $H = T$, and $X^\delta <\cdot H^\delta$.

Proposition 1.3. *Let K be a maximal subgroup of the group L and suppose that K is a cyclic p-group. Then either L is metacyclic or $L = Q \rtimes K$, where Q is an elementary abelian Sylow subgroup of L of order q^α* with $\alpha > 1$ *on which K acts irreducibly.*

Proof. Assume that L is not metacyclic. Then K is a Sylow p-subgroup of L with $\mathcal{N}(K) = K$, so that $L = Q \rtimes K$ for some subgroup Q of L. But $K <\cdot L$ implies that Q is an elementary abelian q-group on which K must act irreducibly.

Corollary 1.4. *Let G be a group in D-situation with $\bar{G}$ and let $K \in \operatorname{Co} G$ be such that $[G/K]$ is a chain. If $H <\cdot K$, then H^δ has the structure of the group L in* 1.3, *and $\delta|[G/H]$ is an anti-isomorphism onto $[H^\delta/1]$ when H^δ is metacyclic.*

Proof. By Lemma 1.2 we have $K^\delta <\cdot H^\delta$, and the result follows by Proposition 1.3 and (2).

Proposition 1.5. *Let G be a simple non-abelian group in D-situation with $\bar{G}$ and let S be a minimal subgroup of G such that $[G/S]$ is a chain of length n. Then $\bar{G}$ is a Frobenius group, with $\overline{S} = S^\delta$ a cyclic subgroup of order p^n (for some prime p) and with Frobenius kernel Q an elementary abelian group of order q^α for some $\alpha > 1$. The number N of maximal subgroups of G is given by $N = 1 + q + \cdots + q^\alpha$.*

Proof. Clearly $\overline{S}$ is cyclic of order p^n for some prime p. If $\bar{G}$ is metacyclic, then the simple non-abelian group G is dual to $\bar{G}$, by (2). According to [15], this is a contradiction. Thus, by Proposition 1.3, $\bar{G} = Q \rtimes \overline{S}$ and Q is an elementary abelian q-group of order q^α, for some $\alpha > 1$, on which $\overline{S}$ acts irreducibly.

Clearly if $n = 1$, then $\bar{G}$ is a Frobenius group. Thus suppose that $n > 1$ and let $S < M <\cdot G$. Since $\mathcal{N}(M) = M$, for $x \in G \backslash M$ we have $M^x \neq M$ and $[G/S^x]$ is also a chain of length n, with $S^x < M^x <\cdot\, G$. Therefore $\overline{S^x}$ is a cyclic group of order p^n, since $n > 1$, and $\bar{S} \cap \overline{S^x} = 1$ since $\bar{M} \cap \overline{M^x} = 1$. But then also $\bar{S} \cap \bar{S}^y = 1$ for any $1 \neq y \in Q$, that is, $\bar{G}$ is a Frobenius group.

Finally N equals the number of minimal subgroups of $\bar{G}$, which is $1 + q + \cdots + q^\alpha$. The result follows.

For future use and as a complementary statement to Proposition 1.1, we prove

Lemma 1.6. *Let G be a group with subgroups R, S, B, P such that $R \lhd P$, $P/R \cong S_3$, $R < S \lhd P$, $R < B < P$ and $[G/B]$ is a Boolean lattice. Assume also that every maximal subgroup of G containing S also contains P. If G is in D-situation with a group $\bar{G}$, then $\bar{R}$ is a Zassenhaus group and the minimal subgroups of $\bar{R}$ generate the square-free cyclic group $\bar{B}$.*

Proof. Note that, by our assumption, we have $P \neq G$. It is clear that the square-free cyclic group $\bar{P} = P^\delta$ is generated by the minimal subgroups of $\bar{S}$ and, since $\bar{P} \lhd\cdot\, \bar{S}$ by Lemma 1.2, $\bar{S}$ is a Zassenhaus group with just one Sylow subgroup of order r^2, where $|\bar{S} : \bar{P}| = r$. We have

$$|\bar{B}| = rp_1 \dots p_t p_{t+1}, \quad |\bar{P}| = rp_1 \dots p_t, \quad |\bar{S}| = r^2 p_1 \dots p_t,$$

for some $t \geqslant 0$. Since $R <\cdot\, B$ and $\bar{P} \lhd\cdot\, \bar{B}$, we have

$$\bar{P} \trianglelefteq \langle \bar{B}, \bar{S} \rangle = \bar{R} \in (C\bar{G})_2.$$

By Proposition 1.3, either $\bar{R}/\bar{P}$ is a P-group of order pq with $p \geqslant q$, or $\bar{R}/\bar{P}$ is a Frobenius group F of order pq^α with $\alpha > 1$ and elementary abelian Frobenius kernel F_q.

Suppose that we have the second case. Whenever $\bar{P} <\cdot\, \bar{X} < \bar{R}$, it follows that $\bar{X}$ is 2-generated. Therefore there exists a unique X satisfying $R < X <\cdot\, P$ such that $\bar{X} = X^\delta$. Then we have a contradiction, since $[P/R]$ has only 4 maximal subgroups. Therefore $\bar{R}/\bar{P}$ is a P-group of order pq with $p \geqslant q$. But $|\bar{S} : \bar{P}| = r$ and $|\bar{B} : \bar{P}| = p_{t+1}$, so that $p \neq q$. Again every $\bar{X} \in [\bar{R}/\bar{P}]$ is 2-generated, and hence $\delta|[P/R]$ is a duality onto $[\bar{R}/\bar{P}]$, so that $\bar{R}/\bar{P} \cong S_3$.

Suppose that $|\bar{R} : \bar{B}| = 3$. Then $|\bar{B} : \bar{P}| = 2$ and hence $2 \nmid |P|$. Thus $|\bar{S} : \bar{P}| = 3$ and so $\bar{R}_3 \cong C_9$ and $\bar{B}_2$ acts non-trivially on $\bar{R}_3$. But then $\bar{B}_2$ acts non-trivially on $\bar{B}_3$, a contradiction since $\bar{B}$ is abelian. Thus we are left with the case when $|\bar{R} : \bar{B}| = 2$. Then $\bar{B} \trianglelefteq \bar{R}$, $(\bar{R})_3$ has order 3 and $|\bar{S} : \bar{P}| = 2$. Hence $r = 2$, the Sylow 2-subgroups of $\bar{R}$ are cyclic of order 4, and $p_{t+1} = 3$. So the Zassenhaus group $\bar{R}$ has order $12p_1 \dots p_t$ and the involution in $\bar{B}$ is central. Therefore the minimal subgroups of $\bar{R}$ are in $\bar{B}$.

Proposition 1.7. *Let G be an indecomposable Zassenhaus group, p a prime divisor of $|G'|$ and φ a projectivity of G. If G is not a P-group of order pq with $p > q$, then φ is regular at p.*

Proof. Let S be a Sylow p-subgroup of G'. If φ is singular at p, then $S \leqslant Z(G)$, by [11, Lemmas 4.2.2 and 4.2.3]. This is a contradiction.

2 Alternating groups and simple Chevalley groups

In this section we deal with alternating groups and simple Chevalley groups in D-situation.

Proposition 2.1. *The alternating group $G = A_n$ admits a D-situation if and only if $n \leqslant 3$.*

Proof. We suppose that $n \geqslant 5$. We denote by G_i the stabilizer of i for each $i \leqslant n$, and by $G_{i,j}$ the intersection $G_i \cap G_j$ for $i \neq j$. Since G is $(n-2)$-transitive, we have $G_{i,j} <\cdot\, G_i <\cdot\, G$.

We show that if $G_{i,j} \leqslant M <\cdot\, G$, then M is not transitive. We may assume M different from G_i. Since $G_{i,j}$ has index $n(n-1)$, if M were transitive, then $|M : M_i| = n$, and $M_i = M \cap G_i = G_{i,j}$. Hence $|G : M| = n - 1 < n$, which is a contradiction. From the classification of the maximal non-transitive subgroups of G (see [7]), it follows that in $[G/G_{i,j}]$ there are only three maximal subgroups: G_i, G_j and $W_{i,j}$, where $W_{i,j}$ is the stabilizer of the subset $\{i, j\}$. Moreover, since $|W_{i,j} : G_{i,j}| = 2$, we have $G_{i,j} <\cdot\, W_{i,j}$. Hence $[G/G_{i,j}]$ is the lattice of subgroups of the four-group. By Corollary 1.4, $\bar{G}_{i,j}$ is a four-group. In particular $[\bar{G}_i, \bar{G}_j] = 1$.

Now let i, j, k be distinct. If $G_{i,j,k} = G_i \cap G_j \cap G_k$, then $\bar{G}_{i,j,k} = \bar{G}_i \times \bar{G}_j \times \bar{G}_k$. On the other hand, $G_{i,j,k} = W_{i,j} \cap W_{i,k}$, so that $\bar{G}_{i,j,k} = \overline{W}_{i,j} \times \overline{W}_{i,k}$, which is a contradiction. The general conclusion follows from [12].

We deal next with Chevalley groups. Let $G = \mathscr{L}(\mathscr{K})$ be a simple Chevalley group associated with the Lie algebra $\mathscr{L}$ of rank l, where K is a field of $q = p^\alpha$ elements with p a prime. If G is in D-situation with $\bar{G}$, then by Proposition 1.1 we know that H has to be 2-generated. On the other hand

$$H \cong (\underbrace{K^* \times \cdots \times K^*}_{l})/T,$$

where K^* is cyclic of order $q-1$ and T is a cyclic group of known order or the four-group (see [1, p. 122]). One concludes that for $|K| \neq 2$, G has to belong to one of the families

$$A_l, B_l, C_l \quad \text{with } l \leqslant 3, \quad G_2 \quad \text{or} \quad D_4(3). \tag{5}$$

We consider each of these possibilities, assuming that $|K| \neq 2$.

Suppose that G has type A_l. Here T is cyclic of order $d = (l+1, q-1)$.

Consider first the case when $l = 3$. So $d = 2$ or 4. If $d = 2$, then for H to be 2-generated, we must have $q - 1 = 2$, i.e. $G = A_3(3)$. We shall deal with this case later. If $d = 4$, then for H to be 2-generated, we must have $q - 1 = 4$. But then H is of exponent 4 and not cyclic, in contradiction to Proposition 1.1 (iii).

Now suppose that $l = 2$. So $d = 1$ or 3. We have two maximal parabolic subgroups P_1, P_2 and a graph automorphism γ of G of order 2 such that $P_1^\gamma = P_2$, $U^\gamma = U$ and $\{r, s\} = \omega(\overline{U})$. We see that $d = 1$ if and only if $3 \nmid q - 1$. Here $H \cong K^* \times K^*$, and since H_r has to be elementary abelian of order r^2, $\overline{U}/\bar{B}$ is a P-group or a four-group. In the former case, $\overline{U}/\bar{B}$ has order rs with $r > s$. Then $\overline{U}$ is a Zassenhaus group of order r^2s^2. Since $\delta|[G/U]$ is a duality onto $\overline{U}$ (by Proposition 1.1), $\bar{\delta}\gamma\delta|\overline{U}$ is singular, in contradiction to Proposition 1.7. If $\overline{U}/\bar{B}$ is the four-group, then $\overline{U} = Q_8 \ltimes C_r$ (with r an odd prime), by Proposition 1.1, and $\bar{\delta}\gamma\delta$ cannot have a 2-singularity on $\overline{U}$. Suppose that $d = 3$. If $q - 1 = 3$, then H is cyclic of order 3, so $|\overline{U}| = r^2t$ and $\bar{\delta}\gamma\delta|\overline{U}$ cannot exist. If $q - 1 \neq 3$, then H_3 is cyclic, so $\overline{U}/\bar{B}$ is isomorphic to the direct product $V \times C$ of the four-group V and a cyclic group C, and we conclude the argument as before.

We are left with the case when $l = 1$. Here $G = \mathrm{PSL}_2(q)$, with $q \neq 2, 3$, $H \cong K^*/C_d$ and $d = (2, q - 1)$. The Borel subgroups are maximal in G and so $\overline{U}/\bar{B}$, $\overline{U^g}/\overline{B^g}$ are cyclic, where $B \cap B^g = H$. It follows that $\overline{U}$, $\overline{U^g}$ and H are cyclic of prime-power order. Since $\mathrm{PSL}_2(5) \cong A_5$ and $\mathrm{PSL}_2(9) \cong A_6$, we may assume $q \neq 2, 3, 5, 9$. But then we have either $H <\cdot \mathcal{N}(H) <\cdot G$ or $H <\cdot B <\cdot G$ (by [14, Example 7 on p. 417]) and $[G/H]$ contains either three atoms or three coatoms. Hence $\overline{H}$ is the four-group, by Corollary 1.4. Therefore $\overline{U} \cong \overline{U^g} \cong C_{2^\alpha}$ with $\alpha \geqslant 2$, and since $\langle \overline{U}, \overline{B^g} \rangle = \bar{G} = \langle \overline{U^g}, \bar{B} \rangle$, we conclude that $\bar{G} \cong C_{2^\alpha} \times C_2$, in contradiction to (2) and the simplicity of G.

Suppose that G has type G_2. Here $d = 1$, $H \cong K^* \times K^*$ and there are two maximal parabolic subgroups P_1, P_2 above B. Hence $\overline{U}/\bar{B}$ is either a P-group of order rt, with $r > t$, or a four-group. Thus $q - 1 = r$ or 2. We distinguish two cases. Let $q = 3$. Then by Proposition 1.1, $\overline{U} = Q_8 \ltimes C_t$ and $G = G_2(3)$. There exists a graph automorphism γ of order 2 such that $P_1^\gamma = P_2$ and $U^\gamma = U$. But $\bar{\delta}\gamma\delta$ cannot exist on $\overline{U}$.

Now suppose that $q > 3$. Then $G = G_2(2^\alpha)$, with $\alpha > 1$, since $q = 1 + r$ where r is an odd prime. Here $\overline{U}$ is a non-cyclic Zassenhaus group of order r^2t^2, with $r > t$, while H is elementary abelian of order r^2. Let $\Pi = \{r_1, r_2\}$ be a fundamental system for G, $L_i = H\langle X_{r_i}, X_{-r_i} \rangle$, and let $P_i = U_{P_i}L_i$ be the Levi decomposition. Since r, but not r^2, divides $|\mathrm{PSL}_2(2^\alpha)|$, we have $L_i > \langle X_{r_i}, X_{-r_i} \rangle \cong \mathrm{PSL}_2(2^\alpha)$. But H normalizes every root subgroup, and hence $L_i' = \langle X_{r_i}, X_{-r_i} \rangle$. We have $P_i/U_{P_i} \cong L_i/1$; in particular $P_i' = UL_i'$ has index r in P_i. Moreover $B \neq UL_i'$, since L_i' is simple. Let $|\bar{P}_1| = r$, $|\bar{P}_2| = t$. From the duality $\delta : [G/U] \to [\overline{U}/1]$ we deduce that there are $r^2 + 1$ maximal subgroups of P_2 containing U. But H acts on these maximal subgoups, fixing at least two of them, namely B and P_2'. Since the orbits have length 1 or r, and r does not divide $r^2 - 1$, we have at least a third maximal subgroup X of P_2 containing U and fixed by H. But then $B = UH \leqslant \mathcal{N}(X)$ and $P = BX$. In particular $X \trianglelefteq P_2$, so that $P_2' \leqslant X$. Hence $X = P_2'$, which is a contradiction.

Suppose that G has type B_2. Let $d = 1$. If $\overline{U}/\bar{B}$ is a non-abelian P-group of order rt, with $r > t$, then H is elementary abelian of order r^2, $q - 1 = r$ and $G = B_2(2^\alpha)$ with $\alpha \geqslant 2$. But G has a graph automorphism γ of order 2 interchanging the two maximal parabolic subgroups above B. Then $\bar{\delta}\gamma\delta$ cannot be an autoprojectivity of $\overline{U}$, by Proposition 1.7. On the other hand, if $\overline{U}/\bar{B}$ is the four-group, then H is the four-group and $q - 1 = 2$, contradicting $d = 1$. Thus suppose that $d = 2$. In this case we have $H \cong (K^* \times K^*)/C_2$ and since H_2 has to be homocyclic, it must be cyclic, i.e. we have $q - 1 = 2$ and $G = B_2(3)$. We postpone the treatment of this group.

Suppose that G has type B_3 or C_3. We must have $d = 2$, since otherwise $H \cong K^* \times K^* \times K^*$. So $H \cong (K^* \times K^* \times K^*)/C_2$ and, by Proposition 1.1, we conclude that $q - 1 = 2$, i.e. $q = 3$. We deal with the groups $A_3(3)$, $B_3(3)$ and $C_3(3)$ simultaneously. Now H is the four-group, and hence $\overline{U} \cong Q_8 \ltimes (C_s \times C_t)$. Let P be the minimal parabolic subgroup associated with the fundamental root r and such that $\overline{U}/\overline{P} \cong Q_8$. Using the Levi decomposition with $L = H\langle X_r, X_{-r}\rangle$ we have one of the following:

$$\langle X_r, X_{-r}\rangle \cong \begin{cases} \mathrm{SL}_2(3) \\ \mathrm{PSL}_2(3) \end{cases}$$

where X_r is isomorphic to C_3 and is a Sylow 3-subgroup of L, by [1, Section 8.5]. In both cases, $[L/X_r]$ contains the interval $[HX_r/X_r]$ which is isomorphic to $[H/1]$, and also the subgroup $\langle X_r, X_{-r}\rangle$ which is not in $[HX_r/X_r]$. This is a contradiction, since, by (1), δ induces a duality from $[P/U]$ onto $[\overline{U}/\overline{P}] \cong [Q_8/1]$, while $[P/U] \cong [L/X_r]$.

Suppose that $G = D_4(3)$. Since H, and so $\overline{U}/\bar{B}$, is the four-group and $|\omega(\overline{U})| = 4$, we have $\overline{U} \cong Q_8 \ltimes (C_r \times C_s \times C_t)$. If P is the minimal parabolic subgroup such that $[\overline{U}/\overline{P}] \cong [Q_8/1]$, then using the same argument as in the previous case, we reach a contradiction.

Suppose finally that G has type $B_2(3)$ $(= C_2(3))$. Referring to [2, p. 26], we see that G has order $2^6 3^4 5$. Let Φ^+ be a positive root system and let Π be the fundamental system contained in Φ^+. We have $\Pi = \{\alpha_1, \alpha_2\}$, and we assume α_1 long, α_2 short. Then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, $U = X_{\alpha_1} X_{\alpha_2} X_{\alpha_1+\alpha_2} X_{\alpha_1+2\alpha_2}$, $B = HU$, $H \cong C_2$ and $B' = X_{\alpha_2} X_{\alpha_1+\alpha_2} X_{\alpha_1+2\alpha_2}$, by [1, p. 175]. Let $P_1 = \langle B, X_{-\alpha_1}\rangle$, $P_2 = \langle B, X_{-\alpha_2}\rangle$ be the minimal parabolic subgroups above B. By [1, p. 186], we have

$$\langle X_{\alpha_1}, X_{-\alpha_1}\rangle \cong \mathrm{SL}_2(3), \quad \langle X_{\alpha_2}, X_{-\alpha_2}\rangle \cong \mathrm{PSL}_2(3),$$

and hence $H \leqslant \langle X_{\alpha_1}, X_{-\alpha_1}\rangle$ and $H \cap \langle X_{\alpha_2}, X_{-\alpha_2}\rangle = 1$. The unipotent radical U_{P_1} of P_1 is $X_{\alpha_2} X_{\alpha_1+\alpha_2} X_{\alpha_1+2\alpha_2}$, and thus it coincides with B'. Moreover, $P_1 = U_{P_1} L_1$, with

$$L_1 = H\langle X_{\alpha_1}, X_{-\alpha_1}\rangle = \langle X_{\alpha_1}, X_{-\alpha_1}\rangle \cong \mathrm{SL}_2(3).$$

Since $\mathrm{SL}_2(3)' = Q_8$, we have

$$P_1' \geqslant \langle B', Q_8\rangle = U_{P_1} Q_8 \quad \text{and} \quad P_1/U_{P_1} Q_8 \cong L_1/Q_8 \cong C_3.$$

Hence $P_1' = U_{P_1} Q_8$ has index 3 in P_1. On the other hand we have $P_2 = U_{P_2} L_2$ and $L_2 = H\langle X_{\alpha_2}, X_{-\alpha_2}\rangle \cong S_4$. In this case $P_2' = \langle U, X_{-\alpha_2}\rangle$ has index 2 in P_2.

Since $l = 2$ and $H \cong C_2$, we have $|\overline{U}| = r^2 p$ and $\bar{B} = C_r C_p$. But

$$[P_2/U] \cong [S_4/(S_4)_3] \cong C_p \times C_r,$$

so that $\bar{U} = C_{r^2} \times C_p$. Moreover it is clear that $|\bar{P}_1| = p$ and $|\bar{P}_2| = r$, since $P_2' > U$, and $P_2' \neq B$.

Suppose that there exists a maximal subgroup M of G containing P_1' but different from P_1. By index considerations, and from the fact that $(P_1)_2 \not\cong (P_2)_2$, we must have $M = P_1^g$ for some $g \notin P_1$. Let $X = (P_1')^{g^{-1}}$. From $P_1' < P_1^g$ it follows that $|P_1 : X| = 3$ and $X \neq P_1'$, since $g \notin P_1 = \mathcal{N}(P_1')$. If $X \trianglelefteq P_1$, then $X = P_1'$, a contradiction. Then $P_1/X_{P_1} \cong S_3$, and there exists $Y \trianglelefteq P_1$ of index 2, again a contradiction. It follows that $[G/P_1']$ is a chain of length 2.

Now H is the center of L_1, and so $Y = U_{P_1}H$ is normal in P_1. We have $[P_1/Y] \cong [L_2/H] \cong [A_4/1]$. Since $|\bar{P}_1| = p$ and $\bar{P}_1'$ is a chain of length 2, we also have $\bar{P}_1' \cong C_{p^2}$, so that $\bar{P}_1' \leqslant Z(\langle \bar{P}_1', \bar{B} \rangle) = Z(\bar{Y})$. Now $\bar{Y}/\bar{P}_1$ is generated by the two minimal subgroups $\bar{B}/\bar{P}_1$ and $\bar{P}_1'/\bar{P}_1$. Also $\bar{B} <\cdot \bar{Y}$. Hence $\bar{Y}/\bar{P}_1$ is either cyclic, a P-group of order pq or a Frobenius group of order pq^α in which $[P_1/Y]$ dually embeds. Since $[P_1/Y] \cong [A_4/1]$, we are left with the case when $\bar{Y}/\bar{P}_1$ is a Frobenius group. But δ induces a bijection between the set of maximal subgroups of P_1/Y and the set of minimal subgroups of $\bar{Y}/\bar{P}_1$. In fact, if $\bar{X}/\bar{P}_1$ is a minimal subgroup of $\bar{Y}/\bar{P}_1$, then $\bar{X}$ is 2-generated, and hence $\bar{X} = X^\delta$ for a unique $X \leqslant G$. But then $X \in [P/Y]$ and is maximal. On the other hand, if $Y < M <\cdot P_1$, then $\bar{P}_1 <\cdot \bar{M}$, by Lemma 1.2. Finally we deduce that P_1/Y has five maximal subgroups. Since there are no solutions to the equation $5 = 1 + q + \cdots + q^\alpha$, we have proved that

a finite simple Chevalley group admits no D-situation for $|K| \neq 2$.

To complete our consideration of finite simple Chevalley groups in D-situation, we are left with the case when the field K has two elements. Here $l \geqslant 2$ and a Borel subgroup B coincides with its unipotent subgroup U. Let P be a minimal parabolic subgroup over B and let $P = U_P L$ be a Levi decomposition of P. Then $L \cong S_3$ and $|U : U_P| = 2$. Set $S = U_P L_3$. Since $S <\cdot P$ we have $S \in (\mathrm{Co}\, G)_2$ and, by [3], S has the following property:

$$\text{if } S \leqslant M <\cdot G \text{ then } P \leqslant M.$$

It follows that if G is in D-situation with $\bar{G}$, then by Lemma 1.6 the minimal subgroups of $\bar{U}_P$ generate the cyclic group $\bar{B}$ of square-free order. We are indebted to B. Stellmacher for the following argument.

Let $\tilde{P}$ be the maximal parabolic subgroup of G such that $P \cap \tilde{P} = U$, and let x be an involution of L not contained in $\tilde{P}$. Now $U_P \leqslant \tilde{P} \cap \tilde{P}^x$ and $x \in \mathcal{N}(\tilde{P} \cap \tilde{P}^x)$. Moreover $x \notin \tilde{P} \cap \tilde{P}^x$. Hence $[G/\tilde{P} \cap \tilde{P}^x]$ contains at least the three distinct elements $\tilde{P}$, $\tilde{P}^x$ and $\langle x \rangle (\tilde{P} \cap \tilde{P}^x)$. This is a contradiction to the fact that $[G/M \cap M_1]$ has just 4 elements for any two distinct maximal subgroups M and M_1 above U_P.

3 Simple twisted groups of Lie type

We begin with twisted groups G^1 obtained from Chevalley groups $G = \mathcal{L}(\mathcal{K})$ whose Dynkin diagram has only single bonds. We denote by K_0 the subfield of fixed points of K under the corresponding field automorphism. Therefore $|K_0| = q$ and $|K| = q^2$ for types ${}^2A_l$, ${}^2D_l$, ${}^2E_6$, and $|K| = q^3$ for ${}^3D_4$, where q is some prime power.

Suppose that we have $G^1 = {}^2A_l(q^2) \cong \mathrm{PSU}_{l+1}(q^2)$ with $l \geqslant 2$. Here we are using [1, Theorem 14.5.1]. We consider separately the cases when $l = 2, 3, 4, 5$ and $l \geqslant 6$.

Let $l = 2$. Then $H^1 \cong K^*/C_d$, where $d = (3, q+1)$. We have $B^1 <\cdot\, G^1$, H^1 cyclic, and since H^1 is dual to $\overline{U}^1/\bar{B}^1$, it is a p-group. Suppose that q is even. Then $q > 2$ since otherwise G^1 is not simple. For $d = 1$ we have $q^2 - 1 = p^a$, an equation with no solution. For $d = 3$, we have $3 \mid (q+1)$, and $q+1 > 3$ because $q > 2$. But then since $(q^2-1)/3 = p^a$, we must have $p \mid (q-1)$ and $p \mid (q+1)$, contradicting the fact that q is even.

Therefore we suppose that q is odd. For $d = 1$, $q^2 - 1 = p^\alpha$ implies $p = 2$, $q = 3$, that is, $G_1 = \mathrm{PSU}_3(9)$ and $H^1 \cong C_8$. For $d = 3$, $(q^2-1)/3 = p^\alpha$ implies $p = 2$, $q - 1 = 2^m$ and $q + 1 = 3.2^n$, $m > 0$ and $n > 0$. If $m = 1$, then $q = 3$, and $3 \nmid (q+1)$. Hence $n = 1$, so that $q + 1 = 6$ and $q = 5$. It follows that $G^1 = \mathrm{PSU}_3(25)$ and $H^1 \cong C_8$. Thus we have to consider $\mathrm{PSU}_3(9)$ and $\mathrm{PSU}_3(25)$.

Let $G^1 = \mathrm{PSU}_3(9)$. We refer to [2, p. 14]. Set $S = (G^1)_7$. There exists only one maximal subgroup above S, namely $M \cong \mathrm{PSL}_2(7)$. We have $S <\cdot\, \mathcal{N}(S) = B <\cdot\, M$, and $[G^1/S]$ is a chain of length 3. By Proposition 1.5, $\bar{G}^1$ is a Frobenius group of order p^3q^α, where $\alpha > 1$. Now G^1 has $N = 190$ maximal subgroups, and so we have $N - 1 = 3^3.7$. If q is a prime divisor of $N - 1$, then $N = 1 + q + \cdots + q^\alpha$ has no solution.

Now let $G^1 = \mathrm{PSU}_3(25)$, so that $|G^1| = 2^4.3^2.5^3.7$. Set $S = (G^1)_2$. There exist four maximal subgroups M_i containing S, and one checks that $\mathcal{N}_{M_i}(S) = S$ for each i, and $S <\cdot\, M_i$ for one i, say $i = 1$. By Corollary 1.4, since $\overline{S}$ has only four minimal subgroups, we have $|\overline{S}| = 3^2.2^2$, with $\overline{S}_3$ cyclic acting irreducibly on the four-group $\overline{S}_2$. Hence we have a duality between $[G^1/S]$ and $[\overline{S}/1]$, by (2). Since $M_1 \cap M_2 = S$ and $\overline{M}_1 \cup \overline{M}_2 < \overline{S}$, we have a contradiction.

Now let $l = 3$. Then $H^1 \cong (K^* \times K_0^*)/C_d$, where $d = (4, q+1)$. Here G^1 has two classes of maximal parabolic subgroups. Again we distinguish cases according to the parity of q. Suppose that $q = 2^n$. In this case $d = (4, q+1) = 1$ and H^1 is cyclic if and only if $q = 2$. But then $\mathrm{PSU}_3(4) \cong B_2(3)$, which has already been excluded in Section 2. We are left with $n > 1$. But by Proposition 1.1, $\overline{U}^1/\bar{B}^1$ is a P-group of order pq, with $p > q$, since $|H^1|$ is odd. Hence $(q-1)^2(q+1) = p^2$, a contradiction.

Now suppose that q is odd. Consider first the case when $d = 2$. Then $(H^1)_2 \geqslant (C_8 \times C_4)/C_2$ since $4 \nmid (q+1)$. Hence $(H^1)_2$ is neither cyclic nor elementary abelian, and this is a contradiction by Proposition 1.1. Therefore we are left with $d = 4$. Suppose that $q > 3$. Then there exists an odd prime p such that $p \mid (q-1)$. Also $(H^1)_p$ is elementary abelian of order p^2 by Proposition 1.1. But then $\overline{U}^1/\bar{B}^1$ is a P-group, in contradiction to $H_2^1 \neq 1$. We are therefore left with $q = 3$, that is, with the group $G^1 = \mathrm{PSU}_4(9)$.

Let P be a maximal parabolic subgroup isomorphic to $3_+^{1+4}.2S_4$ (see [2, p. 52]). Hence $[P/U_P] \cong 2S_4$, so that $[P/U^1] \cong [2S_4/(2S_4)_3]$, where $(2S_4)_3 \cong C_3$ and $(2S_4)_2 \cong Q_{16}$. But now in the interval $[2S_4/(2S_4)_3]$ there is a unique minimal subgroup R, and $[2S_4/R]$ is a diamond, by which we mean the Hasse diagram labelled B_4 in [11, p. 5]. Since $[2S_4/(2S_4)_3]$ is dual to $[\overline{U}^1/\bar{P}]$, the group $\overline{U}^1/\bar{P}$ has only one maximal subgroup, but it is not cyclic. This contradiction completes the case when $l = 3$.

Suppose that $l = 4$. Then $H^1 \cong (K^* \times K^*)/C_d$, where $d = (5, q+1)$. Here G^1 has two classes of maximal parabolic subgroups. If q is odd, then $(H^1)_2$ is neither cyclic nor the four-group and we have a contradiction. Therefore we may assume that q is even. Since there are only two minimal parabolic subgroups above B^1 and $|H^1|$ is odd, H^1 must be either cyclic, or isomorphic to $C_p \times C_p$, with p odd. Hence we are left with $q = 2$, and $G^1 = U_5(2)$ (see [2, p. 72]).

Set $S = (G^1)_{11}$. Then there is a unique maximal subgroup M containing S with $M \cong \mathrm{PSL}_2(11)$. It follows that $[G^1/S]$ is the chain

$$S <\cdot\, B = \mathcal{N}(S) <\cdot\, M <\cdot\, G^1$$

and $\bar{G}^1$ is a Frobenius group of order $p^3 q^\alpha$, with $\alpha > 1$, by Proposition 1.5. Now the number of maximal subgroups of G^1 is $N = 26\,302$, and $N - 1 = 3.11.797$. If q is a prime divisor of $N - 1$, then the equation $N = 1 + q + \cdots + q^\alpha$ has no solution, and we have a contradiction.

Now suppose that $l = 5$. Here $H^1 \cong (K^* \times K^* \times K_0^*)/C_d$, where $d = (6, q+1)$. If q is odd, then $(H^1)_2$ is neither cyclic nor the four-group and this is a contradiction. Thus we suppose that q is even. Suppose that $q > 2$. Then there is a prime p dividing $q - 1$. Hence H^1 contains a subgroup isomorphic to $C_p \times C_p \times C_p$, and this is also a contradiction. We are left with $q = 2$, that is with $G^1 = U_6(2)$. We postpone the treatment of this case.

Finally suppose that $l \geqslant 6$. Then $H^1 \cong R/C_d$, where $R \geqslant K^* \times K^* \times K^*$ and $d = (l+1, q+1)$. If q is odd, then $(H^1)_2$ is neither cyclic nor the four-group and this is a contradiction. Therefore we suppose that q is even. If $q > 2$, then there is a prime p dividing $q - 1$. Hence H^1 contains a subgroup isomorphic to $C_p \times C_p \times C_p$, which is a contradiction. We are left with $q = 2$. Then H^1 contains a subgroup isomorphic to $C_3 \times C_3 \times C_3$, which again is a contradiction.

Suppose that $G^1 = {}^2D_l(K)$. Then

$$H^1 \cong (K^* \times \underbrace{K_0^* \times \cdots \times K_0^*}_{l-2})/N,$$

where $N = 1$, C_2 or $C_2 \times C_2$, and $|N| = (4, q^l + 1)$. If q is odd, then $(H^1)_2$ is non-trivial and is neither cyclic nor the four-group. This is a contradiction, so assume that q is even. Then $N = 1$, and H^1 is 2-generated if and only if $q = 2$, that is, $G = {}^2D_l(4)$. We shall deal with this case later.

Suppose that $G^1 = {}^3D_4(q^3)$. Then $H^1 \cong K^* \times K_0^*$. We consider first the case when q is odd. If $4|(q-1)$, then $(H^1)_2$ is neither cyclic nor elementary abelian, which is a contradiction. Hence $q - 1 = 2m$, where m is odd. Since G^1 has two minimal parabolic subgroups, we have

$$\bar{B}^1 \cong C_2 \times C_r, \quad \bar{U}^1/\bar{B}^1 \cong \bar{V} \times \bar{C},$$

where $\bar{V}$ is the four-group and $\bar{C}$ is cyclic of odd order r^α with $\alpha \geqslant 0$. Hence either

$H^1 \cong V$ or $H^1 \cong V \times C$, where C is a cyclic group of odd order. But then $q - 1 = 2$, since otherwise C is not cyclic. Therefore we are left with $q = 3$, that is, $G^1 = {}^3D_4(3^3)$. In this case, $\bar{U}_1 = Q_8 C_r$. Consider $B^1 <\cdot P <\cdot G^1$ such that $P^\delta = C_r$. Then $[P/U^1]$ is dual to $[Q_8/1]$, but $B^1/U^1 \cong V \times C_{13}$, which is a contradiction.

We now consider the case when q is even. If $q > 2$, then a prime p divides $q - 1$. Thus $H^1 \geqslant C_p \times C_p$, so that $|\bar{U}^1/\bar{B}^1| = pr$, with $p > r$. But then $H^1 = C_p \times C_p$, and we have a contradiction on considering the order of H^1. Therefore we may assume that $q = 2$, that is, $G^1 = {}^3D_4(8)$ (see [2, p. 89]). Set $S = G_{13}$. There exists a unique maximal subgroup M containing S, namely $M = \mathcal{N}(S) = S \rtimes C_4$. Hence $[G/S]$ is a chain of length 3 and $\bar{G}^1$ is a Frobenius group of order $p^3 q^\alpha$, with $\alpha > 1$. Now the number of maximal subgroups of G^1 is $N = 5\,565\,964$, and $N - 1 = 3.13.43.3319$. If q is a prime divisor of $N - 1$, the equation $N = 1 + q + \cdots + q^\alpha$ has no solution, and we have a contradiction, by Proposition 1.5.

Suppose that $G^1 = {}^2E_6(q^2)$. Now $H^1 \cong (K^* \times K^* \times K_0^* \times K_0^*)/C_d$ and $d = (3, q+1)$. If q is odd, then $(H^1)_2$ is neither cyclic nor the four-group, which is a contradiction. Thus assume that q is even. If $q > 2$, then there is a prime p dividing $q - 1$. Hence H^1 contains a subgroup isomorphic to $C_p \times C_p \times C_p$, which again is a contradiction. We are left with $q = 2$, that is, $G^1 = {}^2E_6(4)$.

We postpone consideration of this group until later and continue with the Suzuki and Ree groups.

Suppose that $G^1 = {}^2B_2(2^{2m+1})$ with $m \geqslant 1$. We refer to [5, §7]. Here the interval $[G^1/H^1]$ contains only three maximal subgroups, namely B^1, $\mathcal{N}(H^1) = N^1$ and B^1_{op}. Moreover $|N^1 : H^1| = 2$, so that, by Corollary 1.4, $\bar{H}^1$ is a four-group dual to $[G^1/H^1]$. This is a contradiction, since H^1 is not maximal in B^1.

Suppose that $G^1 = {}^2G_2(3^{2m+1})$. This group is simple for $m \geqslant 1$ (see [1, Theorem 14.4.1]). Also H^1 ($\cong K^*$) is cyclic of order $3^{2m+1} - 1$, and $B^1 <\cdot G^1$, by [5, p. 292]. So $\bar{U}_1$ is a cyclic p-group of order $p^{\alpha+1}$. Therefore $3^{2m+1} - 1 = r^\alpha$ for some prime r, so that $r = 2$, which implies $\alpha = 1$ and $m = 0$. This is a contradiction.

Suppose that $G^1 = {}^2F_4(2^{2n+1})$. This group is simple for $n \geqslant 1$ (see [1, p. 268]). In this case, G^1 has two classes of parabolic subgroups and $H^1 \cong K^* \times K^*$. Hence $\bar{U}^1/\bar{B}^1$ is a P-group and $\bar{U}^1 = C_{p^2} C_{t^2}$, $p = 2^{2n+1} - 1$ and $Z = Z(\bar{U}^1) = C_t$. Let P_1, P_2 be the minimal parabolic subgroups above B^1. Then, by [9], we have $[P_1/U^1] \cong [P_2/U^1]$. This is a contradiction, since $[\bar{U}^1/C_t] \not\cong [\bar{U}^1/C_p]$.

To conclude our consideration of the twisted groups, we deal with the cases which were excluded above, namely ${}^2A_5(4) \cong U_6(2)$, ${}^2D_l(4)$ and ${}^2E_6(4)$. We need a lemma similar to the result used in the case of Chevalley groups over a field of two elements. For this purpose we introduce some notation.

Let G be a simple Chevalley group over K with $|K| = 4$. Let Φ be the set of roots, Φ^+ a set of positive roots and Π the corresponding fundamental system. Let U be the maximal unipotent subgroup corresponding to Π, and B the corresponding Borel subgroup. For every root α, the root subgroup X_α is isomorphic to K. The subgroups B^1, N^1 form a (B, N)-pair of the twisted group G^1, by [1, Theorem 13.5.4]. Let Φ^1 be the corresponding root system of G^1. Fix $r \in \Pi$ such that $r\rho = r$, where ρ is the symmetry of the Dynkin diagram we are considering. By [1, Proposition 13.6.3], we have $X_r^1 \cong K_0$. In particular it follows that

$$\langle X_r^1, X_{-r}^1 \rangle \cong \mathrm{SL}_2(2) \cong S_3.$$

Put $P = \langle B^1, X_{-r}^1 \rangle$. Then P is a minimal parabolic subgroup containing B^1, $U_P = \langle X_s^1 \mid s \in (\Phi^1)^+ \backslash \{r\} \rangle$ is its unipotent radical and $P = U_P L$, with $L = H^1 \langle X_r^1, X_{-r}^1 \rangle$, is a Levi decomposition of P. Since $\langle X_r^1, X_{-r}^1 \rangle \cong S_3$ and $[H^1, X_{\pm r}^1] = 1$, we must have $H^1 \trianglelefteq L$ and $L/H^1 \cong S_3$. Then $H^1 U_P \trianglelefteq P$, and $P/H^1 U_P \cong S_3$. Moreover, P contains $S = U_P L_3$ as a subgroup of index 2, and $S \cap B^1 = H^1 U_P$.

Lemma 3.1. *Every maximal subgroup of G^1 containing S also contains P.*

Proof. We use the corresponding argument in [3], replacing U by B^1.

Now, in order to exclude the remaining cases, we may proceed as we did for the simple Chevalley groups over the field of two elements, replacing U with B^1 and U_P by $H^1 U_P$. Therefore we have proved the following statement:

no finite simple twisted group of Lie type admits a D-situation.

Finally in this section we consider the simple Tits group. Let $G^1 = ({}^2F_4(2))'$. We refer to [2, p. 74]. There exists a maximal parabolic subgroup P such that $B = U <\cdot P = U_P L$, where $P/U_P \cong L \cong S_3$. Set $S = U_P L_3$. Then P is the unique maximal subgroup containing S. By Lemma 1.6, $\overline{S}$ is cyclic of order r^2, and $\overline{U}_P$ is an irreducible Zassenhaus group of order 12, since $\overline{U}_P/\overline{P}$ is isomorphic to S_3. Let $\tilde{P}$ be the other parabolic subgroup above B, and let x be an involution of L not contained in $\tilde{P}$. Now $U_P \leqslant \tilde{P} \cap \tilde{P}^x$ and $x \in \mathcal{N}(\tilde{P} \cap \tilde{P}^x)$. Moreover $x \notin \tilde{P} \cap \tilde{P}^x$. Hence $[G/\tilde{P} \cap \tilde{P}^x]$ contains at least the three distinct elements $\tilde{P}$, $\tilde{P}^x$ and $\langle x \rangle (\tilde{P} \cap \tilde{P}^x)$. This is a contradiction to the fact that $[G/M \cap M_1]$ has just four elements for any two distinct maximal subgroups M and M_1 above U_P.

4 The sporadic groups

This section is devoted to showing that no sporadic simple group admits a D-situation. We consider all 26 of these groups in turn. In each case we include the Atlas reference.

1. $G = M_{11}$ ([2, p. 18]). Set $S = G_{11}$. Then

$$S \underset{5}{<\cdot} \mathcal{N}(S) <\cdot M \cong L_2(11) <\cdot G,$$

and $[G/S]$ is a chain of length 3. Here the integer below the inclusion sign denotes index. Now the number of maximal subgroups of G is $N = 309$ and $N - 1 = 2^2.7.11$. If q is a prime divisor of $N - 1$, then the equation $N = 1 + q + \cdots + q^\alpha$ has no solution and we have a contradiction, by Proposition 1.5.

2. $G = M_{12}$ ([2, p. 31]). Again set $S = G_{11}$. There exist three conjugacy classes of maximal subgroups containing S, represented by

$$M_1 = M_{11}, \quad M_2 = \tilde{M}_{11}, \quad M_3 \cong L_2(11).$$

Moreover $\mathcal{N}(S) = S \rtimes C_5$ and the maximal subgroups containing it are exactly those indicated above. Since $N = \mathcal{N}(S)$ is maximal in M_3, but is not maximal in M_{11}, the group $\overline{N}$ cannot be metacyclic. Hence it is a Frobenius group of order pq^α with $\alpha > 1$, by Proposition 1.3, and this is a contradiction, since $\overline{N}$ contains more than three minimal subgroups.

3. $G = M_{22}$ ([2, p. 39]). Again set $S = G_{11}$. Then

$$S \underset{5}{<\cdot} \mathcal{N}(S) <\cdot M \cong L_2(11) <\cdot G,$$

and $[G/S]$ is a chain of length 3. Now the number of maximal subgroups of G is $N = 2300$ and $N - 1 = 11^2.19$. If q is a prime divisor of $N - 1$, then the equation $N = 1 + q + \cdots + q^\alpha$ has no solution, and we have a contradiction by Proposition 1.5.

4. $G = M_{23}$ ([2, p. 71]). Now set $S = G_{23}$. Then

$$S \underset{11}{<\cdot} \mathcal{N}(S) <\cdot G,$$

and $[G/S]$ is a chain of length 2. Now the number of maximal subgroups of G is $N = 44\,413$ and $N - 1 = 2^2.3.3701$. If q is a prime divisor of $N - 1$, the equation $N = 1 + q + \cdots + q^\alpha$ has no solution, and we have a contradiction, again by Proposition 1.5.

5. $G = M_{24}$ ([2, p. 96]). Again set $S = G_{23}$. There exist two maximal subgroups from different conjugacy classes containing S, namely $M_1 \cong M_{23}$ and $M_2 \cong L_2(23)$. Both contain $\mathcal{N}(S)$ and we have

$$[G/S] = S <\cdot \mathcal{N}(S) <\cdot M_{23}, L_2(23) <\cdot G.$$

Hence S is not the intersection of cocyclic subgroups, and this is a contradiction.

6. $G = J_1$ ([2, p. 36]). Now set $S = G_{19}$. Then there exists a unique maximal subgroup M above S, namely $M = \mathcal{N}(S) \cong S \rtimes C_6$. Therefore $\overline{S}$ has a unique minimal subgroup and is not cyclic. Hence $\overline{S}$ is a quaternion group, which has three maximal subgroups, which is again a contradiction.

7. $G = J_2$ ([2, p. 42] and [4, p. 486]). Set $S = G_7$. There exist two maximal subgroups from different conjugacy classes containing S, namely $M \cong U_3(3)$ and $L \cong \mathrm{PGL}_2(7)$. In fact there are exactly three maximal subgroups above S, namely M, M_1 and L, where M_1 is a conjugate of M. We have

$$M \cap L = M_1 \cap L = F \cong L_2(7) \quad \text{and} \quad |L : F| = 2.$$

Let $N = \mathcal{N}(S)$ and $B = M \cap N$. Then $N \leqslant L$ and $|N : B| = 2$. The interval $[L/B]$ is a diamond. In particular it follows that N is cocyclic and B is maximal in N. By Lemma 1.2 and Proposition 1.3, we obtain information on the structure of $\bar{B}$.

Suppose that $\bar{B}$ is a p-group. We have $\bar{F} <\cdot \bar{B}$ and $\bar{N} <\cdot \bar{B}$, and hence

$$\bar{L} = \bar{F} \cap \bar{N} \trianglelefteq \bar{B}.$$

But $[\bar{B}/\bar{L}]$ is a diamond and this is a contradiction. Suppose that $\bar{B}$ is metacyclic. Then $|\bar{B}| = p^2q$, with $|\bar{N}| = p^2$. But $\bar{F}$ has order pq and has three minimal subgroups, so that $p = q = 2$, and this is a contradiction. Thus we are left with $\bar{B} = \bar{N}Q$, a group of order p^2q^α where $\alpha > 1$, with Q elementary abelian of order q^α. But then $\bar{B}$ has at least four minimal subgroups, which is a contradiction.

8. $G = J_3$ ([2, p. 82]). Set $S = G_{19}$. There exist two maximal subgroups M_1, M_2 containing S from different conjugacy classes, both isomorphic to $L_2(19)$. We have

$$[G/S] = S <\cdot \mathcal{N}(S) <\cdot M_1, M_2 <\cdot G.$$

Hence S is not the intersection of cocyclic subgroups.

9. $G = \mathrm{HS}$ ([2, p. 80]). This time set $S = G_5$. There are two maximal subgroups M_1, M_2 containing S from different conjugacy classes, namely

$$M_1 \cong U_3(5) : 2, \quad M_2 \cong U_3(5) : 2.$$

We have $M_1 \cap M_2 = N = \mathcal{N}(S)$. Let U_i be the subgroup of M_i isomorphic to $U_3(5)$ for $i = 1, 2$. Then $U_1 \cap N = U_2 \cap N = X$ has index 2 in N. Since X is maximal in the cocyclic subgroup N, we have $\overline{N} <\cdot \overline{X}$. Moreover $\overline{N} \trianglelefteq \overline{X}$, since $\overline{N}$ is the subgroup generated by the minimal subgroups of $\overline{X}$. Since $\overline{X}$ has only two minimal subgroups, we have $|\overline{X}| = p^\alpha q^\beta$, with $\alpha \geqslant 2$ and $\beta \geqslant 2$ since both $\overline{U}_1$ and $\overline{U}_2$ are cyclic. On the other hand we have $|\overline{X} : \overline{N}| = r$ for some prime r and $|\overline{N}| = pq$. Hence $|\overline{X}| = pqr$, which is a contradiction.

10. $G = \mathrm{McL}$ ([2, p. 100]). Now set $S = G_{11}$. There exist three maximal subgroups M_i containing S from different conjugacy classes. Here $M_1 \cong M_{22}$, $M_2 \cong M_{11}$, $M_3 \cong M_{22}$. In each M_i there exists a unique maximal subgroup containing S and also $\mathcal{N}(S)$ (each isomorphic to $L_2(11)$). It follows that S is not the intersection of cocyclic subgroups.

11. $G = \mathrm{Suz}$ ([2, p. 131]). Here set $S = G_{13}$. There exist four maximal subgroups M_i containing S from different conjugacy classes:

$$M_1 \cong G_2(4), \quad M_2 \cong L_3(3) : 2, \quad M_3 \cong L_3(3) : 2, \quad M_4 \cong L_2(25).$$

Let $B = \mathcal{N}_{M_2}(S)$. We have the following inclusions:

$$B <\cdot L \cong L_2(13) <\cdot M_1 <\cdot G, \quad B <\cdot M_2 <\cdot G,$$
$$B < T \cong U_3(4) : 2 <\cdot M_1 <\cdot G.$$

In particular L and T are cocyclic and $L \cap T = B$. It follows that $\bar{B} = \langle \bar{L}, \bar{T} \rangle$ and $\overline{M}_2 <\cdot \bar{B}$. Now $\bar{B}$ is not metacyclic, since the Dedekind chain condition is not satisfied, and hence it is a Frobenius group of order pq^α, where $\alpha > 1$, with an elementary abelian Sylow q-subgroup, by Proposition 1.3. We have $|\overline{M}_2| = p$ and $|\overline{M}_1| = q$ since $\overline{M}_1$ is not maximal in $\bar{B}$. It follows that $\bar{L}$ and $\bar{T}$ are both contained in $\bar{B}_q$, which is a contradiction, since $\bar{B} = \langle \bar{L}, \bar{T} \rangle$.

12. $G = \mathrm{He}$ ([2, p. 104]). Set $S = G_{17}$. There exists a unique maximal subgroup M containing S, namely $M = H : 2$, with $H \cong S_4(4)$. In H there are exactly two maximal subgroups L and L^* containing S and both are isomorphic to $L_2(16) : 2$. Let $N_1 = \mathcal{N}_H(S)$. Then we have

$$N_1 <\cdot L, \quad N_1 <\cdot L^*, \quad N_1 = L \cap L^*.$$

Since $\overline{N}_1$ has a unique minimal subgroup it follows that it is a generalized quaternion group. Now we have

$$\bar{L} <\cdot \overline{N}_1, \quad \bar{L}^* <\cdot \overline{N}_1, \quad \overline{M} <\cdot \overline{H} = \bar{L} \cap \bar{L}^* \trianglelefteq \overline{N}_1,$$

so that $\overline{N}_1/\overline{H}$ is a four-group, and this is a contradiction because $\overline{N}_1/\overline{H}$ has only two minimal subgroups, $\bar{L}/\overline{H}$ and $\bar{L}^*/\overline{H}$.

13. $G = \mathrm{Ru}$ ([2, p. 126]). Now set $S = G_{29}$. There exists a unique maximal subgroup $M \cong L_2(29)$ containing S as well as $N = \mathcal{N}(S) \cong S \rtimes C_{14}$. We conclude the argument as for $G = J_1$.

14. $G = \mathrm{Co}_1$ ([2, p. 180] and [6, p. 304]). Here set $S = G_{13}$. There exist two maximal subgroups M_i containing S from different conjugacy classes, namely

$$M_1 \cong (3.\mathrm{Suz}) \rtimes C_2, \quad M_2 \cong (A_4 \times G_2(4)) \rtimes C_2.$$

We have $\mathcal{N}(S) \leqslant M_2$ and $\mathcal{N}(S) \cong ((S \rtimes C_6) \times A_4) \rtimes C_2$. Let $H = S \times V_4 \leqslant \mathcal{N}(S)$ ($V_4 \leqslant A_4$). We claim that $H \not\leqslant M_1$. In fact we know that $|\mathcal{N}_{\mathrm{Suz}}(S) : S| = 6$ and $D_{26} \leqslant \mathcal{N}_{\mathrm{Suz}}(S)$. Hence no involution in $3.\mathrm{Suz}$ centralizes S. Therefore $V_4 \not\leqslant M_1$, since V_4 centralizes S. It follows that M_2 is the unique maximal subgroup of G above H. Hence $\overline{H}$, having a unique minimal subgroup, is a cyclic p-group or a generalized quaternion group. But $[G/H]$ is not a chain, since $((S \rtimes C_6) \times V_4)/H \cong C_6$. On the other hand, if $\overline{H}$ is generalized quaternion, then all its subgroups are 2-generated, and $\delta|[G/H]$ is a duality onto $\overline{H}$. But $\overline{H}$ has three maximal subgroups, which is a contradiction, since in $[G/H]$ there are only two minimal subgroups.

15. $G = \mathrm{Co}_2$ ([2, p. 154]). Set $S = G_{23}$. There exists a unique maximal subgroup $M \cong M_{23}$ containing S, and $[G/S]$ is the chain

$$S <\cdot \mathcal{N}(S) <\cdot M <\cdot G,$$

where $\mathcal{N}(S) \cong S \rtimes C_{11}$. Hence $\bar{G}$ is a Frobenius group of order p^3q^α, with $\alpha > 1$.

Now G has $N = 3\,581\,796\,533$ maximal subgroups, and $N - 1 = 2^2.23.101.385471$. If q is a prime divisor of $N - 1$, one checks that $N = 1 + q + \cdots + q^\alpha$ has no solution.

16. $G = \mathrm{Co}_3$ ([2, p. 134]). Again set $S = G_{23}$. There exists a unique maximal subgroup M above S and it is isomorphic to M_{23}. Also $[G/S]$ is the chain

$$S <\cdot \mathcal{N}(S) <\cdot M <\cdot G,$$

where $\mathcal{N}(S) \cong S \rtimes C_{11}$. So $\bar{G}$ is a Frobenius group of order $p^3 q^\alpha$, with $\alpha > 1$. Now G has $N = 424\,818\,005$ maximal subgroups, and $N - 1 = 2^2.13^2.23.89.307$. If q is a prime divisor of $N - 1$, then $N = 1 + q + \cdots + q^\alpha$ has no solution.

17. $G = \mathrm{Fi}_{22}$ ([2, p. 156]). Here set $S = G_{11}$. There are three maximal subgroups containing S from different conjugacy classes, namely

$$M_1 \cong 2.U_6(2), \quad M_2 \cong 2^{10}.M_{22}, \quad M_3 \cong M_{12}.$$

We have $|\mathcal{N}_{M_i}(S)| = 5.11$ for $i = 2, 3$ and $\mathcal{N}_{M_1}(S) = \mathcal{N}(S)$ has order 2.5.11. Now $[G/\mathcal{N}(S)]$ has only one maximal subgroup M_1, and M_1 has at least five maximal subgroups above $T = \mathcal{N}(S)$, namely

$$2 \cdot M_{12}, \quad 2 \cdot (S_3 \times U_4(2)), \quad 2 \cdot U_5(2), \quad 2 \cdot M_{22}, \quad 2 \cdot M_{22}.$$

Since $\bar{T}$ is a generalized quaternion group, it has only three groups covering $\overline{M_1}$, and this is a contradiction.

18. $G = \mathrm{Fi}_{23}$ ([2, p. 177] and [6, p. 304]). Now set $S = G_{23}$. There are two maximal subgroups containing S from different conjugacy classes, namely $M_1 \cong 2^{11} \cdot M_{23}$ and $M_2 \cong L_2(23)$. Let $B = \mathcal{N}_{M_2}(S)$. Then we know that $B = S \rtimes C_{11}$, and $S <\cdot B <\cdot M_2$. On the other hand, let $N \trianglelefteq M_1$ be such that $N \cong 2^{11}$ and $M_1/N \cong M_{23}$. Then SN/N is a Sylow 23-subgroup of M_1/N, and we know that $\mathcal{N}_{M_1/N}(SN/N) \cong SC_{11}$. Replacing M_1 and M_2 by conjugates if necessary, it follows that we may assume that $\mathcal{N}(S) \leqslant M_1$. In particular M_1 is the unique maximal subgroup conjugate to M_1 containing S.

Since $B <\cdot M_2 <\cdot G$, by Proposition 1.3, $\bar{B}$ is either metacyclic or it is a Frobenius group. But we have $B < NB <\cdot M_1 <\cdot G$, so that $\bar{B}$ is not metacyclic. Therefore $\bar{B}$ is a Frobenius group of order pq^α with $\alpha \geqslant 2$. It follows that $\overline{NB}$ is elementary abelian of order q^β with $\beta \geqslant 2$, since $\overline{NB}$ is not cyclic. Let M be a maximal subgroup containing NB. Then M is conjugate either to M_1 or to M_2. By order considerations, it must be conjugate to M_1, and therefore $M = M_1$. This is a contradiction, since $\overline{NB}$ has more than one minimal subgroup.

19. $G = \mathrm{Fi}_{24}'$ ([2, p. 200] and [6, p. 304]). Set $S = G_{29}$. There exists a unique maximal subgroup M of G containing S, namely $M = \mathcal{N}(S) = S \rtimes C_{28}$. Hence $\bar{S}$ is a generalized quaternion group, and we have a contradiction.

20. $G = \mathrm{O'N}$ ([2, p. 132]). Set $S = G_{31}$. There are two maximal subgroups M_i containing S. Both are isomorphic to $L_2(31)$ and contain $N = \mathcal{N}(S)$. We have

$$N = S \rtimes C_{15}, \quad N <\cdot M_1, \quad N <\cdot M_2, \quad M_1 \cap M_2 = N.$$

Let $X \in [N/S]$ be the subgroup of index 5 in N. Then X is not the intersection of cocyclic subgroups, since $[M_1/X]$ and $[M_2/X]$ are chains and X is not cocyclic.

21. $G = \mathrm{Ly}$ ([2, p. 174]). This time set $S = G_{67}$. There exists a unique maximal subgroup M containing S, namely $M = \mathcal{N}(S) = S \rtimes C_{22}$, and we have the usual contradiction.

22. $G = J_4$ ([2, p. 190] and [6, p. 304]). Here set $S = G_{43}$. There exists a unique maximal subgroup M containing S, namely $M = \mathcal{N}(S) = S \rtimes C_{14}$, and we conclude the argument as before.

23. $G = \mathrm{HN}$ ([2, p. 164]). Now set $S = G_{19}$. There exists a unique maximal subgroup M containing S, namely $M \cong U_3(8) : 3$. Let $N_2 = \mathcal{N}(S)$, H be the normal subgroup of M isomorphic to $U_3(8)$ and $N_1 = \mathcal{N}_H(S)$. Then $N_1 \lhd N_2 <\cdot M$. Suppose that $N_1 < X < M$ with $H \neq X \neq N_2$. Then $H \cap X = N_1 \trianglelefteq X$, so that $N_1 \trianglelefteq \langle X, N_2 \rangle = M$. Hence $N_1 \trianglelefteq H$, which is a contradiction, since H is simple. But then $\overline{N}_1$ has a unique minimal subgroup but it is neither cyclic nor a quaternion group, and this is a contradiction.

24. $G = \mathrm{Th}$ ([2, p. 176] and [6, p. 304]). Set $S = G_{31}$. There are two maximal subgroups containing S from different conjugacy classes, namely

$$M_1 = \mathcal{N}(S) \cong S \rtimes C_{15}, \quad M_2 \cong 2^5 \cdot L_5(2).$$

Also M_2 has a minimal normal subgroup N with $M_2/N \cong L_5(2)$. Set $T = \mathcal{N}_{M_2}(S)$. Then $|T : S| = 5$ and $|M_1 : T| = 3$. Consider the subgroup NT of M_2. Thus $T < NT < M_2$. Since $T <\cdot M_1$, the Dedekind chain condition does not hold in $\overline{T}$ and $\overline{T}$ is a Frobenius group of order pq^α with $\alpha > 1$ and $|\overline{M}_1| = p$. But there are only four maximal subgroups of G containing T, namely M_1 and 3 subgroups conjugate to M_2. This is a contradiction, since $\overline{T}$ has at least seven minimal subgroups.

25. $G = B$ ([16]). Set $S = G_{47}$. There exists a unique maximal subgroup containing S, namely $M = \mathcal{N}(S) = S \rtimes C_{23}$. It follows that $[G/S]$ is a chain of length 2. Hence, by Proposition 1.5, $\overline{G}$ is a Frobenius group of order p^2q^α with $\alpha > 1$. There exists in G a maximal subgroup M_1 of order $2^5.3.5.31$. Hence $M \cap M_1 = 1$, so that $\overline{G} = \langle \overline{M}, \overline{M}_1 \rangle$, which is a contradiction, since $\langle \overline{M}, \overline{M}_1 \rangle$ is contained in a subgroup of order pq^α of $\overline{G}$.

26. $G = M$ ([2, p. 220] and [6, p. 305]). Here

$$|G| = 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.47.59.71.$$

We set $S = G_{59}$. There exists a maximal subgroup $M = \mathcal{N}(S) = S \rtimes C_{29}$. Assume there exists another maximal subgroup M_1 containing S. Then M_1 and M are not conjugate. It follows that $\mathcal{N}_{M_1}(S) = S$ and, by a theorem of Burnside, there exists a normal complement K of S in M_1. For each prime p dividing $|K|$, S normalizes a Sylow p-subgroup of M_1 and acts faithfully on it since $\mathcal{C}(S) = S$. Therefore $p \neq 17, 19, 23, 29, 31, 47, 71$. Let P be one of these Sylow p-subgroups, so that $p \in \{2, 3, 5, 7, 11, 13\}$. Consider the chief factors of SP below P. Then the smallest values of n for which $\mathrm{GF}(p^n)$ has a 59th root of 1 are 29 and 58, and $n = 58$ for $p = 2$. Therefore S cannot act faithfully, and we have a contradiction. Hence M is the unique maximal subgroup containing S.

There exists a maximal subgroup $M_1 = \mathcal{N}(G_{71}) \cong G_{71} \rtimes C_{35}$, and clearly we have $M \cap M_1 = 1$. Then we may conclude the argument as for $G = B$.

We have completed the examination of all sporadic groups. Taking into account the results from the previous sections we have therefore proved our Theorem.

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